1 Fast/Disrete Fourier Transform

- *Algorithm:* Cooley-Tukey algorithm (algo. 1)
- *Input:* A finite sequence f of length n which is a equally-spaced sample of a function.
- *Complexity:* $\mathcal{O}(n \log n)$
- *Data structure compatibility:* N/A
- *Common applications:* Spectral Analysis, data compression, filter bank, solving partial differential equations, etc.

Problem. Fast/Disrete Fourier Transform

Converting a finite sequence of equally-spaced samples of a function in the original domain into a same-length sequence of equally-spaced samples in the frequency domain

Description

Fourier transform is a way to convert a function in its original domain (usually time domain) into frequency domain. The main strategy is to see functions as an accumulation of sine waves in different frequency, which can also be denoted that as

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{2i\pi sx} ds$$

in which $F(s) = \int_{-\infty}^{\infty} f(x) e^{-2i\pi sx} dx$.

The formula is written in the exponential way to express sine functions, where the functions of s and $-s$ will be combined into a sine wave. By Fourier transform the ditribution of the function in different domains can be found easily, thus spectral analysis and some signal processing diveces such as filters can be implemented more easily. This can also be used in Mathematics to analyze the behaviour of a function or in solving differential equations.

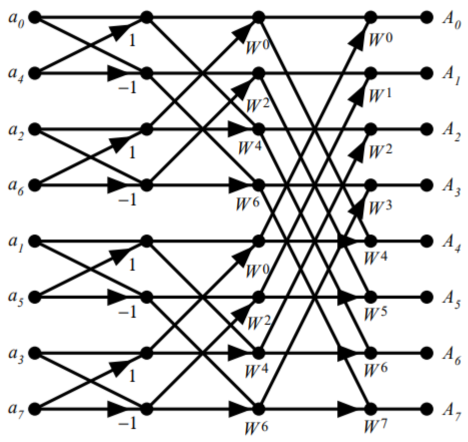
However, since computers does not have the ability to store continuous data in the memory and it also cannot generate a continuous sequence of data which represents the frequency-domain function, a new method sohuld be used such that the implementation only concerns with discrete time data which computers can collect and generate, and that is why discrete fourier transform is introduced. Discrete Fourier transform requires a finite sequence f as an input, which indicates a equally-spaced sample of data of the function. The output of discrete fourier transform is also a sequence F which also represent a sample of data in the frequency domain. The formula is given below.

$$\begin{aligned} F[k] &= \sum_{x=0}^{n-1} f[x] \cdot e^{-2ki\pi x/n} dx \\ &= \sum_{x=0}^{n-1} f[x] \cdot [cos(2\pi kx/n) - i \cdot sin(2\pi kx/n)] \end{aligned}$$

From the given formula, we can see that if brute force is applied to calculate discrete fourier transform, calculating a certain value in F needs an $\mathcal{O}(n)$ complexity to add up all terms. And since there are n elements in F, the brute force algorithm is of $\mathcal{O}(n^2)$ time complexity.

And actually this algorithm can be optimized, and so fast fourier transform is introduced. The optimizing way is based on the idea of divide and conquer. If the Fourier transform of two halves of the sequence is generated, then due to some mathematical basis, the Fourier transform of the whole sequence can be deduced more easily.

Here one of the methods called Cooley-Tukey algorithm will be discussed and for convenience, let n be a power of 2 so that it can be easily be broken int o halves repeatedly. The state diagram of the algorithm is shown below, taking $n = 8$ as an example:



Those a_i on the left are the input and A_i on the right are output. It can be seen that the inputs are not aligned in original order. In fact they are ordered in bit-reverse incrementing order, which will make the diagram more clear.

It also can be seen that each node in the middle part has two edges on the left which points two it. It means a weighted addition with respect to the value on the tip of the arrow. For example, the value of the second node in the second stage is $a_0 - a_4$. The value W in the latter stages is the value $e^{-2i\pi/n}$, and here it is $e^{-2i\pi/8} = (1 - i)/\sqrt{2}$. In the k_{th} last stage, the order of W will increment by 2^{k-1} along the nodes and the modulo of n will be taken. Of course in the first stage the order will be incrementing by $n/2$, then the number should be $1, -1, 1, -1, \ldots$. In this way, the discrete Fourier transform can be derived.

Because for each stage, there will be n numbers to be calculated, which costs $\mathcal{O}(n)$ complexity, and there should be totally $log_2 n$ stages in total, so the total complexity is $\mathcal{O}(n \log n)$

Algorithm 1: Cooley-Tukey Algorithm

```
Input  : A sequence $f$ with $n$ elements
Output: The discrete Fourier transform of $f$
for $i \leftarrow 0$ to $n-1$ do
    if $i < bit\_reverse(i)$ then
        swap($f[i], f[bit\_reverse(i)]$);
end for
W $\leftarrow e^{-2i\pi/N}$;
for $k \leftarrow 1$ to $log_2 n$ do
    b $\leftarrow$ a sequence of length $n$;
    w $\leftarrow W^{n/2^k}$;
    for $m \leftarrow 0$ to $n-1$ do
        if $mod(m, 2^k) < 2^{k-1}$ then
            $b[m] \leftarrow f[m] + w^m f[m+2^{k-1}]$;
        else
            $b[m] \leftarrow f[m-2^{k-1}] + w^m f[m]$;
        end if
        $f \leftarrow b$;
    end for
end for
return f;
```

References

- https://www.cs.cmu.edu/afs/andrew/scs/cs/15-463/2001/pub/www/notes/fourier/fourier.pdf
- James W. Cooley, Peter A. W. Lewis, and Peter W. Welch, "Historical notes on the fast Fourier transform"
- https://en.wikipedia.org/wiki/Fast_Fourier_transform