Lab 2: Loss and risk

Welcome to the second Data 102 lab!

The goal of this lab is to introduce loss functions in hypothesis testing problems.

The code and responses you need to write are commented out with a message **"TODO: fill ..."**. There is additional documentation for each part as you go along.

Collaboration Policy

Data science is a collaborative activity. While you may talk with others about the labs, we ask that you write your solutions individually. If you do discuss the assignments with others please include their names in the cell below.

Submission

To submit this assignment, rerun the notebook from scratch (by selecting Kernel > Restart & Run all), and then print as a pdf (File > download as > pdf) and submit it to Gradescope.

For full credit, this assignment should be completed and submitted before Wednesday, Sep 15, 2021 at 11:59 PM. PST

Collaborators

Write the names of your collaborators in this cell.

<Collaborator Name> <Collaborator e-mail>

```
In [37]:
```

```
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd
import seaborn as sns
from scipy.stats import norm
import itertools
from ipywidgets import interact, interactive

import hashlib
%matplotlib inline

sns.set(style="dark")
plt.style.use("ggplot")

def get_hash(num):
    """Helper function for assessing correctness"""
    return hashlib.md5(str(num).encode()).hexdigest()
```

Question 1: Custom Loss Functions for Hypothesis Testing

The first question looks at a medical diagnostic decisions. For each person that undergoes testing, the null hypothesis is that they don't have the virus, and the alternative is that they do.

The null hypothesis H_0 : Person X does not have the virus.

The alternative hypothesis H_1 : Person X has the virus.

Suppose that you devise a test which takes some measurements from each test subject and then computes a corresponding p-value.

Last week we looked at several approaches for controllling False Positive Rate (FPR), Family Wise Error Rate (FWER) and False Discovery Rate (FDR). However, they all have some drawbacks for medical decision making: FPR and FWER do not depend on the prevalence of the disease and neither of them allows a decision maker to consider different misclassification costs arising from false-negative and false-positive diagnoses.

When making medical decisions, wrong diagnoses carry different costs. Deciding that a patient does not have the virus when in reality they do is a **False Negative**. The potential consequences of such decision are severe: lack of treatment, risk of infecting others, and even premature death.

On the other hand, deciding that a patient has the virus when in reality they don't is a **False Positive**. The potential consequences of that include distress, unnecesary treatment and costs of subsequent testing. This is certainly not ideal, but less severe than the consequences of a false negative.

We've previously evaluated decisions in terms of their TPR and FPR, and showed how ROC curves demonstrate the trade-off curve between the two quantities. We saw that it is not always clear how to choose the best trade-off.

A very popular way of choosing the trade-off, and simultaneously comparing different procedures, is the idea of **risk** that we learnt in Lecture 5. Here, the analyst constructs a loss function by specifying the **cost** of making each type of mistake.

Let's assume that we estimate the cost of making a false negative mistake to be k-times larger than the cost of a false positive. We can express that via a **loss function**:

Checkpoint: fill in the missing value here

$$\begin{cases} l(D=1|R=0) = 1\\ l(D=0|R=1) = K\\ l(D=0|R=0) = l(D=1|R=1) = 0 \end{cases}$$

Checkpoint: Does FPR and FWER depend on the prevalence of the disease? They do not depend on prevalence of the disease.

Checkpoint: Above, you were given one example of when the consequences of a False Negative is more sever than the consequences of a False Positive. Come up with one example of the opposite: when the consequences of a False Positive is more severe than the consequences of a False Negative.

Please include the null and alternate hypothesis as well as what FP and FN would correspond to & what are the consequences of FP and FN with respect to your null and alternate hypothesis.

Null: The heart surgeon is not qualified. Alternative: The heart surgeon is qualified. FP: The heart surgeon doctor is not qualified but we think he's qualified, this will risk people's life. FN: The heart surgeon is qualified but we

think he is not, this will cause us to miss one ualified candidate.

1.a: Compute average empirical loss

In [38]:

TODO: fill in

Test passed!

Assume we have a sample of N patients for which a test outputs the following confusion matrix:

```
        Decision = 0
        Decision = 1

        Reality = 0
        TN
        FP

        Reality = 1
        FN
        TP
```

Compute the average loss this procedure incurs by summing up the losses for every mis-diagnosis and then dividing by the total number of tests.

```
def compute_average_loss(results_dictionary, factor_k):
              """ Function that computes average loss for a given confusion matrix and a multiplical
                  k_factor that compares the consequences of false nagatives and false positives.
                  Inputs:
                      results_dictionary : a dictionary with the counts of TP, FP, TN and FN
                      k_factor : float, quantifies the ratio of the negative consequences of
                                 false negatives compared to false positives
                  Outputs:
                      average_loss : float
              0.00
              TP_count = results_dictionary['TP_count']
              FP_count = results_dictionary['FP_count']
              TN_count = results_dictionary['TN_count']
              FN_count = results_dictionary['FN_count']
              average_loss = (FP_count+FN_count*factor_k)/(FP_count+FN_count+TP_count+TN_count)
              return(average_loss)
In [39]:
          # Running validation tests: Do not modify
          res_dict = {'TP_count': 100, 'FP_count': 20, 'TN_count':450, 'FN_count':30}
          k_{factors} = [0, 10, 100]
          hash_list = ['ab51c1986c756154d0e3feb4f5a5e829','91f25a837f3db78306b0ad2ef0437ff9','3fc18@
          for i, k in enumerate(k_factors):
              average_loss = compute_average_loss(res_dict, k)
              print("For k = {}), the average loss is {:.3f}".format(k, average_loss))
              assert hash_list[i] == get_hash(average_loss)
          print('Test passed!')
         For k = 0, the average loss is 0.033
         For k = 10, the average loss is 0.533
         For k = 100, the average loss is 5.033
```

1.b: Compute the average loss (empirical risk) with respect to various levels α

In this part we will use a simple test that rejects the null hypothesis whenever the p-value of a patient is smaller than some level α .

Our goal is to investigate the performance of the test at different levels with respect to the custom loss defined in **1.a**.

Hint: Recall the naive test from Lab 1.

```
In [40]: # Note: don't make any changes to this function, this is exatly the naive thresholding you
def alpha_threshold_decisions(p_values, alpha):
    """
    Returns decisions on p-values using naive thresholding.

Inputs:
    p_values: array of p-values
    alpha: threshold (significance level)

Returns:
    decisions: binary array of same length as p-values, where `decisions[i]` is 1
    if `p_values[i]` is deemed significant at level `alpha`, and 0 otherwize
    """
    decisions = p_values <= alpha
    return decisions</pre>
```

Let's also bring in from Lab 01 the function that computes the counts of TP, TN, FP, FN by comparing the decision to the reality.

```
In [41]:
          # Note: don't make any changes to this function, this is the report_results function you
          def report_results(decisions, reality):
              Produces a dictionary with counts for the true positives, true negatives,
              false negatives, and false positives from the input `decisions`
              and `reality` arrays.
              Inputs:
                decision: array of 0/1 values where 1 indicates that patient has the virus.
                reality: array of 0/1 values where 1 indicates a draw from the alternative.
              Outputs: a dictionary of TN, TP, FN, and FP counts.
              TP_count = np.sum(decisions*reality)
              TN_count = np.sum((1-decisions)*(1-reality))
              FP_count = np.sum((decisions)*(1-reality))
              FN_count = np.sum((1-decisions)*(reality))
              results_dictionary = {"TN_count": TN_count,
                                    "TP_count": TP_count,
                                    "FN_count": FN_count,
                                    "FP_count": FP_count,
              return results_dictionary
```

First, we will generate ground truth data.

Assume there are N subjects, out of which a fraction trully have the virus. This fraction is known as **prevalence**: $\mathbb{P}\{R=1\}$.

The function below generates p-values associated with each test subject.

```
In [42]: # Note: don't make any changes to this function
def generate_ground_truth(N, prevalence):
    """ Creates simulated p-values corresponding to N subjects at a
```

```
rs = np.random.RandomState(1)
    reality = rs.binomial(1, prevalence, N)
    p_values = 1 - norm.cdf(rs.randn(N) + reality)
    return(p_values, reality)

In [43]:
# Generate p-values: Do not modify
N = 10000
prevalence = 0.1
```

In the cell below complete the function that computes the average loss (empirical risk) for an α level test.

specified disease prevalence level"""

p_values, reality = generate_ground_truth(N, prevalence)

```
In [44]:
          # TODO: complete the function
          def compute_alpha_average_loss(p_values, reality, alpha, factor_k):
              Computes the observed average loss for an alpha level test.
              Inputs:
                  p_values: array of floats, p-value[i] is the p-values associated with test subject
                  reality: array of 0/1s
                  alpha: float, threshold for rejecting the null hypothesis
                  factor_k: float, quantifies the ratio of the negative consequences of
                            false negatives compared to false positives
              Outputs:
                  average_loss: float, average observed loss
              # HINT: Your code should take advantage of functions already defined in this notebook.
              #our_decision = alpha_threshold_decisions(p_values, alpha)
              #our_result = compute_average_loss(decision, factor k)
              average_loss = compute_average_loss( report_results(alpha_threshold_decisions(p_values
              return average_loss
```

```
In [45]:
           # Validation tests do not modify
           alpha_values = [0.05, 0.1, 0.2, 0.4]
           k_{factors} = [0, 10, 100]
           inputs = list(itertools.product(alpha_values, k_factors))
           outputs = [compute_alpha_average_loss(p_values, reality, *inp) for inp in inputs]
           hash_list = ['8d0a5f87d0563079e70969b527b22b2c', '1370c7dde8b12484e1f9f3376ec72693',
                         'f106401a6d5ceaa1a89d99bcc773f8db', 'd92f2ab695c640d68b7b0f055704d892',
                         '066517035812371facdef5f27ff5c7c9', '91709ca4894f058b311f697527479564', '49f12f9c625759e7bb9e3f2fe5cee530', '38888496421ef7c75527678f1e718c19',
                         '701a4e33513e8e55f91eac027a42b6fe', 'a6107b1736a62ac0c1c79546cc4ec85f',
                         'd1d940b705c4e66fc40cd8ea1c2f7c57', 'a57b99387860e7d6b6de25ea47201a0b']
           for i, inp in enumerate(inputs):
               print('At level alpha={} and k={} the average loss is {:.3f}'.format(*inp, outputs[i])
               assert(get_hash(outputs[i])==hash_list[i])
           print('Test passed!')
          At level alpha=0.05 and k=0 the average loss is 0.047
```

At level alpha=0.05 and k=10 the average loss is 0.743 At level alpha=0.05 and k=100 the average loss is 7.007 At level alpha=0.1 and k=0 the average loss is 0.097 At level alpha=0.1 and k=10 the average loss is 0.680 At level alpha=0.1 and k=100 the average loss is 5.936

```
At level alpha=0.2 and k=0 the average loss is 0.186 At level alpha=0.2 and k=10 the average loss is 0.614 At level alpha=0.2 and k=100 the average loss is 4.466 At level alpha=0.4 and k=0 the average loss is 0.369 At level alpha=0.4 and k=10 the average loss is 0.594 At level alpha=0.4 and k=100 the average loss is 2.619 Test passed!
```

1.c Investigate the average loss plot for different levels of disease prevalence

The function below generates ground truth for a sample of 10000 subjects for a given disease prevalence. It then computes the average loss for diagnostic decisions at level α , where $\alpha \in [0,1]$. Finally, it plots the resulting average loss (y axis) at a level α (x axis).

```
In [46]:
          # Run this as is after completing the `compute_alpha_average_loss` function
          # Do not modify
          def plot_average_loss(prevalence, factor_k):
              N = 10000
              # generate ground truth
              p_values, reality = generate_ground_truth(N, prevalence)
              # vary alpha from 0 to 1
              alpha_array = np.arange(0,1, 0.05)
              # compute average loss for each alpha
              average_loss_array = [compute_alpha_average_loss(p_values, reality, alpha, factor_k) 1
              optimal_alpha = alpha_array[np.argmin(average_loss_array)]
              plt.figure(figsize=(10, 6))
              plt.plot(alpha_array, average_loss_array, label = 'Average Loss')
              plt.axvline(x=optimal_alpha, ls='--', label = 'Optimal $\\alpha$', c='green')
              plt.xlabel('$\\alpha$ level')
              plt.ylabel('Average Loss')
              plt.legend()
              plt.show()
```

```
In [47]: # Visualize interactive plot: Do not modify
interactive_plot = interactive(plot_average_loss, prevalence=(0.001, 0.11, 0.01), factor_k
interactive_plot
```

1.c (i) Fix the prevalence of the disease at 0.05 (5%). Using the slider in the interactive plot above, try out different values for the multiplicative factor_k . What do you notice? How would you adjust your diagnosic procedure based on the value of factor_k ? What combination of factor_k and α gives you the lowest possible loss, and why?

When Factor_k become bigger, the average loss starts at a higher level and decays exponentially as alpha becomes bigger, so we need a bigger alpha when Factor_k is big. This makes sense because if Factork is big, this means the consequence of flase negative is bigger than false positive so we need to make more discoveries to balance out the consequence of false negative. When factor k become smaller, the average loss starts at a smaller value and increases exponentially as alpha increases, so we need to choose a smaller alpha when Factor_k is small. This makes sense because if Factor_k is small, this meanshe consequence of false negative is lesser than that of false positive, so we need to be more conservative when making a decision to prevent too much false positives.

1.c (ii) Fix factor_k at 50 (meaning that the negative consequence of a false negative are 50 times larger than the negative consequences of a false positive). Using the slider in the interactive plot above, try out different values for the true prevalence of the disease. What do you notice? How would you adjust your

diagnostic procedure based on the prevalence of the disease? What combination of prevalence and α gives you the lowest possible loss, and why?

When we have a very big prevalence, the average loss function decays exponentially as alpha increases, so we need a larger alpha when prevalence is big. This makes sense because if prevalence is high, that maeans most cases are positive in reality, so we need to be more liberal towards discoveries, so our overall false negative count is smaller. When the prevalence is very small, say 0.01, the loss function increases exponentially as alpha increases, so we need a smaller alpha when prevalence is big. This makes sense because if prevalence is small, that means most cases are negative in reality, so we need to be more conservative when making discoveries to limit the amount of false positives.

Question 2.

In the previous question you played the role of a test designer or device manufacturer that needs to find an appropriate way to calibrate the test such that it minizes some desired loss. In this part put yourself in the shoes of a medical professional who is using this testing device without really knowing all the internals of it.

The test kit claims a certain specificity (1 - FPR) and sensitivity (1 - FNR).

Assume you have a new patient that came in and tested positive (you have only the binary output of the test). Your goal is to determine whether or not to put this patient through treatment.

To answer this we weill make the following assumptions:

- Assume as in part 1, that getting a false negative is k times worse than getting a false positive.
- Assume that you know the prevalence of this disease.
- Assume that the test has a certain specificity and sensitivity.

2.a Compute the posterior

Complete function below to compute the posterior probability that the patient truly has the disease conditioned on a positive test result: namely, compute $\mathbb{P}\{R=1|D=1\}$ as a function of prevalence, sensitivity and specificity.

```
In [48]:
# TODO: complete the function
def compute_posterior_probability(prevalence, sensitivity, specificity):
    """
    Computes the posterior probability that the patient trully has the disease
    conditioned on a positive test result.

Inputs:
    prevalence: float, fraction of the population that has the disease
    sensitivity: float, 1 - false negative rate
    specificity: float, 1 - false positive rate

Outputs:
    posterior probability: probability that the patient has the disease given a positium
posterior_probability = sensitivity*prevalence/ ((sensitivity*prevalence)+((1-specifireturn round(posterior_probability, 3))
```

Hint: We've already seen this in HW1 and discussion 2

```
# Let's compute a few posterior probabilities
prevalence = [0.001, 0.1]
sensitivity = 0.98
specificity = 0.99
print('Conditioned on a positive test result, with sensitivity {} and specificity {}:'.for
print('For a low prevalence disesase ({}), the posterior probability that the test subject
    format(prevalence[0], compute_posterior_probability(prevalence[0], sensitivity, spec
print('For a high prevalence disesase ({}), the posterior probability that the test subject
    format(prevalence[1], compute_posterior_probability(prevalence[1], sensitivity, spec)
```

Conditioned on a positive test result, with sensitivity 0.98 and specificity 0.99: For a low prevalence disesase (0.001), the posterior probability that the test subject is truly positive is 0.089
For a high prevalence disesase (0.1), the posterior probability that the test subject is truly positive is 0.916

```
In [50]:
           # Validation tests: Do not modify
           prevalences = [0.001, 0.01, 0.1]
           sensitivities = [1, 0.95, 0.9, 0.8]
           specificities = [1, 0.99, 0.95, 0.9]
           inputs = list(itertools.product(prevalences, sensitivities, specificities))
           hash_list = ['e4c2e8edac362acab7123654b9e73432','02d032eccec8949f6af6e04e6cdd3a8d',
            '0d16a6cb2ca34d65da949608a7bc01d1','04817efd11c15364a6ec239780038862',
            'e4c2e8edac362acab7123654b9e73432','fdfff99d8a4957f3fccd95640f0a42e6','9535d704a247fcd285782be1c551ecbd','45e6e163a7efd52d15f22976ec0f69a5',
            'e4c2e8edac362acab7123654b9e73432','9068cf4574981fea8addf5d891282c1b'
            '4d861ace718e35d16925a986b74e459f','45e6e163a7efd52d15f22976ec0f69a5',
            'e4c2e8edac362acab7123654b9e73432','93cf3ca031fd3a5694d40fc4c0a2d0b0'
            'ba4b140343c1a0b1ad0dd842f6b6a4f6','e0d088d01b309e29e7e12e7b657d04c6'
            'e4c2e8edac362acab7123654b9e73432','a7473b0d7fc57e553ee66811867c7c5a',
            'c61adf9c548ef03bc82f1758a5290f50','052fbf0fbad7b4a752d34e1dc0e76cfa'
            'e4c2e8edac362acab7123654b9e73432', 'ab01272978f4106a12ede6fe2a850be4'
            '924df509ef4290478c98958694da991e','3733db24881f3632a5c68823d705077c'
            'e4c2e8edac362acab7123654b9e73432','9e89f1ca90178f2aba46ef74daf55f2e',
            'c119c1e8ac146629b48d7e173155659d','9068cf4574981fea8addf5d891282c1b'
'e4c2e8edac362acab7123654b9e73432','66bf30c49ea8026353fd6b17fc915dab'
            '711ad51d1d8b437c5425ca486f7e4d3c', '8afae07169cd20be8708f022eec2ca75',
            'e4c2e8edac362acab7123654b9e73432','277a3442230767b6c34ff69f1caafbfa',
            '2ecbf51392f8c6f3ea44bd7fc2b00efc','061dcd6671866898f6104fdc9ee4907d'
'e4c2e8edac362acab7123654b9e73432','ac9ded11a9d24e0a4098f94ef04d5bd5'
            '33a9cca7782132c27407c8afff0e533a','4181aff438aab51cc6bd64da498f3f24',
            'e4c2e8edac362acab7123654b9e73432','4d94e4ce9554bd92202036432dcd774c'
            '27df4064836a51dbcd313df8d81e02ed','d310cb367d993fb6fb584b198a2fd72c'
            'e4c2e8edac362acab7123654b9e73432', 'a6fb83d730a6b986b3d7715c97485cdd',
            '100dd920921e3c509b9afd9dff04f149','b18a4559a7442eb1c25d3fb936e0a159']
           for inp, hash_val in zip(inputs, hash_list):
                output = compute_posterior_probability(*inp)
                assert (get_hash(output) == hash_val)
           print('Test passed!')
```

Test passed!

2.b Compute expected loss function with respect to posterior distribution

Given that the test returned a positive result (that D=1), you can make one of two posible decisions:

- T=1: start the treatment
- T=0: do not start the treatment despite the positive test result

Similarly to Question 1, let's assume that we estimate the cost of not treating a truly sick patient to be k times larger than the cost of treating a patient that is not truly sick.

Recall from lecture that a loss function takes in a hidden state of the world θ (in this case, that's the reality R: whether or not the patient is sick), and a decision δ (in this case, that's T: whether or not to treat). Our loss function has the formula:

$$\begin{cases} \ell(R=0,T=1)=1\\ \ell(R=1,T=0)=k\\ \ell(R=0,T=0)=\ell(R=1,T=1)=0 \end{cases}$$

Compute the expected loss for each treatment decision, given that someone tested positive:

$$\mathbb{E}[l(R, T = 0)|D = 1] = ?$$

 $\mathbb{E}[l(R, T = 1)|D = 1] = ?$

Hint: Think carefully about what is random here. What's it's distribution?

```
# TODO: complete the function
def compute_expected_loss(treatment, posterior_probability, factor_k):
    """
    Compute the expected loss for the treatment.

Inputs:
          treatment: int 0/1 (0-no treatment, 1-treatment)
          posterior_probability: float, probability that the patient is truly sick given post k_factor: float, quantifies the ratio of the negative consequences of false negatives compared to false positives

"""
    expected_loss =((1-posterior_probability)**treatment)*(posterior_probability*factor_k)
    return(expected_loss)
```

```
In [52]:
          # Run validation tests: Do not modify
          treatments = [0, 1]
          k_{factors} = [0, 10, 100]
          posterior_probabilities = [0.1, 0.5, 0.9]
          inputs = list(itertools.product(treatments, posterior_probabilities, k_factors))
          hash_list = ['30565a8911a6bb487e3745c0ea3c8224', 'e4c2e8edac362acab7123654b9e73432',
                        '43a1437f7f656cd8be7c996c58719e0a', '30565a8911a6bb487e3745c0ea3c8224',
                        '336669dbe720233ed5577ddf81b653d3', '88bce6f1bd04b8521f1167b5a6dec118'
                        '30565a8911a6bb487e3745c0ea3c8224', 'cf5f238fca8b6e155078aa41c175743a'
                        'a5efe444b4090e202d78340494947f63', 'a894124cc6d5c5c71afe060d5dde0762',
                        'a894124cc6d5c5c71afe060d5dde0762', 'a894124cc6d5c5c71afe060d5dde0762'
                        'd310cb367d993fb6fb584b198a2fd72c', 'd310cb367d993fb6fb584b198a2fd72c', 'ec705edd9065ac64dc3985903df2e2e6',
                        'ec705edd9065ac64dc3985903df2e2e6', 'ec705edd9065ac64dc3985903df2e2e6']
          for i, inp in enumerate(inputs):
               assert (get_hash(compute_expected_loss(*inp)) == hash_list[i])
          print('Test passed!')
```

Test passed!

2.c Loss and Risk

Is the quantity you computed above a frequentist risk, a Bayesian posterior risk, or neither? Explain why in two sentences or less.

We are dealing with Bayesian Risk, since we are only given the decision, but the reality here is random

2.d Decide whether or not to administer the treatment by comparing the expected losses in each case.

Compare the cost for treatment T=1 and no treatment T=0 and choose the option with lower expected loss.

```
In [53]:
          # TODO: complete the function
          def make_decision(posterior_probability, factor_k):
              Make a decisions to adminster or not the treatment: T=1 or T=0 .
              Inputs:
                  posterior_probability: float, probability that the patient is truly sick given post
                  k_factor : float, quantifies the ratio of the negative consequences of
                             false negatives compared to false positives
              Outputs:
                  treatment: int, 0/1
              if compute_expected_loss(1, posterior_probability, factor_k)<compute_expected_loss(0,</pre>
                  return 1
              else:
                  return 0
              return treatment
In [54]:
          k_{factors} = [0, 10, 100]
          posterior_probabilities = [0.1, 0.5, 0.9]
          inputs= list(itertools.product(posterior_probabilities, k_factors))
          hash_list = ['cfcd208495d565ef66e7dff9f98764da', 'c4ca4238a0b923820dcc509a6f75849b',
                        'c4ca4238a0b923820dcc509a6f75849b', 'cfcd208495d565ef66e7dff9f98764da',
                       'c4ca4238a0b923820dcc509a6f75849b', 'c4ca4238a0b923820dcc509a6f75849b',
                       'cfcd208495d565ef66e7dff9f98764da', 'c4ca4238a0b923820dcc509a6f75849b',
                       'c4ca4238a0b923820dcc509a6f75849b']
          for i, inp in enumerate(inputs):
              assert (get_hash(make_decision(*inp)) == hash_list[i])
          print("All tests passed! You are awesome!")
          import matplotlib.image as mpimg
          img = mpimg.imread('baby_otter.jpg')
          imgplot = plt.imshow(img)
          imgplot.axes.get_xaxis().set_visible(False)
          imgplot.axes.get_yaxis().set_visible(False)
          plt.show()
```

All tests passed! You are awesome!



In []:		