ZETA FUNCTION AT ZERO FOR SURFACES WITH BOUNDARY

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Abstract. We study the Ruelle zeta function at zero for negatively curved oriented surfaces with boundary. At zero, the zeta function has a zero and its multiplicity is shown to be determined by the Euler characteristic of the surface. This is shown by considering certain Ruelle resonances and identifying their multiplicity with dimensions of the relative cohomology of the surface.

1. Introduction

Consider a compact smooth Riemannian surface (Σ, g) without boundary and everywhere strictly negative curvature. The Ruelle zeta function [Rue76] provides a differential geometric analogy to the Riemann zeta function by replacing a count over primes with a count over primitive closed geodesics $\{\gamma^\#\}$ whose respective lengths are $\{T_\gamma^\#\}$:

$$\zeta_R(\lambda) := \prod_{\gamma^\#} \left(1 - e^{-\lambda T_\gamma^\#}\right).$$

Negative curvature implies this product converges for $\operatorname{Re}\lambda \gg 1$.

This zeta function is related to the Selberg zeta function [Sel56, Sma67]

$$\zeta_R(\lambda) = \frac{\zeta_S(\lambda)}{\zeta_S(\lambda+1)}, \qquad \zeta_S(\lambda) := \prod_{\gamma^\#} \prod_{k \in \mathbb{N}_0} \left(1 - e^{-(\lambda+k)T_\gamma^\#}\right)$$

so results may be translated between the two settings. We will consider only the Ruelle zeta function.

The meromorphic extension of this zeta function has long been known in the setting of constant curvature thanks to the relationship with the Selberg zeta function [Fri86]. Only recently however has the meromorphic extension been obtained in the setting of variable curvature. This result first appeared in [GLP13] and soon after, using a microlocal approach, in [DZ16]. In the constant curvature setting, the zeta function vanishes at $\lambda = 0$ and its order of vanishing is $-\chi(\Sigma)$ where $\chi(\Sigma)$ is the Euler characteristic of the surface [Fri86]. This result holds true in variable curvature indicating the topological invariance of the order of vanishing of the zeta function at the origin [DZ17]. Unlike the constant curvature setting [Fri86, DGRS18], the value of the first non-trivial term in the power series representation of the zeta function about the origin is not understood in the variable curvature setting.

Consider now a compact connected Riemannian surface (Σ, g) with strictly convex boundary $\partial\Sigma$ and everywhere strictly negative curvature. In this open setting, one defines the Ruelle zeta function exactly as in the closed case. Strict convexity of the boundary is geometrically appealing as it ensures that closed geodesics do not touch the boundary. Again, negative curvature implies the convergence of the product for $\operatorname{Re}\lambda \gg 1$.

For constant curvature, the meromorphic extension of the zeta function has also been understood via the Selberg zeta function and its order of vanishing at $\lambda = 0$ is $1-\chi(\Sigma)$ [PP01, BJP05]. See also [Bor16, GHW18]. (The one exception to this is if the surface has vanishing Euler characteristic. In this case, the surface is a hyperboloid and the zeta function is a finite product consisting of the two primitive geodesics – of equal length but opposite direction – hence the zeta function has a zero of order 2 at the origin.)

For variable curvature, the zeta function has a meromorphic extension due to [DG16] which considerably extends the microlocal analysis performed in [DZ16] by analysing dynamics at both spatial and frequency infinities. This result allows us to consider the zeta function near the origin. Here, we show that the result concerning the order of vanishing discovered in the constant curvature setting holds true in variable curvature.

Theorem. *Let (Σ, g) be an oriented connected Riemannian surface of negative curvature with strictly convex boundary and negative Euler characteristic $\chi(\Sigma)$. Then the Ruelle zeta function $\zeta_R(\lambda)$ has a zero at $\lambda = 0$ of multiplicity precisely $1 - \chi(\Sigma)$.*

As with the closed setting, the attractive problem of studying the precise value of the first non-trivial term in the power series representation remains untouched.

We conclude this introduction explaining the method. We also comment on the closed setting for context.

Consider Σ a negatively curved compact surface with strictly convex boundary. Let $M = S^*\Sigma$ be the unit cotangent bundle, $\varphi_t : M \to M$ the geodesic flow, and $X \in C^\infty(M; TM)$ the generator of said flow. Let $\Lambda^k T_0^* M$ be the kernel of ι_X inside $\Lambda^k T^* M$. The results of [DG16] imply that one may construct the resolvents

$$(\mathcal{L}_X + \lambda)^{-1} : L^2(M; \Lambda^k T_0^* M) \to L^2(M; \Lambda^k T_0^* M)$$

which are well-defined for $\operatorname{Re}\lambda \gg 1$ and which extend meromorphically to $\lambda \in \mathbb{C}$ (upon a delicate change in the domain and range of said operators). For fixed $k \in \{0, 1, 2\}$ and a pole λ of the meromorphic extension of the resolvent, the associated residue is a projection operator of finite rank whose image defines (generalised) resonant states. These are distributions (or currents for $k > 0$) satisfying certain wave-front conditions, support conditions, and are in the kernel of some power of $\mathcal{L}_X + \lambda$. Simultaneously the rank of the projection operator is precisely the order of vanishing at λ for a certain zeta function ζ_k associated with $\mathcal{L}_X$ acting on $\Lambda^k T_0^* M$.

The relevance of this result is via a factorisation of ζ_R giving

$$\zeta_R(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda)\zeta_2(\lambda)}$$

hence one can study the order of vanishing of ζ_R by studying the space of generalised resonant states. Before proceeding, we remark that in all cases of interest, the poles at $\lambda = 0$ are simple hence all generalised resonant states are in the kernel of $\mathcal{L}_X$ (rather than a power thereof); a result known in [DZ17] as semisimplicity to which we will return shortly. Due to semisimplicity, we drop the adjective generalised. Denote by $m_k(0)$ the multiplicity of the zero of ζ_k at $\lambda = 0$.

Remark 1. Let us briefly comment on the closed manifold setting of [DZ17]. Resonant states at $\lambda = 0$ are

$$\{u \in \mathcal{D}'(M; \Lambda^k T_0^* M) : \mathrm{WF}(u) \subset E_u^*, \mathcal{L}_X u = 0\}.$$

Here, E_u^* is the unstable subbundle of T^*M associated with the Anosov flow X. For $k = 0$, the only possible resonant states are constant functions. Moreover, if α denotes the contact form associated with X then an algebraic argument using $d\alpha$, which is parallel with respect to $\mathcal{L}_X$, immediately implies $m_2(0) = m_0(0)$. Hence $m_0(0) = b_0$ and $m_2(0) = b_2$, where b_k are the Betti numbers of Σ. A slightly more difficult task is identifying $m_1(0)$ with $\dim H^1(M)$ (which by Gauss-Bonnet is equal to b_1). Up to a semisimplicity argument, the result follows.

Returning to the present setting of an open manifold. Resonant states at $\lambda = 0$ are

$$\{u \in \mathcal{D}'(M; \Lambda^k T_0^* M) : \mathrm{supp}(u) \subset \Gamma_+, \mathrm{WF}(u) \subset E_+^*, \mathcal{L}_X u = 0\}.$$

Here, Γ_+ is the set of points trapped in M in backward time with respect to φ_t, and E_+^* is an extension of the unstable bundle E_u^* (which is only defined on the trapped set) from the trapped set to Γ_+. Due to negative curvature, the volume $V(t)$ of points in M which remain in M after application of φ_t decreases exponentially with respect to time. For $k = 0$ this implies $(\mathcal{L}_X + \lambda)^{-1}$ does not have a pole at $\lambda = 0$. Hence $m_0(0) = 0$. The same algebraic argument from the closed setting using $d\alpha$ then implies $m_2(0) = 0$. It remains to study the space of resonant states for $k = 1$. This is done by considering relative cohomology and building an isomorphism between the space of resonant states and $H^1(M, \partial M)$. A key analytic construction allowing this identification is Lemma 6 providing a step between resonant states which are currents and smooth differential forms. (Gauss-Bonnet and Lefschetz duality then imply $m_1(0) = 1 - \chi(\Sigma)$.)

The final step is showing simplicity of the pole at $\lambda = 0$ for $k = 1$. This requires a regularity result very much in the spirit of [DZ17, Lemma 2.3] however the argument requires a subtle adaption using ideas from [DG16].

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2. Notation

2.1. **Geometry.** Let (Σ, g) be an oriented connected Riemannian surface of negative curvature with strictly convex boundary $\partial\Sigma$. We will also denote by g, the genus of Σ, and by n, the number of connected components of the boundary. The Euler characteristic of Σ is $\chi(\Sigma) = 2 - 2g - n$ which we take to be negative. Denote by M the unit cotangent bundle of Σ:

$$M := S^*\Sigma = \{(y, \eta) \in T^*\Sigma : g(\eta, \eta) = 1\}.$$

Let $\alpha \in \Omega^1(M)$ be the pull-back of the canonical one-form on T^*M. Then α is a completely non-integrable contact form and we set

$$d\mathrm{vol}_M := \alpha \wedge d\alpha.$$

The associated Reeb vector field $X \in C^\infty(M; TM)$, which is uniquely determined by

$$\iota_X \alpha = 1, \qquad \iota_X(d\alpha) = 0,$$

is the generator of the geodesic flow $\varphi_t : M \to M$.

Let Σ_{ch} denote the convex hull of Σ. That is, the boundary of Σ_{ch} is totally geodesic. Set $M_{\mathrm{ch}} := S^*\Sigma_{\mathrm{ch}}$.

We construct a global frame for T^*M [ST67, GK80]. Denote by $V \in C^\infty(M; TM)$ the generator of the $\mathbb{S}^1$ fibres of M over Σ. If we denote by $e^{\frac{\pi}{2}V} : M \to M$ the map given by anticlockwise rotation by $\pi/2$ in the $\mathbb{S}^1$ fibres, then define

$$\beta := (e^{\frac{\pi}{2}V})^*\alpha \in \Omega^1(M).$$

Complete the frame by denoting the connection one-form $\omega \in \Omega^1(M)$. This is the unique one-form satisfying

$$\iota_V \omega = 1, \qquad d\alpha = \omega \wedge \beta, \qquad d\beta = \alpha \wedge \omega.$$

Note that $d\mathrm{vol}_M = \alpha \wedge d\alpha = -\alpha \wedge \beta \wedge \omega$, that $d\omega = K\alpha \wedge \beta$ where K is the Gaussian curvature of the surface, and that $\alpha \wedge \beta$ is the pull-back of the area form $d\mathrm{vol}_\Sigma$ determined by the metric g.

2.2. **Topology.** We use relative cohomology à la Bott and Tu [BT82]. The vector spaces are $\Omega^k(M) \oplus \Omega^{k-1}(\partial M)$ with differential

$$d(v^{(k)}, h^{(k-1)}) := (dv^{(k)}, j^*v^{(k)} - dh^{(k-1)})$$

where $j : \partial M \to M$ is inclusion. The cohomology spaces are denoted $H^k(M, \partial M)$.

The first homology group of Σ is of rank $2g + n - 1 = 1 - \chi(\Sigma)$. Lefschetz duality then implies $H^1(\Sigma, \partial\Sigma)$ is also of rank $1 - \chi(\Sigma)$. The Gauss-Bonnet theorem provides

Lemma 2. *Let Σ be an oriented connected surface with boundary whose Euler characteristic is negative. Then $H^1(M, \partial M)$ has rank $1 - \chi(\Sigma)$ where $M := (T^*\Sigma \backslash 0)/\mathbb{R}^+$.*

Proof. We may suppose the surface has a metric whose boundary is totally geodesic, thus we prove the proposition using $(\Sigma_{\mathrm{ch}}, g)$ and M_{ch}. We denote by $\pi : M_{\mathrm{ch}} \to \Sigma_{\mathrm{ch}}$ the projection and show that $\pi^* : H^1(\Sigma_{\mathrm{ch}}, \partial\Sigma_{\mathrm{ch}}) \to H^1(M_{\mathrm{ch}}, \partial M_{\mathrm{ch}})$ is an isomorphism. Let j denote both inclusions $\partial\Sigma_{\mathrm{ch}} \hookrightarrow \Sigma_{\mathrm{ch}}$ and $\partial M_{\mathrm{ch}} \hookrightarrow M_{\mathrm{ch}}$. As $\partial\Sigma_{\mathrm{ch}}$ is totally geodesic, the Gauss-Bonnet theorem reads simply

$$\int_{\Sigma_{\mathrm{ch}}} K d\mathrm{vol}_\Sigma = 2\pi\chi(\Sigma).$$

Injectivity. Let $[(w, k)] \in H^1(\Sigma_{\mathrm{ch}}, \partial\Sigma_{\mathrm{ch}})$ satisfy $\pi^*[(w, k)] = 0$. That is, there exists $f \in \Omega^0(M_{\mathrm{ch}})$ for which $\pi^* w = df$ and $\pi^* k = j^* f$. Vertical fibres are generated by V and we note $\mathcal{L}_V f = df(V) = (\pi^* w)(V) = w(\pi_* V) = 0$ so $f = \pi^* \tilde{f}$ for $\tilde{f} \in \Omega^0(\Sigma_{\mathrm{ch}})$. Therefore $w = d\tilde{f}$ and $k = j^* \tilde{f}$ whence (w, k) is the Bott and Tu differential of $\tilde{f}$ and so $[(w, k)] = 0$.

Surjectivity. Let $[(v, h)] \in H^1(M_{\mathrm{ch}}, \partial M_{\mathrm{ch}})$ and we search for a candidate $[(w, k)] \in H^1(\Sigma_{\mathrm{ch}}, \partial\Sigma_{\mathrm{ch}})$. It suffices to find $f \in \Omega^0(M)$ such that $\iota_V v = -\mathcal{L}_V f$. (This condition and $dv = 0$ imply that $v + df \in \pi^*\Omega^1(\Sigma_{\mathrm{ch}})$, from which we obtain $v = \pi^* w - df$. Similarly, as this condition implies $\mathcal{L}_V(h + j^* f) = 0$, we may define k by $h = \pi^* k - j^* f$. Therefore $(v - \pi^* w, h - \pi^* k)$ is given by the Bott and Tu differential of $-f$.) Such an f may be constructed if v integrates to zero over the $\mathbb{S}^1$ fibres. We denote this integration as $\pi_* v$ and remark that $\pi_* v$ is constant by Stokes' theorem since all fibres are homotopic. Lifting the Gauss-Bonnet formula to M_{ch} gives

$$2\pi\chi(\Sigma) \cdot \pi_* v = \int_{M_{\mathrm{ch}}} K v \wedge \alpha \wedge \beta = \int_{M_{\mathrm{ch}}} -v \wedge d\omega = \int_{\partial M_{\mathrm{ch}}} v \wedge \omega = \int_{\partial M_{\mathrm{ch}}} dh \wedge \omega.$$

To complete the calculation, we take local coordinates for one component of ∂M_{ch}. Near such a component, the manifold appears as $[0, 1]_\rho \times \partial M_{\mathrm{ch}} \simeq [0, 1]_\rho \times \partial\Sigma \times \mathbb{S}^1 \simeq [0, 1]_\rho \times \mathbb{S}^1_t \times \mathbb{S}^1_\theta$. And as $\partial\Sigma_{\mathrm{ch}}$ is a geodesic boundary, $j^*\omega = d\theta$. Therefore $dh \wedge \omega$ is the total derivative $d(h\omega)$ hence vanishes upon integration over ∂M_{ch}. As the Euler characteristic does not vanish, we conclude $\pi_* v = 0$ as required. ☐

2.3. **Dynamics.** Let ρ be a boundary defining function on M (that is, $\rho \in C^\infty(M)$ such that $\rho > 0$ on M°, $\rho = 0$ on ∂M, and $d\rho \neq 0$ on ∂M). We suppose that ∂M is strictly convex with respect to X. That is, we have have the implication

$$x \in \partial M, (X\rho)(x) = 0 \quad \Longrightarrow \quad (X^2\rho)(x) < 0,$$

(which is independent of the chosen boundary defining function). The boundary ∂M decomposes into incoming/tangent/outgoing directions:

$$\partial M = \partial_- M \cup \partial_0 M \cup \partial_+ M$$

where

$$\partial_\pm M := \{x \in \partial M : \pm d\rho(X_x) < 0\}, \qquad \partial_0 M := \{x \in \partial M : d\rho(X_x) = 0\}.$$

Define the outgoing/incoming tails $\Gamma_\pm \subset M$ and the trapped set K by

$$\Gamma_\pm := \bigcap_{\pm t \geq 0} \varphi_t(M), \qquad K := \Gamma_+ \cap \Gamma_-.$$

The flow is hyperbolic on K. That is, there exists a continuous splitting with respect to $x \in K$ of the cotangent bundle into neutral/stable/unstable bundles each of rank 1 and which is invariant under the flow:

$$T_x^*M = E_n^*(x) \oplus E_s^*(x) \oplus E_u^*(x), \qquad E_n^*(x) = \mathbb{R}\alpha.$$

Given a scalar product on T^*M, there are constants $C_1, C_2 > 0$ such that

$$|\varphi_{-t}{}^*\xi| \leq C_1 e^{-C_2|t|}|\xi|, \qquad \begin{cases} \xi \in E_s^* & t \geq 0; \\ \xi \in E_u^* & t \leq 0. \end{cases}$$

The bundles E_s^*, E_u^* may be extended to Γ_-, Γ_+, respectively. Specifically, there exist subbundles of rank 1, $E_\pm^* \subset T_{\Gamma_\pm}^*M$, which are in the annihilator of X, invariant under the flow, depend continuously on $x \in \Gamma_\pm$, and $E_+^*|_K = E_u^*$ and $E_-^*|_K = E_s^*$. Moreover, if $x \in \Gamma_\pm$ and $\xi \in E_\pm^*$, then as $t \to \mp\infty$,

$$|\varphi_{-t}{}^*\xi| \leq C_1' e^{-C_2'|t|}|\xi|$$

for constants C_1', C_2' independent of (x, ξ) [DG16, Lemma 1.10].

Upon restriction to ∂M, the tails $\Gamma_\pm$ are contained in $\partial_\pm M$. Using a metric on M, giving a distance function $d(\cdot,\cdot)$, define

$$\Gamma_\pm^\delta := \{x \in M : d(\Gamma_\pm, x) \leq \delta\}.$$

By taking δ sufficiently small, we may assume that

$$\Gamma_\pm^\delta \cap \partial M \subset \partial_\pm M.$$

3. Zeta function and Pollicott-Ruelle resonances

3.1. **Zeta functions.** Let $\{\gamma\}$ denote the set of geodesics in M and let $\{\gamma^\#\}$ denote the set of primitive geodesics. Given a geodesic γ, denote respectively by T_γ and $T_\gamma^\#$ the length of γ and the length of the corresponding primitive geodesic. The Ruelle zeta function is denoted

$$\zeta_R(\lambda) := \prod_{\gamma^\#} \left(1 - e^{-\lambda T_\gamma^\#}\right)$$

Denote by T_0^*M the subbundle of TM which annihilates X. The pullback $\varphi_t{}^*$ respects the splitting $TM = \mathbb{R}\alpha \oplus T_0^*M$. Given a geodesic γ of length T_γ and a point $x \in \gamma \subset M$, we introduce the linearised Poincaré map

$$\mathcal{P}_{\gamma,x} := \varphi_{-T_\gamma}{}^* : (T_0^*M)_x \to (T_0^*M)_x.$$

As the endomorphism is conjugate to any other $\mathcal{P}_\gamma(x')$ for $x' \in \gamma$, its determinant and trace are independent of x and under such circumstances we will drop the notation of x. We have the following (linear algebra) expression:

$$\det(I - \mathcal{P}_\gamma) = \sum_{k=0}^{2} (-1)^k \operatorname{tr} \Lambda^k \mathcal{P}_\gamma.$$

A standard manipulation using the preceding expression (as well as the Taylor series for $\log(1-x)$ and, more subtly, the orientability of the stable and unstable bundles) converts the Ruelle zeta function into an alternating product of zeta functions:

$$\zeta_R(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda)\zeta_2(\lambda)}$$

where

$$\log \zeta_k(\lambda) := -\sum_\gamma \frac{T_\gamma^\# e^{-\lambda T_\gamma} \operatorname{tr} \Lambda^k \mathcal{P}_\gamma}{T_\gamma \left|\det(I - \mathcal{P}_\gamma)\right|}.$$

3.2. **Pollicott-Ruelle resonances.** The Lie derivative with respect to X acting on $\Omega^k(M)$ respects the decomposition $T^*M = \mathbb{R}\alpha \oplus T_0^*M$. Restricting to T_0^*M, we consider the transfer operator

$$e^{-t\mathcal{L}_X} : C_0^\infty(M; \Lambda^k T_0^* M) \to C^\infty(M; \Lambda^k T_0^* M).$$

Given $f \in C^\infty(M), u \in C^\infty(M; \Lambda^k T_0^* M)$, we have $\mathcal{L}_X(fu) = (\mathcal{L}_X f)u + f(\mathcal{L}_X u)$ from which the transfer operator satisfies

$$e^{-t\mathcal{L}_X}(fu) = (\varphi_{-t}^* f)(e^{-t\mathcal{L}_X} u).$$

(The flow is not complete on the manifolds with boundary considered in this text. We avoid this irritation by interpreting φ_{-t}^* at $x \in M$ as the zero-operator whenever there is $0 \le T \le t$ such that $\varphi_T(x) \in \partial M$.) After having fixed a smooth inner product on T_0^*M (not necessarily invariant under the flow), we have

$$e^{-t\mathcal{L}_X} : L^2(M; \Lambda^k T_0^* M) \to L^2(M; \Lambda^k T_0^* M).$$

Due to the existence of $C_0 > 0$ such that

$$\|e^{-t\mathcal{L}_X}\|_{L^2(M;\Lambda^k T_0^* M) \to L^2(M;\Lambda^k T_0^* M)} \le e^{C_0 t}, \qquad t \ge 0,$$

we may define the resolvent $(\mathcal{L}_X + \lambda)^{-1}$ on $L^2(M; \Lambda^k T_0^* M)$ for $\operatorname{Re} \lambda > C_0$ by the formula

$$(\mathcal{L}_X + \lambda)^{-1} := \int_0^\infty e^{-t(\mathcal{L}_X + \lambda)} dt.$$

A principal result of [DG16], is that the restricted resolvent

$$R_k(\lambda) = (\mathcal{L}_X + \lambda)^{-1} : C_0^\infty(M; \Lambda^k T_0^* M) \to \mathcal{D}'(M; \Lambda^k T_0^* M)$$

has a meromorphic continuation to $\mathbb{C}$ whose poles are of finite rank. The poles of which are called Pollicott-Ruelle resonances. Moreover, for each $\lambda_0 \in \mathbb{C}$, we have the expansion

$$R_k(\lambda) = R_k^H(\lambda) + \sum_{j=1}^{J(\lambda_0)} \frac{(-1)^{j-1}(\mathcal{L}_X + \lambda_0)^{j-1}\Pi_{\lambda_0}}{(\lambda - \lambda_0)^j}$$

where R_k^H is holomorphic near λ_0 and

$$\Pi_{\lambda_0} : C_0^\infty(M; \Lambda^k T_0^* M) \to \mathcal{D}'(M; \Lambda^k T_0^* M)$$

is a finite rank projector. The range of Π_{λ_0} defines generalised resonant states. They are characterised as

$$\begin{aligned}\operatorname{Res}_k(\lambda_0) &:= \operatorname{Ran}\Pi_{\lambda_0} \\ &= \{u \in \mathcal{D}'(M, \Lambda^k T_0^* M) : \operatorname{supp}(u) \subset \Gamma_+, \operatorname{WF}(u) \subset E_+^*, (\mathcal{L}_X + \lambda_0)^{J(\lambda_0)} u = 0\}.\end{aligned}$$

A generalised resonant state is called simply a resonant state if it is in $\ker(\mathcal{L}_X + \lambda_0)$. Similarly, if $J(\lambda_0) = 1$ then the pole is simple and the adjective "generalised" is superfluous. Finally, it is shown that poles of the meromorphic continuation correspond to zeros of the zeta function ζ_k, and that the rank of the projector Π_{λ_0} equals the multiplicity of the zero, denoted $m_k(\lambda_0)$.

Let $\mathcal{T}(t) \subset M^\circ$ be the set of points $x \in M^\circ$ such that $\varphi_{-s}(x) \in M^\circ$ for all $s \in [0, t]$, and let $V(t) := \operatorname{Vol}(\mathcal{T}(t))$ be the non-escaping mass function. In our setting, the escape rate

$$Q := \limsup_{t\to\infty} \frac{1}{t} \log V(t)$$

is strictly negative [Gui17, Proposition 2.4] thanks to the hyperbolicity of the trapped set, and the strict convexity of the boundary. Hence $V(t)$ decays exponentially fast and [Gui17, Propostion 4.4] provides

Proposition 3. *The resolvent $R_0(\lambda)$ does not have a pole at $\lambda = 0$.*

We observe that $\operatorname{Res}_2(0) = \{0\}$. Indeed, suppose that $u^{(2)} \in \operatorname{Res}_2(0)$ is a resonant state, that is $\mathcal{L}_X u^{(2)} = 0$. Since $\Lambda^2 T_0^* M = \mathbb{R} d\alpha$, we have $u^{(2)} =: u^{(0)} d\alpha$ for $u^{(0)} \in \mathcal{D}'(M)$ with $\operatorname{supp}(u^{(0)}) \subset \Gamma_+$ and $\operatorname{WF}(u^{(0)}) \subset E_+^*$. Moreover $\mathcal{L}_X u^{(0)} = 0$ because $\mathcal{L}_X d\alpha = 0$ hence $u^{(0)} \in \operatorname{Res}_0(0) = \{0\}$. We have proved

Proposition 4. *The resolvent $R_2(\lambda)$ does not have a pole at $\lambda = 0$.*

In Sections 5, 6, we prove

Proposition 5. *The resolvent $R_1(\lambda)$ has a simple pole at $\lambda = 0$.*

Assuming Proposition 5, we note that $\mathrm{Res}_1(0)$ consists only of resonant states:

$$\mathrm{Res}_1(0) = \{u \in \mathcal{D}'(M, T_0^*M) : \mathrm{supp}(u) \subset \Gamma_+, \mathrm{WF}(u) \subset E_+^*, \mathcal{L}_X u = 0\}.$$

and the Theorem follows if we can show

$$\dim \mathrm{Res}_1(0) = \dim H^1(M, \partial M).$$

4. Identification of resonances with relative cohomology

4.1. **Construction of map.** We begin with an analytical result to be proved in Section 6

Lemma 6. *Let $u \in \mathrm{Res}_1(0)$. There exists $f \in \mathcal{D}'(M)$ and $v \in \Omega^1(M)$ such that*

$$\mathrm{supp}(f) \subset \Gamma_+^\delta, \qquad \mathrm{WF}(f) \subset E_+^*, \qquad \mathcal{L}_X f \in C_0^\infty(M),$$

and $v = u - df$ with $v \in \ker d$.

Here, $\Omega^\bullet(M)$ are smooth differential forms up to, and including on, the boundary, while $C_0^\infty(M)$ denotes smooth functions whose support is contained in the interior of M.

In order to identify a candidate relative cohomology class, consider $u \in \mathrm{Res}_1(0)$, and construct v, f as in Lemma 6. We seek an $h \in \Omega^0(\partial M)$ such that $[(v, h)] \in H^1(M, \partial M)$. To this end we first prove

Lemma 7. *For $u \in \mathrm{Res}_1(0)$ the constructed $v = u - df \in \Omega^1(M)$ of Lemma 6 is exact upon pull-back to ∂M.*

Proof. We simplify the exposition by supposing ∂M consists of a single connected component (which is isomorphic to a torus). Noting that $\pi_1(\partial M) = \pi_1(\partial\Sigma) \times \pi_1(\mathbb{S}^1) = \mathbb{Z}^2$, it then suffices to show that $\int_{\gamma_i} v = 0$ where γ_1, γ_2 are two simple closed curves which generate $\pi_1(\partial M)$. Take γ_1 to be the curve which corresponds to the generator of $\pi_1(\partial\Sigma)$, and γ_2 corresponding to the $\mathbb{S}^1$ fibre.

We now take local coordinates similar to Lemma 2. For the moment, we work near ∂M rather than ∂M_{ch}. The manifold appears as $[0,1] \times \partial\Sigma \times \mathbb{S}^1 \simeq [0,1]_\rho \times \mathbb{S}_t^1 \times \mathbb{S}_\theta^1$. We may also identify the θ coordinate with the dynamical properties: $\partial_+ M = \{0 < \theta < \pi\}$, $\partial_- M = \{\pi < \theta < 2\pi\}$.

We may choose γ_1 such that $\gamma_1(\rho, t, \theta) = \gamma_1(0, t, \frac{3\pi}{2})$ so that its image is entirely contained in $\partial_- M$. As the restrictions of u, f are contained in $\Gamma_+, \Gamma_+^\delta \subset \partial_+ M$ respectively, we obtain immediately

$$\int_{\gamma_1} v = 0.$$

In order to show $\int_{\gamma_2} v$ also vanishes, we work with M_{ch}. Recall the push-forward map $\pi_* : \Omega^1(M) \to \Omega^1(\Sigma)$ which, as π, j commute, provides a push-forward π_* :

$\Omega^1(\partial M_{\text{ch}}) \to \Omega^1(\partial \Sigma_{\text{ch}})$. As the $\mathbb{S}^1$ fibres are homotopic to each other, $\int_{\gamma_2} v = \pi_* v$. Gauss-Bonnet over Σ_{ch} lifts to M_{ch} as in Lemma 2:

$$2\pi\chi(\Sigma) \cdot \pi_* v = \int_{M_{\text{ch}}} -v \wedge d\omega = \int_{\partial M_{\text{ch}}} v \wedge \omega.$$

With coordinates $[0,1]_\rho \times \mathbb{S}^1_t \times \mathbb{S}^1_\theta$, the curvature form restricted to ∂M_{ch} is simply $d\theta$. Therefore, writing $v = v_\rho d\rho + v_t dt + v_\theta d\theta$,

$$2\pi\chi(\Sigma) \cdot \pi_* v = \int_{\partial M_{\text{ch}}} v_t dt \wedge d\theta = \int_0^{2\pi} \left(\int_0^{2\pi} v_t dt \right) d\theta = \int_0^{2\pi} 0 d\theta = 0$$

because $\int_0^{2\pi} v_t dt$ vanishes from the prior calculation showing $\int_{\gamma_1} v = 0$. As $\chi(\Sigma) < 0$, we conclude

$$\int_{\gamma_2} v = 0.$$

Therefore $[j^* v] = 0 \in H^1(\partial M)$ implying the existence of the required $h \in \Omega^0(\partial M)$ such that $j^* v = dh$. □

Remark 8. The previous lemma ensures that it is possible to define an $h \in \Omega^0(\partial M)$ such that $[(v,h)] \in H^1(M, \partial M)$. However there are $n-1$ degrees of freedom in the choice of h due to the n connected components of ∂M. (An overall constant would not be seen by relative cohomology.) These degrees of freedom are fixed by the following declaration: The form $j^* v$ has support contained in $\Gamma_+^\delta \subset \partial_+ M$ therefore $dh = 0$ on $\partial_- M$ and is therefore constant on $\partial_- M$. We declare that h must be chosen to vanish on $\partial_- M$ whence we may assume $\operatorname{supp}(h) \subset \partial_+ M$.

Proposition 9. *Lemma 6, Lemma 7, and Remark 8 establish a well-defined map*

$$\operatorname{Res}_1(0) \ni u \longmapsto [(v,h)] \in H^1(M, \partial M).$$

Proof. Suppose Lemma 6 provides f_i and $v_i = u - df_i$ for $i \in \{1,2\}$. Then Lemma 7 and Remark 8 provide h_i with $j^* v_i = dh_i$ and h_i vanish on $\partial_- M$. We aim to construct $k \in \Omega^0(M)$ such that

$$(dk, j^* k) = (v_1 - v_2, h_1 - h_2)$$

in order to verify that the relative cohomology class is independent of the choices made.

Set $k := f_2 - f_1$. As $dk = v_1 - v_2$ is smooth, we conclude k itself is smooth. Next,

$$d(j^* k) = j^* d(f_2 - f_1) = j^*(v_1 - v_2) = d(h_1 - h_2)$$

so $j^* k = h_1 - h_2 + c$ where c is a function constant on each connected component of ∂M. As h_i vanish on $\partial_- M$ and $\operatorname{supp}(k) \subset \Gamma_+^\delta$, we conclude the function c is the zero function. □

4.2. **Injectivity.** Given the notation established from the previous subsection, suppose that for a given $u \in \mathrm{Res}_1(0)$, we obtain $[(v,h)] = 0 \in H^1(M, \partial M)$. This implies the existence of $k \in \Omega^0(M)$ whose Bott and Tu differential gives (v,h). That is, $(dk, j^*k) = (v,h)$. This implies that $u = d(f+k)$. However as u vanishes away from Γ_+, we know that $f+k$ is smooth on $M \backslash \Gamma_+$ and, as $\mathcal{L}_X(f+k) = \iota_X u = 0$, we also know that $f+k$ is constant on each connected component of $M\backslash\Gamma_+$. There are n connected components of $M\backslash\Gamma_+$ and each component may be identified with the n connected components of $\partial_- M$ (upon following geodesics in backward time until they reach the boundary). So the value of $f+k$ on a connected component is determined by its value upon restriction to the corresponding component of $\partial_- M$. Now $\mathrm{supp}(f) \subset \Gamma_+^\delta$ and $j^*k = h$ which by Remark 8 vanishes on $\partial_- M$. Therefore $f+k = 0$ on $M\backslash\Gamma_+$ and upon observing

$$\mathrm{supp}(f+k) \subset \Gamma_+, \qquad \mathrm{WF}(f+k) \subset E_+^*, \qquad \mathcal{L}_X(f+k) = \iota_X u = 0,$$

we conclude $f+k \in \mathrm{Res}_0(0) = \{0\}$ hence $u = 0$.

4.3. **Surjectivity.** Consider an element of $H^1(M, \partial M)$. Suppose it takes the form $[(\tilde{v}, \tilde{h})]$. We first remark that $\tilde{h}$ may be extended to a smooth function on M whose Bott and Tu differential gives $0 \in H^1(M, \partial M)$ and which may be subtracted from our original element. We may therefore assume the element of $H^1(M, \partial M)$ takes the form $[(\tilde{v}, 0)]$ for some modified $\tilde{v}$.

By [Gui17, Section 4], there exists $\tilde{f} \in \mathcal{D}'(M)$ with $\mathrm{WF}(\tilde{f}) \subset E_+^*$ subject to the boundary value problem

$$\begin{cases} \mathcal{L}_X \tilde{f} = -\iota_X \tilde{v}; \\ \tilde{f}|_{\partial_- M} = 0. \end{cases}$$

Set $u := \tilde{v} + d\tilde{f}$. Immediately, $\iota_X u = 0$ and since $\tilde{v}$ is closed, $\mathcal{L}_X u = 0$. It remains to obtain a support condition on u to conclude that u is a resonant state. To this end, consider a point $x \in \partial_- M$ and U a neighbourhood in $\partial_- M$ of x. Locally near x, X is transversal to $\partial_- M$ and is incoming. We may thus consider a chart $[0,\varepsilon)_\rho \times U_{(t,\theta)}$ on which X takes the form ∂_ρ. Writing

$$u|_{[0,\varepsilon)\times U} = u_\rho d\rho + u_t dt + u_\theta d\theta$$

we see that $u_\rho = 0$ (since $\iota_X u = 0$) and that u_t, u_θ are independent of ρ (since $du = 0$). So u_t, u_θ are determined by their values on $\{0\} \times U$ but by the initial condition of the boundary value problem

$$j^* u|_{\partial_- M} = j^*(\tilde{v} + d\tilde{f})|_{\partial_- M} = d(0 + j^*\tilde{f})|_{\partial_- M} = 0.$$

Therefore u vanishes on a neighbourhood of any point in $\partial_- M$. Moreover u is smooth away from Γ_+ and in the kernel of $\mathcal{L}_X$. Therefore $\mathrm{supp}(u) \subset \Gamma_+$ hence $u \in \mathrm{Res}_1(0)$.

To be completely at peace, we ought check that u gives back the original cohomology element following Proposition 9. The argument is that of the proof of Proposition 9:

Suppose Lemma 6 provides f and $v = u - df$. Then Lemma 7 and Remark 8 provide h with $j^*v = dh$ and h vanishes on $\partial_- M$. We must construct $k \in \Omega^0(M)$ such that

$$(dk, j^*k) = (v - \tilde{v}, h).$$

Set $k := \tilde{f} - f$. As $dk = v - \tilde{v}$ is smooth, so too is k. Also $d(j^*k) = dh$ and both k and h vanish on $\partial_- M$ so $j^*k = h$.

5. Semisimplicity

Proposition 5 states that that the pole of $R_1(\lambda)$ at $\lambda = 0$ is simple. The proof consists of two parts; a microlocal argument and a geometric argument. Lemma 10 announces the microlocal result and is proved in Section 6.

Lemma 10. *Let $f' \in \mathcal{D}'(M)$ with $\mathrm{WF}(f') \subset E_+^*$, $\mathrm{supp}(j^*f') \subset \partial_+ M$, and $\mathcal{L}_X f' \in C_0^\infty(M)$. If*

$$\mathrm{Re} \int_M (\mathcal{L}_X f')\, \overline{f'}\, d\mathrm{vol}_M = 0,$$

then $f' \in C^\infty(M)$.

Consider a generalised resonant state $\tilde{u}$ associated with the resonance $\lambda = 0$. That is, $\tilde{u} \in \mathcal{D}'(M; T_0^*M)$ with $\mathrm{supp}(\tilde{u}) \subset \Gamma_+$ and $\mathrm{WF}(\tilde{u}) \subset E_+^*$. In order to show that all generalised resonant states are true resonant states it suffices to assume $\mathcal{L}_X^2 \tilde{u} = 0$ and show that $\mathcal{L}_X \tilde{u} = 0$. To this end suppose $u := \mathcal{L}_X \tilde{u} \in \ker \mathcal{L}_X$.

We start by observing $\alpha \wedge u$ is exact. Define $\tilde{f}$ by $\tilde{f} d\mathrm{vol}_M := \alpha \wedge d\tilde{u}$. Then $(\mathcal{L}_X \tilde{f}) d\mathrm{vol}_M = \alpha \wedge du = 0$ since $du \in \mathrm{Res}_2(0) = \{0\}$. So $\mathcal{L}_X \tilde{f} = 0$ hence $\tilde{f} \in \mathrm{Res}_0(0) = \{0\}$. We conclude that $\alpha \wedge d\tilde{u}$ vanishes. Applying ι_X to this expression yields $0 = \iota_X(\alpha \wedge d\tilde{u}) = d\tilde{u} - \alpha \wedge \mathcal{L}_X \tilde{u}$ hence $\alpha \wedge u$ is indeed exact:

$$d\tilde{u} = \alpha \wedge u.$$

As $u \in \ker \mathcal{L}_X$, we consider a representative of the relative cohomology class obtained from u as constructed in Section 4. Following Section 4, introduce $f \in \mathcal{D}'(M)$, $v \in \Omega^1(M)$, and $h \in C^\infty(\partial M)$ such that

$$\mathrm{supp}(f) \subset \Gamma_+^\delta, \qquad \mathrm{WF}(f) \subset E_+^*, \qquad \mathcal{L}_X f \in C_0^\infty(M),$$

and $v = u - df$ with $j^*v = dh$. Note $dv = 0$ and $\mathrm{supp}(\iota_X v) \subset M^\circ$. Moreover $\mathrm{supp}(h) \subset \partial_+ M$ by Remark 8.

We extend h to $h' \in C^\infty(M)$ such that $\mathcal{L}_X h' \in C_0^\infty(M)$ and $j^*h' = h$ in the following way. Let $U \subset \partial M$ be open and relatively compact in $\partial_+ M$ such that $\mathrm{supp}(h) \subset U$. As X is transversal and outgoing on U, there exists a chart

$$[0, \varepsilon)_\rho \times U_{(t,\theta)} \subset M$$

on which $X = -\partial_\rho$. Consider a cutoff $\chi \in C^\infty([0, \varepsilon); [0, 1])$ with $\mathrm{supp}(\chi) \subset [0, \varepsilon/2]$ and $\chi|_{[0,\varepsilon/3]} = 1$. Now define $h' \in C^\infty(M)$ by declaring

$$h'(\rho, t, \theta) := \chi(\rho) \cdot h(t, \theta)$$

on $[0,\varepsilon)\times U$ and $h'=0$ elsewhere. Then h' has the desired properties. We remark $[(dh',h)]=0$ in $H^1(M,\partial M)$ since (dh',h) is the Bott and Tu differential of h'.

Motivated by the equality $[(v,h)]=[(v,h)]-[(dh',h)]$, set

$$f' := f+h', \qquad v' := u-df' = v-dh'.$$

Now v' is compactly supported in M°. The argument is precisely that as given for the form u in Subsection 4.3 on surjectivity. We use that $\iota_X v' = -\mathcal{L}_X f' \in C_0^\infty(M)$, that $dv'=0$, and that $j^*v'=0$. Note also $\operatorname{supp}(j^*f')\subset \partial_+M$. In order to place ourselves in the setting of Lemma 10 we calculate

$$\begin{aligned}
\int_M (\mathcal{L}_X f')\,\overline{f'}\,\mathrm{dvol}_M &= -\int_M \overline{f'}\,\iota_X v'\,\mathrm{dvol}_M \\
&= -\int_M \overline{f'}\,v'\wedge d\alpha \\
&= -\int_M \overline{df'}\wedge v'\wedge\alpha \\
&= -\int_M \overline{u-v'}\wedge v'\wedge\alpha
\end{aligned}$$

The third equality is obtained by integration by parts. The distributional nature of v' causes no problems because it is supported away from ∂M. Passing to the real part of the preceding equality, the term involving $\overline{v'}\wedge v'$ vanishes. Using the exactness of $\alpha\wedge u$ (and for a second time that $dv'=0$) we integrate by parts (again justifiably due to the support of v'), we conclude

$$\operatorname{Re}\int_M \mathcal{L}_X f'\,\overline{f'}\,\mathrm{dvol}_M = -\operatorname{Re}\int_M \overline{u}\wedge v'\wedge\alpha = -\operatorname{Re}\int_M v'\wedge\overline{d\tilde{u}} = 0.$$

By Lemma 10, we deduce f' is smooth hence u is smooth. Therefore u is forced to vanish as $\operatorname{supp}(u)\subset\Gamma_+$. This finishes the proof as it shows that the generalised resonant state $\tilde{u}$ is actually a true resonant state.

6. All things analysis

This final section tidies up the loose analytical threads and proves Lemmas 6 and 10. Due to the microlocal nature of the proofs, it is easier to work on either a compact manifold without boundary (in the spirit of [DG16]) or on an open manifold (in the spirit of [Gui17]) rather than on a compact manifold with boundary. We choose to work with the open manifold M°.

The proofs of Lemmas 6, and 10 use parametrices and microlocal analysis. The microlocal and semiclassical notation used is standard with short and sufficient introductions given in [DZ16, DG16] or more substantially in [Hör07, Zwo12, DZ19]. We do however make some brief comments.

In the proof of Lemma 6, we use parametrices for the Laplacian acting on ∂M and M°. The parametrices do not enlarge the wavefront sets of the distributions on which they act. In the proof of Lemma 10, we use operators drawn from $\Psi_h^{\mathrm{comp}}(M^\circ)$ which are operators whose wavefronts are compactly supported in T^*M°. See [DZ19, Appendix E.2].

We consider the operator $P := -i\mathcal{L}_X$ which is formally self-adjoint on $L^2(M^\circ)$ with respect to the volume form $d\mathrm{vol}_M$. Let p denote its symbol, $p(x,\xi) = \xi(X_x)$ and let H_p denote the associated Hamiltonian vector field on T^*M. The flow which generates H_p is the lift of φ_t from M to T^*M and is denoted Φ_t.

6.1. **Proof of Lemma 6.** To begin the proof we establish some geometric structure around Γ_+ near the original boundary ∂M. We choose an open set $U \subset \partial M$ containing $\Gamma_+^\delta \cap \partial M$ such that on the chart $V := [0,\varepsilon)_\rho \times U \subset M$ the geodesic flow takes the form $-\partial_\rho$. Denote by π the projection V to U (and also V to ∂M). Transversality provides first that the unstable bundle E_+^* over Γ_+ in M may be restricted to ∂M giving $E_+^*\big|_{\partial M}$ and second that $\pi^*(E_+^*\big|_{\partial M}) = E_+^*$ on V.

Let $\chi \in C^\infty(M;[0,1])$ with $\mathrm{supp}(\chi) \subset \Gamma_+^\delta$ and $\chi = 1$ on $\Gamma_+^{\delta/2}$. We may assume that $\mathcal{L}_X\chi$ vanishes on V. We split χ, writing $\chi = \chi_1 + \chi_2$ with

$$\mathrm{supp}(\chi_1) \subset V, \qquad \mathrm{supp}(\chi_2) \cap \{\rho \le \tfrac{\varepsilon}{2}\} = \varnothing.$$

Now choose a metric on M and ask that on V, the metric takes the product structure $d\rho^2 + g_\partial$ for some metric g_∂ on ∂M. Construct parametrices Q_∂, Q for the Hodge Laplacian acting on $\Omega^1(\partial M), \Omega^1(M^\circ)$, and denote the divergences on $\partial M, M^\circ$ by d_∂^*, d^* respectively.

We are now ready to take a resonant state $u \in \mathcal{D}'(M, \Lambda^1 T_0^* M)$. As in Subsection 4.3, we conclude that $u = \pi^* u_\partial$ on V. Here $u_\partial \in \mathcal{D}'(\partial M; \Lambda^1 T^*\partial M)$ with $\mathrm{supp}(u_\partial) \subset \Gamma_+$ and $\mathrm{WF}(u_\partial) \subset E_+^*\big|_{\partial M}$. Construct

$$f_\partial := d_\partial^* Q_\partial u_\partial, \qquad \tilde f := d^* Q u, \qquad f := \chi_1 \pi^* f_\partial + \chi_2 \tilde f.$$

We claim that f satisfies the conclusions of Lemma 6. The support condition $\mathrm{supp}(f) \subset \Gamma_+^\delta$ is immediate thanks to the presence of χ_1, χ_2. For the wave-front condition, as Q does not enlarge wave-fronts, we first obtain $\mathrm{WF}(\pi^* \tilde f) \subset E_+^*$. Second, as Q_∂ also does not enlarge wave-fronts, $\mathrm{WF}(f_\partial) \subset E_+^*\big|_{\partial M}$ and so by construction of V, this gives $\mathrm{WF}(\pi^* f_\partial) \subset E_+^*$. Therefore $\mathrm{WF}(f) \subset E_+^*$. The support condition $\mathcal{L}_X f \in C_0^\infty(M)$ follows after remembering that $\mathcal{L}_X \chi_1$ vanishes on $\{\rho \le \frac{\varepsilon}{2}\}$.

It remains to show that $v := u - df$ is smooth. Calculating a little provides

$$\begin{aligned} u - df &= (\chi_1 + \chi_2)u - d(\chi_1 \pi^* f_\partial + \chi_2 \tilde f) \\ &= \chi_1(u - d\pi^* f_\partial) + \chi_2(u - d\tilde f) - (\pi^* f_\partial) d\chi_1 - \tilde f d\chi_2 \\ &= \chi_1 \pi^*(u_\partial - df_\partial) + \chi_2(u - d\tilde f) + (\tilde f - \pi^* f_\partial) d\chi_1 \end{aligned}$$

and we investigate each of the three terms in the final line separately, showing that each term is smooth.

The first two terms are each smooth by (the proof of) [DZ17, Lemma 2.1] however we reproduce the argument below for convenience in the case of $u - d\tilde f$. (The argument for $u_\partial - df_\partial$ follows by appropriately subscripting the following argument with "∂".) First,

$$u - d\tilde{f} = u - dd^*Qu = (1-\Delta Q)u + d^*dQu = d^*dQu + \Omega^1(M^\circ)$$

and to show d^*dQu is smooth, it suffices, by elliptic regularity, to show Δd^*dQu is smooth on M°. Working on M°, this follows by commutation of Δ with d^*, d and because du is smooth (in fact it vanishes because u is a resonant state):

$$\Delta d^*dQu = d^*d\Delta Qu = d^*d(u + \Omega^1(M^\circ)).$$

The third term $(\tilde{f} - \pi^* f_\partial)d\chi_1$ is smooth due to the product structure of the metric imposed on V. Working on V°, where $d\chi_1$ is non-zero, we have

$$\begin{aligned}\Delta(\tilde{f} - \pi^* f_\partial) &= \Delta(d^*Qu - \pi^* d_\partial^* Q_\partial u_\partial)\\ &= d^*(u + \Omega^1(V^\circ)) - \pi^* d_\partial^*(u_\partial + \Omega^1(U))\\ &= (d^*u - \pi^* d_\partial^* u_\partial) + \Omega^0(V^\circ).\end{aligned}$$

The second equality is obtained due to the product structure of the metric on V implying $\Delta \pi^* d_\partial^* Q_\partial u_\partial = \pi^* d_\partial^* \Delta_\partial Q_\partial u_\partial$. The final line in the preceding display is smooth as the product metric also implies $d^*u = \pi^* d_\partial^* u_\partial$ on V. Elliptic regularity ensures $\tilde{f} - \pi^* f_\partial$ is smooth on V°.

6.2. **Proof of Lemma 10.** Consider a boundary defining function ρ of Σ and $\varepsilon > 0$ small enough such that (on M°)

$$\mathrm{supp}(Pf') \subset \{\rho > \varepsilon\}, \qquad \mathrm{supp}(f') \subset D_\varepsilon := \{\rho > \varepsilon\} \cup \{X\rho < -\varepsilon\}.$$

In semi-classical notation,

$$\mathrm{WF}_h(Pf') \subset \{\rho > \varepsilon\} \times 0, \qquad \mathrm{WF}_h(f') \subset E_+^* \cup (D_\epsilon \times 0)$$

Where $U \times 0$ for arbitrary $U \subset M$ notates $\{(x,\xi) \in T^*M \,|\, x \in U, \xi = 0\}$. In order to show f' is smooth and compactly supported we show $\mathrm{WF}_h(f') \subset \{\rho > \varepsilon\} \times 0$. It is sufficient to show

- Given $A \in \Psi_h^{\mathrm{comp}}(M^\circ)$ with $\mathrm{WF}_h(A) \cap (\{\rho \geq \varepsilon\} \times 0) = \varnothing$, there exists B (of the same properties as A) such that $\|Af'\|_{L^2(M)}^2 \leq Ch\|Bf'\|_{L^2(M)}^2 + \mathcal{O}(h^\infty)$.

Inductively, this property shows $\|Af'\|_{L^2(M)} = \mathcal{O}(h^\infty)$ for all A with the above conditions. To this end, consider such an A. Specifically, there exist $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$, and $0 < \delta_1 < \delta_2$ such that

$$\begin{aligned}\mathrm{WF}_h(A) \cap \{(x,\xi) \,|\, \rho(x) \geq \varepsilon_1, |\xi| \leq \delta_1\} = \varnothing,\\ \mathrm{WF}_h(A) \subset \{(x,\xi) \,|\, \rho(x) \geq \varepsilon_2, |\xi| \leq \delta_2\}.\end{aligned}$$

We build an escape function $g \in C_0^\infty(T^*M; [0,1])$ (in the spirit of [FS11]) such that:

- $g = 1$ near $\{\rho \geq \varepsilon\} \times 0$;
- $H_p g < 0$ on $\mathrm{WF}_h(A) \cap (E_+^* \cup (\overline{D_\varepsilon} \times 0))$;
- $H_p g \leq 0$ near $E_+^* \cup (\overline{D_\varepsilon} \times 0)$.

To build g, consider $\chi_1, \chi_2 \in C^\infty([0,\infty);[0,1])$. First, χ_1 is increasing with $\operatorname{supp}(\chi_1) \subset (0,\infty)$, and $\chi_1' > 0$ near $[\varepsilon_2, \varepsilon_1]$, and $\chi_1 = 1$ near $[\varepsilon, \infty)$, Second, χ_2 is decreasing with $\chi_2 = 1$ near 0, and $\chi_2' < 0$ near $[\delta_1, \delta_2]$, and $\operatorname{supp}(\chi_2) \subset [0, \delta_2 + 1)$. One can then verify that the following function satisfies the three desired properties:

$$g(x,\xi) := \chi_1(\rho(x)) \cdot \chi_2(|\xi|).$$

Quantise the escape function g to give a self-adjoint operator G and we note that $-\frac{1}{2} H_p g = \sigma_h(\frac{1}{2i}[P,G]) = -\operatorname{Im} GP$. The second and third properties of the escape function allow us to choose $A_1 \in \Psi_h^{\mathrm{comp}}(M^\circ)$ such that

$$\mathrm{WF}_h(A_1) \cap (E_+^* \cup (\overline{D_\varepsilon} \times 0)) = \varnothing, \qquad \tfrac{-1}{C} A^*A - \tfrac{1}{2} H_p g + A_1^* A_1 \geq 0$$

for some $C > 0$. We can now choose $B \in \Psi_h^{\mathrm{comp}}(M^\circ)$ with $\mathrm{WF}_h(B) \cap (\{\rho \geq \varepsilon\} \times 0) = \varnothing$ and

$$\mathrm{WF}_h(I - B) \cap \big(\mathrm{WF}_h(A) \cup \mathrm{WF}_h(A_1) \cup \mathrm{WF}_h([P,G])\big) = \varnothing.$$

The sharp Gårding inequality provides

$$\big\langle \big(\tfrac{-1}{C} A^*A + \tfrac{1}{2i}[P,G] + A_1^* A_1\big) Bf', Bf' \big\rangle_M \geq -C'h \, \|Bf'\|_{L^2(M)}^2$$

for some $C' > 0$. The conditions on $\mathrm{WF}_h(I-B)$ imply $ABf' = Af'$, $A_1 B f' = A_1 f'$, and $B^*[P,G]B = [P,G]f'$ modulo $\mathcal{O}(h^\infty)_{C^\infty(M)}$. Also, the condition on $\mathrm{WF}_h(A_1)$ implies $A_1 f' = \mathcal{O}(h^\infty)_{L^2(M)}$. So rearranging the above display and absorbing the constants provides

$$\|Af'\|_{L^2(M)}^2 \leq Ch \|Bf'\|_{L^2(M)}^2 - C \operatorname{Im}\langle GPf', f' \rangle_M + \mathcal{O}(h^\infty).$$

The first property of the escape function ensures $(I-G)Pf' = \mathcal{O}(h^\infty)_{C^\infty}$ so the proof finishes upon observation that $\operatorname{Im}\langle Pf', f'\rangle_M = \operatorname{Re} \int_M (\mathcal{L}_X f') \, \overline{f'} \, d\mathrm{vol}_M = 0$.

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