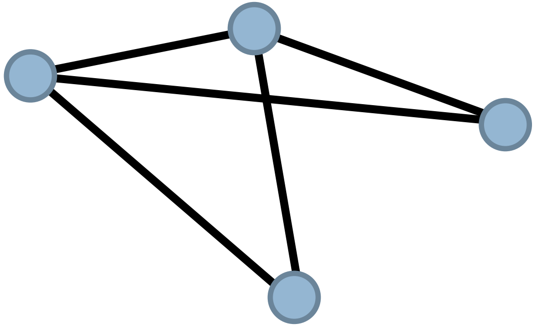
Embedding Planar Graphs

John C. Hart
Department of Computer Science
University of Illinois
at Urbana-Champaign

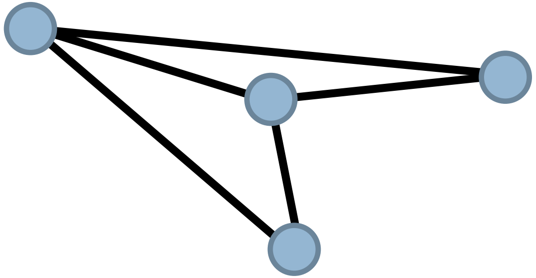
Graphs

non-planar embedding



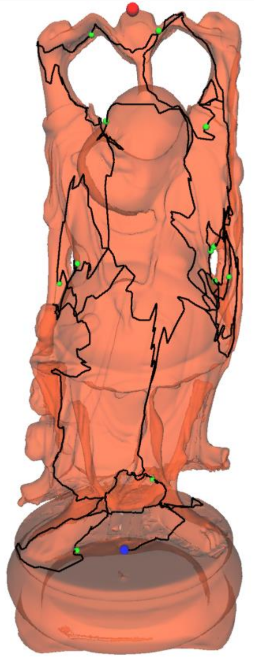
planar embedding

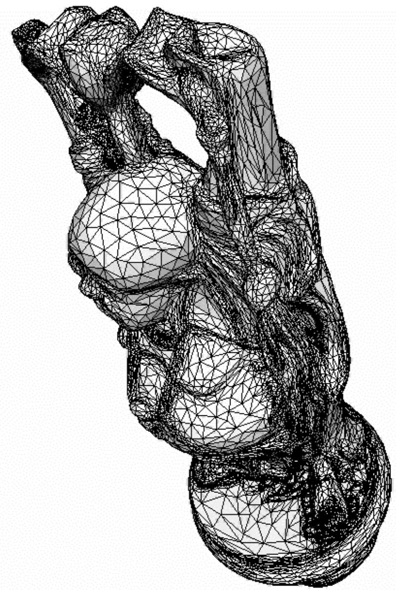
none of the edges crosses each other



Layout of a Large Graph

- “Happy Buddha” 3-D mesh model
- 50,000 nodes
- 100,000 faces
- Can be cut (with six cuts) into a simply connected graph
- Can we find a planar layout of this graph?

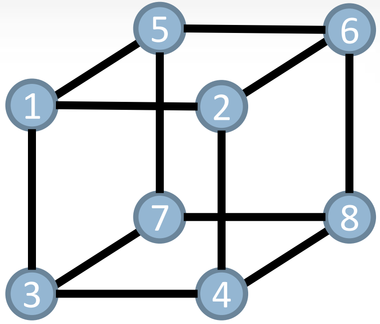


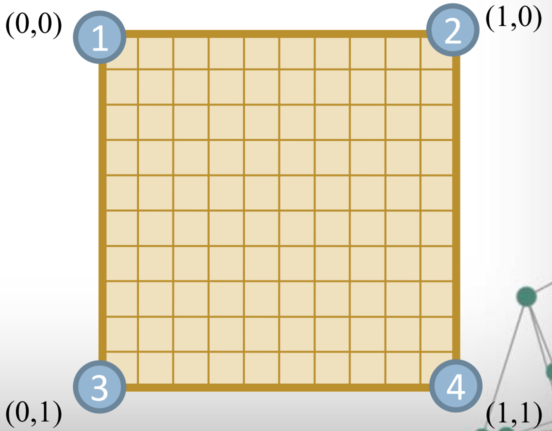


Graph Embedding

Tutte, How to draw a graph, Proc. London Math Soc. 13, 1963, pp. 743-767

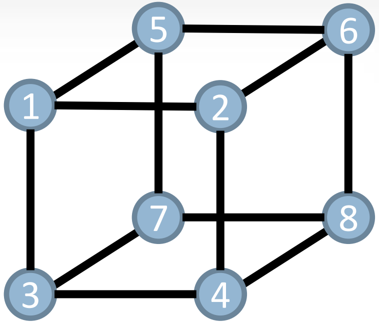
- Input is a graph G consisting of nodes (1,…,N) and edges (i,j) ($1 \leq i < j \leq N$)
- Identify some of the nodes as *boundary nodes* and assign them 2-D positions
- In this example, we have nodes (1,2,3,4,5,6,7,8) and edges (1,2), (1,3), (2,4), (3,4), (1,5), (2,6), (3,7), (4,8), (5,6), (5,7), (6,8) and (7,8)
- Identify nodes 1,2,3 and 4 as boundary nodes, and assign them coordinates (0,0), (1,0), (0,1) and (1,1)





Graph Embedding

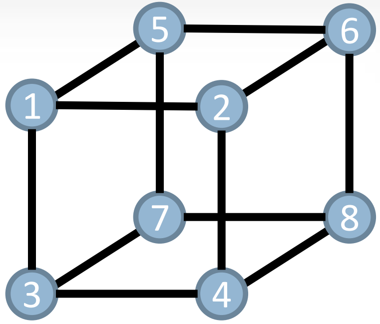
- Create the graph Laplacian matrix
- Adjacency matrix with elements $L_{ij} = 1/\deg(i)$ for an edge between node i and node j



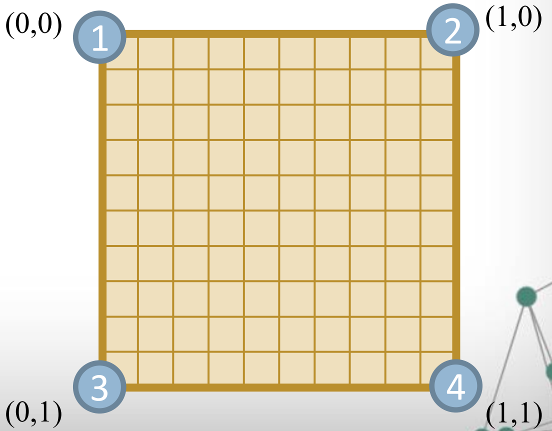
$$L = \begin{bmatrix} 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \end{bmatrix}$$

Graph Embedding

- Zero out the rows for the nodes we have already positioned
- Subtract it from the identity matrix

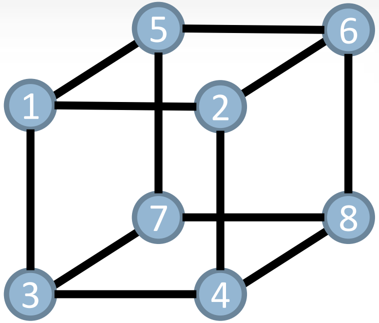


$$A = I - L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -{}^1/_3 & 0 & 0 & 0 & 1 & -{}^1/_3 & -{}^1/_3 & 0 \\ 0 & -{}^1/_3 & 0 & 0 & -{}^1/_3 & 1 & 0 & -{}^1/_3 \\ 0 & 0 & -{}^1/_3 & 0 & -{}^1/_3 & 0 & 1 & -{}^1/_3 \\ 0 & 0 & 0 & -{}^1/_3 & 0 & -{}^1/_3 & -{}^1/_3 & 1 \end{bmatrix}$$

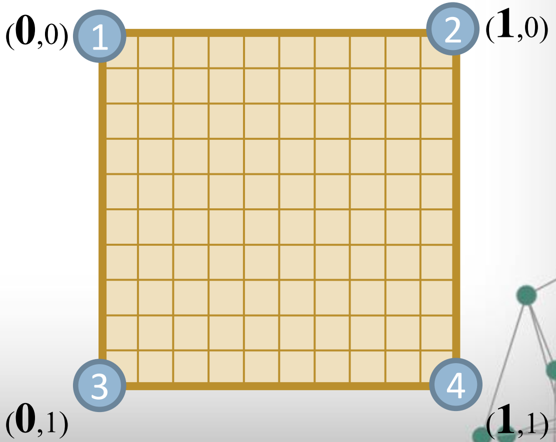


Graph Embedding

- Create linear systems of equations
- Solve $A\mathbf{x} = \mathbf{b}_x$ for the x coordinates

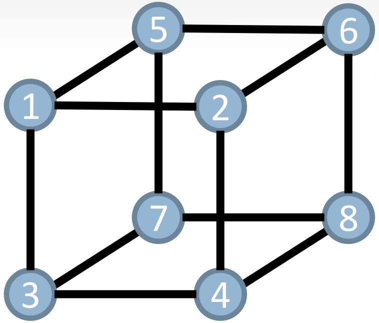


$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1/3 & 0 & 0 & 0 & 1 & -1/3 & -1/3 & 0 \\
0 & -1/3 & 0 & 0 & -1/3 & 1 & 0 & -1/3 \\
0 & 0 & -1/3 & 0 & -1/3 & 0 & 1 & -1/3 \\
0 & 0 & 0 & -1/3 & 0 & -1/3 & -1/3 & 1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

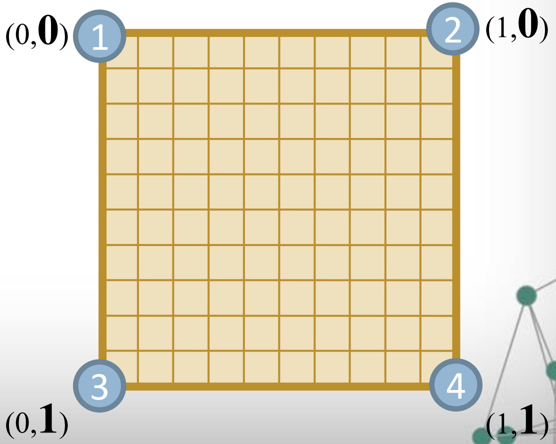


Graph Embedding

- Create linear systems of equations
- Solve $A\mathbf{x} = \mathbf{b}_x$ for the x coordinates
- Solve $A\mathbf{y} = \mathbf{b}_y$ for the y coordinates

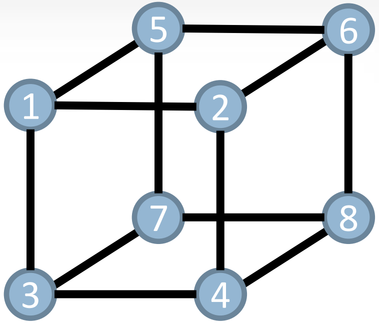


$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/3 & 0 & 0 & 0 & 1 & -1/3 & -1/3 & 0 \\ 0 & -1/3 & 0 & 0 & -1/3 & 1 & 0 & -1/3 \\ 0 & 0 & -1/3 & 0 & -1/3 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & -1/3 & 0 & -1/3 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Graph Embedding

- Create linear systems of equations
- Solve $A\mathbf{x} = \mathbf{b}_x$ for the x coordinates
- Solve $A\mathbf{y} = \mathbf{b}_y$ for the y coordinates



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1/3 & 0 & 0 & 0 & 1 & -1/3 & -1/3 & 0 \\ 0 & -1/3 & 0 & 0 & -1/3 & 1 & 0 & -1/3 \\ 0 & 0 & -1/3 & 0 & -1/3 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & -1/3 & 0 & -1/3 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0, \quad x_2 = 1$$
$$x_3 = 0, \quad x_4 = 1$$
$$x_5 = (x_1 + x_6 + \mathrm{x}_7)/3$$
$$x_6 = (x_2 + x_5 + \mathrm{x}_8)/3$$
$$x_7 = (x_3 + x_5 + \mathrm{x}_8)/3$$
$$x_8 = (x_4 + x_6 + \mathrm{x}_7)/3$$

Graph Embedding

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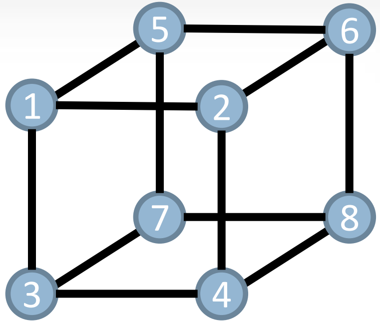
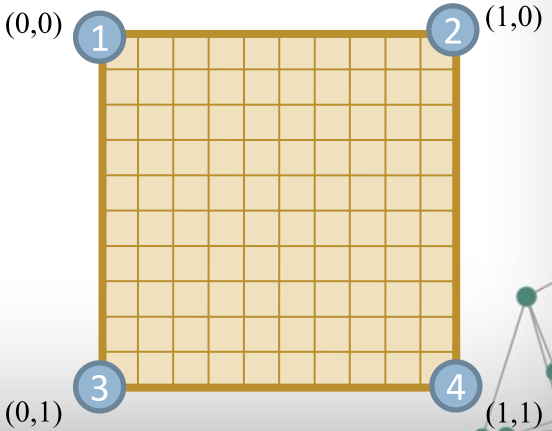


$$
\begin{aligned}
&x_1,y_1 = (0,0), \qquad x_2,y_2 = (1,0)\\
&x_3,y_3 = (0,1), \qquad x_4,y_4 = (1,1)\\
&x_5,y_5 = ((x_1,y_1) + (x_6,y_6) + (x_7,y_7))/3\\
&x_6,y_6 = ((x_2,y_2) + (x_5,y_5) + (x_8,y_8))/3\\
&x_7,y_7 = ((x_3,y_3) + (x_5,y_5) + (x_8,y_8))/3\\
&x_8,y_8 = ((x_4,y_4) + (x_6,y_6) + (x_7,y_7))/3
\end{aligned}
$$



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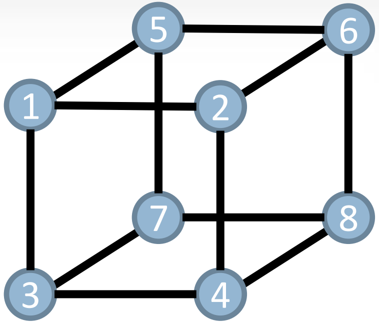

$$
\begin{aligned}
&x_1,y_1 = (0,0), \quad x_2,y_2 = (1,0)\\
&x_3,y_3 = (0,1), \quad x_4,y_4 = (1,1)\\
&x_5,y_5 = ((x_1,y_1) + (x_6,y_6) + (x_7,y_7))/3 \quad = (1/3,1/3)\\
&x_6,y_6 = ((x_2,y_2) + (x_5,y_5) + (x_8,y_8))/3 \quad = (2/3,1/3)\\
&x_7,y_7 = ((x_3,y_3) + (x_5,y_5) + (x_8,y_8))/3 \quad = (1/3,2/3)\\
&x_8,y_8 = ((x_4,y_4) + (x_6,y_6) + (x_7,y_7))/3 \quad = (2/3,2/3)
\end{aligned}
$$