In this study we are interested by the station design in term of parameter accuracy. In short, in presence of only pure coherent acoustic waves, it is clear that the best design is to locate sensors at far as possible. For which reason could we take into account an upper bound on the aperture ? Regarding the size of the station and the distance of the source, the only parameter which can induce a limitation on the previous design is the possible loss of coherence on distant pairs of sensors, in presence of coherent signal. By definition, two signals arriving on a two different locations are said non-coherent signals, or simply noises, if they are uncorrelated.

Therefore to determine a good design, it is necessary to take into account the trade-off between the distance and the possible loss of coherence. A very naive model consists to take:

$$\mathrm{LOC}(f) \quad = \quad e^{-\alpha \frac{d}{\lambda}}$$

where λ denotes the wavelength and d the distance between 2 sensors. If $\alpha = 0$ there is no loss of coherence. This very simple model has to be validated but here we assume that it is.

Remark 1 (on the noise) *Typical station aperture is 2 km. Noise is mainly due to the wind.*

- *Therefore we assume that the noises are spatially uncorrelated regarding the sensor inter-distances.*
- *On the other hand, we assume that the noise levels are identical on all sensors. Although that is not realistic, it is worth to notice that the noise level is not directly related to the inter-distances between sensors. It follows that, for the station design understanding, we can consider there is no loss of generalities to assume that.*

Remark 2 (on the coherent source) *To be able to study the LOC, we need a permanent source. The microbarom plays this role in the following.*

1 LOC model

We consider a station with M sensors. There is only one acoustic source faraway from the station, in such a way it can be considered as planar wave. Therefore the M-ary signal writes:

$$x(t) \quad = \quad \underbrace{s(t;\theta)}_{\text{acoustic signal}} + \underbrace{w(t)}_{\text{noise}}$$

where θ denotes the 3D slowness vector. Under assumption of pure delay we can write for the m-th sensor located in r_m:

$$s_m(t;\theta) \quad = \quad s(t - r_m^T\theta) \tag{1}$$

It follows that, under the assumption that $s(t)$ is stationary random process with spectral density $\gamma_s(f)$, the spectral matrix of the M-ary process $s(t)$, whose the m-th entry is (1), writes:

$$\Gamma_s(f) \;=\; \gamma_s(f)d(f)d^H(f)$$

where $d(f)$ is a M-ary vector whose the m-entry writes $e^{-2j\pi f r_m^T\theta}$. Clearly the matrix $\Gamma_s(f)$ is of rank 1, and that is the definition of the coherence. Loss of coherence (LOC) means that $\Gamma_s(f)$ is of rank greater than 1.

Discrete domain

All signals are real and sampled at the sampling frequency $F_s = 20$ Hz. After sampling we obtain $x_n = x(n/F_s)$. For $k = 0$ to $(N-1)$ we consider the M-ary discrete Fourier transform:

$$X_k \;=\; \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x_n\, e^{-2j\pi n f_k} \quad \text{where} \quad f_k = kF_s/N$$

We let $K = N/2$. For K great enough, $X_1, \ldots, X_K$ is a sequence of M-ary independent circularly gaussian random vectors with zero-mean and respective covariance:

$$\Gamma_k(\alpha) \;=\; \gamma_k d_k(\theta)d_k^H(\theta) + \sigma^2 I_M \tag{2}$$

where $d_k(\theta)$ is an M-ary vector whose the m-th entry writes $D_{k,m} = e^{-2j\pi f_k r_m^T\theta}$ and where $\gamma_k = \gamma_s(f_k)$.

The expression (2) can be rewritten:

$$\Gamma_k(\alpha) \;=\; \gamma_k D_k(\theta)\, C_k\, D_k^H(\theta) + \sigma^2 I_M \tag{3}$$

where $D_k(\theta)$ is an M-ary diagonal matrix whose the m-th diagonal entry writes $D_{k,m} = e^{-2j\pi f_k r_m^T\theta}$ and where $C_k = \mathbb{1}\mathbb{1}^T$ which is a rank 1 matrix.

Under LOC, C_k is no more a rank 1 projector. In the following to take into account the LOC we consider that

$$C_{k,\ell\ell'}(\beta) \;=\; e^{-\beta\frac{d_{\ell,\ell'}}{\lambda_k}} = e^{-\beta\frac{d_{\ell,\ell'}f_k}{c}}$$

where $\lambda_k = c/f_k$ is the wavelength, $d_{\ell,\ell'}$ the distance between 2 locations ℓ and ℓ' in the 3D space, and β a positive constant. For $\beta = 0$, $C(\beta) = \mathbb{1}\mathbb{1}^T$ leading to a perfect coherence. If $\beta \approx +\infty$, $C(\beta) = I_M$ and there is no coherence between any 2 locations.

Summarizing we can write that:

$$(X_1, \ldots, X_K) \;\sim\; \prod_{k=1}^{K} \mathcal{N}_c(x_k; 0_M, \Gamma_k(\alpha))$$

and the likelihood writes:

$$\mathcal{L}(\alpha) \;=\; \sum_{k=1}^{K} \log\det\Gamma_k(\alpha) + \mathrm{trace}\left(\Gamma_k^{-1}(\alpha) X_k X_k^T\right) \tag{4}$$

where the parameter α consists of

$$\alpha \;=\; \{\gamma_1, \ldots, \gamma_K, \theta_1, \theta_2, \theta_3, \beta, \sigma^2\} \in \mathbb{R}^{+K} \times \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^+ \tag{5}$$

Its size is $K + 5$.

Remark 3 *To avoid intractable computation, we assume in the following that* $\gamma_1, \ldots, \gamma_K, \beta$ *are known, leading to an uncertainty on only* θ_1*,* θ_2*,* θ_3 *and* σ^2*. Now the unknown parameter has dimension 4.*

2 Cramer-Rao

The (ℓ, ℓ') entry of the Fisher information matrix associated to the parameter α writes:

$$\mathrm{FIM}_{\ell,\ell'}(\alpha) \;=\; \sum_{k=1}^{K} \mathrm{trace}\left(\Gamma_k^{-1} \times \partial_\ell \Gamma_k \times \Gamma_k^{-1} \times \partial_{\ell'} \Gamma_k\right) \tag{6}$$

where $\ell, \ell' \in \{1, 2, 3, 4\}$ and where $\partial_\ell \Gamma_k$ is the partial derivative w.r.t. θ_1, θ_2, θ_3 and σ^2. It is clear that the CRB terms relative to the correlation of θ and σ^2 are 0. We let:

$$d_{k,\ell}(\theta) \;=\; \begin{bmatrix} -2j\pi f_k r_{1,\ell}\, e^{-2j\pi f_k r_1^T \theta} \\ \vdots \\ -2j\pi f_k r_{M,\ell}\, e^{-2j\pi f_k r_M^T \theta} \end{bmatrix}$$

Then for $\ell = 1, 2, 3$:

$$\begin{aligned} \partial_\ell \Gamma_k &= \gamma_k \mathrm{diag}\left(d_{k,\ell}(\theta)\right) C_k(\beta) D_k^H(\theta) + D_k(\theta) C_k(\beta) \mathrm{diag}\left(d_{k,\ell}(\theta)\right)^H \\ &= 2\,\mathcal{R}\left(\mathrm{diag}\left(d_{k,\ell}(\theta)\right) C_k(\beta) D_k^H(\theta)\right) \end{aligned}$$

and

$$\partial_4 \Gamma_k \;=\; I_M$$

It is worth to notice that, if $\beta \approx +\infty$, $C_k = I_M$ and $\partial_\ell \Gamma_k = 0$ which is normal because in this case Γ_k does not depend on θ. Using (6) we derived the CRB as

$$\mathrm{CRB} \;=\; \mathrm{FIM}^{-1} \tag{7}$$

The CRB w.r.t. the azimuth, elevation and velocity is derived from (7) using the Jacobian of the one-to-one mapping between the slowness and the vector (a, e, c) where a, e and c denote respectively the azimut, the elevation and the velocity.