Math 3900 Spring 2019 Homework 1 Instructor: J. Haga

Due: Friday January 18 by 11:59PM. The first two problems of this homework set depend heavily on the following example.

Example 1. In this example we use MATLAB to determine the value(s) of c guaranteed by the Mean Value Theorem for the function $f(x) = x^3 - x$ on the interval $[-1.5, 1.2]$. In this example, f is a cubic, and since f' is quadratic, we can solve this problem exactly using calculus and algebra:

$$f(x) = x^3 - x$$

$$f(1.2) = 0.528, \quad f(-1.5) = -1.875 \quad \rightarrow \quad \frac{f(1.2) - f(-1.5)}{1.2 - (-1.5)} = 0.89$$

$$f'(x) = 3x^2 - 1$$

$$f'(c) = 3c^2 - 1 = 0.89 \qquad \rightarrow \qquad c = 0.7937,\ -0.7937$$

For more general functions over more general intervals it may not be easy or possible to analytically determine [even approximations of] the desired value of c. We will therefore utilize MATLAB to find approximate values of c, via a method which can be applied to a wide variety of functions on a wide variety of intervals.

We begin by defining symbolic variable x, and the symbolic variable f defined in terms of x:

```
>> syms f(x)
>> f(x) = x^3-x

f(x) =

x^3 - x
```

We have MATLAB determine the derivative of f:

```
>> df(x)=diff(f,x)

df(x) =

3*x^2 - 1
```

MATLAB can automatically determine the desired value(s) of c:

```
>> A=solve(df(x)==(f(1.2)-f(-1.5))/(1.2-(-1.5)))

A =

```

```
  -(3*7^(1/2))/10
   (3*7^(1/2))/10
```

We desire decimal approximations of c rather than algebraic expressions:

```
>> double(A)

ans =

   -0.7937
    0.7937
```

From this we see that we have arrived at the same values of c we obtained analytically. We may thus use MATLAB to determine the equations of the lines tangent to f at each of the values of c given:

```
>> df(A)

ans =

 89/100
 89/100

>> f(A)

ans =

   (111*7^(1/2))/1000
  -(111*7^(1/2))/1000
```

Again, we prefer decimal approximations:

```
>> double(df(A))

ans =

    0.8900
    0.8900

>> double(f(A))

ans =

    0.2937
   -0.2937
```

From this we can determine the equations of the two tangent lines:

$$t_1(x) = 0.89(x + 0.7937) + 0.2937$$
$$t_2(x) = 0.89(x - 0.7937) - 0.2937,$$

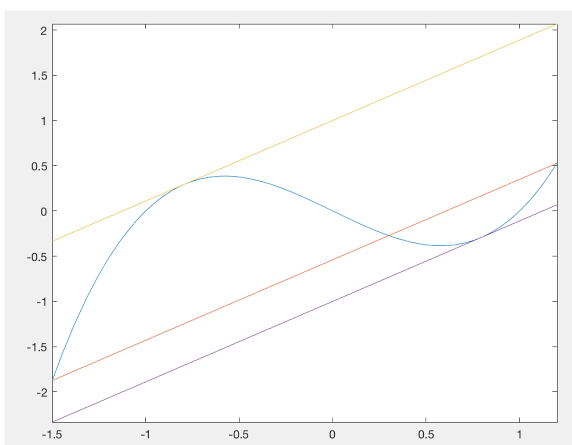
as well as the equation of the line segment connecting both ends of our graph:

$$M(x) = \left(\frac{f(1.2) - f(-1.5)}{(1.2 - (-1.5))}\right)(x - 1.2) + f(1.2),$$

and we can command MATLAB to graph these on the same axes:

```
>> M(x)=(f(1.2)-f(-1.5))/(1.2-(-1.5))*(x-1.2)+f(1.2)

M(x) =

 (89*x)/100 - 27/50

>> B=[f;M;df(A).*([x;x]-A)+f(A)]

B(x) =

                          x^3 - x
              (89*x)/100 - 27/50
  (89*x)/100 + (189*7^(1/2))/500
  (89*x)/100 - (189*7^(1/2))/500

>> fplot(B,[-1.5,1.2])
```

We obtain the following output:



1. (a) (5 points) What is the advantage of the command

```
>> A=solve(df(x)==(f(1.2)-f(-1.5))/(1.2-(-1.5)))
```

versus the possibility

```
>> solve(df(x)==(f(1.2)-f(-1.5))/(1.2-(-1.5)))
```

in line 12 of the command window shown above?

Solution: The top command stores the solution in variable A, which allows you to do operations with the solution afterwards.

(b) (5 points) Why do we use `df(A).*([x;x]-A)` versus `df(A)*([x;x]-A)` (i.e. without the period before the multiplication) in line 56 of the command window above?

Solution: We use `df(A).*([x;x]-A)` because it does multiplication on each element of the matrix `df(A)` whereas using just `*` does matrix multiplication.

2. Utilize commands similar to the ones given above to:

(a) (10 points) determine the value(s) of c guaranteed by the Mean Value Theorem for the function $f(x) = \dfrac{x(x-1)(x-2)(x-3)(x-4)}{2}$ on the interval $[0.2, 3.7]$,

Solution:

Here we set the function and derive.

```
68 >> syms x
69 >>
70 >> f(x)=(x*(x-1)*(x-2)*(x-3)*(x-4))/2
71
72 f(x) =
73
74     (x*(x - 1)*(x - 2)*(x - 3)*(x - 4))/2
75
76 >> df(x)=diff(f,x)
77
78 df(x) =
79
80     (x*(x - 1)*(x - 2)*(x - 3))/2 + (x*(x - 1)*(x - 2)*(x - ...
       4))/2 + (x*(x - 1)*(x - 3)*(x - 4))/2 + (x*(x - 2)*(x ...
       - 3)*(x - 4))/2 + ((x - 1)*(x - 2)*(x - 3)*(x - 4))/2
```

Here we solve using `df(x)` and the threshold values, thus finding the values of c guaranteed by the Mean Value Theorem.

```
>> A=solve(df(x)==(f(.2)-f(3.7))/(.2-(3.7)))

A =

   2 - (6 - 10711^(1/2)/25)^(1/2)/2
   (6 - 10711^(1/2)/25)^(1/2)/2 + 2
   2 - (10711^(1/2)/25 + 6)^(1/2)/2
   (10711^(1/2)/25 + 6)^(1/2)/2 + 2

>> double(A)

ans =

   1.3180
   2.6820
   0.4079
   3.5921
```

(b) (10 points) determine the equation of the line segment $M(x)$ connecting the end-points of the graph described in part (a),

Solution: We use point slope form to determine the equation of the line $M(x)$ between the two points.

```
>> M(x) = ((f(3.7)-f(0.2))/(3.7-(0.2))) * (x - 3.7) + f(3.7)

M(x) =

   172161/100000 - (3789*x)/4000
```

The MATLAB calculation gives us the line (to 4 decimal places):

$$M(x) = -0.9473x + 1.7216$$

(c) (10 points) determine the equations of the tangent lines at each of the values of c determined in part (a) (there are 4 of them for this function on this interval), and

Solution: We used MATLAB to calculate out our tangent lines $T(x)$ (solutions cut for brevity):

```
>> T(x) = (df(A) .* ([x;x;x;x] - A)) + f(A)

T(x) =
   (x + (6 - 10711^(1/2)/25)^(1/2)/2 - 2)* ...
   ((6 - 10711^(1/2)/25)^(1/2)* ...
   (x + (10711^(1/2)/25 + 6)^(1/2)/2 - 2)* ...
   ((10711^(1/2)/25 + 6)^(1/2)* ...
```

Cleaned up, our tangent line functions look like:

$$T_1(x) = -0.9473x + 0.6037$$

$$T_2(x) = -0.9473x + 3.1853$$

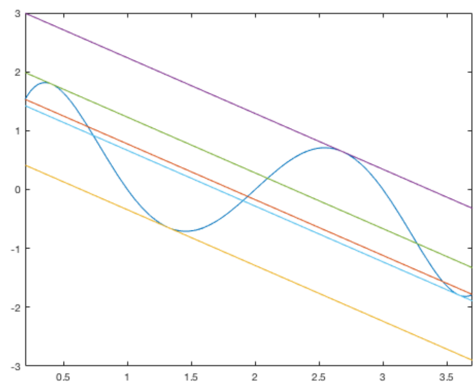
$$T_3(x) = -0.9473x + 2.1765$$

$$T_4(x) = -0.9473x + 1.6125$$

(d) (10 points) plot f, M, t_1, t_2, t_3, and t_4 on the same axes (in MATLAB).

Solution: We then concatenated all of our functions into one array and plotted that array on our bounds:

```
>> B=[f;M;T];
>> fplot(B, [0.2,3.7]);
```



Be sure to format your responses in a manner similar to the example above. Include [a cleaned up version of] your command window, and the image of the plotted functions in your submission.

3. (20 points) Use a Taylor polynomial to approximate $\cos(42°)$ accurate to within 10^{-6}. Note: the approximation is worth 5 points. The remaining points are earned by demonstrating that you appropriately utilized the remainder formula

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

to decide which Taylor polynomial $P_n(x)$ will approximate the desired value to within the given tolerance. Make sure to compare your approximation to the approximation given by the Calculator application on your computer.

Solution: We decided to do all of our calculations in radians, which means that instead of $42°$, we are using $\frac{7\pi}{30}$. We chose an expansion point (x_0) of $\frac{\pi}{4}$ because it is a known value for both cos and sin and it is very close to the value we are trying to approximate. We started out trying to bound our remainder function,

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}.$$

Because $f^{(n+1)}(x)$ will always be $sin(x)$, $cos(x)$, $-sin(x)$, or $-cos(x)$, we know that the max value of any of those functions for $\xi(x)$ will be $cos(42°)$. We know this because $cos(x)$ starts at 1 and decreases, whereas $sin(x)$ starts at 0 and increases, and $cos(x) = sin(x)$ at $x = \frac{\pi}{4}$. Therefore, any value of x less than $42°$ for $cos(x)$ will bind the numerator, $f^{(n+1)}(\xi(x))$, of $R_n(x)$. We chose $\frac{\pi}{6}$, as it has a known value of $\frac{\sqrt{3}}{2}$ for $cos(x)$.

Plugging in x and x_0 with our bound, we get

$$R_n(x) = \frac{\sqrt{3}/2}{(n+1)!}(-\frac{\pi}{4} - \frac{7\pi}{30})^{n+1}.$$

Using this remainder function, we were able to show that we need T_3, as R_3 is equal to $2.7 * 10^{-7}$, which would make T_3 within 10^{-6}.

n	$T_n(x)$	$T_n(\frac{7\pi}{30})$	$R_n(x)$
0	$\cos(\frac{\pi}{4})$	0.7071067812	.04534
1	$\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})(x - \frac{\pi}{4})$	0.7441308057	.00119
2	$\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})(x - \frac{\pi}{4}) - \frac{cos(\frac{\pi}{4})(x-\frac{\pi}{4})^2}{2}$	0.743161519	$2.1 * 10^{-5}$
3	$\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{4})(x - \frac{\pi}{4}) - \frac{cos(\frac{\pi}{4})(x-\frac{\pi}{4})^2}{2} + \frac{sin(\frac{\pi}{4})(x-\frac{\pi}{4})^3}{6}$	0.7431446017	$2.7 * 10^{-7}$

Thus our approximation of $\cos(42°)$ is **0.7431446017**.

The actual value of $\cos(42°)$ is **0.7431448255**, which is within 10^{-6} of our approximation.

4. A function $f : [a, b] \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* with Lipschitz constant L on $[a, b]$ if, for every $x, y \in [a, b]$, we have $|f(x) - f(y)| \le L|x - y|$.

   (a) (5 points) Show that if f satisfies a Lipschitz condition with Lipschitz constant L on an interval $[a, b]$, then $f \in C[a, b]$.

   Solution: Given that f satisfies a Lipschitz condition, then for every $x, y \in [a, b]$, we have $|f(x) - f(y)| \le L|x - y|$. Given this, we can take the limit as x approaches y. This leaves us with $\lim_{x \to y} |f(x) - f(y)| \le \lim_{x \to y} L|x - y|$. The limit as x approaches y of $|x - y|$ is 0, so the right side of the inequality is 0. This leaves us with

   $$\lim_{x \to y} |f(x) - f(y)| \le 0.$$

   For a function to be continuous in an interval I, $\lim_{x \to a} f(x) = f(a)$ needs to be true for all points $x \in I$. Given that the distance from $f(x)$ and $f(y)$ needs to be ≤ 0, and distance cannot be less than 0, therefore $f(x) - f(y)$ must equal 0, as long as f satisfies a Lipschitz condition. This gives us that $lim_{x \to y} f(x) = f(y)$, which you could rewrite as $lim_{x \to a} f(x) = f(a)$. This proves that f is continuous on the interval $[a, b]$, as f satisfies a Lipschitz condition on $[a, b]$.

   (b) (5 points) Show that if f has a derivative that is bounded on $[a, b]$ by L, then f satisfies a Lipschitz condition with Lipschitz constant L on $[a, b]$.

   Solution: The definition for f to have a derivative on $[a, b]$ is that for every x value, the limit

   $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   must exist. If it is bounded by L, that would mean that for all values x,

   $$\lim_{x \to a} \frac{|f(x) - f(a)|}{|x - a|} \le L$$

   must be true. Rearranging that inequality, you get

   $$\lim_{x \to a} |f(x) - f(a)| \le \lim_{x \to a} L|x - a|.$$

   This proves that all slopes of f on the interval $[a, b]$ will be $\le L$, which proves that if the derivative is bounded by L, it also satisfies a Lipschitz condition with that L.

(c) (10 points) Give an example of a function that is continuous on a closed interval, but does not satisfy a Lipschitz condition on the interval. (5 points for a correct example, and 5 points for an explanation of why the function fails to satisfy a Lipschitz condition).

> **Solution:** A function that is continuous on a closed interval and does not satisfy Lipschitz condition is $f(x) = \sqrt{|x|}$ on the interval $[0, 1]$. This function fails to satisfy Lipschitz condition because the $\lim_{x \to 0} f'(x) \to \infty$ which cannot be bounded by a constant L. More specifically when $a = 0$, for any b close to 0 you can find a b even smaller that would require a larger constant L to make the distance between the x values larger than the distance between the y values, hence the slope going to ∞.

5. (a) (5 points) Show that $x \cos x - 8x^2 + 30x - 7 = 0$ has a solution on $[0.2, 0.3]$, and another solution on $[3, 4]$.

> **Solution:** All the terms in the above polynomial are differentiable therefore the whole polynomial is continuous which allows us to use the Intermediate Value Theorem to prove that f has a point c that equals 0 on the interval $[a, b]$ if $f(a) < 0 < f(b)$ which is true for $f(0.2) = -1.124 < 0 < f(0.3) = 1.5666$ and $f(4) = -17.61 < 0 < f(3) = 8.03$.

(b) (5 points) Suppose that $f(x) = 1 - e^x + (e - 1)\sin((\pi/2)x)$. Show that $f'(x)$ is 0 at least once on the interval $[0, 1]$.

> **Solution:** $f'(x)$ is 0 at at least one point on the interval $[0, 1]$ because of Rolle's Theorem. Rolle's Theorem states that if $f(a) = f(b)$, and f is differentiable on the interval $[a, b]$, then there exists a point c such that $f'(x) = 0$. In this problem, $f(0)$ and $f(1)$ both equal 0. $f(x)$ is also differentiable on the interval because all of the terms in the function can be derived. Therefore, it satisfies Rolle's Theorem, which means that there is a point where $f'(x)$ is 0 on the interval $[0, 1]$.