TWISTOR SPACES OF RIEMANNIAN MANIFOLDS WITH EVEN CLIFFORD STRUCTURES

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ABSTRACT. In this paper we introduce the twistor space of a Riemannian manifold with an even Clifford structure. This notion generalizes the twistor space of quaternion-Hermitian manifolds and weak-Spin(9) structures. We also construct almost complex structures on the twistor space for parallel even Clifford structures and check their integrability. Moreover, we prove that in some cases one can give Kähler and Nearly-Kähler metrics to these spaces.

1. INTRODUCTION

The notion of twistor space was first introduced by Roger Penrose in [12]. Following the ideas of Penrose, the twistor construction for a 4-dimensional Riemannian manifold was developed in [2]. This was later generalized for even dimensional manifolds in [11]. The twistor space Z of an even-dimensional Riemannian manifold admits a natural almost complex structure, and it is well known that such a twistor space is complex if and only if the manifold is self-dual for $\dim(M) = 4$ and locally conformally flat for $\dim(M) \geq 6$ [2] and [11]. A converse theorem (the so called reverse Penrose construction) in dimension 4 has been used to construct half-conformally flat Einstein manifolds.

In another generalization, the twistor space Z of quaternion-Kähler manifolds was defined in [13]. This is an $\mathbb{S}^2$-bundle of pointwise Hermitian structures compatible with the quaternionic structure. It is well known that this bundle admits two almost complex structures $\mathcal{J}$ and $\tilde{\mathcal{J}}$, one of which is always integrable and the other is never integrable [6]. Moreover, the manifold Z admits two Einstein metrics h and $\tilde{h}$ such that $(Z, \mathcal{J}, h)$ is Kähler-Einstein [3, 13] and $(Z, \tilde{\mathcal{J}}, \tilde{h})$ is nearly-Kähler [1].

In [7], the twistor space was defined in the context of weak Spin(9) structures on 16-dimensional Riemannian manifolds, which correspond to rank 9 even Clifford structures [9]. Additionally, this twistor construction was studied for $\mathbb{R}^{16}$, which carries a parallel flat even Clifford structure, the Cayley plane $F_4/\mathrm{Spin}(9)$, which carries a parallel non-flat even Clifford structure, and $\mathbb{S}^1 \times \mathbb{S}^{15}$, which carries a non-parallel even Clifford structure. In the first two cases, the twistor space admits a Kähler metric and in the last case the twistor space is a complex manifold which does not admit a Kähler metric.

In this paper, we generalize these constructions to even Clifford structures of arbitrary rank $r \geq 3$, noting that ranks 3 and 9 constitute two of the aforementioned constructions. We construct a $\tilde{\mathrm{Gr}}(2, r)$-bundle of pointwise Hermitian structures. This bundle admits an almost complex structure and we prove theorems analogous to those in [3, 13] and [1].

The paper is organized as follows. Section 2 is devoted to explain the notion of even Clifford structures. In section 3 we explain the construction of the twistor space of Riemannian manifolds with even Clifford structures and its almost complex structures and check their integrability. The main results of the paper are Theorems 3.1 and 3.2 as well as Corollary 3.1.

2. Even Clifford structures

The definition of a rank r even Clifford structure (or Cl_r^0 structures) on a Riemannian manifold was given in [9]:

Definition 2.1. *A rank r even Clifford structure on a Riemannian manifold (M, g) is an oriented rank r Euclidean bundle E over M together with a non-vanishing algebra bundle morphism, called a* Clifford morphism*, $\varphi : \mathrm{Cl}^0(E) \to \mathrm{End}(TM)$ which maps $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$.*

This definition contains almost Hermitian structures and quaternion-Hermitian structures as particular cases, for rank 2 and 3 respectively. An even Clifford structure (M, g, E) is called *parallel*, if there exists a metric connection ∇^E on E such that φ is connection preserving, i.e.

$$\varphi(\nabla_X^E \sigma) = \nabla_X^g \varphi(\sigma)$$

for every tangent vector $X \in TM$ and section σ of $\mathrm{Cl}^0(E)$. For instance, a manifold with a parallel Cl_2^0 structure is actually a Kähler manifold, while a manifold with a parallel Cl_3^0 structure is a quaternion-Kähler manifold. Manifolds with $\mathrm{Spin}(7)$ holonomy correspond to 8-dimensional manifolds with Cl_7^0 parallel structures. Thus, one can hope this definition gives a more general framework in which to study these geometries.

Definition 2.2. *A parallel even Clifford structure (M, E, ∇^E) is called* flat *if the connection ∇^E is flat.*

The list of complete simply connected Riemannian manifolds M carrying a parallel rank r even Clifford structure was found in [9] and is given in the tables below.

r	M	dimension of M
2	Kähler	$2m,\ m \geq 1$
3 and 4	hyper-Kähler	$4q,\ q \geq 1$
4	reducible hyper-Kähler	$4(q^+ + q^-),\ q^+ \geq 1,\ q^- \geq 1$
arbitrary	Cl_r^0 representation space	multiple of $N_0(r)$

Table 1. Manifolds with a flat even Clifford structure

r	M	dimension of M
2	Kähler	$2m,\ m \geq 1$
3	quaternion-Kähler (QK)	$4q,\ q \geq 1$
4	product of two QK manifolds	$4(q^+ + q^-)$
5	QK	8
6	Kähler	8
7	Spin(7) holonomy	8
8	Riemannian	8
5	$\mathrm{Sp}(k+2)/\mathrm{Sp}(k) \times \mathrm{Sp}(2)$	$8k,\ k \geq 2$
6	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$8k,\ k \geq 2$
8	$\mathrm{SO}(k+8)/\mathrm{SO}(k) \times \mathrm{SO}(8)$	$8k,\ k \geq 2$
9	$\mathbb{O}\mathbb{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$	16
10	$(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/\mathrm{Spin}(10) \cdot \mathrm{U}(1)$	32
12	$(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$	64
16	$(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_8/\mathrm{Spin}^+(16)$	128

Table 2. Manifolds with a parallel non-flat even Clifford structure

For the sake of simplicity, in Table 2 the non-compact duals of the compact symmetric spaces have been omitted. $N_0(r)$ denotes the dimension of the irreducible representations of Cl_r^0. For further details on even Clifford structures, we refer to [8, 9].

3. The twistor space of an even Clifford structure

Let (M, g, E) be a manifold with even Clifford structure. Let φ denote the Clifford map. Given $x \in M$, let $J_{ij} := \varphi(e_i \cdot e_j)$ where $\{e_1, \ldots, e_r\}$ is an orthonormal basis for E_x and $\cdot$ denotes Clifford multiplication. For each x we consider the subspace Z_x of

$\mathrm{End}(T_x M)$ where

$$Z_x = \left\{ J = \sum_{1 \leq i < j \leq r} a_{ij} J_{ij} \,\middle|\, J^2 = -\mathrm{Id}_{T_x M},\, a_{ij} \in \mathbb{R} \right\}$$

and define the twistor space of the even Clifford structure to be the disjoint union

$$Z = \bigsqcup_{x \in M} Z_x.$$

We will denote by π the projection onto M. This is a bundle of pointwise orthogonal complex structures. For a parallel Cl_3^0 structure this coincides with the definition of the twistor space of a quaternion-Kähler manifold, where the fibre is homeomorphic to $\mathbb{S}^2$. It is not hard to see that for a Cl_4^0 structure the fibre of the twistor space is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, which corresponds to the isomorphism between $\mathrm{Spin}(4)$ and $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$. In general the fibre at each point is isomorphic to $\tilde{\mathrm{Gr}}(2, r)$, the Grassmannian of oriented 2-planes in $\mathbb{R}^r$, as we see in the next lemma.

Lemma 3.1. *Let $A \in \mathfrak{spin}(r) \subset \mathrm{Cl}_r^0$, then $A^2 = -1$ if and only if there exist $v_1, v_2 \in \mathbb{R}^r$ orthonormal vectors such that $A = v_1 \cdot v_2$.*

Proof. Let $A \in \mathfrak{spin}(r) \subset \mathrm{Cl}_0^r$ and $\{e_1, \ldots, e_r\}$ an orthornormal basis for $\mathbb{R}^r$, then under a change of basis, we can suppose that $A = \sum_{i=1}^{[\frac{r}{2}]} a_i e_{2i-1} \cdot e_{2i}$. The condition $A^2 = -1$ yields the equations

$$\sum_{i=1}^{[\frac{r}{2}]} a_i^2 = 1, \qquad a_i a_j = 0,\, (i < j).$$

The solutions of these equations are the $r-$tuples $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, \pm 1)$. Therefore $A = \pm e_{2i-1} \cdot e_{2i}$ for some $1 \leq i \leq [\frac{r}{2}]$. Conversely, if $A = v_1 \cdot v_2$ with v_1 and v_2 orthonormal, then $A^2 = v_1 \cdot v_2 \cdot v_1 \cdot v_2 = -v_1^2 v_2^2 = -1$, which proves the assertion. $\qquad\square$

Remark: Another way to prove this is by using that an element A in $\Lambda^2 \mathbb{R}^r$ is decomposable if and only if $A \wedge A = 0$.

3.1. Almost complex structures on the twistor space.

Consider $\tilde{\mathrm{Gr}}(2, r)$ as a Hermitian-symmetric space. Its complex structure can be given using Clifford multiplication. Let $z \in \tilde{\mathrm{Gr}}(2, r)$, which for a suitable frame can be written as $z = e_1 \wedge e_2 = e_1 \cdot e_2$. The tangent space $T_z \tilde{\mathrm{Gr}}(2, r)$ can be identified with

$$\mathrm{span}(e_i \cdot e_j \,|\, i \in \{1, 2\}, j \in \{3, \ldots, r\}) = \left\{ \sum_{s=3}^{r} \alpha_s e_1 \cdot e_s + \beta_s e_2 \cdot e_s \,\middle|\, \alpha_s, \beta_s \in \mathbb{R} \right\}.$$

The complex structure is then given by $\tilde{J}_z(v) = z \cdot v$.

The Levi-Civita connection on M induces a connection on Z. For each $S \in Z$, the connection gives a splitting $T_S Z = V_S \oplus H_S$ where $V_S = \ker(\pi_*)$ is isomorphic to $T_{\pi(S)} \tilde{\mathrm{Gr}}(2, r)$, the isomorphism given by the differential of the Clifford map, and H_S, the horizontal subspace, is isomorphic to $T_{\pi(S)} M$. We recall the usual construction of almost complex structures on Z. Given $U \in V_S$ and $X \in H_S$ we define

$$\mathcal{J}(U + X)_S = \hat{J}(U) + \pi_*^{-1}(S \pi_*(X))$$

where $\hat{J}(U) = \varphi_* \tilde{J} \varphi_*^{-1}(U) = SU$.

For rank 3, this is the construction of the almost complex structure for quaternion-Kähler manifolds, so $(Z, \mathcal{J})$ is a complex manifold, see [3, 13]. For a rank 4 parallel even Clifford structures, the manifold is locally a product of two quaternion-Kähler manifolds [9]. On the other hand, $\tilde{\mathrm{Gr}}(2, 4)$ is isomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ as Kähler manifolds, so in this case the twistor space is the product of the twistor spaces of two quaternion-Kähler manifolds. In particular, it is a complex manifold.

From now on, we will suppose that (M, g) carries a parallel even Clifford structure of rank $r \geq 5$. We treat the 8-dimensional case first. In this case, the rank should be $5, 6, 7$ or 8 and the following holds:

- $r = 5$: In this case the manifold is known to be quaternion-Kähler in the case of positive curvature [9]. The twistor space has fibre isomorphic to $\mathrm{Sp}(2)/\mathrm{U}(2)$ and has been considered in [5]. The twistor space is complex exactly when M is locally symmetric, hence locally isometric to either quaternionic projective space or to $\tilde{\mathrm{Gr}}(2, 6) \cong \mathrm{Gr}_{\mathbb{C}}(2, 4)$.
- $r = 6$: In this case the manifold is known to be Kähler [9]. The twistor space has fibre $\tilde{\mathrm{Gr}}(2, 6)$ and has been considered in [11, 5]. The twistor is complex exactly when the Bochner tensor of M vanishes.
- $r = 7$: In this case the manifold has $\mathrm{Spin}(7)$ holonomy. The twistor space has fiber $\mathrm{SO}(7)/\mathrm{SO}(5) \times \mathrm{SO}(2)$. According to [5], the twistor space in this case is never complex.
- $r = 8$: In this case the manifold is Riemannian. The twistor fibre is isomorphic to $\mathrm{SO}(8)/\mathrm{U}(4)$ which is the usual fibre of the twistor space defined for even dimensional Riemannian manifolds. As mentioned in the introduction, the twistor space is complex if and only if M is conformally flat.

Now we will assume that $n \neq 8$.

Lemma 3.2. *Let M be a Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then for every $S \in Z$ and $X, Y \in T_{\pi(S)} M$, the curvature R of the Levi-Civita connection satisfies*

$$[R_{SX,SY}, S] - S[R_{SX,Y}, S] - S[R_{X,SY}, S] - [R_{X,Y}, S] = 0.$$

Proof. It suffices to prove the proposition for $S = J_{12}$. If the parallel even Clifford structure is flat, then by Theorem 2.9 in [9] the manifold is flat, so $R(X,Y) = 0$ for all $X, Y \in T_pM$. If the parallel even Clifford structure is not flat and $n \neq 8$, the proof of Proposition 2.10 in [9], explicitly Equation (15), implies the existence of a non-zero constant κ such that

$$[R_{X,Y}, J_{12}] = \kappa \sum_{s>2} g(J_{s1}X, Y)J_{s2} - g(J_{s2}X, Y)J_{s1}. \tag{3.1}$$

Using the properties of the endomorphisms J_{ij}, Lemma 2.4 [9], specifically, $J_{ij} \circ J_{ik} = J_{jk}$ for i, j, k mutually distinct, the result follows upon summing the following four calculations.

$$[R_{J_{12}X, J_{12}Y}, J_{12}] = \kappa \sum_{s>2} -g(J_{s1}X, Y)J_{s2} + g(J_{s2}X, Y)J_{s1}$$

$$-J_{12}[R_{J_{12}X, Y}, J_{12}] = \kappa \sum_{s>2} -g(J_{s2}X, Y)J_{s1} + g(J_{s1}X, Y)J_{s2}$$

$$-J_{12}[R_{X, J_{12}Y}, J_{12}] = \kappa \sum_{s>2} -g(J_{s2}X, Y)J_{s1} + g(J_{s1}X, Y)J_{s2}$$

$$-[R_{X,Y}, J_{12}] = \kappa \sum_{s>2} -g(J_{s1}X, Y)J_{s2} + g(J_{s2}X, Y)J_{s1} \qquad \square$$

Theorem 3.1. *Let M be a Riemannian manifold of dimension $n \neq 8$ carrying a parallel even Clifford structure of rank $r > 4$, then the almost complex structure $\mathcal{J}$ on Z is integrable.*

Proof. We proceed as in 14.68 of [4]. For an arbitrary vector field W, we let $\mathcal{V}(W)$ denote the vertical part of W and $\mathcal{H}(W)$ the horizontal part of W. Let $N_{\mathcal{J}}$ be the Nijenhuis tensor of $\mathcal{J}$. Let U and V be vertical vector fields and X and Y basic horizontal vector fields.

Let us first check that $N_{\mathcal{J}}(U, V) = 0$. Since U and V are vertical, $\mathcal{J}(U)$ and $\mathcal{J}(V)$ are also vertical vector fields. Thus $N_{\mathcal{J}}(U, V) = N_{\hat{J}}(U, V) = 0$, since $\hat{J}$ is a complex structure.

Now we will check that $N_{\mathcal{J}}(X, U) = 0$. From the two facts that the horizontal transport of the horizontal distribution respects $\hat{J}$, and that $[X, U]$ is vertical if U is, we obtain $[X, \mathcal{J}U] = \mathcal{J}[X, U]$. This reduces the Nijenhuis tensor to $N_{\mathcal{J}}(X, U) = \mathcal{J}([\mathcal{J}(X), U]) - [\mathcal{J}(X), \mathcal{J}(U)]$. The vertical part of this vanishes by noting that both terms in $\mathcal{V}([\mathcal{J}(X), J(U)]) = \mathcal{J}(\mathcal{V}[J(X), U])$ are tensorial in X. Finally, for the horizontal part of $N_{\mathcal{J}}(X, U)$ observe first that $\pi_*([J(X), U]) = -U\pi_*X$ from which we

obtain

$$\begin{aligned}
\pi_*(\mathcal{J}[\mathcal{J}(X), U]) &= \pi_*(\mathcal{J}\mathcal{H}[\mathcal{J}(X), U]) \\
&= \pi_*(\pi_*^{-1} S \pi_*[\mathcal{J}(X), U]) \\
&= -SU\pi_* X
\end{aligned}$$

By the same reasoning

$$\begin{aligned}
\pi_*([\mathcal{J}(X), \mathcal{J}(U)]) &= -\mathcal{J}(U)\pi_* X \\
&= -SU\pi_* X
\end{aligned}$$

and so $N_{\mathcal{J}}(X, U) = 0$.

Finally, we check that $N_{\mathcal{J}}(X, Y) = 0$. This is done by considering the horizontal and vertical components separately. For the horizontal component, we consider S in Z with $\pi(S) = x$ as a section, also denoted S, of Z about x and demand that $\nabla S = 0$ at x. This gives a local almost complex structure on a neighborhood of x which has an associated Nijenhuis tensor N_S. A direct calculation gives agreement, on the neighbourhood of x, between the two Nijenhuis tensors considered, explicitly,

$$\pi_*(N_{\mathcal{J}}(X, Y)_S) = N_S(\pi_*(X), \pi_*(Y)).$$

The tensor N_S is then seen to vanish at x as ∇ is torsion-free and, at x, ∇S vanishes. Studying the vertical component one recalls O'Neill's formulas for Riemannian submersions (see Chapter 9, [4]). In particular, $\mathcal{V}[X, Y]_{\pi(S)} = -[R_{\pi_* X, \pi_* Y}, S]$, which implies $\mathcal{V}(N_{\mathcal{J}}(X, Y)) = 0$ precisely by Lemma 3.2. $\qquad\square$

Theorem 3.2. *The twistor space $(Z, \mathcal{J})$ of a Riemannian manifold of dimension $n \neq 8$ with a parallel even Clifford structure of rank $r > 4$ and $\mathrm{Ric} > 0$ admits a Kähler metric.*

Proof. In this case the manifold (M, g) is Einstein with $\mathrm{Ric} = \kappa(n/4 + 2r - 4)$ (Proposition 2.10 [9]). Using the condition that $\mathrm{Ric} > 0$, we choose a metric h on Z such that π is a Riemannian submersion with totally geodesic fibres isometric to $\tilde{\mathrm{Gr}}(2, r)$ with Kähler metric and $\mathrm{Ric} = 2r\kappa$, so that the collection $\{J_{ij}\}$ forms a mutually orthogonal frame and $\|J_{ij}\|^2 = 1/\kappa$. Let U and V be vertical vector fields and X and Y basic horizontal vector fields. The theorem follows a similar argument to that given in 14.81 of [4]. We consider seperately the four cases coming from $(\nabla_E \mathcal{J})F$ where E, F may be horizontal or vertical.

First we show $\nabla_U \mathcal{J} = 0$. Restricting to its action on a vertical field, we immediately get $(\nabla_U \mathcal{J})V = 0$ as the fibre is Kähler and totally geodesic. In order to prove $(\nabla_U \mathcal{J})X = 0$, it suffices to consider only the horizontal component (again as the fibres are totally geodesic). By appropriately choosing a local orthonormal frame for E, we may assume that $S = J_{12}$ and $U = \lambda J_{s1}$ with $s > 2$. The Koszul formula and

the relationship between the vertical component of the Lie bracket and the curvature
mentioned in the previous proof give, at S,

$$2h(\nabla_U X, Y) = -h([X,Y], U)$$
$$= \lambda h([R_{\pi_* X, \pi_* Y}, J_{12}], J_{s1}).$$

Recalling Equation 3.1 we deduce $h(\nabla_U X, Y) = -\frac{1}{2}\lambda g(J_{s2}\pi_* X, \pi_* Y)$. Using this result,
$\pi_*(\nabla_U X)_S = -\frac{1}{2}\lambda J_{s2}\pi_* X$, we obtain

$$\pi_*(\mathcal{J}\nabla_U X)_S = -\frac{1}{2}J_{12}\lambda J_{s2}\pi_* X$$
$$= \frac{1}{2}U\pi_* X.$$

Similarly, one proves that

$$\pi_*(\nabla_U \mathcal{J}X) = \frac{1}{2}U\pi_* X,$$

from which we conclude $\pi_*((\nabla_U \mathcal{J})X) = 0$.

   Second, we show $\nabla_X \mathcal{J} = 0$. Recall O'Neill's A tensor

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F$$

where E and F are arbitrary vectors. We show, as an initial calculation, that $A_X(\mathcal{J}Y) =
\mathcal{J}(A_X Y)$ and $\mathcal{J}(A_X U) = A_X(\mathcal{J}U)$. In our situation we note the following decompo-
sition into horizontal and vertical components

$$\nabla_X U = \mathcal{V}\nabla_X U + A_X U$$
$$\nabla_X Y = A_X Y + \mathcal{H}\nabla_X Y.$$

By Proposition 9.24 in [4], we have $A_X Y = \frac{1}{2}\mathcal{V}[X, Y]$ so at $S = J_{12}$, we get $A_X Y =
-\frac{1}{2}[R_{\pi_* X, \pi_* Y}, J_{12}]$ and the claim that $A_X(\mathcal{J}Y) = \mathcal{J}(A_X Y)$ is equivalent to

$$[R_{X', J_{12}Y'}, J_{12}] = J_{12}[R_{X',Y'}, J_{12}]$$

where, for the sake of notation, we have denoted $X' = \pi_* X$ and $Y' = \pi_* Y$. Equation 3.1
gives the result as

$$J_{12}[R_{X',Y'}, J_{12}] = J_{12}(\kappa \sum_{s>2} g(J_{s1}X', Y')J_{s2} - g(J_{s2}X', Y')J_{s1})$$
$$= \kappa \sum_{s>2} -g(J_{s1}X', Y')J_{s1} - g(J_{s2}X', Y')J_{s2}$$

and similarly

$$[R_{X', J_{12}Y'}, J_{12}] = \kappa \sum_{s>2} g(J_{s1}X', J_{12}Y')J_{s2} - g(J_{s2}X', J_{12}Y')J_{s1}$$
$$= \kappa \sum_{s>2} -g(J_{s2}X', Y')J_{s2} - g(J_{s1}X', Y')J_{s1}.$$

For the second claim, we use the skew symmetry of A, $h(A_X Y, U) = -h(Y, A_X U)$ to obtain

$$h(\mathcal{J} A_X U, Y) = h(U, A_X \mathcal{J} Y)$$
$$= h(U, \mathcal{J} A_X Y)$$
$$= h(A_X \mathcal{J} U, Y).$$

Therefore $\mathcal{J}(A_X U) = A_X(\mathcal{J} U)$.

We apply this result to $(\nabla_X \mathcal{J}) U$ where

$$(\nabla_X \mathcal{J}) U = \nabla_X (\mathcal{J} U) - \mathcal{J} \nabla_X U$$
$$= \mathcal{V} \nabla_X (\mathcal{J} U) + A_X \mathcal{J} U - \mathcal{J} \mathcal{V} \nabla_X U - \mathcal{J} A_X U$$
$$= \mathcal{V} \nabla_X (\mathcal{J} U) - \mathcal{J} \mathcal{V} \nabla_X U$$

Taking the inner product of each term with V and studying the respective Koszul formulas gives the result that $(\nabla_X \mathcal{J}) U = 0$. By a similar calculation for $(\nabla_X \mathcal{J}) Y$,

$$(\nabla_X \mathcal{J}) Y = \mathcal{H} \nabla_X (\mathcal{J} Y) - \mathcal{J} \mathcal{H} \nabla_X Y.$$

As this is horizontal we may use a similar idea to that presented in the preceding proof. Specifically, we may consider $S \in Z$ with $x = \pi(S)$ as a section over a neighborhood of x with $\nabla S = 0$ at x. Studying the appropriate Koszul formulas one concludes that, at x,

$$(\nabla_X \mathcal{J}) Y = (\nabla_{\pi_* X} S) \pi_* Y.$$

The result now follows since, at x,

$$(\nabla_X \mathcal{J}) Y = \pi_*^{-1} (\nabla_{\pi_* X} (S \pi_* Y) - S \nabla_{\pi_* X} \pi_* Y)$$
$$= \pi_*^{-1} ((\nabla_{\pi_* X} S) \pi_* Y) = 0. \qquad \square$$

The following tables summarize our findings.

r	M	$\dim(M)$	fibre of Z	type of Z
r	Cl_r^0 representation	$N_0(r)n$	$\tilde{\mathrm{Gr}}(2, r)$	complex, Kähler
3	QK manifold	$4n$	$\mathbb{S}^2$	complex, Kähler if Ric > 0
4	$M_1 \times M_2$, M_i QK	$4(n_1 + n_2)$	$\mathbb{S}^2 \times \mathbb{S}^2$	complex, Kähler if Ric$(M_i) > 0$
5	QK	8	Sp(2)/U(2)	complex if locally symmetric
6	Kähler	8	U(4)/U(2) $\times$ U(2)	complex if Bochner tensor $\equiv 0$
7	Spin(7) holonomy	8	$\tilde{\mathrm{Gr}}(2, 7)$	not complex
8	Riemannian	8	SO(8)/U(4)	complex if Weyl tensor $\equiv 0$

Table 3. Twistor spaces for low rank and dimension 8

r	M	$\dim(M)$	fibre of Z	type of Z
5	$\mathrm{Sp}(k+2)/(\mathrm{Sp}(k) \times \mathrm{Sp}(2))$	$8k$, $k \geq 2$	$\mathrm{Sp}(2)/\mathrm{U}(2)$	Kähler
6	$\mathrm{SU}(k+4)/\mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(4))$	$8k$, $k \geq 2$	$\mathrm{U}(4)/\mathrm{U}(2) \times \mathrm{U}(2)$	Kähler
8	$\mathrm{SO}(k+8)/(\mathrm{SO}(k) \times \mathrm{SO}(8))$	$8k$, $k \geq 2$	$\mathrm{SO}(8)/\mathrm{U}(4)$	Kähler
9	$\mathrm{F}_4/\mathrm{Spin}(9)$	16	$\tilde{\mathrm{Gr}}(2,9)$	Kähler
10	$\mathrm{E}_6/(\mathrm{Spin}(10) \cdot \mathrm{U}(1))$	32	$\tilde{\mathrm{Gr}}(2,10)$	Kähler
12	$\mathrm{E}_7/(\mathrm{Spin}(12) \cdot \mathrm{SU}(2))$	64	$\tilde{\mathrm{Gr}}(2,12)$	Kähler
16	$\mathrm{E}_8/\mathrm{Spin}^+(16)$	128	$\tilde{\mathrm{Gr}}(2,16)$	Kähler

Table 4. Twistor spaces for higher rank

For the non-compact dual spaces of these symmetric spaces, the twistor space is only complex as the negative curvature obstructs the construction of an appropriate metric on the fibres.

Finally, we use the following observation of Nagy [10] to construct nearly Kähler metrics on the twistor space. Consider a Riemannian submersion with totally geodesic fibres

$$F \to (Z, h) \to M$$

and let $TZ = V \oplus H$ be the corresponding splitting of TZ. Suppose that Z admits a complex structure $\mathcal{J}$ compatible with h and preserving V and H such that $(Z, \mathcal{J}, h)$ is a Kähler manifold. Consider now the Riemannian metric on Z defined by

$$\tilde{h}(X, Y) = \frac{1}{2}h(X, Y) \text{ for } X, Y \in V,$$

$$\tilde{h}(X, Y) = h(X, Y) \text{ for } X, Y \in H.$$

The metric $\tilde{h}$ admits a compatible almost complex structure $\tilde{\mathcal{J}}$ given by $\tilde{\mathcal{J}}_{|V} = -\mathcal{J}$ and $\tilde{\mathcal{J}}_{|H} = \mathcal{J}$. The next proposition is proved in [10].

Proposition 3.1. [10] *The manifold $(Z, \tilde{\mathcal{J}}, \tilde{h})$ is nearly Kähler.*

$\square$

Corollary 3.1. *The twistor space Z of a Riemannian manifold with a parallel even Clifford structure of rank $r \geq 3$ and $\mathrm{Ric} > 0$, admits an almost complex structure $\tilde{\mathcal{J}}$ and a metric $\tilde{h}$ such that $(Z, \tilde{\mathcal{J}}, \tilde{h})$ is nearly Kähler.*

$\square$

In our case, using the definition of the almost complex structure, one can easily check that this almost complex structure is never integrable.

We conclude by pointing out that even though a classification of parallel even Clifford strucures was given in [9], and one can try to deal with each of these cases seperately, our approach does not rely on this classification (except for dimension 8 in which the

curvature condition is not automatically satisfied). Furthermore, the constructions above can be studied in a more general context. One could check integrability conditions of these twistor spaces for manifolds with non parallel even Clifford structures, as in [7]. In fact, in order for the twistor space to be complex, Lemma 3.2 should be satisfied for every S in the twistor space. One nice example is given by $\mathbb{S}^1 \times \mathbb{S}^{15}$, which admits a non parallel Cl_9^0 structure but its twistor space is a complex manifold which cannot be Kähler since its first Betti number is odd.

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