INFO-F521 - Graph Theory - Homework
Treewidth

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1 Properties

1. *Treewidth is monotone under the operation of taking minors, that is, $tw(H) \leq tw(G)$ if H is a minor of G.*
   We have that $\forall H \subseteq G$, the tree decomposition of H is $(T, (V_t \cap V(H))_{t \in T})$ where $|(V_t \cap V(H)_{t \in T})|$ will never exceed $|V_t|$. Therefore, as $tw(G)$ is defined as the minimum value that $\max_{x \in V(T)} |\{v \in V(G) : x \in V(T_v\}| - 1\}$, $|(V_t \cap V(H)_{t \in T})| \leq |V_t|$, we have $tw(H) \leq tw(G) \;\; \forall H \subseteq G$.
2. *Subtrees of a tree have the Helly property.*
   Let's take $k = 3$ subtrees T_1, T_2 and T_3 and 3 vertices a, b and c. Now, a connects T_1 and T_2, b connects T_2 and T_3, c connects T_3 and T_1. As T_3 is connected to T_1 and T_2 via b and c, it contains those, and their intersection a. Then, the intersection of all 3 subtrees is at least a. This can be generalised easily to any k, the pairwise connections will form a cycle in which trees "contain" others.
3. *If C is a clique of a graph G and $(T, \{T_v\}_{v \in V(G)})$ is a tree decomposition of G then there exists a node $x \in V(T)$ such that x belongs to all subtrees T_v of vertices $v \in C$.*
4. $\omega(G) \leq tw(G) + 1 \;\; \forall G$.
   In the example featured in the statement, we can observe that $\omega(G) = 3$ and $tw(G) = 2$. Using the definition of tree decompositions, we can generalise this in the way that all cliques of size t will turn into vertices (in the tree decomposition) listing t vertices (from the original graph G). As $tw(G)$ corresponds to the greatest number of original vertices listed in the tree decomposition vertices minus 1, we have $\omega(G) \leq tw(G) + 1 \;\; \forall G$.
5. *[...]*
6. *[...]*
7. *[...]*
8. $\chi(G) \leq tw(G) + 1 \;\; \forall G$.
   We already know $\omega(G) \leq tw(G) + 1 \;\; \forall G$. Also, $\chi(G) \geq \omega(G)$ (but I don't think this will help).
9. *[...]*
10. *If G is obtained from H by adding a universal vertex to H then $tw(G) = tw(H) + 1$.*
    Intuitively, I think this will have the same effect as plugging a new tree root to the previous graph. And this new vertex is connected to all other vertices and subtrees - that will raise tw by one.
11. *[...]*
12. *[...]*
13. *[...]*
14. *[...]*
15. *[...]*
16. *[...]*