Ve 280

Programming and Introductory Data Structures

Fibonacci Heap

Learning Objectives:

- Know how a Fibonacci heap is implemented
- Know the performance advantage of Fibonacci heap over binary heap

Fibonacci Heap

- A **mergeable heap,** which supports the following operations
  - **`insert`**: insert a new item into the heap
  - **`getMin`**: get item with **min** key
  - **`extractMin`**: remove and return an item with **min** key
  - **`makeHeap`**: create a new empty heap
  - **`union(`H_1`,`H_2`)`** : create and return a new heap that contains all the elements of heaps H_1 and H_2. Heaps H_1 and H_2 are destroyed by this operation
- Additionally, Fibonacci heap supports
  - **`decreaseKey(Node x, Key k)`** : decrease the key of node x to a smaller value k and restore the heap property

Runtime Complexity Comparison

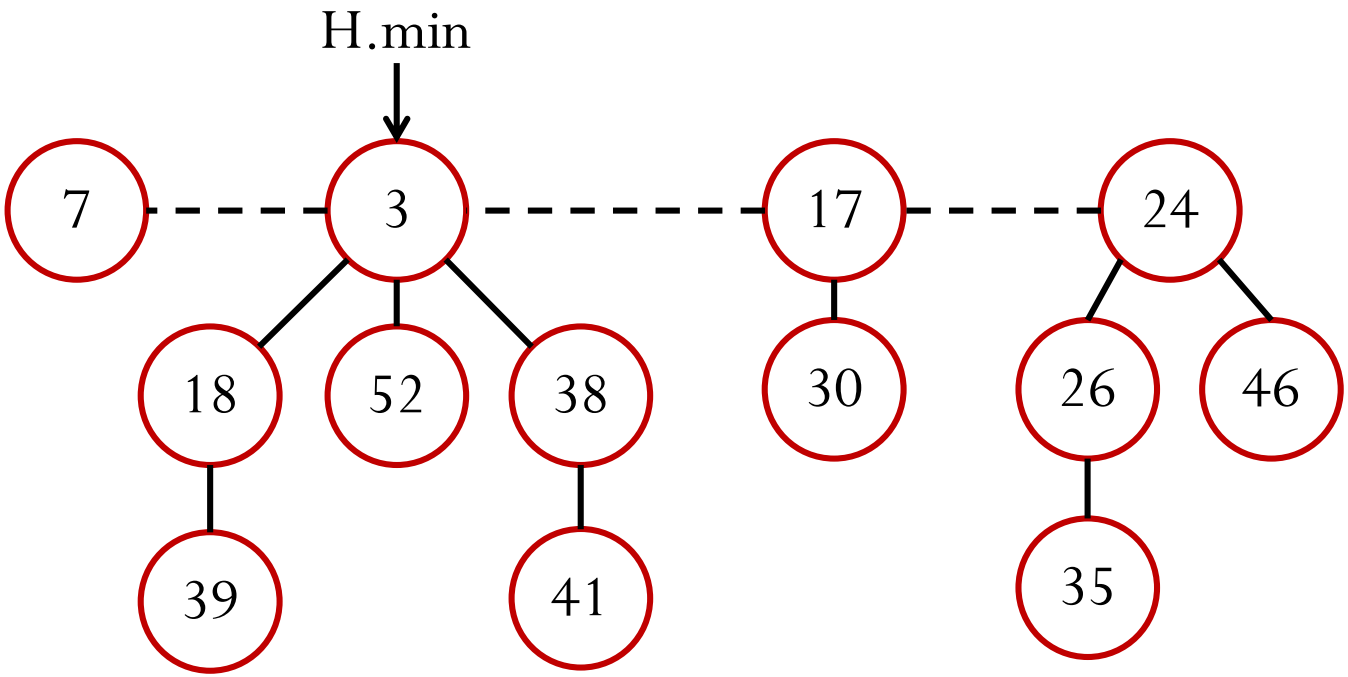
Operation	Binary Heap (worst case)	Fibonacci Heap (amortized analysis)
insert	$\Theta(\log n)$	$\Theta(1)$
extractMin	$\Theta(\log n)$	$O(\log n)$
getMin	$\Theta(1)$	$\Theta(1)$
makeHeap	$\Theta(1)$	$\Theta(1)$
union	$\Theta(n)$	$\Theta(1)$
decreaseKey	$\Theta(\log n)$	$\Theta(1)$

Application

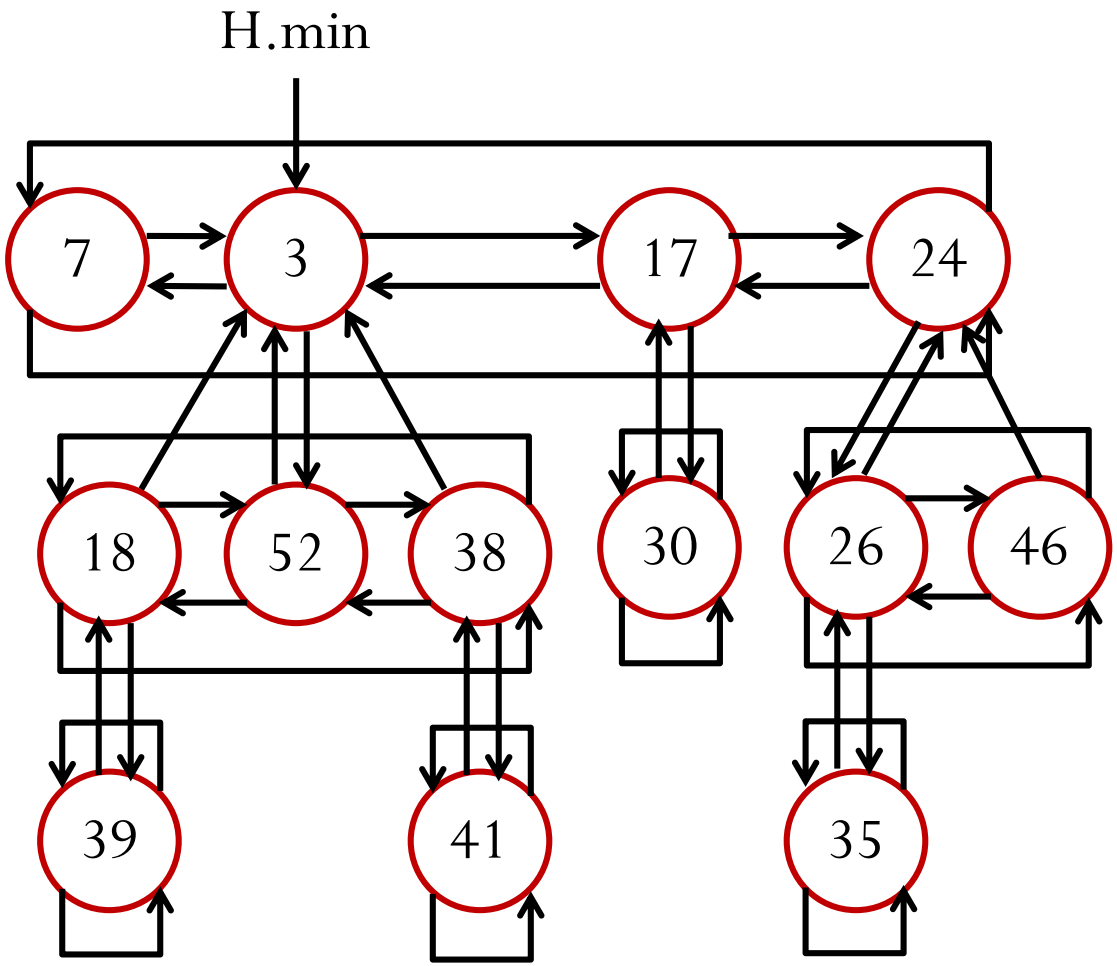
- Fast algorithms for problems such as computing minimum spanning trees and finding single-source shortest paths make essential use of Fibonacci heaps
  - For example, in single-source shortest path problem, we need to extract minimum and decrease key.

Fibonacci Heap: First Look

- A collection of rooted trees, each as a min heap
  - However, the min heap here can have degree > 2

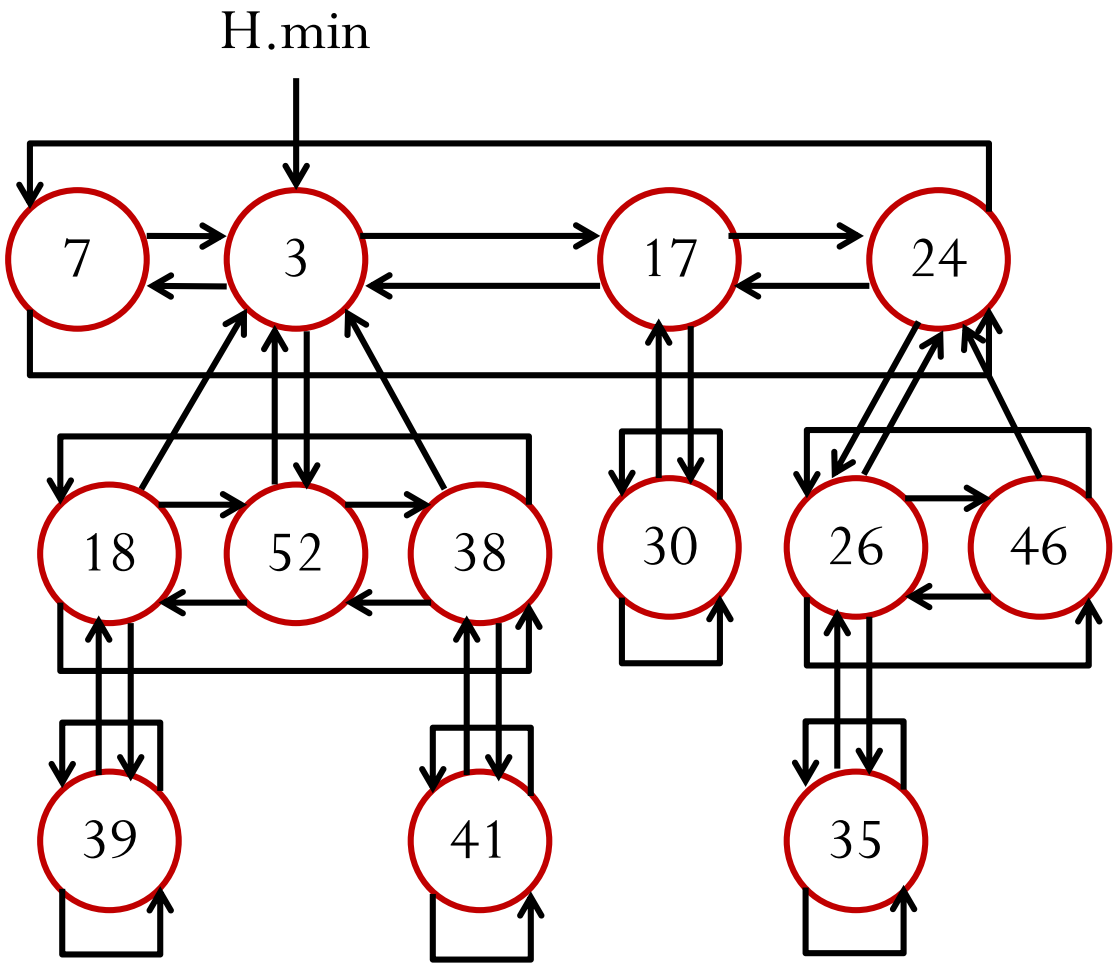


Fibonacci Heap: Implementation Details



- Each node has
  - a pointer to its parent
  - a pointer to one of its children
  - degree (# of children)
- Children are linked by circular, doubly linked list
  - If y is the only child, then `y.prev=y.next=y`
  - Why circular, doubly linked list? O(1) for node insertion, node removal, and list concatenation

Fibonacci Heap: Implementation Details

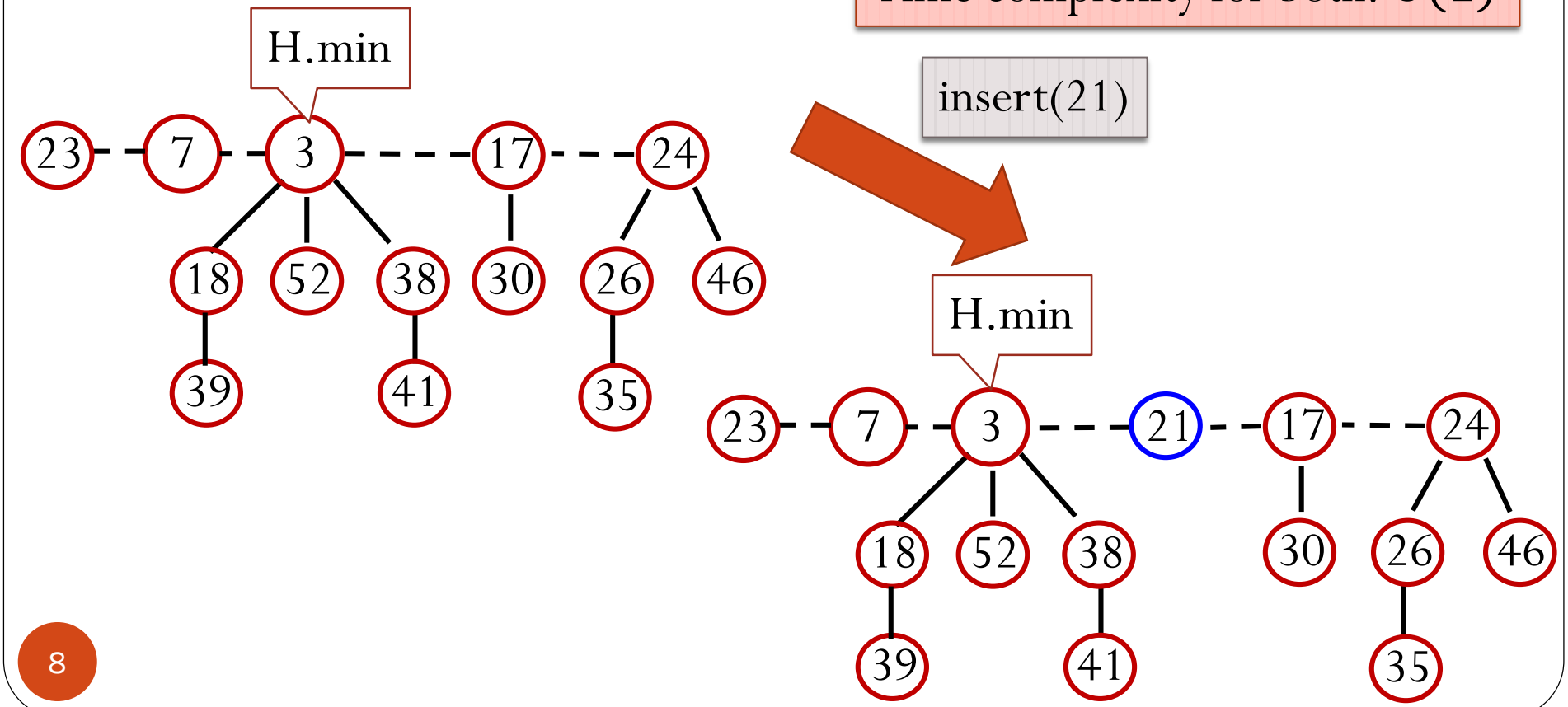


- Roots of the trees are also connected as a circular, doubly linked list
  - called **root list**
- `H.min` points to the minimum root
- `H.n` stores the number of nodes in `H`

Implementation of Some Operations

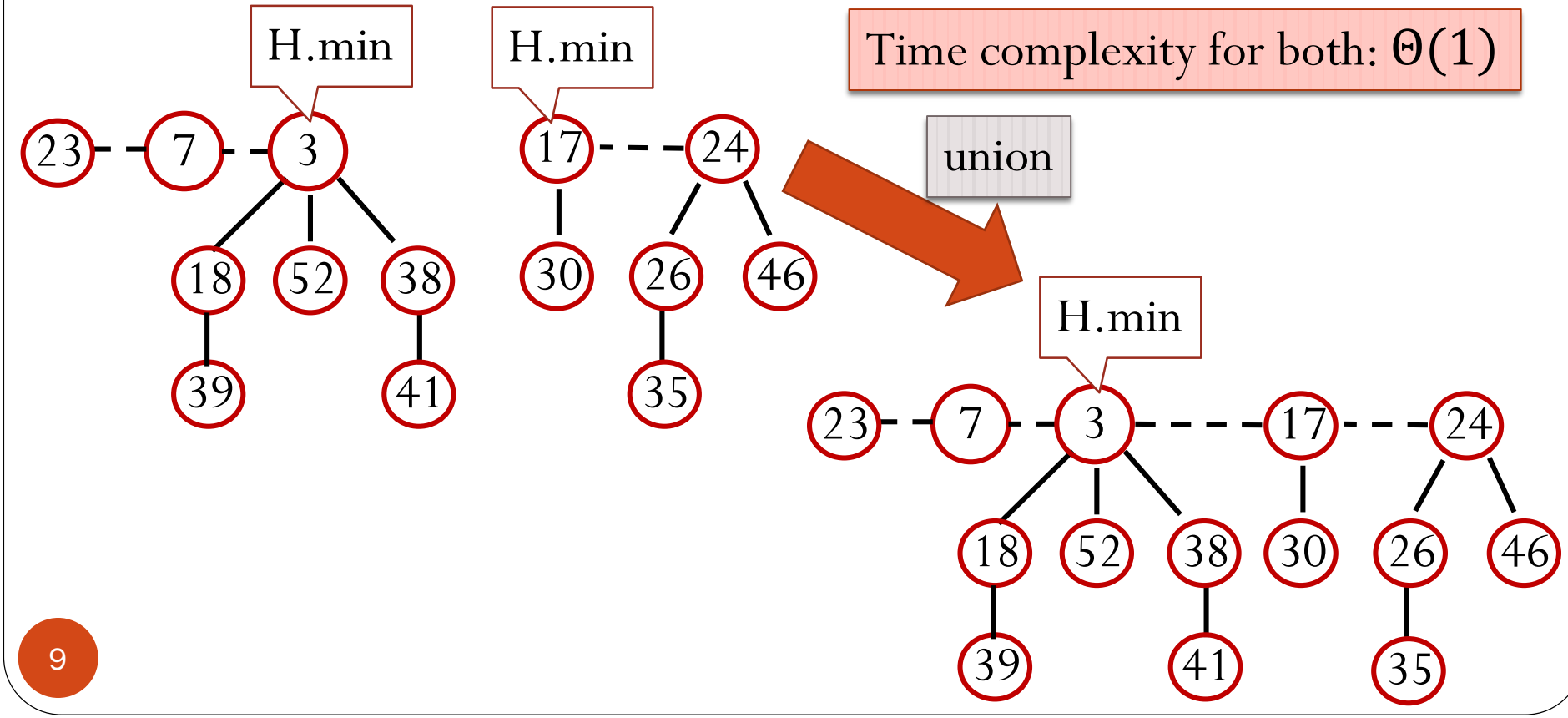
- **`makeHeap`**: set H.min = NULL and H.n = 0
- **`insert`**: Simply put the node into the **root list**
  - Update the H.min if necessary

Time complexity for both: $\Theta(1)$



Implementation of Some Operations

- **`getMin`**: return H.min
- **`union(`H_1`,`H_2`)`** : concatenate the root lists of H_1 and H_2 and then determine the new minimum node

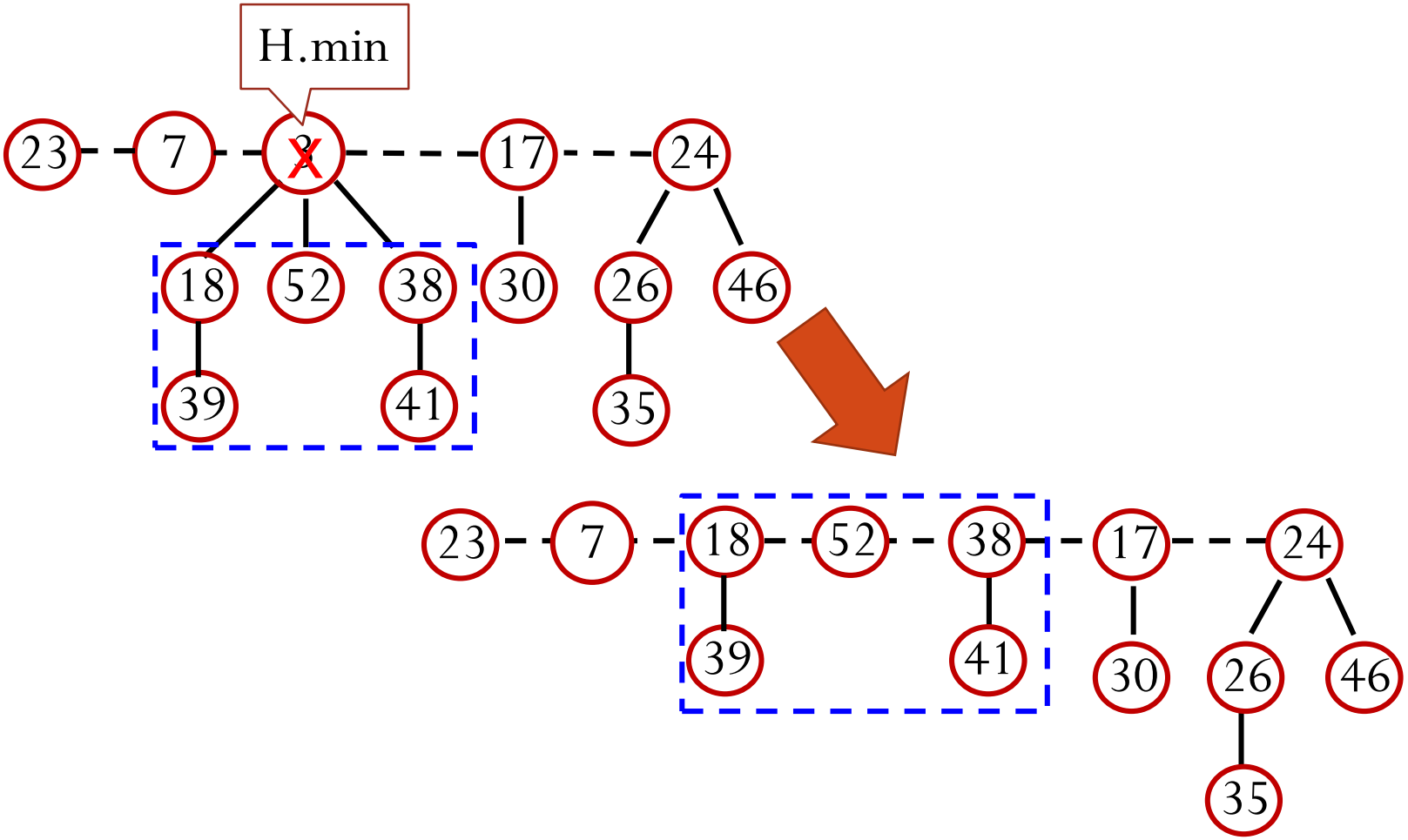


insert and extractMin

- If we start with an empty Fibonacci heap and then insert k items, the Fibonacci heap would consist of just a root list of k nodes
  - If nothing else is done, this degrades to a linked list
- Fortunately, when we perform an extractMin operation, it will go through the entire root list and consolidate nodes to reduce the size of the root list
- Overall idea: the operations on Fibonacci heaps delay work as long as possible

extractMin

- Step 1: remove min and concatenate its children into root list

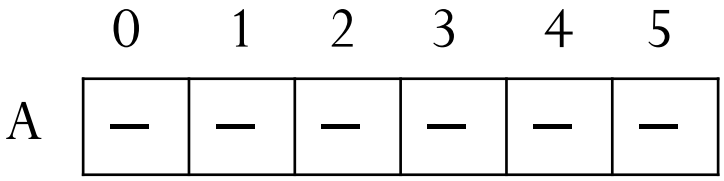


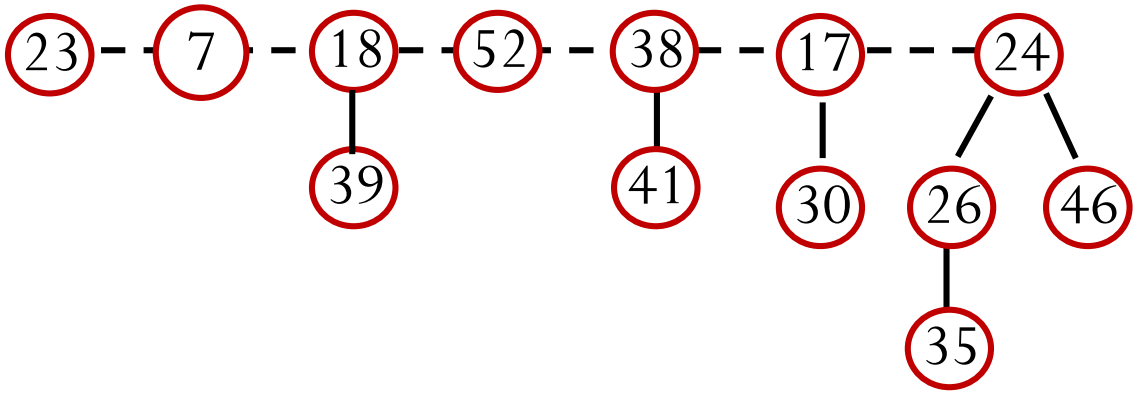
extractMin

- Step 2: **consolidate** the root list
  - Target: merge trees until every root in the root list has a **distinct** degree
- Consolidation iterates over all roots in the root list
  - If find two roots x and y with the same degree and assume $x.key \leq y.key$, remove y from the root list and make it a child of x
- Use an auxiliary array A, where A[i] is either null or storing a root with degree i
  - Size of A is $D(n) + 1$, where $D(n)$ is the maximum degree of any node in an n-node Fibonacci heap.
    - $D(n) = \lfloor \log_\phi n \rfloor$, where $\phi = (1 + \sqrt{5})/2 \approx 1.618$

Consolidating Illustration

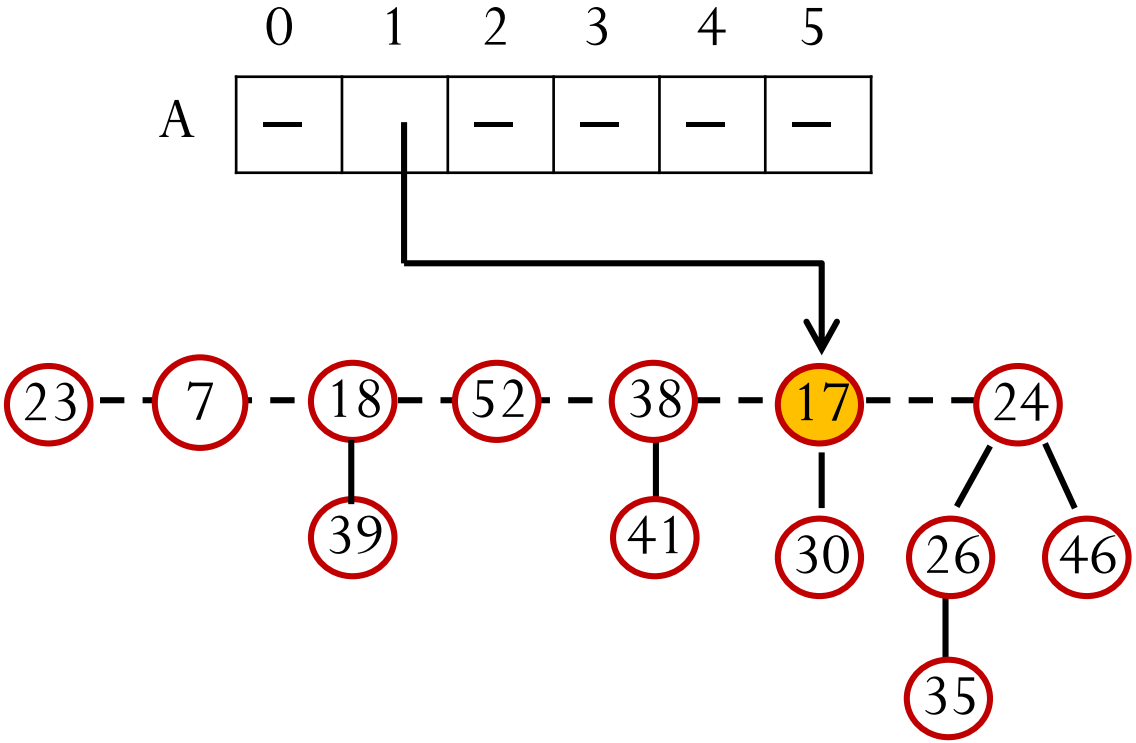
- $D(n) = \lfloor \log_\phi 13 \rfloor = 5$





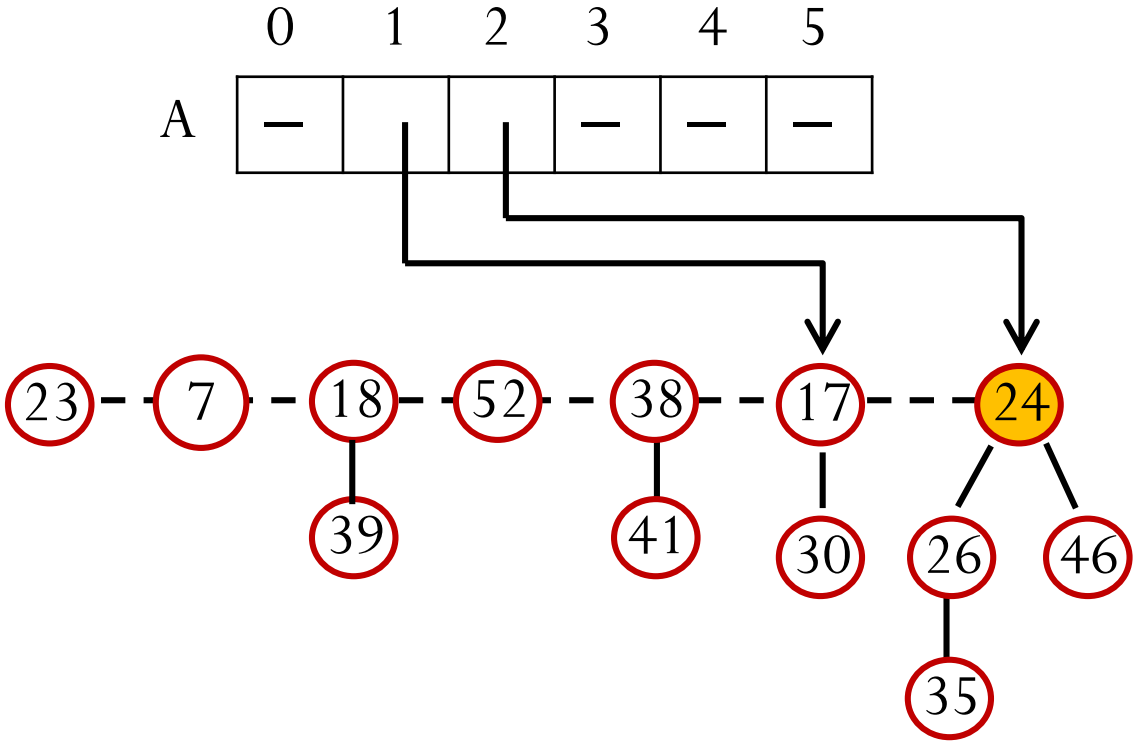
Consolidating Illustration

- Start from the right node of the original H.min, i.e., root 17



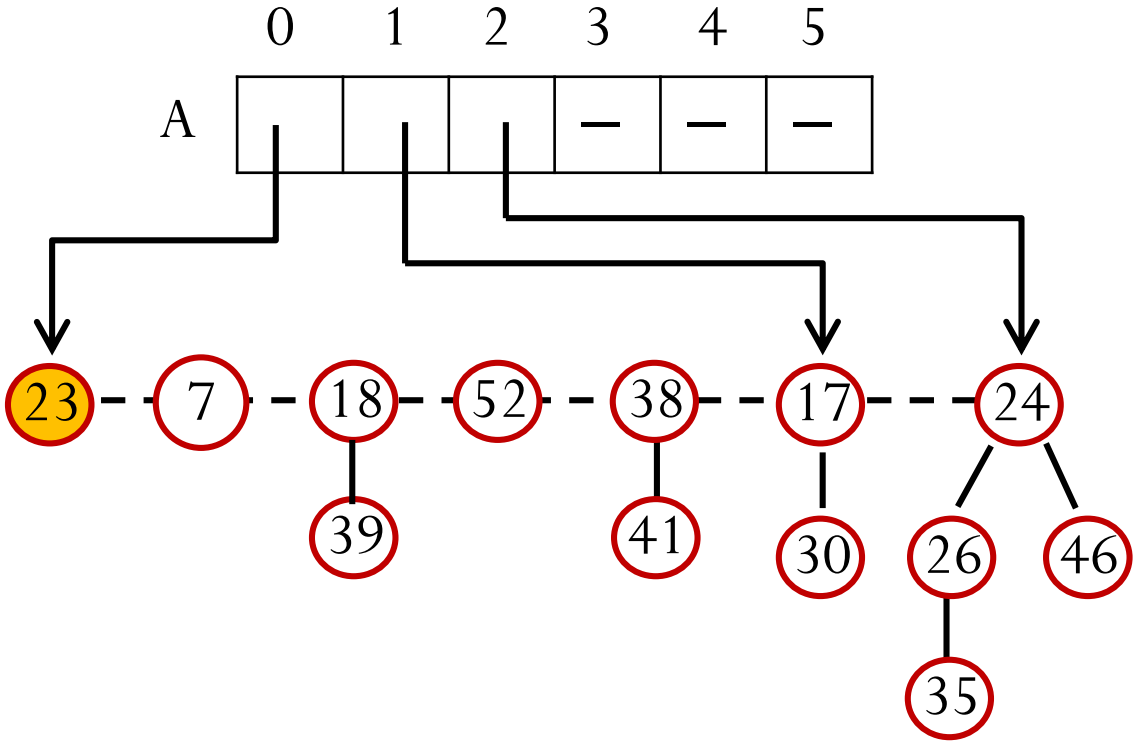
Consolidating Illustration

- Next root to check is 24



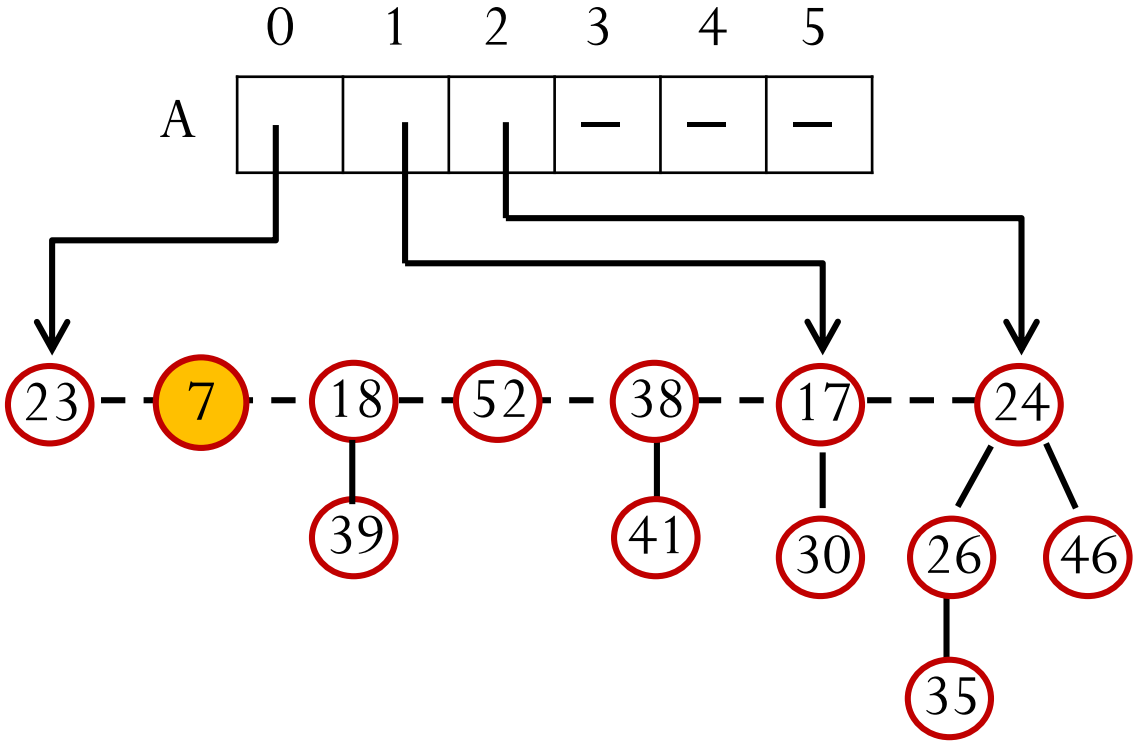
Consolidating Illustration

- Next root to check is 23



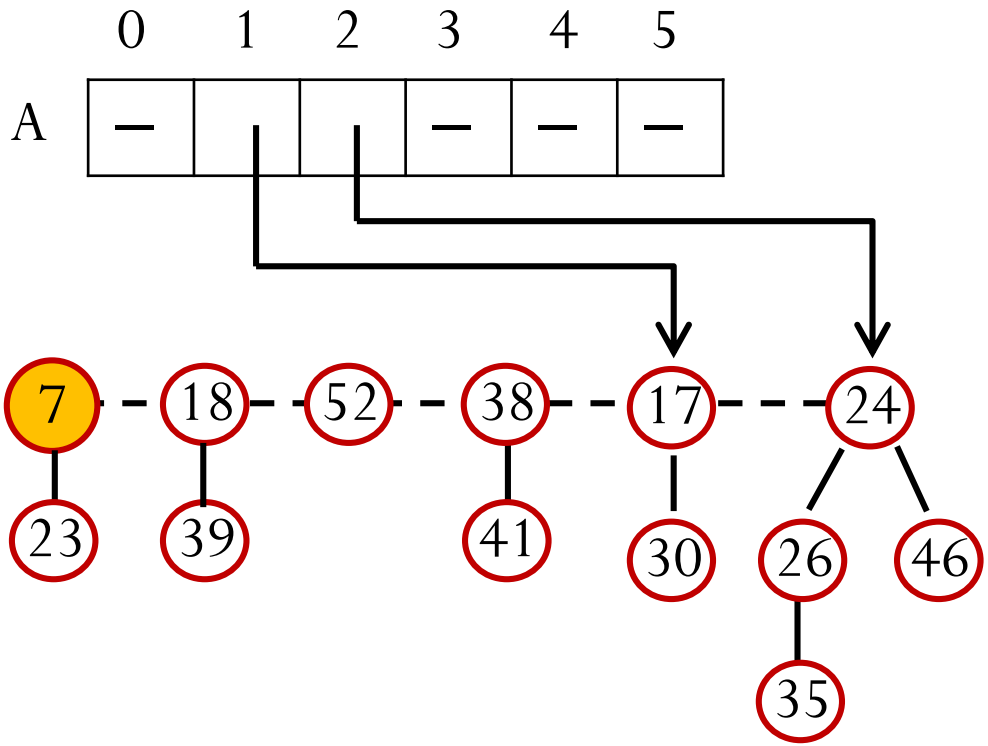
Consolidating Illustration

- Next root to check is 7, with degree 0
  - but we already have a root with degree 0 , i.e., 23. So, merge



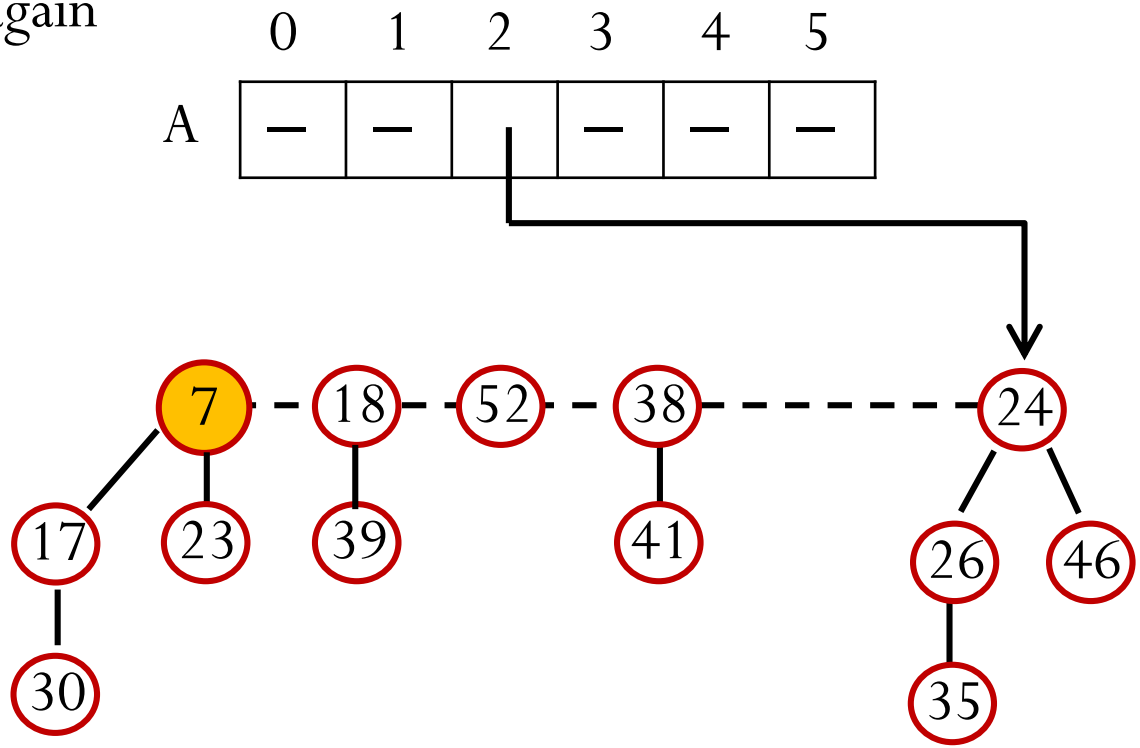
Consolidating Illustration

- Merge tree 23 with tree 7. This creates a root of degree 1
  - but we already have a root with degree 1, i.e., 17. So, merge again



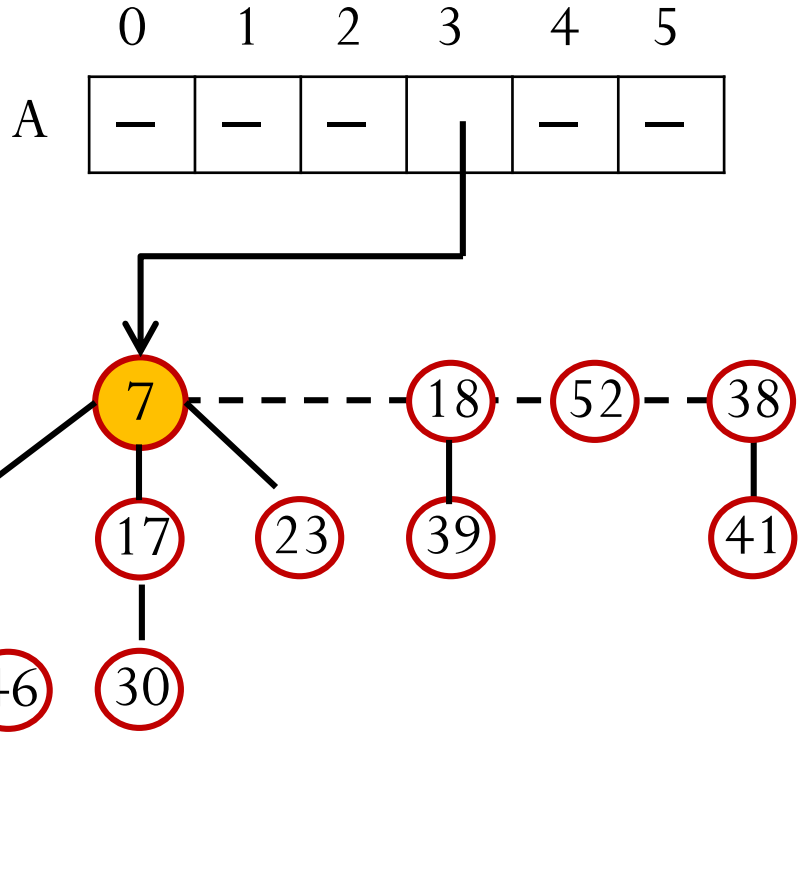
Consolidating Illustration

- Merge tree 17 with tree 7. This creates a root of degree 2
  - but we already have a root with degree 2, i.e., 24. So, merge again



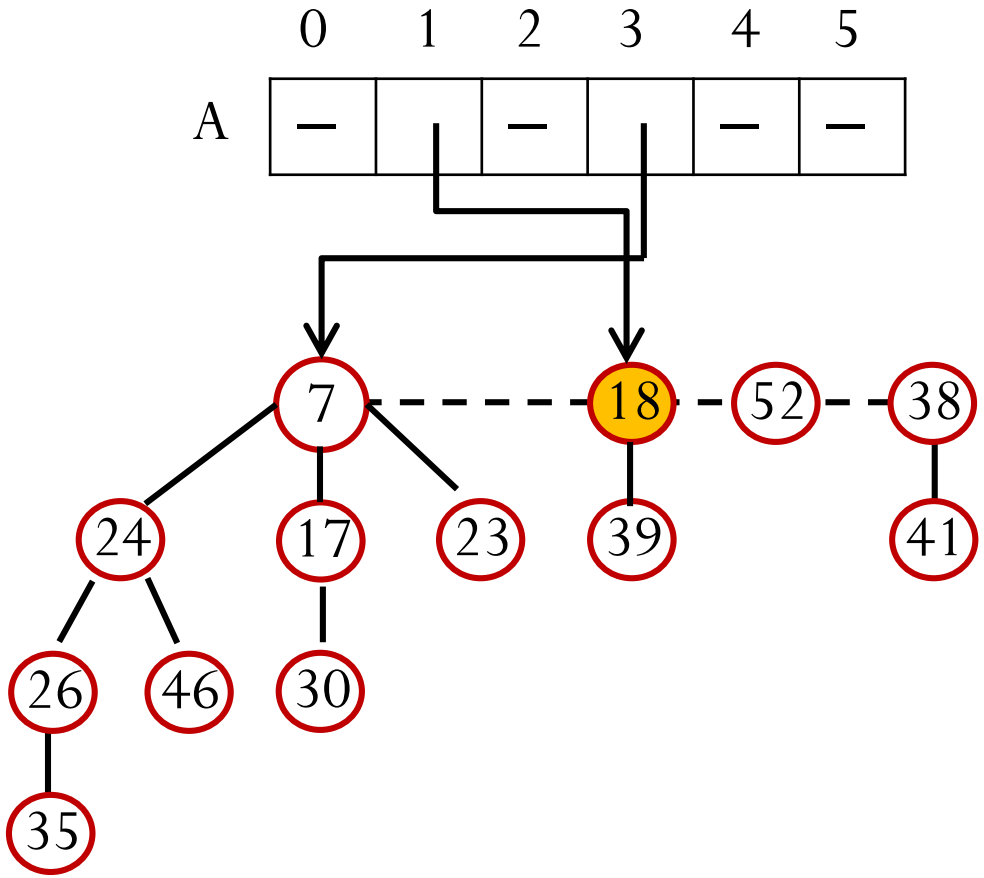
Consolidating Illustration

- Merge tree 24 with tree 7. This creates a root of degree 3
  - It is unique. So, we put the new root into A[3]



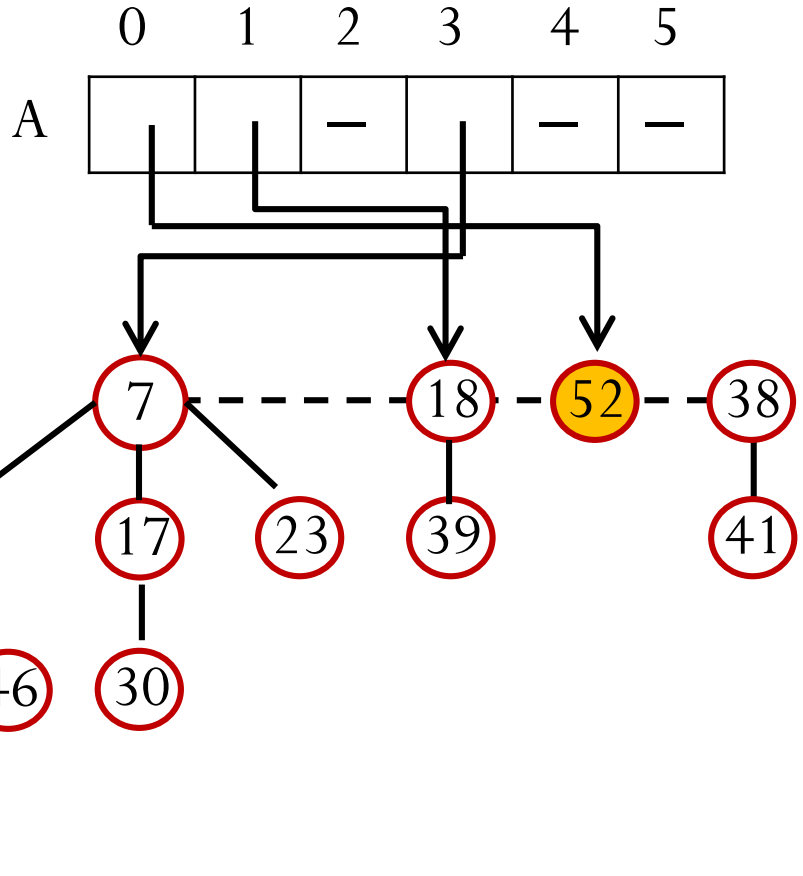
Consolidating Illustration

- Next root to check is 18, with degree 1



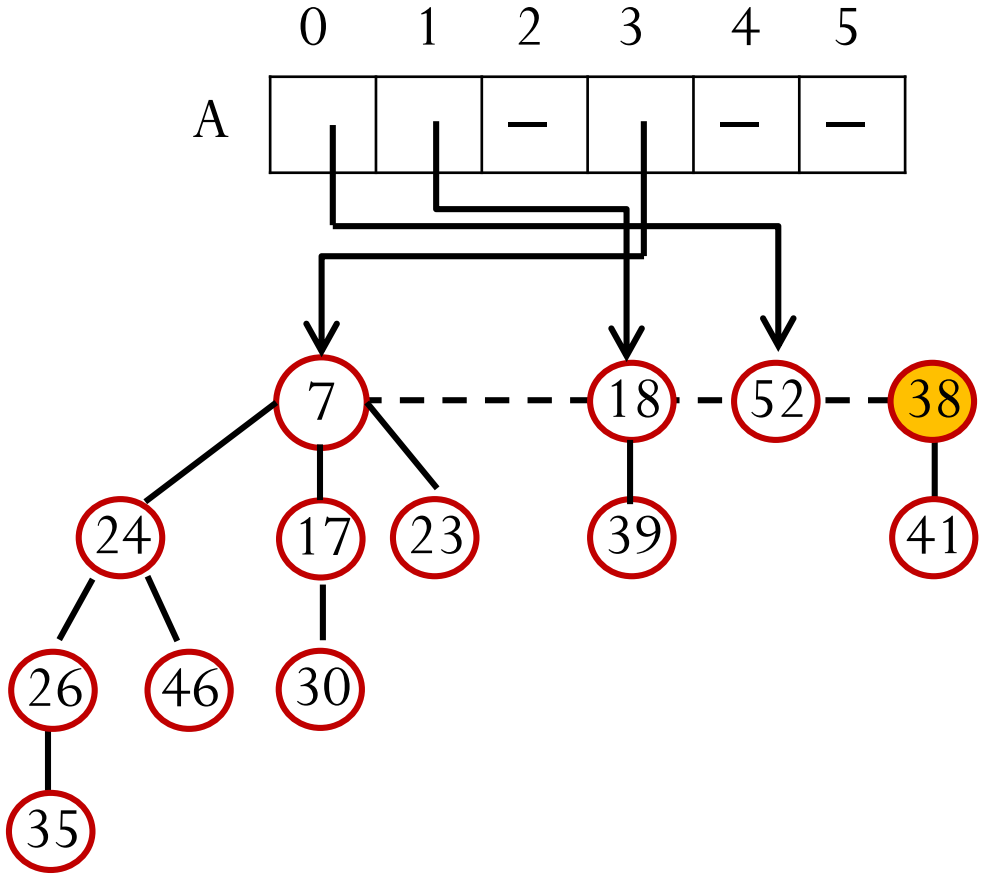
Consolidating Illustration

- Next root to check is 52, with degree 0



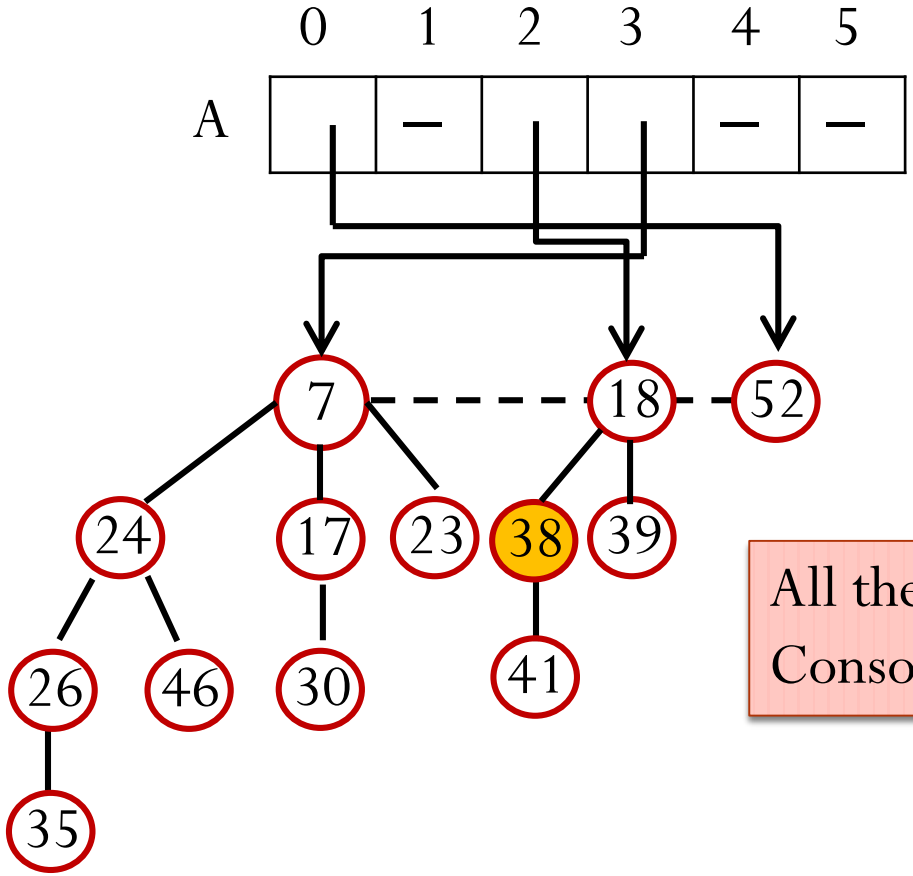
Consolidating Illustration

- Next root to check is 38, with degree 1
  - but we already have a root with degree 1, i.e., 18. So, merge



Consolidating Illustration

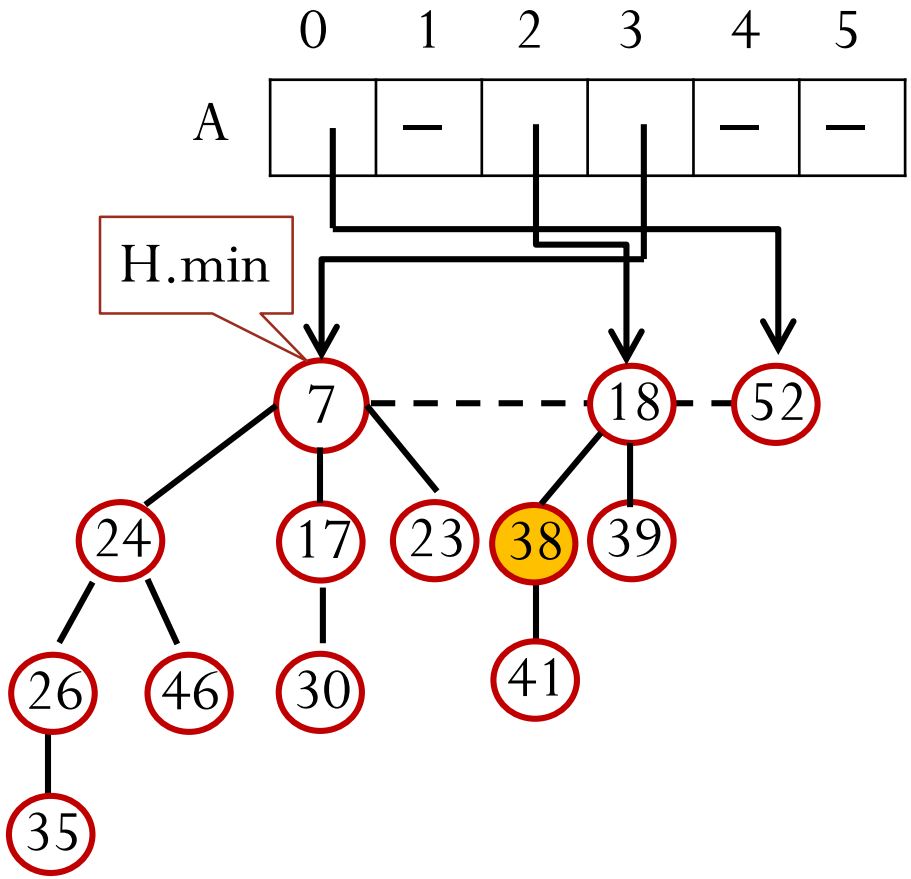
- Merge tree 38 with tree 18. This creates a root of degree 2
  - It is unique. So, we put the new root into A[2]



All the roots have been visited. Consolidation completes

extractMin

- Step 3: link all the roots in array A together; update H.min



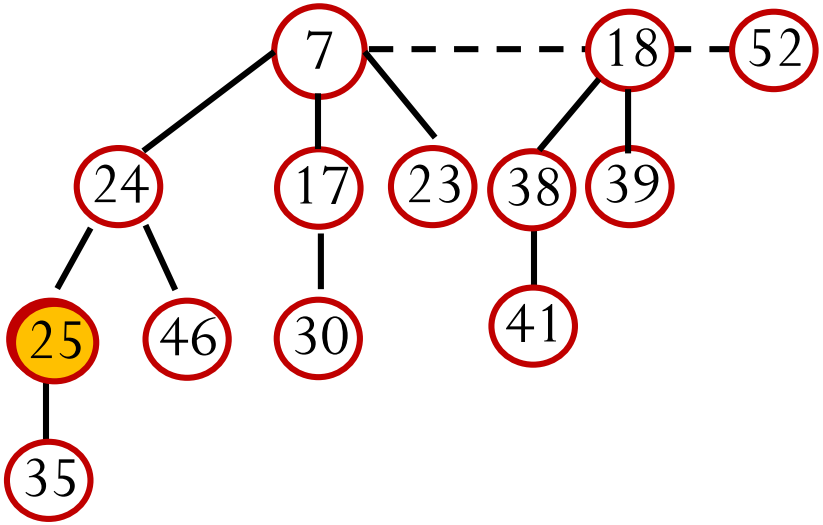
extractMin: Summary

- Step 1: remove min and concatenate its children into root list
- Step 2: consolidate the root list
  - Target: merge trees until every root in the root list has a distinct degree
- Step 3: link all the roots in array A together; update H.min

- Amortized time complexity: $O(\log n)$

decreaseKey

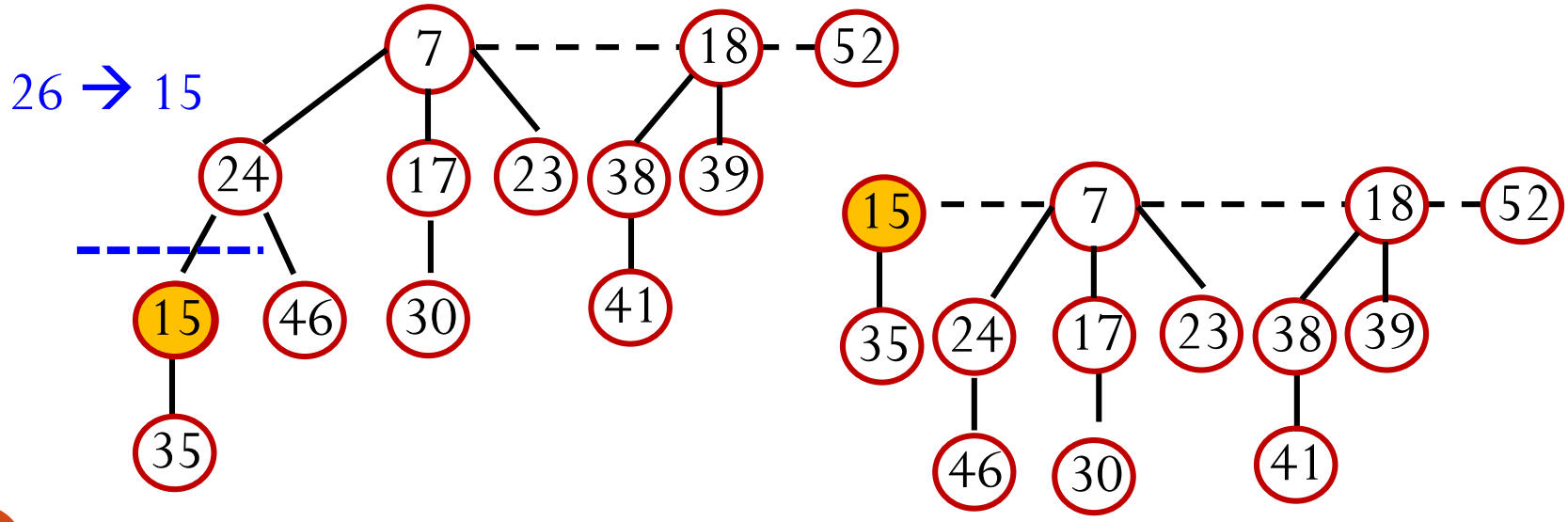
- Case 1: min heap property not violated
  - Only need to change H.min pointer if necessary



26 → 25

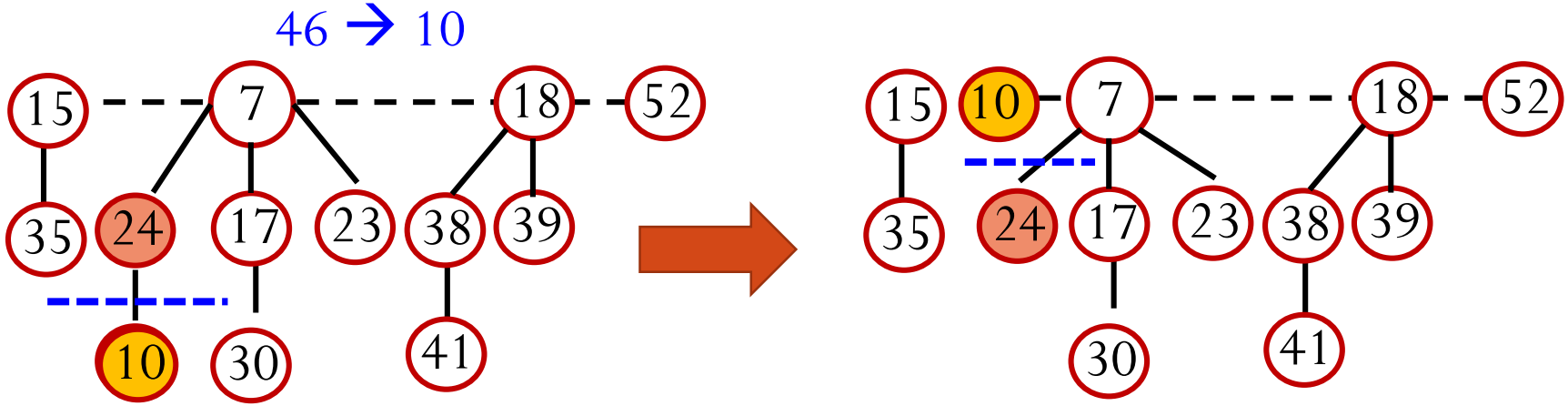
decreaseKey

- Case 2: min heap property **violated**
- Solution
  - Step 1: Cut between the node and its parent
  - Step 2: Move the subtree to the root list
  - Change H.min pointer if necessary



decreaseKey

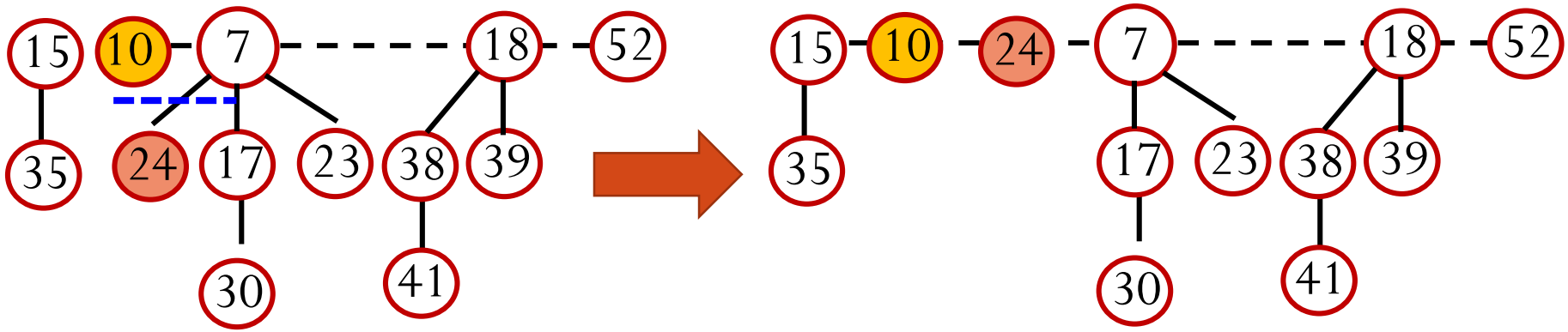
- A special note: if a node *n* not in the root list has lost a child for the **second time**, the subtree rooted at *n* should also be cut from *n*'s parent and move to the root list



24 has already lost one child

Now 24 has lost child for the second time, we need to cut it

decreaseKey



- Note 1: in general case, it may recurse, since the parent may also lose child for the second time
- Note 2: to indicate whether a node has lost child for the second time, a **mark** flag could be used
  - First time losing a child, set mark to **true**
  - When a child is lost and **mark is true**, this indicates losing child for the second time
- Note 3: amortized time $O(1)$