VE281

Data Structures and Algorithms

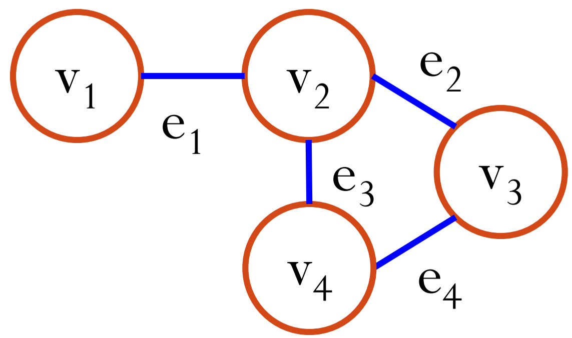
Graphs

Learning Objectives:

- Know some basics about graph
- Know how to represent graphs in computer

Graphs

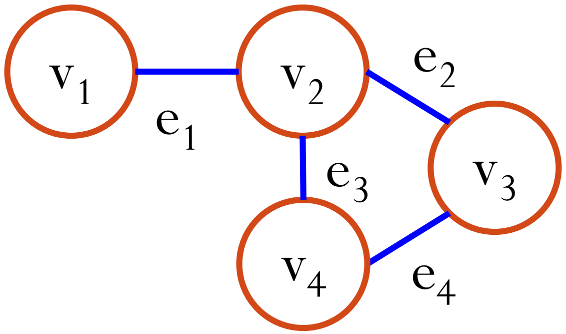
- A **graph** is a set of **nodes** $V = \{v_1, v_2, \ldots, v_n\}$ and **edges** $E = \{e_1, e_2, \ldots, e_m\}$ that connects pairs of nodes.
  - Nodes also known as **vertices**.
  - Edges also known as **arcs**.



- Two nodes are **directly connected** if there is an edge connecting them, e.g., v_1 and v_2 are directly connected, but not v_1 and v_3.

Graphs

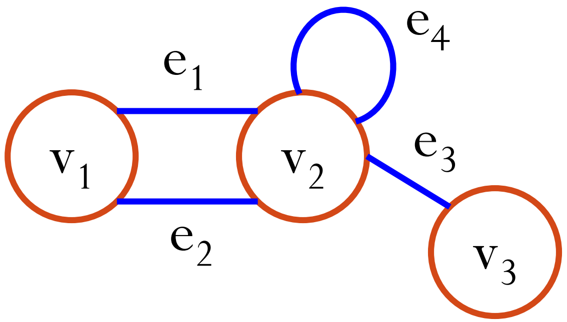
- Directly connected nodes are **adjacent** to each other (e.g., v_1 and v_2), and one is the **neighbor** of the other.



- The edge directly connecting two nodes are **incident** to the nodes, and the nodes **incident** to the edge.

Simple Graphs

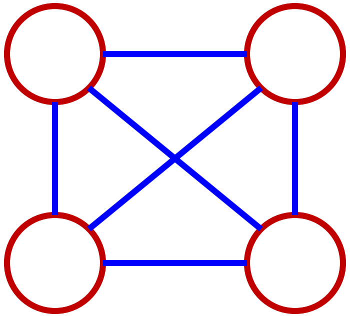
- Two nodes may be directly connected by more than one **parallel edges**, e.g., e_1 and e_2.



- An edge connecting a node to itself is called a **self-loop**, e.g., e_4.
- A **simple graph** is a graph without parallel edges and self-loops.
  - Unless otherwise specified, we will work only with simple graphs in this course.

Complete Graphs

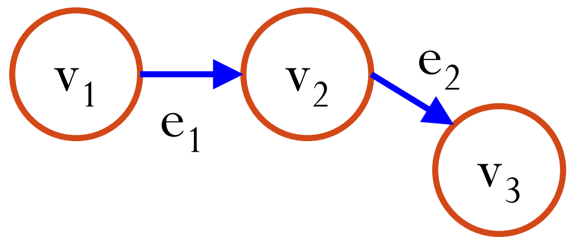
- A **complete graph** is a graph where every pair of nodes is directly connected.



- How many edges are there in a complete graph of N nodes?

Directed Graphs

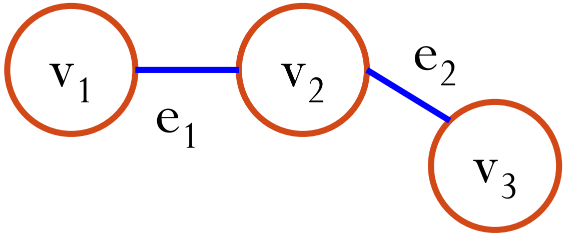
- **Directed graph** (digraph): edges are directional.



- Nodes incident to an edge form an **ordered** pair.
  - $e = (v_1, v_2)$ means there is an edge **from** v_1 **to** v_2. However, there is no edge **from** v_2 **to** v_1.
- Examples: rivers and streams, one-way streets, provider-customer relationships.

Undirected Graphs

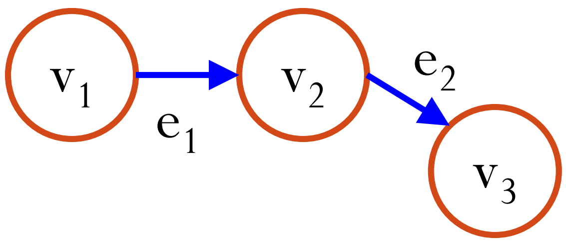
- **Undirected graph**: all edges have no orientation.



- There is no ordering of nodes on edges.
  - $e = (v_1, v_2)$ means there is an edge **between** v_1 **and** v_2.
- Examples: friendship and two-way roads.

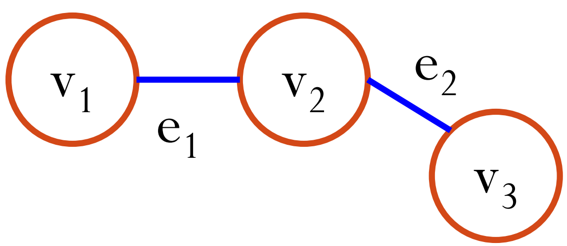
Paths

- A **path** is a series of nodes $v_1, \ldots, v_n$ that are connected by edges.
  - For a directed graph, if $v_1, \ldots, v_n$ is a path, then there is an edge **from** v_i **to** v_{i+1} for each i.



v_1, v_2, v_3 is a path.
v_3, v_2, v_1 is **not** a path.

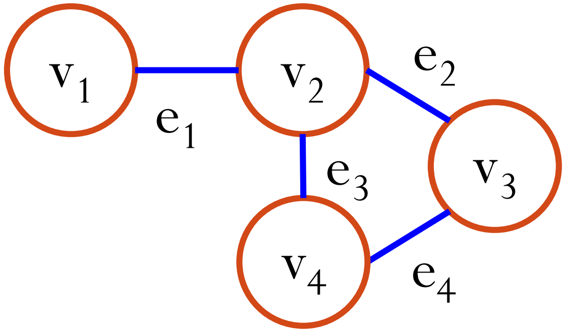
  - For an undirected graph, if $v_1, \ldots, v_n$ is a path, then there is an edge **between** v_i **and** v_{i+1} for each i.



v_1, v_2, v_3 is a path.
v_3, v_2, v_1 is **also** a path.

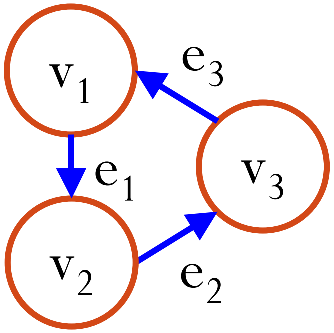
Simple Paths

- A **simple path** is a path with no node appearing twice
  - e.g., v_1, v_2, v_3 is a simple path; v_1, v_2, v_3, v_4, v_2 is not.



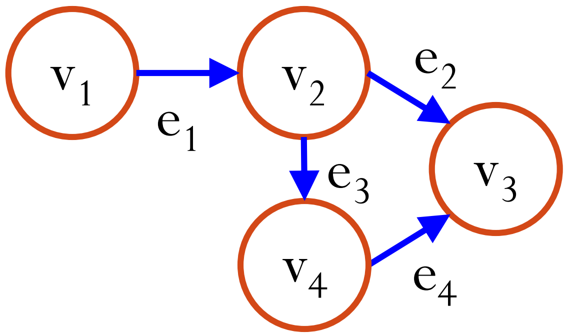
Connected Graphs

- A **connected graph** is a graph where a simple path exists between all pairs of nodes.
- A directed graph is **strongly connected** if there is a simple **directed path** between any pair of nodes.



Connected Graphs

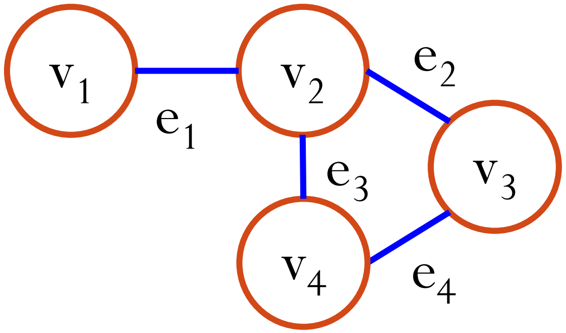
- A directed graph is **weakly connected** if there is a simple path between any pair of nodes in the underlying undirected graph.



The directed graph is weakly connected, but not strongly connected.

Node Degree

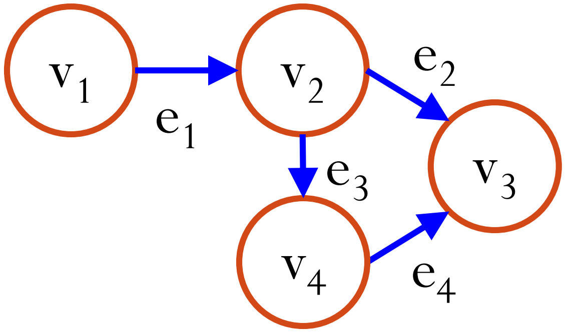
- The **degree** of a node is the number of edges incident to the node, e.g., degree(v_2) = 3, degree(v_3) = 2.



- What is the relationship between the sum of degrees of all nodes and the number of edges?
  - Sum(degrees) = 2 * Number(edges)

Node Degree for Directed Graphs

- For directed graphs, we differentiate between **incoming** edges and **outgoing** edges of a node. Thus we differentiate between a node's **in-degree** and its **out-degree**.
  - in-degree: number of incoming edges of a node
  - out-degree: number of outgoing edges of a node

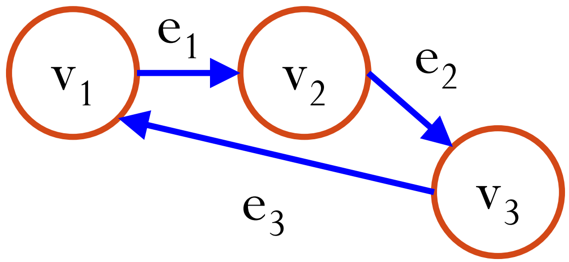


in-degree(v_2) = 1
out-degree(v_2) = 2

- Nodes with zero in-degree are **source** nodes, e.g., v_1.
- Nodes with zero out-degree are **sink** nodes, e.g., v_3.
- What is the sum of in-degrees/out-degrees of all nodes?

Cycles and Directed Acyclic Graphs

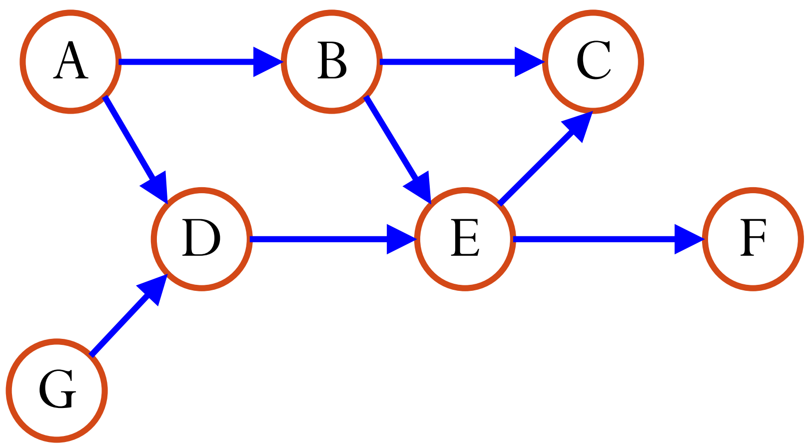
- A **cycle** is a path starting and finishing at the same node.
  - A self-loop is a cycle of length 1.
  - A **simple cycle** has no repeated nodes, except the first and the last node, e.g., v_1, v_2, v_3, v_1.

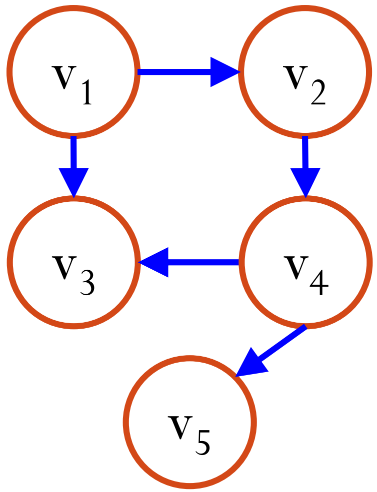


- A graph with no cycle is called an **acyclic graph**.
- A directed graph with no cycles is called a **directed acyclic graph**, or **DAG** for short.

Directed Acyclic Graphs (DAG)

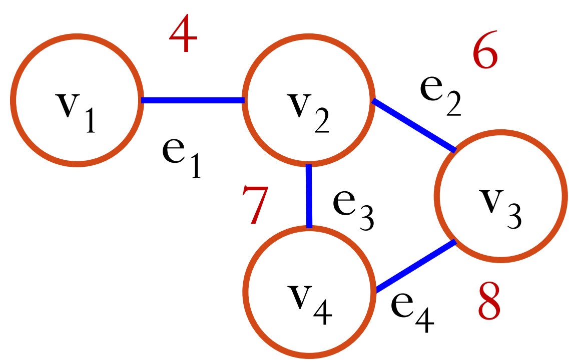
- Are the following graphs DAGs?





Weighted Graphs

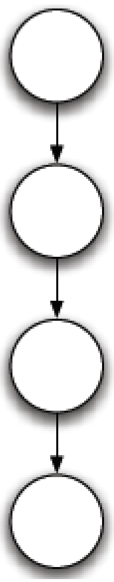
- Weighted graph: edges of a graph may have different costs or weights.
  - For example, the weights on edges represent the distance between two nodes.



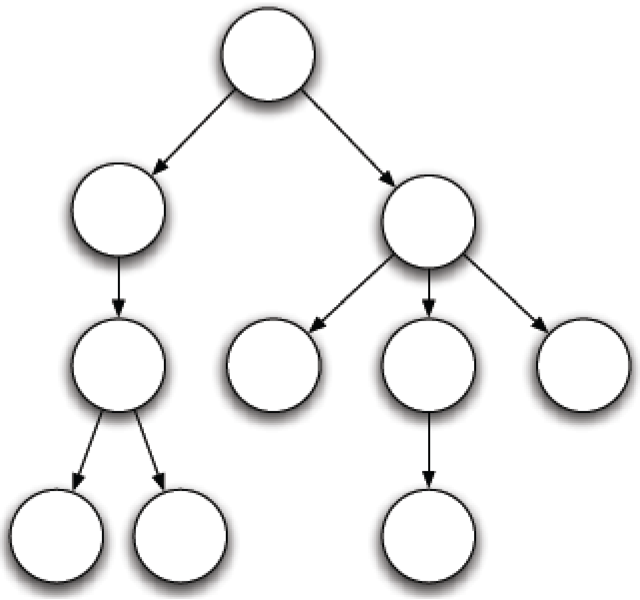
Graph Size and Complexity

- The size of a graph and the complexity of a graph algorithms are usually defined in terms of
  - number of edges $|E|$
  - number of vertices $|V|$
  - or both
- **Sparse** graph: a graph with few edges.
  - $|E| \ll |V|^2$ or $|E| \approx \Theta(|V|)$
  - Example: tree
- **Dense** graph: a graph with many edges.
  - $|E| \approx \Theta(|V|^2)$
  - Example: complete graph

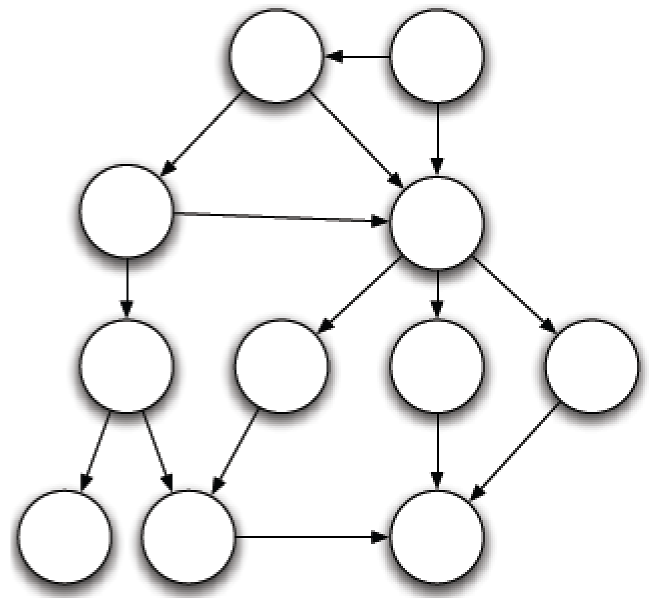
Linked Lists, Trees, and Graphs



Linked List

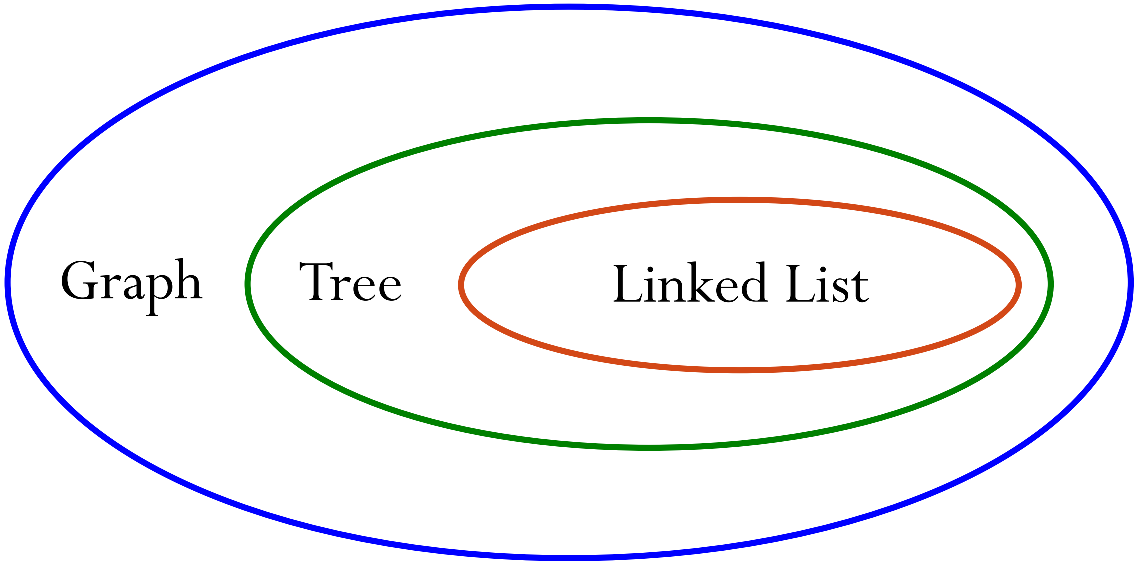


Tree



Graph

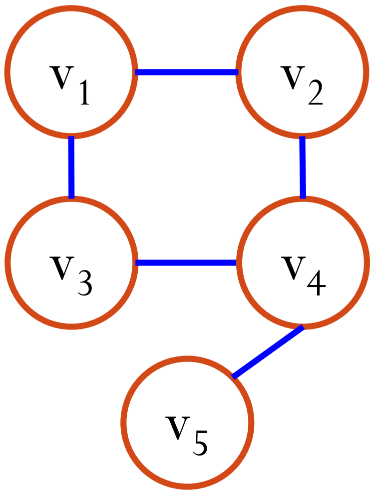
Linked Lists, Trees, and Graphs



Graph Representation

Adjacency Matrix

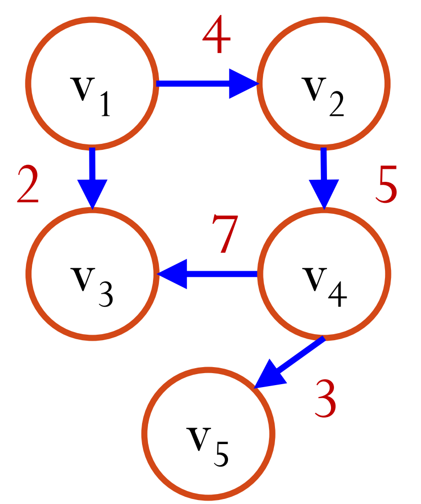
- Adjacency matrix: a $|V| \times |V|$ matrix representation of a graph.
- $A(i,j) = 1$, if (v_i, v_j) is an edge; otherwise $A(i,j) = 0$.



	1	2	3	4	5
1	0	1	1	0	0
2	1	0	0	1	0
3	1	0	0	1	0
4	0	1	1	0	1
5	0	0	0	1	0

Adjacency Matrix for Weighted Graph

- If (v_i, v_j) is an edge and its weight is w_{ij}, then $A(i,j) = w_{ij}$; otherwise $A(i,j) = \infty$.



	1	2	3	4	5
1	∞	4	2	∞	∞
2	∞	∞	∞	5	∞
3	∞	∞	∞	∞	∞
4	∞	∞	7	∞	3
5	∞	∞	∞	∞	∞

Question: why not use 0 to represent a missing edge?

Adjacency Matrix

Properties

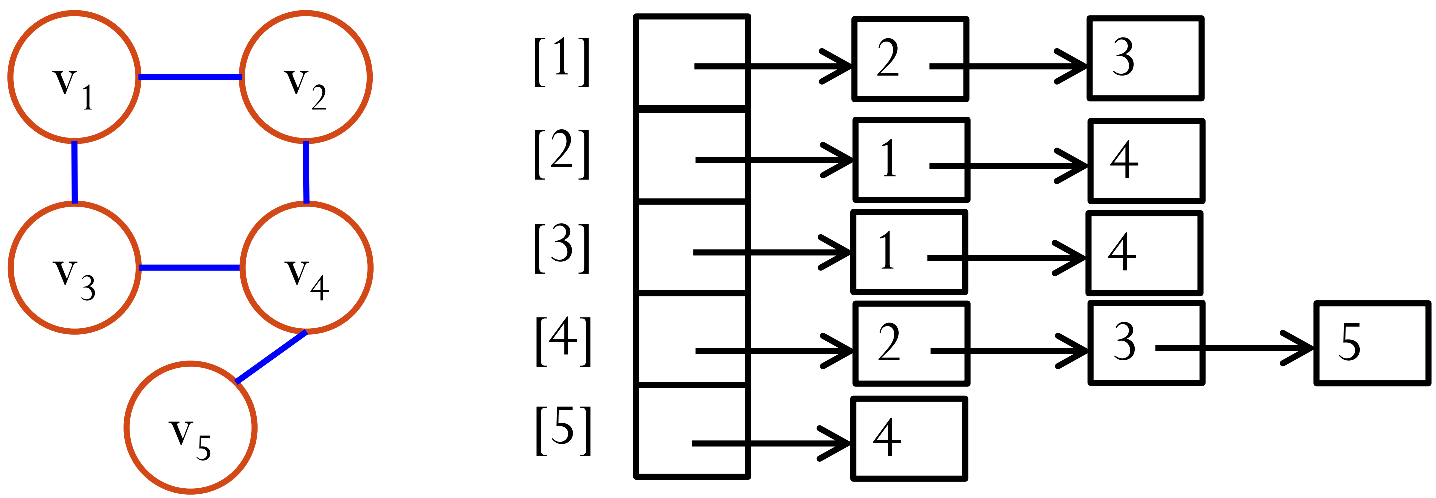
- Space complexity: $|V|^2$ units
  - For an undirected graph, may store only the lower or upper triangle. Thus, $(|V|-1)|V|/2$ units.

- What is the time complexity for finding if node v_i is adjacent to node v_j?
  - $O(1)$
- What is the time complexity for finding **all** nodes adjacent to a given node v_i?
  - $O(|V|)$

Graph Representation

Adjacency List

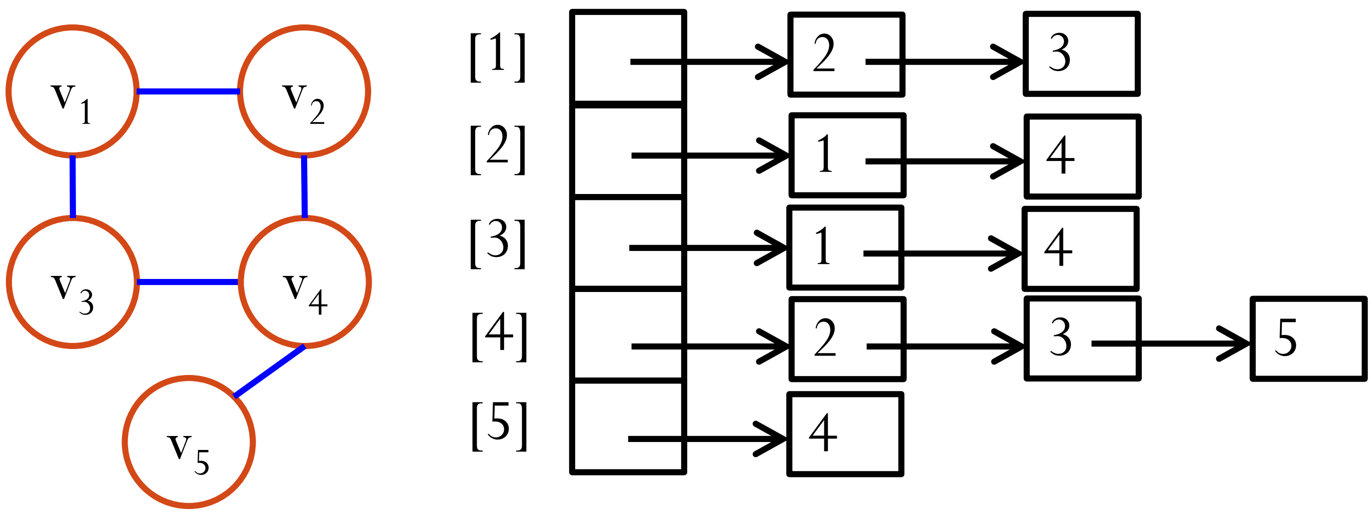
- Adjacency list: an array of $|V|$ linked lists.
  - Each array element represents a node and its linked list represents the node's neighbors.



Graph Representation

Adjacency List

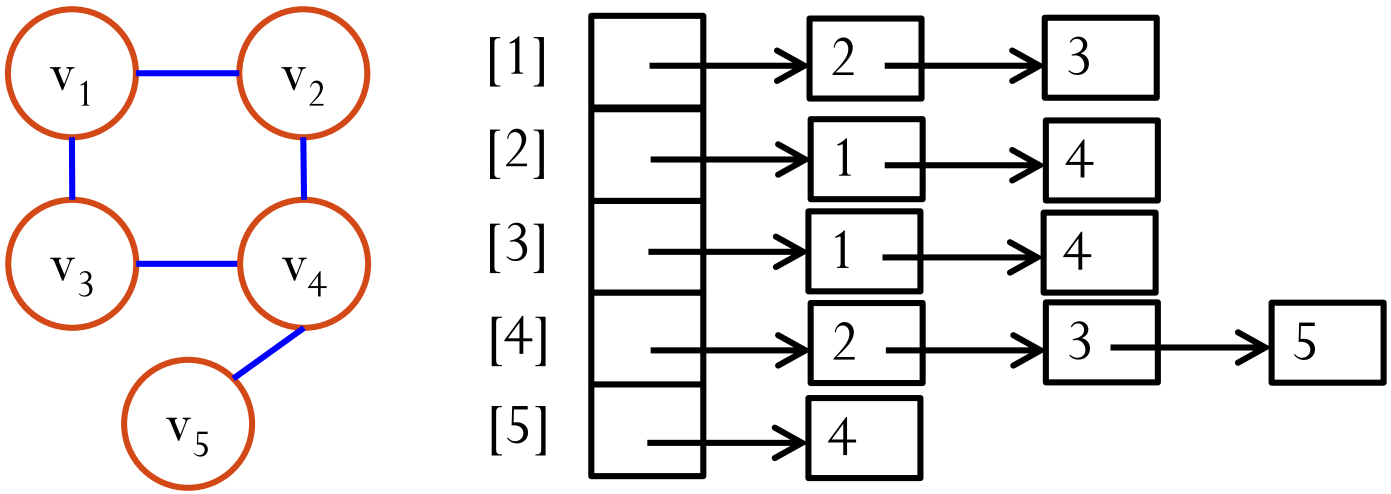
- Each edge in an undirected graph is represented twice.
  - Each edge is treated as **bidirectional**.



- Each edge in a directed graph is represented once.
- Weighted graph stores edge weight in linked-list node.

Adjacency List

Properties



- What is the space complexity? $O(|E| + |V|)$
- What is the **worst case** time complexity for checking if node v_i is adjacent to node v_j? $O(|V|)$
- What is the **worst case** time complexity for finding all nodes adjacent to a given node v_i? $O(|V|)$

Comparison of Graph Representation

- Worst case time complexity for two common operations:

1. Determine whether v_i is adjacent to v_j
   - Adjacency matrix: $O(1)$; Adjacency list: $O(|V|)$
2. Determine all the nodes adjacent to v_i
   - Both adjacency matrix and adjacency list: $O(|V|)$

- Adjacency list often requires less space than adjacency matrix.
- Dense graphs are more efficiently represented as adjacency matrices and sparse graphs as adjacency lists.

Sample Graph Problems

- Path finding problems
  - Find if there exists a path between two given nodes.
  - Find the shortest path between two given nodes.

- Connectedness problems
  - Find if the graph is a connected graph.