1. Suppose $f(x) = 3^x$.

   (a) (15 points) Use the data

$$f(-1) = \frac{1}{3} \qquad f(0) = 1 \qquad f(1) = 3 \qquad f(2) = 9$$

to construct a cubic Lagrange interpolating polynomial P for f. Express P in the standard form $P(x) = \frac{a_3}{b_3}x^3 + \frac{a_2}{b_2}x^2 + \frac{a_1}{b_1}x + \frac{a_0}{b_0}$ where $a_i, b_i \in \mathbb{Z}$ for $0 \leq i \leq 3$.

Solution: Our unsimplified solution starts as:

$$P(x) = \frac{1}{3} \times \frac{(x)(x-1)(x-2)}{(-1)(-1-1)(-1-2)} + 1 \times \frac{(x+1)(x-1)(x-2)}{(1)(0-1)(0-2)} +$$

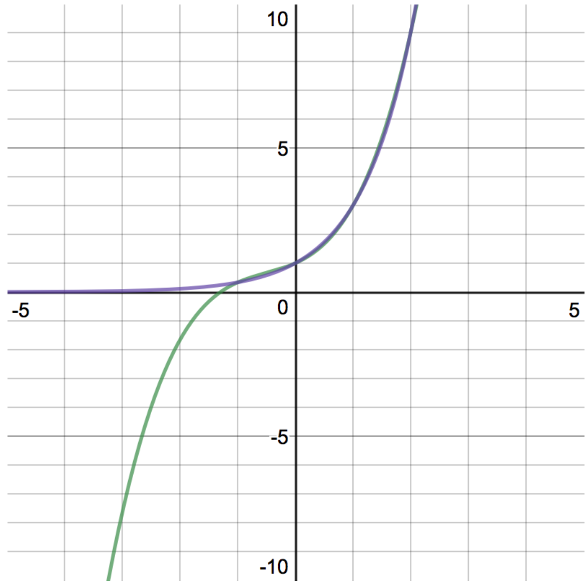
$$3 \times \frac{(x+1)(x)(x-2)}{(1+1)(1)(1-2)} + 9 \times \frac{(x+1)(x)(x-1)}{(2+1)(2)(2-1)}$$

Simplified into stardard form, it looks like:

$$P(x) = \frac{4}{9}x^3 + \frac{2}{3}x^2 + \frac{8}{9}x + 1$$

   (b) (5 points) Use Desmos to produce a graph of both f and P in the window $-10 \leq y \leq 10$, $-5 \leq x \leq 5$ (use the Share Graph feature to export a .png of this graph and embed it in your pdf). Comment qualitatively on the degree to which P fits f on this rectangle.

Solution: It fits well for most of the graph in this rectangle. For $x < -1$ P starts to become a really poor fit for f.



(Purple is f, Green is P)

(c) (5 points) Use $P(1/2)$ to determine a rational approximation of $\sqrt{3}$; express this approximation as a fraction in lowest terms.

Solution:

$$P(\frac{1}{2}) = \frac{4}{9} \times (\frac{1}{2})^3 + \frac{2}{3} \times (\frac{1}{2})^2 + \frac{8}{9} \times \frac{1}{2} + 1$$
$$P(\frac{1}{2}) = \frac{5}{3}$$

(d) (5 points) Using the single precision representation $\sqrt{3} = 1.73205080756887$, estimate the absolute and relative error in approximating $p = \sqrt{3}$ by $p^* = P(1/2)$.

Solution: Absolute error:

$$|p^* - p| = |\frac{5}{3} - 1.73205080756887| = 0.0653841409$$

Relative error:

$$\frac{|p^* - p|}{p} = |\frac{0.0653841409}{1.73205080756887}| = 0.03774955135$$

(e) (10 points) Compare the absolute error from the previous step with the error bound guaranteed by Theorem 3.3.

Solution: Theorem 3.3 guarantees that the remainder term will be

$$\frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Given that our n is 3 and x is $\frac{1}{2}$, we can plug in all of our x values and n:

$$\frac{f^4(\xi(\frac{1}{2}))}{(4)!}(\frac{1}{2}+1)(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)$$

The fourth derivative of 3^x is $ln^4(3)\times 3^x$. Knowing this, we can bound $f^4(\xi(\frac{1}{2})$ by $f^4(2)$, because $f^4(x)$ is an increasing function. Therefore, we get

$$\frac{ln^4(4)\times 3^2}{24}(0.5625),$$

which means that our remainder is guaranteed to be bounded by 0.3073 by Theorem 3.3.

Our absolute error above agrees with that, as it is less than 0.3073.

2. Suppose that $f(x)=\sqrt{4-x^2}$.

   (a) (10 points) Express the Lagrange interpolating polynomial $P(x)$ for $f(x)$ using $x_0=-2$, $x_1=y$, and $x_2=2$, where $y\in(-2,2)$, in the standard form

$$P(x)=a_0+a_1x+\cdots+a_kx^k$$

   (for the appropriate choice of k). (Your answer will involve the variable y).

Solution: We begin by finding the y values of these points, not to be confused with the y variable above:

$$f(x_0)=0,\ f(x_1)=\sqrt{4-y^2},\ f(x_2)=0$$

Now lets list the points:

$$(-2,0),\ (y,\sqrt{4-y^2}),\ (2,0)$$

Now time to write out the LIP, notice how two terms of the sum cancel to 0:

$$P(x)=(\sqrt{4-y^2})\frac{(x+2)(x-2)}{(y+2)(y-2)}$$

And lets simplify:

$$P(x) = \frac{(\sqrt{4-y^2})(x^2-4)}{(y^2-4)}$$

$$P(x) = \frac{(\sqrt{4-y^2})(x^2-4)}{(-1)(4-y^2)}$$

$$P(x) = \frac{(-1)(\sqrt{4-y^2})(x^2-4)}{(\sqrt{4-y^2})(\sqrt{4-y^2})}$$

$$P(x) = \frac{(-1)(x^2-4)}{\sqrt{4-y^2}}$$

$$P(x) = \frac{-x^2+4}{\sqrt{4-y^2}}$$

$$P(x) = \frac{-x^2}{\sqrt{4-y^2}} + \frac{4}{\sqrt{4-y^2}}$$

(b) (10 points) Find the largest value of $y \in (-2, 2)$ for which $P(0) - f(0) = 0.25$.

Solution: Lets start by getting $f(0)$ and $P(0)$.

$$f(0) = 2$$

$$P(0) = \frac{4}{\sqrt{4-y^2}}$$

Now lets plug into the function:

$$\frac{4}{\sqrt{4-y^2}} - 2 = \frac{1}{4}$$

$$\frac{4}{\sqrt{4-y^2}} = \frac{9}{4}$$

$$\frac{16}{9} = \sqrt{4-y^2}$$

$$(\frac{16}{9})^2 = 4 - y^2$$

$$y^2 = 4 - (\frac{16}{9})^2$$

$$y = ^{+}_{-} \sqrt{4 - (\frac{16}{9})^2}$$

We just want the largest y so we take the positive value:

$$y = \sqrt{\frac{68}{81}}$$

$$y \approx 0.9162$$

3. Let $f(x) = |x|$. Suppose that $P(x)$ is the quintic Lagrange interpolating polynomial for f determined using the points

$$(-1, 1), (-0.15, 0.15), (-0.005, 0.005), (0.005, 0.005), (0.15, 0.15), (1, 1).$$

```
function output = neville(x, y, z)
%{
    Inputs:
        x: array of x coords
        y: array of y coords
        z: point to evaluate
    Outputs:
        output: the final value from nevilles
%}
n = length(x);
p = zeros(n, n);

for i = 1:n
    p(i, 1) = y(i);
end

for i=2:n
    for j=2:i
        p(i,j) = (((z-x(i-j+1)) * p(i, j-1)) - ((z-x(i)) * ...
            p(i-1, j-1))) / (x(i) - x(i-j+1));
    end
end

output = p(n, n);
```

(a) (10 points) Write/execute a MATLAB script that utilizes Neville's method to evaluate $P(0.014)$.

Solution: Using the matlab script above, we get the output:

```
>> syms f(n);
>> f(n) = abs(n);
>> x = [-1.0000; -0.1500; -0.0050; 0.0050; 0.1500; 1.0000];
>> y = f(x);

>> neville(x, y, 0.014)

ans =

    0.0061
```

Thus, $P(0.014)$ is 0.0061

(b) (5 points) Determine the absolute and relative errors in using $P(0.015)$ to approximate $f(0.014)$.

Solution:

```
>> neville(x, y, 0.015)

ans =

    0.0063

>> double(abs(neville(x, y, 0.015) - f(0.014)))

ans =

    0.0077

>> double(abs((neville(x, y, 0.015) - f(0.014)) / f(0.014)))

ans =

    0.5489
```

The absolute error in using $P(0.015)$ to approximate $f(0.014)$ is 0.0077.
The relative error in using $P(0.015)$ to approximate $f(0.014)$ is 0.5489.

(c) (10 points) Write/execute a MATLAB script that utilizes Neville's method to evaluate $P(0.75)$.

Solution:

```
>> neville(x, y, 0.75)

ans =

    1.9384
```

$P(0.75) = 1.9384$

(d) (5 points) Determine the absolute and relative errors in using $P(0.75)$ to approximate $f(0.75)$.

Solution:

```
>> double(abs(neville(x, y, 0.75) - f(0.75)))

ans =

    1.1884

>> double(abs((neville(x, y, 0.75) - f(0.75)) / f(0.75)))

ans =

    1.5845
```

The absolute error in using $P(0.75)$ to approximate $f(0.75)$ is 1.1884.
The relative error in using $P(0.75)$ to approximate $f(0.75)$ is 1.5845.

(e) (10 points) Why is it that we cannot use Theorem 3.3 to estimate the error in approximating f by P on $[-1, 1]$?

Solution: We cannot use Theorem 3.3 to estimate the error in f by P because f, which is $|x|$, is not differentiable on the closed interveral from -1 to 1.