ANALYSIS I - NOTES

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Contents

1. Proofs

In this course we have to learn how to make mathematical proofs. Example:

Proposition 1.1. *$\sqrt{2}$ is not a rational number*

Proof. Assume it is, that is we may write

$$\sqrt{2} = \frac{a}{b}$$

for some integers a and $b \neq 0$. As $\sqrt{2} > 0$, a and b should have the same sign. If they are both negative, by multiplying both by -1 we may assume that they are positive. Furthermore, by dividing by their common factors we may assume that a and b are relatively prime. By multiplying the above equation by b we obtain

$$b\sqrt{2} = a.$$

Taking square of the equation yields

$$b^2 \cdot 2 = a^2$$

Hence a is even, so $a = 2r$. But then are equation becomes:

$$b^2 \cdot 2 = (2r)^2 = 4r^2$$

So, $b^2 = 4r^2$, from which we see that b is also even, contradicting the relatively prime assumption on a and b. We obtained a contradiction with our original assumption, hence $\sqrt{2}$ is not a rational number. □

Remark 1.2. This proof is nice, but what is $\sqrt{2}$? Is it a real number? What is a real number anyways? We will get back to these questions.

But be careful, it is easy to write a wrong proof. Here is an example showing that 1 is the largest natural number:

Example 1.3. WRONG PROOF: We claim that 1 is the largest natural number. Indeed, let n be the largest natural number. Then $n \geq n^2$, so $0 \geq n^2 - n = n(n-1)$, so either n or $(n-1)$ is at most 0. So, $n \leq 0$ or $n \leq 1$.

Of course, this is all craziness after the absurd assumption in the first sentence. In fact, there does not need to be a largest element in a set of natural numbers.

Analysis mostly treats infinity. A great basic example is:

Proposition 1.4. $0.999999999\ldots = 1$

Proof. We give two proofs none of which are completely correct. Nevertheless we explain carefully the issues with them, and how these will be cleared up during this course.

(1) First an elementary proof:

$$9 * 0.99\ldots = (10-1) * 0.99\ldots = 10 * 0.99\ldots - 1 * 0.99\ldots = 9.99\ldots - 0.99\ldots = 9$$

So, $0.99\ldots = 9/9 = 1$.

This proof is reasonably OK. However, it assumes that we know what $0.99\ldots$ is. It also assumes that we can manipulate it as usually algebraically. None of these are that clear if you think about it deeper.

So, what is $0.99\ldots$?

What algebraic manipulations are allowed with it?

(2) Now, analysis defines $0.99\ldots$ precisely. $0.99\cdots := \sum_{i=1}^{\infty} \frac{9}{10^i}$.

But, what is $\sum_{i=1}^{\infty} \frac{9}{10^i}$?

By definition it is $\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{9}{10^i} \right)$. Unfortunately we have not learned what lim is precisely, so we cannot quite continue in a precise way from here, nevertheless we continue the argument for completeness. If you are not comfortable with it now, it is completely OK, just skip to the next thing. However, before we proceed, we need to show an identity for the sum of elements in a geometric series:

$$a + a^2 + a^n = \frac{a - a^{n+1}}{1-a} \tag{1.4.a}$$

To prove this equality, we just multiply the left side by $1-a$ to obtain:

$$\begin{aligned}(a + a^2 + \cdot + a^n)(1-a) &= a - a \cdot a + a^2 - a^2 \cdot a + a^3 - \cdots - a^{n-1} \cdot a + a^n - a^n \cdot a \\ &= a - a^{n+1}\end{aligned}$$

This shows that (1.4.a) indeed holds.

And then we can proceed showing the statement:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{9}{10^i} = 9 \cdot \sum_{i=1}^{\infty} \frac{1}{10^i} = 9 \cdot \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{10^i} \right) &= 9 \cdot \lim_{n\to\infty} \left(\frac{\frac{1}{10} - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} \right) \\ &= 9 \cdot \frac{\frac{1}{10} - \lim_{n\to\infty} \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} = 9 \cdot \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 9\frac{1}{9} = 1.\end{aligned}$$

□

2. Sets

There are a couple of sets that we are going to work with:

(1) $\mathbb{N}$: set of natural numbers $\{0, 1, 2, \ldots\}$. $\mathbb{N}$ is well ordered, that is, all its subsets contain a smallest element.
(2) $\mathbb{N} \subseteq \mathbb{Z}$: set of integer numbers $\{\ldots, -1, 0, 1, \ldots\}$ ($\subseteq$ means that $\mathbb{N}$ is a subset of $\mathbb{Z}$, that is, each element of $\mathbb{N}$ is contained in $\mathbb{Z}$).
(3) $\mathbb{Z} \subseteq \mathbb{Q}$: set of rational number of the form $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$.
(4) $\mathbb{Q} \subseteq \mathbb{R}$: the set of real numbers. It is not easy to actually construct it. As you have seen above in case of $0.99\ldots$, it is not clear how to make sense of it. In fact, in this course we do not construct it, although there are many equivalent constructions (you can just go to the wikipedia page "construction of the real numbers").

So, instead of construction, in this course we list certain properties that uniquely defines $\mathbb{R}$ (we also do not prove this unicity, please believe it):

i $\mathbb{R}$ is an *ordered field*, that is, there is division, multiplication, addition, subtraction and comparison in $\mathbb{R}$ (see page 2 of the book for a precise list of axioms).
ii it satisfies the infimum axiom (to be discussed soon).

(5) generally if S is a set, one can always define $\{s \in S|\text{“ condition”}\}$ meaning the set of those elements of S that satisfy the given condition. For example:
i invertible elments: $\mathbb{R}^* := \{x \in \mathbb{R}|x \neq 0\}$.
ii half lines: $\mathbb{R}_+ := \{x \in \mathbb{R}|x > 0\}$, $\mathbb{R}_- := \{x \in \mathbb{R}|x < 0\}$
iii open bounded intervals: if $a < b \in \mathbb{R}$, then
$$]a, b[:= \{x \in \mathbb{R}|a < x < b\}$$
iv open ball centered around a with radius δ: if $a \in \mathbb{R}, \delta \in \mathbb{R}_+$, then
$$B(a, \delta) :=]a - \delta, a + \delta[$$
v closed bounded intervals: if $a < b \in \mathbb{R}$, then
$$[a, b] := \{x \in \mathbb{R}|a \leq x \leq b\}$$
vi half closed half open intervals: $]a, b]$ and $[a, b[$, guess the definition or look it up in 1.1.5 in the book
vii emptyset $\emptyset$: it has no elements

3. Bounds

For this section let $S \neq \emptyset$ denote a subset of $\mathbb{R}$.

Definition 3.1. $a \in \mathbb{R}$ is an *upper* (resp. *lower*) bound of S if $x \leq a$ (resp. $x \geq a$) holds for all $x \in S$.

If S has an upper/lower bound then it is called bounded from above/from below.

If S is bounded both from above and below, then it is called bounded.

The upper and lower bounds are not unique.

Example 3.2. (1) $\{\sin(n^2)|n \in \mathbb{Z}\}$ is bounded; examples of lower bounds are -5 and -13, example of upper bounds are 1 and 27;
(2) $\{n^2|n \in \mathbb{Z}\}$ is not bounded, but it is bounded from below
(3) $\{n^3|n \in \mathbb{Z}\}$ is not bonded from above/below

Definition 3.3. The maximum (resp. minimum) of S denoted by $\max S$ (resp. $\min S$) is an upper (resp. lower) bound of S which is contained in S

Example 3.4. (1) $]1, 2[$ does not have a minimum or a maximum
(2) $[1, 2]$ has both
(3) $S := \left\{ \frac{n-1}{n} \middle| n \in \mathbb{Z}_+ \right\}$ has a minimum, as the elements are larger as n increases, so $\frac{1-1}{1} = 0$ is the minimum. However it does not have a minimum, because:
i if $a < 1$, then $1 - \frac{1}{n} = \frac{n-1}{n} > a$ whenever $n > \frac{1}{1-a}$, so a cannot even be an upper bound
ii however, if $a \geq 1$, then $a \notin S$.

The last example hints that we might need a new notion, as 1 is almost like the maximum of S, just it is not in S. This is given by the next definition:

Definition 3.5. The *supremum* $\operatorname{Sup} S$ (resp. infimum $\inf S$) is the smallest upper bound (resp. largest lower bound), if it exists at all.

Example 3.6. (1) $\operatorname{Sup} \left\{ \frac{n-1}{n} \middle| n \in \mathbb{Z}_+ \right\} = 1$.
(2) $\operatorname{Sup}]a, b[= b$.
(3) $\operatorname{Inf}]a, b[= a$.
(4) $\operatorname{Inf}\{n^3|n \in \mathbb{Z}\}$ and $\operatorname{Sup}\{n^3|n \in \mathbb{Z}\}$ do not exist.

Axiom 3.7. Infimum axiom Each non-empty subset of $\mathbb{R}_+^*$ admits an infimum.

Corollary 3.8. *Each non-empty bounded from above (resp. below) subset $S \subseteq \mathbb{R}$ admits a supremum (resp. infimum).*

Proof. IDEA: shift/reflect S across the origin to turn it into a subset of $\mathbb{R}^*_+$. Then apply the infimum axiom. □

Proposition 3.9. *If $S \subseteq \mathbb{N}$, then* $\operatorname{Inf} S = \operatorname{Min} S$.

Proof. Set $d := \operatorname{Inf} S$. We have to show that $d \in S$. So, assume $d \notin S$. Then, as $\operatorname{Inf} S$ is the largest lower bound of S, for each $\varepsilon > 0$, $d + \varepsilon$ is not a lower bound. Hence:

(3.9.a) For each $\varepsilon > 0$, there is $s_\varepsilon \in S$, such that $s_\varepsilon < d + \varepsilon$.

Apply (3.9.a) for $\varepsilon' := \frac{1}{2}$. This yields $s_{\varepsilon'}$ such that

$$d < s_{\varepsilon'} < d + \varepsilon'$$

Apply then again the above property of S, but now for $\varepsilon'' := s_{\varepsilon'} - d$. We obtain $s_{\varepsilon''} \in S$ such that

$$d < s_{\varepsilon''} < d + \varepsilon'' = s_{\varepsilon'} < d + \varepsilon' = d + \frac{1}{2}.$$

In particular, $s_{\varepsilon''} < s_{\varepsilon'} < s_{\varepsilon''} + 1$. □

4. $\mathbb{Q}$ VS $\mathbb{R}$

4.1. *Integer part*

Let x be a positive real number, for example $\pi^2 + \pi$. According to Corollary 3.8 and Proposition 3.9, the set $S := \{n \in \mathbb{N} | n > x\}$ has an infimum, say N. Then $N - 1$ is not in S, we call it the *integer part* of x and we denote it by $[x]$. We call $\{x\} = x - [x]$ the *fractional part* of x.

For example $[\pi^2 + \pi] = 12$, and $\{\pi^2 + \pi\}$ is what it is, not a number that a human can write down. For rational numbers, things are a bit easier. For example, $[\frac{3}{2}] = 1$ and $\{\frac{3}{2}\} = \frac{1}{2}$.

If x is negative and it is not in $\mathbb{Z}$, we take $[x] = -[-x] - 1$; if x is in $\mathbb{Z}$ we take $[x] = x$. For example $[-7.5] = -8$ and $-7.5 = 0.5$.

4.2. $\sqrt{2}$ *is a real number*

Here is why $\sqrt{2}$ is a real number: it is $\operatorname{Sup}\{x \in \mathbb{R} | x^2 \leq 2\}$ (note this is obviously non-empty as 1 is in it, also it is bounded above by 2, so Sup exists; moreover, the Sup has to be bigger than 1, so it is positive). We do not prove here that the above Sup is indeed equal to $\sqrt{2}$, as it is a quite particular computation (see page 9 of the book).

4.3. *Rational numbers are everywhere*

Proposition 4.1. *If $a < b$ are real numbers, then there is a rational number c, such that $a < c < b$.*

Proof. Easy case, we assume $a = 0$:

We have $[\frac{1}{b}] + 1 > \frac{1}{b}$, morever $[\frac{1}{b}] + 1$ is a positive integer. We conclude that

$$0 < \frac{1}{[\frac{1}{b}] + 1} < b,$$

so we can take $\frac{1}{[\frac{1}{b}]+1}$ as c.

General case:

$$n := \left[\frac{1}{b-a}\right] + 1 \Rightarrow n > \frac{1}{b-a} \Rightarrow \frac{1}{n} < b - a$$

$$a = \frac{an}{n} < \frac{[an] + 1}{n} < \frac{an + 1}{n} = a + \frac{1}{n} < a + b - a = b$$

Furthermore, $\frac{[an]+1}{n}$ is a rational number. We conclude that we can take $c = \frac{[an]+1}{n}$ (this is not the unique rational number between a and b, it is just one example of rational number between a and b). $\square$

4.4. *Irrational numbers are everywhere*

Proposition 4.2. *If $a < b$ are real numbers, then there is an irrational number c, such that $a < c < b$.*

Proof. Apply Proposition 4.1 to $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. This yields a rational number d such that $\frac{a}{\sqrt{2}} < d < \frac{b}{\sqrt{2}}$. In particular, $a < \sqrt{2}d < b$. Furthermore, $\sqrt{2}d$ is irrational, as if it was rational, then $\sqrt{2}$ would also be rational. $\square$

5. Absolute value

The absolutely value $|x|$ of a real number x is equal to x is $x \geq 0$, and $-x$ if $x \leq 0$. For example $|3| = 3$, $|-5| = 5$, $|-\pi| = \pi$, $|0| = 0$ and $|5| = 5$. It is usefull to draw the graph. Another way to introduce the absolute value is the distance between x and 0 on the real line.

The absolute value behaves well with respect to the multiplication; for example, on one hand one has that $|5 \cdot (-3)| = |-15| = 15$, on the other hand $|5| \cdot |-3| = 5 \cdot 3 = 15$, this means that $|5 \cdot (-3)| = |5| \cdot |-3|$. Simlarly, $|(-\sqrt{2}) \cdot (-4)| = |4\sqrt{2}| = 4\sqrt{2}$ and $|-\sqrt{2}| \cdot |-4| = \sqrt{2} \cdot 4 = 4\sqrt{2}$, this means that $|(-\sqrt{2}) \cdot (-4)| = |-\sqrt{2}| \cdot |-4|$ This is true for any pair of real numbers, in symbols: $|xy| = |x||y|$. Analoguesly, one has $|\frac{5}{-4}| = \frac{|5|}{|-4|}$.

The absolute value is also need to relate powers and roost. For example $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$ and $\sqrt{(7)^2} = \sqrt{49} = 7 = |7|$. In symbols: $\sqrt{x^2} = |x|$.

Last but not least, let us remark that $-\sqrt{3}$ is between $-|-\sqrt{3}|$ and $|-\sqrt{3}|$. This is true for any number, and in symbols one writes that $-|x| \leq x \leq |x|$.

A deep propoerty of the absolute value is the triangle inequality. Can you draw a triangle with sides of lenght 1, 4, and 600 ? I do not think so. But you can draw a triangle of sides 3,4 and 6 (give it a try, you might need a compass).

The reason is that for every triangle, the sum of the lenght of two edgese is always bigger then the leg of the third edge. This implies a triangle inequality for the absolute value, we will understand better the realtion with triangles when dealing with complex numbers, let us give a couple of examples now:

$$|3 + (-7)| \leq |3| + |-7|$$

and

$$|(-5) + (-4)| \leq |-5| + |-4|$$

In general one has the following result

Proposition 5.1 (Triangle inequality)**.** *For any pairs of real numbers x and y one has*

$$|x + y| \leq |x| + |y|$$

Proof. Recall that $x \leq |x|$ and $y \leq |y|$. If $x + y \geq 0$, then $|x + y| = x + y \leq |x| + |y|$.

Simlarly, $x \geq -|x|$ and $y \geq -|y|$, so ff $x + y \leq 0$, then $|x + y| = -x - y \leq |x| + |y|$. $\square$

One has also the reverse triangle inequality

$$|x - y| \geq ||x| - |y||$$

you can look it up on the book at page 10.

6. Extended real number line

The extended real line is the set

$$\overline{\mathbb{R}} := \{-\infty, +\infty\} \cup \mathbb{R}$$

The symbols $+\infty$ and $-\infty$ are called plus infinity and minus infinity, *they are not numbers*, just symbols, so be very carefull to do no treat them as numbers.

Later in the course we will use extensively these symbols. For the time being, we just want to use them to define the following subsets of $\mathbb{R}$. For any real number a, the set $[a, +\infty[$ is the set of all real number x greater or equal than a. Similarly

$$]a, +\infty[= \{x \text{ such that } x > a\}$$

$$]-\infty, a] = \{x \text{ such that } x \leq a\}$$

$$]-\infty, a[= \{x \text{ such that } x < a\}$$

and finally $]-\infty, +\infty[$ is the full set of real numbers $\mathbb{R}$. These sets are also called interval, or extended intervals.

7. Complex numbers

To deifne the complex number we have to introduce a new number called i, the immaginary unit. This number is the square root on -1, so it has the property that $i^2 = -1$. The introduction of this new number can somehow be compared with the introduction of 0, or of the negative numbers. Let us give a more formal definition.

Definition 7.1. A complex number is an expression of the form $x + yi$, where x and y are real numbers, and i is the immaginary unit. The set of complex numbers is denoted by $\mathbb{C}$. Often elements of $\mathbb{C}$ are denoted with the letter z, so we will often write $z = x + yi$.

Taking $y = 0$, one sees that $\mathbb{R} \subset \mathbb{C}$, so for example 0, 3, and $-\pi$ are complex numbers. Other examples of complex numbers are $5 - i$, $3i$, $-2i$ and $\frac{1}{2} + \sqrt{2}i$. Complex numbers are not ordered, it makes no sense to ask if a complex number is bigger than another; in particular, it does not make sense to ask if a compex number is positive or negative

It is remarkable that the equation $x^2 = -1$ has no solution in the set of real numbers, but two distinct solutions in the set of complex numbers, namely i and $-i$.

We can add and multiply compex numbers using the standard formal properties of adddition and multiplication, always remebering that $i^2 = -1$.

Example 7.2.
- $(5 + 3i) + (2 - i) = (2 + 5) + (3 - 1)i = 7 + 2i$
- $(1 - 2i)(3 + 4i) = 3 - 6i + 4i - 8i^2 = 3 - 6i + 4i + 8 = 11 - 2i$

It is very important to identify the complex number with the plane, which is the called the complex plane. We have to use x and y as cartesian co-ordinates. The real line can be identified with the lines $\{y = 0\}$. Complex numbers become vectors, and the sum of complex numbers is equal to the sum of vecotrs. Multiply a complex number by a positive real number corresponds to scale the lenght of the vector.

A complex number z has a real part and an imaginary part, denoted by $\Re(z)$ and $\Im(z)$, these are the x and y co-ordinate in the plane.

Example 7.3.
- $\Re(5 + 3i) = 5$, $\Im(5 + 3i) = 3$
- $\Re(-3i) = 0$, $\Im(3i) = -3$

The modulus $|z|$ of a complex number z is its distance from the origin in the complex plane, it can be computed using Pythagorean Theorem.

Example 7.4.
- $|3 - 4i| = \sqrt{9^2 + 4^2} = \sqrt{25} = 5$,
- $|-3i| = 3$

The formula is $|x+yi| = \sqrt{x^2+y^2}$. Of course we have a triangle inequality

$$|z+w| \leq |z| + |w|$$

The conjugate of a complex number z is denoted by $\overline{z}$ is obtined by changing the sign of the immaginary part, so $\overline{x+iy} = x - iy$. It is important to understand that geometrically in the complex plane this correspond to reflect about the real line.

Example 7.5.
- $\overline{3-4i} = 3+4i$,
- $\overline{3i} = -3i$,
- $\overline{1} = 1$.

Remark that, by explicit computation, we have

$$|z|^2 = z\overline{z}$$

We also have the following relation between conjugation, real part and immaginary part

$$\Re(z) = \frac{1}{2}(z+\overline{z}) \quad \text{and} \quad \Im(z) = \frac{1}{2}(z-\overline{z})$$

We can associate to every non-zero complex number an angle, called argument or phase, in the following way. In the compex plane we we haave the half line $\mathbb{R}_+$ of the positive real numbers. Given a non-zero complex number z, we can take the half line L_z starting from the origin and passing trough z. The argument or phase of z is the angle between $\mathbb{R}_+$ and L_z, moving in the anti-clockwise direction. The argumentof z is denoted by $\arg(z)$.

Example 7.6. The argument of 3 is zero. The argument of i is $\frac{\pi}{2}$, the argument of $\frac{\sqrt{2}}{2}(1+i)$ is $\frac{\pi}{4}$.

More generally, for any positive real number R and any angle α, we have that the argument of $R(\cos(\alpha) + \sin(\alpha)i)$ is α, draw a picture!!!!!

Take now a non-zero complex number z, we have seen that its distance from the origin is $|z|$, and let α be its argument. The number $\frac{z}{|z|}$ has distance 1 from the orign, so it lies on the trigonometric circle, hence its x and y co-ordinates are just $\cos(\alpha)$ and $\sin(\alpha)$. We conclude that

$$z = |z|(\cos(\alpha) + \sin(\alpha)i)$$

Thisi is called the *polar form* of z, and it is very important. (Conversly, when we write a complex number as $x+iy$, we can say that we are using the cartesian form, or cartesian representation)

Example 7.7. The polar form of $1+i$ is $\sqrt{2}(\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})i)$

The multiplication of complex numbers become particularly easy if we use the polar form, from this poin ov view the multiplication is equivalent to add the angles and multiply the absolute values.

Example 7.8. Let α and β be two numbers. Then

$$(5(\cos(\alpha) + \sin(\alpha)i))(3(\cos(\beta) + \sin(\beta)i)) =$$
$$= 15(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + (\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta))i =$$
$$= 15(\cos(\alpha+\beta) + \sin(\alpha+\beta)i)$$

Example 7.9.
- The modulus of $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is 1, and the argument is $\frac{\pi}{3}$, so

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{2017} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

because $1^{2017} = 1$, so the absolute values does not change; then $2017 = 336 \cdot 6 + 1$, so $2017 \cdot \frac{\pi}{3} = 336 \cdot 2\pi + \frac{\pi}{3}$, so also the argument does not change.

The above example show that the polar form is really useful!!

Division of two complex numbers. We would like to take two complex number z and w, with $w \neq 0$, and write $\frac{z}{w}$ in the form $x + yi$. A key idea is to make the denominator a real number by multiplying with the conjugate.

Example 7.10.

$$\frac{2-3i}{5+i} = \frac{(2-3i)(5-i)}{(5+i)(5-i)} = \frac{7-17i}{26} = \frac{7}{26} - \frac{17}{26}i$$

In fact, we may write down a general formula using $w\overline{w} = |w|^2$, which we have used already in the above example. So, we have

$$\frac{z}{w} = \frac{z\overline{w}}{\overline{w} \cdot w} = \frac{z\overline{w}}{|w|^2}.$$

Example 7.11.

$$\frac{1}{3-\sqrt{3}i} = \frac{3+\sqrt{3}i}{12} = \frac{1}{4} + \frac{\sqrt{3}}{4}i,$$

or

$$\frac{i}{1-i} = \frac{i(1-i)}{2} = \frac{1}{2} + \frac{1}{2}i$$

We can also use the polar form to divide complex numbers. As with multiplication the moduli (plural of the modulus) multiplied and the arguments added up, with division, we have to do the inverse. That is, moduli are divided and arguments are subtracted:

$$|z|(\cos(\alpha) + \sin(\alpha)i)/|w|(\cos(\beta) + \sin(\alpha)i) = \frac{|z|}{|w|}(\cos(\alpha - \beta) + \sin(\alpha - \beta)i)$$

Note that because of the presence of cos and sin, one can add any multiple of 2π to the argument on the right hand side.

Example 7.12. The inverse of

$$3\left(\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)\right)$$

is

$$\frac{1}{3}\left(\cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right)\right)$$

7.1. *Euler formula*

The following is a formal definition, called the Euler formula. It is very important that you do not try to understand it as the powers of something, as we have not defined i-th powers. So, just think about it as a shortcut for the argument part of the polar form.

Definition 7.13. Let α be a real number, then

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

We can now write

$$|z|(\cos(\alpha) + \sin(\alpha)i) = |z|e^{i\alpha}$$

Example 7.14.

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

and now we can write out the multiplication as

$$zw = \left(|z|e^{i\alpha}\right)\left(|w|e^{i\beta}\right) = |z||w|e^{i(\alpha+\beta)}$$

7.2. *Finding solutions of equations amongst complex numbers*

The main importance of complex numbers is that any polynomial equation has a solution amongst the complex numbers. This is called the *fundamental theorem of algebra.* We are not going to formally state it and prove it, but we will see a few examples.

We have already learned that the equation $z^2 = -1$ has no solutions in the real numbers, but two solutions in the complex number, i and $-i$. We are going to learn how to solve slightly more general equations, thanks to the polar form.

Suppose that we want to solve the equation

$$z^n = Re^{i\alpha}$$

where z is the unknown, n is a positive integer, R is a positive real number, and α is an angle. Then the solution of this equation are always of the form

$$z = \sqrt[n]{R}e^{i\beta}$$

where β is an angle such that $n\beta = \alpha$, as an angle. In particular, β has to be equal to $\frac{\alpha}{n} + \frac{2k\pi}{n}$, where k is an integer between 0 and $n-1$, so that the above equation has always exactly n distinct solutions.

Example 7.15. The equation

$$z^2 = 3e^{i\frac{\pi}{5}}$$

has two solutions: $\sqrt{3}e^{i\frac{\pi}{10}}$ and $\sqrt{3}e^{i\left(\frac{\pi}{10}+\pi\right)}$

The equation

$$z^3 = 27e^{i\frac{\pi}{7}}$$

has three solutions: $3e^{i\frac{\pi}{21}}$, $3e^{i\left(\frac{\pi}{21}+\frac{2\pi}{3}\right)}$ and $3e^{i\left(\frac{\pi}{21}+\frac{4\pi}{3}\right)}$

One can also use the usual quadratic formula (the proof is the same as usually, complete the square, etc.).

Example 7.16. For example the solutions of

$$z^2 + 2z + 3 = 0$$

are

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$$

In any case, the message is that everything works as usually, you just always have solutions, contrary to the real case, and for some equations (say $z^n = a$, where $a \neq 0$) it is even guaranteed that there are different solutions.

8. Sequences

Definition 8.1. A sequence is a function $x : \mathbb{N} \to \mathbb{R}$, where we (by tradition) denote the value of n by x_n, and the whole sequence by (x_n).

The mother of all sequences is the following:

Example 8.2. *Arithmetic progression*: $x_0 = a, x_1 = a + b, x_2 = a + 2b, \dots, x_n = a + nb, \dots$

For example, the arithmetic progression given by $a = 1$ and $b = 2$ is $x_0 = 1, x_1 = 3, x_2 = 5, \dots$, that is, the sequence goes through all the odd numbers.

The best is to visualize a sequence as a function (that takes values only on integers).

Definition 8.3. Let (x_n) be a sequence.

We say (x_n) is *bounded from below*/*bounded from above*/*bounded*, if the set

$$\{x_n | n \in \mathbb{N}\}$$

of its values is bounded from below/bounded from above/bounded.

A sequence (x_n) is *constant*/*increasing*/*strictly increasing*/*decreasing*/*strictly decreasing*, if for each $n \in \mathbb{N}$, $x_n = x_{n+1}$ / $x_n \leq x_{n+1}$ / $x_n < x_{n+1}$ / $x_n \geq x_{n+1}$ / $x_n > x_{n+1}$

A sequence is *monotone/strictly monotone* if it increases or decreases/strictly increases or strictly decreases.

Example 8.4. An arithmetic progression $x_n = a + nb$ is

(1) bounded from above $\Leftrightarrow b \leq 0 \Leftrightarrow$ decreasing
(2) bounded from below $\Leftrightarrow b \geq 0 \Leftrightarrow$ increasing
(3) bounded $\Leftrightarrow b = 0 \Leftrightarrow$ constant
(4) strictly increasing $\Leftrightarrow b > 0$
(5) strictly decreasing $\Leftrightarrow b < 0$

Example 8.5. A *geometric progression* is of the form $x_n = aq^n$ for some real numbers a and q.

For example, if $a = 1$ and

(1) $q = \frac{1}{2}$, then $x_n = \frac{1}{2^n}$.
(2) $q = \frac{1}{2}$, then $x_n = \frac{1}{(-2)^n}$
(3) $q = -1$, then $x_n = (-1)^n$
(4) $q = 1$, then $x_n = 2^n$
(5) $q = -2$, then $x_n = (-2)^n$

Here are how geometric progressions work with respect to the above properties

(1) x_n is bounded $\Leftrightarrow |q| \leq -1$
(2) x_n is increasing $\Leftrightarrow q \geq 1$ and $a \geq 0$ or $0 \geq q \leq 1$ and $a \leq 0$
(3) x_n is strictly increasing $\Leftrightarrow q > 1$ and $a > 0$ or $0 < q < 1$ and $a < 0$
(4) x_n is decreasing $\Leftrightarrow 0 \leq q \leq 1$ and $a \geq 0$ or $q \geq 1$ and $a \leq 0$
(5) x_n is strictly decreasing $\Leftrightarrow 0 < q < 1$ and $a < 0$ or $q > 1$ and $a < 0$
(6) x_n is constant $\Leftrightarrow q = 1$ or $a = 0$
(7) x_n is bounded $\Leftrightarrow q \leq 1$ or $a = 0$
(8) x_n is bounded from above $\Leftrightarrow$ bounded or $q > 1$ and $a \leq 0$
(9) x_n is bounded from below $\Leftrightarrow$ bounded or $q > 1$ and $a \geq 0$

8.1. *Recursive sequences*

Example 8.6. Fibonachi sequence: $x_{n+2} = x_{n+1} + x_n$, $x_0 = 1$, $x_1 = 1$.

Then we have $x_2 = 2$, $x_3 = 3$, $x_4 = 5$, $x_5 = 8$, ...

Example 8.7. $x_n = \sqrt{4 + x_{n-1}}$, $x_0 = 1$. Is this sequence bounded?

The answer to the above question can be given using *induction*. This is a method of proving something for all natural numbers n, by proving first that it holds for $n = 0$, and then that it holds for n assuming that it holds for $n - 1$. The latter step is called the induction step, and the assumption that the statement holds for $n - 1$ is the induction hypothesis. Below is the example:

Proposition 8.8. *The sequence $x_n = \sqrt{4 + x_{n-1}}$, $x_0 = 1$ is bounded.*

Proof. As a warm-up we do first the easy case, that is, x_n is bounded from below. We show that $0 \leq x_n$ for each n. Indeed, for $x_0 = 1 \geq 0$, so we only have to prove the induction step. We assume that $0 \leq x_{n-1}$, and we need to show that $0 \leq x_n$. However, this is straightforward:

$$x_n = \sqrt{4 + x_{n-1}} \geq \sqrt{4 + 0} = 2 \geq 0,$$

where at the first inequality we used our induction hypothesis.

Now, comes the real deal, showing that x_n from above. In fact, here the hardest is probably finding the correct upper bound. For example 2 is obviously not an upper bound, as $x_2 = \sqrt{5} > 2$. We claim that 3 is an upper bound, and we show it by induction. Indeed, $x_0 = 1 < 3$, so we are only left to show the induction step. That is, assume that $x_{n-1} < 3$. Then:

$$x_{n+1} = \sqrt{4 + x_{n-1}} < \sqrt{4 + 3} = \sqrt{7} < 3,$$

where at the first inequality we used our induction hypothesis.

$\square$

Example 8.9. $x_0 = 0$, $x_n = x_{n-1} + (-1)^n n^2$. Equivalently, $x_n = \sum_{i=1}^n (-1)^i i^2$.
Is x_n bounded (in any direction)?

Proposition 8.10. *For the above sequence, $x_{2m} = (2m+1)m$ for every $m \in \mathbb{N}$.*

Proof. We prove by induction.
$x_0 = 0 = (2 \cdot 0 + 1) \cdot 0$, so this is OK.
We need to show then the induction step, so we assume that $x_{2(m-1)} = (2m-1)(m-1)$. Then

$$x_{2m} = x_{2(m-1)} - (2m-1)^2 + (2m)^2 == \underbrace{x_{2(m-1)} - (2m)^2 + 4m - 1 + (2m)^2}_{\text{foiling out the middle square}}$$

$$= \underbrace{x_{2(m-1)} + 4m - 1}_{\text{cancelling the two } (2m)^2 \text{ terms}} = \underbrace{(2m-1)(m-1) + 4m - 1}_{\text{using the induction hypothesis}} = \underbrace{2m^2 - 3m + 1 + 4m - 1}_{\text{foiling out the paretheses}}$$

$$= 2m^2 + m = (2m+1)m$$

□

Example 8.11. Getting back to our example, we see by the previous proposition that $x_{2m} = (2m+1)m$, so x_n is not bounded from above. To see the boundedness from below, we compute also

$$x_{2m+1} = x_{2m} - (2m+1)^2 = (2m+1)m - (2m+1)^2 = -(2m+1)(m+1),$$

and we see that it is also not bounded from below.

There more examples of proof by induction on page 252 of the book. Also, there are further examples of sequences on page 16 of the book, which I suggest you take a look at.

8.1.1. *Binomial expansion.*

Definition 8.12. If $0 \leq k \leq n$ are integers, then $\binom{n}{k}$ is the number of possible ways one can choose a subset of k elements from a set of n elements. With formulas:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot \dots \cdot 1}.$$

One can show using induction also:

Proposition 8.13. *For any $x, y \in \mathbb{R}$, we have $(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$.*

Proof. See pages , 253 and 254 from the book. □

Corollary 8.14. Bernoulli inequality *If $q > 1$ is a real number, then $q^n > 1 + n(q-1)$*

Proof. Simply apply the binomial formula

$$(1 + (q-1))^n = \sum_{i=0}^n \binom{n}{i} (q-1)^i > \binom{n}{0} (q-1)^0 + \binom{n}{1} (q-1)^1 = 1 + n(q-1).$$

□

8.2. *Limit of a sequence*

Definition 8.15. Let (x_n) be a sequence. We say that x_n is *convergent* to some $x \in \mathbb{R}$, if for each $0 < \varepsilon \in \mathbb{R}$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|x_n - x| < \varepsilon$. The x in the definition exists then it is called the limit of x_n, and we denote it by $\lim_{n \to \infty} x_n$.

If (x_n) is not convergent, it is called *divergent*.

Proposition 8.16. *If a sequence (x_n) converges, then its limit is unique.*

Proof. Assume that $x \neq y \in \mathbb{R}$ are two limits. Then, for each $0 < \varepsilon \in \mathbb{R}$ there are $n_\varepsilon^x, n_\varepsilon^y \in \mathbb{N}$ such that for all $n \geq n_\varepsilon^x$ we have:

$$|x - x_n| < \varepsilon$$

and for all $n \geq n_\varepsilon^y$ we have

$$|y - x_n| < \varepsilon.$$

So, if we set $n_\varepsilon := \max\{n_\varepsilon^x, n_\varepsilon^y\}$, then both of the above inequalities hold for all integers $n \geq n_\varepsilon$. In particular, for such n, we have

$$|y - x| \geq \underbrace{|y - x_n| + |x_n - x|}_{\text{triangle inequality}} \geq \varepsilon + \varepsilon = 2\varepsilon$$

Since, this holds for all $0 < \varepsilon \in \mathbb{R}$, we obtain that $y = x$. □

Example 8.17. A convergent sequence is $x_n := 1 - \frac{1}{\sqrt{n}}$.

Indeed, $\lim_{n \to \infty} 1 - \dfrac{1}{\sqrt{n}} = 1$, because for any $0 < \varepsilon \in \mathbb{R}$:

$$\left| 1 - \frac{1}{\sqrt{n}} - 1 \right| = \left| \frac{1}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}} < \varepsilon,$$

if $\sqrt{n} > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon^2}$. So, we may set $n_\varepsilon := \left\lceil \frac{1}{\varepsilon^2} \right\rceil$

Example 8.18. A divergent sequence is $x_n := (-1)^n$. Indeed, if x_n was divergent with limit x, then for $\varepsilon := \frac{1}{2}$ there would exist $n_{\frac{1}{2}} \in \mathbb{N}$ such that for all integers $n \geq n_{\frac{1}{2}}$, we would have $|x_n - x| \leq \frac{1}{2}$. In particular, if $n' \geq n_{\frac{1}{2}}$ is any other integer, then we would have:

$$|x_n - x_{n'}| \leq \underbrace{|x_n - x| + |x - x_{n'}|}_{\text{triangle inequality}} \leq \frac{1}{2} + \frac{1}{2} = 1$$

However, in our sequence $|x_n - x_{n+1}| = 2 > 1$. This is a contradiction.

Remark 8.19. In fact, the above argument shows that if (x_n) is a convergent sequence, then for all $0 < \varepsilon \in \mathbb{R}$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n, n' \geq n_\varepsilon$, $|x_n - x_{n'}| < \varepsilon$. We will call this the Cauchy criterion for convergence and we will learn it later more in detail.

Also, with similar arguments are above we may show that:

Proposition 8.20. *If (x_n) is convergent, then it is bounded.*

Proof. Set $x := \lim_{n \to \infty} x_n$. By definition of the limit, for $\varepsilon := 1$ we have an $n_1 \in \mathbb{N}$, such that for all integers $n \geq n_1$, $|x_n - x| \leq 1$. Set

$$R := \max\{|x_0|, |x_1|, \ldots, |x_{n_1 - 1}|, |x + 1|, |x - 1|\}.$$

Then, R is an upper and $-R$ is a lower bound. Indeed, they are bounds for $x_0, \ldots, x_{n_1 - 1}$ just because R is at least as big as the absolute values of all these elements of the sequence, by definition of R. Furthermore, R and $-R$ are also bounds for the other elements of the sequence, because these elements are lying in the interval $[x - 1, x + 1]$, which is again bounded by $-R$ and R, by definition of R. □

Example 8.21. $x_n = n^2$ cannot be convergent as it is not bounded

Example 8.22. The backwards direction of the above proposition is not true. That is, if a sequence (x_n) is bounded, then it is not necessary convergent. An example is $x_n := (-1)^n$. On the other hand, a little later we will see that a monotone, bounded sequence is convergent.

8.2.1. *Limits and algebra.*

Proposition 8.23. *Let (x_n) and (y_n) be two convergent sequences. Set $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$. Then:*

(1) $(x_n + y_n)$ is also convergent, and $\lim_{n\to\infty} x_n + y_n = x + y$,
(2) $(x_n \cdot y_n)$ is also convergent, and $\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$,
(3) if $y \neq 0$, then $\left(\frac{x_n}{y_n}\right)$ is also convergent, and $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$, and
(4) if there is an $n_0 \in \mathbb{N}$, such that $x_n \leq y_n$ for each integer $n \geq n_0$, then $x \leq y$.

Proof. We prove only the first one and we refer to (2.3.3 and 2.3.6 in the book for the proofs of the others).

So, fix $0 < \varepsilon \in \mathbb{R}$. We want to prove that for big n, $|(x_n + y_n) - (x + y)|$ is smaller than ε. However, all we know that $|x_n - x|$ and $|y_n - y|$ are small for big n. Luckily,

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \underbrace{|x_n - x| + |y_n - y|}_{\text{triangle inequality}}. \tag{8.23.a}$$

So, to make $|(x_n + y_n) - (x + y)|$ be smaller than ε, we have to make the sum of $|x_n - x|$ and $|y_n - y|$ smaller than ε. This we can attain for example if we make both $|x_n - x|$ and $|y_n - y|$ smaller than $\frac{\varepsilon}{2}$ (however, this is an arbitrary division, the proof would work with any two positive numbers that add up to ε, for example with $\frac{\varepsilon}{3}$ and $\frac{2\varepsilon}{3}$).

So, after this initial discussion we can make a formal proof: there are integers $n^x_{\frac{\varepsilon}{2}}$ and $n^y_{\frac{\varepsilon}{2}}$, such that

(1) whenever $n \geq n^x_{\frac{\varepsilon}{2}}$, then $|x - x_n| \leq \frac{\varepsilon}{2}$, and
(2) whenever $n \geq n^y_{\frac{\varepsilon}{2}}$, then $|y - y_n| \leq \frac{\varepsilon}{2}$.

Set $n_\varepsilon := \max\left\{n^x_{\frac{\varepsilon}{2}}, n^y_{\frac{\varepsilon}{2}}\right\}$. Then, (8.23.a) tells us that for every $n \geq n_\varepsilon$ we have

$$|(x_n + y_n) - (x + y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $(x_n + y_n)$ converges and the limit is $x + y$. □

Example 8.24. With the above machinery we can already compute the limits of fractions of polynomials, called rational functions.

(1) $x_n := \frac{n^2+2n+3}{4n^2+5n+6}$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^2 + 2n + 3}{4n^2 + 5n + 6} = \underbrace{\lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{3}{n^2}}{4 + \frac{5}{n} + \frac{6}{n^2}}}_{\substack{\text{dividing both the numerator}\\ \text{and the denominator by } n}} = \underbrace{\frac{\lim_{n\to\infty}\left(1 + \frac{2}{n} + \frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(4 + \frac{5}{n} + \frac{6}{n^2}\right)}}_{\text{using the above division rule for limits}}$$

$$= \underbrace{\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{3}{n^2}}{\lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{5}{n} + \lim_{n\to\infty} \frac{6}{n^2}}}_{\text{the above addition rule of limits}} = \underbrace{\frac{1 + \lim_{n\to\infty} \frac{2}{n} + \left(\lim_{n\to\infty} \frac{3}{n}\right)^2}{4 + \lim_{n\to\infty} \frac{5}{n} + \left(\lim_{n\to\infty} \frac{6}{n}\right)^2}}_{\substack{\text{the above product rule of}\\ \text{limits, and limits of constant}\\ \text{sequences}}} = \frac{1 + 0 + 0}{4 + 0 + 0} = \frac{1}{4}$$

Comments:

i dividing the numerator and the denominator by n is an operation that one cannot perform for $n = 0$. So, after the second equality sign the sequence does not make sense for $n = 0$. But this is fine, as the 0-th term (and in fact, even any finitely many first terms) of the sequence does not matter for the limit computation, so you can think about the 0-th term being anything (for example 0) after the second equality sign.

The same issue shows up many times later in our computations when we are computing limits of sequences of the form $\frac{2}{n}$, or $\frac{3}{n^2}$

ii for any number $c \in \mathbb{R}$: $\lim\limits_{n\to\infty} \frac{c}{n} = 0$, as for $0 < \varepsilon \in \mathbb{R}$ we may choose $n_\varepsilon := \left\lceil \frac{c}{\varepsilon} \right\rceil$, and for this choice we have for each integer $n \geq n_\varepsilon$:

$$\left| \frac{c}{n} \right| < \frac{c}{\frac{c}{\varepsilon}} = \varepsilon$$

iii in the step where we use that limits behave well with respect to fractions, we should check first that the limit of the denumerator is not 0. However, following our argument, we see that this limit is 4, so we are fine.

(2) $x_n = \frac{n+2}{3n^2+4n+5}$. Here we will not give the above explanations again (as they are the same):

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n+2}{3n^2+4n+5} = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{2}{n^2}}{3 + \frac{4}{n} + \frac{5}{n^2}} = \frac{0+0}{3+0+0} = 0$$

(3) $x_n = \frac{n^2+2n+3}{4n+5}$. For $n \geq 1$, we have $0 \leq \frac{3}{n}$ and $1 \geq \frac{5}{n}$. Hence, for $n \geq 1$:

$$x_n = \frac{n^2+2n+3}{4n+5} = \frac{n+2+\frac{3}{n}}{4+\frac{5}{n}} \geq \frac{n+2}{5}$$

This shows that (x_n) is not bounded and hence cannot be convergent by Proposition 8.20.

Using the method of the above exercise one can show (see page 22 for a precise proof, although there is an unnecessary assumption in the book about $y_n \neq 0$ for all $n \in \mathbb{N}$, in fact it is enough if $y_n \neq 0$ for some $n \in \mathbb{N}$):

Proposition 8.25. *If (x_n) and (y_n) are sequences given by formulas*

$$x_n = a_0 + a_1 n + \cdots + a_p n^p, \text{ with } a_p \neq 0$$

and

$$y_n = b_0 + b_1 n + \cdots + b_q n^q, \text{ with } b_q \neq 0,$$

then

(1) if $p \leq q$, then $\left(\frac{x_n}{y_n}\right)$ is convergent, and

i if $p = q$, then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a_p}{b_q}$,

ii if $p < q$, then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 0$,

(2) if $p > q$, then $\left(\frac{x_n}{y_n}\right)$ is divergent.

Theorem 8.26. Squeeze Theorem *Let x_n, y_n and z_n be three sequences satisfying:*

(1) (x_n) and (z_n) are convergent, with the same limit, say a,

(2) there is an $n_0 \in \mathbb{N}$, such that for all integers $n \geq n_0$, $x_n \leq y_n \leq z_n$

then (y_n) is also convergent, with limit a.

Proof. For each $\varepsilon > 0$, there are natural numbers n_ε^x and n_ε^z, such that for each integer $n \geq n_\varepsilon^x$, we have $a - \varepsilon < x_n$ and for each integer $n \geq n_\varepsilon^y$ we have $a + \varepsilon > z_n$.

Set $n_\varepsilon := \max\{n_\varepsilon^x, n_\varepsilon^y, n_0\}$. Then, for each integer $n \geq n_\varepsilon$ we have:

$$a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon$$

which in particular implies that $|y_n - a| < \varepsilon$.

□

Example 8.27. In general a geometric sequence $x_n = aq^n$ is convergent if and only if $a = 0$ or $-1 < q \leq 1$. Indeed:

The $a = 0$, the $q = 1$, the $q = 0$ and the $q = -1$ cases are clear, so we may assume that $a \neq 0$, and $|q| \neq 0, 1$. Then:

(1) In the $|q| > 1$ case, we show that $|aq^n|$ is not bounded, from where we see that x_n is not bounded and hence divergent. For this we just use the Bernoulli inequality (that is, Corollary 8.14) :

$$|aq^n| = |a||q|^n > |a|(1 + n(|q| - 1)) > n|a|(|q| - 1)$$

As $|a|(|q| - 1) > 0$, this shows that $|aq|^n$ is not bounded. As for any $M \in \mathbb{R}_+$, we have $|aq|^n > M$ whenever

$$n > \frac{M}{|a|(|q| - 1)}.$$

(2) In the $|q| < 1$ case, we show that $\lim_{n\to\infty} aq^n = 0$. For that we should understand when $|aq|^n < \varepsilon$ for any $0 < \varepsilon \in \mathbb{N}$:

$$|aq|^n < \varepsilon \Leftrightarrow \frac{|a|}{\varepsilon} < \left(\frac{1}{|q|}\right)^n \tag{8.27.b}$$

According to the Bernoulli inequality,

$$\left(\frac{1}{|q|}\right)^n \geq 1 + n\left(\frac{1}{|q|} - 1\right) > n\left(\frac{1}{|q|} - 1\right) \tag{8.27.c}$$

putting (8.27.b) and (8.27.c) together we see that $|aq|^n < \varepsilon$ holds as soon as we guarantee that $\frac{|a|}{\varepsilon} < n\left(\frac{1}{|q|} - 1\right)$, or equivalently $n \geq n_\varepsilon$ if we set

$$n_\varepsilon = \left\lceil \frac{\frac{|a|}{\varepsilon}}{\left(\frac{1}{|q|} - 1\right)} \right\rceil.$$

Example 8.28. Now, we are ready to state our first squeeze example.

We claim that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. Indeed, we have for all integers $n \geq 3$:

$$0 \leq \frac{2^n}{n!} \leq \frac{2^n}{2 \cdot 3^{n-1}} = \frac{3}{2} \cdot \left(\frac{2}{3}\right)^n$$

Furthermore $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{3}{2} \cdot \left(\frac{2}{3}\right)^n = \frac{3}{2} \cdot \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = \frac{3}{2} \cdot 0 = 0$ by the previous example. So, squeeze theorem concludes our claim.

Example 8.29. $x_n = \sqrt[n]{n}$ (there is a different proof in the book, on page 24, check it out too):

We squeeze x_n with $1 \leq x_n \leq y_n := 1 + \frac{1}{\sqrt{n}}$.

$$\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}} \Leftrightarrow n \leq \left(1 + \frac{1}{\sqrt{n}}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{(\sqrt{n})^i}$$

Note that the sum on the right hand side for $i = 4$ is $\frac{n(n-1)(n-2)(n-3)}{24} \frac{1}{(\sqrt{n})^4} = \frac{n(n-1)(n-2)(n-3)}{24n^2}$. So, we have $\sqrt[n]{n} \leq 1 + \frac{1}{\sqrt{n}}$ as soon as $n \geq 4$ and

$$n \leq \frac{n(n-1)(n-2)(n-3)}{24n^2} \Leftrightarrow \frac{24n^2}{n(n-1)(n-2)(n-3)} \leq 1$$

(and the denominator is not zero). However, we have just learned that

$$\lim_{n\to\infty} \frac{24n^2}{n(n-1)(n-2)(n-3)} = 0,$$

so there is an integer n_1, such that for each $n \geq n_1$,

$$\left|\frac{24n^2}{n(n-1)(n-2)(n-3)}\right| \leq 1$$

Corollary 8.30. *If $\lim_{n\to\infty} x_n = 0$ and (y_n) is bounded, then $\lim_{n\to\infty} x_n y_n = 0$.*

Proof. Note that showing $\lim x_n y_n = 0$ is equivalent to showing $\lim |x_n y_n| = 0$ (by the definition, as $|x_n y_n - 0| = ||x_n y_n| - 0|$). Similary, from the assumption $\lim_{n\to\infty} x_n = 0$ we obtain that also that $\lim_{n\to\infty} |x_n| = 0$

As y_n is bounded, there is an integer $M > 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Hence, we may squeeze $|x_n y_n|$:

$$0 \leq |x_n y_n| \leq |x_n| M,$$

where both sides converge to 0. This shows that so does $|x_n y_n|$. □

Example 8.31. $\lim_{n\to\infty} \frac{1}{n^2} \sin(n) = 0$. Note that here $\sin(n)$ does not converge in itself. So, we may not apply the previous multiplication rule of limits. However, we may apply the previous corollary, as $\lim_{n\to\infty} \frac{1}{n^2} = 0$, and $\sin(n)$ is bounded (by -1 and 1).

Example 8.32. Define the recursive sequence $x_{n+1} = \frac{\sin(x_n)}{2}$, $x_0 = 1$. Then we have:

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\frac{|\sin(x_n)|}{2}}{|x_n|} \leq \frac{1}{2},$$

as $\frac{|\sin(x)|}{|x|} \leq 1$ for all $x \in \mathbb{R}$ ($|x|$ measures the length of the circle segment of angle x, where we count multiple revolutions too, and $|\sin(x)|$ gives the absolute value of the y-coordinate of the endpoint of the circle segment).

In particular, we have $|x_n| \leq \frac{1}{2^n}$. So, we may use squeeze to show that $\lim_{n\to\infty} x_n = 0$:

$$0 \leq x_n \leq \frac{1}{2^n}.$$

Corollary 8.33. Quotient criterion *Let (x_n) be a sequence such that*

$$q = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists. If $q < 1$, then (x_n) converge to 0, and if $q > 1$, then (x_n) diverge.

Proof. We show here only the $q < 1$ case, the other one is similar, and is left a homework. Note that as q is the limit of non-negative numbers, it is automatically non-negative. So, we have $0 \leq q < 1$.

Set $\varepsilon := \frac{1-q}{2}$. Then, there is a $n_\varepsilon \in \mathbb{N}$, such that for all integers $n \geq n_\varepsilon$:

$$q - \varepsilon < \frac{|x_{n+1}|}{|x_n|} < q + \varepsilon = q + \frac{1-q}{2} = \frac{q+1}{2} =: \overline{q} < 1.$$

In particular, then $x_{n_\varepsilon + i} \leq x_{n_\varepsilon} \overline{q}^i$ (for all $i \in \mathbb{N}$), so we may squeeze $|x_n|$ (for every integer $n \geq n_\varepsilon$):

$$0 \leq |x_n| \leq |x_{n_\varepsilon}| \overline{q}^{n - n_\varepsilon} \to 0 \text{ (as } n \to \infty),$$

because $|\overline{q}| < 1$. □

Example 8.34. Some examples showing that we are not able to say anything in the $q = 1$ case:

(1) If $x_n := n$, then (x_n) is divergent and $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n+1}{n} = 1$
(2) If $x_n := \frac{n+1}{n}$, then (x_n) is convergent to 1 (see the computation one line above), and $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\frac{n+2}{n+1}}{\frac{n+1}{n}} = \lim_{n\to\infty} \frac{(n+2)n}{(n+1)^2} = 1.$

8.2.2. *Limits of recursive sequences.*

Example 8.35. The Fibonacci seqeunce is $x_0 = x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$. In particular then if we define $y_n := \frac{x_{n+1}}{x_n}$ we obtain $y_{n+1} = 1 + \frac{1}{y_n}$, and $y_0 = 1$. We call this the sequence of Fibonacci quotients.

Proposition 8.36. *If (y_n) is the sequence of Fibonacci quotients, then for each integer $n > 0$ we have $1 \leq y_n \leq 2$.*

Proof. We prove the above statements by induction, that is, we prove by induction on n that $y_n > 0$ and that .

The $n = 0$ case: by definition we have $2 \geq y_0 = 1$.

So, assume we know that statement for n and then we prove it for $n+1$ below:

$$y_{n+1} = 1 + \frac{1}{y_n} > 1 + \frac{1}{3} \geq 1,$$

and

$$y_{n+1} = 1 + \frac{1}{y_n} \leq 1 + \frac{1}{1} = 2,$$

$\square$

Example 8.37. Let us continue with our Fibonacci quotient example.

Let us assume now that (y_n) is convergent, and let y be the limit. Then, as $y_n \geq 1$, $y \geq 1$ and we have

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = \underbrace{\lim_{n\to\infty} 1 + \frac{1}{y_n}}_{\text{definition of } y_{n+1}} = \underbrace{1 + \frac{1}{\lim_{n\to\infty} y_n}}_{\text{algebraic rules of limit}} = 1 + \frac{1}{y}$$

This yields an equation which we are able to solve:

$$y = 1 + \frac{1}{y} \Leftrightarrow y^2 = y + 1 \Leftrightarrow y = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

As we have seen that $y \geq 1$, the only possibility that can happen is $y = \frac{1+\sqrt{5}}{2}$.

However, we do not know at this point that (y_n) converges. As we have figured out that if it converges the only possible limit is $\frac{1+\sqrt{5}}{2}$, we may show that (y_n) converges by showing that $z_n := \left| y_n - \frac{1+\sqrt{5}}{2} \right|$ converges to 0.

$$z_{n+1} = \left| y_{n+1} - \frac{1+\sqrt{5}}{2} \right| = \left| 1 + \frac{1}{y_n} - 1 - \frac{1}{\frac{1+\sqrt{5}}{2}} \right| = \left| \frac{1}{y_n} - \frac{1}{\frac{1+\sqrt{5}}{2}} \right| = \frac{\left| y_n - \frac{1+\sqrt{5}}{2} \right|}{y_n \frac{1+\sqrt{5}}{2}} \leq \frac{|z_n|}{\frac{1+\sqrt{5}}{2}}$$

So, we obtain that

$$0 \leq z_n \leq \frac{|z_0|}{\left(\frac{1+\sqrt{5}}{2}\right)^n} \to 0, \text{ as } n \to \infty$$

So, we showed that for the Fibonacci sequence (x_n),

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}$$

This number is also called the Golden ratio.

Summarizing, the general approach of finding the limit of a recursive sequence x_n is:

(1) showing some kind of upper and lower bounds
(2) assuming that there exists a limit, computing what it can be using the algebraic rules of limits
(3) let x be the number guessed in the previous point, then showing that $\lim_{n\to\infty} |x_n - x| = 0$.

Example 8.38. This method of finding the limit does not always work. For example:

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$$

gives $x = \frac{1}{2}(x + x)$.

Example 8.39. On the other hand, if the above limit finding step has no solution, we automatically know that (y_n) is divergent. For example if we change one sign in the Fibonacci quotient sequence, we obtain:

$$y_n = 1 - \frac{1}{y_{n-1}}.$$

For simplicity (so that we do not have to worry about when $y_{n-1} = 0$), let us consider the related recursive sequence:

$$y_n^2 = y_{n-1} - 1.$$

Let us assume y_n is convergent to y, and take limits. Then we obtain

$$y^2 - y + 1 \Leftrightarrow y = \frac{1 \pm \sqrt{(-1)^2 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2}.$$

Since there is no real solution, the limit cannot exist.

8.2.3. *Approaching infinities.*

Definition 8.40. We say that a sequence (x_n) approaches $+\infty$ (resp. $-\infty$) if for all real numbers $A \in \mathbb{R}$ there is an $n_A \in \mathbb{N}$ such that for all integers $n \geq n_A$, $x_n \geq A$ (resp. $x_n \leq A$).

In the first case we write $\lim\limits_{n\to\infty} x_n = +\infty$, and $\lim\limits_{n\to\infty} x_n = -\infty$ in the second case. In both cases the sequence is DIVERGENT.

Example 8.41.
(1) $\lim\limits_{n\to\infty} \frac{n!}{2^n} = +\infty$
(2) $\lim\limits_{n\to\infty} - \frac{n!}{2^n} = -\infty$
(3) but we cannot say anything along these lines about $(-1)^n \frac{n!}{2^n}$

Check out page 29 and 30 of the book for the algebraic identities concerning approaching infinities.

8.3. *Monotone sequences*

We call a sequence (x_n) monotone if it is increasing or decreasing.

Theorem 8.42. *If x_n is bounded and increasing (resp. decreasing), then (x_n) is convergent and*

$$\lim_{n\to\infty} x_n = \operatorname{Sup}\{x_n | n \in \mathbb{N}\} \text{ (resp. } \lim_{n\to\infty} x_n = \operatorname{Inf}\{x_n | n \in \mathbb{N}\}).$$

Proof. We prove only the increasing case. We leave as a homework to change the words in it to obtain a proof for the decreasing case.

Set $S := \operatorname{Sup}\{x_n | n \in \mathbb{N}\}$ and let $0 < \varepsilon \in \mathbb{R}$ be arbitrary. By definition, S is the smallest upper bound, so $S - \varepsilon$ is not, hence there is an $n_\varepsilon \in \mathbb{N}$ such that $S - \varepsilon < x_{n_\varepsilon}$. Hence for any integer $n \geq n_\varepsilon$:

$$S - \varepsilon \underbrace{< x_{n_\varepsilon}}_{\text{prev. line}} \underbrace{\leq x_n}_{(x_n) \text{ is montone}} \underbrace{\leq S}_{S \text{ is an upper bound}} < S + \varepsilon.$$

□

Example 8.43. INTRODUCTION OF e.

Let $n \in \mathbb{Z}_+$. We claim that $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$. Indeed:

$$\text{(8.43.a)} \quad \left(1 + \frac{1}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n^i} = \sum_{i=0}^{n} \frac{1}{i!} \frac{n(n-1) \cdot \dots \cdot (n-i+1)}{n^i}$$
$$= \sum_{i=0}^{n} \frac{1}{i!} \frac{n(n-1) \cdot \dots \cdot (n-i+1)}{n^i} = \sum_{i=0}^{n} \frac{1}{i!} \left(1 - \frac{1}{n}\right) \cdot \dots \cdot \left(1 - \frac{i-1}{n}\right)$$

Similarly,
(8.43.b)

$$\left(1+\frac{1}{n+1}\right)^{n+1} = \sum_{i=0}^{n+1}\frac{1}{i!}\left(1-\frac{1}{n+1}\right)\cdot\dots\cdot\left(1-\frac{i-1}{n}\right) > \sum_{i=0}^{n}\frac{1}{i!}\left(1-\frac{1}{n+1}\right)\cdot\dots\cdot\left(1-\frac{i-1}{n+1}\right)$$

So, to prove our claim it is enough to show that each term on the right side of (8.43.a) is at most as big as the corresponding term on the right side of (8.43.b). However, that clear as $\frac{j}{n} > \frac{j}{n+1}$ for any $j \in \mathbb{Z}$.

Having proved our claim, we see that $x_n := \left(1+\frac{1}{n}\right)^n$ (set $x_0 = 1$ as for $n = 0$ the expression does not have a meaning) is a monotone increasing sequence, with $x_0 = 1$, $x_1 = 2$. Is it bounded? Well yes, because:

$$\left(1+\frac{1}{n}\right)^{n} = \sum_{i=0}^{n}\frac{1}{i!}\left(1-\frac{1}{n}\right)\cdot\dots\cdot\left(1-\frac{i-1}{n}\right) \leq \sum_{i=0}^{n}\frac{1}{i!} \leq 1+\sum_{i=1}^{n}\frac{1}{2^{i-1}} = 3-\frac{1}{2^{n-1}} \leq 3,$$

where in the last step we used the formula that we have already shown earlier (for $a = \frac{1}{2}$) that

$$(1+\dots+a^{n-1}) = \frac{1-a^n}{1-a}.$$

So, indeed, (x_n) is not only increasing, but also bounded by above by 3. So, there exists $\lim_{n\to\infty} x_n$.

Definition 8.44. We define $x_n := \left(1+\frac{1}{n}\right)^n$. We define $e := \lim_{n\to\infty} x_n$.

Example 8.45. Consider the recursive sequence $x_0 = 2$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$.

First we clam that $x_n \geq 0$ for all integers $n \in \mathbb{N}$. This is certainly true for $n = 0$, and if we assume it for $n-1$ then the recursive formula gives it to us also for n. Hence, indeed, it is true for all $n \in \mathbb{N}$. In particular, the division in the definition does make sense.

Next, we clam that $x_n \geq 1$ for all integers $n \geq 1$. Indeed,

$$\frac{1}{2}\left(x_n+\frac{1}{x_n}\right) \geq 1 \Leftrightarrow x_n+\frac{1}{x_n} \geq 2 \Leftrightarrow x_n^2+1 \geq 2x_n \Leftrightarrow (x_n-1)^2 \geq 0,$$

where we used that we already know that $x_n > 0$, when we multiplied by x_n.

Next, we claim that x_n is decreasing. Indeed,

$$x_n - \frac{1}{2}\left(x_n+\frac{1}{x_n}\right) = \frac{1}{2}\left(x_n-\frac{1}{x_n}\right) \geq 0,$$

where we obtained the last inequality using that $x_n \geq 1 \geq \frac{1}{x_n}$.

So, (x_n) is decreasing (hence bounded from above) and also bounded from below by 1. In particular, x_n is convergent, and $\lim_{n\to\infty} x_n \geq 1$. Hence to find the actual limit we may just apply limit to the recursive equation to obtain that if y is the limit, then

$$y = \frac{1}{2}\left(y+\frac{1}{y}\right) \Leftrightarrow \frac{y}{2} = \frac{1}{2y} \Leftrightarrow y^2 = 1$$

As we also know that $y \geq 1$, $y = 1$ has to hold. So, $\lim_{n\to\infty} x_n = 1$.

Example 8.46. The

8.4. *Liminf, limsup*

Definition 8.47. Let (x_n) be a bounded sequence. We define a new sequence

$$(y_n) := \operatorname{Sup}\{x_k | n \leq k \in \mathbb{N}\} \text{ (resp. } \operatorname{Inf}\{x_k | n \leq k \in \mathbb{N}\})$$

This is a decreasing, (resp. increasing) sequence, as Sup (resp. Inf) is taken over smaller and smaller sets as n increases. Hence, y_n is convergent, and we call the limsup (resp. liminf) of x_n. We denote it by $\lim_{n\to\infty} \sup x_n$ (resp. $\lim_{n\to\infty} \inf x_n$).

Example 8.48. $x_n = (-1)^n$. Then, $\lim_{n\to\infty} \sup x_n = 1$, and $\lim_{n\to\infty} \inf x_n = -1$.

Example 8.49. $x_n = (-1)^n \left(1+\frac{1}{n}\right)^n$. Then, $\lim_{n\to\infty} \sup x_n = e$, and $\lim_{n\to\infty} \inf x_n = -e$.

8.5. *Subsequences*

Definition 8.50. If (x_n) is a sequence, then a subsequence is a sequence $k \mapsto (x_{n_k})$, where $k \mapsto n_k$ is a strictly increasing function of k.

Example 8.51. $x_n := (-1)^n$, then both the constant 1 and the constant -1 are subsequences

Example 8.52. $x_n = n^2$, then $x_k = k^6$ is a subsequence by setting $n_k = k^3$.

Example 8.53. $x_n = \left(1+\frac{2}{n}\right)^n$ and $n = 2k$, then we get $x_k = \left(1+\frac{2}{2k}\right)^{2k} = \left(\left(1+\frac{1}{1k}\right)^k\right)^2 \to e^2$. But is the same true for x_n? This will be an exercise.

Btw, what is the connection between the convergence of a sequence and a subsequence?

Proposition 8.54. *If $\lim_{n\to\infty} x_n = a \in \mathbb{R}$, then all subsequence (x_{n_k}) converges also to a.*

Theorem 8.55. Bolzano-Weierstrass *Every bounded sequence contains a convergent subsequence.*

Proof. We define n_k with induction on k. We set $n_0 = 0$. So, assume n_{k-1} is defined. Set then $s_k := \text{Sup}\{x_n | n > n_{k-1}\}$. Then there is a integer $n_k > n_{k-1}$ such that

$$x_{n_k} > s_k - \frac{1}{k}.$$

We claim that (x_{n_k}) is convergent. Indeed, this follows from the squeeze principle, as we have:

$$s_k - \frac{1}{k} < x_{n_k} < s_k,$$

where $\lim_{k\to\infty} s_k = \lim_{n\to\infty} \sup x_n$ as it is a subsequence of $\text{Sup}\{x_{n'} | n' > n\}$, and

$$\lim_{k\to\infty} s_k - \frac{1}{k} = \lim_{k\to\infty} s_k - \lim_{k\to\infty} \frac{1}{k} = \lim_{n\to\infty} \sup x_n - 0 = \lim_{n\to\infty} \sup x_n.$$

□

Example 8.56. Sometimes, it is possible to write down explicitly the convergent subsequences, for example for $x_n = \cos\left(\frac{n\pi}{4}\right)\left(1+\frac{1}{n}\right)$, if we set

(1) $n = 8k+1$, then $\lim_{k\to\infty} x_{n_k} = \frac{1}{\sqrt{2}} e$,
(2) $n = 8k+2$, then $\lim_{k\to\infty} x_{n_k} = e$,
(3) $n = 8k+5$, then $\lim_{k\to\infty} x_{n_k} = -\frac{1}{\sqrt{2}} e$,
(4) etc.

however, sometimes this is not quite possible, and we just know that the convergent subsequence exists, for example for $x_n = \cos(n)\left(1+\frac{1}{n}\right)$

Exercise 8.57. True or false?

(1) If (x_n) is convergent, then it can have subsequences converging to different limits.
(2) Every sequence has a convergent subsequence.
(3) If (x_n) is bounded and $\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \inf x_n$, then (x_n) is convergent.

Example 8.58. Let $a > 0$ be an integer. Then for $x_n = \left(1+\frac{a}{n}\right)^n$, we may consider the subsequence $n_k = ak$ to obtain:

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \left(1+\frac{a}{ak}\right)^{ak} = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{ak} = \lim_{k\to\infty} \left(\left(1+\frac{1}{k}\right)^k\right)^a = e^a.$$

There is an exercise on the exercise sheet that x_n is increasing and bounded for $a = 2$ (where the method should be similar than for $a = 1$ using the binomial expansion). In particular, x_n is convergent. However, if it is convergent we may find the limit as the limit of any of its subsequence. So, $\lim_{n\to\infty} x_n = e^a$.

8.6. *Cauchy convergence*

Definition 8.59. A sequence (x_n) is Cauchy convergent, if for every $0 < \varepsilon \in \mathbb{R}$ there is an $n_\varepsilon \in \mathbb{N}$ such that for every integer $n, m \geq n_\varepsilon$, $|x_n - x_m| \leq \varepsilon$.

Example 8.60. $x_n := 1 - \frac{1}{n}$, then

$$|x_n - x_m| = \left|1 - \frac{1}{n} - 1 + \frac{1}{m}\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon$$

if $n, m \geq \frac{2}{\varepsilon}$, as then

$$\frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, x_n is Cauchy convergent.

Theorem 8.61. *(x_n) is convergent if and only if it is Cauchy convergent.*

Proof.
(1) First we assume that (x_n) is convergent, and then we show that it is Cauchy convergent. Let $x := \lim_{n\to\infty} x_n$ and $0 < \varepsilon \in \mathbb{R}$ arbitrary. Then there is an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all integers $n \geq n_{\frac{\varepsilon}{2}}$, we have $|x_n - x| \leq \frac{\varepsilon}{2}$. Then, for any integers $n, m \geq n_{\frac{\varepsilon}{2}}$ we have
$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
(2) For the other direction, assume that (x_n) is Cauchy convergent. We first claim that then (x_n) is bounded. Indeed, there is an $n_1 \in \mathbb{N}$ such that for all integers $n \geq n_1$, $|x_n - x_m| \leq 1$. Then, an upper bound for $|x_n|$ is
$$\max\{|x_0|, \dots, |x_{n_1-1}|, |x_{n_1}| + 1\}.$$
So, (x_n) is bounded. Hence it contains a convergent subsequence x_{n_k} converging to $x \in \mathbb{R}$. Fix then a $0 < \varepsilon \in \mathbb{R}$. As (x_n) is Cauchy, there is an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all integers $n, m \geq n_{\frac{\varepsilon}{2}}$,
$$|x_n - x_m| < \frac{\varepsilon}{2}.$$
Now, there is a k such that $n_k \geq n_{\frac{\varepsilon}{2}}$ and $|x_{n_k} - x| \leq \frac{\varepsilon}{2}$. For this value of k and any integer $n \geq n_{\frac{\varepsilon}{2}}$ we have:
$$|x_n - x| \leq |(x_n - x_{n_k}) + (x_{n_k} - x)| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
□

9. Series

Definition 9.1. A series is a sequence S_n associated to another sequence (x_n) using the formula

$$S_n = \sum_{k=0}^{n} x_k$$

Example 9.2.
(1) $\displaystyle\sum_{k=0}^{n} \frac{1}{2^k} = \frac{1 - \frac{1}{2^{k+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{k+1}}\right)$
(2) $\displaystyle\sum_{k=0}^{n} \frac{1}{k!}$
(3) $\displaystyle\sum_{k=0}^{n} \frac{1}{k}$
(4) $\displaystyle\sum_{k=0}^{n} (-1)^k \frac{1}{k}$

(5) $\displaystyle\sum_{k=0}^{n} \frac{1}{k^2}$

(6) $\displaystyle\sum_{k=0}^{n} \frac{1}{k^s}$

Definition 9.3. A series $S_n = \sum_{k=0}^{n} x_k$ is

$$\left.\begin{array}{r} \text{convergent} \\ \text{divergent} \\ \text{absolute convergent} \end{array}\right\}, \text{ if } \left\{\begin{array}{l} (S_n) \text{ is convergent} \\ (S_n) \text{ is divergent} \\ (S'_n) \text{ is absolute convergent, where } S'_n = \sum_{k=0}^{n} |x_k| \end{array}\right.$$

If it is convergent, then $\displaystyle\sum_{k=0}^{\infty} x_k := \lim_{n\to\infty} S_n$.

Example 9.4.

$$\sum_{k=0}^{n} \frac{1}{2^k} = 2\left(1 - \frac{1}{2^{k+1}}\right) \Rightarrow \sum_{k=0}^{\infty} \frac{1}{2^k} = \lim_{k\to\infty} 2\left(1 - \frac{1}{2^{k+1}}\right) = 2$$

By Cauchy's convergence criterion we have:

Proposition 9.5. *$\displaystyle\sum_{k=0}^{\infty} x_k$ is convergent if and only if for every $0 < \varepsilon \in \mathbb{R}$, there is an $n_\varepsilon \in \mathbb{N}$ such that for every integer $m > n \geq n_\varepsilon$,*

$$\left|\sum_{k=n+1}^{m} x_k\right| < \varepsilon$$

Example 9.6.
$\displaystyle\sum_{k=0}^{\infty} \frac{1}{k}$ is not convergent. Indeed, assume that for $\varepsilon = \frac{1}{4}$, Cauchy's criterion is satisfied with $n_1 \in \mathbb{N}$. Then, for $n := n_1$ and $m = 2n$,

$$\frac{1}{4} = \varepsilon > \left|\sum_{k=n+1}^{2n} x_k\right| = \left|\sum_{k=n+1}^{2n} \frac{1}{k}\right| \geq \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{1}{2}.$$

This is a contradiction

An immediate consequence of the Cauchy convergence criterion (Proposition 9.5) is the following:

Proposition 9.7. *If $\displaystyle\sum_{k=0}^{\infty} x_n$ is convergent, then $\displaystyle\lim_{n\to\infty} x_n = 0$.*

Proof. Indeed, by Proposition 9.5, for every $0 < \varepsilon \in \mathbb{R}$, there is an $n_\varepsilon \in \mathbb{N}$ such that for all integers $m > n \geq n_\varepsilon$,

$$\left|\sum_{k=n+1}^{m} x_k\right| \leq \varepsilon.$$

In particular, if we choose $m = n + 1$, then we obtain that

$$\varepsilon \geq \left|\sum_{k=n+1}^{n+1} x_k\right| = |x_{n+1}|.$$

This implies that $\displaystyle\lim_{n\to\infty} x_n = 0$. □

Example 9.8. $\sum_{k=0}^{\infty} \cos(n)$ is not convergent. Indeed, for this it is enough to see that $x_n := \cos(n)$ does not converge to 0. Assume it does. Then, so does all its subsequence. However, consider the subsequence given by $n_k := \lfloor 2k\pi \rfloor$. We have

$$x_{n_k} = \cos(\lfloor 2k\pi \rfloor) \geq \cos(2k\pi - 1) = \cos(-1) > 0.$$

This contradicts the earlier assumption that all subsequences of x_n converge to 0.

Definition 9.9. If $0 < s$ is a rational number, say $s = \frac{a}{b}$ then one defines $n^s := \sqrt[b]{n^a}$ for all $n \in \mathbb{N}$. This does not depend on the representation of s as $\frac{a}{b}$. That is, if we replace $\frac{a}{b}$ by $\frac{ca}{cb}$ (where $c \in \mathbb{N}$), then:

$$\sqrt[cb]{n^{ca}} = \sqrt[b]{\sqrt[c]{n^{ca}}} = \sqrt[b]{n^c}.$$

Example 9.10. $2^{\frac{2}{3}} = \sqrt[3]{4}$

Example 9.11. If $0 < s < 1$ is a rational number, then $\sum_{k=0}^{\infty} \frac{1}{k^s}$ is divergent. Indeed, by the Cauchy convergence criterion we have to find a $0 < \varepsilon \in \mathbb{R}$, such that for any $n_\varepsilon \in \mathbb{N}$ there are integers $m > n \geq n_\varepsilon$ such that

$$\left| \sum_{k=n+1}^{m} x_k \right| > \varepsilon. \tag{9.11.a}$$

We claim that $\varepsilon = \frac{1}{4}$ is a good choice for this. Indeed, for any $n_\varepsilon \in \mathbb{N}$ and $n \geq n_\varepsilon$, if we set $m = 2n$, then we see that (9.11.a) holds. That is:

$$\left| \sum_{k=n+1}^{2n} \frac{1}{k^s} \right| \geq \underbrace{\left| \sum_{k=n+1}^{2n} \frac{1}{k} \right|}_{s<1} \geq \left| \sum_{k=n+1}^{2n} \frac{1}{2n} \right| = n\frac{1}{2n} = \frac{1}{2} > \frac{1}{4}.$$

Example 9.12. Let $s > 1$ be a rational number. Then:

$$\begin{aligned} S_n := \sum_{k=1}^{n} \frac{1}{k^s} \leq \sum_{k=1}^{2n+1} \frac{1}{k^s} = 1 + \sum_{k=1}^{n} \frac{1}{(2k)^s} + \sum_{k=1}^{n} \frac{1}{(2k+1)^s} \\ \leq 1 + \sum_{k=1}^{n} \frac{1}{(2k)^s} + \sum_{k=1}^{n} \frac{1}{(2k)^s} = 1 + \frac{2}{2^s} S_n = 1 + \frac{1}{2^{s-1}} S_n \end{aligned}$$

So, we have

$$S_n \leq \frac{1}{1 - \frac{1}{2^{s-1}}}$$

Hence, S_n is bounded from above and increasing. Hence, it is convergent.

Example 9.13. According to the previous example, $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is not absolute convergent ($\left|(-1)^k \frac{1}{k}\right| = \frac{1}{k}$).

However, it is convergent:

$$\begin{aligned} S_{2n+1} := -1 + \sum_{k=2}^{2n+1} (-1)^k \frac{1}{k} &= -1 + \sum_{k=1}^{n} \left((-1)^{2k} \frac{1}{2k} + (-1)^{2k+1} \frac{1}{2k+1} \right) \\ &= -1 + \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{2k+1} \right) = -1 + \sum_{k=1}^{n} \frac{1}{2k(2k+1)} \\ &\leq -1 + \sum_{k=1}^{n} \frac{1}{2k \cdot 2k} = -1 + \frac{1}{4} \sum_{k=1}^{n} \frac{1}{k^2} \leq \underbrace{-1 + \frac{1}{1 - \frac{1}{2^{2-1}}}}_{\text{by the previous example}} = 1 \end{aligned}$$

So, S_{2n+1} is an increasing bounded sequence, hence it is convergent. The question, is what happens when we put in the even terms. We have:

$$S_{2n} = S_{2n+1} + \frac{1}{2n+1} \geq S_{2n+1}$$

and furthermore

$$S_{2n} = \sum_{k=1}^{2n} (-1)^k \frac{1}{k} = \sum_{k=1}^{n} (-1)^{2k-1} \frac{1}{2k-1} + (-1)^{2k} \frac{1}{2k} = \sum_{k=1}^{n} \frac{-1}{2k-1} + \frac{1}{2k} = \sum_{k=1}^{n} \frac{-1}{(2k-1)2k}.$$

So, (S_{2n}) is decreasing, (S_{2n+1}) is increasing, and furthermore the second one is

The previous example generalizes to:

Proposition 9.14. LEIBNIZ CRITERION: *if (x_n) is decreasing, and $\lim_{n\to\infty} x_n = 0$, then $\sum_{k=0}^{n} (-1)^n x_n$ is convergent.*

Proposition 9.15. $\sum_{n=0}^{\infty} x_n$ *is absolute convergent, then it is convergent.*

Proof. If we apply the Cauchy criterion for convergence in both cases, then in the absolute convergence case we have to show that

$$\sum_{k=n+1}^{m} |x_k| \leq \varepsilon \tag{9.15.b}$$

and in the case of "usual" convergence

$$\left| \sum_{k=n+1}^{m} x_k \right| \leq \varepsilon. \tag{9.15.c}$$

As we have by the triangle equality

$$\left| \sum_{k=n+1}^{m} x_k \right| \geq \sum_{k=n+1}^{m} |x_k|,$$

we see that (9.15.b) implies (9.15.c). □

Proposition 9.16. BERNOULLI INEQUALITY (NEGATIVE CASE).
If $-1 < x < 0$, then $(1+x)^n \geq 1 + nx$

Proof.

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = (1 + (n+1)x + nx^2) \geq 1 + (n+1)x$$

□

Example 9.17. In the last example, we compute $\sum_{k=0}^{\infty} \frac{1}{k!}$. First, $S_n := \sum_{k=0}^{n} \frac{1}{k!}$ is definitely an increasing sequence. Furthermore, it is bounded, because

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n} \frac{1}{2^k} \leq 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 3.$$

So, $\sum_{k=0}^{\infty} \frac{1}{k!}$ is convergent.

This is one of the rare occasions when we are actually able to compute an infinite sum. We will use that we have showed earlier that

$$\sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdot \dots \cdot \left(1 - \frac{k-1}{n}\right) = e,$$

where the above notation is a bit loose for the $k=0$ and $k=1$ terms, as there we mean that the terms with the big parentheses do not exist at all. So the $k=0$ and $k=1$ terms are both 1.

Having recalled the above infinite sum, we may use it to find the limit of our present sum too:

$$\sum_{k=0}^{n} \frac{1}{k!} \geq \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdot \dots \cdot \left(1-\frac{k-1}{n}\right) \geq 2+\sum_{k=2}^{n} \frac{1}{k!}\left(1-\frac{k-1}{n}\right)^{k-1}$$
$$\geq \underbrace{2+\sum_{k=2}^{n} \frac{1}{k!}\left(1-\frac{(k-1)^2}{n}\right)}_{\text{Proposition 9.16}} = \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{n}\sum_{k=2}^{n} \frac{(k-1)^2}{k!}$$
$$\geq \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{n}\sum_{k=2}^{n} \frac{1}{(k-2)!} \geq \sum_{k=0}^{n} \frac{1}{k!} - \frac{3}{n}$$

So, we see that $\sum_{k=0}^{\infty} \frac{1}{k!} \geq e$ and that $e \geq \sum_{k=0}^{\infty} \frac{1}{k!} - \lim_{n\to\infty} \frac{3}{n} = \sum_{k=0}^{\infty} \frac{1}{k!}$. Then, it follows that $\sum_{k=0}^{\infty} \frac{1}{k!} = e$

Last there are two theorems in the book that you should know, although for most examples that they are used, squeeze is usually easier to use.

Proposition 9.18. Cauchy convergence/divergence criterion

(1) If $\left(\sqrt[n]{|x_n|}\right)$ is bounded and $\lim_{n\to\infty} \sup \sqrt[n]{|x_n|} < 1$, then $\sum_{k=0}^{\infty} x_n$ is absolutely convergent.

(2) If $\left(\sqrt[n]{|x_n|}\right)$ is bounded and $\lim_{n\to\infty} \sup \sqrt[n]{|x_n|} > 1$ or if $\left(\sqrt[n]{|x_n|}\right)$ is not bounded, then $\sum_{k=0}^{\infty} x_n$ is divergent.

Proposition 9.19. Alembert's criterion

Let (x_n) be a sequence such that $\left(\frac{|x_{n+1}|}{|x_n|}\right)$ is convergent and let the limit be ρ.

(1) if $\rho < 1$, then $\sum_{k=0}^{\infty} x_n$ is absolutely convergent.

(2) if $\rho > 1$, then $\sum_{k=0}^{\infty} x_n$ is divergent.

The ideas behind the proofs. (1) For the convergence statements, for both sequences one tries to do the squeeze given by $0 \leq |x_n| \leq q$ for some $q < 1$. The only question is what q should one choose. In the case of Cauchy criterion we obtain that if ρ is the limsup, then $\sqrt[n]{|x_n|} \leq \rho + \varepsilon$ after finitely many steps, so, then $|x_n| \leq (\rho+\varepsilon)^n$. So, we may set $q = \rho + \varepsilon$ in this case. The case of Alembert is similar. There $\frac{x_{n+1}}{x_n} \leq \rho + \varepsilon$ after finitely many steps, say after $n \geq n_\varepsilon$. So, we have $|x_n| \leq q^{n-n_\varepsilon} |x_{n_\varepsilon}|$ if we set $q = \rho+\varepsilon$ here too.

(2) For the divergence statements one just shows that $|x_n|$ does not converge to 0. In the case of Cauchy there are infinitely many elements with $\sqrt[n]{|x_n|} \geq 1$ which is equivalent to $|x_n| \geq 1$. In the case of Alembert, $\frac{|x_{n+1}|}{|x_n|} > 1$ after finitely many steps, that is, $|x_{n+1}| > |x_n|$.

$\square$

10. Real functions of 1-variable

We are going to consider functions $f : E \to \mathbb{R}$ where E is a subset of $\mathbb{R}$. Being a function here means that f assigns a single element $f(x)$ to each $x \in E$. We call E the *domain* of f. The set of values of f:

$$R(f) := \{f(x) \in \mathbb{R} | x \in E\}$$

is called the *range* of f.

Whenever $E' \subseteq E$ is a smaller set, we can restrict f to E', regard it as a function $f|_{E'} : E' \to \mathbb{R}$. The latter is called the restriction of f over E'.

10.1. *Basic properties of functions*

If $f, g : E \to \mathbb{R}$ are functions with the same domain, then we say that $f < g$ or $f \le g$ if for all points of E the value of f is smaller/smaller or equal than the value of g. (With formula: for all $x \in E$, we have $f(x) < g(x)$ (resp. $f(x) \le g(x)$)).

If $f : E \to \mathbb{R}$ is a function then the absolute value function $|f|$ is defined by $|f|(x) := |f(x)|$. Geometrically we reflect the negative part across the x-axis.

Example 10.1. $f(x) = x^3$

An *even* function is the one the graph of which can be reflected onto the y-axis, so $f(x) = f(-x)$.

An *odd* function is the one the graph of which can be reflected onto the y-axis, so $f(x) = -f(-x)$.

Example 10.2. $\cos(x)$ is an even function, and $\sin(x)$ is an odd function.

x^3 is an odd function, $|x^3|$ is an even function, but $|x^3 + 1|$ neither even nor odd.

A *periodic* function is one such that $f(x) = f(x + P)$ for some real number $P > 0$ (which we call the period).

Example 10.3. $\cos(x)$ and $\sin(x)$ are periodic with period 2π, and $\{x\}$ is periodic with period 1

The inverse function is the function f^{-1} such that $f^{-1}(x) = y \Leftrightarrow f(y) = x$. Most functions DO NOT have an inverse!!

Example 10.4. $\sin(x)$ does not have an inverse, however $\sin(x)|_{[\frac{-\pi}{2},\frac{\pi}{2}]}$ does have one, which is denoted by $\arcsin(x)$.

A number y_0 is an upper (resp. lower) bound for $f : E \to \mathbb{R}$ if y_0 is an upper (resp. lower) bound for the range $R(f)$ of f. This then defines when a function is bounded (resp. bounded from above, from below)

A number y_0 is a supremum (resp. infimum) for $f : E \to \mathbb{R}$ if y_0 is the supremum (resp. infimum) of the range $R(f)$ of f. We write

$$\mathrm{Sup}_{x\in E} f(x), \text{ and } \mathrm{Inf}_{x\in E} f(x)$$

Example 10.5. $\frac{3}{2}$ is an upper bound of $\sin(x)$ and $\cos(x)$, but

$$\mathrm{Sup}_{x\in\mathbb{R}} \cos(x) = 1, \text{ and } \mathrm{Sup}_{x\in\mathbb{R}} \sin(x) = 1.$$

Also,

$$\mathrm{Sup}_{x\in\mathbb{R}_+} 1 - \frac{1}{x} = 1$$

Definition 10.6. A function $f : \mathbb{R} \supseteq E \to \mathbb{R}$ has a *local maximum* at $x_0 \in D$ if there is a real number $\delta > 0$ such that for every $x \in E$ if $|x - x_0| \le \delta$ then $f(x) \le f(x_0)$.

Example 10.7. (1)

(2) $\cos(x)$ has a local maximum at $2k\pi$ and a local minimum at $(2k + 1)\pi$ for any $k \in \mathbb{Z}$.

(3) We have not learned yet how to compute, but we will learn that $x(x-1)(x+1) = x^3 - x$ has a local maximum at $x = \frac{1}{\sqrt{3}}$

(4) We have also not learned yet how to compute, but we will learn that $\sin(x) + \frac{1}{\sqrt{2}}x$ has a local maximum at $2k\pi + \frac{3\pi}{2}$ and a local minimum at $2k\pi + \frac{5\pi}{2}$

A function is *injective* if for every $y \in \mathbb{R}$ there is a single $x \in X$ such that $f(x) = y$. With other words every horizontal line intersects the graph of f in at most 1 point.
increasing/decreasing

10.2. *Limits of functions and continuity*

The starting point of our investigation is what happening with the function $f(x) := \frac{\sin(x)}{x}$ in $x = 0$. For the first sight, $f(0)$ is not defined so there is nothing to be discussed. However, if we approach $x = 0$ with the sequence $\frac{1}{n}$, then we obtain a limit. Indeed,

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$$

So, if we set $f(0) = 1$ then maybe $f(x)$ becomes a nice "continuous" function. We will make sense of this below. We will define precisely why $f(0) = 1$ makes $f(x)$ continuous.

Definition 10.8. If $f : E \to \mathbb{R}$ is defined on a pointed neighborhood of $x_0 \in \mathbb{R}$ if there is an interval of the form $]x_0 - \alpha, x_0 + \alpha[$ contained in $E \cup \{x_0\}$.

Definition 10.9. Assume $f : E \to \mathbb{R}$ is defined on a pointed neighborhood of $x_0 \in \mathbb{R}$. Then, $\lim_{x\to x_0} f(x) = l$ if one of the following two equivalent definitions holds:

(1) For every $0 < \varepsilon \in \mathbb{R}$ there is a $0 < \delta \in \mathbb{R}$ such that whenever $0 < |x - x_0| \leq \delta$, then $|f(x) - l| < \varepsilon$.
(2) For every sequence $(x_n) \subseteq E \setminus \{x_0\}$ for which $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = l$.

With everyday language the two definitions mean the following:

(1) whenever x is close to x_0, $f(x)$ is also close to l (More precisely: for every $\varepsilon > 0$ there is a $\delta > 0$ such that if x is closer to x_0 than δ then $f(x)$ is closer to l than ε.)
(2) whenever (x_n) converges to x_0, then $(f(x_n))$ converges to l.

We comment a bit about why the above two definitions are equivalent.

- $(i) \Rightarrow (ii)$: Take a sequence (x_n) for which $\lim_{n\to\infty} x_n = x_0$. We have to show that $\lim_{n\to\infty} f(x_n) = l$. So, fix $\varepsilon > 0$. Then, this yields a $\delta > 0$ as in definition (i). For this δ, there is an n_δ such that $|x - x_n| \leq \delta$ for $n \geq n_\delta$, and hence for all such n, $|l - f(x_n)| < \varepsilon$.
- $NOT(i) \Rightarrow NOT(ii)$: The negation of (i) is that there is an $\varepsilon > 0$ such that for each $\delta > 0$ there is an $y_\delta \in [x_0 - \delta, x_0 + \delta]$ such that $|f(y_\delta) - l| > \varepsilon$. So, $x_n := y_{\frac{1}{n}}$ converges to x_0, but all $f(x_n)$ have distance at least ε from l, so $(f(x_n))$ does not converge to l.

In general, we try to use definition (ii) more as it is simpler. Luckily, it is almost always enough for proving that a limit does not exist. We usually use definition (i) only when (ii) does not work.

Example 10.10. $\lim_{x\to 2} x^2 = 4$. Indeed, for any $\varepsilon > 0$ we have to give $\delta > 0$ such that for any $|x - 2| \leq \delta$, $|x^2 - 4| \leq \varepsilon$. For that note that

$$(x - 2)(x + 2) = (x^2 - 4)$$

and furthermore, if $|x - 2| \leq 1$, then $5 \geq x + 2 \geq 3$, and so

$$|x^2 - 4| = |x - 2||x + 2| \leq 5|x - 2|.$$

In particular, $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ works, as if

$$|x - 2| \leq 1, \text{ and } \frac{\varepsilon}{5},$$

then we have

$$|x^2 - 4| \leq 5|x - 2| \leq 5\frac{\varepsilon}{5} = \varepsilon.$$

Example 10.11. The above example is not hard to show also with definition (ii) (which usually does not work for trickier limits such as $\frac{\sin(x)}{x}$ at $x = 0$). So, assume that (x_n) is a sequence such that $\lim_{n\to\infty} x_n = 2$. Then, by the algebraic properties of the limit we know that $\lim_{n\to\infty} x_n^2 = 2^2 = 4$.

Example 10.12. The same argument as above shows that for any $x_0 \in \mathbb{R}$, $\lim_{x\to x_0} x^2 = x_0^2$.

Or even better if $p(x)$ is a polynomial of 1 variable, so something of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r$, then $\lim_{x\to x_0} p(x) = p(x_0)$.

Definition 10.13. $f : E \to \mathbb{R}$ is continuous at $x_0 \in E$, if $\lim_{x\to x_0} f(x) = f(x_0)$ (including that).

Corollary 10.14. *Every polynomial is continuous on* $\mathbb{R}$*.*

Example 10.15. Set

$$f(x) := \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

Then we have $\lim_{x\to 0} f(x) = 0$, as in the definition we assumed $0 < |x - x_0| \leq \delta$, so the function value 1 for $x_0 = 0$ does not cause any problem.

Example 10.16. We claim that $\lim_{x\to 0} \cos(x) = 1$. Indeed, let (x_n) be a sequence converging to 0. Then,

$$0 \leq |\cos(x_n) - 1| = \left|\sin^2\left(\frac{x_n}{2}\right)\right| \leq \frac{x_n^2}{4},$$

using the inequality $|\sin(x)| \leq |x|$. So, squeeze theorem tells us that $\lim_{n\to\infty} |\cos(x_n) - 1| = 0$.

Example 10.17. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

and let x_n and y_n be two sequences defined by

$$x_n := \frac{1}{n}, \text{ and } y_n := \frac{\sqrt{2}}{n}.$$

Then:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty} f(x_n) = 1, \text{ but } \lim_{n\to\infty} f(y_n) = -1$$

So, we exhibited to sequences converging to 0, for which the associated function value sequences have different limits. In particular, the limit of f in 0 does not exist.

Example 10.18. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist. Indeed, consider the sequences $x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}$ and $x_n'; = \frac{1}{\pi\left(2n+\frac{3}{2}\right)}$. Then, first $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} x_n' = 0$. However,

$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{\frac{1}{\pi\left(2n+\frac{1}{2}\right)}}\right) = \sin\left(\pi\left(2n + \frac{1}{2}\right)\right) = 1,$$

but

$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n'}\right) = \sin\left(\frac{1}{\frac{1}{\pi\left(2n+\frac{3}{2}\right)}}\right) = \sin\left(\pi\left(2n + \frac{3}{2}\right)\right) = -1.$$

So, definition (ii) of the limit is not satisfied, and hence the

Definition (ii) of the limit of a function also gives rise to moving all the statements about limits of sequences to limits of functions. Indeed, say we are working at x_0, and we want to prove that if l and k are the limits of $f(x)$ and $g(x)$ (at x_0), then $l+k$ is the limit of $(f+g)(x)$. So take a sequence (x_n) converging to x_0. We know that $\lim_{n\to\infty} f(x_n) = l$ and $\lim_{n\to\infty} g(x_n) = k$. But then we have learned about sequences that $\lim_{n\to\infty} f(x_n) + f'(x_n) = l + k$, which exactly shows that $\lim_{n\to\infty} (f + f')(x_n) = l + k$.

We collect all the statements one can show along the same arguments:

Proposition 10.19. *Let f and g be two functions such that a pointed neighborhood of x_0 is in the domain of both f and g. Assume that the limits of f and g at x_0 exist and they are l and k, respectively. Then,*

(1) the limit of $f+g$ exists at x_0 and $\lim_{x\to x_0} (f+g)(x) = l+k$
(2) the limit of $f \cdot g$ exists at x_0 and $\lim_{x\to x_0} (f\cdot g)(x) = l\cdot k$
(3) if $k \neq 0$, then the limit of $\frac{f}{g}$ exists at x_0 and $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{l}{k}$
(4) if $f(x) \le g(x)$ for any x in a pointed neighborhood of x_0, then $l \ge k$.
(5) if there is a third function $h(x)$ such that there is also a pointed neighborhood of x_0 in the domain of h, and:
  i on some pointed neighborhood of x_0 we have $f(x) \le h(x) \le g(x)$, and
  ii $l = k$,
  then $\lim_{x\to x_0} h(x) = l$.

Example 10.20. The main example is that $\lim_{x\to x_0} \dfrac{\sin(x)}{x} = 1$. Indeed, by geometric reasons we have

$$\frac{\sin(x)}{x} \le 1, \text{ and } \tan(x) \ge x \Rightarrow \frac{\sin(x)}{x} \ge \cos(x).$$

So, squeeze tells us the limit, as

$$\cos(x) \le \frac{\sin(x)}{x} \le 1.$$

Example 10.21. $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$. Indeed, we can squeeze it with

$$-x \le x \sin\left(\frac{1}{x}\right) \le x.$$

The above proposition has all the nice consequences about continuity:

Proposition 10.22. *If $f, g : E \to \mathbb{R}$ are continuous functions at $x_0 \in E$, then so are*

(1) $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{R}$,
(2) $f \cdot g$, and
(3) if $g|_E$ is nowhere zero (meaning that for all $x \in E : g(x) \neq 0$), then $\frac{f}{g}$ too.

Example 10.23.
- $f(x) := \frac{1}{x}$ is continuous on $\mathbb{R} \setminus 0$.
- $\frac{x}{x^2-3x+1}$ is continuous on $\mathbb{R} \setminus \left\{\frac{3\pm\sqrt{5}}{2}\right\}$,
- In general, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ is continuous on $\{x \in \mathbb{R} | q(x) \neq 0\}$ (which is the whole real line minus finitely any points).

Proposition 10.24. *Let $f : E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ be functions and let $x_0 \in E$ be point such that*

(1) $f(E) \subseteq G$,
(2) $\lim_{x\to x_0} f(x_0) = y_0$,
(3) $\lim_{y\to y_0} g(y_0) = l$

(4) *there is a pointed neighborhood $B(x_0, r) \setminus \{x_0\} \subseteq E$ such that for every x in this neighborhood, $f(x) \neq y_0$.*

Then: $\lim_{x \to x_0} (g \circ f)(x) = l$

The idea of the proof of the proposition is simple: one takes a sequence $(x_n) \subseteq E \setminus \{x_0\}$ converging to x_0. Then $\lim_{n \to \infty} f(x_n) = y_0$, by our assumption (2), and then $\lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} g(f(x_n)) = l$ by (3). See page 71 of the book for the details.

Example 10.25. Condition (4) in the above proposition is necessary, because of the following example. Set:

$$g(x) = \begin{cases} 0 & , \text{for } x \neq 0 \\ 1 & , \text{for } x = 0 \end{cases} \qquad f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & , \text{for } x \neq 0 \\ 0 & , \text{for } x = 0 \end{cases}$$

. Then, $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} g(x) = 0$. However, $\lim_{x \to 0} (g \circ f)(x) \neq 0$, because the following two sequences induce function value sequences with different limits:

$$x_n := \frac{1}{\pi n} \qquad y_n := \frac{1}{\pi n + \frac{\pi}{2}}$$

as

$$\lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} 1 = 1, \text{ and } \lim_{n \to \infty} (g \circ f)(y_n) = \lim_{n \to \infty} 0 = 0.$$

Also, note that condition (4) is not satisfied in this example, as $f(x) = 0$ for $x = \frac{1}{\pi n}$, so there is no pointed neighborhood of 0 such that the function value avoids 0.

Example 10.26. A positive example for applying the above proposition about composition is that $\lim_{x \to 0} \dfrac{\sin(x^2)}{x^2} = 1$. Indeed, if we set $g(x) := \frac{\sin(x)}{x}$, and $f(x) = x^2$, then condition (4) is satisfied, as $f(x) \neq 0$ for $x \neq 0$, so we may apply the proposition.

Definition 10.27. A (pointed) neighborhood of $+\infty$ (resp. $-\infty$) is an interval of the form $]a, +\infty]$ (resp. $[-\infty, a[$).

Definition 10.28. Let x_0 and l be either a real number, or $+\infty$, or $-\infty$, and let $f : E \to \mathbb{R}$ be a function such that a pointed neighborhood of x_0 is contained in E. We say that the limit of $f(x)$ at x_0 is l, which we denote by $\lim_{x \to x_0} f(x) = l$, if for each sequence $(x_n) \subseteq E \setminus \{x_0\}$, whenever $\lim_{n \to \infty} x_n = x_0$ we have $\lim_{n \to \infty} f(x_n) = l$.

Example 10.29. $\lim_{x \to 0} \frac{1}{x^2} = +\infty$. Indeed, if $\lim_{n \to \infty} x_n = 0$, then $\lim_{n \to \infty} x_n^2 = +\infty$.

Example 10.30. $\lim_{x \to 0} \frac{1}{x}$ does not exist, as for $x_n = \frac{1}{n}$ we have $\lim_{n \to \infty} \frac{1}{x_n} = \lim_{n \to \infty} n = +\infty$, and for $y_n = \frac{-1}{n}$ we have $\lim_{n \to \infty} \frac{1}{x_n} = \lim_{n \to \infty} -n = -\infty$.

10.2.1. *Algebra and infinite limits.*

Proposition 10.31. *Let x_0 be either a real number, or $\pm\infty$, and let $f, g : E \to \mathbb{R}$ be functions.*

(1) Addition rule. *If*
- $\lim_{x \to x_0} f(x) = +\infty$ *(resp. $-\infty$), and*
- $g(x)$ *is bounded from below (resp. above)*

then $\lim_{x \to x_0} (f + g)(x) = +\infty$ *(resp. ∞).*

(2) Product rule. *If*
- $\lim_{x \to x_0} |f(x)| = +\infty$,
- $|g(x)|$ *is bounded from below by a positive number (that is, there is a $\delta > 0$ such that $|g(x)| \geq \delta$ for all $x \in E$), and*
- $f(x)g(x) > 0$ *(resp. < 0) for all $x \in E$,*

then $\lim_{x \to x_0} f(x)g(x) = +\infty$ *(resp. $-\infty$).*

(3) First division rule. *If*
- $f(x)$ *is bounded,*
- $g(x)$ *is nowhere zero, and*
- $\lim_{x\to x_0} |g(x)| = +\infty$.

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$

(4) Second division rule. *If*
- $\lim_{x\to x_0} g(x) = 0$,
- $|f(x)|$ *is bounded from below by a positive number (that is, there is a* $\delta > 0$ *such that* $|f(x)| \geq \delta$ *for all* $x \in E$*), and*
- $f(x)/g(x) > 0$ *(resp.* < 0*) for all* $x \in E$,

then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = +\infty$ *(resp.* $-\infty$*).*

(5) Squeeze. *If* $f(x) \leq g(x)$*, and*
- *i if* $\lim_{x\to x_0} f(x) = +\infty$*, then* $\lim_{x\to x_0} g(x) = +\infty$
- *ii if* $\lim_{x\to x_0} g(x) = -\infty$*, then* $\lim_{x\to x_0} f(x) = -\infty$

Example 10.32.
- $\lim_{x\to 0} \frac{1}{x^2} + \cos(x) = +\infty$
- $\lim_{x\to +\infty} \cos(x) - x = -\infty$.

Example 10.33. The above assumptions for the addition law are important, as otherwise we can have all different kinds of limits. We give examples of this using the functions $f(x) = x^3$, $g(x) = x^2$ and $h(x) = x^3 + 1$. We have
- $\lim_{x\to +\infty} \pm f(x) = \lim_{x\to +\infty} \pm x^3 = \pm\infty$,
- $\lim_{x\to +\infty} \pm h(x) = \lim_{x\to +\infty} \pm x^3 + 1 = \pm\infty$, and
- $\lim_{x\to +\infty} \pm g(x) = \lim_{x\to +\infty} \pm x^2 = \pm\infty$.

On the other hand:
- $\lim_{x\to +\infty} f(x) - g(x) = \lim_{x\to +\infty} x^3 - x^2 = \lim_{x\to +\infty} x^2(x-1) = +\infty$,
- similarly $\lim_{x\to +\infty} g(x) - f(x) = -\infty$, and
- $\lim_{x\to +\infty} f(x) - h(x) = -1$.

Intuitively we can say, don't apply product law in the $+\infty, -\infty$ case, but then does not cover the non-limit cases (the addition law allows the second sequence not to have a limit).

Example 10.34. The above assumptions for the product law are also important. We give examples of this using the functions $f(x) = x$, $g(x) = \frac{\cos(x)}{x}$ and $h(x) = (-1)^{[x]}$. We have
- $\lim_{x\to +\infty} |f(x)| = +\infty$,
- $|g(x)|$ is not bounded from below, and
- $|h(x)|$ is bounded from below, but $f(x)h(x) \not> 0$.

Then:
- $\lim_{x\to +\infty} f(x)g(x) = \lim_{x\to +\infty} \cos(x)$ does not exist, and
- $\lim_{x\to +\infty} f(x)h(x) = \lim_{x\to +\infty} x(-1)^{[x]}$ does not exist, on the other hand
- the product law applies to $(f(x)h(x))(h(x))$ and yields $\lim_{x\to +\infty} (f(x)h(x))(h(x)) = +\infty$

Never try to use product rule to limits of the type $0 \cdot \infty$.

Example 10.35. The assumptions of the first division rule are also important. One can can show that in the $\frac{\pm\infty}{\pm\infty}$ case anything can happen for example using $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$,$(-1)^{\left[\frac{1}{x}\right]}\frac{1}{x}$ with limit at 0:

(1) $\lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0} x = 0$,

(2) $\lim_{x\to 0} \frac{(-1)^{\left[\frac{1}{x}\right]}\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0}(-1)^{\left[\frac{1}{x}\right]}$ does not exist,
(3) $\lim_{x\to 0} \frac{\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}$ does not exist, and
(4) $\lim_{x\to 0} \frac{\frac{1}{x^3}}{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x^2} = +\infty$.

Similar examples show that the assumptions are important for the second division rule. Never try to use division rules to limits of the form $\frac{\pm\infty}{\pm\infty}$ and $\frac{0}{0}$.

10.2.2. *One sided limits.* The main question is how to make sense of limits such as at 0 of $\sqrt{x}$, as here the domain does not contain a pointed neighborhood of 0. The solution for this is the notion of one sided limit.

Definition 10.36. A function $f : E \to \mathbb{R}$ is defined on the left (resp. right) of $x_0 \in E$, if E contains an interval of the form $]x_0 - \delta, x_0[$ (resp. $]x_0, x_0 + \delta[$).

Definition 10.37. Let $f : E \to \mathbb{R}$ be a function, such that $x_0 \in E$, and f is defined on the left (resp. right) of x_0. Let l be either a real number or $\pm\infty$. Then, $\lim_{x\to x_0^-} f(x) = l$ (resp. $\lim_{x\to x_0^+} f(x) = l$) if for all sequences $(x_n) \subseteq \{x \in E|x < x_0\}$ (resp. $(x_n) \subseteq \{x \in E|x > x_0\}$) converging to x_0 we have $\lim_{n\to\infty} f(x_n) = l$.

Example 10.38. $f(x) := \sqrt{x} : \{x \in \mathbb{R}|x \geq 0\} \to \mathbb{R}$. We claim that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Indeed, fix a sequence $(x_n) \subseteq \mathbb{R}_+$ such that $\lim_{n\to\infty} x_n = 0$. We have to show that then $\lim_{n\to\infty} \sqrt{x_n} = 0$ too. So, we need to show that for each $\varepsilon > 0$, there is an n_0 such that for every integer $n \geq n_0$, $\sqrt{x_n} \leq \varepsilon$. However, we know that $\lim_{n\to\infty} x_n = 0$. So, we know that there is an n_0 such that $|x_n| < \varepsilon^2$ for all $n \geq n_0$. But then, for any such n we also have $\sqrt{x_n} < \varepsilon$.

Proposition 10.39. *If $f : E \to \mathbb{R}$ is a function such that there is a pointed neighborhood of x_0 contained in E, and*

$$l = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x),$$

then $\lim_{x\to x_0} f(x) = l$.

10.2.3. *Monotone functions.*

Proposition 10.40. *If $f : E \to \mathbb{R}$ is monotone, then at each point $x_0 \in E$:*

(1) if f is defined on the left of x_0, $\lim_{x\to x_0^-} f(x)$ exists,
(2) if f is defined on the right of x_0, $\lim_{x\to x_0^+} f(x)$ exists, and
(3) if f is defined in a neighborhood of $\pm\infty$, then $\lim_{x\to\pm\infty} f(x)$ exists.

Proof. We treat only the increasing case, as the decreasing one follows from that by regarding $-f$ instead of f. Also, we treat only the first case as the others are similar. Set:

$$l := \mathrm{Sup}\{f(x)|x \in E, x < x_0\}.$$

Let $(x_n) \subseteq \{x \in E|x < x_0\}$ such that $\lim_{n\to\infty} x_n = x_0$. We want to show that $\lim_{n\to\infty} f(x_n) = l$. Fix a $\varepsilon > 0$. Then, by the definition of l, there is an $x' \in \{x \in E|x < x_0\}$, such that $f(x') > l - \varepsilon$. Then, there is an $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$ we have $x_n > x'$. However, then using that f is increasing we obtain that for all such n,

$$l \geq f(x_n) \geq f(x') \geq l - \varepsilon.$$

This shows that $\lim_{n\to\infty} f(x_n) = l$ indeed.

□

Example 10.41. (1)
$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{, if } x > 0 \\ 0 & \text{, if } x = 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

(2) $[x]$,
(3) $\{x\}$

10.2.4. *More on continuity.* First, we note that continuity can happen at no points, or just one:

Example 10.42. (1) example which is nowhere continuous
$$f(x) := \begin{cases} 1 & \text{, for } x \in \mathbb{Q} \\ -1 & \text{, for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
Indeed, to each point, we can converge both with rational and both with irrational numbers (because we learned at the beginning of the semester that between each two real number there is a rational and an irrational number as well), the function value can converge to each point with 1's and -1's too.

(2) an example, which is continuous at only one point:
$$f(x) := \begin{cases} 1 & \text{, for } x \in \mathbb{Q} \\ x & \text{, for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
This is continuous exactly at 1

Addition to algebraic rules of continuity (we already discussed addition, multiplication and division)

Proposition 10.43. *If $f, g : E \to \mathbb{R}$ are functions that are continuous at $x_0 \in E$, then so is*
(1) $|f|$,
(2) $\max\{f, g\}$, *where*
$$\max\{f, g\}(x) := \max\{f(x), g(x)\}$$
(3) $\min\{f, g\}$ *(defined similarly),*
(4) $f^+ := \max\{f, 0\}$,
(5) $f^- := \min\{f, 0\}$.

Furthermore, as the function value in the definition of continuity is required to be the same as the limit value, so now all our problems with composition vanishes.

Corollary 10.44. *If $f : E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ be two functions, that are continuous at x_0 and y_0, respectively, and $f(x_0) = y_0$. Then, $g \circ f$ is continuous at x_0.*

We will show soon that $\cos(x)$ is continuous, and then the above corollary will imply that $\cos(x^2)$, $\cos^2(x)$, etc. are continuous.

Definition 10.45. A function $f : E \to \mathbb{R}$ is *uniformly* continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in E$ we have:
$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Proposition 10.46. *If $f : E \to \mathbb{R}$ is uniformly continuous then it is continuous.*

Example 10.47. $x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous (but it is continuous as it is a polynomial). Indeed, we have:
$$|x^2 - y^2| = |x + y| \cdot |x - y|.$$
So, for any $\varepsilon > 0$ and $\delta > 0$ we may find x and y such that $|x+y| > \frac{2\varepsilon}{\delta}$, and $|x-y| = \frac{\delta}{2}$ (that is, chose two numbers that are very big but very close to each other). However, then $|x - y| \leq \delta$, but
$$|x^2 - y^2| > \frac{2\varepsilon}{\delta}\frac{\delta}{2} = \varepsilon.$$

Example 10.48. $\cos(x) : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and hence continuous.

$$|\cos(x) - \cos(y)| = 2\left|\sin\left(\frac{x+y}{2}\right)\right|\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 2|x-y|$$

So, if we set $\delta = \frac{\varepsilon}{2}$, then we have

$$|x-y| \leq \delta \Rightarrow |\cos(x) - \cos(y)| \leq 2|x-y| \leq 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$$

Definition 10.49. Let $f : E \to \mathbb{R}$ be a function, and $x_0 \in E$.

(1) f is continuous from the left, if $\lim_{x \to x_0^-} f(x) = f(x_0)$.
(2) f is continuous from the right, if $\lim_{x \to x_0^+} f(x) = f(x_0)$.

Definition 10.50. A function $f : [a,b] \to \mathbb{R}$ is continuous if it is continuous for at c for all $a < c < b$, it is continuous from the left at b and from the right at a.

Example 10.51. $\sqrt{1-x^2} : [-1,1]$ is continuous. Indeed:

(1) if $-1 < c < 1$, then $\sqrt{1-x^2}$ at c is continuous because $\sqrt{1-x^2}$ is the composition of $\sqrt{y}$ and $1-x^2$, and the latter is continuous at c and the former is continuous at $1-c^2$ (as $1-c^2 > 0$).
(2) $\sqrt{1-x^2}$ is continuous from the left at 1, because for all (x_n) converging to 1 from the left we have $\lim_{n\to\infty} \sqrt{1-x_n^2} = 1$, as $\lim_{n\to\infty} 1 - x_n^2 = 0$, and $\lim_{x\to 0^+} \sqrt{y} = 0$.
(3) $\sqrt{1-x^2}$ is continuous from the right at -1 for similar reasons.

10.2.5. *Consequences of Bolzano-Weierstrass.*

Theorem 10.52. *If $f : [a,b] \to \mathbb{R}$ is continuous for some $a, b \in \mathbb{R}$, then there is are $c, d \in [a,b]$ such that*

$$M := \max_{x\in[a,b]} f(x) = f(c) \qquad m := \min_{x\in[a,b]} f(x) = f(d)$$

Proof. There is a sequence $(x_n) \subseteq$ such that $f(x_n) \geq M - \frac{1}{n}$. In particular, $\lim_{n\to\infty} f(x_n) = M$. According to Bolzano-Weierstrass, there is a convergent subsequence x_{n_k}. Set $c := \lim_{k\to\infty} x_{n_k}$. Then $c \in [a,b]$, and by continuity of f

$$f(c) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = M.$$

□

Example 10.53. The above theorem is not true, if the domain is not $[a,b]$ for $a, b \in \mathbb{R}$. For example, take $f(x) = \pm\frac{1}{x^2+1} : \mathbb{R} \to \mathbb{R}$, then f does not attain its minimum/maximum as the function is nowhere 0, but it converges to 0 as x goes to $\pm\infty$.

Theorem 10.54. *If $f : [a,b] \to \mathbb{R}$ is continuous for some $a, b \in \mathbb{R}$, then f is uniformly continuous.*

Idea. The proof of the latter theorem is also an application of the Bolzano-Weierstrass theorem. Hence, we do not prove it here, and we refer to page 80-81 of the book. The main idea though is not horribly difficult and it is as follows.

Assume that f is not uniformly continuous. Then there is a $\varepsilon > 0$ such that for every $\frac{1}{n}$ there are x_n and $y_n \in [a,b]$ for such that $|x_n - y_n| \geq \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon$. By Bolzano-Weierstrass we may assume that both of the above sequences converge to $x_0 \in [a,b]$, which then using continuity implies that $|f(x_0) - f(x_0)| \geq \varepsilon$, a contradiction. □

Example 10.55. Again, it is important that the domain is $[a,b]$ for $a, b \in \mathbb{R}$. We have seen that $x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous on $\mathbb{R}$.

Theorem 10.56. Intermediate value theorem *If $f : [a, b] \to \mathbb{R}$ is continuous then it takes each value between $M := \max_{x\in[a,b]} f(x)$ and $m := \min_{x\in[a,b]} f(x)$ at least once. That is, for each $c \in [m, M]$ there is a $d \in [a, b]$ such that $f(d) = c$.*

Idea. Again we give only the idea and we refer to the precise proof to page 81-82 of the book.

We know by the above theorem that there are $a', b' \in [a, b]$ such that $m = f(a')$ and $M = f(b')$. Hence, by replacing a with a' and b with b' (and some algebraic manipulation in the case when $b' < a'$), we may assume that $f(a) = m$, $f(b) = M$ and $m < c < M$. Then, the idea is to consider

$$S := \{x \in [a, b]|f(x) < c\}$$

Set $d := \operatorname{Supp} S$. By the definition of Supp, there is a sequence $(x_n) \subseteq S$ converging to d from the left and let y_n be any sequence converging to d from the right. Applying continuity to the first sequence shows that $f(d) \leq c$, and by applying it to the second one shows that $f(d) \geq c$. So, $f(d) = c$.

□

Example 10.57. With other words, the above theorem says that $f([a, b]) = [m, M]$. So, the image of $[1, 2]$ for example via a continuous function is $[1, 2] \cup [3, 4]$.

Corollary 10.58. Banach fixed point theorem for closed intervals

If $f : [a, b] \to [a, b]$ is a continuous function (where $a, b \in \mathbb{R}$), then it has a fixed point (that is $x \in [a, b]$ such that $f(x) = x$).

Proof. Set $g(x) := f(x) - x$. Then $g(a) = f(a) - a \geq 0$ and $g(b) = g(b) - b \leq 0$. So, by the intermediate value theorem, there is a real number $c \in [a, b]$ such that $0 = g(c)$, which means that $f(c) = c$. □

Example 10.59. $\cos(x) : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ can be regarded as $\cos(x) : \left[0, \frac{\pi}{2}\right] \to \left[0, \frac{\pi}{2}\right]$. Then the above theorem says that there is a fixed point x for which $\cos(x) = x$.

Definition 10.60. $f : E \to \mathbb{R}$ is strictly increasing (resp. decreasing) if $f(x) < f(y)$ (resp. $f(y) > f(x)$) for every $x < y$ in E.

$f : E \to \mathbb{R}$ is strictly monotone, if it is strictly increasing or decreasing.

Corollary 10.61. *If $f :]a, b[\to \mathbb{R}$ is strictly monotone and continuous then the image of every open interval is an open interval.*

Theorem 10.62. *Let $f : E \to F$ be a continuous function. Then, f is strictly monotone if and only if it is injective.*

Proof. If it is strictly monotone, it is injective by definition.

So, suppose that f is injective. By replacing f with $-f$ we may assume that $f(a) < f(b)$. Then, we need to show that f is strictly increasing. Indeed, if it is not, then there are $a < x < y < b$ such that $f(x) > f(y)$. We have then either $f(a) < f(x)$ or $f(b) > f(y)$, as otherwise the following inequalities yield a contradiction with $f(a) < f(b)$:

$$f(a) > f(x) > f(y) > f(b).$$

Let us assume the first case. Then, either $f(a) > f(y)$ or $f(a) < f(y)$. Again assume the first case. Then, the intermediate value theorem tell us that there is a $x < z < y$ such that $f(z) = f(a)$, which contradicts injectivity. The other cases can be concluded similarly, and we leave them as homeworks (there are 4 cases as we made two choices of cases).

□

Theorem 10.63. *If $f : E \to F$ is continuous, strictly monotone and surjective, then f^{-1} (which exists by the previous theorem) is also continuous.*

Example 10.64. $\operatorname{Arcsin}(x)$, $\operatorname{Arccos}(x)$, $\operatorname{Arctg}(x)$, $\operatorname{Arccotg}(x)$.

11. Differentiation

Let $f : E \to \mathbb{R}$ be a real valued one variable function. We would like to approximate it with a linear one. That is, we would like to write

$$f(x) = f(x_0) + a(x - x_0) + r(x), \tag{11.0.a}$$

where a is a real number, and $r(x)$ is small in a neighborhood of x_0. The question is how small we would like it to be that we can also attain. Definitely, we would like it to be smaller than linear. The precise mathematical wording of this is that

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0. \tag{11.0.b}$$

Considering the graph of the function, $f(x_0) + a(x - x_0)$ would be a tangent line to the graph at $(x_0, f(x_0))$.

Now, the question is if the above a and $r(x)$ exist, which again visually means whether there is a tangent line to the graph of the function at $(x_0, f(x_0))$. Equation (11.0.a) implies that

$$\frac{r(x)}{x - x_0} + a = \frac{f(x) - f(x_0)}{x - x_0} \tag{11.0.c}$$

By applying limit to both sides of this equation, using (11.0.b), we obtain that

$$a = \underbrace{\lim_{x \to x_0} \frac{r(x)}{x - x_0}}_{\text{by (11.0.b)}} + a = \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{\text{by (11.0.c)}}.$$

So, the existence of the real number a, which we call the derivative of f at x_0, and of the behavior of the error term described in (11.0.b) is equivalent to the existence of the latter limit. So, instead of using equations (11.0.a) and (11.0.b), we may equivalently define the derivative also by:

Definition 11.1. $f : E \to \mathbb{R}$ is a function is differentiable at $x_0 \in E$, if the function (of x)

$$\frac{f(x) - f(x_0)}{x - x_0} : E \setminus \{x_0\} \to \mathbb{R}$$

admits a limit in x_0. In this case we call the limit the derivative of f at x_0 and we denote it by $f'(x_0)$. The function $x_0 \mapsto f(x_0)'$ is called the derivative function of f (where the domain is the set of all points of E where the above limit exists).

We say that f is differentiable if it is differentiable at all $x_0 \in E$.

Remark 11.2. By the above introduction, $f'(x_0)$ can be also defined to be the unique number that satisfies

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + r(x), \tag{11.2.d}$$

such that $\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0$.

Example 11.3. We show that $(x^2)' = 2x$.

Indeed, for this we need to find at each x_0 the limit

$$\lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} x + x_0 = 2x_0$$

Example 11.4. Similarly, if $a \in \mathbb{Z}_+$, then $(x^a)' = a x^{a-1}$.

Indeed,

$$\lim_{x \to x_0} \frac{x^a - x_0^a}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^{a-1} + x^{a-2} x_0 + \dots + x^1 x_0^{a-2} + x_0^{a-1})}{x - x_0}$$
$$= \lim_{x \to x_0} x^{a-1} + x^{a-2} x_0 + \dots + x^1 x_0^{a-2} + x_0^{a-1} = a x_0^{a-1}$$

Example 11.5. $\sin(x)' = \cos(x)$.
Indeed,

$$\lim_{x\to x_0} \frac{\sin(x)-\sin(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x-x_0} = \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right)\cdot \lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = \cos(x_0)$$

Example 11.6. $\cos(x)' = -\sin(x)$.
Indeed,

$$\lim_{x\to x_0} \frac{\cos(x)-\cos(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{-2\sin\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x-x_0} = \lim_{x\to x_0} -\sin\left(\frac{x+x_0}{2}\right)\cdot \lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = -\sin(x_0)$$

Proposition 11.7. *If f is differentiable at x_0, then it is continuous at x_0.*

Proof. This is a consequence of the following computation:

$$\lim_{x\to x_0} f(x) = \underbrace{\lim_{x\to x_0} f(x_0) + (x-x_0)f(x_0)' + r(x)}_{\text{by (11.2.d)}} = f(x_0) + \lim_{x\to x_0} r(x).$$

$$= f(x_0) + \lim_{x\to x_0} \frac{r(x)}{x-x_0} \cdot \lim_{x\to x_0} (x-x_0) = f(x_0).$$

□

Example 11.8. The above implication is not true backwards, that is, if f is continuous at x_0, it does not have to be differentiable. For example, consider $f(x) := |x|$. This is continuous at $x_0 = 0$ (as we have already showed it), but it is not differentiable at 0, because that would mean that $\lim_{x\to 0} \frac{|x|}{x}$ exists at least, however:

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \neq 1 = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} \frac{|x|}{x}.$$

11.1. *Algebraic properties of derivation*

11.1.1. *Addition.*

Proposition 11.9. *If $f, g : E \to \mathbb{R}$ are differentiable at x_0, then so is $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{R}$, and furthermore*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0)$$

Proof.

$$(\alpha f + \beta g)'(x_0) = \lim_{x\to x_0} \frac{(\alpha f + \beta g)(x) - (\alpha f + \beta g)(x_0)}{x-x_0}$$

$$= \alpha \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \beta \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = \alpha f'(x_0) + \beta g'(x_0)$$

□

Example 11.10. $(5x^3 + 6x^2)' = 15x^2 + 12x$

11.1.2. *Multiplication.*

Proposition 11.11. *If $f, g : E \to \mathbb{R}$ are differentiable at x_0, then so is $f \cdot g$, and furthermore*

$$(f\cdot g)'(x_0) = (fg' + f'g)$$

Proof.

$$\begin{aligned}(f\cdot g)'(x_0) &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0}\frac{f(x)(g(x)-g(x_0))+(f(x)-f(x_0))g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0} f(x)\frac{g(x)-g(x_0)}{x-x_0}+\lim_{x\to x_0} g(x_0)\frac{f(x)-f(x_0)}{x-x_0} = f(x_0)g'(x_0)+g(x_0)f'(x_0)\end{aligned}$$

□

Example 11.12.

$$(x^2\cos(x))' = 2x\cos(x)+x^2(-\sin(x)) = 2x\cos(x)-x^2\sin(x)$$

11.1.3. *Division.*

Proposition 11.13. *If $f, g : E \to \mathbb{R}$ are differentiable at x_0, and $g(x_0)\neq 0$, then so is $\frac{f}{g}$ is also differentiable at x_0, and furthermore*

$$\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf'-fg'}{g^2}\right)(x_0)$$

In particular,

$$\left(\frac{1}{g}\right)'(x_0) = \left(\frac{-g'}{g^2}\right)(x_0)$$

Proof.

$$\begin{aligned}\lim_{x\to x_0}\frac{\frac{f}{g}(x)-\frac{f}{g}(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x)}{g(x)g(x_0)(x-x_0)}\\ &= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x_0)+f(x_0)g(x_0)-f(x_0)g(x)}{g(x)g(x_0)(x-x_0)}\\ &= \lim_{x\to x_0}\frac{g(x_0)}{g(x_0)g(x)}\frac{f(x)-f(x_0)}{x-x_0} - \lim_{x\to x_0}\frac{f(x_0)}{g(x)g(x_0)}\frac{g(x)-g(x_0)}{x-x_0} = \left(\frac{gf'-fg'}{g^2}\right)(x_0)\end{aligned}$$

□

Example 11.14. If $b > 0$ is an integer, then:

$$\left(\frac{1}{x^b}\right)' = -\frac{bx^b-1}{x^{2b}} = -b\frac{1}{x^{b+1}}.$$

That is, by setting $a=-b$ we obtain $(x^a)' = ax^{a-1}$. In particular, then the formula $(x^a)' = ax^{a-1}$ holds for all integer a (not only the non-negative ones).

Example 11.15.

$$\tan(x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos(x)^2} = \frac{1}{\cos^2(x)}$$

11.1.4. *Inversion of functions.*

Proposition 11.16. *If $f : I =]a,b[\to F$ is a bijective continuous function (so f is strictly monotone, and f^{-1} exists and is continuous), and $x_0\in I$ such that $f'(x_0)\neq 0$, then f^{-1} is differentiable at $y_0 := f(x_0)$, and we have*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

The idea behind the proof of the proposition is simple: if we set $y = f(x)$ and $y_0 = f(x_0$, we have $\frac{f(x)-f(x_0)}{x-x_0} = \frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)} = \frac{1}{\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}}$. Check page 109 for the precise proof.

Example 11.17. If $f(x) = x^b$ for some integer $b \geq 1$, then $f^{-1}(y) = \sqrt[b]{y} = y^{\frac{1}{b}}$. So, $f'(x) = bx^{b-1}$, and

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{b\left(y^{\frac{1}{b}}\right)^{b-1}} = \frac{1}{b}y^{-\frac{b-1}{b}} = \frac{1}{b}y^{\frac{1}{b}-1}.$$

So, for $c = \frac{1}{b}$, the formula for $(y^c)'$ is the same as for c being an integer. By applying a fraction rule we see that we do not even have to assume that b is positive for this.

Example 11.18. If $f(x) = \sin(x)$, then $f^{-1}(y) = \mathrm{Arcsin}(y)$. Then, $f'(x) = \cos(x)$, and so

$$\mathrm{Arcsin}'(y) = \frac{1}{\cos(\mathrm{Arcsin}(y))} = \frac{1}{\sqrt{1-\sin^2(\mathrm{Arcsin}(y))}} = \frac{1}{\sqrt{1-y^2}}$$

Proposition 11.19. *If $f : E \to F$ is differentiable at $x_0 \in E$, $g : G \to H$ is differentiable at $f(x_0)$, and $f(E) \subseteq G$, then $g \circ f : E \to H$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = ((g' \circ f) \cdot f')(x_0)$$

Idea of the proof.

$$\lim_{x\to x_0} \frac{g(f(x)) - g(f(x))}{x - x_0} = \lim_{x\to x_0} \frac{g(f(x)) - g(f(x))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0)) \cdot f'(x_0)$$

□

Example 11.20.

$$\cos(x^2)' = -\sin(x^2)2x$$

Example 11.21. Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, then

$$\left(x^{\frac{a}{b}}\right)' = \left((x^a)^{\frac{1}{b}}\right)' = \frac{1}{b}(x^a)^{\frac{1}{b}-1} ax^{a-1} = \frac{a}{b}x^{\frac{a}{b}-a+a-1} = \frac{a}{b}x^{\frac{a}{b}-1}$$

So, the formula is the same for x^r, where r is any rational number, as it was for r an integer.

11.2. *One sided derivatives*

Definition 11.22. If $f : E \to \mathbb{R}$ is a function and $x_0 \in E$, then we say that the left (resp. right) derivative of f exists at x_0 if the function

$$\frac{f(x) - f(x_0)}{x - x_0}$$

admits a left (resp. right limit). The value of this limit is then the left (resp. right) derivative.

Example 11.23. For $|x|$ at 0 the left derivative is -1 and the right derivative is 1.

Proposition 11.24. *Let $f : E \to \mathbb{R}$ be a function and $x_0 \in E$ a real number. Then f is differentiable at a point x_0 if and only if both its left and right derivatives exist and they agree. Furthermore, then the value of the derivative is the same as the common value of the left and the right derivatives.*

11.3. *Higher derivatives*

We may iterate derivation, obtaining second, third, etc. derivatives.

Example 11.25.

$$\arcsin\left(\frac{1}{x}\right)'' = \left(-\frac{1}{\sqrt{1-x^{-2}}}x^{-2}\right)' = -\left((x^4-x^2)^{-\frac{1}{2}}\right)' = \frac{1}{2}(x^4-x^2)^{-\frac{3}{2}}(4x^3-2x)$$

Example 11.26. Set $f(x) := |x^3|$. Then f' exists, and it is:

$$f'(x) = \begin{Bmatrix} 3x^2 & \text{for } x \geq 0 \\ -3x^2 & \text{for } x < 0 \end{Bmatrix}$$

Then, also f'' exists and it is $6|x|$. However, then f''' does not exist.

Definition 11.27. $f : E \to F$ is called a C^n function if its n-th derivatives exists at all $x_0 \in E$ and they are all continuous.

Example 11.28. (1) x, x^2, etc. are C^n for all n

(2) $|x| : \mathbb{R} \to \mathbb{R}$ is not C^1

(3) $|x^3|$ is C^2 but not C^3.

11.4. *Local extrema*

Proposition 11.29. *If $f : E \to \mathbb{R}$ is differentiable at x_0 and it admits a local extremum (that is, a local minimum or a local maximum) at x_0, then $f'(x_0) = 0$.*

Example 11.30. For $f(x) = x^3$, $f'(x) = 3x^3$, so $f'(0) = 0$. However, $f(0) = 0$ is not a local extremum,

Definition 11.31. If $f : E \to \mathbb{R}$ is a function such that $f'(x_0) = 0$ for some $x_0 \in E$, then we call x_0 a

So, if $f'(x_0) = 0$, we cannot be sure that there is a local extrema there. However, if the domain of f is a closed, bounded interval $[a, b]$, then we know that there is also a global maximum and minimum, and using the above proposition we know that these points can be only either in a point x_0, where $f'(x_0) = 0$ or in a or b. This gives an algorithmic way of finding global extrema, an example of which is shown below:

Example 11.32. $f(x) = \frac{4}{3}x^3 + \frac{3}{2}x^2 - x + 2$ $f'(x) = 4x^2 + 3x^2 - 1$

$x = \frac{-3 \pm \sqrt{25}}{8} = \frac{-3 \pm 5}{8} = -1$, or $\frac{1}{4}$

x	$f(x)$
-2	$-\frac{4}{3}8 + \frac{3}{2}4 + 2 + 2 = \frac{36-64}{6} + 4 = 4 - \frac{28}{6} = -\frac{4}{6}$
-1	$-\frac{4}{3} + \frac{3}{2} + 1 + 2 = \frac{9-8}{6} + 3 = 3 + \frac{1}{6}$
$\frac{1}{4}$	$\frac{4}{3 \cdot 64} + \frac{3}{2 \cdot 16} - \frac{1}{4} + 2 = \frac{4+18-48}{192} + 2 = \frac{-26}{192} + 2 = 2 - \frac{13}{96}$
$\frac{1}{2}$	$\frac{4}{3 \cdot 8} + \frac{3}{2 \cdot 4} - \frac{1}{2} + 2 = \frac{4+9-12}{24} + 2 = 2 + \frac{1}{24}$

So, $f(x)$ on $\left[-2, \frac{1}{2}\right]$ takes its minimumg at $x = -2$ and amximum at $x = -1$.

11.5. *Rolle's theorem and consequences*

Theorem 11.33. ROLLE'S THEOREM *If $f : [a, b] \to \mathbb{R}$ is a continuous function (where a and b are real numbers), such that $f|_{]a,b[}$ is differentiable, and $f(a) = f(b)$, then there is a $c \in]a, b[$ for which $f'(c) = 0$.*

Proof. If f is constant, then $f'(x) = 0$.

If f is not constant, then, it has either a maximum or a minimum which is not equal to $f(a) = f(b)$ at a point $a < c < b$. Hence f is differentiable at c and then as it is a (local) extremum, $f'(c) = 0$. □

Example 11.34. Differentiable is needed here. For example if one takes $f(x) = |x|$ on $[-1, 1]$, then $f(-1) = f(1)$, but there is no point, where the derivative is 0.

Theorem 11.35. MEAN VALUE THEOREM (FOR THE DERIVATIVE) *If $f : [a, b] \to \mathbb{R}$ is a continuous function (where a and b are real numbers), such that $f|_{]a,b[}$ is differentiable, then there is a $c \in]a, b[$ for which $f'(c)(b - a) = f(b) - f(a)$.*

Proof. Apply the previous theorem to $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$ □

Corollary 11.36. *If $f, g : [a, b] \to \mathbb{R}$ are continuous functions (where a and b are real numbers) that are differentiable over $]a, b[$ such that $f'(x) = g'(x)$ for each $x \in]a, b[$, then there is a real number c such that $f(x) = g(x) + c$.*

Proof. By regarding $f - g$, it is enough to show the other statement that $f'(x) = 0$ (for all $x \in]a, b[$) implies that f is a constant function. Assume it is not. Then, there it has two different function values, say at c and $d \in [a, b]$. By replacing a and b with c and d we may assume that this is at the endpoints. However, then the mean value theorem for the derivative tells us that then there has to be a $c \in]a, b[$, such that $f'(c) = \frac{f(b)-f(a)}{b-a} \neq 0$. This is a contradiction. $\square$

11.5.1. *Monotone functions and differentials.*

Corollary 11.37. *If $f : [a, b] \to \mathbb{R}$ is a continuous function (where a and b are real numbers), such that $f|_{]a,b[}$ is differentiable, then*

(1) f is increasing (resp. decreasing) if and only if $f'(x) \geq 0$ (resp. ≤ 0)
(2) if $f'(x) > 0$ (resp. < 0), then f is strictly increasing (resp. strictly decreasing)

Proof. We only prove the increasing case of (1), as the others are similar.

- First we assume that f is increasing. Then
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$
- Second, let us assume that $f'(x) \geq 0$ everywhere, and we also assume that f is not increasing. Hence, there are $a \leq c < d \leq d$, such that $f(c) > f(d)$. However, then the intermediate value theorem for derivatives tell us that then there has to be a $c < e < d$ such that $f'(e) = \frac{f(d)-f(c)}{d-c} < 0$.

$\square$

Example 11.38. Strictly increasing does not imply that $f'(x) > 0$! For example, $f(x) = x^3$ is strictly increasing, but $f'(0) = 3 \cdot 0^2 = 0$.

Example 11.39. For which real numbers a, is $f(x) := \sin(x) + ax$ monotone. As $f(x)$ is differentiable on $\mathbb{R}$ this happens if and only if $f'(x)$ is everywhere non-negative or everywhere non-positive. So, let us compute $f'(x)$:
$$f'(x) = -\cos(x) + a$$
So:

- f is increasing if and only if $a \geq 1$
- f is decreasing if and only if $a \leq 1$

11.5.2. *L'Hospital's rule.*

Theorem 11.40. *Assume we are in the following situation:*

(1) $f, g : I :=]a, b[\to \mathbb{R}$ are differentiable functions,
(2) $x_0 \in I$ ($a, b = \pm\infty$ is allowed),
(3) $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in I \setminus \{x_0\}$.
(4) $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \alpha$ for $\alpha = 0$ or $\pm\infty$
(5) $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} = \mu \in \overline{\mathbb{R}}$.

Then, $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \mu$.

Furthermore, we can take $a = x_0$ if we replace all limits by right limits, and we can take $b = x_0$ if we replace all limits by left limits.

Proof. We prove only the $\alpha = 0$ and $x_0 \in]a, b[$ case, and we refer to page 121-122 of the book for the rest.

As f and g are differentiable at x_0 they are also continuous there, and hence $f(x_0) = \lim\limits_{x \to x_0} f(x) = \alpha = 0$ and $g(x_0) = \lim\limits_{x \to x_0} g(x) = \alpha = 0$. So, by the mean value theorem for derivatives, there is a real number $c(x)$ between x and x_0 such that $f'(c(x)) = \frac{f(x)-f(x_0)}{x-x_0}$. In particular, $c(x) : I \setminus x_0 \to I \setminus x_0$ is a function such that $\lim\limits_{x \to x_0} c(x) = x_0$.

$$\mu = \lim_{x \to x_0} \frac{f'(c(x))}{g'(c(x))} = \lim_{x \to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$$

□

Example 11.41.

$$\lim_{x \to 0} \frac{\text{Arcsin}(x)}{\sin(x)} = \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos(x)} = 1$$

Example 11.42.

$$\lim_{x \to +\infty} \frac{e^x}{x^2} = \lim_{x \to +\infty} \frac{e^x}{2x} = \lim_{x \to +\infty} \frac{e^x}{2} = +\infty$$

So, e^x goes to $+\infty$ as x goes to $+\infty$ quicker than x^2 (in fact a similar argument shows that it goes faster than any polynomial).

Example 11.43.

$$\lim_{x \to 0^+} \frac{\text{Log}(x)}{\frac{-1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x \to 0^+} x = 0$$

and

$$\lim_{x \to +\infty} \frac{\text{Log}(x)}{x} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{1} = 0$$

So, Log goes to $-\infty$ as x goes to 0 and to $+\infty$ as x goes to $+\infty$ slower than $\frac{1}{x}$ and x, respectively.

11.5.3. *Taylor expansion.*

Definition 11.44. Let $f : E \to \mathbb{R}$ be a function such that there is a neighborhood of $a \in E$ which is contained in the domain (so in E). The n-th order expansion of f around a is an equality

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n \epsilon(x)$$

where a_i are real number, and $\varepsilon(x)$ is a function $E \to \mathbb{R}$ such that $\lim\limits_{x \to a} \epsilon(x) = 0$.

Proposition 11.45. *In the above situation, if the n-th order expansion around a exists, then a_i are uniquely determined.*

Proof. Let a_i and $\epsilon(x)$ be giving one and let a_i' and $\epsilon'(x)$ be giving another expansion. We show by induction on i that $a_i = a_i'$. For $i = 0$ this is given by passing to the limit as x goes to a of the two expansion:

$$\begin{aligned} a_0 = \lim_{x \to a} a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n \epsilon(x) = \lim_{x \to a} f(x) \\ = \lim_{x \to a} a_0' + a_1'(x-a) + a_2'(x-a)^2 + \cdots + a_n'(x-a)^n + (x-a)^n \epsilon'(x) = a_0' \end{aligned}$$

Then, we have to prove the induction step. So, let us assume that we know that $a_j = a_j'$ for $j = 0, \ldots, i-1$. Then we have:

$$\begin{aligned} a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n \epsilon(x) = f(x) \\ = a_0' + a_1'(x-a) + a_2'(x-a)^2 + \cdots + a_n'(x-a)^n + (x-a)^n \epsilon'(x) \\ = a_0 + a_1(x-a) + \cdots + a_{i-1}(x-a)^{i-1} + a_i'(x-a)^i + \cdots + a_n'(x-a)^n + (x-a)^n \epsilon'(x) \end{aligned}$$

So, take the two endpoints of this stream of equalities, and subtract from both $a_0 + a_1(x-a) + \cdots + a_{i-1}(x-a)^{i-1}$ and then divide both by $(x-a)^i$. This yields that

$$a_i + a_{i+1}(x-a) + \cdots + a_n(x-a)^n + (x-a)^n \epsilon(x) = a_i' + a_{i+1}'(x-a) + \cdots + a_n'(x-a)^n + (x-a)^n \epsilon'(x).$$

Taking limit of this equation as x goes to a yields that $a_i = a_i'$, which concludes the induction step.

So, we know that $a_i = a_i'$ for each i. However, then it follows also that $\epsilon(x) = \epsilon'(x)$ for each $x \in E$. $\square$

Theorem 11.46. *Let $n \geq 0$ be a real number. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is $n+1$ times differentiable on I, and let $a \in I$ be an arbitrary real number. Then, for each $x \in I$ there is a real number $0 < \theta_{x,a} < 1$ such that*

$$f(x) = \left(\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \right) + f^{(n+1)}(a + \theta_{x,a}(x-a)) \frac{(x-a)^{n+1}}{(n+1)!}.$$

Proof. To understand the proof, note that the statement for $n = 0$ is just the mean value theorem for the derivative. Indeed, that says that there is a y between a and x such that $f'(y) = \frac{f(x)-f(a)}{x-a}$. If we reorder this equation and we write y as $a + \theta_{x,a}(x-a)$, then we obtain

$$f(x) = f(a) + f'(a + \theta_{x,a}(x-a))(x-a).$$

You might recall that this proof was an application of Rolle's theorem to $g(y) := f(y) - f(a) - \frac{f(x)-f(a)}{x-a}(y-a)$, and the main feature was that $g(a) = g(x)$. And furthermore, $g'(y)$ was equal to $f'(y) - \frac{f(x)-f(a)}{x-a}$. So, $g'(y)$ being 0 yielded the statement of the intermediate value theorem for the derivative.

Define

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

And consider

$$g(y) = f(y) - P_n(y) + \frac{P_n(x) - f(x)}{(x-a)^{n+1}}(y-a)^{n+1}$$

Then, we have $0 = g(x) = g(a) = g'(a) = \cdots = g^{(n)}(a)$. This means that there is a y_1 between a and x such that $g'(y_1) = 0$ by Rolle's theorem. But then applying Rolle's theorem again we obtain a y_2 between a and y_1 such that $g^{(2)}(y_2) = 0$. Iterating this process we obtain y_{n+1} between a and x such that $g^{(n+1)}(y_{n+1}) = 0$. In particular, we may write $y_{n+1} = a + \theta_{x,a}(x-a)$, and then we obtain that

$$0 = g^{(n+1)}(a + \theta_{x,a}(x-a)) = f(a + \theta_{x,a}(x-a)) + \frac{P_n(x) - f(x)}{(x-a)^{n+1}}(n+1)!$$

Reorganizing the latter equation yields exactly the statement of the theorem. $\square$

Corollary 11.47. *Let $n \geq 0$ be a real number. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is n times continuously differentiable on I, and let $a \in I$ be an arbitrary real number. Then, the n-th order expansion of f around a exists and is*

$$\sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(x-a)^j.$$

The idea behind the proof of the corollary is that by the previous theorem the error term is $f^{(n+1)}(a + \theta_{x,a}(x-a)) - f^{(n+1)}(x)$, which converges to zero as x goes to a as $f^{(n)}$ is continuous. For the precise proof we refer to page 126 of the book.

Example 11.48.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \epsilon(x)$$

as we have

$$\left(\frac{1}{1-x} \right)^{(n)} = n!(1-x)^{n+1} \Rightarrow f^{(n)}(0) = n!.$$

Also,

$$e^x = \sum_{j=0}^{n} \frac{x^n}{n!} + x^n \epsilon(x),$$

as $(e^x)^{(n)} = e^x$, and hence $f^{(n)}(0) = 1$. Similarly, the $2n+1$-th order expansion of $\cos(x)$ is

$$\cos(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j}}{2j!} + x^{2n+1}\epsilon(x)$$

and the $2n+2$-th order expansion of $\sin(x)$ is

$$\sin(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j+1}}{(2j+1)!} + x^{2n+2}\epsilon(x)$$

Note that

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \dots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \dots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = \cos(x) + i\sin(x) \end{aligned}$$

which gives a way to figure out the expansions for $\cos(x)$ and $\sin(x)$.

Look up $\frac{1}{\sqrt{1-x}}$ from the book (page 127).

Example 11.49. One can also figure out expansions of products, sums, compositions, etc. For example the 3-rd order expansion of $\sin(\cos(x))$ is as follows:

$$\begin{aligned} \cos(\sin(x)) &= \sin\left(x - \frac{x^3}{6} + x^3\epsilon_1(x)\right) \\ &= 1 + \frac{\left(x - \frac{x^3}{6} + x^3\epsilon_1(x)\right)^2}{2} + \left(x - \frac{x^3}{6} + x^3\epsilon_{\sin}(x)\right)^3 \varepsilon_2(\sin(x)) = 1 + \frac{x^2}{2} + x^3\varepsilon(x), \end{aligned}$$

where $\varepsilon(x)$ is the sum of many terms of the form x^3 times something going to 0 as x goes to 0. In particular, $\lim_{x\to 0}\varepsilon(x) = 0$ and hence the above is indeed the 3-rd order expansion. One warning should be given though here: the base-point of the expansion of the outside function should be the value of the inside function at the base-point of the inside expansion. So, for example, $\sin(\cos(x))$ at 0 is not easy to compute this way, because one would need the expansion of sin around $\cos(0) = 1$, for which there is no nice formula.

Similarly, one can write the order 3 expansion of $\frac{1}{1-x} \cdot e^x$ around 0:

$$\frac{1}{1-x} \cdot e^x = (1+x+x^2+x^3+x^3\varepsilon_1(x))\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \varepsilon_2(x)\right) = 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^2+x^3\varepsilon(x)$$

See pages 127-131 for more examples.

Example 11.50. One can use expansions also to replace L'Hospital arguments. For example, if one would like to determine $\lim_{x\to 0} \frac{(e^x-1-x)+x\sin(x)}{\cos(x)-1}$, then can compute the 2-nd order expansions first:

$$\begin{aligned} (e^x - 1 - x) + x\sin(x) &= \left(1 + x + \frac{x^2}{2} + x^2\epsilon_1(x) - 1 - x\right) + x(x + x^2\epsilon_2(x)) \\ &= \frac{x^2}{2} + x^2 + x^2\underbrace{(\epsilon_1(x) + \epsilon_2(x))}_{=:\epsilon_3(x)} = \frac{3}{2}x^2 + x^2\epsilon_3(x) \end{aligned}$$

$$\cos(x) - 1 = 1 - \frac{x^2}{2} + x^2\epsilon_4(x) - 1 = -\frac{x^2}{2} + x^2\epsilon_4(x)$$

Then we have:

$$\lim_{x\to 0} \frac{(e^x - 1 - x) + x\sin(x)}{\cos(x) - 1} = \lim_{x\to 0} \frac{\frac{3}{2}x^2 + x^2\epsilon_3(x)}{-\frac{x^2}{2} + x^2\epsilon_4(x)} = \lim_{x\to 0} \frac{\frac{3}{2} + \epsilon_3(x)}{-\frac{1}{2} + \epsilon_4(x)} = -3.$$

11.5.4. *Application of Taylor expansion to local extrema and inflection points.* We have seen that if f has a local extremum at a, then $f'(a) = 0$. However, we have also seen that there is no backwards implication. Nevertheless is there some sufficient conditions for local extrema?

Let us assume that we have a function f whose first many derivatives are 0 at a and the n-th one is the first that is not-zero, for some even number n. Let us assume that $f^{(n)}(a) > 0$. Then, we have an n-th order expansion: $f(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n \epsilon(x)$. So, there will be a small neighborhood of a where $|\varepsilon(x)| < \frac{1}{2} \cdot \frac{f^{(n)}(a)}{n!}$ holds. In particular, on this neighborhood we have:

$$\frac{1}{2} \cdot \frac{f^{(n)}(a)}{n!}(x)(x-a)^n \leq f(x) \leq \frac{3}{2} \cdot \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This shows that a is a local minimum. There is a similar argument for local maximum too yielding the following:

Theorem 11.51. *Let $n \geq 2$ be an even integer. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is n times differentiable on I, and let $a \in I$ be an arbitrary real number. If $f'(a) = \cdots = f^{(n-1)}(a) = 0$ and*

(1) *if $f^{(n)} > 0$, then f has a local minimum, and*
(2) *if $f^{(n)} < 0$, then f has a local maximum.*

Example 11.52. $f(x) = \sin(x) + \frac{1}{2}x|_{[0,2\pi]}$

We have $f'(x) = 0$ if and only if $\cos(x) = -\frac{1}{2}$ if and only if $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Whether or not we have a maximum or minimum at these points is decided by the sign of $f''(x) = -\sin(x)$.

- At $x = \frac{2\pi}{3}$, $f(x) < 0$, so $f(x)$ has a local maximum, and
- At $x = \frac{4\pi}{3}$, $f(x) > 0$, so $f(x)$ has a local minimum.

Question 11.53. What if we have n odd in the above statement instead of even?

Then, our function locally looks like $(x-a)^3$, $(x-a)^5$, or $(x-a)^7$, etc. This looks opposite to what a local maximum is. That is on one side we have exactly bigger than 0 and on the other side smaller than that.

Definition 11.54. If $f : E \to \mathbb{R}$ is a function such that f is differentiable at $a \in E$, then we say that f has an inflection point at a if there is an $\delta > 0$ such that either

(1) $\{x \in E | a < x < a + \delta\} \Rightarrow f(x) - f(a) - f'(a)(x-a) > 0$, and
$\{x \in E | a - \delta < x < a\} \Rightarrow f(x) - f(a) - f'(a)(x-a) < 0$, or
(2) $\{x \in E | a < x < a + \delta\} \Rightarrow f(x) - f(a) - f'(a)(x-a) < 0$, and
$\{x \in E | a - \delta < x < a\} \Rightarrow f(x) - f(a) - f'(a)(x-a) > 0$.

Theorem 11.55. *Let $n \geq 3$ be an odd integer. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is n times differentiable on I, and let $a \in I$ be an arbitrary real number. If $f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then f has an inflection point at a.*

The reason why this is true is the same as for the above theorem on local maximum/minimum.

11.5.5. *Asymptotes.* There are three kinds of asymptotes:

(1) If for some $c \in \mathbb{R}$, $\lim_{x \to c^-} f(x) = \pm\infty$ or $\lim_{x \to c^+} f(x) = \pm\infty$, then f has a vertical asymptote.
(2) If for some $c \in \mathbb{R}$, $\lim_{x \to +\infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$, then f has a horizontal asymptote.
(3) If for some $a \neq 0, b \in \mathbb{R}$, $\lim_{x \to +\infty} f(x) - ax = b$ or $\lim_{x \to -\infty} f(x) - ax = b$, then we say that f has a slant asymptote.

Example 11.56. (1) vertical asymptote: $f(x) = \frac{1}{1-x}$ at $x = 1$.
(2) horizontal asymptote: $f(x) = 2 - e^{-x}$ at $y = 2$,
(3) $f(x) = 2 + 3x + \frac{1}{x^2}$ for $a = 3$ and $b = 2$.

11.5.6. *Convex and concave functions.*

Definition 11.57. A function $f : I \to \mathbb{R}$ on an open interval is called convex (resp. concave) if for every $a, b \in I$ and every $\lambda \in [0,1]$ we have:

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b).$$

(resp. $f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda)f(b)$).

Geometrically, the above definition means the following: we may assume that $a < b$ and denote $x := \lambda a + (1-\lambda)b$, that is x is a point between a and b. Then, there are two possibilities for the slopes for any function f any fixed x as above:

(1) either $f(x)$ is below the line segment connecting $(a, f(a))$ and $(b, f(b))$ (this is characterized by the fact that the slope of f between a and b is at least as big as the slope of f between a and x and at most as big as the slope of f between x and b), or
(2) either $f(x)$ is above the line segment connecting $(a, f(a))$ and $(b, f(b))$ (this is characterized by the fact that the slope of f between a and b is at most as big as the slope of f between a and x and at least as big as the slope of f between x and b), or

Now, f is convex, if and only if, the first case holds for all choices of $x = \lambda a + (1-\lambda)b$ (where $\lambda \in]0,1[$).

Theorem 11.58. *Let $f : I \to \mathbb{R}$ be a differentiable function on an open interval. Then f is convex if and only if $f' : I \to \mathbb{R}$ is an increasing function.*

Proof. We prove only the statements about convexity, as f is convex if and only if $-f$ is concave.

(1) First, let us assume that f is convex. Let $a < b$ be points of I. We want to prove that $f'(a) \leq f'(b)$. By the above characterization of convexity we have

$$\frac{f(b) - f(\lambda a + (1-\lambda)b)}{b - (\lambda a + (1-\lambda)b)} \geq \frac{f(b) - f(a)}{b-a}, \text{ and } \frac{f(\lambda a + (1-\lambda)b) - f(a)}{(\lambda a + (1-\lambda)b) - a} \leq \frac{f(b) - f(a)}{b-a}.$$

Now, as λ goes to 0, the left side of the first inequality converges to $f'(b)$, and as λ goes to 1 the left side of the second inequality converges to $f'(a)$. This yields:

$$f'(b) \geq \frac{f(b) - f(a)}{b-a} \geq f'(a)$$

(2) For, the other direction let us assume that f' is increasing. Fix $a < b \in I$. Set $x := \lambda a + (1-\lambda)b$ for any $\lambda \in]0,1[$ (for $\lambda = 0$ and 1 the convexity inequality is automatic). Then, the mean value theorem tells us that there is $x < y < b$ such that $\frac{f(x)-f(a)}{x-a} = f'(x_1)$ and $\frac{f(b)-f(x)}{b-x} = f'(x_2)$. In particular, by our assumption that the derivative is increasing it follows that $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x}$. But this shows that f is convex by the above characterization of convexity in terms of slopes.

□

If f' is differentiable (or equivalently, f'' is differentiable twice), then f' being increasing is equivalent to f'' being at least 0.

Corollary 11.59. *Let $f : I \to \mathbb{R}$ be a two times differentiable function on an open interval. Then f is convex if and only if $f''(x) \geq 0$ for all $x \in I$.*

Example 11.60. $(e^x)'' = e^x$ so e^x is convex

Example 11.61. $\mathrm{Log}(x)'' = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$, so Log is concave.

11.6. *Lipschitz continuity*

Definition 11.62. Let $k > 0$ be a real number. A function $f : E \to \mathbb{R}$ is k-Lipschitz if for every $x, y \in E$, $|f(x) - f(y)| \leq k|x - y|$. If $k < 1$, then we call k-Lipschitz functions k-contractions.

Proposition 11.63. *$f : I \to \mathbb{R}$ a function on an interval is k-Lipschitz (for any $k > 0$), then it is uniformly continuous on I.*

Theorem 11.64. Banach fixed point theorem *If $f : \mathbb{R} \to \mathbb{R}$ is a k-contraction, then it has a fixed point.*

Idea of the proof. ☐

11.7. *Hyperbolic trigonometric functions*

12. Integration

12.1. *Definition*

Definition 12.1. A partition $\sigma = (x_i)$ of a bounded interval $[a, b]$ is an ordered collection $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of points of $[a, b]$. The norm of σ is $\max_{1 \leq i \leq n} x_i - x_{i-1}$. A refinement $\sigma' = (x'_i)$ of σ is a partition such that each value of x_i shows up amongst x'_i.

The regular partition of length n is $x_i := a + i\frac{b-a}{n}$.

Proposition 12.2. *Each two partitions have a common refinement, and each partition can be refined to another one with arbitrarily small norm.*

Definition 12.3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\sigma = (x_i)$ a partition of $[a, b]$. Then, the upper Darboux sum of f with respect to σ is

$$\overline{S}_\sigma = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $M_i := \operatorname{Sup}_{x \in [x_{i-1}, x_i]} f(x)$. The lower Darboux sum of f with respect to σ is

$$\underline{S}_\sigma = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

where $m_i := \operatorname{Sup}_{x \in [x_{i-1}, x_i]} f(x)$.

Example 12.4. Let us consider a constant unction $f(x) = c$. Then for any partition σ,

$$\underline{S}_\sigma = \overline{S}_\sigma = \sum_{i=1}^{n} c(x_i - x_{i-1}) = \underbrace{cx_n - cx_0}_{\text{telescopic sum}} = c(b - a).$$

Example 12.5. Consider $f := x|_{[a,b]}$, and $\sigma_n = \left(a + \frac{i}{b-a}\right)$ the regular partition. Then

$$\overline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} = a(b-a) + \frac{n(n+1)}{2}\frac{(b-a)^2}{n^2}$$

and

$$\underline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(a + (i-1)\frac{b-a}{n}\right)\frac{b-a}{n} = a(b-a) + \frac{(n-1)n}{2}\frac{(b-a)^2}{n^2}$$

Note that $\lim_{n\to\infty} \overline{S}_{\sigma_n} = \lim_{n\to\infty} \underline{S}_{\sigma_n} = a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2}{2} - \frac{a^2}{2}$.

Proposition 12.6. *Let M and m be upper and lower bounds for $f : [a, b] \to \mathbb{R}$. Then, for any partition σ of $[a, b]$, $m(b - a) \leq \overline{S}_\sigma, \underline{S}_\sigma \geq M(b - a)$. In particular, the sets*

$$\{\overline{S}_\sigma | \sigma \text{ is a partition of } [a, b]\}$$

and

$$\{\underline{S}_\sigma | \sigma \text{ is a partition of } [a, b]\}$$

are bounded.

Proof. Immediate from the definition □

Definition 12.7. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, then the upper Darboux integral of f (on $[a, b]$) is

$$\overline{S} := \text{Inf}\{\overline{S}_\sigma | \sigma \text{ is a partition of } [a, b]\}$$

and the lower Darboux integral of f (on $[a, b]$) is

$$\underline{S} := \text{Sup}\{\underline{S}_\sigma | \sigma \text{ is a partition of } [a, b]\}$$

Example 12.8. Using the above computation for the constant function Example 12.4, we see that if f is the constant function on $[a, b]$, then $\overline{S} = \underline{S} = (b - a)c$.

Proposition 12.9. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(1) If σ is a partition of $[a, b]$ and σ' is a refinement of σ, then:

$$\underline{S}_\sigma \leq \underline{S}_{\sigma'}, \text{ and } \overline{S}_\sigma \geq \overline{S}_{\sigma'}.$$

(2) If σ is a partition of $[a, b]$, then:

$$\underline{S}_\sigma \leq \overline{S}_\sigma.$$

Corollary 12.10. *If $f : [a, b] \to \mathbb{R}$ is a bounded function, then $\underline{S} \leq \overline{S}$.*

Proof. It is enough to prove that $\underline{S}_{\sigma_1} \leq \overline{S}_{\sigma_2}$ for any partitions σ_1 and σ_2 of $[a, b]$. However, this follows straight from Proposition 12.9. Indeed, if σ is a common refinement of σ_1 and σ_2, then Proposition 12.9 yields that

$$\underline{S}_{\sigma_1} \leq \underbrace{\underline{S}_\sigma}_{\text{Proposition 12.9.(1)}} \leq \underbrace{\underline{S}_\sigma}_{\text{Proposition 12.9.(2)}} \leq \underbrace{\overline{S}_{\sigma_2}}_{\text{Proposition 12.9.(1)}} .$$

□

Definition 12.11. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We say that f is *integrable*, if $\overline{S} = \underline{S}$, in which case this common value is called the *integral of f on between a and b*, and it is denoted by $\int_a^b f(x)dx$.

Remark 12.12. Using Corollary 12.10, f is integrable if one shows a sequence σ_n of partitions such that $\lim_{n\to\infty} \overline{S}_{\sigma_n} = \lim_{n\to\infty} \underline{S}_{\sigma_n}$. Indeed, this follows immediately as,

(12.12.a) $$\lim_{n\to\infty} \underline{S}_{\sigma_n} \leq \underline{S} \leq \overline{S} \leq \lim_{n\to\infty} \overline{S}_{\sigma_n}.$$

Example 12.13. Above we have seen that constant functions are integrable.

Example 12.14. Using Remark 12.12 and the above computation for $f(X) := x|_{[a,b]}$, we see that $x|_{[a,b]}$ is integrable.

Example 12.15. Consider the function $[0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 3 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, for all partition σ, $\overline{S}_\sigma = 6$, and $\underline{S}_\sigma = 0$. So, $\overline{S} = 6$, $\underline{S} = 0$, and hence f is not integrable.

Proposition 12.16. *If $f : [a, b] \to \mathbb{R}$ is continuous then it is integrable. More precisely, we know that the above assumptions imply uniform continuity of f. So, fix $\varepsilon > 0$. Let $\delta > 0$ be the constant in the definition of uniform continuity associated to $\frac{\varepsilon}{b-a}$ (that is, $|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{b-a}$), and let σ be a partition of $[a, b]$ with norm at most δ. Then $\overline{S}_\sigma - \underline{S}_\sigma \leq \varepsilon$.*

Proof. The statement is the direct consequence of the definition:

$$\overline{S}_\sigma - \underline{S}_\sigma = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \frac{\varepsilon}{b - a}(x_i - x_{i-1}) = \underbrace{\frac{\varepsilon}{b - a}(b - a)}_{\text{telescopic sum}} = \varepsilon$$

□

12.2. ***Basic properties***

Proposition 12.17. *If $f, g : [a,b] \to \mathbb{R}$ is integrable and $\alpha, \beta \in \mathbb{R}$, then*

(1) If f extends over $[b,c]$ for some $b < c \in \mathbb{R}$ and it is also integrable over $[b,c]$, then it is integrable over $[a,b]$, and

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

(2)

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(3) If $f \leq g$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(4) $|f|$ is then also integrable, and

$$\int_a^b |f(x)|dx = \left| \int_a^b f(x)dx \right|$$

Proof. All these statements are the same. One writes up the inequalities for lower and for upper Darboux sums for fixed partitions. Then these remain valid when taking Sup or Inf. This gives inequalities in both direction, which then implies equalities.

For example, let us look at how this goes in the case of point (1) (we leave the rest to the reader). Let σ, and τ be partitions for $[a,b]$ and $[b,c]$ respectively. Then these induce a partition ρ for $[a,c]$ and by definition we have

$$\overline{S}_\rho^{[a,c]} = \overline{S}_\sigma^{[a,b]} + \overline{S}_\tau^{[b,c]}, \text{ and } \underline{S}_\rho^{[a,c]} = \underline{S}_\sigma^{[a,b]} + \underline{S}_\tau^{[b,c]}.$$

As this is true for all σ and τ, we obtain by taking Inf and Sup that

$$\overline{S}^{[a,c]} \leq \overline{S}^{[a,b]} + \overline{S}^{[b,c]}, \text{ and } \underline{S}^{[a,c]} \geq \underline{S}^{[a,b]} + \underline{S}^{[b,c]}.$$

However, as f is integrable both on $[a,b]$ and on $[a,c]$, we have $\int_a^b f(x)dx = \overline{S}^{[a,b]} = \underline{S}^{[a,b]}$ and $\int_b^c f(x)dx = \overline{S}^{[b,c]} = \underline{S}^{[b,c]}$. So, the last displayed line yields that

$$\int_a^b f(x)dx + \int_b^c f(x)dx \leq \underline{S}^{[a,c]} \leq \overline{S}^{[a,c]} \leq \int_a^b f(x)dx + \int_b^c f(x)dx.$$

□

Example 12.18.

$$\int_a^b (1+x)dx = (b-a) + \frac{b^2 - a^2}{2}$$

12.3. ***Fundamental theorem of calculus***

Definition 12.19. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. A function $G : [a,b] \to \mathbb{R}$ is called the anti-derivative of f if G is differentiable on $]a,b[$, and $G'(x) = f(x)$ for all $x \in]a,b[$.

Remark 12.20. According to Corollary 11.36 the anti-derivative if exists it is determined up to adding a constant. For that reason, many times the anti-derivative of f is denoted by $\int f(x)dx + c$. Also, sometimes it is also called the indefinite integral, and what we defined as the integral sometimes is called the definite integral. We use the integra/anti-derivative naming in this course.

Theorem 12.21. MEAN VALUE THEOREM FOR INTEGRALS *If $f : [a,b] \to \mathbb{R}$ is continuous, then there is a $c \in [a,b]$, such that*

$$\int_a^b f(x)dx = f(c)(b-a).$$

Proof. Set $M := \max_{x\in[a,b]} f(x)$ and $m := \min_{x\in[a,b]} f(x)$. Then, (as $[a,b]$ is closed and f is continuous), f takes all values in $[m,M]$. However, by Proposition 12.6, we have

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M,$$

so there is a $c \in [a,b]$ such that $f(c)$ equals the above fraction, which is exactly the statement of the theorem. □

So, far we defined $\int_a^b f(x)dx$ only for $a<b$. If $a=b$, then we define it to be 0, and if $a>b$, then we define $\int_a^b f(x) := -\int_b^a f(x)dx$. With these notations our previously proven rules give that if $f:[a,b]\to\mathbb{R}$ is continous, and $c,d\in[a,b]$ are any points, then

$$\int_a^c f(x)dx + \int_c^d f(x)dx = \int_a^d f(x)dx.$$

We use this notation in the following proof.

Theorem 12.22. Fundamental theorem of calculus I *Let $f:[a,b]\to\mathbb{R}$ be continuous. Then,*

$$F(x) := \int_a^x f(t)dt$$

is an anti-derivative of f.

Proof. Fix $x_0 \in]a,b[$. Then, for any $x_0 \neq x \in]a,b[$:

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^x f(t)dt = f(c(x)),$$

for a real number $c(x)$ between x and x_0. Hence:

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = \lim_{x\to x_0} f(c(x)) = \underbrace{\lim_{x\to x_0} f(x)}_{\lim_{x\to x_0} c(x)=x_0} = \underbrace{f(x_0)}_{f \text{ is continuous}}.$$

□

Theorem 12.23. Fundamental theorem of calculus II *Let $f:[a,b]\to\mathbb{R}$ be continuous and let G be an anti-derivative of f. Then,*

$$\int_a^b f(x)dx = G(b)-G(a).$$

Proof. We have already shown in Theorem 12.22 that $F(x)=\int_a^x f(t)dt$ is an anti-derivative of f. As both F and G are anti-derivatives, they differ in a constant, say c. So, $G+c=F$. Then:

$$G(b)-G(a) = (G(b)+c)-(G(a)+c) = F(b)-F(a) = \int_a^b f(x)dx - \int_a^a f(x)dx = \int_a^b f(x)dx.$$

□

12.4. *Substitution*

Theorem 12.24. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function, and let $\phi:[\alpha,\beta]\to[a,b]$ be a continuously differentiable function. Then:*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt \tag{12.24.a}$$

Proof. Define $G(x) := \int_a^x f(u)du$. By Theorem 12.22, G is an anti-derivative of f, and hence Theorem 12.23 tells us that the left side of the equation equals $G(\phi(\beta))-G(\phi(\alpha))$. So, it is enough to show that the right side equals the same quantity.

For that, note that by the chain rule $G(\phi(t))' = G'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$. However, then Theorem 12.23 applied to the integral on the right side of the equation we obtain that this integral equals $G(\phi(\beta)) - G(\phi(\alpha))$. □

Example 12.25. First an example, where we go from the right side of (12.24.a) to the left side.

$$\int_0^1 \sqrt{e^x} e^x dx = \underbrace{\int_1^e \sqrt{u} du}_{u=e^x;(e^x)'=e^x} = \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_{u=1}^{u=e} = \frac{2}{3}\left(e^{\frac{3}{2}} - 1\right)$$

Example 12.26. The above integral computes the area of a quarter of a circle of radius 1, so the result should be $\frac{\pi}{4}$. Indeed, the above computation shows that are train of thought is correct. Note that, opposite to the previous example, in this argument at our first substitution we go from the left side of (12.24.a) to the right side.

$$\int_0^1 \sqrt{1-x^2} dx = \underbrace{\int_0^{\frac{\pi}{2}} \sqrt{1-(\sin(t))^2}\cos(t) dt}_{x=\sin(t);\sin(t)'=\cos(t)} = \int_0^{\frac{\pi}{2}} \cos(t)\cos(t) dt = \int_0^{\frac{\pi}{2}} \frac{\cos(2t)+1}{2} dt$$

$$= \underbrace{\int_0^{\pi} \frac{\cos(u)+1}{2}\frac{1}{2} du}_{t=\frac{u}{2}} = \frac{1}{4}\int_0^{\pi} \cos(u) + 1 du = \frac{1}{4}\left(\sin(u)+u\right)|_{u=0}^{u=\pi} = \frac{\pi}{4}$$

Example 12.27. A similar exercise is the following which will use hyperbolic trigonometric functions, that is, hyperbolic sine and hyperbolic cosine:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \text{ and } \cosh(x) = \frac{e^x - e^{-x}}{2}$$

These satisfy the following identities (check it!):

$$\sinh(x)' = \cosh(x),\ \cosh(x)' = \sinh(x),\ 1 + (\sinh(x))^2 = (\cosh(x))^2.$$

Futhermore, note that $\sinh(x)$ is an odd function and it is strictly increasing (indeed, $\sinh(x)' = \cosh(x) > 0$. In particular, it has an inverse, which we denote by $\operatorname{Arcsinh}(x) : \mathbb{R} \to \mathbb{R}$.

With this we may compute similarly

$$\int_0^1 \sqrt{1+x^2} dx = \underbrace{\int_0^{\operatorname{Arcsinh}(1)} \sqrt{1+(\sinh(t))^2}\cosh(t) dt}_{x=\sinh(t);\sinh(t)'=\cosh(t)} = \int_0^{\operatorname{Arcsinh}(1)} \cosh(t)\cosh(t) dt$$

$$= \int_0^{\operatorname{Arcsinh}(1)} \frac{\cosh(2t)+1}{2} dt = \underbrace{\int_0^{2\operatorname{Arcsinh}(1)} \frac{\cosh(u)+1}{2}\frac{1}{2} du}_{t=\frac{u}{2}} = \frac{1}{4}\int_0^{2\operatorname{Arcsinh}(1)} \cosh(u) + 1 du$$

$$= \frac{1}{4}\left(\sinh(u)+u\right)|_{u=0}^{u=2\operatorname{Arcsinh}(1)} = \frac{\sinh(2\operatorname{Arcsinh}(1)) + 2\operatorname{Arcsinh}(1)}{4}$$

$$= \frac{2\sinh(\operatorname{Arcsinh}(1))\cosh(\operatorname{Arcsinh}(1)) + 2\operatorname{Arcsinh}(1)}{4}$$

$$= \frac{2\sinh(\operatorname{Arcsinh}(1))\sqrt{1+\sinh(\operatorname{Arcsinh}(1))^2} + 2\operatorname{Arcsinh}(1)}{4}$$

$$= \frac{2\cdot 1\cdot\sqrt{1+1^2} + 2\operatorname{Arcsinh}(1)}{4} = \frac{2\sqrt{2} + 2\operatorname{Arcsinh}(1)}{4}$$

12.5. *Integration by parts*

Theorem 12.28. *If $f, g : I \to \mathbb{R}$ two continuously differentiable functions on an open interval, and $a < b$ elements of I, then*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx,$$

Proof. Equivalently, it is enough to show that

$$\int_a^b (f(x)g'(x) + f'(x)g(x))dx = f(x)g(x)|_a^b .$$

However, this follows immediately from Theorem 12.23 as

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x),$$

by the product rule.

□

Generally integration by parts are useful for products. The main question is how you distribute f and g. There is a rule which works in most cases (but not always!). The idea that on this list you find the first type of function that you have in your product (of two different type of functions), and you assign g' to be that:

(1) E(xponential)
(2) T(rigonometric)
(3) A(lgebraic, that is, polynomial)
(4) L(ogarithm)
(5) I(nverse trigonometric).

Here are some examples (omitting the limits, so put $(\_)|_{x=a}^b$ around every function not between an integral and dx, and put limits a and b on every integral sign). Or alternatively you can treat these as the rules for anti-derivatives.

Example 12.29.

$$\int xe^x dx = \underbrace{xe^x - \int e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=x, f'(x)=1} = xe^x - e^x = (x-1)e^x$$

Example 12.30.

$$\int \sin(x)e^x dx = \underbrace{\sin(x)e^x - \int \cos(x)e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=\sin(x), f'(x)=\cos(x)} = \underbrace{\sin(x)e^x - \cos(x)e^x + \int(-\sin(x))e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=\cos(x), f'(x)=-\sin(x)}$$

So, comparing the two endpoints we have the equality:

$$\int \sin(x)e^x dx = \frac{e^x(\sin(x) - \cos(x))}{2}.$$

Example 12.31.

$$\int \mathrm{Log}(x)dx = \underbrace{x\,\mathrm{Log}(x) - \int 1dx}_{f(x)=\mathrm{Log}(x), f'(x)=\frac{1}{x}, g'(x)=1, g(x)=x} = x\,\mathrm{Log}(x) - x$$

Example 12.32.

$$\int \mathrm{Arctg}(x) = \underbrace{x\,\mathrm{Arctg}(x) - \int \frac{x}{1+x^2}dx}_{f(x)=\mathrm{Arctg}(x), f'(x)=\frac{1}{1+x^2}, g'(x)=1, g(x)=x} = \underbrace{x\,\mathrm{Arctg}(x) - \frac{1}{2}\int \frac{1}{u}du}_{u(x)=1+x^2, u(x)'=2x}$$

$$= x\,\mathrm{Arctg}(x) - \frac{1}{2}\ln|u| = x\,\mathrm{Arctg}(x) - \frac{1}{2}\ln|1+x^2|$$

12.6. *Integrating rational functions*

That is, functions of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

By a theorem called the FUNDAMENTAL THEOREM OF ALGEBRA (which we do not prove as it is really-really hard), $Q(x)$ can be written as

$$Q(x) = (x - a_1)^{k_1} \dots (x - a_n)^{k_n} (x^2 + 2b_1 x + c_1)^{l_1} \dots (x^2 + 2b_m x + c_m)^{l_m},$$

where there is no real number x_0 such that $x_0^2 + 2b_m x_0 + c_m = 0$

Hence, one can write $\frac{P(x)}{Q(x)}$ as the sum of terms of the form:

(1) polynomial
(2) $\frac{1}{(x-r)^p}$
(3) $\frac{x+c}{(x^2+2rx+x)^p}$

Example 12.33.

$$\frac{4x^3 + 9x^2 + 11x + 8}{(x^2 + x + 1)^2} = \frac{Ax + B}{(x^2 + x + 1)^2} + \frac{Cx + D}{x^2 + x + 1} = \frac{Ax + B + (Cx + D)(x^2 + x + 1)}{(x^2 + x + 1)^2}$$
$$= \frac{Cx^3 + (C + D)x^2 + (A + C + D)x + (B + D)}{(x^2 + x + 1)^2}$$

So, we have

$$\begin{aligned} C &= 4 \\ C + D &= 9 \\ A + C + D &= 11 \\ B + D &= 8 \end{aligned}$$

This yields

$$C = 4 \Rightarrow 4 + D = 9 \Rightarrow D = 5 \Rightarrow A + 4 + 5 = 11; B + 5 = 8 \Rightarrow A = 2; B = 3$$

That is,

$$\frac{4x^3 + 9x^2 + 11x + 8}{(x^2 + x + 1)^2} = \frac{2x + 3}{(x^2 + x + 1)^2} + \frac{4x + 5}{x^2 + x + 1}.$$

So, then the question is how we integrate these terms separately:

Example 12.34. ∘ $p > 1$, then

$$\int \frac{1}{(x - r)^p} dx = \frac{(x - r)^{1-p}}{1 - p}$$

∘ $p = 1$, then

$$\int \frac{1}{(x - r)} dx = \operatorname{Log} |x - r|$$

Example 12.35.

$$\int \frac{x + c}{(x^2 + 2rx + s)^p} = \frac{1}{2} \int \frac{2(x + r)}{(x^2 + 2rx + s)^p} dx + \int \frac{c - r}{(x^2 + 2rx + s)^p} dx,$$

where

$$\int \frac{2(x + r)}{(x^2 + 2rx + s)^p} = \begin{cases} \operatorname{Log} |x^2 + 2rx + s| & \text{if } p = 1 \\ \frac{(x^2+2rx+s)^{1-p}}{1-p} & \text{if } p > 1 \end{cases}$$

So, the question boils down to computing

$$\int \frac{1}{(x^2+2rx+s)^p}dx = \int \frac{1}{((x+r)^2+(s-r^2))^p}dx = \underbrace{\frac{1}{(s-r^2)^p}\int \frac{1}{\left(\left(\frac{x+r}{\sqrt{s-r^2}}\right)^2+1\right)^p}dx}_{s-r^2>0,\text{ as } x^2+2rx+s \text{ has no real roots}}$$
$$= \underbrace{\frac{1}{(s-r^2)^{p-\frac{1}{2}}}\int \frac{1}{(u^2+1)^p}du}_{u=\frac{x+r}{\sqrt{s-r^2}}}$$

So, the question further boils down to computing:

$$I_p := \int \frac{1}{(u^2+1)^p}du$$

for every $p > 0$. Luckily, we have $I_1 := \operatorname{Arctg}(u)$, and if $p \geq 1$, then

$$I_p = \int \frac{1}{(u^2+1)^p}du = \underbrace{\frac{u}{(u^2+1)^p} - \int \frac{-pu\cdot 2u}{(u^2+1)^{p+1}}du}_{\text{Integration by parts with } f(u)=(u^2+1), g'(u)=1}$$
$$= \frac{u}{(u^2+1)^p} + 2p\int \frac{u^2+1-1}{(u^2+1)^{p+1}}du = \frac{u}{(u^2+1)^p} + 2pI_p - 2pI_{p+1}$$

So, by looking at the two ends of the equation, we obtain the recursive equality:

$$I_{p+1} = \frac{\frac{u}{(u^2+1)^p} + (2p-1)I_p}{2p}.$$

Be careful, there is an error on page 201 of the book about this, where intead of $2p-1$, $2(p-1)$ is written!!

Also, let us compute I_2 for example:

$$I_2 = \frac{\frac{u}{u^2+1}+I_1}{2} = \frac{1}{2}\left(\frac{u}{u^2+1} + \operatorname{Arctg}(u)\right)$$

Example 12.36. Let us get back to our actual example above:

$$\begin{aligned}
\int \frac{4x^3+9x^2+11x+8}{(x^2+x+1)^2}dx &= \int \frac{2x+3}{(x^2+x+1)^2}dx + \int \frac{4x+5}{x^2+x+1}dx \\
&= \int \frac{2x+3}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dx + \int \frac{4x+5}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx \\
&= \int \frac{2}{\sqrt{3}} \frac{\frac{2}{\sqrt{3}}(2x+1)+\frac{4}{\sqrt{3}}}{\left(\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right)^2}dx + \int \frac{2}{\sqrt{3}} \frac{\frac{2}{\sqrt{3}}(4x+2)+\frac{6}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}dx \\
&= \underbrace{\int \frac{2u+\frac{4}{\sqrt{3}}}{(u^2+1)^2}du + \int \frac{4u+\frac{6}{\sqrt{3}}}{u^2+1}du}_{u=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)} \\
&= \int \frac{2u}{(u^2+1)^2}du + \frac{4}{\sqrt{3}} \int \frac{1}{(u^2+1)^2}du + 2\int \frac{2u}{u^2+1}dx + \frac{6}{\sqrt{3}} \int \frac{1}{u^2+1}du \\
&= \frac{-1}{u^2+1} + \frac{2}{\sqrt{3}}\left(\frac{u}{u^2+1} + \operatorname{Arctg}(u)\right) + 2\operatorname{Log}|u^2+1| + \frac{6}{\sqrt{3}}\operatorname{Arctg}(u) \\
&= \frac{-1}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \frac{2}{\sqrt{3}}\left(\frac{\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \operatorname{Arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right) \\
&\qquad + 2\operatorname{Log}\left|\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right| + \frac{6}{\sqrt{3}}\operatorname{Arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)
\end{aligned}$$

12.6.1. *Polynomials in exponentials.*

Example 12.37.

$$\int \frac{1}{e^x+1}dx = \underbrace{\int \frac{1}{(t+1)t}dt}_{t=e^x} = \int \frac{1}{t} - \frac{1}{t+1}dt = \ln|t| - \ln|t+1| = \ln|e^x| - \ln|e^x+1|$$

12.6.2. *Polynomials in roots.*

Example 12.38.

$$\int \frac{1}{\sqrt{x}+1}dx = \underbrace{\int \frac{1}{t+1}2tdt}_{t=\sqrt{x}} = 2 - \int \frac{2}{t+1}dt = 2 - 2\operatorname{Log}|t+1| = 2 - 2\operatorname{Log}|\sqrt{x}+1|$$

There is a similar substitution if we have a rational function of $\sqrt[n]{x}$ for any n (check out the book!).

12.7. *Improper integrals*

The question is how to make sense of integrals of the form $\int_1^\infty \frac{1}{x^2}dx$. Or more generally, we have a real valued function f that is continuous on an interval I of the form $[a,b[$, $]a,b]$ or $]a,b[$, where $a,b \in \overline{\mathbb{R}}$, but either f does not extend continuously to $[a,b]$, or (in the case when a or b are $\pm\infty$) if $[a,b]$ does not exist at all.

Definition 12.39. In the above case we define the improper integral of f on I as

(1) if $I = [a, b[$, then
$$\int_a^{b-} f(t)dt := \lim_{x\to b-}\left(\int_a^x f(t)dt\right)$$
(2) if $I =]a, b]$, then
$$\int_{a^+}^{b} f(t)dt := \lim_{x\to a+}\left(\int_x^b f(t)dt\right)$$
(3) if $I =]a, b[$,
$$\int_{a^+}^{b^-} f(t)dt := \int_{a^+}^{c} f(t)dt + \int_c^{b-} f(t)dt$$
for any $c \in I$ (it is an easy exercise that the sum does not depend on c),

assuming that the above limits exist. Furthermore, if the limits exist we say that the integrals converge.

If the above limits diverge, we say that the corresponding improper integral is divergent.

Remark 12.40. By abuse of notation many times the $+$ and the $-$ is forgot from the lower and upper limits.

Example 12.41.
$$\int_{0^+}^{1} \frac{1}{\sqrt{t}}dt = \lim_{x\to 0^+}\left(\left.\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right|_{t=x}^{t=1}\right) = \lim_{x\to 0^+} 2 - 2\sqrt{x} = 2$$

Example 12.42.
$$\int_{0^+}^{1} \frac{1}{t}dt = \lim_{x\to 0^+}\left(\mathrm{Log}(t)|_{t=x}^{t=1}\right) = \lim_{x\to 0^+} -\mathrm{Log}(x) = -\infty$$
So, $\int_{0^+}^{1} \frac{1}{t}dt$ is divergent.

$$\int_{0^+}^{1} \mathrm{Log}(t)dt = \lim_{x\to 0^+} (\mathrm{Log}(t)t - t)|_{t=x}^{t=1} = -1 - \lim_{x\to 0^+} (\mathrm{Log}(x)x - x) = -1 - \lim_{x\to 0^+} (\mathrm{Log}(x)x)$$
Where we may compute $\lim_{x\to 0^+}(\mathrm{Log}(x)x)$ using L'Hospital's rule:
$$\lim_{x\to 0^+} (\mathrm{Log}(x)x) = \lim_{x\to 0^+} \frac{\mathrm{Log}(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$
Hence, $\int_{0^+}^{1} \mathrm{Log}(t)dt = -1$.

Definition 12.43. In the situations of Definition 12.39, we say that the integral is absolute convergent if the integrals with f replaced by $|f|$ are also convergent.

The following is an immediate consequence of Cauchy's convergence criterion:

Proposition 12.44. *If an improper integeral is absolute convergent, then it is also convergent.*

Example 12.45. The backwards implication of the above proposition does not holds, as shown by the next example.
$$\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin(t)}{t}dt = \lim_{x\to+\infty} \frac{-\cos(x)}{x} - \frac{-\cos\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{+\infty} \frac{-\cos(t)}{-t^2}dt = -\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos(t)}{t^2}dt$$
So, $\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin(t)}{t}dt$ is convergent, if so is $\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos(t)}{t^2}dt$. However, the latter is convergent because it is absolute convergent:
$$\int_{\frac{\pi}{2}}^{+\infty} \left|\frac{\cos(t)}{t^2}\right| dt \leq \int_{\frac{\pi}{2}}^{+\infty} \frac{1}{t^2}dt = \lim_{x\to+\infty}\left(\left.\frac{-1}{t}\right|_{\frac{\pi}{2}}^{x}\right) = -\frac{2}{\pi} + \lim_{x\to+\infty} \frac{1}{x} = -\frac{2}{\pi}.$$

This yields that $\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin(t)}{t} dt$ is convergent.

However, be careful, the fact that $\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos(t)}{t^2} dt$ is absolute convergent, does not mean that so is $\int_{\frac{\pi}{2}}^{+\infty} \frac{\sin(t)}{t} dt$. That is, although the two integrals are equal, the computation which shows this equality does not work for $\frac{\sin(t)}{t}$ replaced by $\left|\frac{\sin(t)}{t}\right|$. And, in fact, $\int_{\frac{\pi}{2}}^{+\infty} \frac{\cos(t)}{t^2} dt$ is not absolute convergent:

$$\int_{\frac{\pi}{2}}^{+\infty} \left|\frac{\sin(t)}{t}\right| dt \geq \sum_{n=0}^{\infty} \frac{1}{\sqrt{2}} \frac{\frac{1}{\sqrt{2}}\frac{\pi}{2}}{\pi n + \frac{5\pi}{4}} = \sum_{n=0}^{\infty} \frac{1}{4n+5} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{n+\frac{5}{4}} \geq \frac{1}{4}\sum_{n=2}^{\infty} \frac{1}{n}$$

where the latter is divergent.

13. Power series

Definition 13.1. A *power series* centered at $x_0 \in \mathbb{R}$ is an expression of the form $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ for some sequence a_k of real numbers.

The *domain of convergence* of $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is

$$D := \{x \in \mathbb{R} | \sum_{k=0}^{\infty} a_k(x-x_0)^k \text{ is convergent}\}$$

Note that $x_0 \in D$ always, as the only non-zero term in $\sum_{k=0}^{\infty} a_k(x_0-x_0)^k$ is a_0.

Theorem 13.2. *There is an R (called* radius of convergence*) such that*

$$if \begin{Bmatrix} |x-x_0| < R \\ |x-x_0| > R \end{Bmatrix}, then \begin{Bmatrix} \sum_{k=0}^{\infty} a_k(x-x_0)^k \text{ is convergent} \\ \sum_{k=0}^{\infty} a_k(x-x_0)^k \text{ is divergent} \end{Bmatrix}$$

Proof. Let D be the domain of convergence. We have to show that if $x' \in D$, then for all x with $0 < |x-x_0| < |x'-x_0|$, $x \in D$.

So, we assume htat $\sum_{k=0}^{\infty} a_k(x'-x_0)^k$ is convergent. In particular, $\lim_{k\to\infty} a_k(x'-x_0)^k \to 0$, and hence the sequence $b_k := a_k(x'-x_0)^k$ is bounded. So, there is a real number B, such that $|b_k| \leq B$. However, then if we set $y := \frac{x-x_0}{x'-x_0}$, which is smaller than 1 by our assumption, we get that

$$\sum_{k=0}^{\infty} |a_k(x-x_0)^k| = \sum_{k=0}^{\infty} |b_k y^k| \leq \sum_{k=0}^{\infty} B|y|^k$$

Since the latter is a geometric series, it is convergent, and then we see that $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is absolutely convergent, and hence also convergent.

□

Theorem 13.3. *Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series, with radius of convergence R.*

(1) If $l := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists for some $l \in \overline{\mathbb{R}}$, then $R = \frac{1}{l}$ (for $l = 0$ and $l = +\infty$, then $R = +\infty$ and $R = 0$).

(2) If $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists for some $L \in \overline{\mathbb{R}}$, then $R = \frac{1}{L}$.

Proof. Let $x \neq x_0$. We want to decide when $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent

(1) We use the quotient criterion:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-x_0| l$$

So, by the quotient criterion $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent if $|x-x_0| < \frac{1}{l}$ and it is divergent if $|x-x_0| > \frac{1}{l}$. Let $x \neq x_0$. We want to decide when $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent

(2) We use the Alembert's criterion:

$$\lim_{n\to\infty}\left|\sqrt[n]{a_n(x-x_0)^n}\right| = \lim_{n\to\infty}|\sqrt[n]{a_n}|\,|x-x_0| = l|x-x_0|$$

So, by Alembert's criterion $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent if $|x-x_0| < \frac{1}{l}$ and it is divergent if $|x-x_0| > \frac{1}{l}$.

□

Example 13.4. We have seen earlier that

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

is convergent for all $x \in \mathbb{R}$, which we can verify now easier:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

So, indeed, the radius of convergence of the above series is ∞.

13.1. *Taylor series*

Definition 13.5. If $f : I \to \mathbb{R}$ is a function on an open interval such that it is differentiable n-times for all integer $n > 0$ at $x_0 \in I$ (we say that f is infinitely many times differentiable at x_0), then the Taylor series of f is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Example 13.6. We have seen that the Taylor series of $\mathrm{Log}(x)$ is

$$\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}(x-1)^k.$$

Let us determine the radius of convergence:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}}\right| = 1,$$

so the radius is 1. In particular $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}(x-1)^k$ converges for whenever $|x-1| < 1$ and diverges whenever $|x-1| > 1$. However, still not immediate that the above series gives $\mathrm{Log}(x)$. That is, whether

$$\mathrm{Log}(x) = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k}(x-1)^k\right)$$

This is equivalent to showing that the error term $\varepsilon(x)$ of the Taylor expansion of degree n converges to 0 as n goes to infinity. This can be shown here, but we do not go into details. So, in particular $\mathrm{Log}(x) = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}(x-1)^k$ whenever $|x-1| < 1$.

We note that this last step sometimes fails, and so even if the Taylor series converges it might not be equal to the original function. The famous example is

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then, f is infinitely many times differentiable everywhere, and its Taylor series around 0 is the constant 0 function. So, the radius of convergence is $+\infty$, but apart from 0 there is no point where the Taylor series equals to the function. We do not prove these claims in this course, they are here just to make the picture complete.

Example 13.7. Similar example as above is the Taylor series for $\frac{1}{x+1}$. It turns out that the series is $\sum_{k=0}^{\infty}(-1)^k x^k$ (we have seen it). The above criteria show that the radius of convergence is 1, and then one can show with an error term argument that in fact, for $]-1,1[$, $\frac{1}{x+1} = \sum_{k=0}^{\infty}(-1)^k x^k$.

Theorem 13.8. *Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series, let R be its radius of convergence and set $D := \{x \in \mathbb{R} | |x-x_0| < R\}$. Then, we may define the function $f(x) : D \to \mathbb{R}$ as $f(x) := \sum_{k=0}^{\infty} a_k(x-x_0)^k$.*

Then, for every x for which $|x-x_0| < R$ we have

(1)
$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1},$$

(2)
$$\int_{x_0}^{x} f(t)dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-x_0)^{k+1}.$$

Example 13.9. If we want to take the derivative and integral of $\operatorname{Log}(x)$, then we can take it over $]0,2[$ also term by term as a power series. So, for $x \in]0,2[$:

$$\operatorname{Log}(x)' = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k}(x-1)^{k-1} = \sum_{k=1}^{\infty}(-1)^{k-1}(x-1)^{k-1} = \sum_{k=0}^{\infty}(-1)^k(x-1)^k = \frac{1}{1-(x-1)} = \frac{1}{x}.$$

and

$$\begin{aligned}\int_1^x \operatorname{Log}(t)dt = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}(x-1)^{k+1} &= \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{k} - \frac{1}{k+1}\right)(x-1)^{k+1} \\ &= (x-1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(x-1)^k + \sum_{k=1}^{\infty}(-1)^{k+2}\frac{1}{k+1}(x-1)^{k+1} \\ &= (x-1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(x-1)^k + \sum_{k=2}^{\infty}(-1)^{k+1}\frac{1}{k}(x-1)^k \\ &= (x-1)\operatorname{Log}(x) + (\operatorname{Log}(x) - (x-1)) = x\operatorname{Log}(x) - x + 1\end{aligned}$$

where the result of the second computation is exactly the anti-derivative we know up to adding a constant to it.