Finding the Edge Length Minimizers for Right Regular Pyramids and Platonic Solids

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Abstract

This research paper investigates what the edge length minimizers are for two different families of polyhedrons: the right regular pyramids and the platonic solids. This was done using the ratio $R = \frac{EL^3}{V}$, where EL is edge length and V is volume. This ratio was first proposed by Z.A. Melzak, who sought the polyhedron that minimizes R. For our research, we set volume equal to 1 for all polyhedrons investigated, and edge length was formulated for each family. We found the edge length minimzer for the right regular pyramids to be one with a triangular base, with $R \approx 1842.77$. The edge length minimzer for the platonic solids was the cube, with $R = 1728$. Therefore, it was concluded that the edge length minimizer of the union of these two families of polyhedrons is the cube.

1 Introduction

During the mid 1900's, Z.A. Melzak proposed a problem concerning geometric optimization; the problem seeks a polyhedron that minimizes the ratio $R = \frac{EL^3}{V}$ where EL represents edge length and V represents volume[3]. The ratio, R, can additionally be represented as $R = \frac{P}{3\sqrt{V}}$. However, it is better suited to cube the edge length to demonstrate the total edge length of a polyhedron as taking the cube root of a constant volume of $1m^3$ will not demonstrate the differences in R values. Melzak's problem is a constrained optimization problem where the volume can be no larger than a cubic meter and aims to minimize a singular length of a curve[9][1].

Melzak's problem as also been referred to as the waste storage problem[1]. The problem states that a cubic meter of waste needs to be stored in a container built from plates[1]. However, the plates cannot bend or fold, and the sealant that is connecting the plates itself is weak[1]. Hence it is crucial to minimize the amount of sealant used in order to construct the waste storage container[10]. Melzak's problem is not only limited to waste storage but is also applicable to cell membranes[11]. As cells contain edge energy which is proportional to its edge length[11]. However, cells cannot occupy a large amount of volume or else the cell is incapable of completing its physiological task[7]. While the goal of Melzak's problem is to find the most optimal polyhedron concerning edge length with a restricted volume; there is no guarantee for an utmost minimizer[1]. It may be possible to achieve a two dimensional shape in an attempt to minimize edge length[1]. Additionally, there is a possibility of producing a curved container as the faces get smaller[1]. However, Melzak believes that the right triangular prism is the minimizing solution[3].

A simple concept pertaining to related rates in Calculus I is that the first derivative of a circle's area with respect to the radius produces the circumference of the circle. However, the circle is not unique in this aspect. In the work of Dorff and Hall, the first derivative of a polygon's area was found to yield the perimeter such as a square [5]. It was also found that this could be applied to polyhedrons [6]. If one took the first derivative of a regular polyhedron's volume with respect to its radius, the area could be calculated[5]. It has also been show that the total edge length of any convex polyhedron containing a ball in three dimensions is always at least as large as the total edge length of a cube circumscribed[6]. Additionally previous work has shown that that the some of the lengths of the edges L of a convex polyhedron containing a sphere of unit diameter satisfies $12 \leq L$[8].

It is known from geometry that the area of a regular pyramid is $V = \frac{BH}{3}$ where B is area of base and H is the height of the pyramid[1]. However a regular pyramid is capable of containing varying bases such as a triangle, octagon, pentagon, etc meaning that the area of a base is subject to change. Previous work has shown that by creating triangles from the vertices to the center a more general formula for an area of a regular polygonal base can be given[5]. Building off of this, the edge length of a right regular pyramid can be generated from using the Pythagorean theorem via the hypotenuse of a triangle created in the area the base of pyramid combined with the height to produce the edge length[2].

The questions that we seek to answer during this research are the following: what is the edge length minimizer of the platonic solids, right regular pyramids, and of these two families and other families of polyhedrons researched, what group of family is the most optimal shape via the ratio of R.

Additionally, in mathematical optimization, the usage of Lagrange Multipliers is an approach for finding the local maxima and minima when subject to a constrain. More specifically, the constrain observed will be a volume of $1m^3$ In doing so, the strategy allows for optimization to be solve without parameterization in terms of the constraints[10].

Throughout the following paper we will demonstrate a proof for the volume of a regular pyramid, derive and define the variables used for the right regular pyramid and platonic solids, derive and interpret the results, and finally arrive at a conclusion.

2 Mathematical Background

Given that there are an infinite number of polyhedrons, this paper will focus on two specific families: right regular pyramids and platonic solids.

The concern of determining the volume of a pyramid was brought by the Greek mathematician Democritus, but was proven by Eudoxus; he reasoned that a pyramid can be approximated to a collection of slabs reducing size[7].

Proof. Consider any pyramid, perpendicular height h and with area of base A. Imagine that the pyramid is split into n layers. The kth layer will have a base with dimensions $\frac{k}{n}$ as a fraction of the original base. So the area of the kth base will be $(\frac{k}{n})^2 A$. Hence the area of the base of each layer will be $(\frac{1}{n})^2 A$, $(\frac{2}{n})^2 A$, ...$(\frac{n}{n})^2$. If each layer is considered to be a prism, each prism will be $\frac{h}{n}$ units tall and the volume of the kth slab will be $(\frac{h}{n})$ $(\frac{k}{n})^2 A$. Thus, the volume of the pyramid can be approximated. $V = (\frac{Ah}{n^3})(1^2 + 2^2 + 3^2 + ... + n^2)$ Hence, the result $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ we get $V \approx (\frac{Ah}{n^3})\frac{1}{6}n(n+1)(2n+1) = (\frac{Ah}{6})(1 + \frac{1}{n})(2 + \frac{1}{n})$ With more layers, or a larger value of n, $\frac{1}{n}$ tends to zero. Thus V tends towards $(\frac{Ah}{6})(1+0)(2+0) = \frac{Ah}{3}$.[7] $\square$

3 Mathematical Formulation

3.0.1 Right Regular Pyramid Formulation

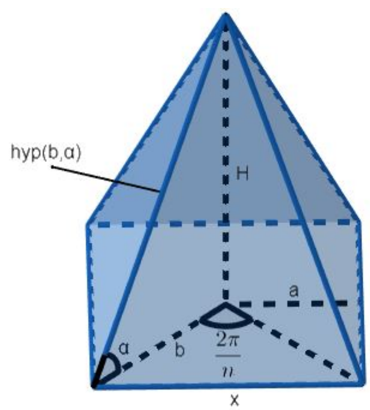

Figure 1: A square pyramid.

To begin the endeavor of determining the edge length minimizer for right regular pyramids, it was determined to formulate total edge length, EL in terms of its simplest variables and use Lagrange Multipliers. Total edge length of right regular pyramids was determined to be

$$EL = nx + nhyp$$

where n represents number of sides of the pyramid, x is the length of a side of the regular base, and hyp represents edge length of the right regular pyramid.

Additionally, the central angle of the right regular pyramid was determined to be $\frac{2\pi}{n}$, with a representing the apothem, b representing the leg of the base, and α representing the angle from the leg to the hypotenuse.

To minimize EL, our edge length formula will ultimately need to be in terms of one variable, n. In order to accomplish this, we will first need to set EL in terms of two variables. In this case, α and b are the two variables that produce the easiest formula to work with. Hence we must find x and hyp in terms of these two variables. This will require applying some basic rules of trigonometry and some mathematical manipulation. Consider Figure 2 below, which depicts a triangular sector of our regular polygonal base.

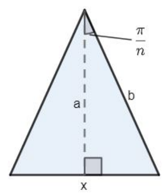

Figure 2: A triangular sector of a regular polygon.

From Figure 2, we know $\sin(\frac{\pi}{n}) = \frac{\frac{x}{2}}{b}$. Therefore $x = 2b\sin(\frac{\pi}{n})$. Also, we can see that $a = b\cos(\frac{\pi}{n})$.

Now, to find hyp in terms of α and b, consider Figure 3 below, which displays one of the triangles that extends upwards to the apex of the right regular pyramid.

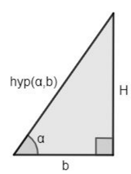

Figure 3: A vertical triangle of a right regular pyramid.

From Figure 3, we know $\cos(\alpha) = \frac{b}{hyp}$. Thus $hyp = \frac{b}{\cos(\alpha)}$. We can also see that $H = b\tan(\alpha)$.

Now we have both x and hyp in terms of α and b. Also to further simplify our formulas, we will let $s = \sin(\frac{\pi}{n})$ and $c = \cos(\frac{\pi}{n})$ from this point on. We can now write our edge length formula as

$$EL = 2nbs + \frac{nb}{\cos(\alpha)}$$

.

Now we must introduce our constraint, volume, for the Lagrange multiplier problem. As proven earlier, we know that, for right regular pyramids,

$$V = \frac{AH}{3} = 1$$

where A is the area of the base and H is the height. To find A, we can split up the base into n triangles similar to the triangle in Figure 2 and take the sum of the areas of the triangles. Using information from Figure 2, the area of one of these triangles will be given by $\frac{xbc}{2}$, so $A = \frac{nxbc}{2}$. From Figure 3, $H = b\tan(\alpha)$, and from Figure 2 we know $x = 2bs$. Through substitution, our volume formula simplifies to

$$V = \frac{nb^3sc\tan(\alpha)}{3} = 1$$

.

Treating n as a parameter, we will now take partial derivatives of EL and V with respect to b and α and then apply Lagrange multipliers:

$$\frac{\partial EL}{\partial b} = 2ns + \frac{n}{\cos(\alpha)}, \tag{1}$$

$$\frac{\partial EL}{\partial \alpha} = \frac{nb\sin(\alpha)}{\cos^2(\alpha)}, \tag{2}$$

$$\frac{\partial V}{\partial b} = nb^2sc\tan(\alpha), \tag{3}$$

$$\frac{\partial V}{\partial \alpha} = \frac{\frac{n}{3}b^2sc}{\cos^2(\alpha)}. \tag{4}$$

Applying our Lagrange multipliers will give

$$\frac{\partial EL}{\partial b} = \lambda\frac{\partial V}{\partial b}, \tag{5}$$

$$\frac{\partial EL}{\partial \alpha} = \lambda\frac{\partial V}{\partial \alpha}, \tag{6}$$

$$V = 1 = \frac{n}{3}b^3sc\tan(\alpha). \tag{7}$$

Then substituting for the partial derivatives yields the following system of equations:

$$2ns + \frac{n}{\cos(\alpha)} = \lambda(nb^2sc\tan(\alpha)), \tag{8}$$

$$\frac{nb\sin(\alpha)}{\cos^2(\alpha)} = \lambda\frac{nb^3sc}{3\cos^2(\alpha)}, \tag{9}$$

$$V = 1 = \frac{n}{3}b^3sc\tan(\alpha). \tag{10}$$

Solving for λ in Equation (9) yields $\lambda = \frac{3\sin(\alpha)}{b^2sc}$. Then substituting this into Equation (8) gives

$$2ns + \frac{n}{\cos(\alpha)} = \left(\frac{3\sin(\alpha)}{b^2sc}\right)(nb^2sc\tan(\alpha)).$$

Using several trigonometric identities, we can simplify this to

$$3\cos^2(\alpha) + 2s\cos(\alpha) - 2 = 0,$$

a polynomial in $\cos(\alpha)$. Using the quadratic formula to solve for $\cos(\alpha)$ gives

$$\cos(\alpha) = \frac{-2s \pm \sqrt{4s^2 - 4(3)(-2)}}{6}.$$

We know that, since the angle α (see Figure 1) must be less than $90°$, $0 < \alpha < \frac{\pi}{2}$. Hence

$$\cos(\alpha) = \frac{-2s + \sqrt{4s^2 + 24}}{6},$$

which can be simplified to

$$\cos(\alpha) = \frac{-s + \sqrt{s^2 + 6}}{3}.$$

This gives us $\cos(\alpha)$ for our EL formula, solely in terms of n.

Now we must solve for b in terms of n. We can obtain b from Equation (10), but only if we have $\tan(\alpha)$. Using more trigonometric identities, we know that

$$\sin(\alpha) = \sqrt{1 - \cos^2(\alpha)}.$$

Substituting what we found for $\cos(\alpha)$ gives

$$\sin(\alpha) = \frac{\sqrt{3 - 2s^2 + 2s\sqrt{s^2 + 6}}}{3}.$$

This now allows us to find $\tan(\alpha)$, since $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$. So

$$\tan(\alpha) = \frac{\sqrt{3 - 2s^2 + 2s\sqrt{s^2 + 6}}}{-s + \sqrt{s^2 + 6}}.$$

Now, from Equation (10), we know

$$b^3 = \frac{3}{nsc\tan(\alpha)} = \frac{-3s + 3\sqrt{s^2 + 6}}{nsc\sqrt{(3 - 2s^2 + 2s\sqrt{s^2 + 6})}}.$$

Therefore

$$b = \sqrt[3]{\frac{-3s + 3\sqrt{s^2 + 6}}{nsc\sqrt{(3 - 2s^2 + 2s\sqrt{s^2 + 6})}}}.$$

Now that we have both $\cos(\alpha)$ and b in terms of n, we can write our EL formula as

$$EL = \sqrt[3]{\frac{-3s + 3\sqrt{s^2 + 6}}{nsc\sqrt{(3 - 2s^2 + 2s\sqrt{s^2 + 6})}}}\left(2ns + \frac{3n}{-s + \sqrt{s^2 + 6}}\right).$$

This is now a function of n alone, which is crucial because we can now theoretically take its derivative and minimize it over $n \in \mathbb{N}$, $n \geq 3$. This will provide us with the n that minimizes edge length, with which we can then obtain the optimal values for α and b as well.

3.1 Platonic Solid Formulation

In order to determine total edge length for the platonic solids, the formula for a single edge length of any platonic solid needed to be produced. By determining the volumes of each platonic solid in terms of its side length, s, as well as maintaining a constrained volume of $1m^3$, the following was determined:

Polyhedron	**Volume**
Tetrahedron	$\frac{\sqrt{2}}{12}s^3$
Cube	s^3
Octahedron	$\frac{\sqrt{2}}{3}s^3$
Dodecahedron	$\frac{15+7\sqrt{5}}{4}s^3$
Icosahedron	$\frac{15+5\sqrt{5}}{12}s^3$

Table 1: Platonic Solid volumes in terms of s

Not only this, but total edge length of a platonic solid was formulated such that $EL = \frac{(NoF)(EPF)(s)}{2}$ where NoF is number of faces, EPF is edges per face, and s is side length[14].

4 Mathematical Results

4.1 Right Regular Pyramid Results

We used the computer program Maple to produce the derivative of our EL formula. The result, as expected, was an extremely complicated function. Therefore, instead of setting EL' equal to zero and solving for n, we decided to graph EL' in Maple and find where the function was equal to zero. The graph of EL' is shown below in Figure 4.

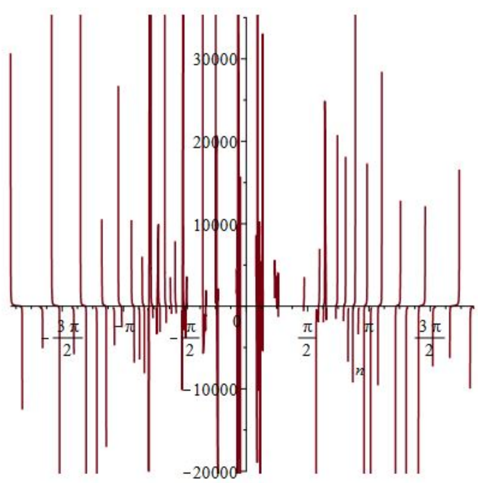

Figure 4: The first derivative graph of EL.

To get a clearer idea of where EL' equals zero, we zoomed in on Figure 4. As mentioned previously, $n \in \mathbb{N}$ and $n \geq 3$ because any polygon must have at least 3 sides, and the number of sides will obviously be a natural number. Figure 5 below displays the portion of the graph with the solution for n.

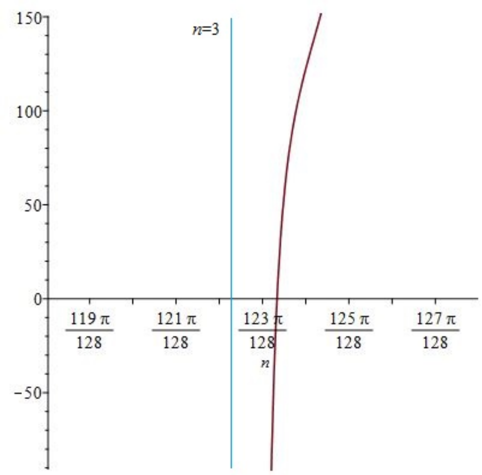

Figure 5: The first derivative graph of EL, zoomed-in.

As shown in Figure 5, the graph of EL' equals zero at approximately $n \approx 3.01$, so we can conclude $n = 3$ is the optimal number of sides of our base to result in minimal edge length.

We can use our newly found n value to produce the α and b of the edge length minimizer as well. We already found that $\sin(\alpha) = \frac{\sqrt{3-2s^2+2s\sqrt{s^2+6}}}{3}$, so substituting

$$s = \sin(\frac{\pi}{n}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

yields

$$\sin(\alpha) = \frac{\sqrt{(3 - 2(\frac{3}{4}) + \sqrt{3}\sqrt{\frac{3}{4} + 6})}}{3}.$$

Then applying arcsin to this function to solve for α gives us $\alpha \approx 54.73$. To solve for b, we can substitute $n = 3$ into our formula that we previously found for b, $\sqrt[3]{\frac{-3s+3\sqrt{s^2+6}}{nsc\sqrt{(3-2s^2+2s\sqrt{s^2+6})}}}$. This substitution gives us $b \approx 1.18$.

Therefore, we can conclude that the edge length minimizer for the right regular pyramids is a triangular pyramid with α angles of approximately 54.73 degrees and legs with lengths of approximately 1.18. Using the formula $EL = 2nbs + \frac{nb}{\cos(\alpha)}$, these values will result in an edge length of approximately 12.26. Thus, since our volume is equal to 1, R ratio for this triangular pyramid is $EL^3 \approx 1842.77$.

4.2 Platonic Solid Results

Table 2: Edge Length Results of Platonic Solids and R values.

Polyhedron	EL	R
Tetrahedron	12.2379	1832.82
Cube	12	1728
Octahedron	14.74	3203.75
Dodecahedron	38.3436	563737.98
Icosahedron	121.2396	1782105.80

Of the calculated R values, the cube was determined to be the edge length minimizer of the platonic solids as its R value was the smallest. This suggests that an utmost minimizer for Melzak's conjecture is not necessarily a polyhedron with the least amount of faces.

5 Conclusions

Table 3: Edge Length and R values of Platonic Solids and Right Regular Pyramids.

Polyhedron	EL	R
Tetrahedron	12.2379	1832.82
Cube	12	1728
Pyramid(4)	13.42	2416.89369
Pyramid(5)	13.91	2697.2282
Octahedron	14.74	3203.75
Dodecahedron	38.3436	563737.98
Icosahedron	121.2396	1782105.80

After comparing sample bases of the right regular pyramid to the platonic solids, it was determined that the edge length minimizer of the platonic solids and the right regular pyramids was the cube as its R value was the lowest.

Pressing this topic further, an avenue to explore would be attempting to solve for the right regular pyramid edge length but in terms of x, n. This was attempted early on to solve for edge length, however, the the partial derivatives began to be cumbersome. Likewise other possible polyhedron to explore would entail the prisms as they are a main concern for Melzak's conjecture. Additionally, the zapezohedran as well as the trisoctahedran may be investigated. These two polyhedron will more than likely not be the edge length minimizer, but will provide more insight into the problem.

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