Goal: to show that $E[\frac{1}{2}(y-\hat{y})^2]$ is minimized by

$f^*(x) = E[Y \mid X = x]$ for any given x.

(1) $E[\frac{1}{2}(y-\hat{y})^2]$ is minimized when $E[(y-\hat{y})^2]$ is min.

(2) $E[(y-\hat{y})^2] = E[(\hat{y} - E[Y|X=x] + E[Y|X=x] - y)^2$

(3) let $a = \hat{y} - E[Y|X=x]$ and $b = E[Y|X] - y$.

(4) then $(2) = E[(a+b)^2] = E[a^2] + E[2ab] + E[b^2]$

Starting with $E[b^2]$;

$E[b^2] = E[(E[Y|X=x] - y)^2]$. This is invariable; not dependent on $f^*(x)$

(5) $E[2ab] = 2E[ab] = 2E[\hat{y}E[Y|X=x] - \hat{y}y - E[Y|X=x]^2 + yE[Y|X=x]]$

$E[\hat{y}] = f^*(x) = E[Y|X=x]$

$\Rightarrow (5) = 2[E[Y|X=x]^2 - E[Y|X=x]y - E[Y|X=x]^2 + yE[Y|X=x]]$

$= 0$

Therefore $E[\frac{1}{2}(y-\hat{y})^2]$ is min when $E[a^2]$ is min

$E[a^2] = E[(\hat{y} - E[Y|X=x])^2]$

$\underset{\hat{y}}{\arg\min}((\hat{y} - E[Y|X=x])^2) = E[Y|X=x]$