# 1 Karatsuba's multiplication

- *Algorithm:* Karatsuba's multiplication (algo. 1)
- *Input:* Two integers $a, b$, number of digits in an integer $n$
- *Complexity:* $\mathcal{O}(n^{log_2 3})$
- *Data structure compatibility:* N/A
- *Common applications:* Large number multiplication

**Problem.** Karatsuba's multiplication
Give the precise product of two integers which might have many digits

## Description

In practice there are many scenarios when multiplication of two large numbers is needed. Since the original multiplication function implemented in computer might be limited with regard to precision, algorithms wit hhigh precision to perform large number operation is needed.

One intuitive way is to perform multiplication like the school kids do: multiply every digit in one number with every digit from another, adding proper number of zeros after the product and then sum all results. It is a correct result, but when the number of digits is large enough, this algorithm turns out to be much too time consuming.

Karatsuba's multiplication uses the idea of divide and conquer to solve this problem. Instead of being divided into single digits, the two numbers will first be divided into two parts with same length.

Then by properly perform addition/subtraction of the four parts, the number of recursive call of multiplication can be reduced from 4 to 3. Since multiplication's time cost is much more than that of addition, this will save a lot of time.

Because of the feature that the recursive call is much more time consuming, we can just focus on the time of multiplication. Using recurrance relationship it can be seen that the time of multiplying two n-digit numbers is 3 times that of two $\frac{n}{2}$-digit numbers. So the time of multiplying two n-digit numbers in Karatsuba's algorithm can be roughly presented as

$$T(n) = T(1) \times 3^{log_2 n} = T(1) \times n^{log_2 3} = O(n^{log_2 3})$$

**Note:** for convenience, $n$ is set to be a power of 2. If necessary, the two numbers can be prefixed with zeros.

**Algorithm 1:** Karatsuba's multiplication

**Input** : Integers $a$, $b$,number of digits $n$

**Output:** Product of $a$ and $b$

**Function** `Karatsuba`($a$, $b$, $n$)**:**

    **if** $n$=$1$ **then**

        **return** $a \times b$;

    sign $\leftarrow$ 1;

    **if** $a < 0$ **then**

        $a \leftarrow -a$;

        $sign \leftarrow -sign$;

    **if** $b < 0$ **then**

        $b \leftarrow -b$;

        $sign \leftarrow -sign$;

    a1 $\leftarrow \lfloor \frac{a}{10^{\frac{n}{2}}} \rfloor$;

    b1 $\leftarrow \lfloor \frac{b}{10^{\frac{n}{2}}} \rfloor$;

    a2 $\leftarrow a - a1 \times 10^{\frac{n}{2}}$;

    b2 $\leftarrow b - b1 \times 10^{\frac{n}{2}}$;

    U $\leftarrow$ `Karatsuba`($a1$, $b1$, $\frac{n}{2}$);

    V $\leftarrow$ `Karatsuba`($a2$, $b2$, $\frac{n}{2}$);

    W $\leftarrow$ `Karatsuba`($a1 - a2$, $b1 - b2$, $\frac{n}{2}$);

    Z $\leftarrow$ U+V-W;

    **return** $(10^n U + 10^{\frac{n}{2}} Z + V) \times sign$

**end**

**return** `Karatsuba`($a$, $b$, $n$)

## References


- http://people.cs.uchicago.edu/~laci/HANDOUTS/karatsuba.pdf
- A. A. Karatsuba (1995). "The Complexity of Computations" (PDF). Proceedings of the Steklov Institute of Mathematics.
- https://en.wikipedia.org/wiki/Karatsuba_algorithm