INFO-F521 - Graph Theory - List of Theorems, Lemmas, Corollaries and Observations

1 Matchings

Theorem 1 (König's, 1931) *The maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover.*

Corollary 1 *The minimum size of a vertex cover is $\geq$ the maximum size of a matching.*

Theorem 2 (Hall, 1935) *If G is bipartite with partitions A, B, it contains a matching of A if and only if $|N(S) \geq |S|$ for all $S \subseteq A$).*

Theorem 3 (Tutte, 1947) *A graph G has a 1-factor if and only if $q(G - S) \leq |S|$ for all $S \subseteq V(G)$.*

Observation 1 *If H is a graph and H' is a subgraph obtained by deleting some edge e of H, then $q(H') \geq q(H)$.*

Corollary 2 *If H' is a spanning subgraph of H then $q(H') \geq q(H)$.*

Corollary 3 *If S is a bad set for H then S is bad for all spanning subgraphs of H.*

2 Connectivity

Theorem 1 (ear decomposition of 2-connected graphs) *Let G be 2-connected $\iff$ G can be built starting with a cycle and iteratively adding ears.*

Theorem 2 (Tutte) *If G is 3-connected and $|V(G)| \geq 5$, then $\exists e \in E(G)$ such that G/e is still 3-connected.*

Theorem 3 (Menger) *Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint $A - B$ paths in G.*

Theorem 4 (Menger, global version) *A graph G is k-connected $\iff$ every pair a, b of distinct vertices are linked by k independent paths (no internal vertex in common, they only meet at endpoints).*

3 Planar Graphs

Lemma 1 *Let $F \in E(G)$. Then:*

1. $\partial F \subseteq G$
2. $\forall e \in E(G)$, *either* $e \in \partial F$, *or* $\partial F \cap int(e) = \emptyset$
3. *If $e \in E(G)$ is on a cycle $C \in G$, then e is contained in the boundary of exactly two faces of G, one inside C and the other outside C.*
4. *If $e \in E(G)$ is not included in any cycle, then e appears in the boundary of exactly one face.*

Lemma 2 *If G is a 2-connected plane graph, then the boundary of every face is a cycle of G.*

Lemma 3 *A plane graph on at least 3 vertices is maximally plane $\iff$ it is a plane triangulation*

Theorem 1 (Euler) *For G, a connected plane graph with n vertices, m edges and f faces: $n - m + f = 2$.*

Corollary 1 *If G is an n-vertex plane graph with n being at least 3, then G has at most $3n-6$ edges.*

Corollary 2 *K_5 is not planar.*

Lemma 4 *If G is an n-vertex plane graph ($n \geq 3$) and has no triangle (i.e cycle of length 3 or 3 vertices that are pairwise adjacent), then G has at most $2n-4$ edges.*

Corollary 3 *$K_{3,3}$ is not planar.*

Corollary 4 *No planar graph contains K_5 or $K_{3,3}$ as a minor.*

Theorem 2 (Kuratowski, 1930) *A graph G is planar $\iff$ G it has neither K_5 nor $K_{3,3}$ as minor.*

4 Coloring

Theorem 1 (Kempe and Heawood, 1890) *If G is planar, $\chi_G \leq 5$.*

Theorem 2 (Erdö, 1950) *For all $k \geq 1$, there exists a graph G such that $\chi(G) \geq k$ and girth$(G) \geq k$*

Theorem 3 (Lovász, 1972) *G is perfect $\iff$ $\alpha(H)\omega(H) \geq |V(H)|$ $\forall$ induced subgraphs H of G.*

5 Random graphs

Theorem 1 *Every graph G has a bipartite subgraph H with $|E(H)| \geq |E(G)|/2$.*

Theorem 2 (Erdö, 1956) *$\forall k \geq 0, \exists \mathcal{G}$ with girth$(G) > k$ and $\chi(G) > k$.*

Lemma 1 *For $G \in \mathcal{G}(n,p)$, let X_k be the random variable counting the number of cycles of length k in G. Then: $E(X_k) = \frac{n!}{2k(n-k)}p^k$*

Lemma 2 *Let $n \geq k \geq 3$ and $p \in [0,1]$ with $p \geq (6k \ln n)/n$. Then, for $G \in \mathcal{G}(n,p)$: $\lim_{n\to\infty} P[\alpha(G) \geq \frac{n}{2k}] = 0$*

Lemma 3 (Lovász, local lemma symmetric version) *Given events $A_1, ...A_n$ and a dependency digraph with max outdegree d, if $P(A_i) \leq p$ $\forall i$ and $ep(d+1) \leq 1$, then: $P(\cap \bar{A}_i) > 0$*

Lemma 4 (Lovász, local lemma asymmetric version) *Given events $A_1, ...A_n$ and a dependency digraph D. Suppose that $\exists x_1, ..., x_n \in [0,1]$ such that $P(A_i) \leq \prod_{(i,j)\in E(D)}(1-x_j)\forall i$. Then $P(\bigcap_i \bar{A}_i) \geq \prod_i (1-x_i) > 0$.*

Theorem 3 *If A is a k-uniform hypergraph and each edge intersects $\leq 2^{k-1}/2$ others, then $\exists 2-$coloring of H.*