VE281

Data Structures and Algorithms

Linear Time Selection

Learning Objective:

- Understand randomized selection algorithm
- Understand deterministic selection algorithm
- Know how to analyze their runtime complexity

Outline

- Randomized selection algorithm
- Deterministic selection algorithm

The Selection Problem

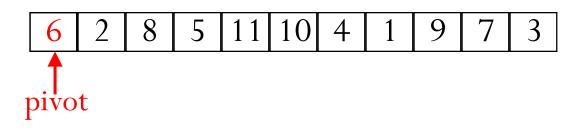
- Input: array A with n distinct numbers and a number i
 - "Distinct" for simplicity
- Output: *i*-th smallest element in the array
 - Assume index starts from 1
- Example: A = (6, 3, 5, 4, 2), i = 3
 - Should return 4
- Special cases
 - i = 1: the smallest item. Runtime: O(n)
 - i = n: the largest item. Runtime: O(n)
 - i = n/2: the median

Solution: Reduction to Sorting

- Step 1: Do merge sort
- Step 2: output the i-th element of the sorted array
- Time complexity is $O(n \log n)$
- Can we do better?
 - This essentially asks whether selection is fundamentally easier than sorting
 - Answer: Yes!
 - We will show an O(n) time randomized algorithm by modifying quick sort
 - Also will show an O(n) time deterministic algorithm (However, not as practical as the randomized algorithm)

Recall: Partitioning in Quick Sort

- Pick a pivot
- Put all elements < pivot to the left of pivot
- Put all elements ≥ pivot to the right of pivot
- Move pivot to its correct place in the array





Basic Idea

- Suppose we are looking for 6th smallest item in an array of length 12. We do partition.
 - Suppose the pivot is at position 4. Then we only need to focus on the sub-array right of the pivot and look for the 2nd item in the array
 - Suppose the pivot is at position 8. Then we only need to focus on the sub-array left of the pivot and look for the 6th item in the array
 - In both cases, recurse!

Randomized Selection

```
Rselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
  if(n == 1) return A[1];
  Choose pivot p from A uniformly at random;
  Partition A using pivot p;
  Let j be the index of p;
  if(j == i) return p;
  if(j > i) return Rselect(1st part of A, j-1, i);
  else return Rselect(2nd part of A, n-j, i-j);
}
```



Which Statements Are Correct?

Given a fixed input array, consider the runtime of the randomized selection algorithm to choose the i-th smallest element

- **A.** The runtime depends on the pivot sequence
- **B.** When i = n/2, the worst-case runtime is $\Theta(n^2)$
- C. When i = n/2, the worst case happens when the pivot sequence is the sorted version of the input array
- **D.** For any given i, the best-case runtime is $\Theta(1)$



Average Runtime of Rselect

- Theorem: for every input array of length n, the average runtime of Rselect is O(n)
 - Holds for every input data (no assumption on data)
 - "Average" is over random pivot choices made by the algorithm

Average Runtime Analysis

- Note: Rselect uses $\leq cn$ operations outside of recursive call (from partitioning)
- Observation: the length of the array the algorithm works on decreases
- Definition: We say Rselect is in phase j if current array size is between $(\frac{3}{4})^{j+1}n$ and $(\frac{3}{4})^{j}n$
- X_i denote the number of recursive calls in phase j
- $runtime \leq \sum_{j} X_{j} \cdot c \cdot (\frac{3}{4})^{j} n$ We need to further get $E[X_{j}]$

$$E[runtime] \le E\left[\sum_{j} X_{j} \cdot c \cdot \left(\frac{3}{4}\right)^{j} n\right] = cn \sum_{j} \left(\frac{3}{4}\right)^{j} E[X_{j}]$$

Average Runtime Analysis

- <u>Claim</u>: If Rselect chooses a pivot so that the <u>left sub-array</u>'s size is am, where $a \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and m is the old length, then the current phase ends
 - Because new sub-array length is at most 75% of the old length
 - "Good pivot"
- What is the probability of $a \in \left[\frac{1}{4}, \frac{3}{4}\right]$ (i.e., good pivot)?
 - Answer: 0.5
- Claim: $E[X_j] \le$ Expected number of times you need to get a good pivot
 - Same as the expected number of times you flip a fair coin to get a "head". (Heads: good pivot; tails: bad pivot)

Coin Flipping Analysis

- Let *N* be the number of coin flips until you get heads
 - N is a geometric random variable: $P(N = k) = \frac{1}{2^k}$, k = 1,2,...

#flips when 1st is head #flips when 1st is tail

•
$$E[N] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + E[N]) \Rightarrow E[N] = 2$$

Prob. 1st flip is head

Therefore, $E[X_j] \leq E[N] = 2$

Prob. 1st flip is tail

Average Runtime Analysis

$$E[runtime] \le E\left[\sum_{j} X_{j} \cdot c \cdot \left(\frac{3}{4}\right)^{j} n\right]$$

$$= cn \sum_{j} \left(\frac{3}{4}\right)^{j} E[X_{j}] \le 2cn \sum_{j} \left(\frac{3}{4}\right)^{j} \le 2cn \frac{1}{1 - \frac{3}{4}}$$

$$= 8cn = O(n)$$

Outline

• Randomized selection algorithm

• Deterministic selection algorithm

A Good Pivot

- Best pivot: the median
 - But, this is a circular problem
- Goal: find pivot guaranteed to be good enough
- Idea: use "median of medians"

A Deterministic ChoosePivot

ChoosePivot(A, n)

- A subroutine called by the deterministic selection algorithm
- Steps:
- 1. Break A into n/5 groups of size 5 each
- 2. Sort each group (e.g., use insertion sort)
- 3. Copy n/5 medians into new array C
- 4. Recursively compute median of C
 - By calling the deterministic selection algorithm!
- 5. Return the median of C as pivot

Deterministic Selection Algorithm

```
Dselect(int A[], int n, int i) {
      // find i-th smallest item of array A of size n
        if(n == 1) return A[1];
        Break A into groups of 5, sort each group;
        C = n/5 \text{ medians};
        p = Dselect(C, n/5, n/10);
                                          ChoosePivot
        Partition A using pivot p;
        Let j be the index of p;
Same as
        if(j == i) return p;
Rselect
        if(j > i) return Dselect(1st part of A, j-1, i);
        else return Dselect(2nd part of A, n-j, i-j);
```

The function has two recursive calls

Runtime of Dselect

- Theorem: For every input array of length n, Dselect runs in O(n) time
- Warning: not as good as Rselect in practice
 - Worse constants
 - Not-in-place



What's the Runtime of Step 2?

```
Dselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
1 if(n == 1) return A[1];
2 Break A into groups of 5, sort each group;
3 C = n/5 \text{ medians};
4 p = Dselect(C, n/5, n/10);
5 Partition A using pivot p;
6 Let j be the index of p;
7 if(j == i) return p;
8 if(j > i) return Dselect(1st part of A, j-1, i);
9 else return Dselect(2nd part of A, n-j, i-j);
                     B. \Theta(n^2)
A. \Theta(n)
C. \Theta(n \log n)
                     D. \Theta(n \log \log n)
```

Runtime of Dselect

Assume the runtime is T(n)

```
Dselect(int A[], int n, int i) {
// find i-th smallest item of array A of size n
1 if(n == 1) return A[1];
2 Break A into groups of 5, sort each group; \Theta(n)
3 C = n/5 medians; \Theta(n)
4 p = Dselect(C, n/5, n/10); T(n/5)
5 Partition A using pivot p; \Theta(n)
6 Let j be the index of p;
7 if(j == i) return p;
8 if(j > i) return Dselect(1st part of A, j-1, i);
9 else return Dselect(2nd part of A, n-j, i-j);
T(?)
```

Recurrence

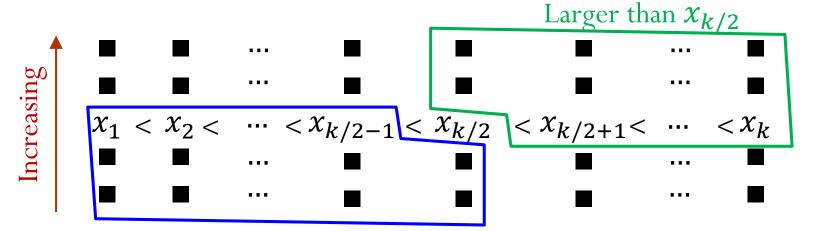
- There exists a positive constant *c* such that
 - $T(1) \leq c$
 - $T(n) \le cn + T\left(\frac{n}{5}\right) + T(?)$
- The next question is what is the size of the array of the second recursive call

Lemma on Size

- Lemma: 2^{nd} recursive call guaranteed to be on an array of size $\leq 0.7n$ (roughly)
- (Rough) proof:
 - Let k = n/5: number of groups
 - Let x_i be the i-th smallest of the k medians
 - Thus, the pivot is $x_{k/2}$
 - Goal
 - >=30% of input array smaller than $x_{k/2}$
 - >=30% of input array larger than $x_{k/2}$

Proof of Lemma

• Imagine we layout elements of A in a 2-D grid



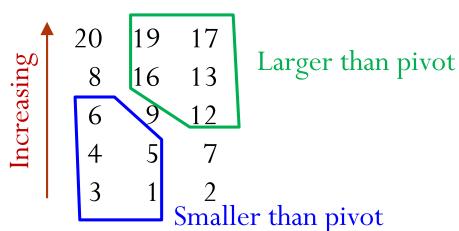
Smaller than $x_{k/2}$

- At least $\sim (3/5)*(1/2) = 30\%$ elements smaller than $x_{k/2}$
- At least $\sim 30\%$ elements larger than $x_{k/2}$
- Result: Number of elements $< x_{k/2}$ is in between 30% and 70%. The same for number of elements $> x_{k/2}$

Example

• Input:

After sorting each group of 5 elements



Recurrence

- There exists a positive constant *c* such that
 - $T(1) \leq c$
 - $T(n) \le cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$
- <u>Note</u>: different-sized sub-problems. Cannot use master method!
- How can we solve this?
 - <u>Strategy</u>: Hope and check
- Hope: there is a constant a (independent of n) such that $T(n) \le an$ for all n > 1
 - Then T(n) = O(n)
- We choose a = 10c

Proof T(n) = O(n)

- <u>Claim</u>: suppose there exists a positive constant *c* such that
 - 1. $T(1) \le c$

2.
$$T(n) \le cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)$$

Then $T(n) \le 10cn$

- Proof by induction
 - Base case: $T(1) \leq 10c$
 - Inductive step: inductive hypothesis $T(k) \leq 10ck$, $\forall k < n$. Then

$$T(n) \le cn + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \le cn + 2cn + 7cn = 10cn$$

Dselect runs in linear time