1 Gaussian elimination

- *Algorithm:* Gaussian elimination (algo. 1)
- *Input:* Number of unknowns n, Augmented matrix A
- *Complexity:* $\mathcal{O}(n^3)$
- *Data structure compatibility:* N/A
- *Common applications:* Solving linear system, finding the rank of a matrix, finding the inverse of a matrix

Problem. Gaussian elimination
Use elementary row operation to transform an augmented matrix of a linear system into row echelon form

Description

Linear system is a euqation system in the form

$$\sum_{k=1}^{n} c_{1k}x_k = b_1$$
$$\sum_{k=1}^{n} c_{2k}x_k = b_1$$
$$\vdots$$
$$\sum_{k=1}^{n} c_{nk}x_k = b_1$$

where x_k are unknowns, c_ij coefficients and b_k constant terms in the equation system. If the coefficients are aligned in a coefficient matrix C, unknowns in a vector x and constants in a vector b, the system can also be represented as

$$Cx = b$$

The augmented matrix of a system is generated by concatenating the constant vector to the right of the coefficient matrix, which can be represented as

$$A = [\ C \mid b\ ]$$

The problem of elimination is to use elementary row operations to transform the augmented matrix(input) into row echelon form. Elementary row operations consist of three types: interchanging two rows (E_{ij}), row scaling ($E_{(\alpha)i}$), and adding a scaled row to another ($E_{(\alpha)i,j}$).

Row echelon form is a special shape of matrix in which all rows of zeros lie in the bottom of the matrix and for rows with nonzero entries, the leading nonzero entry is strictly at the right of the leading non zero entries on rows above it. When an augmented matrix is transformed in to row echelon form, solving methods such as back substitution can be easily performed.

For a linear system with n unknowns, performing Gaussian elimination requires $\frac{n(n+1)}{2}$ divisions, $\frac{2n^3+3n^2-5n}{6}$ multiplications and $\frac{2n^3+3n^2-5n}{6}$ subtractions, so the complexity of this algorithm is $O(n^3)$

Besides solving linear systems, Gaussian elimination has many other applications.If the constant vector b is set to be the identity matrix, using Gaussian elimination can also solve the inverse of a given matrix. If no constant value is concatenated, barely performing Gaussian elimination to a matrix can identify the rank of the matrix.

Algorithm 1: Gaussian elimination

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Input  : Number of unknowns n, augmented matrix A
Output: Row echelon form of A
Function GaussianElimination(n, A):
    prow ← 1;
    pcolumn ← 1;
    while prow≤n  and pcolumn ≤ n+1 do
        rmax ← row with largest entry on pcolumn_th column;
        if A[rmax,pcolumn]=0 then
            pcolumn ← pcolumn+1;
        else
            interchange prow_th row and rmax_th row;
            for i ← prow+1 to n do
                f ← A[i,pcolumn]/A[prow,pcolumn];
                A[i,pcolumn] ← 0;
                for j ← pcolumn+1 to n+1 do
                    A[i,j] ← A[i,j]-A[prow,j]*f;
                end for
            end for
            prow ← prow+1;
            pcolumn ← pcolumn+1;
        end if
    end while
    return A
end
return GaussianElimination(n, A)
```

References

- Gilbert Strang, Introduction to Linear Algebra (4th edition)
- Kaw, Autar; Kalu, Egwu (2010). "Numerical Methods with Applications: Chapter 04.06 Gaussian Elimination" . University of South Florida.
- https://en.wikipedia.org/wiki/Gaussian_elimination