Contents

1 Initial Equations **1**

2 Equations with No Shear or Viscosity **1**
2.1 A standard thermodynamic derivation of the stress energy tensor 1
2.2 An Effective field theory for Fluids 2
2.3 Conservation equations, and equivalence to Naiver-Stokes 3
2.4 Evolution equations with no curvature or source 3
2.4.1 Initial Equations Expanded 3
2.4.2 Evolution equations in 1+1 5
2.5 Evolution Equations including a Source Term and Curvature 6
2.5.1 Equations in 1+1 with source but no curvature 7

3 Scalar field source **8**
3.1 A Phenomenological Dissipative Coupling 8

4 Equations with Viscosity/Shear **8**
4.1 Stress-Energy Tensor 8

5 Gravitational Wave Generation **9**

1 Initial Equations

We start out by noting we will be working in a $(-1, 1, 1, 1)$ signature. We also note that any fluid should behave in a time-like manner is used as well:

$$U^\mu U_\mu = -1 \tag{1}$$

or, taking the derivative of both sides,

$$U^\mu D_\nu U_\mu = 0 \tag{2}$$

where we will use a capital D to denote a covariant derivative. We also note that our stress-energy tensor will be conserved modulo a source term,

$$D_\mu T^{\mu\nu} = j^\nu \quad , \tag{3}$$

an equation from which we will derive equations used to evolve the fluid. We will also be considering fluids with an equation of state $p = w\epsilon$, with w a parameter determined by the physical situation we wish to consider.

2 Equations with No Shear or Viscosity

2.1 A standard thermodynamic derivation of the stress energy tensor

For this special case, we only have degrees of freedom coming from the fluid velocityU^μ , in addition to it's associated energy density ϵ and pressure p. Following Schutz (1970), we can write down an action for the special case of a fluid at rest:

$$\begin{aligned} S &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_{fluid}\right) \quad . & (4)\\ &= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\kappa} + p\right) & (5)\end{aligned}$$

Here, p is a function of the normalized inertial mass $\mu \equiv \frac{\epsilon+p}{\epsilon_0}$ (where ϵ is the density of the fluid relative to a background density ϵ_0) and metric $g^{\mu\nu}$. We also have a relationship between dp and $d\mu$: $dp = \epsilon_0 d\mu - \epsilon_0 T dS$, where S is the entropy of the fluid and T is the temperature. Keeping entropy constant, we have that

$\frac{\partial p}{\partial \mu} = \epsilon_0 = \frac{\epsilon+p}{\mu}$. Lastly, we have that the fluid velocity can be written as a function of mass and other fundamental quantities (typically scalar fields and derivatives thereof, but these are not of direct interest so we hide then in our fluid momentum P^ν): $U^\nu = \frac{1}{\mu} P^\nu$. From normalization of U^ν, we have that $\mu^2 = g^{\mu\nu} P_\mu P_\nu$.

Varying the action with respect to the metric gives us:

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \sqrt{-g} p}{\delta g^{\mu\nu}} = p g^{\mu\nu} + -2 \frac{\delta p}{\delta g^{\mu\nu}} \ ,$$

where

$$\frac{\delta p}{\delta g^{\mu\nu}} = \frac{\partial p}{\partial \mu} \frac{\delta \mu}{\delta g^{\mu\nu}} = -\frac{1}{2} \frac{\rho + p}{\mu^2} P_\mu P_\nu \ ,$$

so

$$T^{\mu\nu} = (\epsilon + p) U^\mu U^\nu + p g^{\mu\nu} = (1+w)\epsilon U^\mu U^\nu + w\epsilon g^{\mu\nu} \ . \tag{6}$$

We note that P_μ can be written in terms of scalar fields $\{\phi, \theta, \alpha, \beta\}$ and entropy S:

$$P_\mu = \phi_{,\mu} + \alpha \beta_{,\mu} + \theta S_{,\mu} \ .$$

The equations of motion for the fields themselves will be simple conservation laws along the direction of motion of the fluid ($U^\mu \partial_\mu \alpha = 0$), with the only two non-conserved fields being:

$$\begin{aligned} U^\mu \partial_\mu \phi &= -\mu \\ U^\mu \partial_\mu \theta &= T \end{aligned}$$

2.2 An Effective field theory for Fluids

Perhaps more fundamentally, we can consider Dubovsky et al, 2011 (arXiv:1107.0731). We label the coordinates of a fluid volume element using three fields ϕ_I, where in equilibrium at a fixed reference pressure, $\phi^I = x^I$. The quantity $b = \det(B_{IJ})$, where $B_{IJ} = \partial_\mu \phi_I \partial^\mu \phi_J$ is the unique term that can be written down in an action that preserves all the symmetries a fluid is expected to obey:

- $\phi^I \to \phi^I + a^I$ for constant a^I,
- $\phi^I \to R^I_J \phi^J$ for $R \in SO(3)$, and
- $\phi^I \to \xi^I(\phi)$ where $\det(\partial \xi^I / \partial \phi^J) = 1$.

We note that varying this with respect to a field results in:

$$\frac{\delta b}{\delta \phi_K} = b \left(B^{-1}\right)^{IJ} \frac{\delta B_{IJ}}{\delta \phi_K}$$

where B^{-1} is not a trivial quantity, since $B_{IJ} B^{JK} \neq \delta^K_I$, unlike in the analagous identity used when varying the metric tensor.

Varying a general action

$$\mathcal{S} = \int \mathrm{d}^3 x F(b)$$

with respect to the metric gives the stress-energy tensor:

$$T^{\mu\nu} = -F_{,b} b B^{-1}_{IJ} \partial_\mu \phi^I \partial_\nu \phi^J + F \eta_{\mu\nu}$$

where the fluid energy density is$\epsilon = -F$ and pressure is $p = F - F_{,b} b$.

2.3 Conservation equations, and equivalence to Naiver-Stokes

As $T^{\mu\nu}$ is conserved, we can immediately write down a continuity equation. First we write that, placing any source terms in j^{ν},

$$\begin{aligned} D_\mu T^{\mu\nu} &= j^\nu \\ &= (1+w)D_\mu(\epsilon U^\mu U^\nu) + wD^\nu\epsilon \\ &= (1+w)\left(\epsilon U^\mu D_\mu U^\nu + U^\nu D_\mu(\epsilon U^\mu)\right) + wD^\nu\epsilon \quad , \end{aligned}$$

then contracting with U_ν, we see that

$$\begin{aligned} U_\nu j^\nu &= (1+w)\left(\epsilon U^\mu U_\nu D_\mu U^\nu - D_\mu(\epsilon U^\mu)\right) + wU_\nu D^\nu\epsilon \\ &= wU_\mu D^\mu\epsilon - (1+w)D_\mu(\epsilon U^\mu) \end{aligned}$$

where we have used the identities $U_\nu D_\mu U^\nu = 0$ and $U_\nu U^\nu = -1$. We can then write this in the following form, assuming $w \neq -1$:

$$D_\mu(\epsilon U^\mu) = \left(\frac{1}{1+w}\right)(wU^\mu D_\mu\epsilon - U_\mu j^\mu) \quad . \tag{7}$$

In the case of a non-relativistic fluid consisting of ordinary matter in flat space ($U^i = \gamma\vec{v} \simeq \vec{v}$, $w = 0$, and $D_\mu = \partial_\mu$) this reduces to the normal continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = \sigma$$

where $\sigma = -U_\mu j^\mu$ denotes any fluid sources or sinks.

We can also write down equations for the evolution of the fluid. The degrees of freedom will include the energy density field $\epsilon(x^\mu)$, fluid velocity vector field $U^\mu(x^\mu)$, and any degrees of freedom the source may contain. We can constrain one of the components of U^μ using equation 1, reducing a degree of freedom.

2.4 Evolution equations with no curvature or source

2.4.1 Initial Equations Expanded

Given we are working with first-order equations, to evolve the fluid and energy density, we need to find $\partial_t U^\mu$ and $\partial_t\epsilon$ in terms of U^μ and ϵ and their spatial derivatives only. We will denote spatial components, as is typical, with latin indices.

We can first start by attempting to eliminate one degree of freedom from consideration, so we only need to evolve spatial components of U. We start with $U^\mu\partial_\nu U_\mu = 0$, and taking the case when $\nu = t$, we can write down that

$$\partial_t U^t = \frac{U^i}{U^t}\partial_t U_i \quad .$$

This will help elimiate time derivatives of U^t from our equations later. We also note that because $U^\mu U_\mu = -1$, we have that

$$U^t = \sqrt{1 + U^i U^i}$$

In the sourceless case with no curvature, we have that the continuity equation is simply

$$\partial_\mu(\epsilon U^\mu) = \frac{w}{1+w}U^\mu\partial_\mu\epsilon \quad . \tag{8}$$

We also have the conservation equation

$$\epsilon U^\mu\partial_\mu U^\nu + U^\nu\partial_\mu(\epsilon U^\mu) + \frac{w}{1+w}\partial^\nu\epsilon = 0 \tag{9}$$

which becomes, when considering a spatial component ($\nu = i$),

$$\epsilon U^\mu\partial_\mu U^i + U^i\partial_\mu(\epsilon U^\mu) = -\frac{w}{1+w}\partial^i\epsilon \quad .$$

Expanding this and working through to obtain an equation for $\partial_t U^i$, we have that

$$\partial_t U^i = -\frac{1}{U^t}\left(\frac{w}{1+w}\frac{\partial^i \epsilon}{\epsilon} + U^j\partial_j U^i + \frac{U^i}{\epsilon}\partial_\mu(\epsilon U^\mu)\right) \ ,$$

and substituting in the continuity equation to simplify and eliminate the $\partial_\mu(\epsilon U^\mu)$ term,

$$\begin{aligned}\partial_t U^i &= -\frac{1}{U^t}\left(\frac{w}{1+w}\frac{\partial^i \epsilon}{\epsilon} + U^j\partial_j U^i + U^i\frac{w}{1+w}U^\mu\frac{\partial_\mu \epsilon}{\epsilon}\right) \\ &= -\frac{1}{U^t}\frac{w}{1+w}\left(U^iU^t\partial_t \ln(\epsilon) + U^iU^j\partial_j \ln(\epsilon) + \partial^i \ln(\epsilon)\right) - \frac{U^j}{U^t}\partial_j U^i\end{aligned}$$

Here we still have an expression for $\partial_t U^i$ that depends on time derivatives of ϵ. To eliminate these, we can expand the continuity equation and solve for $\partial_t \epsilon$ as follows.

$$\begin{aligned}\partial_\mu(\epsilon U^\mu) &= \frac{w}{1+w}U^\mu\partial_\mu\epsilon \\ \partial_i(\epsilon U^i) + U^t\partial_t\epsilon + \epsilon\partial_t U^t &= \frac{w}{1+w}\left(U^t\partial_t\epsilon + U^i\partial_i\epsilon\right) \\ \frac{1}{1+w}\left(U^t\partial_t\epsilon\right) &= \frac{w}{1+w}U^i\partial_i\epsilon - \partial_i(\epsilon U^i) - \epsilon\frac{U^i}{U^t}\partial_t U_i \\ \partial_t\epsilon &= \frac{1+w}{U^t}\left(\frac{w}{1+w}U^i\partial_i\epsilon - \partial_i(\epsilon U^i) - \epsilon\frac{U^i}{U^t}\partial_t U_i\right)\end{aligned}$$

Plugging this back in, we have that

$$\begin{aligned}\partial_t U^i &= \frac{-1}{U^t}\frac{w}{1+w}\left(U^i\frac{1+w}{\epsilon}\left(\frac{w}{1+w}U^j\partial_j\epsilon - \partial_j(\epsilon U^j) - \epsilon\frac{U^j}{U^t}\partial_t U_j\right) + U^iU^j\partial_j \ln(\epsilon) + \partial^i \ln(\epsilon)\right) - \frac{U^j}{U^t}\partial_j U^i \\ &= \frac{-1}{U^t}\left(\frac{w}{1+w}\partial^i \ln(\epsilon) - U^iw\left(\partial_j U^j + \frac{U^j}{U^t}\partial_t U_j\right) + U^j\partial_j U^i\right) \\ &= \frac{-1}{U^t}\left(\frac{w}{1+w}\partial^i \ln(\epsilon) - U^iw\left(\partial_j U^j\right) + U^j\partial_j U^i\right) + \frac{wU^iU^j}{(U^t)^2}\partial_t U_j\end{aligned}$$

or,

$$\partial_t U^i - \frac{wU^iU^j}{(U^t)^2}\partial_t U_j = \frac{1}{U^t}\left(wU^i\partial_j U^j - U^j\partial_j U^i - \frac{w}{1+w}\partial^i \ln(\epsilon)\right) \tag{10}$$

This gives us an equation we can, in principle, evolve. In 1+1 dimensions, the time derivative of the spatial component is easily separated. When working in more than one spatial dimension, the time derivatives $\partial_t U^i$ are still coupled with the fluid velocity U^i. To solve for the time derivatives, the above equation can be written as:

$$(\delta^i_j - M^i_j(U^k))\partial_t U^j = B^i(U^k, \epsilon) \ ,$$

where M^i_j is a matrix (which is a function of U^k) serving to mix spatial components of U^i, and B^i is just a vector function of U^k and ϵ. It is only needed to invert the matrix $\delta^i_j - M^i_j$ on LHS, and we have solved for any component $\partial_t U^j$. Thus we have that:

$$\partial_t U^j = \left[(\delta - M)^{-1}\right]^j_i B^i \ .$$

We can explicitly invert this matrix by considering the form an inverse should take. We have that:

$$\left(\delta^i_j - \frac{wU^iU_j}{(U^t)^2}\right)\left(\delta^j_k + \lambda U^jU_k\right) = \delta^i_k + \left(\lambda - \frac{w}{(U^t)^2} - \lambda\frac{wU^jU_j}{(U^t)^2}\right)U^iU_k \ ,$$

for some parameter λ. For the second term on the LHS to be an inverse matrix, the RHS must only be an identity matrix, so the following should be true:

$$\left(\lambda - \frac{w}{(U^t)^2} - \lambda\frac{wU^jU_j}{(U^t)^2}\right) = 0 \ ,$$

or

$$\lambda = \frac{w}{(1+(1-w)U^jU_j)} \quad .$$

So we find that

$$\left[(\delta - M)^{-1}\right]^i_j = \delta^i_j + \frac{w}{(1+(1-w)U^kU_k)}U^iU_j \quad . \tag{11}$$

We can now write down an explicit expression for $\partial_t U^i$:

$$\begin{aligned}
\partial_t U^i &= \frac{1}{U^t}\left(\delta^i_j + \frac{w}{(1+(1-w)U^kU_k)}U^iU_j\right)\left(wU^j\partial_k U^k - U^k\partial_k U^j - \frac{w}{1+w}\partial^j \ln(\epsilon)\right) \\
&= \frac{1}{U^t}\left(\left(\frac{1+U^kU_k}{1+(1-w)U^kU_k}\right)wU^i\partial_k U^k - \left(\delta^i_j + \frac{wU^iU_j}{(1+(1-w)U^kU_k)}\right)\left(U^k\partial_k U^j + \frac{w}{1+w}\partial^j \ln(\epsilon)\right)\right) \quad ,
\end{aligned}$$

so

$$\partial_t U^i = \frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[\partial_k U^k - \frac{U_j}{(U^t)^2}\left(U^k\partial_k U^j + \frac{w}{1+w}\frac{\partial^j\epsilon}{\epsilon}\right)\right] - \frac{1}{U^t}\left(U^k\partial_k U^i + \frac{w}{1+w}\frac{\partial^i\epsilon}{\epsilon}\right) \tag{12}$$

It now remains to find $\partial_t\epsilon$ independent of $\partial_t U^i$:

$$\begin{aligned}
\partial_t\epsilon &= \frac{1+w}{U^t}\left(\frac{w}{1+w}U^i\partial_i\epsilon - \partial_i(\epsilon U^i)\right) \\
&\quad -\frac{1+w}{U^t}\epsilon\frac{U^i}{U^t}\left(\frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[\partial_k U^k - \frac{U_j}{(U^t)^2}\left(U^k\partial_k U^j + \frac{w}{1+w}\frac{\partial^j\epsilon}{\epsilon}\right)\right] - \frac{1}{U^t}\left(U^k\partial_k U^i + \frac{w}{1+w}\frac{\partial^i\epsilon}{\epsilon}\right)\right) \quad ,
\end{aligned}$$

or simplifying, we have that:

$$\partial_t\epsilon = -\frac{(1+w)U^t}{(1+(1-w)U^kU_k)}\left(\frac{1-w}{1+w}U^i\partial_i\epsilon + \epsilon\partial_i U^i - \epsilon\frac{U_jU^k}{(U^t)^2}\partial_k U^j\right) \quad . \tag{13}$$

Letting $u^i = \gamma v^i$ and $U^t = \gamma$ (so $\partial_\mu u^i = \gamma^3\partial_\mu v^i$), we have that the energy evolution equation reduces to a generalized conservation equation:

$$\partial_t\epsilon = -\frac{1+w}{1-wv^iv_i}\left(\frac{1-w}{1+w}v^i\partial_i\epsilon + \epsilon\gamma^2\left(\partial_i v^i - v_jv^k\partial_k v^j\right)\right)$$

2.4.2 Evolution equations in 1+1

The most simple case, we can write down evolution equations for U^i and ϵ. We start by noting that there will only be one independent component of U^i to worry about, which we will notate as $U^\mu = (U^t, U^x) = (\sqrt{1+u^2}, u)$. The one component of $(\delta - M)^{-1}$ is $(1 - w\frac{u^2}{1+u^2})^{-1} = \frac{(U^t)^2}{1+u^2(1-w)}$. Taking the single spatial component of U^μ and using eq. 10 (simpler than 12) we can write down that

$$\partial_t u = \left(\frac{U^t}{1+u^2(1-w)}\right)\left((w-1)u\partial_x u - \frac{w}{1+w}\frac{1}{\epsilon}\partial_x\epsilon\right) \quad .$$

We can also write down how the energy density evolves using 13:

$$\partial_t\epsilon = -\frac{(1+w)U^t}{(1+(1-w)u^2)}\left(\frac{1-w}{1+w}u\partial_x\epsilon + \frac{\epsilon}{1+u^2}\partial_x u\right) \quad .$$

For the simple case when $w = 0$, we have the following evolution equations:

$$\begin{aligned}
\partial_t u &= -\frac{u}{\sqrt{1+u^2}}\partial_x u \\
\partial_t\epsilon &= -\frac{1}{\sqrt{1+u^2}}\left(u\partial_x\epsilon + \epsilon\frac{1}{1+u^2}\partial_x u\right)
\end{aligned}$$

Which, in terms of velocity v (where $u^i = \gamma v^i$ and $\partial_\mu u^i = \gamma^3 \partial_\mu v^i$):

$$\begin{aligned}\partial_t v &= v\partial_x v \\ \partial_t \epsilon &= -\partial_x (v\epsilon)\end{aligned}$$

The first equation here is a simple form of the Navier-Stokes equation, the Inviscid Burgers' Equation, and the second one is simply the continuity equation. We note that the relativistic form reduces exactly to the non-relativistic form, a consequence of setting $w = 0$. In an ultrarelativistic $w = 1/3$ large γ fluid, we have that the evolution equations become:

$$\begin{aligned}\partial_t u &= -\left(\frac{3U^t}{3+2u^2}\right)\left(\frac{2}{3}u\partial_x u + \frac{1}{4\epsilon}\partial_x \epsilon\right) \\ \partial_t \epsilon &= -\frac{4U^t}{(3+2u^2)}\left(\frac{1}{2}u\partial_x \epsilon + \frac{\epsilon}{1+u^2}\partial_x u\right)\end{aligned}$$

or in terms of v, we have that

$$\begin{aligned}\partial_t v &= ... \\ \partial_t \epsilon &= ...\end{aligned}$$

2.5 Evolution Equations including a Source Term and Curvature

Here, we additionally want to include a source term and curvature in our calculations. Similar to the Minkowski case, we have that:

$$\begin{aligned}D_t U^t &= \frac{U^i}{U^t} D_t U_i \\ D_i U^t &= -\frac{U_j}{U_t} D_i U^j\end{aligned}$$

We also still have that our continuity equation holds:

$$D_\mu(\epsilon U^\mu) = \frac{w}{1+w} U^\mu D_\mu \epsilon - \frac{1}{1+w} U_\mu j^\mu \ ,$$

which we can write as:

$$\epsilon D_\mu U^\mu = -\frac{1}{1+w} U^\mu \left(D_\mu \epsilon + j_\mu\right) \ .$$

As before, we can expand this to get:

$$\epsilon(D_t U^t + D_i U^i) = -\frac{1}{1+w}\left(U^i D_i \epsilon + U^\mu j_\mu\right) - \frac{1}{1+w} U^t D_t \epsilon$$

$$\Rightarrow D_t \epsilon = -\frac{1+w}{U^t}\left[\frac{1}{1+w} U^i D_i \epsilon + \epsilon D_i U^i + \epsilon \frac{U^i}{U^t} D_t U_i + \frac{1}{1+w} U^\mu j_\mu\right]$$

Which is the same as before in the case that $j_\mu = 0$. We also have from conservation of $T^{\mu\nu}$ that

$$j^\nu = (1+w)\left(\epsilon U^\mu D_\mu U^\nu + U^\nu D_\mu(\epsilon U^\mu)\right) + w D^\nu \epsilon$$

Taking $\nu = i$ as before and combining equations, we have that:

$$\frac{j^i}{(1+w)} = \epsilon U^t D_t U^i + \epsilon U^j D_j U^i + U^i \left(\frac{w}{1+w} U^\mu D_\mu \epsilon - \frac{1}{1+w} U_\mu j^\mu\right) + \frac{w}{(1+w)} D^i \epsilon \ ,$$

or that

$$D_t U^i = -\frac{1}{\epsilon U^t}\frac{w}{1+w}\left[U^i U^t D_t \epsilon + U^i U^j D_j \epsilon + D^i \epsilon\right] + \frac{1}{\epsilon U^t}\frac{1}{1+w}\left[j^i + U^i U_\mu j^\mu\right] - \frac{U^j}{U^t} D_j U^i$$

Which is again the same as before if $j^\mu = 0$. We again need to remove the $D_t\epsilon$, which after subbing in, leaves us with:

$$\begin{aligned} D_t U^i &= \frac{w}{\epsilon U^t}\left[\epsilon U^i D_j U^j + \epsilon U^i \frac{U^j}{U^t} D_t U_j - \frac{1}{1+w} D^i\epsilon + \frac{j^i}{w(1+w)} + \frac{1}{w} U^i U^\mu j_\mu\right] - \frac{U^j}{U^t} D_j U^i \\ \left(\delta^i_j - \frac{wU^iU_j}{(U^t)^2}\right) D_t U^j &= \frac{1}{U^t}\left[wU^i D_j U^j - U^j D_j U^i - \frac{w}{1+w}\frac{D^i\epsilon}{\epsilon}\right] + \frac{1}{\epsilon U^t}\left(\frac{j^i}{1+w} + U^i U^\mu j_\mu\right) \ . \end{aligned}$$

Which is the same as before, but with the source term. We can look at how the matrix inverse acts on this term, and add it on to our earlier work. It doesn't simplify much, so we'll just write it down and hide it in another term.

$$J^i = \frac{1}{\epsilon U^t}\left(\delta^i_j + \frac{w}{(1+(1-w)U^kU_k)} U^i U_j\right)\left(\frac{j^j}{1+w} + U^j U^\mu j_\mu\right)$$

We can now just add this in to (need to covariantize) eq. 12:

$$D_t U^i = \frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[D_k U^k - \frac{U_j}{(U^t)^2}\left(U^k D_k U^j + \frac{w}{1+w}\frac{D^j\epsilon}{\epsilon}\right)\right] - \frac{1}{U^t}\left(U^k D_k U^i + \frac{w}{1+w}\frac{D^i\epsilon}{\epsilon}\right) + J^i \ .$$

or looking at how the log of energy density evolves (and j is replaced by j/ϵ),

$$D_t U^i = \frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[D_k U^k - \frac{U_j}{(U^t)^2}\left(U^k D_k U^j + \frac{w}{1+w} D^j \ln(\epsilon)\right)\right] - \frac{1}{U^t}\left(U^k D_k U^i + \frac{w}{1+w} D^i \ln(\epsilon)\right) + J^i$$

We then have that the energy density evolution equation will pick up an extra factor of

$$-(1+w)\frac{U^iJ_i}{(U^t)^2} - \frac{1}{U^t} U^\mu j_\mu \ ,$$

so (need to actually covariantize... certainly missing some metric factors here):

$$D_t\epsilon = -\frac{(1+w)U^t}{(1+(1-w)U^kU_k)}\left(\frac{1-w}{1+w} U^i D_i\epsilon + \epsilon\partial_i U^i + \epsilon\frac{U_jU^k}{(U^t)^2} D_k U^j\right) - (1+w)\,\epsilon\frac{U^iJ_i}{(U^t)^2} - \frac{1}{U^t} U^\mu j_\mu \ .$$

(or evolving the log:)

$$D_t \ln(\epsilon) = -\frac{(1+w)U^t}{(1+(1-w)U^kU_k)}\left(\frac{1-w}{1+w} U^i D_i \ln(\epsilon) + \partial_i U^i + \frac{U_jU^k}{(U^t)^2} D_k U^j\right) - (1+w)\frac{U^iJ_i}{(U^t)^2} - \frac{1}{U^t} U^\mu \frac{j_\mu}{e^{\ln(\epsilon)}} \ .$$

—Verified above here, need to verify below equs (solved in Mathematica for those... consistency check)

2.5.1 Equations in 1+1 with source but no curvature

We can as before write this down in 1+1, but now with a source term. We start by writing $J^i = J$. We have that $j^\mu = (j^o, j)$, so:

$$J = \frac{U^t}{\epsilon}\frac{1}{(1+(1-w)u^2)}\left(\frac{j}{1+w} + u^2 j - uU^t j^t\right) \ .$$

So then, we have that:

$$\partial_t u = \frac{U^t}{1+u^2(1-w)}\left[(w-1)\,u\partial_x u - \frac{w}{1+w}\frac{\partial_x\epsilon}{\epsilon}\right] + J$$

and

$$\partial_t \epsilon = -\frac{U^t}{(1+(1-w)u^2)}\left((1-w)u\partial_x\epsilon + \frac{(1+w)}{1+u^2}\epsilon\partial_x u + \left(\frac{1}{U^t} - 2U^t\right)j^t + 2uj\right) \quad .$$

Again for the simple case $w=0$, we have that:

$$J = \frac{u^2(1-U^t)}{U^t}\frac{j}{\epsilon}$$

$$\begin{aligned}\partial_t u &= -\frac{u}{\sqrt{1+u^2}}\partial_x u + J \\ \partial_t \epsilon &= -\frac{1}{\sqrt{1+u^2}}\left(u\partial_x\epsilon + \epsilon\left(\frac{1}{1+u^2}\right)\partial_x u\right) - \frac{u}{(U^t)^2}J - \frac{uj}{U^t} + j^0\end{aligned}$$

3 Scalar field source

3.1 A Phenomenological Dissipative Coupling

Here we consider a basic scalar field theory we want to include in our source.

$$\begin{aligned}S = S_{free} + S_v &= \int \mathrm{d}^4x\left[-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right] \\ &= \int \mathrm{d}^4x\left[\tfrac{1}{2}\phi\Box\phi + V(\phi)\right]\end{aligned}$$

so that

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \Rightarrow \Box\phi + V'(\phi) = 0$$

for the field. For EWSB, a coupling term between the field and fluid is not introduced into an action in the literature, but rather comes in as a friction term connecting applied to the separate equations of motion for the field and fluid. Friction terms are in general non-local, as they generally aren't seen in closed systems. We can write down that$\partial_\mu T^{\mu\nu}_{fluid} = \delta^\nu$ and $\Box\phi + V'(\phi) = \delta$, and following (ref...?) pick the model-independent, phenomonological frictional term $\delta^\nu = \frac{1}{\gamma\Gamma}U^\mu\partial_\mu\phi\partial^\nu\phi$, and $\delta = \frac{1}{\gamma\Gamma}U^\mu\partial_\mu\phi$, which gives us a set of coupled evolution equations for ϕ and U^μ.

4 Equations with Viscosity/Shear

4.1 Stress-Energy Tensor

We start by noting the form the stress-energy tensor will take on the form:

$$T^{\mu\nu} = pg^{\mu\nu} + (e+p)U^\mu U^\nu + \tau^{\mu\nu} \quad ,$$

where $\tau^{\mu\nu}$ will contain all information about viscosity. It must take the form (Weinberg, LL, etc):

$$\begin{aligned}\tau^{\mu\nu} &= -2\eta\left(D^{(\mu}U^{\nu)} - U^\alpha(D_\alpha U^{(\mu})U^{\nu)}\right) - \left(\zeta - \frac{2}{3}\eta\right)(D_\alpha U^\alpha)(g^{\mu\nu} + U^\mu U^\nu) \\ &= -\eta\left(2D^{(\mu}U^{\nu)} - U^\alpha D_\alpha(U^\mu U^\nu)\right) - \left(\zeta - \frac{2}{3}\eta\right)(D_\alpha U^\alpha)(g^{\mu\nu} + U^\mu U^\nu)\end{aligned}$$

To relate this to our continuity equation, we look at the divergence of τ:

$$\begin{aligned}D_\mu\tau^{\mu\nu} &= -\eta\left(D_\mu D^{(\mu}U^{\nu)} + (D_\mu U^\alpha)(D_\alpha U^{(\mu})U^{\nu)} + U^\alpha D_\mu(D_\alpha U^{(\mu})U^{\nu)}\right) \\ &\quad -\left(\zeta - \frac{2}{3}\eta\right)\left[(D_\mu D_\alpha U^\alpha)(g^{\mu\nu} + U^\mu U^\nu) + (D_\alpha U^\alpha)(g^{\mu\nu} + D_\mu(U^\mu U^\nu))\right] \\ &= \ldots\end{aligned}$$

5 Gravitational Wave Generation

For now, just some formulae.

Synchronous gauge (linearized). Here we haven't so far considered expansion of space, but h_{ij} can perhaps even contain $a(t)$ information if $a(t) \simeq 1 + \mathcal{O}(h \times a(t))$.

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j$$

From this, after finding the Riemann tensor, etc, then finding the Einstein equations to first order in h and picking the transverse/traceless gauge choice

$$h^i_i = \partial_i h^{ij} = 0$$

(or equivalently $\tilde{h}^i_i = k_i \tilde{h}^{ij} = 0$), it follows that the perturbations will evolve as $\Box h_{ij} = S^{TT}_{ij}$where S^{TT}_{ij} is the transverse, traceless part of the spatial part of the stress-energy tensor. The traceless stress-energy tensor is found by taking the trace of the Einstein equations and replacing the Ricci scalar with the trace of the stress-energy tensor (gives the $1/3$ factor), or:

$$S^T_{ij} = T_{ij} - \frac{1}{3}\delta_{ij} T^k_k.$$

Projecting out the transverse part of this is done by the operator $\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}$. Finally, in fourier space then, we have that

$$\int \frac{\mathrm{d^3k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\left(-\frac{d^2}{dt^2} + \vec{k^2} \right) \tilde{h}_{ij}(\vec{k},t) - \tilde{S}^{TT}_{ij}(\vec{k},t) \right] = 0$$

which is solvable as an ODE, though it requires that the stress-energy tensor be transformed each iteration. The projection operator is no longer nonlocal, becoming $\delta_{ij} - \frac{k_i k_j}{k^2}$, so the above can be instead written as

$$\left(-\frac{d^2}{dt^2} + \vec{k^2} \right) \tilde{h}_{ij}(\vec{k},t) - \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \left(\tilde{T}_{ij} - \frac{1}{3}\delta_{ij} \tilde{T}^k_k \right) = 0.$$

Converting this to a power spectrum is done looking at the 0-0 component of the stress-energy tensor $T^{00}_{gw} = \rho_{gw} = \frac{m^2_{pl}}{32\pi} \left\langle \dot{h}^2_{ij} \right\rangle$, which is converted into momentum space via parsevals theorem, and manipulating some more, we have that:

$$\Omega_{gw,0} h^2 = \Omega_{r,0} h^2 \left(\frac{g_0}{g_*} \right)^{1/3} \frac{1}{\rho_{tot,e}} \frac{d\rho_{gw,e}}{d\ln k},$$

where

$$\frac{d\rho_{gw,e}}{d\ln k} = \frac{m^2_{pl} k^3}{32\pi} \frac{1}{V} \sum_{i,j} \int \mathrm{d}\Omega \left| \dot{\tilde{h}}^{TT}_{ij}(\vec{k},t) \right|^2.$$

There are a few potential optimizations we can employ here. The transverse-traceless gauge conditions mean in principle only 2 of the components of h_{ij} need to be evolved, and the rest are simply rational functions of each other in momentum space. It is possible only calculating two components results in a loss of stability, although each component is evolved independently, so this possibility should be mitigated. There is also a tradeoff in computation time between calculating the different components of h from S^{TT} and from other components of h, but the relative difficulty of computation doesn't seem significantly different, perhaps a bit in favor of not evolving using S^{TT}. But the factor of 3 reduction in storage space is instead taken advantage of.

For simplicity, numerical work is also performed using Hartley transforms rather than dealing with complex/fourier transforms.