3D point correspondences in the LiDAR Frame (or point cloud registration)

$$P = [\underline{p}_1, \underline{p}_2, \underline{p}_3, \dots, \underline{p}_n] \quad , \quad \underline{p}_i \in \mathbb{R}^3$$

$$Q = [\underline{q}_1, \underline{q}_2, \dots, \underline{q}_n] \quad , \quad \underline{q}_i \in \mathbb{R}^3$$

n : No of points in each frame.

*Let P and Q be 2 sets of corresponding points in $\mathbb{R}^3$.
Note that here we know that $\underline{p}_i$ maps to $\underline{q}_i$: we want to find a rigid transformation that optimally aligns these two sets in least squares sense,

=> we seek a rotation R and a translation $\underline{t}$ such that:

$$(R, \underline{t}) = \underset{\substack{R \in SO(3), \\ \underline{t} \in \mathbb{R}^3}}{\operatorname{argmin}} \sum_{i=1}^{n} \| (R\underline{p}_i + \underline{t}) - \underline{q}_i \|^2$$

* Compute translation

$$F(\underline{t}) = \sum_{i=1}^{n} \| R\underline{p}_i + \underline{t} - \underline{q}_i \|^2$$

$$0 = \frac{\partial F}{\partial \underline{t}} = 2 \sum_{i=1}^{n} (R\underline{p}_i + \underline{t} - \underline{q}_i) = R \sum_{i=1}^{n} \underline{p}_i + \underline{t} \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} \underline{q}_i$$

(or)

$$\boxed{\underline{t} = \frac{1}{n} \sum_{i=1}^{n} \underline{q}_i - R \frac{1}{n} \sum_{i=1}^{n} \underline{p}_i}$$

(or)

$$\boxed{\underline{t} = \bar{Q} - R\bar{P}}$$

we can see that our problem reduces to.

$$(R, \underline{t}) = \underset{\substack{R \in SO(3), \\ \underline{t} \in \mathbb{R}^3}}{\operatorname{argmin}} \sum_{i=1}^{n} \| R\underline{a}_i - \underline{b}_i \|$$

$$\underline{a}_i = \underline{p}_i - \bar{P}$$

$$\underline{b}_i = \underline{q}_i - \bar{Q}$$

either make centroids of P & Q equal to zero Or make centroid of P equal to centroid of Q.

* Compute rotation

Substitute $t = \bar{Q} - R\bar{P}$ into the initial equation

$$R = \underset{R \in SO(3)}{\arg\min} \sum_{i=1}^{n} \| R(p_i - \bar{P}) - (q_i - \bar{Q}) \|^2$$

Let $\underset{1\times 3}{X_i} = \underline{p_i} - \underline{\bar{P}}$ and $\underset{1\times 3}{Y_i} = \underline{q_i} - \underline{\bar{Q}}$

$$X_i' = R X_i$$

$$\sum_{i=1}^{n} \| X_i' - Y_i \|^2 = \text{Trace}\left((X' - Y)^T (X' - Y) \right)$$

Derivation: $\| X_i - Y_i \|^2 = \sum_{j=1}^{3} (x_{ij} - y_{ij})^2$

$$X - Y = [\, X_1 - Y_1 \quad X_2 - Y_2 \quad \cdots \quad X_n - Y_n \,]_{3\times n}$$

$$X_i - Y_i = \begin{bmatrix} x_{i1} - y_{i1} \\ x_{i2} - y_{i2} \\ x_{i3} - y_{i3} \end{bmatrix}$$

$$[X - Y]^T [X - Y] = \begin{bmatrix} [X_1 - Y_1]^T_{1\times 3} \\ [X_2 - Y_2]^T_{1\times 3} \\ \vdots \\ [X_n - Y_n]^T_{1\times 3} \end{bmatrix} \begin{bmatrix} \underset{3\times 1}{[X_1 - Y_1]} & \underset{3\times 1}{[X_2 - Y_2]} & \underset{3\times}{[X_3 - Y_3]} & \cdots & \underset{3\times 1}{[X_n - Y_n]} \end{bmatrix}$$

$$= \begin{bmatrix} [X_1 - Y_1]^T [X_1 - Y_1] & [X_1 - Y_1]^T [X_2 - Y_2] & \cdots & [X_1 - Y_1]^T [X_n - Y_n] \\ [X_2 - Y_2]^T [X_1 - Y_1] & [X_2 - Y_2]^T [X_2 - Y_2] & \cdots & [X_2 - Y_2]^T [X_n - Y_n] \\ \vdots & \vdots & & \vdots \\ [X_n - Y_n]^T [X_1 - Y_1] & [X_n - Y_n]^T [X_2 - Y_2] & \cdots & [X_n - Y_n]^T [X_n - Y_n] \end{bmatrix}_{n\times n}$$

$$\mathrm{Trace}((X' - Y)^T (X' - Y)) = \mathrm{Trace}(X'^T X') + \mathrm{Trace}(Y^T Y) - 2\,\mathrm{Trace}(Y^T X)$$

R is orthonormal $\Rightarrow |x_i'|^2 = |x_i|^2$

$$\mathrm{Trace}((X' - Y)^T (X' - Y)) = \underbrace{\sum_{i=1}^{n} (|x_i|^2 + |y_i|^2)}_{\text{independent of } R.} - 2\,\mathrm{Tr}(Y^T X')$$

$$\Rightarrow \quad R = \underset{R \in SO(3)}{\mathrm{argmax}} \; \mathrm{Tr}(Y^T X')$$

$$\mathrm{Tr}(Y^T X') = \mathrm{Tr}(Y^T X R) = \mathrm{Tr}(X Y^T R)$$

$$\boxed{X Y^T = U D V^T}$$

SVD

Let A be an $m \times n$ matrix, $m \geq n$.

A can be factorized as,

$$\boxed{A = U D V^T}$$

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U: $m \times n$ matrix with ~~org~~ orthogonal columns. i.e. $U^T U = I_{n \times n}$

D: $n \times n$ diagonal matrix of singular values in descending orders, all ≥ 0

V: $n \times n$ ~~of~~ orthogonal matrix, whose columns are singular vectors corresponding to singular values of D.

$$Tr(XY^T R) = Tr(UDV^T R) = Tr(DV^T RU) = \sum_{i=1}^{3} d_i v_i^T R u_i$$

$$M = V^T R U$$

$$Tr(Y^T X') = \sum_{i=1}^{3} d_i M_{ii} \leq \sum_{i=1}^{3} d_i$$

Maximum value of $Tr(Y^T X')$ occurs when $M = I$.

$$V^T R U = I \qquad\qquad V V^T = U U^T = I$$

$$\Rightarrow \boxed{R = V U^T}$$

properties of this method : OPTIMAL DIRECT

To ensure that the rotation matrix ; $R \in SO(3)$, we need to make sure that $\det(R) = +1$.

if $R = VU^T$ ~~is~~ has $\det(R) = -1$, we nee to find R s.t. $Tr(Y^T X')$ takes the second largest value possible.

~~$Tr(Y^T X')$~~ $Tr(Y^T X') = d_1 M_{11} + d_2 M_{22} + d_3 M_{33}$, $d_1 \geq d_2 \geq d_3$ & $|M_{ii}| \leq 1$

$\Rightarrow$ 2^{nd} Largest value occurs when $M_{11} = M_{22} = 1$ & $M_{33} = -1$

The above observation can be simplified by multiply by a matrix C which ~~at~~ automatically makes R_{33} equal to $-R_{33}$ when needed.

$$\Rightarrow \boxed{R = V C U^T}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(VU^T) \end{bmatrix}$$

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