Maths - Module-3

* Problems on Digraph of relation:

① Let $A = \{1, 2, 3, 4\}$, let R be a relation on A defined by xRy if $x|y$ and $y = 2x$.
i) R as a relation of set of ordered pairs
ii) Digraph of R
iii) Indegrees & outdegrees of the vertices in digraph.

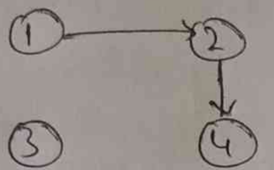
→ i) Here $y = 2x$, $x|y$

when $x = 1 \Rightarrow y = 2$ $(1,1)(1,2)(1,3)(1,4)$

$x = 2 \Rightarrow y = 4.$ $(2,4)(2,2)$

$R = \{(1,2)(2,4)\}$

ii)



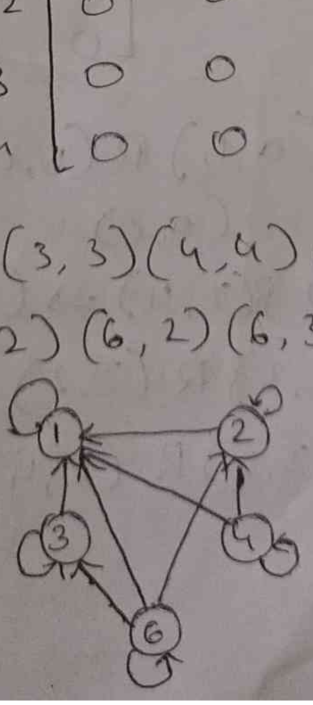
iii)

Vertices	Indegree	Outdegree
1	0	1
2	1	1
3	0	0
4	1	0

iv) $M(R) = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

② $R = \{(1,1)(2,2)(3,3)(4,4)(6,6)(2,1)(3,1)(4,1)$
$(6,1)(4,2)(6,2)(6,3)\}$

digraph:



Vertices	Indegree	Outdegree
1	5	1
2	3	2
3	2	2
4	1	3
6	1	4

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

③. i) $$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

ii) $$[M(R)]^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R^2 = ?$ $(a,b) R (b,c)$

(a,c)

$(1,2) R (2,4) \Rightarrow (1,4)$

$(1,3) R (3,2) \Rightarrow (1,2)$

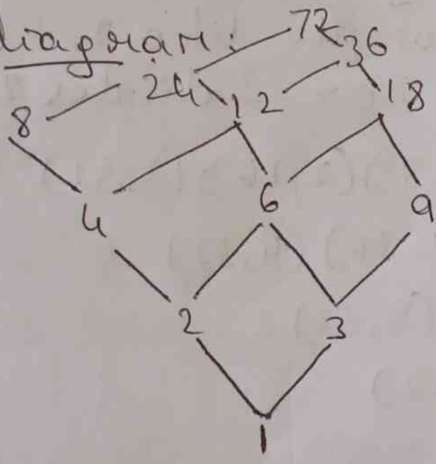
$(3,2) R (2,4) \Rightarrow (3,4)$

* Problems on Hasse diagram:

(5) Let the divisor of 72 is {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}

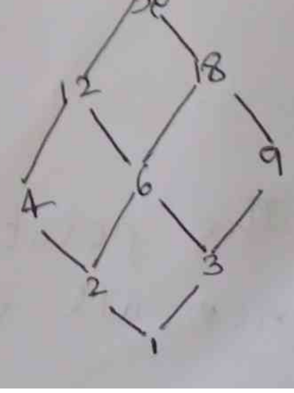
- 1 divides all related no.'s = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}
- 2 divides = {2, 4, 6, 8, 12, 18, 24, 36}
- 3 divides = {3, 6, 9, 12, 18, 24, 36}
- 4 divides = {4, 8, 12, 24, 36, 72}
- 6 divides = {6, 12, 18, 24, 36, 72}
- 8 divides = {8, 24, 72}
- 9 divides = {9, 18, 36, 72}
- 12 divides = {12, 24, 36, 72}
- 18 divides = {18, 36, 72}
- 24 divides = {24, 72}
- 36 divides = {36, 72}
- 72 divides = {72}

Hasse diagram:

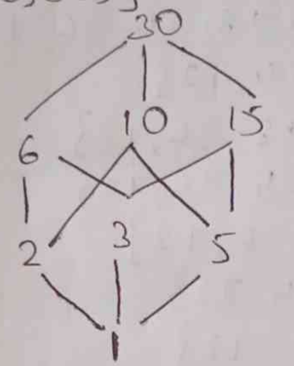


(6) Let the divisor of 36 is {1, 2, 3, 4, 6, 9, 12, 18, 36}

- 1 divides all = {1, 2, 3, 4, 6, 9, 12, 18, 36}
- 2 divides = {2, 4, 6, 12, 18, 36}
- 3 divides = {3, 6, 9, 12, 18, 36}
- 4 divides = {4, ~~8~~ 12, 18, 36}
- 6 divides = {6, 12, 18, 36}
- 9 divides = {9, 18, 36}
- 12 divides = {12, 36}
- 18 divides = {18, 36}
- 36 divides = {36}

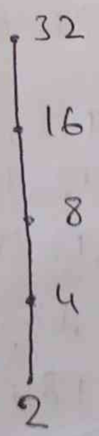


(1) i) Let R = {(1,1) (1,2) (1,3) (1,5) (1,6) (1,10) (1,15) (1,30)
(2,2) (2,6) (2,10) (2,30)
(3,3) (3,6) (3,15) (3,30)
(5,5) (5,10) (5,15) (5,30)
(6,6) (6,30)
(10,10) (10,30)
(15,15) (15,30)
(30,30)}



(A, R) is not totally ordered because if we consider one of example like 2, 3 in A neither 2 divides 3 nor 3 divides 2.

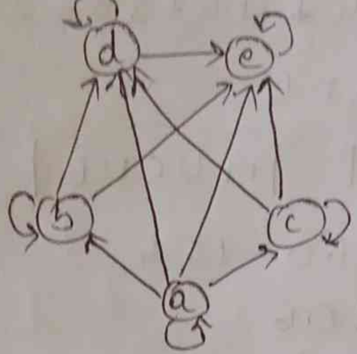
ii) R = {(2,2) (2,4) (2,8) (2,16) (2,32)
(4,4) (4,8) (4,16) (4,32)
(8,8) (8,16) (8,32)
(16,16) (16,32)
(32,32)}



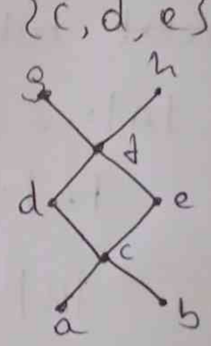
The poset (A, R) is linearly ordered Since all the elements are divisible.

(8) i) R = { (a a) (b b) (c c) (d d) (e e)
(a b) (a c) (a d) (a e)
(b d) (b e)
(c d) (c e)
(d e) }

ii) $M(R) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$



(9) a) i. All upper bounds of B If B = {c, d, e}
= {f, g, h}



ii. all lower bounds of B =
{a, b}

iii. LUB (lower upper bound) of B =
{f}

iv. GLB of B = {c}

b) A = {2, 3, 4, 6, 8, 12, 24} B = {4, 6, 12}

i. {12, 24}

ii. {2}

iii. {12}

iv. {2}

Module-4

* Problems on inclusion and exclusion:

(10). $|U| = 260$, $|M| = 64$, $|C| = 94$, $|E| = 58$, $|M \cap E| = 28$
$|M \cap C| = 26$ $|C \cap E| = 22$ $|M \cap C \cap E| = 14$

i) $|\bar{M} \cap \bar{C} \cap \bar{E}| = |U| - |M \cup C \cup E|$
$= 260 - |M \cup C \cup E|$

Let $|M \cup C \cup E| = |M| + |C| + |E| - |M \cap C| -$
$|C \cap E| - |M \cap E| + |M \cap C \cap E|$
$= 64 + 94 + 58 - 26 - 22 - 28 + 14$
$= 154.$

Now $|U| - |M \cup C \cup E|$
$= 260 - 154$
$= 106.$

Thus the no. of students who had ~~not~~ taken none of 3 courses is 106.

(11). $|A| = 3$, $|B| = 5$, $|C| = 7$

$|A| = \left| \frac{300}{3} \right| = 100$

$|B| = \left| \frac{300}{5} \right| = 60$

$|C| = \left| \frac{300}{7} \right| = |42.8| = 42$

$S_1 = 100 + 60 + 42$
$= 202$

$|A \cap B| = \left| \frac{300}{3 \times 5} \right| = 20$

$|B \cap C| = \left| \frac{300}{5 \times 7} \right| = |8.57| = 8$

$|C \cap A| = \left| \frac{300}{7 \times 3} \right| = |14.28| = 14$

$S_2 = 20 + 8 + 14$
$= 42$

$|A \cap B \cap C| = \left\lfloor \frac{300}{3 \times 5 \times 7} \right\rfloor = \lfloor 2.8 \rfloor = 2$

$S_3 = 2.$

i) exactly two of 3, 5, 7

$E_2 = S_2 - 3C_1 S_3$

$= 42 - 3C_1(2)$

$= 36.$

ii) atleast two of 3, 5, 7

$L_2 = S_2 - 2C_1 S_3$

$= 42 - 2C_1(2)$

$= 38.$

(12). CAR DOG PUN BYTE
A B C D

→ $S_0 = 26!$

$26 - 3 = 23 + 1 = 24$

$|A| = 24!$

$|B| = 24!$

$|C| = 24!$

$|D| = 23!$

$S_1 = |A| + |B| + |C| + |D|$

$= (3 \times 24!) + 23!$

$|A \cap B| = 22!$

$|B \cap C| = 22!$

$|C \cap D| = 21!$

$|D \cap A| = 21!$

$|A \cap C| = 22!$

$|B \cap D| = 21!$

$S_2 = (3 \times 22!) + (3 \times 21)!$

$|A \cap B \cap C| = 20!$

$|B \cap C \cap D| = 19!$

$|C \cap D \cap A| = 19!$ $S_3 = (3 \times 19)! + 20!$

$|A \cap B \cap D| = 19!$

$|A \cap B \cap C \cap D| = 17!$

$S_4 = 17!$

We have,

$|\overline{A \cup B \cup C}| = S_0 - S_1 + S_2 - S_3$

$= 26! - [(3 \times 24)! + 23!]$

$+ [(3 \times 22)! + (3 \times 21)!]$

$- [(3 \times 19)! + 20!]$

$\Rightarrow 4.014 \times 10^{26}$

(13) Here in CORRESPONDENTS we have

$C=1, O=2, R=2, E=2, S=2, P=1, N=2, D=1, T=1$

$$S_0 = \frac{14!}{2!2!2!2!2!} \Rightarrow 2724321600$$

- The pair of consecutive letters are:

R, E, S, N, O

$$|A| = \frac{13!}{1!2!2!2!2!} = 389188800$$

$$|B| = |C| = |D| = |E| = |A|$$

$$S_1 = 5 \times |A| = 1945944000$$

$$|A \cap B| = \frac{12!}{1!1!2!2!2!} = 59875200$$

$|A \cap B| = |B \cap C| = |C \cap D| = |D \cap E| = |E \cap A| = |E \cap B| = |E \cap C| = |D \cap A| = |D \cap B| = |A \cap C|$

$$S_2 = 10 \times 59875200 = 598752000$$

$$|A \cap B \cap C| = \frac{11!}{1!1!1!2!2!} = 9979200$$

$|A \cap B \cap C| = |B \cap C \cap D| = |C \cap D \cap E| = |A \cap B \cap E|$ ~~$|B \cap C \cap A$~~ $|D \cap E \cap A| = |E \cap A \cap B| = |A \cap B \cap D| = |B \cap C \cap E| = |C \cap D \cap A| = |B \cap D \cap A|$

$$S_3 = 10 \times 9979200 = 99792000$$

$$|A \cap B \cap C \cap D| = \frac{10!}{1!1!1!1!2!} = 1814400$$

$|A \cap B \cap C \cap D| = |A \cap B \cap C \cap E| = |B \cap C \cap D \cap E| = |C \cap D \cap E \cap A| = |D \cap E \cap A \cap B|$

$$S_4 = 5 \times 1814400 = 9072000$$

$|A\cap B\cap C\cap D\cap E| = \frac{9!}{1!1!1!1!1!1!} = 362880$

$S_5 = 362880$

$|A\cup B\cup C\cup D\cup E| = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$= 1286046720.$

i) $m=2$

$E_m = S_m - {}^{m+1}C_1 S_{m+1} + {}^{m+2}C_2 S_{m+2}$

$= S_2 - {}^3C_1 S_3 + {}^4C_2 S_4$

$= 353808000$

ii) $m=3$

$L_m = S_m - {}^mC_{m-1} S_{m+1}$

$= S_3 - {}^3C_2 S_4 + {}^5C_2 S_5$

$= 76204800$

(14) $|\overline{A\cup B\cup C}| = S_0 - S_1 + S_2 - S_3$

$\boxed{5}$ $\boxed{4}$ $\boxed{3}$

$|S| = S_0 = \frac{12!}{5!4!3!} = 27720$

$|A| = \frac{8!}{1!4!3!} = 280$

$|B| = \frac{9!}{5!1!3!} = 504$

$|C| = \frac{10!}{5!4!1!} = 1260$

$S_1 = |A| + |B| + |C| = 2044$

$|A\cap B| = \frac{5!}{1!1!3!} = 20$

$|B\cap C| = \frac{7!}{5!1!1!} = 42$

$|C\cap A| = \frac{6!}{1!4!1!} = 30$

$S_2 = 92.$

$|A\cap B\cap C| = \frac{3!}{1!1!1!} = 6$

$|\overline{A\cup B\cup C}| = S_0 - S_1 + S_2 - S_3$

$= 25762$

(15) $|S| = S_0 = \{1, 2, 3 \ldots 100\}$

i) $|A| = 2 \Rightarrow \left|\frac{100}{2}\right| = 50$

$|B| = 3 \Rightarrow \left|\frac{100}{3}\right| = 33$

$|C| = 5 \Rightarrow \left|\frac{100}{5}\right| = 20$

$|A| + |B| + |C| = 103$

$S_1 = 103$

$|A \cap B| = \left|\frac{100}{2 \times 3}\right| = 16$

$|B \cap C| = \left|\frac{100}{3 \times 5}\right| = 6$

$|C \cap A| = \left|\frac{100}{5 \times 2}\right| = 10.$

$|A \cap B| + |B \cap C| + |C \cap A| = 32$

$S_2 = 32$

$|A \cap B \cap C| = \left|\frac{100}{2 \times 3 \times 5}\right| = S_3 = 3$

★ Problems on Derangements:

(16) The permutations of n objects in which none of the object is in original place.

→ 10 people selecting 10 items which is being denoted by d_{10} since item is ≥ 7 we use $n = 7$ i.e $\boxed{d_n = n!\, e^{-1}}$

$d_{10} = 10!\,(0.3679)$

$= 1335035.52$

(17) Let 1, 2, 3, 4, 5 are arranged in 1^{st} 5 position

$$d_5 = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= 5! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$$d_5 = 44.$$

From 6, 7, ... n^{th} term are written in the form $n-5 \Rightarrow d_{n-5}$

$$d_5 \times d_{n-5} = 11660$$

$$d_{n-5} = \frac{11660}{44} \Rightarrow 265$$

For $n \geq 7$, $d_n = n! e^{-1}$

$$d_{n-5} = (n-5)! e^{-1}$$

$$(n-5)! = d_{n-5} \times e$$

$$= \frac{265}{0.3679}$$

$$= 720.344$$

$$(n-5)! = 720$$

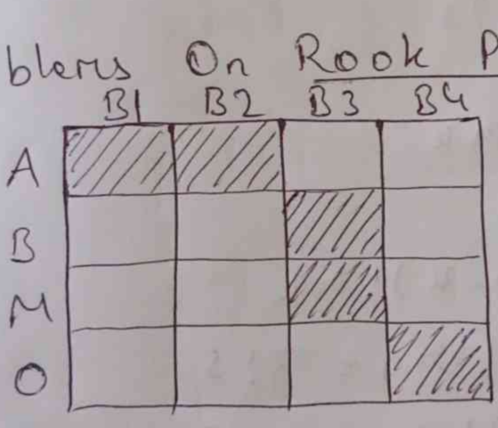
$$= 6!$$

$$n-5 = 6$$

$$\boxed{n = 11}$$

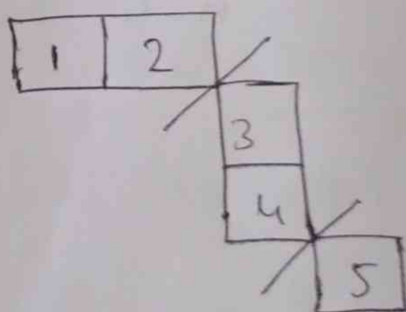
* Problems On Rook Polynomials:

(18)

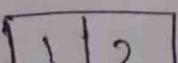


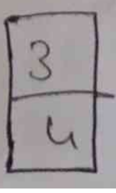
Let us consider the board C.

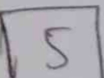
C:



Let the board C is divided into

C_1: 

C_2: 

C_3: 

For the board C_1 :

$r_1 = 2$

Since there is no non capturing rook

Rook polynomial is

$r(C_1, x) = 1+2x$

For the board C_2 :

$r_1 = 2$

$r(C_2, x) = 1+2x$

For the board C_3 :

$r_1 = 1$

$r(C_3, x) = 1+x$

We have,

$r(C, x) = r(C_1, x) \times r(C_2, x) \times r(C_3, x)$

$= (1+2x)^2 (1+x)$

$= (1+4x^2+4x)(1+x)$

$= 1+4x^2+4x+x+4x^3+4x^2$

$= 1+5x+8x^2+4x^3$

Let $r_1 = 5$, $r_2 = 8$, $r_3 = 4$ we have

$\bar{N} = S_0 - S_1 + S_2 - S_3$

$S_0 = n!$

$S_k = (n-k)!\,r_k$

$k = 1, 2, 3, 4$

$n = 4$ $\quad S_k = (n-k)!\,r_k$

$S_1 = 3!\,r_1 = 3!\,5$

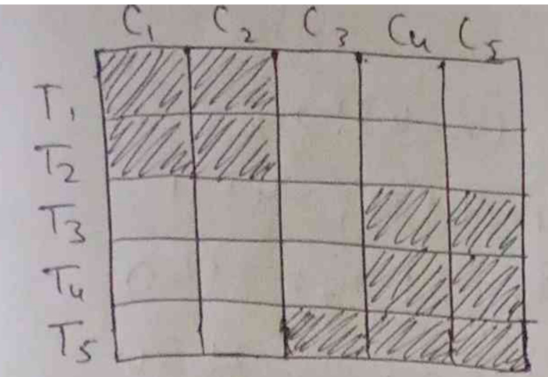
$S_2 = 2!\,r_2 = 2!\,8$

$S_3 = 1!\,r_3 = 1!\,4$

$\bar{N} = 4! - 3!(5) + 2!(8) - 1!\,4$

$\bar{N} = 6.$

19.

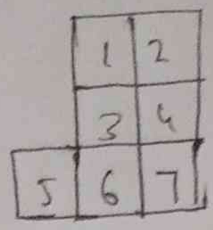


Board divided into C_1 & C_2

C_1 :

1	2
3	4

C_2 :



For the board C_1:

$r_1 = 4$

~~Since~~ The 2 non capturing rooks are $(1,4)(2,3)$

$r(C_1, x) = 1 + 4x + 2x^2$

For the board C_2:

$r_1 = 7$

~~for~~ the two non-capturing rooks are:

$(1,4)\ (2,3)\ (3,7)\ (4,6)\ (4,5)\ (3,5)\ (1,7)\ (1,5)$
$(2,5)\ (2,6)$

$r_2 = 10.$

The 3 non capturing rooks are

$(1\,4\,5)\ (2\,3\,5)$

$r_3 = 2.$

$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

~~Expansion~~ Product formula:

$$r(C, x) = (1 + 4x + 2x^2) \times (1 + 7x + 10x^2 + 2x^3)$$

$$= 1 + 7x + 10x^2 + 2x^3 + 4x + 28x^2 + 40x^3 + 8x^4 + 2x^2 + 14x^3 + 20x^4 + 4x^5$$

$$= 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

$r_1 = 11, \quad r_2 = 40, \quad r_3 = 56, \quad r_4 = 28 \quad r_5 = 4$

We, have $N = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$S_0 = n!$

$S_k = (n-k)!\, r_k$

$k = 1, 2, 3, 4, 5$

$n = 5$ $\qquad S_k = (n-k)!\, r_k$

$S_1 = 4!\, r_1 = 4!\, 11$

$S_2 = 3!\, r_2 = 3!\, 40$

$S_3 = 2!\, r_3 = 2!\, 56$

$S_4 = 1!\, r_4 = 1!\, 28$

$S_5 = 0!\, 4 = 4$

$\bar{N} = 5! - 264 + 240 - 112 + 28 - 4$

$= 8.$

(20) Let us assume 3×3 board.

1	2	3
4	5	6
7	8	9

D :

5	6
8	~~8~~9

E :

	2	3
4	5	6
7	8	9

For board D:

$r_1 = 4$

The 2 non capturing rooks are

$(5, 9)\ (6, 8)$

$r_2 = 2$

The rook polynomial is

$r(D, x) = 1 + 4x + 2x^2$

For board E :

$r_1 = 8$

The 2 non cap

$(2,4)\ (2,6)\ (2,7)\ (2,9)\ (3,4)\ (3,5)\ (3,7)\ (3,8)$

$(4,8)\ (4,9)\ (5,7)\ (5,9)\ (6,7)\ (6,8)$

$r_2 = 14$

The 3 capt rooks are

(2 4 9) (2 6 7) (3 4 8) (3 5 7)

$r_3 = 4$

The rook poly

$r(E, x) = 1 + 8x + 14x^2 + 4x^3$

Expansion formula:

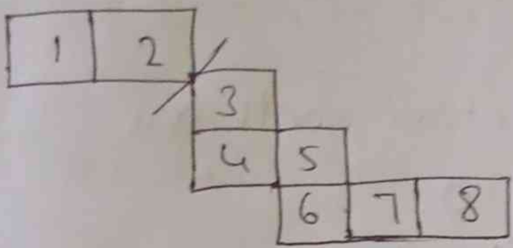
$x\, r(D\, x) + r(E\, x)$

$x[1 + 4x + 2x^2] + [1 + 8x + 14x^2 + 4x^3]$

$x + 4x^2 + 2x^3 + 1 + 8x + 14x^2 + 4x^3$

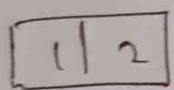
$= 1 + 9x + 18x^2 + 6x^3$

(21)

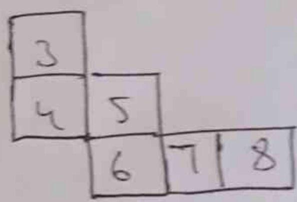


Let us assume the given board as C & divide the board C into 2 disjoint sub boards C_1 & C_2

C_1 :



C_2 :



For board C_1:

$r_1 = 2$ $\qquad r(C, x) = 1 + 2x$

For board C_2

$r_1 = 6$

The 2 non capturing rooks are

(3 5) (3 6) (3 7) (3 8) (4 6) (4 7) (4 8) (5 7) (5 8)

$r_2 = 9$

The 3 non . .

(3 5 7) (3 5 8) $\qquad r(C_2, x) = 1 + 6x + 9x^2 + 2x^3$

$r_3 = 2$

Using product formula:

$$r(c_1, x) \times r(c_2 x)$$

$$= (1+2x)(1+6x+9x^2+2x^3)$$

$$= 1+6x+9x^2+2x^3+2x+12x^2+18x^3+4x^4$$

$$= 1+8x+21x^2+20x^3+4x^4$$

Problems on Recurrance Relation:

23. $a_{n+2} - 3a_{n+1} + 2a_n = 0$, $a_0 = 1$, $a_1 = 6$

The equation is in the form

$$a_n - 3a_{n-1} + 2 = 0$$

$$k^2 - 3k + 2 = 0$$

The absolute equation roots are

$$k^2 - 2k - k + 2 = 0$$

$$k(k-2) - 1(k-2) = 0$$

$$k = 1, 2$$

24. $a_n = n a_{n-1}$ where $n \geq 1$ & $a_0 = 1$

Let $n \geq 1$ i.e $n = 1, 2, 3, \ldots$

$a_1 = 1 . a_0 = 1 \Rightarrow 1 a_0$

$a_2 = 2 . a_1 = 2 a_0$

$a_3 = 3 . a_2 = 6 a_0$

$\vdots$

$a_n = n! \, a_0$

$\boxed{a_n = n!}$.

$a_0 = 1$

25. $F_{n+2} = F_{n+1} + F_n$ where $n \geq 0$ & $F_0 = 0$, $F_1 = 1$.

The eqn is in the form: $F_n - F_{n-1} - F_{n-2} = 0$

The A.E is $k^2 - k - 1 = 0$

The roots are $k = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

The G.S is $F_n = A k_1^n + B k_2^n$

$= A \left(\dfrac{1+\sqrt{5}}{2} \right)^n + B \left(\dfrac{1-\sqrt{5}}{2} \right)^n$

Given $F_0 = 0$ $F_1 = 1$

Put $n = 0$

$F_0 = A \left(\dfrac{1+\sqrt{5}}{2} \right)^0 + B \left(\dfrac{1-\sqrt{5}}{2} \right)^0$

$0 = A + B$

Put $n = 1$

$F_1 = A \left(\dfrac{1+\sqrt{5}}{2} \right) + B \left(\dfrac{1-\sqrt{5}}{2} \right)$

Let us consider $B = A$

$F_1 = A \left(\dfrac{1+\sqrt{5}}{2} \right) + A \left[\dfrac{1-\sqrt{5}}{2} \right)$

$1 = \dfrac{A}{2} \left[1 + \cancel{\sqrt{5}} + 1 - \cancel{\sqrt{5}} \right]$

$1 = \dfrac{A}{\cancel{2}} (\cancel{2})$

$\boxed{A = 1}$

Since $A = -B$

$B = -1/5$

$F_n = \dfrac{1}{\sqrt{5}} \left[\left(\dfrac{1+\sqrt{5}}{2} \right)^n - \dfrac{1}{\sqrt{5}} \left(\dfrac{1-\sqrt{5}}{2} \right)^n \right]$

$F_n = \dfrac{1}{\sqrt{5}} \left[\left(\dfrac{1+\sqrt{5}}{2} \right)^n - \left(\dfrac{1-\sqrt{5}}{2} \right)^n \right]$

(26). $a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad a_0 = 5 \quad a_1 = 12$

It is of the form $k^2 - 6k + 9 = 0$

The roots are

$$k^2 - 3k - 3k + 9 = 0$$

$$k = 3, 3$$

The G.S is $a_n = (A + Bn)k^n$

$$= (A + Bn)3^n$$

Given $a_0 = 5 \quad a_1 = 12$.

Let $n = 0, 1$

Put $n = 0$

$$a_0 = (A + B(0))3^0$$

$$\boxed{A = 5}$$

Put $n = 1$

$$a_1 = (A + B(1))3^1$$

$$= (5 + B)3$$

$$12 = 5 + 3B$$

$$-3B = 3B$$

$$\boxed{B = -1}$$

$$a_n = (5 - n)3^n$$

Module - 5

Problems on abelian group:

(27) A non-empty set G is said to be a group with respect to binary operation if its satisfy the following properties.

1) Closure property: $a * b \in G \quad \forall a, b \in G$ $i^2 = -1$ $-i^2 = 1$

2) Associative: $(a*b)*c = a*(b*c)$
$\forall a, b \in G$

3) Identity: $a * e = e * a = a$
There is an element $e \in G$ such that
$a * e = e * a = a \quad \forall a \in G$

4) Inverse: for all $\forall a \in G$ also the element $a' \in G$ such that $a * a' = a' * a = e$

(28) 1) For any two non-zero real numbers and ~~let a *~~ a, b.
We note that $\frac{ab}{2}$ is a non-zero real number that is for any $a, b \in G$, $a * b \in G$
$\therefore$ G is closed under $*$

~~28.~~ 2) For any $a, b, c \in G$ We have

$$a * (b * c) = a * \left(\frac{1}{4} bc\right)$$
$$= \frac{1}{2}\left(\frac{1}{2} a(bc)\right)$$
$$= \frac{1}{4}(a(bc))$$
$$= \frac{1}{2}\left(\frac{1}{2} a(bc)\right) \Rightarrow \frac{1}{2}((a*b)c)$$

$(a * b) * c$ Thus $*$ is associative

3) For any $a \in G$ we have $a * 2 = 2 * a$

$$= \frac{1}{2} a \times 2$$
$$= a.$$

$\therefore$ 2 is identity under $*$ and $2 \in G$

4) For any $a \in G$ then $a' \in G$

We have $a * a' = a' * a = 2$

$= \frac{1}{2} a'a \Rightarrow 2$

$a'a = 4$

$a' = 4/a$

Thus for every a in G has $a' = 4/a$ as the inverse in G under $*$

29. $a, b \in G \; (ab)^2 = a^2 b^2$

$(ab)^2 = (ab)(ab)$

$= a(ba)b$

$= a(ab)b$

$= (aa)(bb)$

$= a^2 b^2$

Conversly

$(ab)^2 = a^2 b^2$

$(ab)(ab) = (aa)(bb)$

$\cancel{a}(ba)\cancel{b} = \cancel{a}(ab)\cancel{b}$

$\cancel{a(ba)b = a(ba)b}$

Cancellation law

$ab = ba$

Hence Commutative proof G is abelian Group.

$a * b = a + b + ab$

1) Let $a, b \in G$, $a * b \in G$ $\therefore$ G is closed under $*$

2) for $a, b, c \in G$ we have

$$a*(b*c) = a*(b+c+bc)$$

$$= a + (b+c+bc) + a(b+c+bc)$$

$$= a+b+c+bc+ab+ac+abc$$

$$= (a+b+ab)+c + (ac+bc+abc)$$

$$= (a+b+ab)+c + (a+b+ab)c$$

$$= a+b+ab.$$

$$= (a+b+ab) * c.$$

$$a*(b*c) = (a*b)*c$$

$\therefore$ The $*$ is an associative.

3) For $a \in G$ we have $a * 0 = 0 * a = a + 0 + a \cdot 0$

$$= a.$$

Thus 0 is identity in G under $*$

4) For any $a \in G$ and also $a' \in G$ we have $a * a' = a' * a = 0$

$$a * a' = 0$$

$$a + a' + aa' = 0$$

$$a + a'(1+a) = 0$$

$$a' = \frac{-a}{(1+a)}$$

Thes a' is inverse of a under $*$

* Problems on Modulo 'n':

32. $G = \{0, 1, 2, 3, 4, 5\}$ modulo 6.

The primary operation $+_6$ Now construct composition table w.r. to addition.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$1+5 \to 6$ $6)6(1$, 6, 0

7 $6)7(1$, 6, 1

8 $6)8(1$, 6, 2

- Closure law: Clearly from table $\forall a, b \in G$, $a +_6 b \in G$

  $\therefore (G, +_6)$ satisfy closure law.

- Associative law: $a, b, c \in G$ such that

  $a +_6 (b +_6 c) = (a +_6 b) +_6 c$.

- Identity law: w.r.t zero we consider all the elements with zero as e which is the additive identity elements.

- Inverse law: The inverse of zero is zero one is five, 2 is 4, 3 is 3, 4 is 2 & 5 is 1.

33. $G = \{1, 5, 7, 11\}$ $\times_{12}$

$\times_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$5 \times 5 = 25$ $12)25(2$, 24, 1

$5 \times 7 = 35$ $12)35(2$, 24, 11

$5 \times 11 = 55$ $12)55(4$, 48, 7

49 $12)49(4$, 48, 1

- Closure law: Clearly from the table $\forall a, b \in G$,
  $a \times_{12} b \in G$.
  $\therefore$ $(G, \times_{12})$ satisfy closure law.
- Associative law:
  $$a \times_{12} (b \times_{12} c) = (a \times_{12} b) \times_{12} c$$
- Identity: same. ... multiplicative ...
- Inverse: 1 is 1, 5 is 5, 7 is 7, 11 is 11.

* Problems on klein 4-groups:

34.

$A = \{e, a, b, c\}$

W.K.T $a * a = e$, $b * b = e$, $e * e = e$, $c * c = e$,
$a * e = a$, $b * e = b$, $c * e = c$, $a * b = c$, $b * c = a$,
$a * c = b$.

The composition table is

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Since A is a klein Group under the binary operation defined it is observed that e is the identity element in the group & every element is its own inverse

i.e $a^{-1} = a$ $b^{-1} = b$ $e^{-1} = e$ $c^{-1} = c$

$\therefore$ k4 is a abelian group.

35. $A = \{1, 3, 5, 7\}$ w.r.to $\times_8$

$\times_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$8)9(1$, 8, 1 — $8)15(1$, 8, 7 — $8)21(2$, 16, 5 — $8)27(3$, 24, 3 — $8)35(4$, 32, 3

Inverse of $1 = 1$
$2 = 2$
$3 = 3$
$5 = 5$
$7 = 7$

Hence k_4 is a abelian group.

36. In the klein 4 sub group every element in its own inverse thus for any x in this group we have $x^2 = e$ consequetly $x^n = e$ if n is even and $x^n = ex = x$ if n is odd.

~~36.~~ $\therefore$ Every integral power of x is equal to e or x this means that no element x in this group can be a generator of the group. Thus the klein 4 group is not cyclic.

Z_5^* is the multiplicative group of units of modulo 5 i.e set of integers < 5 that are relatively prime to 5. Since 5 is prime

$$Z_5^* = [1][2][3][4]$$

Congruence class.

$[1]^1 = 1$	$[2]^1 = 2$	$[3]^1 = 3$	$[4]^1 = 4$
$[1]^2 = 1$	$[2]^2 = 4$	$[3]^2 = 9 \equiv 4$	$[4]^2 = 16 \equiv 1$
$[1]^3 = 1$	$[2]^3 = 8 \equiv 3$	$[3]^3 = 27 \equiv 2$	$[4]^3 = 64 \equiv 4$
$[1]^4 = 1$	$[2]^4 = 16 \equiv 1$	$[3]^4 = 81 \equiv 1$	$[4]^4 = 256 \equiv 1$
is not generator	$\therefore$ 2 is generator	$\therefore$ 3 is generator	$\therefore$ 4 is not a generator

The elements 1, 2, 3, 4 are called congruence class & the operation in $(Z_5^*, \cdot)$ multiplication modulo 5
Thus in a group $(Z_5^*, \cdot)$ every element is an integral power of the element 2, 3.
$\therefore (Z_5^*, \cdot)$ is a cyclic group with 2 & 3 as a generator.

* Problems of Subgroup:

(37). Given H & K be two subgroup of G
Consider $e \in H$ and $e \in K$ so that
$e \in H \cap K$.
$\therefore H \cap K$ is a subgroup of G.
Also for all $A, B \in H \cap K \to ab^{-1} \in H$ & $ab^{-1} \in K$
$ab^{-1} \in H \cap K$
Hence $H \cap K$ is a subgroup of G.

(38). Let $HK = \{hk, h \in H$ & $k \in K\}$ (sufficient cond^n)
Suppose $hk \in HK$, Since HK is a subgroup.
The inverse $(hk)^{-1} = h^{-1}k^{-1}$ must also be in h, k
But $(k^{-1}h^{-1}) \in KH$ so that
$HK \subseteq KH$
$\therefore HK = KH$
Suppose H and K are two subroots of G
and $HK = KH$.
To show that HK is subgroup of G
We have
$(HK)(HK)^{-1} \in HK$
$(HK)(H^{-1}K^{-1})$
$H(KK^{-1})H^{-1}$
$(HK)H^{-1}$
$K(HH^{-1})$
$HK = KH$
Hence proved

39. Lagrange's Theorem:

(40) $G = S_4$ $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $H = \langle \alpha \rangle$

→ We note that S_4 is multiplicative group of all permutations of the set
$A = \{1, 2, 3, 4\}$
Let us consider the identity element in this group is the identity permutation.
The cyclic group generated by the given α can then written in form
$H = P_0, \alpha, \alpha^2, \alpha^3$ so on...

$$\therefore \quad P_0 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

$$\alpha^2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$\alpha^3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$\alpha^4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} = P_0$$

Let $S_4 = 24$
Find S_4 is a group of all 4 factorial $= 24$ w.r.t permutation.
$\therefore$ Using Lagrange's theorem

$$[G : H] = \frac{G}{H}$$

$$\frac{24}{4} = 6 \text{ ways}$$

(41) $|G| = 660$ $|K| = 66$.

By Lagrange's theorem & condition $K \subset H \subset G$

we have

$|K| = 66$ must divide $|H|$

$|H|$ must divide $|G| = 660$

$\therefore$ Let us consider $\frac{|G|}{|H|}$ $\exists\, q_1 \in G$

$\frac{|G|}{|H|} = q_1 \Rightarrow |G| = |H| q_1$

$660 = |H| q_1 \rightarrow (1)$

$\frac{|H|}{|K|} = \exists\, q_2 \in G$

$\frac{|H|}{|K|} = q_2 \Rightarrow |H| = |K| q_2$

$|H| = 66 q_2 \rightarrow (2)$

$(1) \Rightarrow 660 = 66 q_1 q_2$

$q_1 q_2 = 10$

$|H| = q_1 = 5$ $q_2 = 2$ (or) $q_1 = 2$ $q_2 = 5$

$(2) \Rightarrow |H| = 66 \times 2 = 132$

or

$H = 60 \times 5 = 330$

(42). Let us consider with table

$a^3 = a^2 . a = a . a = a$

$a^4 = a^3 . a = a . a = a$

$a^5 = a^4 . a = a . a = a$

$a^6 = a^5 . a = a . a = a$

$\therefore$ a is not generator

$b^3 = b^2 . b = c . b = d$

$b^4 = b^3 . b = d . b = e$

$b^5 = b^4 . b = e . b = f$

$b^6 = b^5 . b = f . b = a$

$\therefore$ b is generator

$c^3 = c^2 . c = e . c = a$

$c^4 = c^3 . c = a . c = c$

$c^5 = c^4 . c = c . c = e$

$c^6 = c^5 . c = e . c = a$

$\therefore$ c is not generator

$d^3 = d^2 . d = a . d = d$

$d^4 = d^3 . d = d . d = a$

$d^5 = d^4 . d = a . d = d$

$d^6 = d^5 . d = d . d = a$

$\therefore$ d is not generator

$e^3 = e^2 . e = c . e = a$

$e^4 = e^3 . e = a . e = e$

$e^5 = e^4 . e = e . e = c$

$e^6 = e^5 . e = c . e = a$

$\therefore$ e is not generator

$d^3 = d^2 . d = e . f = d$

$d^4 = d^3 . d = d . d = c$

$d^5 = d^4 . d = c . d = b$

$d^6 = d^5 . d = b . d = a$

$\therefore d$ is a generator.

Every element of G is an integral Power of b & d.

$\therefore G^*$ is a cyclic group with b & d as a generator.