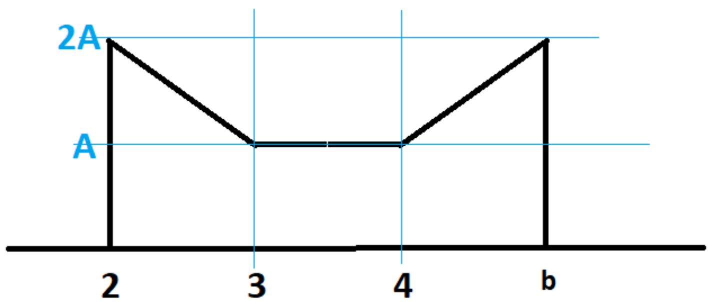
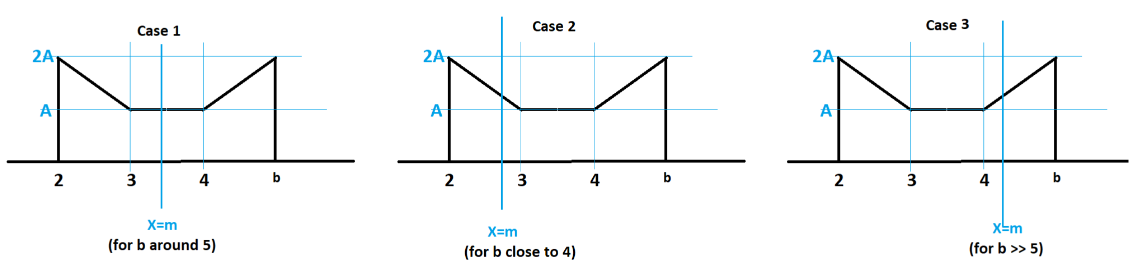
1 Questions/Problems

1-The grades of a class is provided in terms of letters A,B,C,D,F.
a) Present the outcome space. [2 points]
b) Find the number of events (i.e. the size of the event space)? [3 points]

2- The following figure depicts the probability distribution function (pdf) of a continuous-valued random value X.
a) Find A in terms of b [2 points, only for undergrads]
b) Find the median, mean and mode of X (if $b = 5$). [3 points, only for undergrads]
c) Find the median, mean and mode of X (if $b = 6$). [5 points, only for undergrads]
d) Find the median, mean and mode of X (for a general $b > 4$). [10 points, only for grads, 10 extra points for undergrads]



Hint: Note that median is defined as a point that equally splits the area under curve. You would need to consider 3 differenet cases when finding the median for a general case in (b), as follows.



3- Verify (proof) that the Bernoulli distribution with $p(X = 1) = \mu, p(X = 0) = 1 - \mu$ satisfies the following properties:

- $\sum_{x=0}^{1} p(x) = 1$ [2 pts]
- $\mathbf{E}[X] = \mu$ [3 pts]
- $\text{var}[x] = \mu(1 - \mu)$ [5 pts]
- Entropy is defined as
  $H[X] = \sum_{x=0}^{1} p(x) \log(\frac{1}{p(x)}) = -\mu \log \mu - (1 - \mu) \log(1 - \mu)$.
  What value for μ maximizes the entropy? [5 pts]
  Hint: Note that the maximum of a concave function is where the first derivative is zero.

4-Verify (prove mathematically) the following equations for the mean, variance, and mode of the beta distribution $\text{Beta}(\mu|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1 - \mu)^{b-1}$: [optional question. rewards 10 extra points for grads and undergrads]

- $E[\mu] = \frac{a}{a+b}$ [5 pts]

- var[μ] = $\frac{ab}{(a+b)^2(a+b+1)}$ [5 pts]

Hint: Use $\Gamma(a+1) = a\Gamma(a)$ and also the fact that $\int_{\infty}^{\infty} f(u)du = 1$ for any distribution $f(u)$. So try to put the $E[u] = \int_{\infty}^{\infty} uf(u)du$ in a format close to $\int_{\infty}^{\infty} f(u)du$ by the change of variables.

5- [Mixture Models] Let's assume that the length of pine trees is normally distributed as $\mathcal{N}(20, 3)$ and the length of ponderosa trees as as $\mathcal{N}(15, 5)$. Also we know that the population of pine trees in a forest is three times more than that of ponderosa trees. We show the length of a randomly chosen tree with X. Answer the following questions:

- draw the [approximate] pdf of X by hand or in MATLAB/Python. [2 pts]
- Write the expression for the distribution of a randomly chosen tree? [3 pts]
- show that the distribution sums up to 1 [3]
- if we pick a tree by random, what is the probability of having a tree longer than 16 meters? [3]
- what is the expected value of the length of a randomly chosen tree?[4]

Hint: Note that we have f(X)=P(selecting a pine tree) f(X—pine tree is selected) + P(selecting a ponderosa tree) f(X—ponderosa tree is selected)

6- Determine if the following functions are convex, concave or neither [in teh given interval].: [20 pts]

- f(x)= $4x^2 - 2$ for real-valued scalar number x
- f(x) = log(x), $x > 0$
- f(x) = exp(-x)=e^{-x} , for $x > 0$
- f(x) = cos(x)
- f(x) = tan(x) , for $-\pi/2 < x < 0$
- f(x) = x^2 , for $x < 0$
- f(x) = exp(cos(x)) , for $0 < x < \pi/2$
- f(x) = cos(x) for $-pi/2 < x < \pi/2$
- f(x) = $3x^3$
- f(x) = $\log(x^2 + 1)$

Hint: Note that the second derivative is positive for convex functions and negative for concave functions. If the second derivative can be positive or nagative, then the function is considered neither convex nor concave. Note that you need to check the positivity of the second derivative only in the given interval. the interval is $(-\infty, +\infty) if not given$

7- Prove that if f(x) and g(x)are convex functions and f(x) is monotonic increasing, then f(g(x)) is also a convex function. [10 pts, only for grads, rewards extra points for undergrads]

[Answers 1-7 in the end]

8- Let's practice random probabilities with MATLAB or Python.

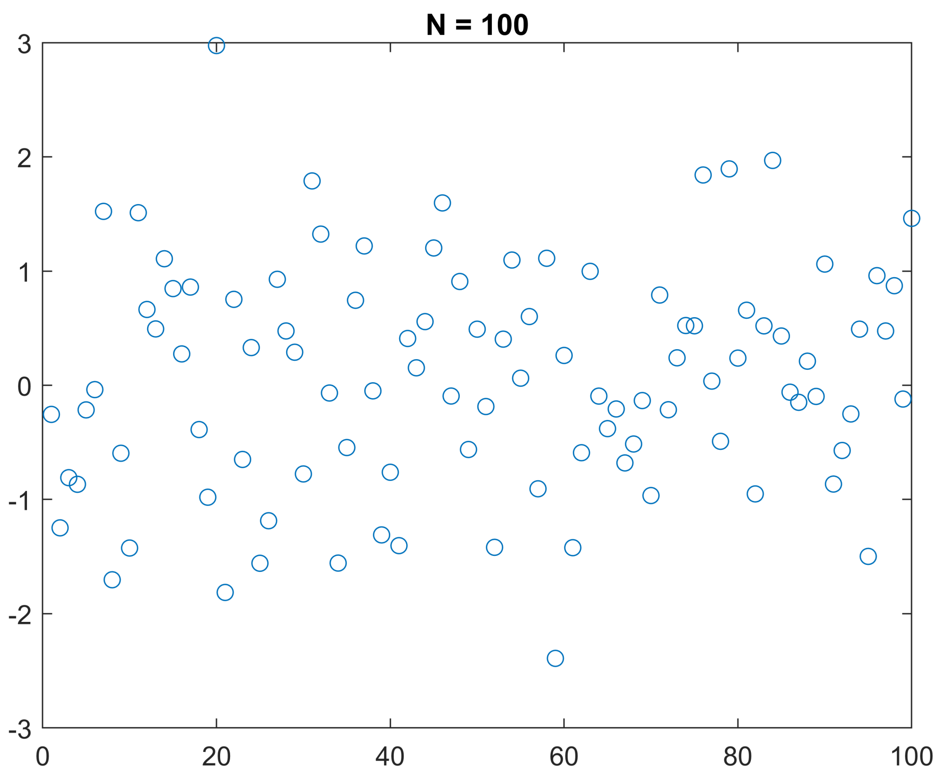
- a) Generate N=100 random variables that are Normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$. Display x.
  Hint: in MATLAB you use x= randn(1,100); [5 pts]
- b) Calculate the sample mean and variance of x using the mean(x) and var(x) commands. [5 pts]
- c) Repeat parts (a) and (b) for N=10000. Interpret the results. Which one provides a mean and variance that is closer to the distribution mean and variance? [5 pts]

- d) Plot the histogram of x for parts a (100 samples) and b (10000 samples). Which histogram is closer to the actual distribution? Does it match your expectation? [5 pts]

Hint: Histogram is the empirical distribution obtained from the samples.

Ans 8)

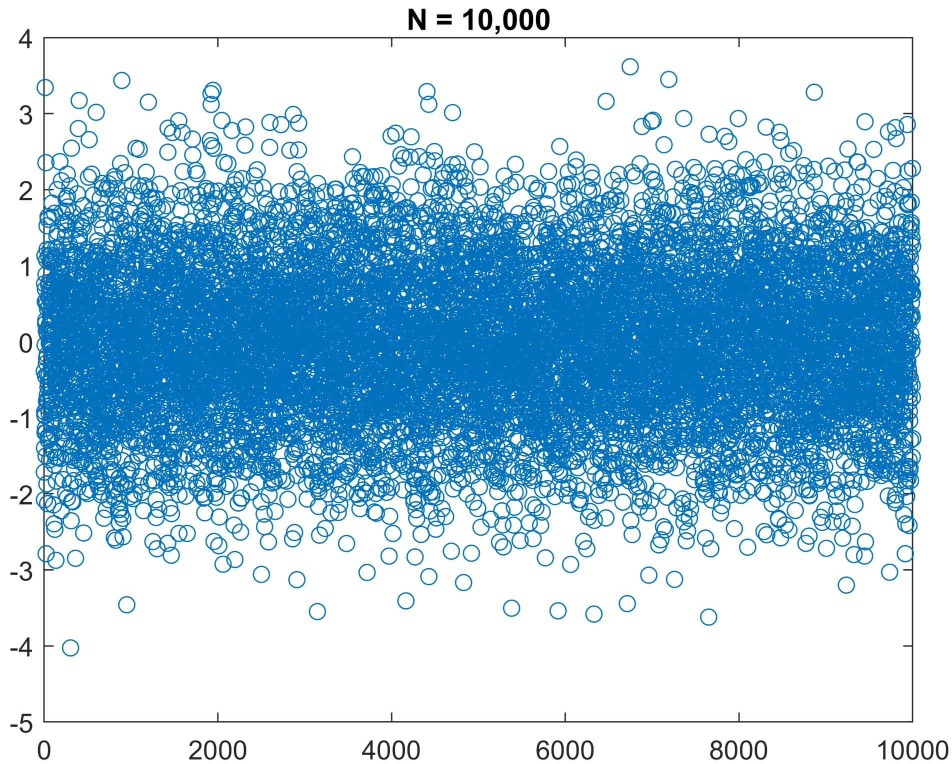
```
x = randn(1,100); % Generating a random array of variable from standard normal
% distribution of size 1X100
plot(x,'o')
title("N = 100")
```




```
X = ["Mean: ",mean(x),"Variance: ", var(x)];
disp(X)
```

"Mean: "

```
y = randn(1,10000); % Generating a random array of variable from standard normal
% distribution of size 1X100
plot(y,'o')
title("N = 10,000")
```

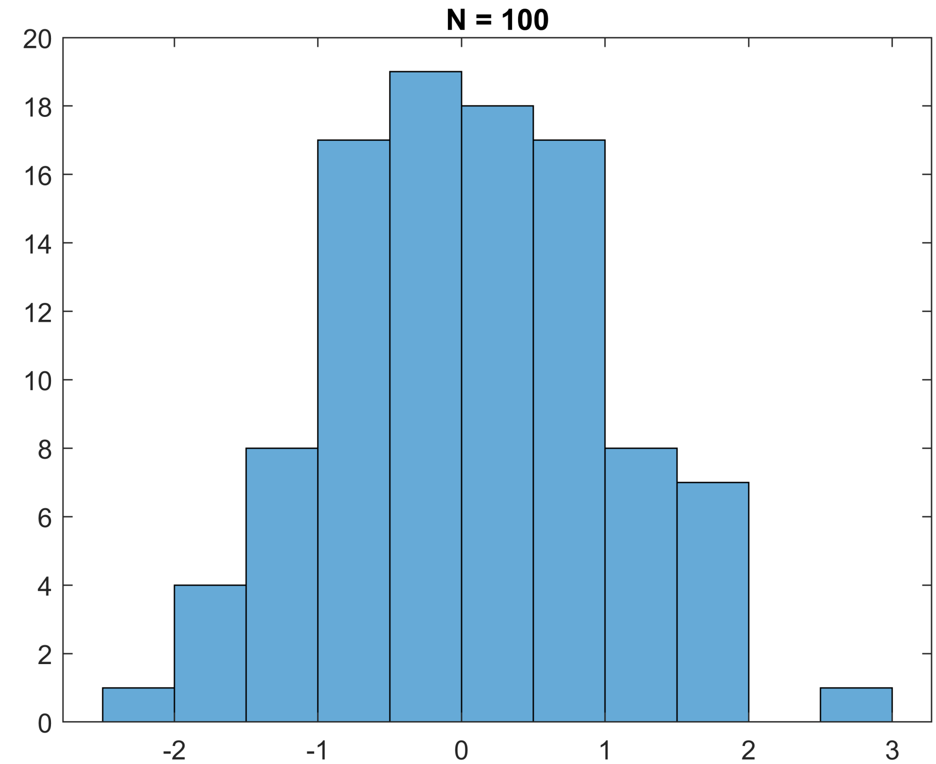



```
Y = ["Mean: ",mean(y),"Variance: ", var(y)];
disp(Y)
```

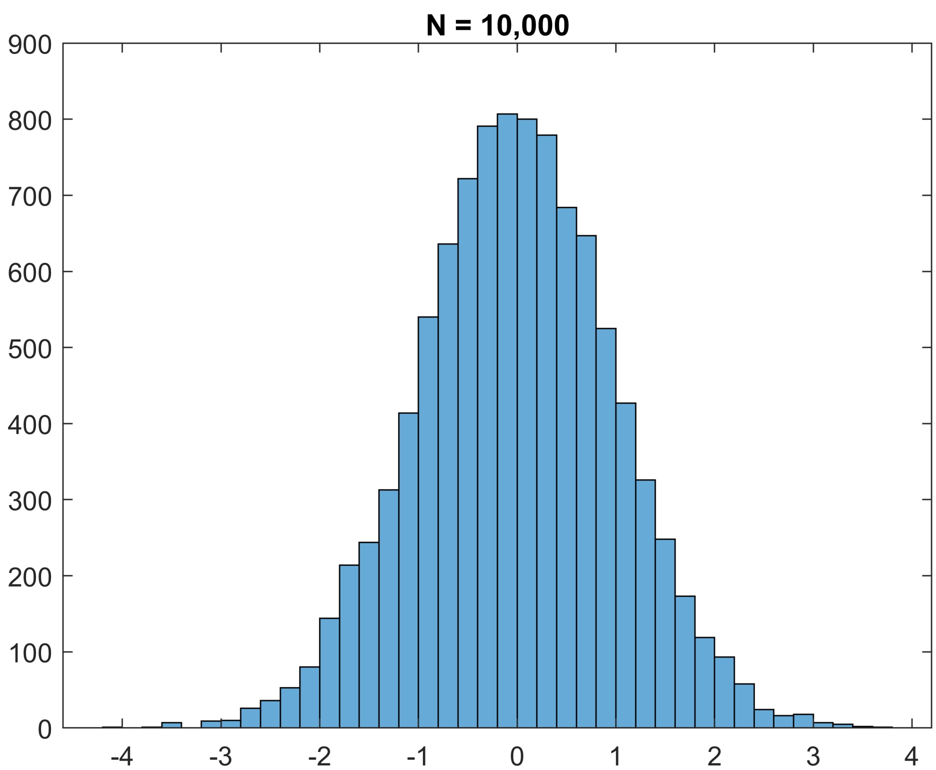
```
    "Mean: "    "-0.011505"    "Variance: "    "0.9867"
```

Majority of the time, sample with N = 10,000 generated a mean and variance that is closer to the distribution mean and variance.

```
histogram(x)
title("N = 100")
```




```
histogram(y)
title("N = 10,000")
```



Histogram with N = 10,000 is closer to the actual distribution. Yes, it matches my expextations as according to the Central Limit Theorem, as the sample size increases, the disribution of random variables approaches normal distribution.

9- Random vectors [10 pts]

(a) Generate N=10000 random vectors that are Normally distributed with the following mean μ and covariance matrix Σ.

$$\mu = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Display the results as dot points in 2D space.

(b) repeat part (a) for

$$\mu = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 2 \end{bmatrix}.$$

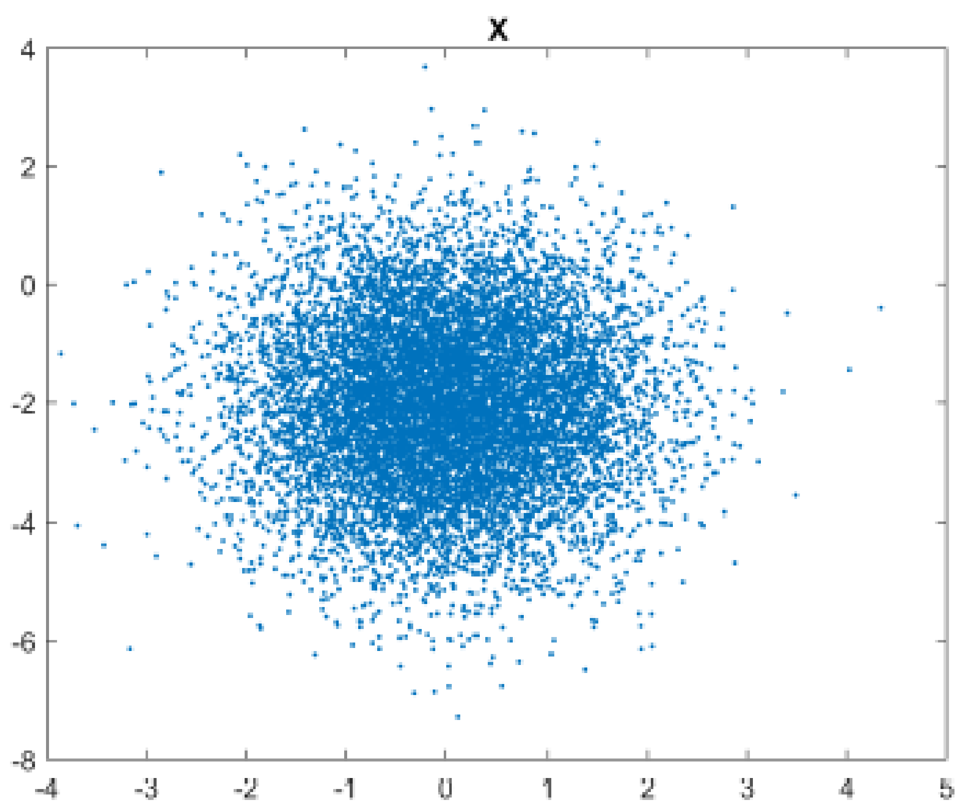
Interpret the results. What is the main difference between the two?

Ans 9)

Generating 10,000 random vector set X normally distributed with:

$$\mu = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

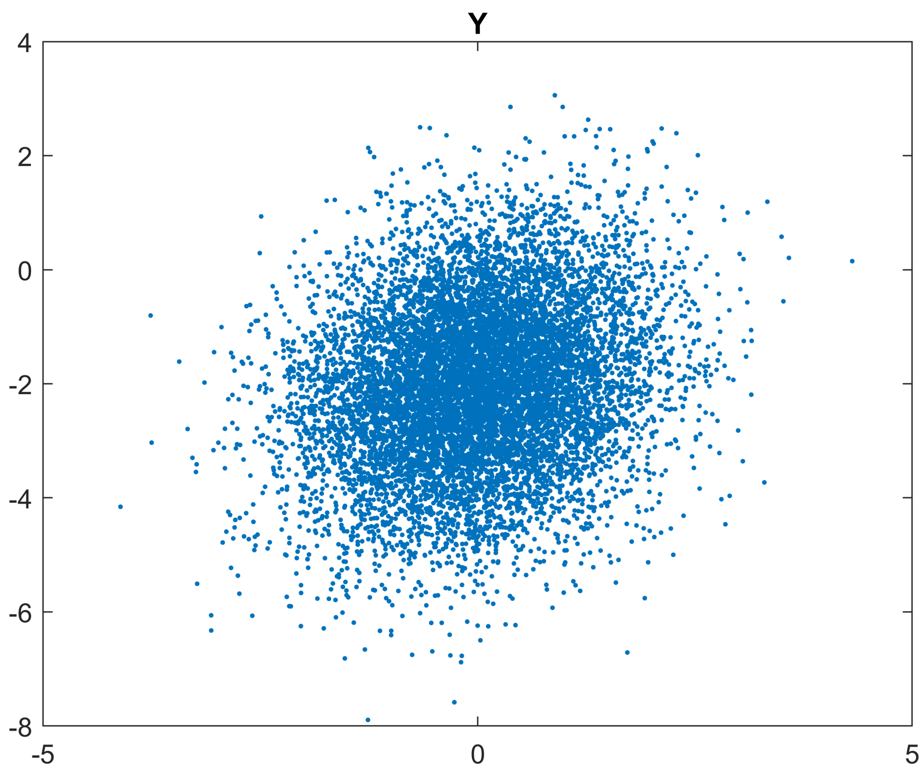
```
x = mvnrnd([0,-2], [1, 0 ; 0, 2], 10000);
plot(x(:,1),x(:,2),'.')
title("X")
```



Generating 10,000 random vector set Y normally distributed with:

$$\mu = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 2 \end{bmatrix}$$

```
y = mvnrnd([0,-2], [1, 0.3 ; 0.3, 2], 10000);
plot(y(:,1),y(:,2),'.')
title("Y")
```



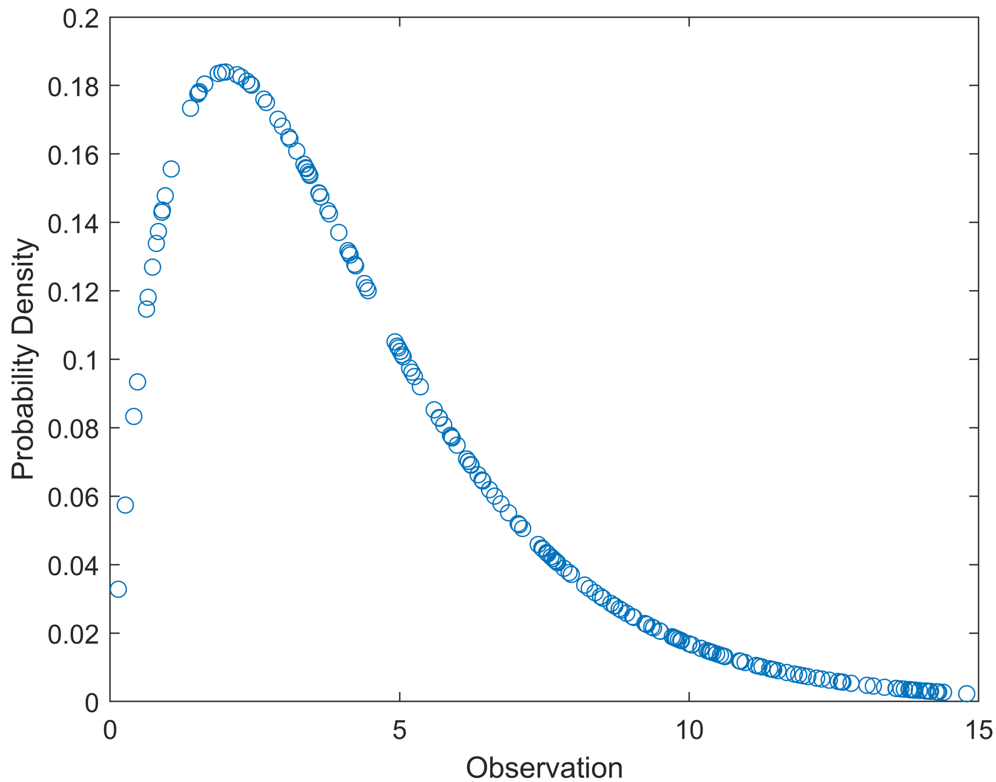
The main difference which can be very evidently seen in from the two plots is that although both sets are concentrated around the same mean [0 -2], the change in covariance changes the distribution of generated vectors.

10- Plot 5 different probability distributions of your choice and display the pdf curves.
Hint: Use MATLAB help for 'pdf' and 'random' commands for more information.

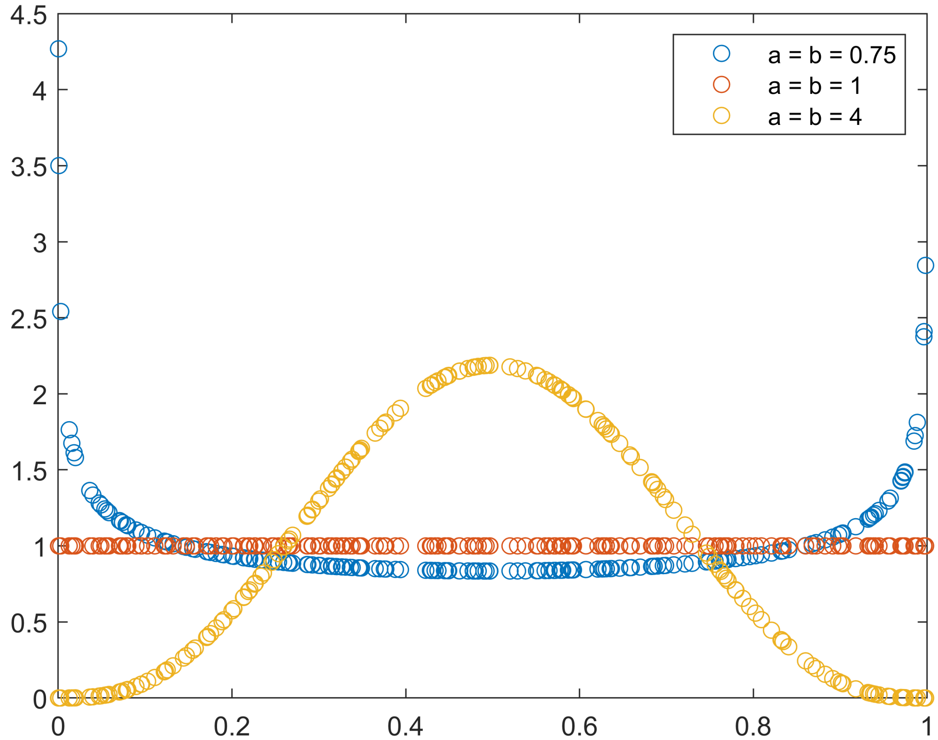
[5 pts]

Ans 10)

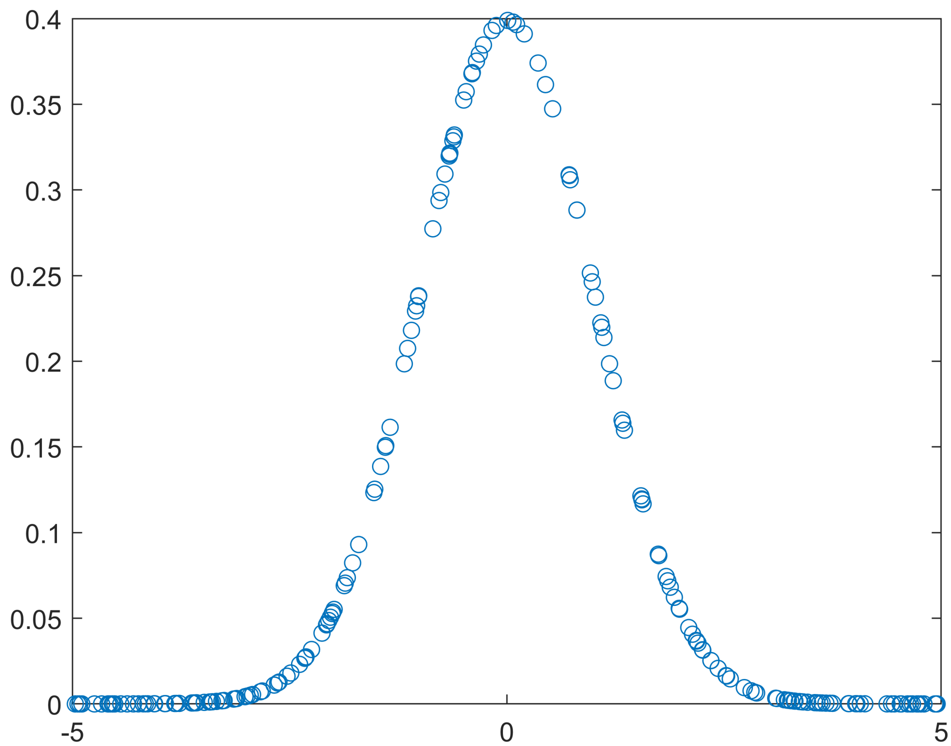
```
% #1 Chi Squared Distribution
x = 0 + (15)*rand(200,1);
y = chi2pdf(x,4);
figure;
plot(x,y,'o')
xlabel('Observation')
ylabel('Probability Density')
```



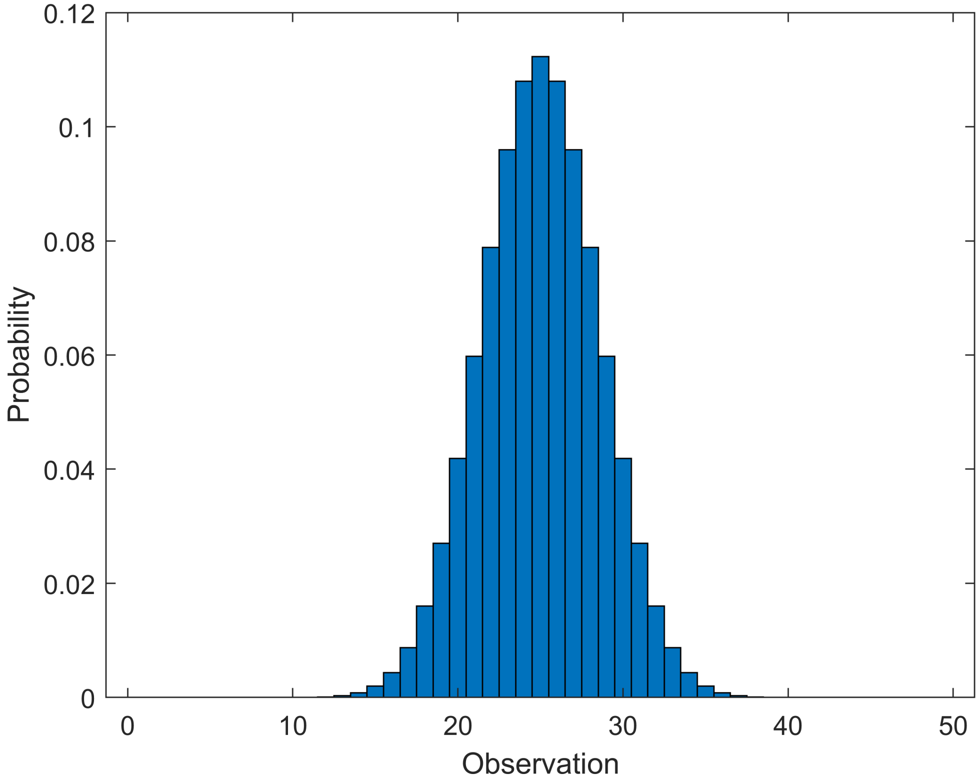

```
% #2 Beta Distribution
x = rand(200,1);
y1 = betapdf(x,0.75,0.75);
y2 = betapdf(x,1,1);
y3 = betapdf(x,4,4);
plot(x,y1,'o')
hold on
plot(x,y2,'o')
plot(x,y3,'o')
legend(["a = b = 0.75","a = b = 1","a = b = 4"]);
hold off
```



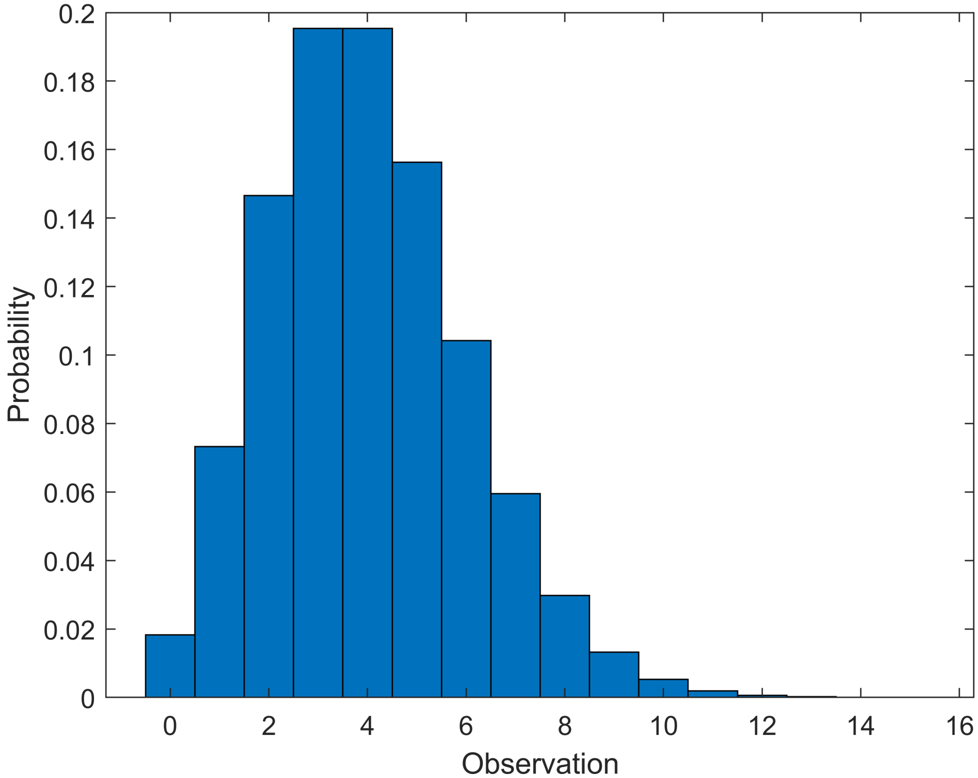

```
% #3 Normal Distribution
x = -5 + (10)*rand(200,1);
y = normpdf(x,0,1);
plot(x,y,'o')
```




```
% #4 Binomial Distribution
x = 0:50;
y = binopdf(x,50,0.5);
figure
bar(x,y,1)
xlabel('Observation')
ylabel('Probability')
```




```
% #5 Poisson Distribution
x = 0:15;
y = poisspdf(x,4);
figure
bar(x,y,1)
xlabel('Observation')
ylabel('Probability')
```



0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Probability
0
2
4
6
8
10
12
14
16
Observation

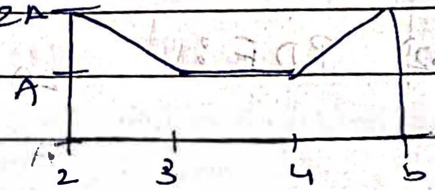
Assignment #4

Ans 1) (a) Outcome space: $\Omega = \{A, B, C, D, F\}$

Event space $= 2^{\Omega} = \underline{32}$

Ans 2) (a) Area under P.D.F. $= 1$

So, area of the figure $= 1$.



$$\frac{1}{2}(A)(1) + A(1) + A(1) + A(b-4) + \frac{1}{2}(b-4)(A) = 1$$

$$\Rightarrow \frac{5}{2}A + (b-4)\left(\frac{3A}{2}\right) = 1$$

$$\Rightarrow A = \frac{2}{3b-7}$$

(d) For a given b, mode $= \{2, b\}$.

$E(x)$ for a given b:

$$E(X) = \int_2^3 f_1(x)\, x\, dx + \int_3^4 x\, f_2(x)\, dx + \int_4^b x\, f_3(x)\, dx$$

$f_1(x) = A(4-x)$, for $x \in [2,3]$; $f_2(x) = A$, for $x \in [3,4]$; $f_3(x) = \frac{A(8-b-x)}{(4-b)}$ for $x \in [4,b]$

$$\Rightarrow A\int_2^3 x(4-x)\,dx + A\int_3^4 x\,dx + \int_4^b \frac{A(8-b-x)}{4-b}\, x\,dx$$

$$\rightarrow A\left[\left(2x^2 - \frac{x^3}{3}\right)\Bigg|_2^3 + \left(\frac{x^2}{2}\right)\Bigg|_3^4 + \frac{8-b}{4-b}\left(\frac{x^2}{2}\right)\Bigg|_4^b - \frac{1}{(4-b)}\left[\frac{x^3}{3}\right]_4^b \right.$$

$$\Rightarrow \quad = \frac{A}{6}\left[43 + \frac{3(8-b)(b^2-2^2) - 2(b^3-4^3)}{(4-b)}\right]$$

$$= \frac{A}{6}\left[43 - 3(8-b)(b+4) - (b^2+4^2+4b)\right]$$

$$= \frac{A}{6}\left[43 - [-4b^2+4b+64]\right]$$

$$\text{Mean} = \frac{A}{6}\left[5b^2-4b-21\right] = E(x)$$

For median: consider 3 cases:

Case 1: $b \approx 4$

Case 2: $b \approx 5$

Case 3: $b >> 5$.

$\Rightarrow$ Case 1: $b \approx 4$

$$A\int_2^m (4-x)\,dx = \frac{1}{2}$$

$$4(m-2) - \frac{(m^2-2^2)}{2} = \frac{1}{2A}$$

$$8m - 16 - m^2 + 4 = \frac{1}{2A}.$$

$$m^2 - 8m + 12 - \frac{1}{2A} = 0$$

Putting $A = \frac{2}{3b-7}$

$$m^2 - 8m + 12 - \frac{(3b-7)}{4} = 0$$

$$\Rightarrow \boxed{4m^2 - 32m + 55 = 3b}$$

PAGE NO.:

For case 2 : $[b \approx 5]$

The graph is symmetric for $b = 5$, so, mean = median = 3.5.

For $b \approx 5$, let median = m

$$\int_2^3 A(4-x)\,dx + \int_3^m A\,dx = \frac{1}{2}$$

$$A\left[4x - \frac{x^2}{2}\right]_2^3 + A[x]_3^m = \frac{1}{2}$$

$$A\left[\left(12 - \frac{9}{2}\right) - \left(8 - \frac{4}{2}\right)\right] + A[m-3] = \frac{1}{2}$$

$$\left(\frac{15}{2} - 6\right) + m - 3 = \frac{1}{2A}$$

$$m = \frac{(3b-7)}{4} + 9 - \frac{15}{2}$$

$$m = \frac{3b-7}{4} + \frac{3}{2} = \frac{3b-1}{4}$$

$$m = \frac{3b-1}{4}$$

For case 3 : $b > 5$

$$\int_m^b \frac{A(8-b-x)}{(4-b)}\,dx = \frac{1}{2} \qquad \frac{A}{(4-b)}\int_m^b (8-b-x)\,dx = \frac{1}{2}$$

$$\cancel{\frac{A(b-m)}{(4-b)}\left[8-b-\frac{(b+m)}{2}\right] = \frac{1}{2}}$$

$$\frac{A}{(4-b)}\left[8x - bx - \frac{x^2}{2}\right]_m^b = \frac{1}{2} \Rightarrow \frac{A}{(4-b)}\left[8b - b^2 - \frac{b^2}{2} - 8m + bm + \frac{m^2}{2}\right] = \frac{1}{2}$$

$$\cancel{\frac{A(b-m)}{(4-b)}\left[16-3b-m\right] = \frac{1}{2A}}$$

(from above eqⁿ we can find m.

$$\frac{A}{(4-b)}\left[8b - \frac{3b^2}{2} + m(b-8) + \frac{m^2}{2}\right] = \frac{1}{2}$$

PAGE NO.:

(b) if $b = 5$

from part a,

$$E(x) = \frac{A}{6}[5b^2 - 4b - 21]$$

$$A = \frac{2}{15-7} = \frac{2}{8} = \frac{1}{4}$$

$$E(x) = \frac{1}{6\times4}[5(25) - 20 - 21]$$

$$\frac{1}{24}[125 - 41]$$

$$= \frac{84}{24} = 3.5$$

as at $b = 5$, graph is symmetric, mean = median $= 3.5$.

Mode = 2 & 5

(c) if $b = 6$,

$$E(x) = \frac{A}{6}[5b^2 - 4b - 21]$$

$$A = \frac{2}{11}$$

$$E(x) = \frac{2}{11\times6}[5\times36 - 24 - 21] = \frac{1}{11\times3}\times135$$

$$= 4.09$$

for median:

area till $x = 4$: $\frac{1}{2}(1)(A) + A(1) + A(1)$

$$\frac{1}{11} + \frac{4}{11} = \frac{5}{11} \approx 0.45 \text{ (i.e. } < 0.5)$$

PAGE NO.:

So, m lies in $f_3(x)$ [in 3rd case].

$$\Rightarrow \int_2^3 A(4-x)\,dx + \int_3^4 A\,dx + \int_4^m \frac{A}{2}(x-2)\,dx = \frac{1}{2}$$

$$\Rightarrow A\left[\left[4x - \frac{x^2}{2}\right]_2^3 + [x]_3^4 + \frac{1}{2}\left[\frac{x^2}{2} - 2x\right]_4^m\right] = \frac{1}{2}.$$

$$\Rightarrow \left[\left[12 - \frac{9}{2}\right] - \left[8 - \frac{4}{2}\right]\right] + [4-3] + \frac{1}{2}\left[\frac{m^2}{2} - 2m\right] - \frac{1}{2}\left[\frac{16}{2} - 8\right]\Bigg] = \frac{1}{2A}$$

$$\Rightarrow \left[\frac{15}{2} - 6 + 1 + \frac{1}{2}\left[\frac{m^2}{2} - 2m\right] - \frac{1}{2}[0]\right] = \frac{1}{2A}.$$

$$\frac{15}{2} - 5 + \frac{1}{2}\left[\frac{m^2}{2} - 2m\right] = \frac{11}{4}.$$

$$\Rightarrow 30 - 20 + 2\left[\frac{m^2}{2} - 2m\right] = 11$$

$$\Rightarrow 10 + m^2 - 4m = 11$$

$$m^2 - 4m - 1 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(-1)}}{2}$$

$$m = \frac{4 \pm \sqrt{20}}{2}$$

$$m = 2 + \frac{\sqrt{20}}{2}$$

$$m \approx 4.24$$

PAGE NO.:

Ans3) $p(X=1) = \mu$

$p(X=0) = 1-\mu$

- $\sum_{x=0}^{1} p(x) = 1$

$p(x=0) + p(x=1) = \cancel{\mu} + 1 - \cancel{\mu} = \underline{\underline{1}}$

hence, $\underline{\sum_{x=0}^{1} p(x) = 1}$.

- $E[X] = \sum_{x=0}^{1} x\, p(x)$

$\Rightarrow 0.(1-\mu) + 1(\mu) = \underline{\underline{\mu}}$

- $var[x] = E[x^2] - [E[x]]^2$

$E[x^2] = \sum_{x=0} x^2 p(x)$

$= 0^2.(1-\mu) + 1^2.\mu = \mu.$

$var[x] = \mu - \mu^2 \Rightarrow \mu(1-\mu)$

- Entropy : $-\mu\log\mu - (1-\mu)\log(1-\mu)$

maximizing entropy: $\frac{d(\text{entropy})}{d\mu} = 0$

$\Rightarrow -\frac{d}{d\mu}\left(\mu\log\mu + (1-\mu)\log(1-\mu)\right) = 0$

$\Rightarrow$ using product rule:

$\left[\log\mu + \cancel{\mu}.\frac{1}{\cancel{\mu}}\right] + \left[-\log(1-\mu)\right] + \left[\frac{-\cancel{(1-\mu)}}{\cancel{1-\mu}}\right] = 0$

$\log\mu - \log(1-\mu) = 0$

$\log\left(\frac{\mu}{1-\mu}\right) = 0$

$\Rightarrow \frac{\mu}{1-\mu} = 1 \Rightarrow \mu = 1-\mu \Rightarrow \boxed{\mu = 1/2}$

Ans 4) $\text{Beta}(\mu|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}$

$E(\mu) = \int_0^1 \mu \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1} du$

$E(\mu) = \int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{(a+1)-1}(1-\mu)^{b-1} du$

$E(\mu) = \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(a+1)}{\Gamma(a+1+b)} \int_0 \frac{\Gamma(a+1+b)}{\Gamma(a+1)\Gamma(b)} \mu^{(\alpha+1)-1}(1-\mu)^{b-1} du$

using $\Gamma(x+1) = \Gamma(x) \cdot x$.

$E(\mu) = \frac{\Gamma(a+b)}{\Gamma(a)} \frac{a\,\Gamma(a)}{(a+b)\Gamma(a+b)} \int_0 \frac{\Gamma(a+1+b)}{\Gamma(a+1)\Gamma(b)} \mu^{(a+1)-1}(1-\mu)^{b-1} dx$

$E(x) = \frac{a}{a+b} \int_0 \frac{\Gamma(a+1+b)}{\Gamma(a+1)\Gamma(b)} \mu^{(a+1)-1}(1-\mu)^{b-1} d\mu.$

$\left(\to = 1\right)$

so, $E(x) = \frac{a}{a+b}$.

For variance:

$\text{var}(\mu) = E(\mu^2) - [E(\mu)]^2$

$E(x^2) = \int_0^1 \mu^2 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1} du$

$= \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(a+2)}{\Gamma(a+2+b)} \int_0 \frac{\Gamma(a+2+b)}{\Gamma(a+2)\Gamma(b)} \mu^{(x+2)-1}(1-\mu)^{b-1} dx.$

Twice applying the relation $\Gamma(x+1) = \Gamma(x) \cdot x$.

$$E(x^2) = \frac{\Gamma(a+b)}{\Gamma(a)} \frac{(a+1)a\,\Gamma(a)}{(a+b+1)(a+b)\Gamma(a+b)} \int_0^1 \frac{\Gamma(a+2+b)}{\Gamma(a+2)\Gamma(b)} \mu^{(a+2)-1}(1-\mu)^{b-1}\,d\mu$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)} \int_0^1 \frac{\Gamma(a+2+b)}{\Gamma(a+2)\Gamma(b)} \mu^{(a+2)-1}(1-\mu)^{b-1}\,d\mu$$

$$E(x^2) = \frac{(a+1)a}{(a+b+1)(a+b)} \int_0^1 Bet(\mu \cdot a+2, b)\,d\mu$$

$$E(x^2) = \frac{(a+1)a}{(a+b+1)(a+b)}$$

$$Var(x) = \frac{(a+1)a}{(a+b+1)(a+b)} - \left(\frac{a}{a+b}\right)^2$$

$$= \frac{(a^2+a)(a+b)}{(a+b+1)(a+b)^2} - \frac{a^2(a+b+1)}{(a+b+1)(a+b)^2}$$

$$= \frac{(a^3+a^2b+a^2+ab) - (a^3+a^2b+a^2)}{(a+b+1)(a+b)^2}$$

$$Var(x) = \frac{ab}{(a+b)^2(a+b+1)}$$

Hence Proved

Ans 5:) distribution of length of pine trees $\sim N(20, 3)$

distribution of length of ponderosa trees $\sim N(15, 5)$

Population of Ponderosa trees = P.

So, population of pine trees = 3P.

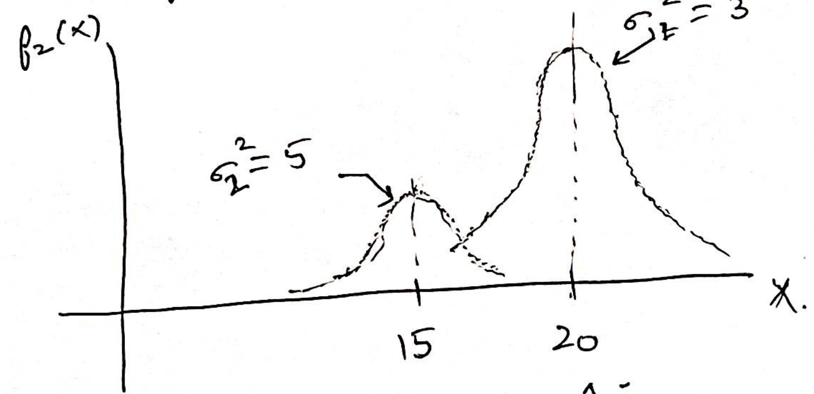
$X \sim$ length of randomly chosen tree.

P.D.F. of X: $\pi_1 f(z \mid \text{Pine trees}) + \pi_2 f(z \mid \text{Ponderosa trees})$

$$f_z(X) = \left(\frac{3P}{3P+P}\right) \cdot \left[\frac{1}{\sqrt{6\pi}} \exp\left(-\frac{(x-20)^2}{6}\right)\right] + \left(\frac{P}{3P+P}\right)\left[\frac{1}{\sqrt{10\pi}} \exp\left(-\frac{(x-15)^2}{10}\right)\right]$$

$$f_z(x) = \left(\frac{3}{4}\right)\left(\frac{e^{-\frac{(x-20)^2}{6}}}{\sqrt{6\pi}}\right) + \left(\frac{1}{4}\right)\left(\frac{e^{-\frac{(x-15)^2}{10}}}{\sqrt{10\pi}}\right)$$ → expression for distribution of randomly chosen tree.

Approximate P.d.f. plot:



Proof that distribution sums up to 1:

$$\int_{-\infty}^{\infty} f_z(x) = \int_{-\infty}^{\infty} \left[\frac{3}{4}(f_z(\text{Pine})) + \frac{1}{4}(f_z(\text{Ponderosa}))\right] dx$$

$$= \frac{3}{4}\int_{-\infty}^{\infty} f_z(\text{Pine})\, dx + \frac{1}{4}\int_{-\infty}^{\infty} f_z(\text{Ponderosa})\, dx$$

we know, $f_z(x)$ function sums to 1, when integrated over the interval $[-\infty, \infty]$.

So, $\int_{-\infty}^{\infty} f_z(x)\, dx = \frac{3}{4} \times 1 + \frac{1}{4} \times 1 = \frac{3}{4} + \frac{1}{4} = \underline{1}$

- $P(X > 16) \Rightarrow \int_{16}^{\infty} \left[\left(\frac{3}{4}\right)\left(\frac{e^{-\frac{(x-20)^2}{6}}}{\sqrt{6\pi}}\right) + \left(\frac{1}{4}\right)\left(\frac{e^{-\frac{(x-15)^2}{10}}}{\sqrt{10\pi}}\right) \right] dx$

$$\Rightarrow \underline{\underline{P(X>16)}} = \left(\frac{3}{4}\right) \int_{16}^{\infty} \frac{e^{-\frac{(x-20)^2}{6}}}{\sqrt{6\pi}} dx + \left(\frac{1}{4}\right) \int_{16}^{\infty} \frac{e^{-\frac{(x-15)^2}{10}}}{\sqrt{10\pi}} dx.$$

- Expected value of length of a randomly chosen tree:

$E[X] = \pi_1 \, E[\text{Pine}] + \pi_2 \, E[\text{Ponderosa}].$

$$E[X] = (0.75)[20] + (0.25)[15].$$
$$= 15 + 3.75$$
$$\underline{E[X] = 18.75}$$

Ans 6.) • $f(x) = 4x^2 - 20$

$f'(x) = 8x$

$f''(x) = 8$ (>0)

so, $f(x)$ is convex.

• $f(x) = \log(x)$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$ (<0)

$f(x)$ is concave.

• $f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$ (>0)

so, $f(x)$ is convex

• $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

as $f''(x)$ can be >0 as well as <0, it is neither concave ~~for~~ nor convex.

• $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2(\sec x)^2 \tan x = \frac{2\sin x}{\cos^3 x}$

for $-\frac{\pi}{2} < x < 0$, $f''(x) \leq 0$

so, $f(x)$ is concave.

- $f(x) = x^2$, for $x<0$

  $f'(x) = 2x$

  $f''(x) = 2 \quad (>0)$

  So, $f(x)$ is convex.

- $f(x) = e^{\cos x}$, $0 < x < \pi/2$

  $f'(x) = -e^{\cos x} \sin x$

  $f''(x) = -\left[-e^{\cos x}(\sin x)^2 + \cos x \, e^{\cos x}\right]$

  $= e^{\cos x}(\sin x)^2 - e^{\cos x}\cos x$

  $= e^{\cos x}\left[(\sin x)^2 - \cos x\right]$

  $f(x)$ is neither convex nor concave

- $f(x) = \cos x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

  $f'(x) = -\sin x$

  $f''(x) = -\cos x$.

  for $(-\pi/2 < x < \pi/2)$, $\cos x > 0$

  i.e. $f''(x) \leq 0$

  So, $f(x)$ is concave.

- $f(x) = 3x^3$, $f'(x) = 9x^2$

  $f''(x) = 18x$

  as $f''(x)$ can be >0 & <0, so $f(x)$ is neither concave nor convex.

PAGE NO.:

- $f(x) = \log(x^2+1)$

$$f'(x) = \frac{2x}{1+x^2}$$

$$f''(x) = \frac{d}{dx}\left(\frac{2}{x+\frac{1}{x}}\right) = \left(\frac{1}{x^2}-1\right)\left(\frac{1}{x+\frac{1}{x}}\right)^2.$$

as $f''(x)$ can be >0 & <0,
so, $f(x)$ is neither concave nor convex

Ans 7.) As $f(x)$ & $g(x)$ are convex functions,
$f''(x)$ & $g''(x) \geq 0$.

and given $f'(x) \geq 0$ (condition for monotonic increase)

To prove: $f(g(x))$ is also convex.
differentiating:
$\frac{d}{dx}[f(g(x))] \Rightarrow$ using chain rule:

$$\Rightarrow f'(g(x)).g'(x)$$

differentiating again using product rule:

$$\Rightarrow \frac{d}{dx}[f'(g(x))].g'(x) + \frac{d}{dx}(g'(x)).f'(g(x))$$

$$\rightarrow f''(g(x)).[g'(x)]^2 + g''(x) f'(g(x))$$

As we can see, all the terms in above eq^n:
$f''(y), [g'(x)]^2, g''(x), f'(y) \geq 0$
$[y = g(x)]$.

So, $f(g(x))$ is also a convex function.