LOCAL GEOMETRY OF EVEN CLIFFORD STRUCTURES ON CONFORMAL MANIFOLDS

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Abstract. We introduce the concept of a Clifford-Weyl structure on a conformal manifold, which consists of an even Clifford structure parallel with respect to the tensor product of a metric connection on the Clifford bundle and a Weyl structure on the manifold. We show that the Weyl structure is necessarily closed except for some "generic" low-dimensional instances, where explicit examples of non-closed Clifford-Weyl structures can be constructed.

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1. Introduction

Even Clifford structures on Riemannian manifolds were introduced in [9] as a natural generalisation of almost Hermitian and almost quaternion Hermitian geometries and consist of a (locally defined) Euclidean vector bundle (E, h) over (M, g) together with an algebra bundle morphism $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ mapping $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(\mathrm{T}M)$. Alternatively, one can define even Clifford structures as structure group reductions, cf. [1] and [4] for details. Several articles appeared recently on this topic, see e.g. [2], in which the twistor space is constructed, [3] which studies even Clifford structures with large automorphism groups, [7] for rigidity and vanishing results, [8] for the classification of homogeneous even Clifford structures, or [11] and [12] for relations with symmetric spaces and the Severi varieties.

The present paper studies even Clifford structures over conformal – rather than Riemannian – manifolds. We introduce the notion of a Clifford-Weyl structure, defined by the following data: a conformal manifold (M, c), equipped with a Weyl structure D and an even Clifford structure (E, h, φ) which is parallel with respect to D and some metric connection ∇^E on E. In the case where D is exact (that is, the Levi-Civita connection of some metric in c), this notion is equivalent to that of a parallel even Cliford structure from [9]. Immediately the natural question to ask is under what conditions does this problem locally reduce to a problem in Riemannian geometry, i.e. under what conditions is D closed? We show that there are six instances (called generic) where the presence of a Clifford-Weyl structure need not force the Weyl connection to be closed, and that in all other cases (called non-generic), the associated Weyl structure of a Clifford-Weyl structure is automatically closed. More precisely, our results can be stated as follows:

Theorem 1. *Suppose a conformal manifold of dimension n carries a rank $r \geq 2$ Clifford-Weyl structure such that (n, r) is different from $(2, 2)$, $(4, 2)$, $(4, 3)$, $(4, 4)$ and $(8, 8)$. Then the associated Weyl connection is closed. The same conclusion holds if $(n, r) = (8, 4)$, provided that the restriction of the Clifford morphism φ to $\Lambda^2 E$ is not injective.*

The cases excluded by this theorem are somehow generic, and they are treated in the following

Proposition 2. *(i) Let D be a Weyl structure on an oriented conformal manifold (M, c) of dimension 2, 4 or 8. Then (M, c) carries a Clifford-Weyl structure of rank $r = 2$ for $n = 2$, $r = 3$ or $r = 4$ for $n = 4$ and $r = 8$ for $n = 8$, whose associated Weyl structure is D.*

(ii) Let D be a Weyl structure on a conformal manifold (M^n, c). Then there exists a Clifford-Weyl structure of rank 2 on (M, c) with associated Weyl structure D if and only if D preserves a complex structure compatible with c. If $n = 4$, every complex structure J compatible with c is preserved by a unique Weyl structure D^J, which is closed if and only if J is locally conformally Kähler.

(iii) Let D be a Weyl structure on a conformal manifold (M^8, c). Then there exists a Clifford-Weyl structure of rank 4 whose Clifford morphism $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}(\mathrm{T}M)$ is injective upon restriction to $\Lambda^2 E$, if and only if D is the adapted Weyl structure of a conformal product structure on (M, c) with 4-dimensional factors (cf. [5]*).*

The paper is organised as follows. Section 2 recalls several notions of even Clifford structures, defines Clifford-Weyl structures and introduces the required differential and algebraic objects from differential and conformal geometry. It finishes with a toy problem from Hermitian geometry which inspires the beginning of the proof of the theorem (and provides the proof for the case $n > 4$, $r = 2$). Section 3 establishes the theorem for large rank $r \geq 5$ structures and Section 4 establishes the theorem in the remaining low rank setting. Section 5 considers the generic cases of Proposition 2 and shows that in each of these cases there are examples of Clifford-Weyl structures with non-closed associated Weyl structures.

2. Preliminaries

Let M be a smooth manifold. It is well-known that there is a one-to-one correspondence between isomorphism classes of oriented Euclidean rank k vector bundles (E, h)

over M and isomorphism classes of principal $\mathrm{SO}(k)$-bundles P over M. By this correspondence, one may identify metric covariant derivatives on (E,h) with connections on P. We denote

$$\mathrm{PSO}(k) := \begin{cases} \mathrm{SO}(k) & \text{if } k \text{ is odd} \\ \mathrm{SO}(k)/\{\pm \mathrm{I}_k\} & \text{if } k \text{ is even} \end{cases}$$

and by a slight abuse of language we set the following

Definition 3. *A* locally defined *oriented Euclidean rank k vector bundle over M is a principal* $\mathrm{PSO}(k)$*-bundle over M.*

The terminology is justified by the fact that the structure group of a principal $\mathrm{PSO}(k)$-bundle can be reduced to $\mathrm{SO}(k)$ over any contractible open neighborhood U of M, and thus gives rise to an oriented Euclidean rank k vector bundle over U.

If E is a locally defined oriented Euclidean rank k vector bundle over M and $\rho : \mathrm{PSO}(k) \to \mathrm{SO}(N)$ is a group morphism, one obtains a rank N oriented Euclidean vector bundle $\rho(E)$ over M by enlarging the structure group of E to $\mathrm{SO}(N)$ and considering the associated vector bundle. In particular, since the even-dimensional tensor powers of the standard representation of $\mathrm{SO}(k)$ on $\mathbb{R}^k$ descend to $\mathrm{PSO}(k)$, the even tensor powers of a locally defined oriented Euclidean vector bundle are globally defined vector bundles.

We can now recall the definition of even Clifford structures on Riemannian manifolds, which have been introduced in [9].

Definition 4. *A rank $r \geq 2$* even Clifford structure *on a Riemannian manifold (M^n, g) is an oriented, locally defined, rank r Euclidean bundle (E,h) over M together with a non-vanishing algebra bundle morphism, called* Clifford morphism, *$\varphi : \mathrm{Cl}^0(E,h) \to \mathrm{End}(\mathrm{T}M)$ which maps $\Lambda^2 E \subset \mathrm{Cl}^0(E,h)$ to the bundle of skew-symmetric endomorphisms* $\mathrm{End}^-(\mathrm{T}M)$

Definition 5. *An even Clifford structure (M, g, E, h, φ), is called* parallel *if there exists a metric connection ∇^E on (E,h) such that φ is connection preserving with respect to ∇^E and the Levi-Civita connection ∇ of (M,g).*

As $\mathrm{End}^-(\mathrm{T}M)$ is invariant under a conformal change of the metric g, the notion of an even Clifford structure extends directly to the setting of a conformal manifold (M,c). The condition of parallelism is transferred by considering Weyl connections giving what we term Clifford-Weyl structures.

Definition 6. *A rank $r \geq 2$* Clifford-Weyl structure *on a conformal manifold (M^n, c), is a tuple $(E, h, \varphi, \nabla^E, D)$ where*

- *(E,h) is an oriented locally defined rank r Euclidean bundle;*

- $\varphi : \mathrm{Cl}^0(E,h) \to \mathrm{End}(\mathrm{T}M)$ *is an algebra bundle morphism sending* $\Lambda^2 E$ *to* $\mathrm{End}^-(\mathrm{T}M)$*;*
- ∇^E *is a metric connection on* E*;*
- D *is a Weyl connection on* (M,c)*,*

such that φ*, seen as a section of* $\mathrm{Cl}^0(E,h)^* \otimes \mathrm{End}(\mathrm{T}M)$*, is parallel with respect to* $\nabla^E \otimes D$*.*

Let (E,h,φ,∇^E,D) be a rank r Clifford-Weyl structure on a conformal manifold (M^n,c) and let L denote the weight bundle of M (the real line bundle associated with the principal bundle of frames via the representation $|\det|^{1/n}$ of $\mathrm{GL}(n;\mathbb{R})$, cf. [5, Section 2]).

Consider a metric g in the conformal class c. Associated with g we have the Levi-Civita connection ∇ as well as the gauge ℓ (a section of L) and Lee form θ defined respectively by

$$c = g \otimes \ell^2, \qquad D\ell = \theta \otimes \ell.$$

This gives the useful formula $Dg = -2\theta \otimes g$. Independent of the choice of metric in the conformal class, we have the Faraday form $F = \mathrm{d}\theta$. The Weyl structure D is closed if and only if D is locally the Levi-Civita connection of a metric in the conformal class. This is equivalent to $F = 0$.

Let $\xi_1,\dots,\xi_r$ be a local oriented orthonormal frame for (E,h). We introduce the collection of connection coefficients and the curvature two-forms

$$\nabla^E \xi_j =: \sum_i \eta_{ij} \otimes \xi_i, \quad R^E \xi_j =: \sum_i \omega_{ij} \otimes \xi_i,$$

(with $\omega_{ij} = \mathrm{d}\eta_{ij} + \sum_k \eta_{ik} \wedge \eta_{kj}$), and define endomorphisms $J_{ij} := \varphi(\xi_i \cdot \xi_j)$ where $\cdot$ denotes Clifford multiplication. Because of the relations in the Clifford algebra, and the fact that φ is an algebra bundle morphism mapping $\Lambda^2 E$ into $\mathrm{End}^-(\mathrm{T}M)$, the endomorphisms J_{ij} are locally defined almost Hermitian structures on M for $i \neq j$. Moreover, for mutually distinct indices i,j,k we have

$$(\xi_i \cdot \xi_j) \cdot (\xi_i \cdot \xi_k) = \xi_j \cdot \xi_k = -(\xi_i \cdot \xi_k) \cdot (\xi_i \cdot \xi_j)$$

and thus J_{ij} anticommutes with J_{i_k}. In particular, this shows that

$$\langle J_{ij}, J_{ik} \rangle = 0, \qquad \forall\, i \neq j \neq k \neq i, \tag{2.1}$$

where $\langle \cdot,\cdot \rangle$ denotes, as usual, minus the trace of the product of two endomorphisms.

Let $e_1,\dots,e_n$ denote a local orthonormal frame for $(\mathrm{T}M,g)$. For each J_{ij} with $i \neq j$, we obtain an associated non-degenerate two-form Ω_{ij}:

$$\Omega_{ij}(\cdot,\cdot) := g(J_{ij}\,\cdot,\cdot).$$

(In the process of establishing (3.2), as well in the the rank $r = 4$ case, the calculations are simplified by summing indiscriminately over subscripts, for this we define $\Omega_{ii} := 0$.) Using the natural scalar product on the bundle of exterior forms induced by g, we obtain the Lefschetz-type operators for $i \neq j$,

$$\mathrm{L}_{ij} := \Omega_{ij} \wedge, \quad \Lambda_{ij} := \mathrm{L}_{ij}^* = \tfrac{1}{2} \sum_{a=1}^{n} J_{ij}(e_a) \lrcorner e_a \lrcorner.$$

For later use, notice that by the usual identification using the metric g of $\Lambda^2(\mathrm{T}^*M)$ with $\mathrm{End}^-(\mathrm{T}M)$, the Lefschetz operator Λ_{ij} acting on $\Lambda^2(\mathrm{T}^*M)$ (with $i \neq j$) is identified with $\frac{1}{2}\langle J_{ij}, \cdot\rangle$ acting on $\mathrm{End}^-(\mathrm{T}M)$.

We finish this section with a toy problem: That when $r = 2$ and $n > 4$, the Weyl connection is closed. This is a standard fact in Hermitian geometry, but the proof below contains, at embryonic state, the main ideas of the proof of Theorem 1.

We choose as before a metric g in the conformal class and drop the superfluous subscripts on J_{12} and Ω_{12}. First, the fact that φ is $\nabla^E \otimes D$-parallel implies

$$DJ = D(\varphi(\xi_1 \cdot \xi_2)) = \varphi(\nabla^E(\xi_1 \cdot \xi_2))$$

and as

$$\nabla^E(\xi_1 \cdot \xi_2) = (\eta_{21} \otimes \xi_2) \cdot \xi_2 + \xi_1 \cdot (\eta_{12} \otimes \xi_1) = -(\eta_{12} + \eta_{21}) 1_{\mathrm{Cl}^0(E,h)} = 0,$$

we conclude J is parallel with respect to D. Differentiating Ω with respect to the Weyl connection

$$(D\Omega)(\cdot,\cdot) = (Dg)(J\cdot,\cdot) + g(DJ\cdot,\cdot)$$

and using the formulae $DJ = 0$ and $Dg = -2\theta \otimes g$ gives $D\Omega = -2\theta \otimes \Omega$ which upon extracting the totally antisymmetric part yields

$$\mathrm{d}\Omega = -2\theta \wedge \Omega.$$

Differentiating this equation gives

$$0 = \mathrm{d}^2\Omega = -2F \wedge \Omega - 4\theta \wedge \theta \wedge \Omega$$

and as Ω is non-degenerate with $n > 4$, the equation $F \wedge \Omega = 0$ forces $F = 0$.

3. Large rank Clifford-Weyl structures

For this section suppose that $(E, h, \varphi, \nabla^E, D)$ is a rank r Clifford-Weyl structure on a conformal manifold (M^n, c), with $r \geq 5$. The structure of Cl_r^0 forces the dimension of the manifold to be a multiple of 8. Apart from the generic situation $n = 8$, $r = 8$, which will be treated later on, we will show that the Weyl connection is closed.

As in the previous section, we consider an arbitrary Riemannian metric g in the conformal class c. Then (E, h) is an even Clifford structure, and we may build Lefschetz-type operators as well as identify $\Lambda^2(\mathrm{T}^*M)$ with $\mathrm{End}^-(\mathrm{T}M)$.

The connection coefficients give for all $i \neq j$

$$\begin{aligned}\nabla^E(\xi_i \cdot \xi_j) &= \sum_k \eta_{ki} \otimes (\xi_k \cdot \xi_j) + \eta_{kj} \otimes (\xi_i \cdot \xi_k) \\ &= \sum_{k \neq i,j} \eta_{ki} \otimes (\xi_k \cdot \xi_j) - \eta_{kj} \otimes (\xi_k \cdot \xi_i)\end{aligned}$$

and as the even Clifford structure is parallel,

$$DJ_{ij} = \sum_{k \neq i,j} \eta_{ki} \otimes J_{kj} - \eta_{kj} \otimes J_{ki}.$$

Differentiating Ω_{ij} with respect to the Weyl connection

$$(D\Omega_{ij})(\cdot,\cdot) = (Dg)(J_{ij}\cdot,\cdot) + g(DJ_{ij}\cdot,\cdot)$$

and using the previous formula as well as the fundamental formula $Dg = -2\theta \otimes g$ provides

$$D\Omega_{ij} = -2\theta \otimes \Omega_{ij} + \sum_{k \neq i,j} \eta_{ki} \otimes \Omega_{kj} - \eta_{kj} \otimes \Omega_{ki}.$$

Taking the totally antisymmetric part of this equation (and recalling $\Omega_{ii} := 0$) gives

$$\mathrm{d}\Omega_{ij} = -2\theta \wedge \Omega_{ij} + \sum_k \eta_{ki} \wedge \Omega_{kj} - \eta_{kj} \wedge \Omega_{ki}.$$

Differentiating this equation and replacing appearances of $\mathrm{d}\Omega_{ij}$ (as well as $\mathrm{d}\Omega_{kj}$ and $\mathrm{d}\Omega_{ki}$) using this same equation yields

$$\begin{aligned}2F \wedge \Omega_{ij} &= \sum_k \Big(2\theta \wedge \eta_{ki} \wedge \Omega_{kj} + \mathrm{d}\eta_{ki} \wedge \Omega_{kj} - \eta_{ki} \wedge \mathrm{d}\Omega_{kj}\Big) - \{i \leftrightarrow j\} \\ &= \sum_k \Big(\mathrm{d}\eta_{ki} \wedge \Omega_{kj} - \eta_{ki} \wedge \sum_\ell \Big(\eta_{\ell k} \wedge \Omega_{\ell j} - \eta_{\ell j} \wedge \Omega_{\ell k}\Big)\Big) - \{i \leftrightarrow j\} \\ &= \sum_k \Big(\mathrm{d}\eta_{ki} + \sum_\ell \eta_{k\ell} \wedge \eta_{\ell i}\Big) \wedge \Omega_{kj} - \{i \leftrightarrow j\}\end{aligned}$$

where $\{i \leftrightarrow j\}$ corresponds to the previously displayed term with indices i and j interchanged. The previous equation simplifies upon introducing the curvature two-forms ω_{ij} of ∇^E into

$$2F \wedge \Omega_{ij} = \sum_k \omega_{ki} \wedge \Omega_{kj} - \omega_{kj} \wedge \Omega_{ki}.$$

Writing this using the Lefschetz-type operators establishes, for all $i \neq j$,

$$2\mathrm{L}_{ij}F = \sum_k \mathrm{L}_{kj}\omega_{ki} - \mathrm{L}_{ki}\omega_{kj}. \tag{3.1}$$

Assuming $i \neq j$, we apply Λ_{ij} to (3.1). Calculating Λ_{ij} applied to the left hand side of (3.1) is aided by the $\mathfrak{sl}(2)$ structure of the Lefschetz-type operators. Specifically $2[\Lambda_{ij}, \mathrm{L}_{ij}]F = (n-4)F$ and $\mathrm{L}_{ij}(2\Lambda_{ij}F)$ identifies, as an endomorphism, with $\langle J_{ij}, F\rangle J_{ij}$.

In order to calculate Λ_{ij} applied to the right hand side of (3.1), we note that the sum in (3.1) may be taken over $k \neq i, j$ and we write

$$\Lambda_{ij} = \tfrac{1}{2} \sum_a J_{ij}(e_a) \lrcorner\, e_a \lrcorner$$

(recall that $e_1, \dots e_n$ denotes a local orthonormal frame for $(\mathrm{T}M, g)$). This gives, for $k \neq i, j$,

$$2\Lambda_{ij} \mathrm{L}_{kj} \omega_{ki} = \sum_a J_{ij}(e_a) \lrcorner \Big(\Omega_{kj}(e_a) \wedge \omega_{ki} + \Omega_{kj} \wedge \omega_{ki}(e_a) \Big)$$

The summands in the previous display consist of four terms, the first of which vanishes because of (2.1). Developing the remaining three terms gives

$$2\Lambda_{ij} \mathrm{L}_{kj} \omega_{ki} = \sum_a -\Omega_{kj}(e_a) \wedge (\omega_{ki} \circ J_{ij})(e_a) + (\Omega_{kj} \circ J_{ij})(e_a) \wedge \omega_{ki}(e_a) + \langle J_{ij}, \omega_{ki} \rangle \Omega_{kj}.$$

Testing against two tangent vectors shows

$$\sum_a -\Omega_{kj}(e_a) \wedge (\omega_{ki} \circ J_{ij})(e_a) = \omega_{ki}(J_{ik}\cdot, \cdot) + \omega_{ki}(\cdot, J_{ik}\cdot)$$
$$\sum_a (\Omega_{kj} \circ J_{ij})(e_a) \wedge \omega_{ki}(e_a) = \omega_{ki}(J_{ik}\cdot, \cdot) + \omega_{ki}(\cdot, J_{ik}\cdot)$$

which, viewed as endomorphisms via the metric, are each precisely $[J_{ki}, \omega_{ki}]$. Therefore, for i, j, k distinct,

$$\Lambda_{ij} \mathrm{L}_{kj} \omega_{ki} = [J_{ki}, \omega_{ki}] + \tfrac{1}{2} \langle_{ij}, \omega_{ki} \rangle \Omega_{kj}$$

which establishes, for $i \neq j$,

$$(n-4)F + \langle J_{ij}, F \rangle \Omega_{ij} = \sum_{k \neq i,j} [J_{ki}, \omega_{ki}] + [J_{kj}, \omega_{kj}] + \tfrac{1}{2} \langle J_{ij}, \omega_{ki} \rangle \Omega_{kj} - \tfrac{1}{2} \langle J_{ij}, \omega_{kj} \rangle \Omega_{ki}. \tag{3.2}$$

Working with (3.2), we apply $2\Lambda_{ij}, 2\Lambda_{ia}, 2\Lambda_{ab}$ for a, b different from i, j. For $2\Lambda_{ij}$ applied to (3.2) we remark that $\langle J_{ij}, J_{ij} \rangle = n$ while J_{ij} is orthogonal to J_{kj} and J_{ki} for i, j, k distinct hence

$$\begin{aligned}(2n-4)\langle J_{ij}, F \rangle &= \sum_{k \neq i,j} \langle J_{ij}, [J_{ki}, \omega_{ki}] \rangle + \langle J_{ij}, [J_{kj}, \omega_{kj}] \rangle \\ &= \sum_{k \neq i,j} \langle [J_{ij}, J_{ki}], \omega_{ki} \rangle + \langle [J_{ij}, J_{kj}], \omega_{kj} \rangle\end{aligned}$$

establishing

$$(2n-4)\langle J_{ij}, F \rangle = 2 \sum_{k \neq i,j} \langle J_{kj}, \omega_{ki} \rangle - \langle J_{ki}, \omega_{kj} \rangle, \tag{3.3}$$

For $2\Lambda_{ia}$ applied to (3.2), we remark that $\langle J_{ia}, J_{ki}\rangle = -n\delta_{ak}$ for $k \neq i, j$ and importantly, as $r \geq 5$, the terms involving $\langle J_{ia}, J_{kj}\rangle$ vanish [9, Equation 2]. Therefore

$$(n-4)\langle J_{ia}, F\rangle = \tfrac{n}{2}\langle J_{ij}, \omega_{aj}\rangle + \sum_{k\neq i,j} \langle [J_{ia}, J_{ki}], \omega_{ki}\rangle + \langle [J_{ia}, J_{kj}], \omega_{kj}\rangle$$

where

$$\sum_{k\neq i,j} \langle [J_{ia}, J_{ki}], \omega_{ki}\rangle = \sum_{k\neq i,j} \langle J_{ka} - J_{ak}, \omega_{ki}\rangle = 2\sum_{k\neq j} \langle J_{ka}, \omega_{ki}\rangle$$

and since J_{ia} commutes with J_{kj} for $k \neq i, j$ except when $k = a$,

$$\sum_{k\neq i,j} \langle [J_{ia}, J_{kj}], \omega_{kj}\rangle = \langle [J_{ia}, J_{aj}], \omega_{aj}\rangle = -2\langle J_{ij}, \omega_{aj}\rangle$$

establishing

$$(n-4)\langle J_{ia}, F\rangle = (\tfrac{n}{2} - 2)\langle J_{ij}, \omega_{aj}\rangle + 2\sum_{k\neq j} \langle J_{ka}, \omega_{ki}\rangle, \tag{3.4}$$

For $2\Lambda_{ab}$ applied to (3.2), we again use the large rank hypothesis $r \geq 5$. Indeed, due to this condition, for $k \neq i, j$, terms involving $\langle J_{ab}, J_{kj}\rangle$ and $\langle J_{ab}, J_{ki}\rangle$ vanish. So

$$\begin{aligned}(n-4)\langle J_{ab}, F\rangle &= \sum_{k\neq i,j} \langle [J_{ab}, J_{ki}], \omega_{ki}\rangle + \langle [J_{ab}, J_{kj}], \omega_{kj}\rangle \\ &= \sum_{k\in\{a,b\}} \langle [J_{ab}, J_{ki}], \omega_{ki}\rangle + \langle [J_{ab}, J_{kj}], \omega_{kj}\rangle\end{aligned}$$

and developing the four terms from the summation in the preceding display establishes

$$(n-4)\langle J_{ab}, F\rangle = 2\langle J_{bi}, \omega_{ai}\rangle - 2\langle J_{ai}, \omega_{bi}\rangle + 2\langle J_{bj}, \omega_{aj}\rangle - 2\langle J_{aj}, \omega_{bj}\rangle. \tag{3.5}$$

Armed with the preceding numbered equations, we may establish the orthogonality between J_{ij} and F. From (3.5), by collecting the first two terms, and collecting the second two terms, we see that $\langle J_{bi}, \omega_{ai}\rangle - \langle J_{ai}, \omega_{bi}\rangle$ is independent of i so

$$(n-4)\langle J_{ab}, F\rangle = 4\langle J_{bi}, \omega_{ai}\rangle - 4\langle J_{ai}, \omega_{bi}\rangle$$

Summing the previous display over $i \neq a, b$ and changing the notation of indices $a, b, i \to i, j, k$ gives

$$(r-2)(n-4)\langle J_{ij}, F\rangle = 4\sum_{k\neq i,j} \langle J_{kj}, \omega_{ki}\rangle - \langle J_{ki}, \omega_{kj}\rangle.$$

Comparing this equation with (3.3) provides the constraint

$$(r-2)(n-4)\langle J_{ij}, F\rangle = 2(2n-4)\langle J_{ij}, F\rangle$$

whence $\langle J_{ij}, F\rangle = 0$ unless $4(2n-4) = 2(r-2)(n-4)$. As $r \geq 5$ and n is a multiple of 8, the only obstructive case is the generic case $n = 8$, $r = 8$, which was excluded. Therefore

$$\langle J_{ij}, F\rangle = 0 \qquad \forall\ i \neq j. \tag{3.6}$$

Updating (3.4) and (3.5) using this orthogonality, we obtain a pair symmetry from (3.5)

$$\langle J_{ia}, \omega_{ja}\rangle = \langle J_{ja}, \omega_{ia}\rangle, \qquad \forall\ i, j, a \text{ distinct.} \tag{3.7}$$

which, upon switching the variables j, a in (3.4), provides

$$(2 - \tfrac{n}{2})\langle J_{ia}, \omega_{ja}\rangle = -2\langle J_{aj}, \omega_{ai}\rangle + 2\sum_k \langle J_{kj}, \omega_{ki}\rangle,$$

giving

$$(2 - \tfrac{n}{4})\langle J_{ia}, \omega_{ja}\rangle = \sum_k \langle J_{ik}, \omega_{jk}\rangle, \qquad \forall\ i, j, a \text{ distinct.} \tag{3.8}$$

Therefore if $n = 8$, the sum on the right hand side vanishes, while if $n \neq 8$, $\langle J_{ia}, \omega_{ja}\rangle$ is independent of $a \neq i, j$ hence

$$(2 - \tfrac{n}{4})\langle J_{ia}, \omega_{ja}\rangle = (r-2)\langle J_{ia}, \omega_{ja}\rangle$$

and so J_{ia} is orthogonal to ω_{ja}. It thus turns out that the sum $\sum_k \langle J_{ik}, \omega_{jk}\rangle$ vanishes no matter what the dimension n is.

As a penultimate result, we remark that upon summation over $j \neq i$ (for i fixed), the final two terms of (3.2) vanish:

$$\sum_{j\neq i}\left(\sum_{k\neq i,j} \langle J_{ij}, \omega_{ki}\rangle J_{kj} - \langle J_{ij}, \omega_{kj}\rangle J_{ki}\right) = 0. \tag{3.9}$$

In fact, the previous display naturally splits into two collections of summations, each collection vanishing independently as we now show. The first collection of summations in (3.9) may be written as the sum over j, k both different from i and from each-other:

$$\sum_{j\neq i}\left(\sum_{k\neq i,j} \langle J_{ij}, \omega_{ki}\rangle J_{kj}\right) = \sum_{\substack{j,k\neq i\\ j\neq k}} \langle J_{ij}, \omega_{ki}\rangle J_{kj}$$

which thus vanishes as $\langle J_{ij}, \omega_{ki}\rangle$ is symmetric in j, k due to (3.7) while J_{jk} is antisymmetric in j, k. Considering the second collection of summations in (3.9), we rearrange the summation,

$$\begin{aligned}
\sum_{j\neq i}\left(\sum_{k\neq i,j}\langle J_{ij},\omega_{kj}\rangle J_{ki}\right) &= \sum_{j\neq i}\left(\sum_{k\neq i}\langle J_{ij},\omega_{kj}\rangle J_{ki}\right)\\
&= \sum_{k\neq i}\left(\sum_{j\neq i}\langle J_{ij},\omega_{kj}\rangle\right) J_{ki}\\
&= \sum_{k\neq i,j}\left(\sum_{j}\langle J_{ij},\omega_{kj}\rangle\right) J_{ki}
\end{aligned}$$

and by the remark following (3.8), the preceding display vanishes and provides (3.9).

We may now establish the result. By defining $S_i := \sum_k [J_{ki}, \omega_{ki}]$, (3.2) now reads

$$(n-4)F = S_i + S_j - 2[J_{ij},\omega_{ij}] + \tfrac{1}{2}\sum_{k\neq i,j}\langle J_{ij},\omega_{ki}\rangle J_{kj} - \langle J_{ij},\omega_{kj}\rangle J_{ki}.$$

Keeping i fixed and summing over $j \neq i$, making use of (3.9), we obtain

$$\begin{aligned}
(r-1)(n-4)F &= \sum_{j\neq i}\left(S_i + S_j - 2[J_{ij},\omega_{ij}]\right)\\
&= (r-4)S_i + \sum_j S_j
\end{aligned}$$

which implies (as $r \neq 4$) that S_i is independent of i and proportional to F:

$$(r-1)(n-4)F = 2(r-2)S_i.$$

Equation (3.2) thus develops to

$$\frac{2(n-4)}{r-2}F = 4[J_{ij},\omega_{ij}] - \sum_{k\neq i,j}\langle J_{ij},\omega_{ki}\rangle J_{kj} - \langle J_{ij},\omega_{kj}\rangle J_{ki}.$$

Commuting F with J_{ij} we see that

$$\begin{aligned}
\frac{2(n-4)}{r-2}FJ_{ij} &= 4(J_{ij}\omega_{ij}J_{ij} + \omega_{ij}) - \sum_{k\neq i,j}\langle J_{ij},\omega_{ki}\rangle J_{ki} + \langle J_{ij},\omega_{kj}\rangle J_{kj},\\
\frac{2(n-4)}{r-2}J_{ij}F &= -4(\omega_{ij} + J_{ij}\omega_{ij}J_{ij}) + \sum_{k\neq i,j}\langle J_{ij},\omega_{ki}\rangle J_{ki} + \langle J_{ij},\omega_{kj}\rangle J_{kj}.
\end{aligned}$$

Therefore F anticommutes with J_{ij} for every $i \neq j$. By taking some k different from both i and j we get that F commutes with $J_{ik}J_{jk} = J_{ij}$. Hence $F = 0$, thus proving Theorem 1 when the rank of the Clifford-Weyl structure is at least 5.

4. Low rank Clifford-Weyl structures

We consider now the remaining cases from Theorem 1. If the rank of the Clifford-Weyl structure is 2, this follows directly from our toy problem described at the end of Section 2.

That D is closed for $r = 3$ and $n \geq 8$ is a standard result in quaternion Hermitian Weyl geometry (or locally conformally quarternion Kähler geometry) [10]. We present a proof which can also be adapted to the case $r = 4$.

Define $\Omega := \Omega_{12}^2 + \Omega_{23}^2 + \Omega_{31}^2$ to be the fundamental four-form (or Kraines form) of quaternion Hermitian geometry. By (3.1), which continues to hold for $r = 3$, we obtain

$$\Omega_{12}^2 \wedge F = \tfrac{1}{2}\Omega_{12} \wedge \Omega_{13} \wedge \omega_{32} - \tfrac{1}{2}\Omega_{12} \wedge \Omega_{23} \wedge \omega_{31}$$

and cyclically commuting $(1, 2, 3)$ gives two similar equations. Upon summation, cancellations give $\Omega \wedge F = 0$ and, as the fundamental four-form is well-known to be non-degenerate (and $n \neq 4$), $F = 0$. (Alternatively, if one follows the derivation of (3.1), one obtains similar equations for $\mathrm{d}\Omega_{ij}$ in terms of the connection coefficients η_{ij} which result in the equation $\mathrm{d}\Omega = -4\theta \wedge \Omega$. Differentiating a second time gives $\Omega \wedge F = 0$.)

Finally, if $(E, h, \nabla^E, \varphi)$ is a rank 4 Clifford-Weyl structure and $n \geq 8$, let us consider $A \in \mathrm{End}(\mathrm{T}M)$ to be the image under φ of the volume element of E. From the properties of φ, A is a symmetric involution, hence the tangent bundle splits into a direct sum $\mathrm{T}M = T^+ \oplus T^-$ of the ± 1 eigenspaces of A. If either T^+ or T^- are of dimension zero, then the rank 4 even Clifford structure is effectively a rank 3 even Clifford structure and the result follows from the previous paragraph. We may thus assume the decomposition of $\mathrm{T}M$ is non-trivial. In particular φ is injective upon restriction to $\Lambda^2 E$, so we only need to consider the case $n \geq 12$.

We construct quaternionic structures on $T^\pm$

$$J_{12}^\pm = \mp\tfrac{1}{2}(J_{14} \pm J_{23}), \quad J_{31}^\pm = \mp\tfrac{1}{2}(J_{13} \mp J_{24}), \quad J_{23}^\pm = \mp\tfrac{1}{2}(J_{12} \pm J_{34})$$

which vanish upon restriction to $T^\mp$. We may thus define two four-forms $\Omega_\pm \in \Lambda^4(T^\pm)^*$ as in the case of quaternion Hermitian geometry and set $\Omega = \Omega_+ + \Omega_-$. We decompose the exterior algebra $\Lambda^* M = \oplus(\Lambda^p(T^+)^* \oplus \Lambda^q(T^-)^*)$ and say that elements of $\Lambda^p(T^+)^* \oplus \Lambda^q(T^-)^*$ are of type (p, q). The decomposition of $\Lambda^2 M$ enables us to write $F = F_+ + F_m + F_-$ where F_+, F_m, F_- are respectively of type $(2, 0)$, $(1, 1)$, $(0, 2)$. Using this we calculate $\Omega \wedge F$ (whose 6 pieces are of distinct type). Meanwhile, we remark that

$$\Omega = \tfrac{1}{2}\sum_{i<j} \Omega_{ij}^2 = \tfrac{1}{4}\sum_{i,j} \Omega_{ij}^2$$

and via (3.1), which continues to hold for $r = 4$,

$$\Omega_{ij}^2 \wedge F = \tfrac{1}{2} \sum_k \Omega_{ij} \wedge \Omega_{ik} \wedge \omega_{kj} - \Omega_{ij} \wedge \Omega_{jk} \wedge \omega_{ki}$$

Summing over i, j the second term in the previous sum, $\sum_{i,j,k} \Omega_{ij} \wedge \Omega_{jk} \wedge \omega_{ki}$ may be written, under a permutation $i, j, k \to k, i, j$, as the first term $\sum_{i,j,k} \Omega_{ij} \wedge \Omega_{ik} \wedge \omega_{kj}$ hence

$$\Omega \wedge F = 0.$$

Since they have different types, each of the six terms in the expansion of $\Omega \wedge F$ also individually vanish. As M is at least 12-dimensional with both subbundles $T^{\pm}$ being non-trivial, we deduce from $\Omega_- \wedge F_+ = 0$ and $\Omega_+ \wedge F_- = 0$ that $F_{\pm} = 0$. And as one of the subbundles $T^{\pm}$ has rank larger than 4 (say T^+) then $F_m = 0$ (from $\Omega_+ \wedge F_m = 0$).

This finishes the proof of Theorem 1 when the rank of the Clifford-Weyl structure is 2, 3 or 4.

5. Generic cases

In this final section we prove Proposition 2 and, in the process, show examples of Clifford-Weyl structures with non-closed associated Weyl covariant derivatives.

(i) If M has dimension 2, we define (E, h) to be the trivial rank 2 Euclidean vector bundle with trivial flat connection ∇^E, and $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$ by $\varphi(\xi_1 \wedge \xi_2) := J$, where ξ_1, ξ_2 is an oriented orthonormal frame of E and J is the rotation in $\mathrm{T}M$ by $\pi/2$ in the positive direction determined by c. Since $DJ = 0$, $(E, h, \varphi, \nabla^E, D)$ is a rank 2 Clifford-Weyl structure.

If M has dimension 4, we define $E := \Lambda^+ M \otimes L^2$ (the bundle of self-dual two-forms of conformal weight 0), ∇^E to be the covariant derivative induced by D on E and h to be the canonical scalar product induced by c on E. Since $\Lambda^2 M \otimes L^2$ is canonically isomorphic to $\mathrm{End}^-(\mathrm{T}M)$, E is in fact a rank 3 sub-bundle of the bundle of skew-symmetric endomorphisms of M. Moreover, since E is oriented, the metric h provides an identification of $\Lambda^2 E$ with E, and thus a map $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$. It is straightforward to check that this map extends to an algebra morphism from $\mathrm{Cl}^0(E, h)$ to $\mathrm{End}(\mathrm{T}M)$ which is tautologically parallel with respect to $\nabla^E \otimes D$, thus defining a rank 3 Clifford-Weyl structure.

Moreover, every rank 3 Clifford-Weyl structure $(E, h, \nabla^E, \varphi, D)$ determines in a tautological way a rank 4 Clifford-Weyl structure $(\tilde{E}, \tilde{h}, \nabla^{\tilde{E}}, \tilde{\varphi}, D)$ where $\tilde{E} = E \oplus \mathbb{R}$ with induced metric $\tilde{h}$ and connection $\nabla^{\tilde{E}}$, and $\tilde{\varphi}$ is defined on $\Lambda^2 \tilde{E} \simeq \Lambda^2 E \oplus E$ by $\tilde{\varphi} = \varphi$ on $\Lambda^2 E$ and $\tilde{\varphi} = \varphi \circ *$ on E, where $*$ denotes the Hodge isomorphism $* : E \to \Lambda^2 E$.

If M has dimension 8, we define $E = \Sigma_0^+ M$ (the bundle of real half-spinors of conformal weight 0, cf. [6]) and ∇^E and h to be the covariant derivative and the scalar product induced on E by D and c. Of course, if M is not spin, E is only locally defined, but $\Lambda^2 E$ is always globally defined. We consider the map $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$ defined by

$$\varphi(\psi \wedge \phi) := X \mapsto -\sum_{i=1}^{8} h(\ell^{-2} e_i \cdot X \cdot \psi, \phi)\, e_i - h(\psi, \phi)\, X,$$

where ℓ is a local section of L and e_i is a local frame of $\mathrm{T}M$ satisfying $c(e_i, e_j) = \ell^2 \delta_{ij}$. The map φ is tautologically parallel with respect to $\nabla^E \otimes D$. Moreover, φ extends to an algebra morphism from $\mathrm{Cl}^0(E, h)$ to $\mathrm{End}(\mathrm{T}M)$. Indeed, in order to check the universality property for the even Clifford algebra ([9, Lemma A.1]), consider local sections ψ, ϕ and ξ of E such that ψ is orthogonal to ϕ and ξ and $h(\psi, \psi) = 1$. Then $\{\ell^{-1} e_i \cdot \psi\}$ is a local orthonormal basis of the zero-weight half-spin bundle $\Sigma_0^- M$ (whose metric is also denoted by h) and thus

$$\begin{aligned}
[\varphi(\psi \wedge \phi) \circ \varphi(\psi \wedge \xi)](X) &= -\sum_{i=1}^{8} h(\ell^{-2} e_i \cdot X \cdot \psi, \xi)\, \varphi(\psi \wedge \phi)(e_i) \\
&= \sum_{i,j=1}^{8} h(\ell^{-2} e_i \cdot X \cdot \psi, \xi) h(\ell^{-2} e_j \cdot e_i \cdot \psi, \phi)\, e_j \\
&= \sum_{i,j=1}^{8} h(\ell^{-2} X \cdot e_i \cdot \psi + 2\ell^{-2} c(e_i, X)\psi, \xi) h(\ell^{-2} e_i \cdot \psi, e_j \cdot \phi)\, e_j \\
&= -\sum_{j=1}^{8} h(\ell^{-2} X \cdot \xi, e_j \cdot \phi)\, e_j \\
&= -\varphi(\xi \wedge \phi)(X) - h(\phi, \xi)\, X \\
&= \varphi(\phi \wedge \xi)(X) - h(\phi, \xi)\, X.
\end{aligned}$$

This shows that $(E, h, \varphi, \nabla^E, D)$ is a rank 8 Clifford-Weyl structure on M.

(ii) With any Clifford-Weyl structure of rank 2 on M one can associate the image, J, of the volume form of E through the Clifford morphism φ. Clearly J is an almost complex structure on M compatible with c and D-parallel. On the other hand, every almost complex structure preserved by a torsion-free connection is integrable.

Conversely, if D is a Weyl structure on (M, c) and J is a D-parallel Hermitian structure, we define as before a rank 2 Clifford-Weyl structure on M by taking (E, h) to be the trivial rank 2 Euclidean vector bundle with trivial flat connection ∇^E, and $\varphi : \Lambda^2 E \to \mathrm{End}^-(\mathrm{T}M)$ defined by the fact that it maps the volume form of E onto J.

For the second point, recall that on 4-dimensional conformal manifolds, every complex structure J compatible with the conformal structure is preserved by a unique Weyl covariant derivative D (see e.g. the proof of [5, Lemma 5.7]) which is closed if and only if (J, c) is locally conformally Kähler.

(iii) If $(E, h, \varphi, \nabla^E, D)$ is a Clifford-Weyl structure of rank 4 with φ injective on $\Lambda^2 E$ on an 8-dimensional conformal manifold (M, c), then the image of the volume form of E through φ is a D-parallel involution of $\mathrm{T}M$ whose eigenbundles are 4-dimensional D-parallel distributions. By [5, Theorem 4.3], (M, c) has a conformal product structure defined by these two distributions.

Conversely, every conformal product structure on (M, c) with 4-dimensional distributions $T^\pm$ defines a unique Weyl connexion D (called the adapted Weyl structure in [5, Definition 4.4]) such that the splitting $\mathrm{T}M = T^+ \oplus T^-$ is D-parallel. We obtain in this way a structure group reduction from $\mathrm{CO}(8)$ to $G := \mathrm{CO}(8) \cap (\mathrm{CO}(4) \times \mathrm{CO}(4))$, that is, a G-principal bundle P over M and a connection induced by D on P. Since $\mathrm{CO}(4) = \mathbb{R}^* \times (\mathrm{Spin}(3) \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(3))$, the projections from $\mathbb{R}^* \times (\mathrm{Spin}(3) \times \mathrm{Spin}(3))$ to the second and third factors respectively define group morphisms i_l and i_r from $\mathrm{CO}(4)$ to $\mathrm{SO}(3)$. Let E denote the (locally defined) rank 4 Euclidean vector bundle over M associated with P via the group morphism $i_l \times i_r$ from G to $\mathrm{SO}(3) \times \mathrm{SO}(3) = \mathrm{PSO}(4)$, and let ∇^E denote the induced covariant derivative. By construction, $\Lambda^2 E$ is globally defined, and isomorphic to the weightless bundle $(\Lambda^+(T^+) \otimes L^{-2}) \oplus (\Lambda^-(T^-) \otimes L^{-2})$. The composition of this isomorphism with the canonical inclusion in $\Lambda^+(\mathrm{T}M) \otimes L^{-2} = \mathrm{End}^-(\mathrm{T}M)$ yields as before a Clifford morphism, which is $\nabla^E \otimes D$-parallel by naturality of the construction.

Examples of conformal products with non-closed adapted Weyl structures can be easily constructed. Take (M_1, g_1) and (M_2, g_2) two 4-dimensional Riemannian manifolds and let $M = M_1 \times M_2$ with conformal class $c = [e^f g_1 + g_2]$ where f is any smooth map on M. Then the adapted Weyl structure of this conformal product structure is closed if and only if there exist functions f_i on M_i such that $f = \pi_1^*(f_1) + \pi_2^*(f_2)$ where $\pi_i : M \to M_i$ are the canonical projections (see [5, Section 6.1] for details).

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