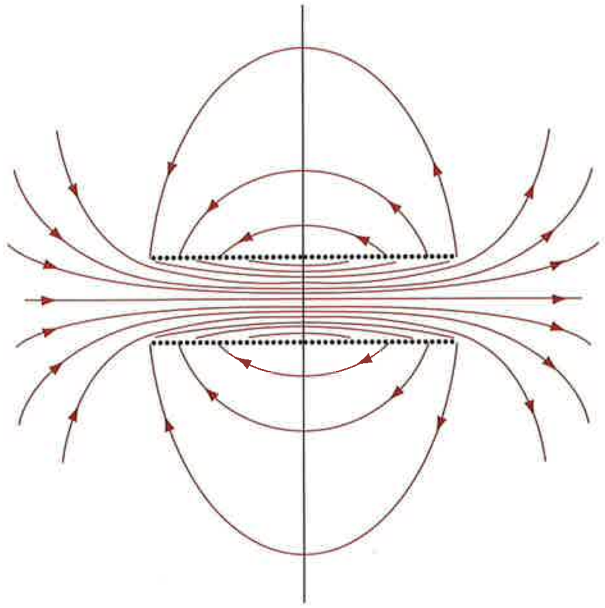

FIGURE 33-19. Magnetic field lines for a solenoid of finite length. Note that the field is stronger (indicated by the greater density of field lines) inside the solenoid than it is outside.

turns (marked ⊗, because the current is into the page), which points to the right near P. In the limiting case of the ideal solenoid, the field outside the solenoid is zero. Taking the external field to be zero is a good approximation for a real solenoid if its length is much greater than its radius and if we consider only external points such as P. Figure 33-19 shows the magnetic field lines for a nonideal solenoid. You can see from the spacing of the field lines that the field exterior to the solenoid is much weaker than the field in the interior, which is very nearly uniform over the cross section of the solenoid.

The solenoid is for magnetic fields what the parallel-plate capacitor is for electric fields: a relatively simple device capable of producing a field that is approximately uniform. In a parallel-plate capacitor, the electric field is nearly uniform if the plate separation is small compared with the dimensions of the plates, and if we are not too close to the edge of the capacitor. In the solenoid, the magnetic field is nearly uniform if the radius is small compared with the length and if we are not too close to the ends. As shown in Fig. 33-17, even for a length that is only 10 times the radius, the magnetic field is within a few percent of the field of the ideal solenoid over the central half of the device.

SAMPLE PROBLEM 33-7. A solenoid has a length $L = 1.23$ m and an inner diameter $d = 3.55$ cm. It has five layers of windings of 850 turns each and carries a current $i = 5.57$ A. What is B at its center?

Solution With $L/R = 69$, we are safe in regarding this as a nearly ideal solenoid. From Eq. 33-28 we have

$$B = \mu_0 ni = (4\pi \times 10^{-7}\ \mathrm{T\cdot m/A})\left(\frac{5 \times 850\ \text{turns}}{1.23\ \mathrm{m}}\right)(5.57\ \mathrm{A})$$

$$= 2.42 \times 10^{-2}\ \mathrm{T} = 24.2\ \mathrm{mT}.$$

Note that Eq. 33-28 applies even if the solenoid has more than one layer of windings because the diameter of the windings does not enter into the equation.

33-5 AMPÈRE'S LAW

Coulomb's law can be considered a fundamental law of electrostatics; we can use it to calculate the electric field associated with any distribution of electric charges. In Chapter 27, however, we showed that Gauss' law permitted us to solve a certain class of problems, those containing a high degree of symmetry, with ease and elegance. Furthermore, we showed that Gauss' law contained within it Coulomb's law for the electric field of a point charge. We consider Gauss' law to be more basic than Coulomb's law, and Gauss' law is one of the four fundamental (Maxwell) equations of electromagnetism.

The situation in magnetism is similar. Using the Biot–Savart law, we can calculate the magnetic field of any distribution of currents, just as we used Eq. 26-6 or Eqs. 26-13 and 26-14 (which are equivalent to Coulomb's law) to calculate the electric field of any distribution of charges. A more fundamental approach to magnetic fields uses a law that (like Gauss' law for electric fields) takes advantage of the symmetry present in certain problems to simplify the calculation of $\vec{\mathbf{B}}$. This law is considered more fundamental than the Biot–Savart law and leads to another of the four Maxwell equations.

This new result is called *Ampère's law* and is written

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 i. \tag{33-29}$$

You will recall that, in using Gauss' law, we first constructed an imaginary closed surface (a Gaussian surface) that enclosed a certain amount of charge. In using Ampère's law we construct an imaginary closed curve (called an *Ampèrian loop*), as indicated in Fig. 33-20. The left side of Eq. 33-29 tells us to divide the curve into small segments of

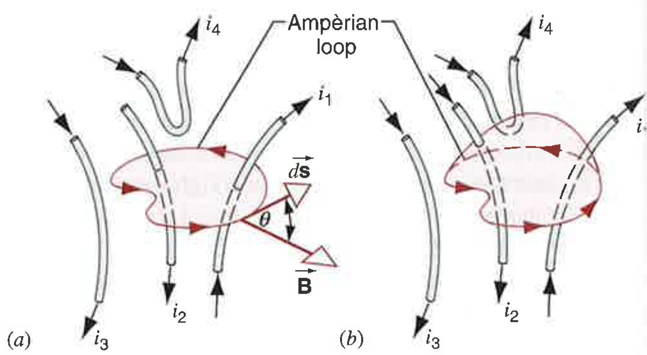

FIGURE 33-20. (*a*) In applying Ampère's law, we integrate around a closed loop. The integral is determined by the net current that passes through the surface bounded by the loop. (*b*) The surface bounded by the loop has been stretched upward.

length $d\vec{s}$. As we travel around the loop (our direction of travel determining the direction of $d\vec{s}$), we evaluate the quantity $\vec{B} \cdot d\vec{s}$ and add (integrate) all such quantities around the loop.

The integral on the left of Eq. 33-29 is called a *line integral.* (Previously we used line integrals in Chapter 11 to calculate work and in Chapter 28 to calculate potential difference.) The circle superimposed on the integral sign reminds us that the line integral is to be evaluated around a *closed* path. Letting θ represent the angle between $d\vec{s}$ and $\vec{B}$, we can write the line integral as

$$\oint \vec{B} \cdot d\vec{s} = \oint B\, ds \cos\theta. \qquad (33\text{-}30)$$

The current i in Eq. 33-29 is the total current "enclosed" by the loop; that is, it is the total current carried by wires that pierce any surface bounded by the loop. In analogy with charges in the case of Gauss' law, currents outside the loop are not included. Figure 33-20a shows four wires carrying current. The magnetic field $\vec{B}$ at any point is the net effect of the currents in all wires. However, in the evaluation of the right side of Eq. 33-29, we include only the currents i_1 and i_2, because the wires carrying i_3 and i_4 do not pass through the surface enclosed by the loop. The two wires that pass through the loop carry currents in opposite direction. A right-hand rule is used to assign signs to currents: with the fingers of your right hand in the direction in which the loop is traveled, currents in the direction of your thumb (such as i_1) are taken to be positive, whereas currents in the opposite direction (such as i_2) are taken to be negative. The net current i in the case of Fig. 33-20a is thus $i = i_1 - i_2$.

The magnetic field $\vec{B}$ at points on the loop and within the loop certainly depends on the currents i_3 and i_4; however, the *integral* of $\vec{B} \cdot d\vec{s}$ around the loop does *not* depend on currents such as i_3 and i_4 that do not penetrate the surface enclosed by the loop. This is reasonable, because $\vec{B} \cdot d\vec{s}$ for the field established by i_1 or i_2 always has the same sign as we travel around the loop; however, $\vec{B} \cdot d\vec{s}$ for the fields due to i_3 or i_4 change sign as we travel around the loop, and in fact the positive and negative contributions exactly cancel one another.

Changing the shape of the surface without changing the loop does not change these conclusions. In Fig. 33-20b the surface has been "stretched" upward so that now the wire carrying current i_4 penetrates the surface. However, note that it does so twice, once moving downward (which would correspond to a contribution $-i_4$ to the total current through the surface, according to our right-hand rule) and once moving upward (which would contribute $+i_4$ to the total). Thus the total current through the surface does not change; this is as expected, because stretching the surface does not change $\vec{B}$ at locations along the fixed loop, and therefore the line integral on the left side of Ampère's law does not change.

Note that including the arbitrary constant of 4π in the Biot–Savart law reduces the constant that appears in Ampère's law to simply μ_0. (A similar simplification of Gauss' law was obtained by including the constant 4π in Coulomb's law.)

For the situation shown in Fig. 33-20, Ampère's law gives

$$\oint B\, ds \cos\theta = \mu_0(i_1 - i_2). \qquad (33\text{-}31)$$

Equation 33-31 is valid for the magnetic field $\vec{B}$ as it varies in both magnitude and direction around the path of the Ampèrian loop. We cannot solve that equation for B unless we can find a way to remove B from the integral. To do so, we use symmetries in the geometry to choose an Ampèrian loop for which B is constant. We used a similar trick in calculating electric fields using Gauss' law.

The following examples illustrate how Ampère's law can be used to calculate magnetic fields in cases with a high degree of symmetry.

Applications of Ampère's Law

A Long, Straight Wire (external points). We can use Ampère's law to find the magnetic field at a distance d from a long, straight wire, a problem we have already solved using the Biot–Savart law.

As illustrated in Fig. 33-21, we choose as our Ampèrian loop a circle of radius d centered on the wire with its plane perpendicular to the wire. From the symmetry of the problem, $\vec{B}$ can depend only on d (and not, for instance, on the angular coordinate around the circle). By choosing a path that is everywhere the same distance from the wire, we know that B is constant around the path.

We know from Oersted's experiments that $\vec{B}$ has only a tangential component. Thus the angle θ is zero, and the line integral becomes

$$\oint B\, ds \cos\theta = B \oint ds = B(2\pi d). \qquad (33\text{-}32)$$

Note that the integral of ds around the path is simply the length of the path, or $2\pi d$ in the case of the circle. The right side of Ampère's law is simply $\mu_0 i$ (taken as positive,

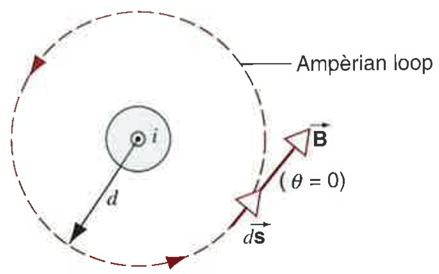

FIGURE 33-21. A circular Ampèrian loop is used to find the magnetic field set up by a current in a long, straight wire. The wire is perpendicular to the page, and the direction of the current is out of the page.

in accordance with the right-hand rule). Ampère's law gives

$$B(2\pi d) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi d}.$$

This is identical with Eq. 33-13, a result we obtained (with considerably more effort) using the Biot–Savart law.

A Long, Straight Wire (internal points). We can also use Ampère's law to find the magnetic field *inside* a wire. We assume a cylindrical wire of radius R in which a total current i is distributed uniformly over its cross section. We wish to find the magnetic field at a distance $r < R$ from the center of the wire.

Figure 33-22 shows a circular Ampèrian loop of radius r inside the wire. Symmetry suggests that $\mathbf{B}$ is constant in magnitude everywhere on the loop and tangent to the loop, so the left side of Ampère's law gives $B(2\pi r)$, exactly as in Eq. 33-32. The right side of Ampère's law involves only the current inside the radius r. If the current is distributed uniformly over the wire, the fraction of the current inside the radius r is the same as the fraction of the area inside r, or $\pi r^2/\pi R^2$. Ampère's law then gives

$$B(2\pi r) = \mu_0 i \frac{\pi r^2}{\pi R^2}, \tag{33-33}$$

where again the right side includes only the fraction of the current that passes through the surface enclosed by the path of integration (the Ampèrian loop).

Solving for B, we obtain

$$B = \frac{\mu_0 i r}{2\pi R^2}. \tag{33-34}$$

At the surface of the wire ($r = R$), Eq. 33-34 reduces to Eq. 33-13 (with $d = R$). That is, both expressions give the same result for the field at the surface of the wire. Figure 33-23 shows how the field depends on r at points both inside and outside the wire.

Equation 33-34 is valid only for the case in which the current is distributed uniformly over the wire. If the current density depends on r, a different result is obtained (see Problem 13). However, Eq. 33-13 for the field outside the wire remains valid whether the current density is constant or a function of r.

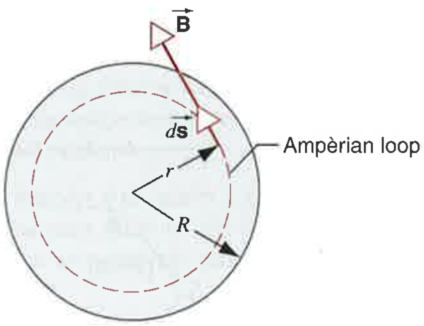

FIGURE 33-22. A long, straight wire carries a current that is emerging from the page and is uniformly distributed over the circular cross section of the wire. A circular Ampèrian loop is drawn inside the wire.

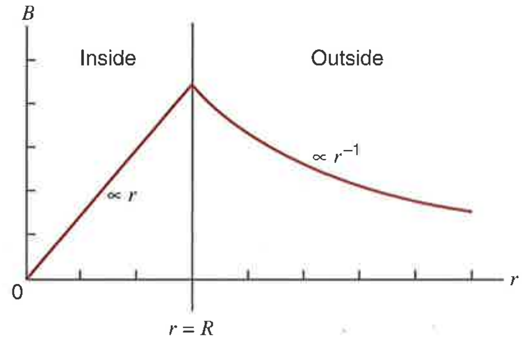

FIGURE 33-23. The magnetic field calculated for the wire shown in Fig. 33-22. Note that the largest field occurs at the surface of the wire.

A Solenoid. We consider an ideal solenoid as shown in Fig. 33-24 and choose an Ampèrian loop in the shape of the rectangle *abcda*. In this analysis we assume that the magnetic field is parallel to the axis of this ideal solenoid and constant in magnitude along line *ab*. As we shall prove, the field is also uniform in the interior (independent of the distance of *ab* from the central axis), as suggested by the equal spacing of the field lines in Fig. 33-24.

The left side of Ampère's law can be written as the sum of four integrals, one for each path segment:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \int_a^b \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_b^c \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_c^d \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} + \int_d^a \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}. \tag{33-35}$$

The first integral on the right is Bh, where B is the magnitude of $\vec{\mathbf{B}}$ inside the solenoid and h is the arbitrary length of the path from a to b. Note that path ab, though parallel to the solenoid axis, need not coincide with it. The second and fourth integrals in Eq. 33-35 are zero because for every element of these paths $\vec{\mathbf{B}}$ is either at right angles to the path (for points inside the solenoid) or is zero (for points outside). In either case, $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ is zero, and the integrals vanish. The third integral, which includes the part of the rectangle that lies outside the solenoid, is zero because we have taken $\vec{\mathbf{B}}$ as zero for all external points for an ideal solenoid.

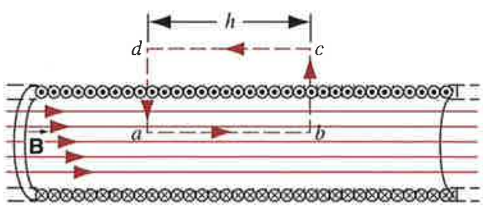

FIGURE 33-24. An Ampèrian loop (the rectangle *abcd*) is used to calculate the magnetic field of this long, idealized solenoid.

For the entire rectangular path, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ has the value Bh. The net current i that passes through the rectangular Ampèrian loop is not the same as the current in the solenoid because the windings pass through the loop more than once. Let n be the number of turns per unit length; then nh is the number of turns inside the loop, and the total current, passing through the rectangular Ampèrian loop of Fig. 33-24 is nhi. Ampère's law then becomes

$$Bh = \mu_0 nhi$$

or

$$B = \mu_0 ni.$$

This result agrees with Eq. 33-28, which referred only to points on the central axis of the solenoid. Because line ab in Fig. 33-24 can be located at any distance from the axis, we can now conclude that the magnetic field inside an ideal solenoid is uniform over its cross section.

A Toroid. Figure 33-25 shows a toroid, which we may consider to be a solenoid bent into the shape of a doughnut. We can use Ampère's law to find the magnetic field at interior points.

From symmetry, the lines of $\vec{\mathbf{B}}$ form concentric circles inside the toroid, as shown in the figure. Let us choose a concentric circle of radius r as an Ampèrian loop and traverse it in the clockwise direction. Ampère's law yields

$$B(2\pi r) = \mu_0 iN,$$

where i is the current in the toroid windings and N is the total number of turns. This gives

$$B = \frac{\mu_0 iN}{2\pi r}. \qquad (33\text{-}36)$$

In contrast to the solenoid, B is not constant over the cross section of a toroid. You should be able to show, from Ampère's law, that $B = 0$ for points outside an ideal toroid and in the central cavity.

Close inspection of Eq. 33-36 justifies our earlier statement that a toroid is "a solenoid bent into the shape of a doughnut." The denominator in Eq. 33-36, $2\pi r$, is the central circumference of the toroid, and $N/2\pi r$ is just n, the number of turns per unit length. With this substitution, Eq. 33-36 reduces to $B = \mu_0 in$, the equation for the magnetic field in the central region of a solenoid.

The direction of the magnetic field within a toroid (or a solenoid) follows from the right-hand rule: curl the fingers of your right hand in the direction of the current; your extended right thumb then points in the direction of the magnetic field.

Toroids form the central feature of the *tokamak,* a device showing promise as the basis for a fusion power reactor. We discuss its mode of operation in Chapter 51.

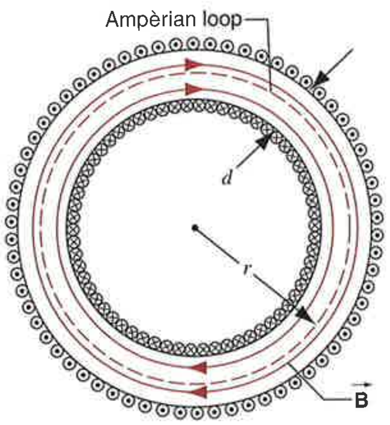

FIGURE 33-25. A toroid. The interior field can be found using the circular Ampèrian loop shown.

The Field Outside a Solenoid (Optional). We have so far neglected the field outside the solenoid, but even for an ideal solenoid, the field at points outside the winding is not zero. Figure 33-26 shows an Ampèrian path in the shape of a circle of radius r. Because the solenoid windings are helical, one turn of the winding pierces the surface enclosed by the circle. The product $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ for this path depends on the tangential component of the field B_t, and thus Ampère's law gives

$$B_t(2\pi r) = \mu_0 i$$

or

$$B_t = \frac{\mu_0 i}{2\pi r}, \qquad (33\text{-}37)$$

which is the same field (in magnitude *and* direction) that would be set up by a straight wire. Note that the windings, in addition to carrying current around the surface of the solenoid, also carry current from left to right in Fig. 33-26, and in this respect the solenoid behaves like a straight wire at points outside the windings.

The tangential field is much smaller than the interior field (Eq. 33-28), as we can see by taking the ratio

$$\frac{B_t}{B} = \frac{\mu_0 i/2\pi r}{\mu_0 ni} = \frac{1}{2\pi rn}. \qquad (33\text{-}38)$$

Suppose the solenoid consists of one layer of turns in which the wires are touching one another, as in Fig. 33-24. Every interval along the solenoid of length equal to the diameter D of the wire contains one turn, and thus the number of turns per unit length n must be $1/D$. The ratio thus becomes

$$\frac{B_t}{B} = \frac{D}{2\pi r}. \qquad (33\text{-}39)$$

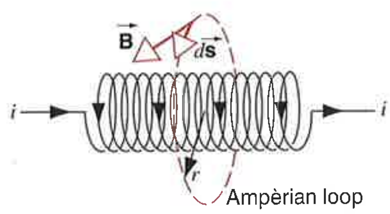

FIGURE 33-26. A circular Ampèrian loop of radius r is used to find the tangential field external to a solenoid.

For a typical wire, $D = 0.1$ mm. The distance r to exterior points must be at least as large as the radius of the solenoid, which might be a few centimeters. Thus $B_t/B \leq 0.001$, and the tangential exterior field is indeed negligible compared with the interior field along the axis. We are therefore safe in neglecting the exterior field.

By drawing an Ampèrian circle similar to that of Fig. 33-26 but with radius smaller than that of the solenoid, you should be able to show that the tangential component of the interior field is zero. ■

33-6 ELECTROMAGNETISM AND FRAMES OF REFERENCE (Optional)

Figure 33-27*a* shows a particle carrying a positive charge q at rest near a long, straight wire that carries a current i. We view the system from a frame of reference S in which the wire is at rest. Inside the wire are negative electrons moving with the drift velocity $\vec{\mathbf{v}}_d$ and positive ion cores at rest. In any given length of wire, the number of electrons equals the number of ion cores, and the net charge is zero. The electrons can instantaneously be considered as a line of negative charge, which sets up an electric field at the location of q according to Eq. 26-17:

$$E = \frac{\lambda_-}{2\pi\epsilon_0 r},$$

where λ_- is the linear charge density of electrons (a negative number). The positive ion cores also set up an electric field given by a similar expression, depending on the linear charge density λ_+ of positive ions. Because the charge densities are of equal magnitude and opposite sign,

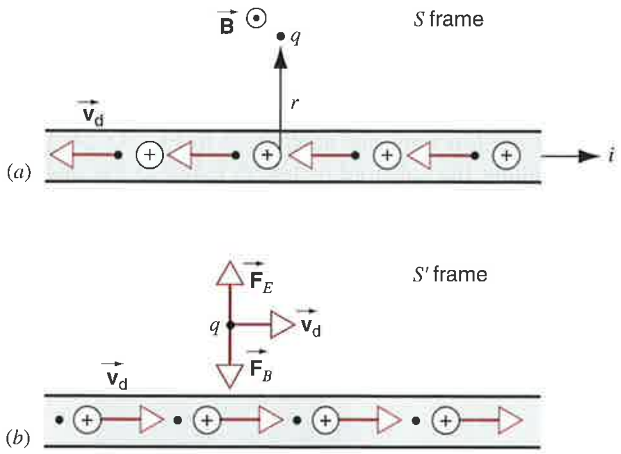

FIGURE 33-27. (*a*) A particle of charge q is at rest in equilibrium near a wire carrying a current i. The situation is viewed from a reference frame S at rest with respect to the particle.
(*b*) The same situation viewed from a frame S' that is moving with the drift velocity of the electrons in the wire. The particle is also in equilibrium in this frame under the influence of the two forces $\vec{\mathbf{F}}_E$ and $\vec{\mathbf{F}}_B$.

$\lambda_+ + \lambda_- = 0$, and the net electric field that acts on the particle is zero.

There is a nonzero magnetic field at the location of the particle, but because the particle is at rest, there is no magnetic force. Therefore no net force of electromagnetic origin acts on the particle in this frame of reference.

Now let us consider the situation from the perspective of a frame of reference S' moving parallel to the wire with velocity $\vec{\mathbf{v}}_d$ (the drift velocity of the electrons). Figure 33-27*b* shows the situation in this frame of reference, in which the electrons are at rest and the ion cores move to the right with velocity $\vec{\mathbf{v}}_d$. Clearly, in this case the particle, being in motion, experiences a magnetic force $\vec{\mathbf{F}}_B$ as shown in the figure.

Observers in different inertial frames must agree that if there is no acceleration of the charge q in the S frame, there must also be no acceleration in the S' frame. The particle must therefore experience no net force in S', and so there must be another force in addition to $\vec{\mathbf{F}}_B$ that acts on the particle to give a net force of zero.

This additional force that acts in the S' frame must be of electric origin. Consider in Fig. 33-27*a* a length L of the wire. We can imagine that length of the wire to consist of two measuring rods, a positively charged rod (the ions) at rest and a negatively charged rod (the electrons) in motion. The two rods have the same length (in S) and contain the same number of charges. When we transform those rods into S', we find that the rod of negative charge has a greater length in S'. In S, this moving rod has its *contracted length*, according to the relativistic effect of length contraction we considered in Section 20-3. In S', it is at rest and has its *proper length*, which is longer than the contracted length in S. The negative linear charge density λ'_- in S' is smaller in magnitude than that in S (that is, $|\lambda'_-| < |\lambda_-|$), because the same amount of charge is spread over a greater length in S'.

For the positive charges, the situation is opposite. In S, the positive charges are at rest, and the rod of positive charge has its proper length. In S', it is in motion and has a shorter, contracted length. The linear density λ'_+ of positive charge in S' is greater than that in S ($\lambda'_+ > \lambda_+$), because the same amount of charge is spread over a shorter length. We therefore have the following relationships for the charge densities:

$$\text{in } S: \qquad \lambda_+ = |\lambda_-|,$$

$$\text{in } S': \qquad \lambda'_+ > |\lambda'_-|.$$

The charge q experiences the electric fields due to a line of positive charge and a line of negative charge. In S', these fields do not cancel, because the linear charge densities are different. The electric field at q in S' is therefore that due to a net linear density of positive charge, and q is repelled from the wire. The electric force $\vec{\mathbf{F}}_E$ on q opposes the magnetic force $\vec{\mathbf{F}}_B$, as shown in Fig. 33-27*b*. A detailed calculation* shows that the resulting electric force is

*See, for example, R. Resnick, *Introduction to Special Relativity* (Wiley, 1968), Chapter 4.