Contents

1 Initial Equations

We start out by noting we will be working in a $(-1,1,1,1)$ signature. We also note that any fluid should behave in a time-like manner is used as well:

$$U^\mu U_\mu = -1 \tag{1}$$

or, taking the derivative of both sides,

$$U^\mu D_\nu U_\mu = 0 \tag{2}$$

where we will use a capital D to denote a covariant derivative. We also note that our stress-energy tensor will be conserved modulo a source term,

$$D_\mu T^{\mu\nu} = j^\nu \quad , \tag{3}$$

an equation from which we will derive equations used to evolve the fluid. We will also be considering fluids with an equation of state $p = w\epsilon$, with w a parameter determined by the physical situation we wish to consider.

2 Equations with No Shear or Viscosity

2.1 A standard thermodynamic derivation of the stress energy tensor

For this special case, we only have degrees of freedom coming from the fluid velocityU^μ , in addition to it's associated energy density ϵ and pressure p. Following Schutz (1970), we can write down an action for the special case of a fluid at rest:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_{fluid}\right) \quad . \tag{4}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2\kappa} + p\right) \tag{5}$$

Here, p is a function of the normalized inertial mass $\mu \equiv \frac{\epsilon+p}{\epsilon_0}$ (where ϵ is the density of the fluid relative to a background density ϵ_0) and metric $g^{\mu\nu}$. We also have a relationship between dp and $d\mu$: $dp = \epsilon_0 d\mu - \epsilon_0 T dS$, where S is the entropy of the fluid and T is the temperature. Keeping entropy constant, we have that $\frac{\partial p}{\partial \mu} = \epsilon_0 = \frac{\epsilon+p}{\mu}$. Lastly, we have that the fluid velocity can be written as a function of mass and other fundamental quantities (typically scalar fields and derivatives thereof, but these are not of direct interest so we hide then in our fluid momentum P^ν): $U^\nu = \frac{1}{\mu}P^\nu$. From normalization of U^ν, we have that $\mu^2 = g^{\mu\nu}P_\mu P_\nu$.

Varying the action with respect to the metric gives us:

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta\sqrt{-g}p}{\delta g^{\mu\nu}} = pg^{\mu\nu} + -2\frac{\delta p}{\delta g^{\mu\nu}} \quad ,$$

where

$$\frac{\delta p}{\delta g^{\mu\nu}} = \frac{\partial p}{\partial \mu}\frac{\delta \mu}{\delta g^{\mu\nu}} = -\frac{1}{2}\frac{\rho+p}{\mu^2}P_\mu P_\nu \quad ,$$

so

$$T^{\mu\nu} = (\epsilon+p)U^\mu U^\nu + pg^{\mu\nu} = (1+w)\epsilon U^\mu U^\nu + w\epsilon g^{\mu\nu} \quad . \tag{6}$$

We note that P_μ can be written in terms of scalar fields $\{\phi, \theta, \alpha, \beta\}$ and entropy S:

$$P_\mu = \phi_{,\mu} + \alpha\beta_{,\mu} + \theta S_{,\mu} \quad .$$

The equations of motion for the fields themselves will be simple conservation laws along the direction of motion of the fluid ($U^\mu \partial_\mu \alpha = 0$), with the only two non-conserved fields being:

$$U^\mu \partial_\mu \phi = -\mu$$
$$U^\mu \partial_\mu \theta = T$$

2.2 An Effective field theory for Fluids

Perhaps more fundamentally, we can consider Dubovsky et al, 2011 (arXiv:1107.0731). We label the coordinates of a fluid volume element using three fields ϕ_I, where in equilibrium at a fixed reference pressure, $\phi^I = x^I$. The quantity $b = \det(B_{IJ})$, where $B_{IJ} = \partial_\mu \phi_I \partial^\mu \phi_J$ is the unique term that can be written down in an action that preserves all the symmetries a fluid is expected to obey:

- $\phi^I \to \phi^I + a^I$ for constant a^I,
- $\phi^I \to R^I_J \phi^J$ for $R \in SO(3)$, and
- $\phi^I \to \xi^I(\phi)$ where $\det(\partial\xi^I/\partial\phi^J) = 1$.

We note that varying this with respect to a field results in:

$$\frac{\delta b}{\delta \phi_K} = b\left(B^{-1}\right)^{IJ}\frac{\delta B_{IJ}}{\delta \phi_K}$$

where B^{-1} is not a trivial quantity, since $B_{IJ}B^{JK} \neq \delta^K_I$, unlike in the analagous identity used when varying the metric tensor.

Varying a general action

$$\mathcal{S} = \int \mathrm{d}^3x F(b)$$

with respect to the metric gives the stress-energy tensor:

$$T^{\mu\nu} = -F_{,b} b B^{-1}_{IJ} \partial_\mu \phi^I \partial_\nu \phi^J + F\eta_{\mu\nu}$$

where the fluid energy density is$\epsilon = -F$ and pressure is $p = F - F_{,b}b$.

2.3 Conservation equations, and equivalence to Naiver-Stokes

As $T^{\mu\nu}$ is conserved, we can immediately write down a continuity equation. First we write that, placing any source terms in j^ν,

$$\begin{aligned} D_\mu T^{\mu\nu} &= j^\nu \\ &= (1+w) D_\mu (\epsilon U^\mu U^\nu) + w D^\nu \epsilon \\ &= (1+w)\left(\epsilon U^\mu D_\mu U^\nu + U^\nu D_\mu(\epsilon U^\mu)\right) + w D^\nu \epsilon \quad , \end{aligned}$$

then contracting with U_ν, we see that

$$\begin{aligned} U_\nu j^\nu &= (1+w)\left(\epsilon U^\mu U_\nu D_\mu U^\nu - D_\mu(\epsilon U^\mu)\right) + w U_\nu D^\nu \epsilon \\ &= w U_\mu D^\mu \epsilon - (1+w) D_\mu (\epsilon U^\mu) \end{aligned}$$

where we have used the identities $U_\nu D_\mu U^\nu = 0$ and $U_\nu U^\nu = -1$. We can then write this in the following form, assuming $w \neq -1$:

$$D_\mu(\epsilon U^\mu) = \left(\frac{1}{1+w}\right)\left(w U^\mu D_\mu \epsilon - U_\mu j^\mu\right) \quad . \tag{7}$$

In the case of a non-relativistic fluid consisting of ordinary matter in flat space ($U^i = \gamma \vec{v} \simeq \vec{v}$, $w = 0$, and $D_\mu = \partial_\mu$) this reduces to the normal continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = \sigma$$

where $\sigma = -U_\mu j^\mu$ denotes any fluid sources or sinks.

We can also write down equations for the evolution of the fluid. The degrees of freedom will include the energy density field $\epsilon(x^\mu)$, fluid velocity vector field $U^\mu(x^\mu)$, and any degrees of freedom the source may contain. We can constrain one of the components of U^μ using equation 1, reducing a degree of freedom.

2.4 Evolution equations with no curvature or source

2.4.1 Initial Equations Expanded

Given we are working with first-order equations, to evolve the fluid and energy density, we need to find $\partial_t U^\mu$ and $\partial_t \epsilon$ in terms of U^μ and ϵ and their spatial derivatives only. We will denote spatial components, as is typical, with latin indices.

We can first start by attempting to eliminate one degree of freedom from consideration, so we only need to evolve spatial components of U. We start with $U^\mu \partial_\nu U_\mu = 0$, and taking the case when $\nu = t$, we can write down that

$$\partial_t U^t = \frac{U^i}{U^t} \partial_t U_i \quad .$$

This will help elimiate time derivatives of U^t from our equations later. We also note that because $U^\mu U_\mu = -1$, we have that

$$U^t = \sqrt{1 + U^i U^i}$$

In the sourceless case with no curvature, we have that the continuity equation is simply

$$\partial_\mu (\epsilon U^\mu) = \frac{w}{1+w} U^\mu \partial_\mu \epsilon \quad . \tag{8}$$

We also have the conservation equation

$$\epsilon U^\mu \partial_\mu U^\nu + U^\nu \partial_\mu(\epsilon U^\mu) + \frac{w}{1+w}\partial^\nu \epsilon = 0 \tag{9}$$

which becomes, when considering a spatial component ($\nu = i$),

$$\epsilon U^\mu \partial_\mu U^i + U^i \partial_\mu(\epsilon U^\mu) = -\frac{w}{1+w}\partial^i \epsilon \ .$$

Expanding this and working through to obtain an equation for $\partial_t U^i$, we have that

$$\partial_t U^i = -\frac{1}{U^t}\left(\frac{w}{1+w}\frac{\partial^i \epsilon}{\epsilon} + U^j \partial_j U^i + \frac{U^i}{\epsilon}\partial_\mu(\epsilon U^\mu)\right) \ ,$$

and substituting in the continuity equation to simplify and eliminate the $\partial_\mu(\epsilon U^\mu)$ term,

$$\begin{aligned}
\partial_t U^i &= -\frac{1}{U^t}\left(\frac{w}{1+w}\frac{\partial^i \epsilon}{\epsilon} + U^j \partial_j U^i + U^i \frac{w}{1+w} U^\mu \frac{\partial_\mu \epsilon}{\epsilon}\right) \\
&= -\frac{1}{U^t}\frac{w}{1+w}\left(U^i U^t \partial_t \ln(\epsilon) + U^i U^j \partial_j \ln(\epsilon) + \partial^i \ln(\epsilon)\right) - \frac{U^j}{U^t}\partial_j U^i
\end{aligned}$$

Here we still have an expression for $\partial_t U^i$ that depends on time derivatives of ϵ. To eliminate these, we can expand the continuity equation and solve for $\partial_t \epsilon$ as follows.

$$\begin{aligned}
\partial_\mu(\epsilon U^\mu) &= \frac{w}{1+w} U^\mu \partial_\mu \epsilon \\
\partial_i(\epsilon U^i) + U^t \partial_t \epsilon + \epsilon \partial_t U^t &= \frac{w}{1+w}\left(U^t \partial_t \epsilon + U^i \partial_i \epsilon\right) \\
\frac{1}{1+w}\left(U^t \partial_t \epsilon\right) &= \frac{w}{1+w} U^i \partial_i \epsilon - \partial_i(\epsilon U^i) - \epsilon \frac{U^i}{U^t}\partial_t U_i \\
\partial_t \epsilon &= \frac{1+w}{U^t}\left(\frac{w}{1+w} U^i \partial_i \epsilon - \partial_i(\epsilon U^i) - \epsilon \frac{U^i}{U^t}\partial_t U_i\right)
\end{aligned}$$

Plugging this back in, we have that

$$\begin{aligned}
\partial_t U^i &= \frac{-1}{U^t}\frac{w}{1+w}\left(U^i \frac{1+w}{\epsilon}\left(\frac{w}{1+w} U^j \partial_j \epsilon - \partial_j(\epsilon U^j) - \epsilon \frac{U^j}{U^t}\partial_t U_j\right) + U^i U^j \partial_j \ln(\epsilon) + \partial^i \ln(\epsilon)\right) - \frac{U^j}{U^t}\partial_j U^i \\
&= \frac{-1}{U^t}\left(\frac{w}{1+w}\partial^i \ln(\epsilon) - U^i w\left(\partial_j U^j + \frac{U^j}{U^t}\partial_t U_j\right) + U^j \partial_j U^i\right) \\
&= \frac{-1}{U^t}\left(\frac{w}{1+w}\partial^i \ln(\epsilon) - U^i w\left(\partial_j U^j\right) + U^j \partial_j U^i\right) + \frac{w U^i U^j}{(U^t)^2}\partial_t U_j
\end{aligned}$$

or,

$$\partial_t U^i - \frac{w U^i U^j}{(U^t)^2}\partial_t U_j = \frac{1}{U^t}\left(w U^i \partial_j U^j - U^j \partial_j U^i - \frac{w}{1+w}\partial^i \ln(\epsilon)\right) \tag{10}$$

This gives us an equation we can, in principle, evolve. In 1+1 dimensions, the time derivative of the spatial component is easily separated. When working in more than one spatial dimension, the time derivatives $\partial_t U^i$ are still coupled with the fluid velocity U^i. To solve for the time derivatives, the above equation can be written as:

$$(\delta^i_j - M^i_j(U^k))\partial_t U^j = B^i(U^k, \epsilon) \ ,$$

where M^i_j is a matrix (which is a function of U^k) serving to mix spatial components of U^i, and B^i is just a vector function of U^k and ϵ. It is only needed to invert the matrix $\delta^i_j - M^i_j$ on LHS, and we have solved for any component $\partial_t U^j$. Thus we have that:

$$\partial_t U^j = \left[(\delta - M)^{-1}\right]^j_i B^i \ .$$

We can explicitly invert this matrix by considering the form an inverse should take. We have that:

$$\left(\delta^i_j - \frac{wU^iU_j}{(U^t)^2}\right)\left(\delta^j_k + \lambda U^jU_k\right) = \delta^i_k + \left(\lambda - \frac{w}{(U^t)^2} - \lambda\frac{wU^jU_j}{(U^t)^2}\right)U^iU_k \ ,$$

for some parameter λ. For the second term on the LHS to be an inverse matrix, the RHS must only be an identity matrix, so the following should be true:

$$\left(\lambda - \frac{w}{(U^t)^2} - \lambda\frac{wU^jU_j}{(U^t)^2}\right) = 0 \ ,$$

or

$$\lambda = \frac{w}{(1+(1-w)U^jU_j)} \ .$$

So we find that

$$\left[(\delta - M)^{-1}\right]^i_j = \delta^i_j + \frac{w}{(1+(1-w)U^kU_k)}U^iU_j \ . \tag{11}$$

We can now write down an explicit expression for $\partial_t U^i$:

$$\begin{aligned}
\partial_t U^i &= \frac{1}{U^t}\left(\delta^i_j + \frac{w}{(1+(1-w)U^kU_k)}U^iU_j\right)\left(wU^j\partial_kU^k - U^k\partial_kU^j - \frac{w}{1+w}\partial^j\ln(\epsilon)\right) \\
&= \frac{1}{U^t}\left(\left(\frac{1+U^kU_k}{1+(1-w)U^kU_k}\right)wU^i\partial_kU^k - \left(\delta^i_j + \frac{wU^iU_j}{(1+(1-w)U^kU_k)}\right)\left(U^k\partial_kU^j + \frac{w}{1+w}\partial^j\ln(\epsilon)\right)\right) \ ,
\end{aligned}$$

so

$$\partial_t U^i = \frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[\partial_kU^k - \frac{U_j}{(U^t)^2}\left(U^k\partial_kU^j + \frac{w}{1+w}\frac{\partial^j\epsilon}{\epsilon}\right)\right] - \frac{1}{U^t}\left(U^k\partial_kU^i + \frac{w}{1+w}\frac{\partial^i\epsilon}{\epsilon}\right) \tag{12}$$

It now remains to find $\partial_t\epsilon$ independent of $\partial_t U^i$:

$$\begin{aligned}
\partial_t\epsilon &= \frac{1+w}{U^t}\left(\frac{w}{1+w}U^i\partial_i\epsilon - \partial_i(\epsilon U^i)\right) \\
&\quad - \frac{1+w}{U^t}\epsilon\frac{U^i}{U^t}\left(\frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[\partial_kU^k - \frac{U_j}{(U^t)^2}\left(U^k\partial_kU^j + \frac{w}{1+w}\frac{\partial^j\epsilon}{\epsilon}\right)\right] - \frac{1}{U^t}\left(U^k\partial_kU^i + \frac{w}{1+w}\frac{\partial^i\epsilon}{\epsilon}\right)\right) \ ,
\end{aligned}$$

or simplifying, we have that:

$$\partial_t\epsilon = -\frac{(1+w)U^t}{(1+(1-w)U^kU_k)}\left(\frac{1-w}{1+w}U^i\partial_i\epsilon + \epsilon\partial_iU^i - \epsilon\frac{U_jU^k}{(U^t)^2}\partial_kU^j\right) \ . \tag{13}$$

Letting $u^i = \gamma v^i$ and $U^t = \gamma$ (so $\partial_\mu u^i = \gamma^3\partial_\mu v^i$), we have that the energy evolution equation reduces to a generalized conservation equation:

$$\partial_t\epsilon = -\frac{1+w}{1-wv^iv_i}\left(\frac{1-w}{1+w}v^i\partial_i\epsilon + \epsilon\gamma^2\left(\partial_iv^i - v_jv^k\partial_kv^j\right)\right)$$

2.4.2 Evolution equations in 1+1

The most simple case, we can write down evolution equations for U^i and ϵ. We start by noting that there will only be one independent component of U^i to worry about, which we will notate as $U^\mu = (U^t, U^x) = (\sqrt{1+u^2}, u)$. The one component of $(\delta - M)^{-1}$ is $(1 - w\frac{u^2}{1+u^2})^{-1} = \frac{(U^t)^2}{1+u^2(1-w)}$. Taking the single spatial component of U^μ and using eq. 10 (simpler than 12) we can write down that

$$\partial_t u = \left(\frac{U^t}{1+u^2(1-w)}\right)\left((w-1)u\partial_xu - \frac{w}{1+w}\frac{1}{\epsilon}\partial_x\epsilon\right) \ .$$

We can also write down how the energy density evolves using 13:

$$\partial_t\epsilon = -\frac{(1+w)U^t}{(1+(1-w)u^2)}\left(\frac{1-w}{1+w}u\partial_x\epsilon + \frac{\epsilon}{1+u^2}\partial_x u\right) \quad .$$

For the simple case when $w=0$, we have the following evolution equations:

$$\begin{aligned}
\partial_t u &= -\frac{u}{\sqrt{1+u^2}}\partial_x u \\
\partial_t\epsilon &= -\frac{1}{\sqrt{1+u^2}}\left(u\partial_x\epsilon + \epsilon\frac{1}{1+u^2}\partial_x u\right)
\end{aligned}$$

Which, in terms of velocity v (where $u^i = \gamma v^i$ and $\partial_\mu u^i = \gamma^3\partial_\mu v^i$):

$$\begin{aligned}
\partial_t v &= v\partial_x v \\
\partial_t\epsilon &= -\partial_x(v\epsilon)
\end{aligned}$$

The first equation here is a simple form of the Navier-Stokes equation, the Inviscid Burgers' Equation, and the second one is simply the continuity equation. We note that the relativistic form reduces exactly to the non-relativistic form, a consequence of setting $w=0$. In an ultrarelativistic $w=1/3$ large γ fluid, we have that the evolution equations become:

$$\begin{aligned}
\partial_t u &= -\left(\frac{3U^t}{3+2u^2}\right)\left(\frac{2}{3}u\partial_x u + \frac{1}{4\epsilon}\partial_x\epsilon\right) \\
\partial_t\epsilon &= -\frac{4U^t}{(3+2u^2)}\left(\frac{1}{2}u\partial_x\epsilon + \frac{\epsilon}{1+u^2}\partial_x u\right)
\end{aligned}$$

or in terms of v, we have that

$$\begin{aligned}
\partial_t v &= \ldots \\
\partial_t\epsilon &= \ldots
\end{aligned}$$

2.5 Evolution Equations including a Source Term and Curvature

Here, we additionally want to include a source term and curvature in our calculations. Similar to the Minkowski case, we have that:

$$\begin{aligned}
D_t U^t &= \frac{U^i}{U^t}D_t U_i \\
D_i U^t &= -\frac{U_j}{U_t}D_i U^j
\end{aligned}$$

We also still have that our continuity equation holds:

$$D_\mu(\epsilon U^\mu) = \frac{w}{1+w}U^\mu D_\mu\epsilon - \frac{1}{1+w}U_\mu j^\mu \quad ,$$

which we can write as:

$$\epsilon D_\mu U^\mu = -\frac{1}{1+w}U^\mu\left(D_\mu\epsilon + j_\mu\right) \quad .$$

As before, we can expand this to get:

$$\epsilon(D_t U^t + D_i U^i) = -\frac{1}{1+w}\left(U^i D_i\epsilon + U^\mu j_\mu\right) - \frac{1}{1+w}U^t D_t\epsilon$$

$$\Rightarrow D_t\epsilon \;=\; -\frac{1+w}{U^t}\left[\frac{1}{1+w}U^i D_i\epsilon + \epsilon D_i U^i + \epsilon\frac{U^i}{U^t}D_t U_i + \frac{1}{1+w}U^\mu j_\mu\right]$$

Which is the same as before in the case that $j_\mu = 0$. We also have from conservation of $T^{\mu\nu}$ that

$$j^\nu = (1+w)\left(\epsilon U^\mu D_\mu U^\nu + U^\nu D_\mu(\epsilon U^\mu)\right) + wD^\nu\epsilon$$

Taking $\nu = i$ as before and combining equations, we have that:

$$\frac{j^i}{(1+w)} \quad = \quad \epsilon U^t D_t U^i + \epsilon U^j D_j U^i + U^i\left(\frac{w}{1+w}U^\mu D_\mu\epsilon - \frac{1}{1+w}U_\mu j^\mu\right) + \frac{w}{(1+w)}D^i\epsilon \ ,$$

or that

$$D_t U^i = -\frac{1}{\epsilon U^t}\frac{w}{1+w}\left[U^iU^tD_t\epsilon + U^iU^jD_j\epsilon + D^i\epsilon\right] + \frac{1}{\epsilon U^t}\frac{1}{1+w}\left[j^i + U^iU_\mu j^\mu\right] - \frac{U^j}{U^t}D_jU^i$$

Which is again the same as before if $j^\mu = 0$. We again need to remove the $D_t\epsilon$, which after subbing in, leaves us with:

$$\begin{aligned}
D_tU^i &= \frac{w}{\epsilon U^t}\left[\epsilon U^iD_jU^j + \epsilon U^i\frac{U^j}{U^t}D_tU_j - \frac{1}{1+w}D^i\epsilon + \frac{j^i}{w(1+w)} + \frac{1}{w}U^iU^\mu j_\mu\right] - \frac{U^j}{U^t}D_jU^i \\
\left(\delta^i_j - \frac{wU^iU_j}{(U^t)^2}\right)D_tU^j &= \frac{1}{U^t}\left[wU^iD_jU^j - U^jD_jU^i - \frac{w}{1+w}\frac{D^i\epsilon}{\epsilon}\right] + \frac{1}{\epsilon U^t}\left(\frac{j^i}{1+w} + U^iU^\mu j_\mu\right) \ .
\end{aligned}$$

Which is the same as before, but with the source term. We can look at how the matrix inverse acts on this term, and add it on to our earlier work. It doesn't simplify much, so we'll just write it down and hide it in another term.

$$J^i = \frac{1}{\epsilon U^t}\left(\delta^i_j + \frac{w}{(1+(1-w)U^kU_k)}U^iU_j\right)\left(\frac{j^j}{1+w} + U^jU^\mu j_\mu\right)$$

We can now just add this in to (need to covariantize) eq. 12:

$$D_tU^i = \frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[D_kU^k - \frac{U_j}{(U^t)^2}\left(U^kD_kU^j + \frac{w}{1+w}\frac{D^j\epsilon}{\epsilon}\right)\right] - \frac{1}{U^t}\left(U^kD_kU^i + \frac{w}{1+w}\frac{D^i\epsilon}{\epsilon}\right) + J^i \ .$$

or looking at how the log of energy density evolves (and j is replaced by j/ϵ),

$$D_tU^i = \frac{wU^tU^i}{(1+(1-w)U^kU_k)}\left[D_kU^k - \frac{U_j}{(U^t)^2}\left(U^kD_kU^j + \frac{w}{1+w}D^j\ln(\epsilon)\right)\right] - \frac{1}{U^t}\left(U^kD_kU^i + \frac{w}{1+w}D^i\ln(\epsilon)\right) + J^i$$

We then have that the energy density evolution equation will pick up an extra factor of

$$-(1+w)\frac{U^iJ_i}{(U^t)^2} - \frac{1}{U^t}U^\mu j_\mu \ ,$$

so (need to actually covariantize... certainly missing some metric factors here):

$$D_t\epsilon = -\frac{(1+w)U^t}{(1+(1-w)U^kU_k)}\left(\frac{1-w}{1+w}U^iD_i\epsilon + \epsilon\partial_iU^i - \epsilon\frac{U_jU^k}{(U^t)^2}D_kU^j\right) - (1+w)\,\epsilon\frac{U^iJ_i}{(U^t)^2} - \frac{1}{U^t}U^\mu j_\mu \ .$$

(or evolving the log:)

$$D_t\ln(\epsilon) = -\frac{(1+w)U^t}{(1+(1-w)U^kU_k)}\left(\frac{1-w}{1+w}U^iD_i\ln(\epsilon) + \partial_iU^i - \frac{U_jU^k}{(U^t)^2}D_kU^j\right) - (1+w)\frac{U^iJ_i}{(U^t)^2} - \frac{1}{U^t}U^\mu\frac{j_\mu}{e^{\ln(\epsilon)}} \ .$$

—Verified above here up to the missing metric factors, need to verify below equs (solved in Mathematica for those above as a consistency check)

2.5.1 Equations in 1+1 with source but no curvature

We can as before write this down in 1+1, but now with a source term. We start by writing $J^i = J$. We have that $j^\mu = (j^o, j)$, so:

$$J = \frac{U^t}{\epsilon}\frac{1}{(1+(1-w)u^2)}\left(\frac{j}{1+w} + u^2 j - uU^t j^t\right) \quad .$$

So then, we have that:

$$\partial_t u = \frac{U^t}{1+u^2(1-w)}\left[(w-1)\,u\partial_x u - \frac{w}{1+w}\frac{\partial_x \epsilon}{\epsilon}\right] + J$$

and

$$\partial_t \epsilon = -\frac{U^t}{(1+(1-w)u^2)}\left((1-w)u\partial_x\epsilon + \frac{(1+w)}{1+u^2}\epsilon\partial_x u + \left(\frac{1}{U^t} - 2U^t\right)j^t + 2uj\right) \quad .$$

Again for the simple case $w = 0$, we have that:

$$J = \frac{u^2(1-U^t)}{U^t}\frac{j}{\epsilon}$$

$$\begin{aligned}
\partial_t u &= -\frac{u}{\sqrt{1+u^2}}\partial_x u + J \\
\partial_t \epsilon &= -\frac{1}{\sqrt{1+u^2}}\left(u\partial_x\epsilon + \epsilon\left(\frac{1}{1+u^2}\right)\partial_x u\right) - \frac{u}{(U^t)^2}J - \frac{uj}{U^t} + j^0
\end{aligned}$$

3 Scalar field source

3.1 A Phenomenological Dissipative Coupling

Here we consider a basic scalar field theory we want to include in our source.

$$\begin{aligned}
S = S_{free} + S_v &= \int \mathrm{d}^4x\left[-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \\
&= \int \mathrm{d}^4x\left[\tfrac{1}{2}\phi\Box\phi - V(\phi)\right]
\end{aligned}$$

so that

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \Rightarrow \Box\phi - V'(\phi) = 0$$

for the field. A coupling term between the field and fluid is not introduced into an action in the literature, but rather comes in as a friction term applied to the separate equations of motion for the field and fluid. Friction terms at the level of the action are in general non-local, as they come from a system that isn't closed; nevertheless we can pick an interaction term at the level of the equations of motion with the correct phenomenology.

We can write down that $\partial_\mu T^{\mu\nu}_{fluid} = \delta^\nu$ and $\Box\phi + V'(\phi) = \delta$, and following (ref...?) pick the phenomonological frictional term $\delta^\nu = \eta U^\mu\partial_\mu\phi\partial^\nu\phi$, and $\delta = \eta U^\mu\partial_\mu\phi$, which gives us a set of coupled evolution equations for ϕ and U^μ.

4 Equations with Viscosity/Shear

4.1 Stress-Energy Tensor

We start by noting the form the stress-energy tensor will take on the form:

$$T^{\mu\nu} = pg^{\mu\nu} + (e+p)U^\mu U^\nu + \tau^{\mu\nu} \quad ,$$

where $\tau^{\mu\nu}$ will contain all information about viscosity. It must take the form (Weinberg, LL, etc):

$$\begin{aligned}\tau^{\mu\nu} &= -2\eta\left(D^{(\mu}U^{\nu)} - U^{\alpha}(D_{\alpha}U^{(\mu})U^{\nu)}\right) - \left(\zeta - \frac{2}{3}\eta\right)(D_{\alpha}U^{\alpha})\left(g^{\mu\nu} + U^{\mu}U^{\nu}\right) \\ &= -\eta\left(2D^{(\mu}U^{\nu)} - U^{\alpha}D_{\alpha}\left(U^{\mu}U^{\nu}\right)\right) - \left(\zeta - \frac{2}{3}\eta\right)(D_{\alpha}U^{\alpha})\left(g^{\mu\nu} + U^{\mu}U^{\nu}\right)\end{aligned}$$

To relate this to our continuity equation, we look at the divergence of τ:

$$\begin{aligned}D_{\mu}\tau^{\mu\nu} &= -\eta\left(D_{\mu}D^{(\mu}U^{\nu)} + (D_{\mu}U^{\alpha})(D_{\alpha}U^{(\mu})U^{\nu)} + U^{\alpha}D_{\mu}(D_{\alpha}U^{(\mu})U^{\nu)}\right) \\ &\quad -\left(\zeta - \frac{2}{3}\eta\right)\left[(D_{\mu}D_{\alpha}U^{\alpha})\left(g^{\mu\nu} + U^{\mu}U^{\nu}\right) + (D_{\alpha}U^{\alpha})\left(g^{\mu\nu} + D_{\mu}(U^{\mu}U^{\nu})\right)\right] \\ &= \ldots\end{aligned}$$

5 The Symmetry-Breaking potential

5.1 Motivation / Origin

From coupling the fluid to a scalar field as in section (3.1), we can pick a generic potential $V_{eff}(\phi) = \frac{m^2}{2}\phi^2 + \eta\phi^3 + \frac{\lambda}{8}\phi^4$. This originates from considering thermal and quantum corrections to a field in a symmetry-breaking potential. Consider a field ϕ, which we can decompose into an average field and fluctuations, $\phi = \bar{\phi} + \delta\phi$, where $\langle\delta\phi\rangle = 0$. If the field is sitting in a general potential $V(\phi)$, and we expand the equations of motion:

$$\Box\bar{\phi} = V'(\bar{\phi}) + \frac{1}{2}V'''(\bar{\phi})\left\langle\delta\phi^2\right\rangle$$

and the fluctuations satisfy (to lowest order):

$$\Box\delta\phi = V''(\bar{\phi})\delta\phi \equiv m^2_{\delta\phi}\delta\phi \ .$$

We can decompose the solution of the above into plane waves with coefficients $a_{\mathbf{k}}$ and dispersion relation $\omega_k = \sqrt{k^2 + m^2_{\delta\phi}}$:

$$\delta\phi = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(e^{-\imath\omega t + \imath\mathbf{kx}}a^-_{\mathbf{k}} + e^{\imath\omega t - \imath\mathbf{kx}}a^+_{\mathbf{k}}\right)$$

so looking at the two-point function gives:

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle &= \int \frac{d^3k d^3k'}{(2\pi)^3}\frac{1}{2\sqrt{\omega_{k'}\omega_k}}\left\langle a^+_{\mathbf{k}}a^-_{\mathbf{k}} + a^-_{\mathbf{k}}a^+_{\mathbf{k}}\right\rangle \\ &= \int \frac{d^3k d^3k'}{(2\pi)^3}\frac{1}{2\sqrt{\omega_{k'}\omega_k}}\left\langle 2a^+_{\mathbf{k}}a^-_{\mathbf{k}} + \left[a^-_{\mathbf{k}}, a^+_{\mathbf{k}}\right]\right\rangle\end{aligned}$$

But since $\left\langle a^+_{\mathbf{k}}a^-_{\mathbf{k}}\right\rangle = n_{\mathbf{k}}\delta(\mathbf{k} - \mathbf{k}')$ and $\left\langle\left[a^-_{\mathbf{k}}, a^+_{\mathbf{k}}\right]\right\rangle = \delta(\mathbf{k} - \mathbf{k}')$, we have that:

$$\begin{aligned}\left\langle\delta\phi^2\right\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\left(n_{\mathbf{k}} + \frac{1}{2}\right) \\ &= \frac{1}{2\pi^2}\int \frac{k^2 dk}{\sqrt{k^2 + m^2_{\delta\phi}}}\left(n_k + \frac{1}{2}\right) \ .\end{aligned}$$

We can then take n_k to be a thermal Bose-Einstein distribution (appropriate for a high-temperature, false-vacuum state with neglected chemical potential):

$$\left\langle\delta\phi^2\right\rangle = \frac{1}{2\pi^2}\int \frac{k^2 dk}{\omega_k(e^{\omega_k T} - 1)} \ ,$$

and defining the last term in the EOM for $\bar{\phi}$ using this, we can define:

$$\frac{1}{2}V'''(\bar{\phi})\left\langle\delta\phi^2\right\rangle = \frac{\partial V_{\delta\phi}(\bar{\phi},T)}{\partial\bar{\phi}} \ .$$

So it ends up that the total/effective potential for $\bar{\phi}$ canbe written as:

$$\begin{aligned} V_{eff} &= V(\bar{\phi}) + V_{\delta\phi}(\bar{\phi},T) \\ &= V(\bar{\phi}) + \frac{m_{\delta\phi}^4}{64\pi^2}\ln\frac{m_{\delta\phi}^2}{\mu^2} + \frac{T^4}{\pi^2}\int_0^{m_{\delta\phi}/T} d\alpha\alpha\int_\alpha^\infty dx\frac{\left(x^2-\alpha^2\right)^{1/2}}{e^x-1} \end{aligned}$$

For a renormalizable quartic potential $V(\bar{\phi}) = \Lambda + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$, the effective potential can be (after renormalization, regularization, and some maths), written down. The 1-loop low-temperature potential appears as (also, assuming $\lambda \gg m^2$):

$$V_{eff} = \Lambda_R + \frac{m_R^2}{2}\bar{\phi}^2 + \frac{\lambda_R}{4}\bar{\phi}^4 + \frac{(3\lambda\bar{\phi})^4}{32\pi^2}\ln\frac{\bar{\phi}}{\bar{\phi}_0} \ ,$$

where ϕ_0 is the renormalization scale, and the low-temperature requirement says that $0 \sim T \ll m_{\delta\phi}^2$. For high temperatures, $T \gg m_{\delta\phi}^2$, the potential can be written:

$$V_{eff} = \Lambda_R + \frac{m_T^2}{2}\bar{\phi}^2 - T(3\lambda)^3\bar{\phi}^3 + \frac{1}{4}\lambda_T\bar{\phi}^4 \ ,$$

where $\lambda_T = \lambda_R + \frac{(3\lambda)^4}{16\pi^2}(\frac{1}{2} + 2\ln\frac{5.8T}{3\lambda})$ has weak temperature dependence, and $m_T^2 = \frac{(3\lambda)^2}{4}(T^2 - \frac{2V''_{eff}(\bar{\phi}_0)}{(3\lambda)^2} + \frac{3(3\lambda)^2\bar{\phi}_0^2}{4\pi^2}) \equiv \frac{(3\lambda)^2}{4}(T^2 - T_0^2)$. If we treat the temperature in the simulation to be constant, and ignore Λ_R (it does not enter the EOMs), then we have (for a redefined λ and m) the toy potential:

$$V_{eff}(\phi) = \frac{m^2}{2}\phi^2 + \eta\phi^3 + \frac{\lambda}{8}\phi^4 \, .$$

5.2 Rescaling to Dimensionless Units

We can further rescale the effective potential and spacetime coordinates by choosing a field and coordinate system:

$$\psi = \frac{m^2}{2\eta}\phi \ , \ \ \bar{x} = mx$$

which allows us to write the potential in the action and EOM as:

$$\bar{V}(\psi) = \frac{1}{2}\psi^2 + \frac{1}{2}\psi^3 + \frac{\alpha}{8}\psi^4 \ .$$

The EOM for the field thus simply becomes

$$\bar{\Box}\psi = \bar{V}(\psi) \ .$$

To represent a symmetry breaking potential, the parameter $\alpha = \frac{\lambda m^2}{4\eta^2}$ can take on values $\alpha \in (0,1)$, with $\alpha = 1$ being a potential with two degenerate minima, and $\alpha = 0$ being a cubic potential.

5.3 Comparison to the 'original' vacuum decay potential

This can be related to the standard Coleman(-De Luccia?) potential:

$$V_{CDL}(\phi) = \frac{\lambda}{8}(\phi^2 - a^2)^2 - \frac{\epsilon}{2a}(\phi - a)$$

where $\lambda > 0$, $a > 0$, $\epsilon < 0$. Expanding around the false vacuum "ϕ_+" in this model and keeping only terms up to $\mathcal{O}(\epsilon)$ gives the earlier effective potential with the relationships:

$$\begin{aligned} m^2 &= \frac{\lambda}{2}(3\phi_+ - a^2) \\ \eta &= \frac{\lambda}{2}|\phi_+| \\ \alpha &= 1 - \frac{\epsilon}{2\lambda\alpha} \ . \end{aligned}$$

6 Gravitational Wave Generation

Working in the synchronous gauge (linearized). Here we haven't so far considered expansion of space, but h_{ij} can perhaps even contain $a(t)$ information if $a(t) \simeq 1 + \mathcal{O}(h \times a(t))$.

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})\, dx^i dx^j$$

From this, after finding the Riemann tensor, etc, then finding the Einstein equations to first order in h and picking the transverse/traceless part of the metric

$$h^i_i = \partial_i h^{ij} = 0$$

(or equivalently $\tilde{h}^i_i = k_i \tilde{h}^{ij} = 0$), it follows that the perturbations will evolve as $\Box h_{ij} - 2\kappa S^{TT}_{ij} = 0$ where S^{TT}_{ij} is the transverse, traceless part of the spatial part of the stress-energy tensor. The traceless stress-energy tensor is found by taking the trace of the Einstein equations and replacing the Ricci scalar with the trace of the stress-energy tensor (gives a 1/3 factor), or:

$$S^T_{ij} = T_{ij} - \frac{1}{3}\delta_{ij} T^k_k .$$

Projecting out the transverse part of this is done by the operator $\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}$, or in fourier space, we have that

$$\int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\left(\frac{d^2}{dt^2} + \vec{k}^2 \right) \tilde{h}_{ij}(\vec{k}, t) - \tilde{S}^{TT}_{ij}(\vec{k}, t) \right] = 0$$

which is solvable as an ODE, though it requires that the stress-energy tensor be transformed each iteration. The projection operator no longer requires an integration, becoming $\delta_{ij} - \frac{k_i k_j}{k^2}$, so the above can be instead written as

$$\left(\frac{d^2}{dt^2} + \vec{k}^2 \right) \frac{\tilde{h}_{ij}(\vec{k}, t)}{2\kappa} = \left(\delta_{il} - \frac{k_i k_l}{k^2} \right) \left(\delta_{jm} - \frac{k_j k_m}{k^2} \right) \tilde{T}^{lm} \ .$$

Converting this to a power spectrum is done looking at the 0-0 component of the stress-energy tensor $T^{00}_{GW} = \rho_{GW} = \frac{1}{4\kappa} \left\langle \dot{h}^2_{ij} \right\rangle$, which is converted into momentum space via parsevals theorem, and manipulating some more, we have that:

$$\Omega_{GW,0} h^2 = \Omega_{r,0} h^2 \left(\frac{g_0}{g_*} \right)^{1/3} \frac{1}{\rho_{tot,e}} \frac{d\rho_{GW,e}}{d \ln k},$$

where

$$\frac{d\rho_{GW,e}}{d \ln k} = \frac{m^2_{pl} k^3}{32\pi} \frac{1}{V} \sum_{i,j} \int \mathrm{d}\Omega \left| \dot{\tilde{h}}_{ij}(\vec{k}, t) \right|^2 .$$

There are a few potential optimizations we can employ here. The transverse-traceless gauge conditions mean in principle only 2 of the components of h_{ij} need to be evolved, and the rest are simply rational functions of each other in momentum space. It is possible only calculating two components results in a loss of stability, although each component is evolved independently, so this possibility should be mitigated. There is also a tradeoff in computation time between calculating the different components of h from S^{TT} and from other components of h.

For computational simplicity, we evolve the quantity $b_{ij}(\vec{k},t) \equiv \frac{\tilde{h}_{ij}(\vec{k},t)}{2\kappa}$, making the EOM simply:

$$\ddot{b}_{ij} + k^2 b_{ij} = S_{ij}^{TT} \ ,$$

and the relationship (averaging over angles/assuming isotropy):

$$\frac{d\rho_{GW,e}}{d\ln k} = \frac{k^3}{L^3} 4\pi\kappa \left|\dot{b}_{ij}(k,t)\right|^2$$

6.1 Rescaling to Dimensionless Units

Here, as earlier, we have defined the dimensionless coordinate system:

$$\psi = \frac{m^2}{2\eta}\phi \ , \ \ \bar{x} = mx \, .$$

Since All terms in the lagrangian must have the same scaling, the terms $(\partial\psi)^2$, $p \sim \rho$, and $\frac{R}{2\kappa} \sim \frac{\partial\partial g}{2\kappa}$ must all transform the same way. Since $\bar{g} = g$ and $\bar{\partial} = \frac{1}{m}\partial$, it follows that:

$$\bar{\rho} = \frac{4\eta^2}{m^6}\rho, \ \ \bar{R} = R/m^2, \ \ \bar{\kappa} = \frac{m^4}{4\eta^2}\kappa \ .$$

From the preceeding section we also need to require that $k \cdot x = \bar{k} \cdot \bar{x}$, so that $\bar{k} = \frac{1}{m}k$. We can then write down how the gravitational wave spectrum will transform:

$$\frac{d\bar{\rho}_{GW}}{\bar{\rho}_{fl}} = \frac{d\rho_{GW}}{\rho_{fl}} \ , \ \ \frac{d\bar{k}}{\bar{k}} = \frac{dk}{k}$$

so that

$$\frac{1}{\bar{\rho}_{fl}}\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} = \frac{1}{\rho_{fl}}\frac{d\rho_{GW,e}}{d\ln k} \ .$$

Defining the ratio between the dimensionless fluid energy density and the VEV difference in the potential $\beta = \frac{\Delta V(\alpha)}{\bar{\rho}_{fl}}$, we have that:

$$\frac{1}{\rho_{fl}}\frac{d\rho_{GW,e}}{d\ln k} = \frac{\beta}{\Delta V(\alpha)}\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} \ .$$

More rigorously, we can work through the derivation at each step. Beginning with how the stress-energy tensor transforms: $\bar{T}^{\mu\nu} = \frac{4\eta^2}{m^6}T^{\mu\nu}$, we have that the transverse-traceless piece $S_{ij}^{TT} = \Lambda_{ij,lm}T_{lm}$ transforms the same way, since $\Lambda_{ij,lm} = (\delta_{il} + k_i k_l/k^2)(\delta_{jm} + k_j k_m/k^2)$ d oes not transform:

$$\bar{S}_{ij}^{TT}(\bar{x}) = \frac{4\eta^2}{m^6}S_{ij}^{TT}(x) \ .$$

Now looking at how the fourier transform of S behaves (making replacements as per above relationships):

$$\begin{aligned}
\tilde{S}(k) = \int d^3x e^{2\pi\imath k\cdot x}S(x) \ &= \int \frac{d^3\bar{x}}{m^3}e^{2\pi\imath\bar{k}\cdot\bar{x}}\frac{m^6}{4\eta^2}\bar{S}(\bar{x}) \\
&= \frac{m^3}{4\eta^2}\int d^3\bar{x}e^{2\pi\imath\bar{k}\cdot\bar{x}}\bar{S}(\bar{x}) \\
&\equiv \frac{m^3}{4\eta^2}\bar{\tilde{S}}(\bar{k}) \ .
\end{aligned}$$

Then starting with the EOM for h and converting to dimensionless variables, we have:

$$\begin{aligned}
\left(-\frac{d^2}{dt^2} + k^2\right)\tilde{h}_{ij} \ &= \ 2\kappa\tilde{S}_{ij}^{TT} \\
m^2\left(-\frac{d^2}{d\bar{t}^2} + \bar{k}^2\right)\tilde{h}_{ij} \ &= \ 2\frac{4\eta^2}{m^4}\bar{\kappa}\frac{m^3}{4\eta^2}\bar{\tilde{S}}(\bar{k}) \\
\left(-\frac{d^2}{d\bar{t}^2} + \bar{k}^2\right)m^3\tilde{h}_{ij} \ &= \ 2\bar{\kappa}\bar{\tilde{S}}(\bar{k}) \ ,
\end{aligned}$$

suggesting the relationships:

$$\tilde{h} = \frac{1}{m^3}\bar{\tilde{h}}, \;\; h = \bar{h},$$

where the fourier transform of h is effectively picking up an extra factor of m^3 due to the momentum integral. The simple relationship between the (non-fourier-transformed) metric perturbations can just be seen as following from the metric being a dimensionless quantity. The time derivative of $\tilde{h}$ which enters into the gravitational wave spectrum equation can then be calculated:

$$\tilde{l} = \frac{d}{dt}\tilde{h} = \frac{m}{m^3}\frac{d}{d\bar{t}}\bar{\tilde{h}} = \frac{1}{m^2}\bar{\tilde{l}} \; ,$$

so the expression for the gravitational wave spectrum can now be translated between dimensionless and dimensionful expressions:

$$\begin{aligned}
\frac{d\rho_{GW,e}}{d\ln k} &= \frac{k^3}{4\kappa}\frac{4\pi}{V}\left|\tilde{l}_{ij}(k,t)\right|^2 \\
&= \frac{m^4}{4\bar{\kappa}4\eta^2}m^3\bar{k}^3\frac{4\pi m^3}{\bar{V}}\left|\frac{1}{m^2}\bar{\tilde{l}}_{ij}(\bar{k},\bar{t})\right|^2 \\
&= \left(\frac{m^6}{4\eta^2}\right)\frac{\bar{k}^3}{4\bar{\kappa}}\frac{4\pi}{\bar{V}}\left|\bar{\tilde{l}}_{ij}(\bar{k},\bar{t})\right|^2 \\
&= \left(\frac{m^6}{4\eta^2}\right)\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} \; .
\end{aligned}$$

Dividing by the energy density of the fluid, we again have that:

$$\frac{1}{\bar{\rho}_{fl}}\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} = \frac{1}{\rho_{fl}}\frac{d\rho_{GW,e}}{d\ln k} \; ,$$

or

$$\frac{1}{\rho_{fl}}\frac{d\rho_{GW,e}}{d\ln k} = \frac{\beta}{\Delta V(\alpha)}\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} \; .$$

6.2 What to Actually Compute

In the actual simulation, we calculate the quantity b instead of h (and 'a' instead of $l = 2\kappa a$). In the fourier transform, fftw computes:

$$\frac{1}{dx^3}\tilde{S} = \sum_{\vec{n}=0}^{N-1} S(\vec{n})e^{-2\pi i\vec{k}\vec{n}/N} \; ,$$

which is unnormalized. To relate this to the normal/continuous fourier transform, a factor of $dx^3 = (L/N)^3$ must also be added in so $\tilde{S}$ is output. Given the normalized $\tilde{S}$ enters the equations in the code, then the output of the code is for the a, so we have:

$$\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} = \bar{k}^3\frac{4\pi\bar{\kappa}}{\bar{L}^3}\left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2$$

Here, a value must be specified for $\bar{\kappa}$. To some degree this need can be substituted: noting that $\bar{\kappa} = \frac{\rho}{\bar{\rho}}\frac{L^2}{\bar{L}^2}\kappa = \kappa\rho_{fl}\frac{\beta}{\Delta V}\frac{L^2}{\bar{L}^2}$, we can write the above as:

$$\frac{d\bar{\rho}_{GW,e}}{d\ln\bar{k}} = \kappa L^2\rho_{fl}\frac{\beta}{\Delta V}\bar{k}^3\frac{4\pi}{\bar{L}^5}\left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2,$$

so we have that:

$$\frac{d\rho_{GW,e}}{d\ln k} = \kappa L^2\rho_{fl}^2\left(\frac{\beta}{\Delta V(\alpha)}\right)^2\frac{4\pi\bar{k}^3}{\bar{L}^5}\left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2 \; ,$$

or that:

$$\frac{d\Omega_{GW,e}}{d\ln k} = \frac{1}{\rho_{crit}}\frac{d\rho_{GW,e}}{d\ln k} = \frac{1}{3}\left(\frac{\kappa L \rho_{fl}}{H}\right)^2 \left(\frac{\beta}{\Delta V(\alpha)}\right)^2 \frac{4\pi\bar{k}^3}{\bar{L}^5} \left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2 \ .$$

For a spatially flat, radiation-fluid-dominated universe, $H^2 = \kappa\rho/3$, and letting $L = nH^{-1}$, we can write this as:

$$\frac{d\Omega_{GW,e}}{d\ln k} = 3n^2 \left(\frac{\beta}{\Delta V(\alpha)}\right)^2 \frac{4\pi\bar{k}^3}{\bar{L}^5} \left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2 \ .$$

Evolving this to the present day (plugging in a formula from Tom's thesis), we have that:

$$\Omega_{GW,0}h^2 = \Omega_{r,0}h^2 \left(\frac{g_0}{g_*}\right)^{1/3} 3n^2 \left(\frac{\beta}{\Delta V(\alpha)}\right)^2 \frac{4\pi\bar{k}^3}{\bar{L}^5} \left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2$$

which taking $n = 1/2$, $\Omega_{r,0} = 8\cdot 10^{-5}$, and $g_0/g_* = 1/100$ gives:

$$\Omega_{GW,0}h^2 \simeq 8\cdot 10^{-6} \left(\frac{\beta}{\Delta V(\alpha)}\right)^2 \frac{\bar{k}^3}{\bar{L}^5} \left|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})\right|^2 \tag{14}$$

Further setting $\beta = 0.1$, $\bar{k} = 2\pi q/\bar{L}$, and $\bar{L} = s\bar{R}_0 = s/(1-\alpha)$, we have:

$$\Omega_{GW,0} \simeq 2\cdot 10^{-4}\frac{q^3}{s^8}A(\alpha)|\bar{\tilde{a}}(\bar{k},\bar{t})|^2$$

where

$$A(\alpha) = \left(\frac{(1-\alpha)^4}{\Delta V(\alpha)}\right)^2$$

Interestingly, this function has a maximum value. This doesn't necessarily imply that $\Omega_{GW,0}h^2$ has a maximum, since $|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})|$ has some nontrivial implicitα-dependence. The function A rapidly goes to 0 as α goes to 0 or 1, as:

$$\lim_{\alpha\to 0} A(\alpha) = \frac{64}{729}\alpha^6 + \mathcal{O}(\alpha^7)\ ,\ \lim_{\alpha\to 1} A(\alpha) = \frac{1}{4}(\alpha-1)^6 + \mathcal{O}((1-\alpha)^7)\ ,$$

which perhaps suggests $|\bar{\tilde{a}}_{ij}(\bar{k},\bar{t})|$ may diverge at a comparable rate so the GW spectrum amplitude will not be 0 at very large/small α.

Naively, we might try to guess how a behaves. Since the stress energy tensor is (very roughly) proportional to $(\partial\phi)^2+\rho$, and the amplitude of terms will scale as $\left(\frac{\sqrt{9-8\alpha}+3}{2\alpha}\right)^2$ and ΔV respectively, we might guess that $\tilde{a}$ behaves as a linear combination of the two. Thus,

$$\Omega_{GW,0}h^2 \propto \left(c_\phi\left(\frac{\sqrt{9-8\alpha}+3}{2\alpha}\right)^2 + c_{fl}\Delta V\right)^2 \left(\frac{(1-\alpha)^4}{\Delta V}\right)^2 \ .$$

The relative strength of c_ϕ/c_{fl} may determine whether or not a maximum will be present: for $c_\phi/c_{fl} > 3/2$, a maximum will form. Relating this back to β, or getting a more exact form for Ω_{GW} may take a lot of work.