Математическое моделирование. Лабораторная 1.

Вариант 3

N1

$\varepsilon=1 \quad a=2 \quad b=4 \quad c=6 \qquad \rho(x,y,z) = -z(x^2+y^2)$

$U|_{x=0} = z\sin y \qquad U_z|_{z=0} = \cos x \sin y$

$U_x|_{x=a} = -z\sin y \sin 2 \qquad U_z|_{z=c} = 6\cos x \sin y$

$U_y|_{y=0} = xz^2(z-6)(x-2)^2 \qquad U|_{y=b} = z\cos x \sin 4 + \sin(5\pi y/4)\cos(7\pi z/12)$

$v_y|_{y=0} = v_y|_{y=b} = v_z|_{z=0} = v_z|_{z=c} = 0$

$v|_{x=0} = y^2(y-4)^2\cos(\pi z/6)$

$v|_{x=a} = \cos(5\pi y/4) + \cos(7\pi z/6) + 2$

• Условия согласованности

$U|_{x=0}\ (0,z) = 0 \qquad U_x|_{y=0}\ (a,z) = 0$

$U|_{x=0}\ (b,z) = z\sin 4 \qquad U|_{y=b}\ (0,z) = z\sin 4$

$U_x|_{x=a}\ (0,z) = 0 \qquad U_x|_{y=b}\ (a,z) = -z\sin 4 \sin 2$

$U_x|_{x=a}\ (b,z) = -z\sin 4 \sin 2 \qquad U_z|_{z=0}\ (x,0) = 0$

$U|_{y=0}\ (0,z) = 0 \qquad U_z|_{z=0}\ (x,b) = \cos x \sin 4$

$U|_{z=c}\ (x,b) = 6\cos x \sin 4 \qquad U|_{z=c}\ (x,0) = 0$

$U_z|_{y=0}\ (x,0) = 0 \qquad U_y|_{y=0}\ (x,c) = 0$

$U_z|_{y=b}\ (x,0) = \cos x \sin 4 \qquad U|_{y=b}\ (x,c) = 6\cos x \sin 4$

• $\Delta U = -\rho(x,y,z)/\varepsilon = z(x^2+y^2)$

$U = v + w$

$w = A(x)\sin(y)\,B(z)$

$v|_{x=0} = z\sin y - A(0)\sin y\, B(z) = 0$

$A(0) = 1 \qquad B(z) = z$

$$U|_{z=c} = c\cos x \sin y - A(x)\sin y \cdot c = 0$$

$$A(x) = \cos x$$

$$\omega = z\cos x \sin y$$

$$\Delta U = (x^2+y^2)z + 2z\cos x \sin y$$

$$U|_{x=0} = U_x|_{x=a} = U_z|_{z=0} = U|_{z=c} = 0$$

$$U|_{y=0} = xz^2(z-c)(x-a)^2$$

$$U|_{y=b} = \sin\frac{5\pi x}{4}\cos\frac{7\pi z}{12}$$

$$U = U_1 + U_2$$

$$\begin{cases} \Delta U_1 = (x^2+y^2)z + 2z\cos x \sin y \\ U_1|_{x=0} = U_{1x}|_{x=a} = U_{1z}|_{z=0} = U_1|_{z=c} = 0 \\ U_1|_{y=0} = U_1|_{y=b} = 0 \end{cases}$$

$$\begin{cases} \Delta U_2 = 0 \\ U_2|_{x=0} = U_{2x}|_{x=a} = U_{2z}|_{z=0} = \\ = U_2|_{z=c} = 0 \\ U_2|_{y=0} = xz^2(z-c)(x-a)^2 \\ U_2|_{y=b} = \sin\frac{5\pi x}{4}\cos\frac{7\pi z}{12} \end{cases}$$

Для U_2:

$$U_2 = X(x)Y(y)Z(z)$$

$$\begin{cases} X'' + \vartheta^2 X = 0 \\ X(0) = X'(a) = 0 \end{cases}$$

$$\begin{cases} Z''(z) + \mu^2 Z = 0 \\ Z'(0) = Z(c) = 0 \end{cases}$$

$$\vartheta_n = \frac{\pi(2n+1)}{2a}$$

$$\mu_m = \frac{\pi(2m+1)}{2c}$$

$$X_n = \sin(\vartheta_n x)$$

$$Z_m = \cos(\mu_m z)$$

$$\lambda_{nm} = \mu_m^2 + \lambda_n^2$$

$$v_2 = \sum_{n,m=0} \left(A_{nm}\, ch(\lambda_{nm} y) + B_{nm}\, sh(\lambda_{nm} y) \right) \sin(\lambda_n x) \cos(\mu_m z)$$

$$v_2\big|_{y=0} = \sum_{n,m=0} A_{nm} \cos\vartheta_n x \cos\mu_m z = x z^2 (z-c)(x-a^2)$$

$$A_{nm} = \frac{4}{ac} \int_0^a \int_0^c x z^2 (z-c)(x-a)^2 \sin\vartheta_n x \cos\mu_m z \, dx\, dz =$$

$$= \frac{4}{ac} \left(\frac{6(-1)^{n+1} 16a^4}{\pi^4 a (2n+1)^4} + \frac{4a^4 8}{\pi^3 (2n+1)^3} \right) \left(\frac{6 \cdot 16 c^4}{\pi^4 (2n+1)^4} + \frac{4c^4 \cdot 8 \cdot (-1)^{m+1}}{\pi^3 (2m+1)^3} \right)$$

$$v_2\big|_{y=b} = \sum A_{nm}\, ch \lambda_{nm} b + B_{nm}\, sh \lambda_{nm} b) \sin\vartheta_n x \cos\mu_m z$$

$$A_{23}\, ch \lambda_{23} b + B_{23}\, sh \lambda_{23} b = 1$$

$$B_{23} = \frac{1}{sh \lambda_{23} b} (1 - A_{23}\, ch \lambda_{23} b)$$

$$A_{nm}\, ch \lambda_{23} b + B_{nm}\, sh \lambda_{nm} = 0$$

$$B_{nm} = -A_{nm} \cancel{cth} \frac{ch \lambda_{nm} b}{sh \lambda_{nm} b}$$

$$v_2 = \left[A_{23}\, ch \lambda_{23} y + \frac{1}{sh \lambda_{23} b} (1 - A_{23}\, ch \lambda_{23} b) \cdot sh \lambda_{23} y \right] \sin\frac{5\pi x}{4} \cos\frac{7\pi z}{12}$$

$$+ \sum' \left[A_{nm} \left(ch \lambda_{nm} y - \frac{ch \lambda_{nm} b}{sh \lambda_{nm} b} \cdot sh \lambda_{nm} y \right) \right] \sin\vartheta_n x \cos\mu_m z$$

Для v_1:

$$v_1 = \sum Y_{nm} \sin\vartheta_n x \cos\mu_m z$$

$$\sum \left(Y''_{nm} - \lambda_{nm}^2 Y \right) \sin[\vartheta_n x] \cos[\mu_m z] = y^2 \sum f_{nm} \cos\mu_m z \cdot \sin\vartheta_n x$$

$$+ \sum g_{nm} \sin\vartheta_n x \cos\mu_m z + 2\sin y \sum K_{nm} \cos\mu_m z \cdot \sin\vartheta_n x$$

$$f_{nm} = \frac{4}{ac} \int_0^a \int_0^c z \cos\mu_m z \cos\vartheta_n x \, dx\, dz = \frac{4 \cdot 2 \cdot a}{a \cdot c \cdot \pi (2n+1)} \cdot \left(\frac{c^2 \cdot 2 (-1)^m}{\pi (2m+1)} - \frac{4c^2}{\pi^2 (2m+1)^2} \right)$$

$$g_{nm} = \frac{4}{ac}\int_0^a\int_0^c x^2 z \cos\mu_m z \sin\vartheta_n x\,dx\,dz = \frac{4}{ac}\cdot\left[\frac{c^2\cdot 2(-1)^m}{\pi(2m+1)} - \frac{4c^2}{\pi^2(2m+1)^2}\right]\cdot$$

$$\cdot\left[\frac{8a^3(-1)^n}{\pi^2(2n+1)^2} - \frac{16a^3}{\pi^3(2n+1)^3}\right]$$

$$k_{nm} = \frac{4}{ac}\int_0^a\int_0^c 2\cos x\cos\mu_m z\sin\vartheta_n x = \frac{4}{ac}\left[\frac{c^2\cdot 2(-1)^m}{\pi(2m+1)} - \frac{4c^2}{\pi^2(2m+1)^2}\right]\cdot$$

$$\cdot\left[\frac{(-1)^{2n+2}\sin a}{2(\vartheta_n+1)} + \frac{(-1)^{2n+3}\sin a}{2(\vartheta_n-1)} + \frac{2\vartheta_n}{2(\vartheta_n^2-1)}\right]$$

$$\begin{cases} Y_{nm}'' - \lambda^2 Y_{nm} = y^2 f_{nm} + g_{nm} + 2\sin y\, k_{nm} \\ Y_{nm}(0) = Y_{nm}(b) = 0 \end{cases}$$

$$Y_{nm}(y) = \tilde{A}_{nm}\operatorname{ch}\lambda_{nm} y + \tilde{B}_{nm}\operatorname{sh}\lambda_{nm} y - \frac{2k_{nm}\sin y}{\lambda_{nm}^2+1} -$$

$$-\frac{f_{nm}y^2}{\lambda_{nm}^2} - \frac{2f_{nm}}{\lambda_{nm}^4} - \frac{g_{nm}}{\lambda^2}$$

$$\tilde{A}_{nm} = \frac{2f_{nm}}{\lambda^4} + \frac{g_{nm}}{\lambda^2}$$

$$\tilde{B}_{nm} = \frac{1}{\operatorname{sh}\lambda_{nm} b}\left(\left(\frac{2f_{nm}}{\lambda_{nm}^4} + \frac{g_{nm}}{\lambda_{nm}^2}\right)(1-\operatorname{ch}\lambda_{nm} b) + \frac{2k_{nm}\sin b}{\lambda_{nm}^2+1} + \frac{f_{nm} b^3}{\lambda_{nm}^2}\right)$$

$$v_1 = \sum\left(\tilde{A}_{nm}\operatorname{ch}\lambda_{nm} y + \tilde{B}_{nm}\operatorname{sh}\lambda_{nm} y - \frac{2k_{nm}\sin y}{\lambda^2+1} - \frac{f_{nm}y^2}{\lambda_{nm}^2} - \frac{2f_{nm}}{\lambda_{nm}^4} - \frac{g_{nm}}{\lambda^2}\right)$$
$$\times\cos\mu_m z\sin\vartheta_n x$$

$$u = v_1 + v_2 + w$$

Теперь рассмотрим ур-ие для υ

$\Delta\upsilon=0 \qquad \upsilon_y|_{y=0}=\upsilon_y|_{y=b}=\upsilon_z|_{z=0}=\upsilon_z|_{z=c}=0$

$\upsilon|_{x=0}=y^2(y-4)^2\cos\frac{\pi z}{6}$

$\upsilon|_{x=a}=\cos\frac{5\pi y}{4}+\cos\frac{7\pi z}{6}+2$

$$\begin{cases} Y''+\nu^2 Y=0 \\ Y'(0)=Y'(b)=0 \end{cases} \qquad \begin{cases} Z''+\mu^2 Z=0 \\ Z'(0)=Z'(c)=0 \end{cases}$$

$\nu_n=\frac{\pi n}{b} \qquad \mu_m=\frac{\pi m}{c}$

$Y_n(y)=\cos\nu_n y \qquad Z_m(z)=\cos\mu_m z$

при $\nu,\mu=0$ $\exists$ нетривиальные решения

$\lambda_{nm}^2=\mu_m^2+\nu_n^2$

$\upsilon=A_{00}x+B_{00}+\sum[A_{nm}\operatorname{ch}\lambda_{nm}x+B_{nm}\operatorname{sh}\lambda_{nm}x]\cdot\cos(\nu_n y)\cos(\mu_m z)$

$\upsilon|_{x=0}=B_{00}+\sum A_{nm}\cos\nu_n y\cos\mu_m z=y^2(y-4)^2\cos\frac{\pi z}{6}$

$B_{00}=0$

$A_{n1}=\frac{2}{b}\int_0^b y^2(y-4)^2\cos\nu_n y\,dy=\frac{24b^4}{\pi^4n^4}((-1)^n-1)$

Для всех остальных $A_{nm}=0$

$A_{01}=\frac{b^4}{30}$

$\upsilon|_{x=a}=A_{00}a+\sum A_{n1}\operatorname{ch}\lambda_{n1}a\cos\nu_n y\ \cancel{}\cos\frac{\pi z}{c}$

$+\sum B_{nm}\operatorname{sh}\lambda_{nm}a\cos\nu_n y\cos\mu_m z=\cos\frac{5\pi y}{b}+\cos\frac{7\pi z}{c}+2$

$A_{00} \cdot a = 2 \quad A_{00} = 1$

$A_{n1} \operatorname{ch} \lambda_{n1} + B_{n1} \operatorname{sh} \lambda_{n1} a = 0 \quad B_{n1} = -A_{n1} \frac{\operatorname{ch} \lambda_{n1} a}{\operatorname{sh} \lambda_{n1} a}$

$B_{50} \cdot \operatorname{sh} \lambda_{50} a = 1 \quad B_{50} = \frac{1}{\operatorname{sh} \lambda_{50} a}$

$B_{07} \cdot \operatorname{sh} \lambda_{07} a = 1 \quad B_{07} = \frac{1}{\operatorname{sh} \lambda_{07} a}$

$U = x + \sum [A_{n1} (\operatorname{ch} \lambda_{n1} x - \operatorname{sh} \lambda_{n1} x - \frac{\operatorname{ch} \lambda_{n1} a}{\operatorname{sh} \lambda_{n1} a})] \cos \nu_n y \cdot \cos \frac{\pi z}{c}$

$+ \cancel{\cos} \frac{\operatorname{sh} \lambda_{50} x}{\operatorname{s} \lambda_{50} a} \cos \frac{5\pi y}{b} + \frac{\operatorname{sh} \lambda_{07} x}{\operatorname{sh} \lambda_{07} a} \cos \frac{7\pi z}{c}$