In this study we are interested by the station design in term of parameter accuracy. In short, in presence of only pure coherent acoustic waves, it is clear that the best design is to locate sensors at far as possible. For which reason could we take into account an upper bound on the aperture ? Regarding the size of the station and the distance of the source, the only parameter which can induce a limitation on the previous design is the possible loss of coherence on distant pairs of sensors, in presence of coherent signal. By definition, two signals arriving on a two different locations are said non-coherent signals, or simply noises, if they are uncorrelated.

Therefore to determine a good design, it is necessary to take into account the trade-off between the distance and the possible loss of coherence. A very naive model consists to take:

$$\mathrm{LOC}(f) \quad = \quad e^{-\beta \frac{d}{\lambda}}$$

where λ denotes the wavelength and d the distance between 2 sensors. If $\beta = 0$ there is no loss of coherence. This very simple model has to be validated but here we assume that it is.

Remark 1 (on the noise) *Typical station aperture is 2 km. Noise is mainly due to the wind.*

- *Therefore we assume that the noises are spatially uncorrelated regarding the sensor inter-distances.*
- *On the other hand, we assume that the noise levels are identical on all sensors. Although that is not realistic, it is worth to notice that the noise level is not directly related to the inter-distances between sensors. It follows that, for the station design understanding, we can consider there is no loss of generalities to assume that.*

Remark 2 (on the coherent source) *To be able to study the LOC, we need a permanent source. The microbarom plays this role in the following.*

Remark 3 (on the isotropy) *In the absence of LOC, the station is isotrope iff*

$$XX^T \quad \propto \quad I_d \tag{1}$$

where X is $d \times M$ matrix whose the m-th colomun if the coordinates of the m-th sensor (in any system of coordinates).

It follows that given an arbitrary array with M sensors in $\mathbb{R}^d$ it is always possible to complete with d locations in such a way the new array is isotrope, i.e. verifies (1)*, see annex.*

Remark 4 (on the geometry) *In the absence of LOC, we will see below that the best performances are associated with inter-distances as large as possible. In this* *case the maximization of the minimal distance leads to locate the sensors on a circle. But in this case the distribution of the distances is not uniform. It follows that in presence of LOC, several sensors are concern with the same LOC. It seems (it is not a proof) that a rule could be to locate the sensors in such a way that the distances are more uniform (see figure 1).*

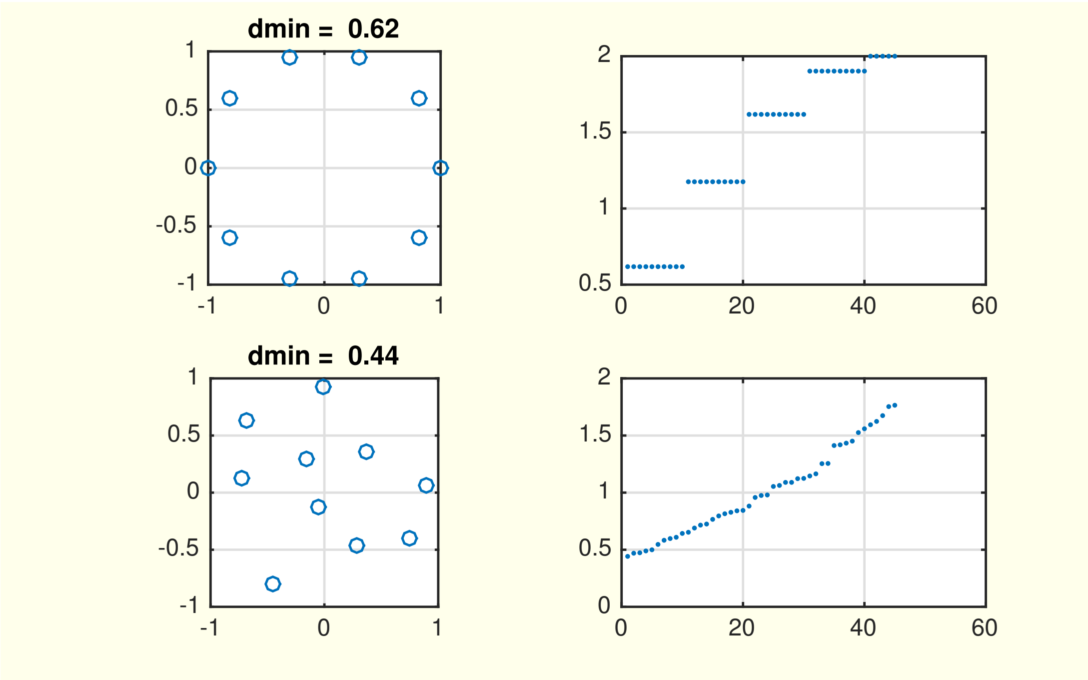

Figure 1: *Sensor locations with inter distances*

1 LOC model

We consider a station with M sensors. There is only one acoustic source faraway from the station, in such a way it can be considered as planar wave. Therefore the M-ary signal writes:

$$x(t) = \underbrace{s(t;\theta)}_{\text{acoustic signal}} + \underbrace{w(t)}_{\text{noise}}$$

where θ denotes the 3D slowness vector. Under assumption of pure delay we can write for the m-th sensor located in r_m:

$$s_m(t;\theta) = s(t - r_m^T\theta) \tag{2}$$

It follows that, under the assumption that $s(t)$ is stationary random process with spectral density $\gamma_s(f)$, the spectral matrix of the M-ary process $s(t)$, whose the m-th entry is (2), writes:

$$\Gamma_s(f) = \gamma_s(f)d(f)d^H(f)$$

where $d(f)$ is a M-ary vector whose the m-entry writes $e^{-2j\pi f r_m^T\theta}$. Clearly the matrix $\Gamma_s(f)$ is of rank 1, and that is the definition of the coherence. Loss of coherence (LOC) means that $\Gamma_s(f)$ is of rank greater than 1.

Discrete domain

All signals are real and sampled at the sampling frequency $F_s = 20$ Hz. After sampling we obtain $x_n = x(n/F_s)$. For $k = 0$ to $(N-1)$ we consider the M-ary discrete Fourier transform:

$$X_k = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x_n\, e^{-2j\pi n f_k} \quad \text{where} \quad f_k = kF_s/N$$

We let $K = N/2$. For K great enough, $X_1, \ldots, X_K$ is a sequence of M-ary independent circularly gaussian random vectors with zero-mean and respective covariance:

$$\Gamma_k(\alpha) = \gamma_k d_k(\theta)d_k^H(\theta) + \sigma^2 I_M \tag{3}$$

where $d_k(\theta)$ is an M-ary vector whose the m-th entry writes $D_{k,m} = e^{-2j\pi f_k r_m^T\theta}$ and where $\gamma_k = \gamma_s(f_k)$.

The expression (3) can be rewritten:

$$\Gamma_k(\alpha) = \gamma_k D_k(\theta)\, C_k\, D_k^H(\theta) + \sigma^2 I_M \tag{4}$$

where $D_k(\theta)$ is an M-ary diagonal matrix whose the m-th diagonal entry writes $D_{k,m} = e^{-2j\pi f_k r_m^T\theta}$ and where $C_k = \mathbb{1}\mathbb{1}^T$ which is a rank 1 matrix.

Under LOC, C_k is no more a rank 1 projector. In the following to take into account the LOC we consider that

$$C_{k,\ell\ell'}(\beta) \quad = \quad e^{-2\pi^2 f_k^2 (r_\ell - r_{\ell'})^T S (r_\ell - r_{\ell'})} \tag{5}$$

where S is a 3×3 covariance matrix. For simplicity we assume that $S \propto I_3$ then

$$C_{k,\ell\ell'}(\beta) \quad = \quad e^{-\beta f_k^2 d_{\ell,\ell'}^2} \tag{6}$$

$d_{\ell,\ell'}$ the distance between 2 locations ℓ and ℓ' in the 3D space, and β a positive constant. For $\beta = 0$, $C(\beta) = \mathbb{1}\mathbb{1}^T$ leading to a perfect coherence. If $\beta \approx +\infty$, $C(\beta) = I_M$ and there is no coherence between any 2 locations.

Summarizing we can write that:

$$(X_1, \ldots, X_K) \quad \sim \quad \prod_{k=1}^{K} \mathcal{N}_c(x_k; 0_M, \Gamma_k(\alpha))$$

and the likelihood writes:

$$\mathcal{L}(\alpha) \quad = \quad \sum_{k=1}^{K} \log\det\Gamma_k(\alpha) + \mathrm{trace}\left(\Gamma_k^{-1}(\alpha) X_k X_k^T\right) \tag{7}$$

where the parameter α consists of

$$\alpha \quad = \quad \{\gamma_1, \ldots, \gamma_K, \theta_1, \theta_2, \theta_3, \beta, \sigma^2\} \in \mathbb{R}^{+K} \times \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^+ \tag{8}$$

Its size is $K + 5$.

Remark 5 *To avoid intractable computation, we assume in the following that β is known, leading to an uncertainty on only θ_1, θ_2, θ_3, σ^2, γ_1, …, γ_K. Now the unknown parameter has dimension $K + 4$.*

2 Validation

Taking the log of (6) we get:

$$\log C_{k,\ell\ell'}(\beta) \quad = \quad -\beta f_k^2 d_{\ell,\ell'}^2 \tag{9}$$

meaning that the the log of MSC is proportional to the frequency, and proportional to the distance.

For the proportionality to the frequency, it is not obvious to test it because we don't dispose of a coherent signal in a large frequency bandwidth of interest.

3 Cramer-Rao bound (CRB)

The (ℓ, ℓ') entry of the Fisher information matrix associated to the parameter α writes:

$$\mathrm{FIM}_{\ell,\ell'}(\alpha) \quad = \quad \sum_{k=1}^{K} \mathrm{trace}\left(\Gamma_k^{-1} \times \partial_\ell \Gamma_k \times \Gamma_k^{-1} \times \partial_{\ell'} \Gamma_k\right) \tag{10}$$

where $\ell, \ell' \in \{1, 2, 3, 4\}$ and where $\partial_\ell \Gamma_k$ is the partial derivative w.r.t. θ_1, θ_2, θ_3 and σ^2. It is clear that the CRB terms relative to the correlation of θ and σ^2 are 0. We let:

$$d_{k,\ell}(\theta) \quad = \quad \begin{bmatrix} -2j\pi f_k r_{1,\ell}\, e^{-2j\pi f_k r_1^T \theta} \\ \vdots \\ -2j\pi f_k r_{M,\ell}\, e^{-2j\pi f_k r_M^T \theta} \end{bmatrix}$$

Then for $\ell = 1, 2, 3$:

$$\begin{aligned} \partial_\ell \Gamma_k &= \gamma_k \mathrm{diag}\left(d_{k,\ell}(\theta)\right) C_k(\beta) D_k^H(\theta) + D_k(\theta) C_k(\beta) \mathrm{diag}\left(d_{k,\ell}(\theta)\right)^H \\ &= 2\,\mathcal{R}\left(\mathrm{diag}\left(d_{k,\ell}(\theta)\right) C_k(\beta) D_k^H(\theta)\right) \end{aligned}$$

It is worth to notice that, if $\beta \approx +\infty$, $C_k = I_M$ and $\partial_\ell \Gamma_k = 0$ which is normal because in this case Γ_k does not depend on θ. For the derivation w.r.t. σ^2 we have:

$$\partial_4 \Gamma_k \quad = \quad I_M$$

A direct consequence is that the FIM is of this shape:

$$\mathrm{FIM} \quad = \quad \begin{bmatrix} F_{123} & 0_{3,1} & 0_{3,K} \\ 0_{1,3} & f_{\sigma^2} & ? \\ 0_{K,3} & ? & F_{\gamma,K,K} \end{bmatrix}$$

Because the CRB on the estimation of θ is given by the 3×3 matrix of the inverse of FIM and because the correlations with other parameter estimates are 0, we are only concern with F_{123} by taking F_{123}^{-1} . Then the CRB w.r.t. the azimuth, elevation and velocity can be derived using the Jacobian of the one-to-one mapping between the slowness and the vector (a, e, c) where a, e and c denote respectively the azimut, the elevation and the velocity. The same calculation can be used to derive the CRB w.r.t. the azimuth and the trace velocity.

Appendix A

Transform an array in isotrope array

We consider an arbitrary array whose locations are given by:

$$X = \begin{bmatrix} x_{1,1} & \dots & x_{1,M} \\ x_{2,1} & \dots & x_{2,M} \\ x_{3,1} & \dots & x_{3,M} \end{bmatrix}$$

It follows that

$$XX^T = \sum_{i=1}^{d} \mu_i \xi_i \xi_i^T$$

where $0 \leq \mu_i \leq \alpha_0$. The new array writes

$$Y = \begin{bmatrix} X & \sqrt{(\alpha_0 - \mu_1)}\xi_1 & \sqrt{(\alpha_0 - \mu_2)}\xi_2 & \sqrt{(\alpha_0 - \mu_3)}\xi_3 \end{bmatrix}$$

It is easy to verify that $YY^T = \alpha_0 I_d$.

Appendix B

Coherence

The LOC is modeled by the random slowness:

$$\Theta = \theta_0 + \epsilon$$

where θ_0 is a 3D deterministic vector and ϵ a 3D random vector with zero-mean and distribution probability density denoted p_ϵ. The spectral matrix of the M-ary random vectors associated to the sensor locations has the following entry:

$$\begin{aligned}
S_{\ell,\ell'}(f) &= \gamma(f) \int_{\mathbb{R}^3} e^{-2j\pi f (r_\ell - r_{\ell'})^T t} p_e(t-\theta_0) dt \\
&= \gamma(f) e^{-2j\pi f (r_\ell - r_{\ell'})^T \theta_0} \int_{\mathbb{R}^3} e^{-2j\pi f (r_\ell - r_{\ell'})^T t} p_e(t) dt \\
&= \gamma(f) e^{-2j\pi f (r_\ell - r_{\ell'})^T \theta_0} \times \Phi_e(2\pi f (r_\ell - r_{\ell'}))
\end{aligned}$$

where $\Phi_e(v)$ denotes the characteristic function of the r.v. e. The deterministic case where $\epsilon = 0$ leads to:

$$S_{\ell,\ell'}(f) = \gamma(f) e^{-2j\pi f (r_\ell - r_{\ell'})^T \theta_0}$$

and therefore $S(f) = d(f) d^H(f)$ where d is a complex vector whose the ℓ-entry writes $e^{-2j\pi f r_\ell^T \theta_0}$. Hence S is a projector of rank 1 and corresponds to a pure coherent case.

In the general case:

$$S(f) = \gamma_s(f) D(f) C(f) D^H(f)$$

$D(f)$ is a diagonal matrix whose the ℓ-th diagonal entry is $e^{-2j\pi f r_\ell^T \theta_0}$ and $C(f)$ a matrix whose the (ℓ, ℓ')-entry writes:

$$C_{\ell,\ell'}(f) = \Phi_e(2\pi f (r_\ell - r_{\ell'}))$$

If ϵ is gaussian with covariance matrix Σ_θ:

$$\Phi_e(2\pi f (r_\ell - r_{\ell'})) = e^{-2\pi^2 f^2 (r_\ell - r_{\ell'})^T \Sigma_\theta (r_\ell - r_{\ell'})}$$

Σ_θ depends on 6 free parameters.

Remark 6 (Unities) *the covariance matrix Σ_θ is in s^2/m^2. It follows that $f^2\Sigma_\theta$ is homogeneous at the inverse of the wavelength square.*

Using (C.1), we can derive an expression in term of azimuth a, elevation e, velocity c. We have:

$$\Sigma_\theta = J\,\Sigma_{\text{aec}}\,J^T$$

Then

$$\begin{aligned}\log\Phi_e(2\pi f(r_\ell - r_{\ell'})) &= -2\pi^2\frac{f^2}{c^2}(r_\ell - r_{\ell'})^T K(a,e,c)\Sigma_{\text{aec}}K^T(a,e,c)(r_\ell - r_{\ell'})\\ &= -2\pi^2(\rho_\ell - \rho_{\ell'})^T K_{\text{aec}}\Sigma_{\text{aec}}K_{\text{aec}}^T(\rho_\ell - \rho_{\ell'})\end{aligned}$$

where $\rho = r/\lambda$ with $\lambda = c/f$ and

$$K_{\text{aec}} = \begin{bmatrix}\cos(a)\cos(e) & \sin(a)\sin(e) & c^{-1}\sin(a)\cos(e)\\ -\sin(a)\cos(e) & -\cos(a)\sin(e) & -c^{-1}\cos(a)\cos(e)\\ 0 & \cos(e) & -c^{-1}\sin(e)\end{bmatrix}$$

A simple case is if we take Σ_{aec} diagonal and hence depending on only 3 free parameters.

Appendix C

One-to-one mappings and Jacobians

θ **to** (a, e, c)

If we consider the one-to-one mapping θ to (a, e, c) in $(0, 2\pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^+$, we can write:

$$\left\{\begin{array}{ll} \theta_1 = -c^{-1}\sin(a)\cos(e) & \theta_1 \in \mathbb{R} \\ \theta_2 = c^{-1}\cos(a)\cos(e) & \theta_2 \in \mathbb{R} \\ \theta_3 = c^{-1}\sin(e) & \theta_3 \in \mathbb{R} \end{array}\right. \quad \left\{\begin{array}{ll} a = \arg(\theta_2 - j\theta_1) & a \in (0, 2\pi) \\ e = \arg\sin(c\theta_3) & e \in (-\pi/2, \pi/2) \\ c = (\theta_1^2 + \theta_2^2 + \theta_3^2)^{-1/2} & c \geq 0 \end{array}\right.$$

whose the Jacobian is

$$\begin{aligned} J(a, e, c) &= \begin{bmatrix} -c^{-1}\cos(a)\cos(e) & c^{-1}\sin(a)\sin(e) & c^{-2}\sin(a)\cos(e) \\ -c^{-1}\sin(a)\cos(e) & -c^{-1}\cos(a)\sin(e) & -c^{-2}\cos(a)\cos(e) \\ 0 & c^{-1}\cos(e) & -c^{-2}\sin(e) \end{bmatrix} \quad \text{(C.1)} \\ &= c^{-1}\begin{bmatrix} \cos(a)\cos(e) & \sin(a)\sin(e) & c^{-1}\sin(a)\cos(e) \\ -\sin(a)\cos(e) & -\cos(a)\sin(e) & -c^{-1}\cos(a)\cos(e) \\ 0 & \cos(e) & -c^{-1}\sin(e) \end{bmatrix} \\ &= c^{-1}K_{\text{aec}} \end{aligned}$$

θ **to** (a, e, v)

If we consider the one-to-one mapping θ to (a, e, v) in $(0, 2\pi) \times (-\pi/2, \pi/2) \times \mathbb{R}^+$, we can write:

$$\left\{\begin{array}{ll} \theta_1 = -v^{-1}\sin(a) & \theta_1 \in \mathbb{R} \\ \theta_2 = v^{-1}\cos(a) & \theta_2 \in \mathbb{R} \\ \theta_3 = v^{-1}\tan(e) & \theta_3 \in \mathbb{R} \end{array}\right. \quad \left\{\begin{array}{ll} a = \arg(\theta_2 - j\theta_1) & a \in (0, 2\pi) \\ e = \arg\tan(v\theta_3) & e \in (-\pi/2, \pi/2) \\ v = (\theta_1^2 + \theta_2^2)^{-1/2} & c \geq 0 \end{array}\right.$$

whose the Jacobian is

$$J(a, e, v) = \begin{bmatrix} -v^{-1}\cos(a) & 0 & v^{-2}\sin(a) \\ -v^{-1}\sin(a) & 0 & -v^{-2}\cos(a) \\ 0 & v^{-1}/\cos^2(e) & -v^{-2}\tan(e) \end{bmatrix} \quad \text{(C.2)}$$

Appendix D

Estimation of parameters

we have

$$
\begin{aligned}
\log C_{k,m,m'} &= -2\pi^2 f_k^2 (r_m - r_{m'})^T \Gamma (r_m - r_{m'}) \\
&= -2\pi^2 f_k^2 (r_{m,1} - r_{m',1})^2 \Gamma_{11} - 2\pi^2 f_k^2 (r_{m,2} - r_{m',2})^2 \Gamma_{22} - 4\pi^2 f_k^2 (r_{m,1} - r_{m',1})(r_{m,2} - r_{m',2}) \Gamma_{12} \\
&= \mu_1(k,m,m')\Gamma_{11} + \mu_2(k,m,m')\Gamma_{22} + 2\mu_3(k,m,m')\Gamma_{12}
\end{aligned}
$$

Enumerating $k = 1:K$, $m = 1:M$, $m' = 1:M$ with $m' > m$ leads to $KM(M-1)/2$ equations with 3 unknowns.