Discrete Mathematics Year 1 Semester 1 — Full Solutions

Section A: Relations and Functions (30 Questions) — Solutions

1. Define a relation on a set.

Answer: A relation R on a set A is a subset of A times A. That is, R subset of A x A and consists of ordere

2. What is the domain and range of a relation?

Answer: For a relation R subset of A x B, the domain is {a in A | exists b in B, (a,b) in R} and the range

3. Define a function. How does it differ from a general relation?

Answer: A function f from A to B is a relation f subset of A x B such that for every a in A there exists ex

4. What is an injective (one-to-one) function?

Answer: A function f:A->B is injective if f(a1)=f(a2) implies a1=a2. Equivalently distinct inputs map to di

5. What is a surjective (onto) function?

Answer: f:A->B is surjective if for every b in B there exists a in A with f(a)=b. The image of f equals B.

6. Define a bijective function. Give an example.

Answer: Bijective means both injective and surjective. Example: f:Z->Z given by f(n)=n+1 is bijective.

7. Define an inverse function. Under what condition does it exist?

Answer: If f:A->B is bijective there exists f^{-1}:B->A defined by f^{-1}(b)=a where f(a)=b. Inverse exists

8. What is the composition of two functions?

Answer: Given f:A->B and g:B->C, the composition g o f : A->C is defined by (g o f)(x)=g(f(x)) for all x in

9. Define reflexive, symmetric, and transitive relations.

Answer: Reflexive: for all a in A, (a,a) in R. Symmetric: for all a,b, (a,b) in R implies (b,a) in R. Trans

10. Define antisymmetric relations and give an example.

Answer: R is antisymmetric if (a,b) in R and (b,a) in R implies a=b. Example: <= on real numbers is antisym

11. Let A = {1, 2, 3}. List all possible relations on A. How many are there?

Answer: Relations are subsets of A x A. |A x A| = 9, so number of relations = 2^9 = 512. Each relation corr

12. Determine whether the relation R = {(1,1), (2,2), (3,3), (1,2)} on A = {1,2,3} is reflexive, symmetric,

Answer: Reflexive: yes (all (a,a) present). Symmetric: no because (1,2) in R but (2,1) not in R. Transitive

13. Is the 'divides' relation on the set of positive integers a partial order? Justify.

Answer: Divides (|) is reflexive (n|n), antisymmetric (if a|b and b|a then a=b for positive integers), and

14. Is the 'less than or equal to' relation an equivalence relation? Why or why not?

Answer: <= is reflexive and antisymmetric and transitive, but not symmetric in general; equivalence relatio

15. Define an equivalence relation. Give an example on the set of integers.

Answer: An equivalence relation is reflexive, symmetric, and transitive. Example on Z: congruence modulo n,

16. Describe the equivalence classes of the relation 'congruence modulo 3' on the set of integers.

Answer: There are three classes: [0] = {...,-6,-3,0,3,6,...}, [1] = {...,-5,-2,1,4,7,...}, [2] = {...,-4,-1

17. Show that if R1 and R2 are transitive relations on a set A, then R1 intersect R2 is transitive.

Answer: Proof: Let (a,b),(b,c) in R1 intersect R2. Then both pairs are in R1 and in R2. Since R1 and R2 are

18. Is the union of two transitive relations necessarily transitive? Prove or disprove.

Answer: Counterexample: on A={1,2,3}, let R1={(1,2)} and R2={(2,3)}. Each is transitive (vacuous). But R1 u

19. Represent the relation R = {(1,2), (2,3), (3,1)} on A = {1,2,3} as a directed graph.

Answer: Directed graph: vertices 1,2,3 with edges 1->2, 2->3, 3->1 forming a 3-cycle.

20. What is the matrix representation of the relation R = {(1,2), (2,3), (3,3)}?

Answer: Adjacency matrix indexed 1..3: [[0,1,0],[0,0,1],[0,0,1]].

21. How many functions are there from a set with 3 elements to a set with 4 elements?

Answer: Each of 3 domain elements has 4 choices in codomain: 4^3 = 64 functions.

22. How many injective functions exist from A = {1,2,3} to B = {a,b,c,d}?

Answer: Number of one-to-one maps = permutations P(4,3) = 4 * 3 * 2 = 24.

23. Let f(x) = 2x + 3 and g(x) = x^2. Find (f o g)(x) and (g o f)(x).

Answer: (f o g)(x) = f(g(x)) = 2(x^2) + 3 = 2x^2 + 3. (g o f)(x) = g(f(x)) = (2x+3)^2 = 4x^2 + 12x + 9.

24. If f: A -> B and g: B -> C are bijective, prove that g o f is bijective.

Answer: Proof: If f,g injective then g o f injective (if g(f(x1))=g(f(x2)) then f(x1)=f(x2) then x1=x2). If

25. Give an example of a relation that is reflexive and symmetric but not transitive.

Answer: On A={1,2,3}, take R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)}. R is reflexive and symmetric but

26. Give an example of a function that is not surjective.

Answer: f:N->N defined by f(n)=n+1 is not surjective if N includes 0 because 0 has no preimage; or f:Z->Z,

27. Find the inverse of f(x) = 3x + 2.

Answer: Solve y = 3x + 2 => x = (y - 2)/3. So f^{-1}(y) = (y - 2)/3.

28. If f(x) = x^2 defined on R, is f injective? How can we restrict the domain to make it bijective?

Answer: On R, f is not injective because f(1)=f(-1). Restrict domain to [0, infinity) to make f: [0, infini

29. Prove that the composition of two injective functions is injective.

Answer: Let f:A->B and g:B->C injective. If (g o f)(x1)=(g o f)(x2) then g(f(x1))=g(f(x2)) so f(x1)=f(x2) (

30. Prove that every equivalence relation on a set partitions the set into disjoint equivalence classes.

Answer: Sketch: For equivalence ~ on A define [a]={x in A | x ~ a}. Show each element is in some class; two

Section B: Proofs in Discrete Mathematics (50 Questions) — Solutions

31. Define a proposition.

Answer: A proposition is a declarative statement that is either true or false (but not both).

32. What is a tautology? Give an example.

Answer: A tautology is a proposition that is always true for all truth-values of its components. Example: p

33. What is a contradiction? Give an example.

Answer: A contradiction is always false. Example: p AND not p.
34. Define the negation of a statement.

Answer: Negation 'not P' is true exactly when P is false; it flips truth value.

35. State De Morgan’s Laws.

Answer: not (P and Q) equivalent to (not P) or (not Q); not (P or Q) equivalent to (not P) and (not Q).

36. Translate 'If it rains, then I take an umbrella' into symbolic logic.

Answer: Let p = 'it rains', q = 'I take an umbrella'. Then p -> q.

37. Write the converse, inverse, and contrapositive of 'If p, then q'.

Answer: Converse: q -> p. Inverse: not p -> not q. Contrapositive: not q -> not p.

38. What is a truth table? Construct one for p -> q.

Answer: A truth table lists truth values for p,q and p->q: TT->T, TF->F, FT->T, FF->T.

39. Determine whether (p or q) -> r and (p -> r) and (q -> r) are logically equivalent.

Answer: They are equivalent. Use distribution or truth tables: (p or q)->r is equivalent to (not p and not

40. Show that not (p and q) equivalent to (not p) or (not q).

Answer: This is De Morgan's law; verify by truth table.

41. Define universal and existential quantifiers.

Answer: Universal: for all x, P(x). Existential: there exists x such that P(x).

42. Negate the statement: 'For every real number x, x^2 > 0.'

Answer: Negation: There exists x in R such that x^2 <= 0.

43. Negate: 'There exists an integer n such that n^2 = 2.'

Answer: Negation: For all integers n, n^2 != 2.

44. Translate 'Every student passed the exam' into logical notation.

Answer: If domain is students and P(s) = 's passed', then for all s, P(s).

45. Explain the difference between for all x there exists y P(x,y) and there exists y for all x P(x,y).

Answer: For all x exists y: y may depend on x. Exists y for all x: same y works for every x. The latter imp

46. What is a direct proof?

Answer: A proof that assumes premises and deduces conclusion by direct logical steps.

47. What is a proof by contraposition?

Answer: To prove P->Q, show not Q -> not P instead.

48. What is a proof by contradiction?

Answer: Assume the negation of the conclusion and derive a contradiction, thus the conclusion holds.

49. Explain proof by cases.

Answer: Split the proof into exhaustive cases and prove the statement in each case.

50. State the principle of mathematical induction.

Answer: If P(0) holds and for all n, P(n) -> P(n+1) holds, then P(n) holds for all n >= 0.

51. Prove that the sum of two even integers is even.
Answer: Let a=2k, b=2m. Then a+b=2(k+m), which is even.

52. Prove that the square of an odd integer is odd.

Answer: Let n=2k+1. Then n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1, which is odd.

53. Prove by contraposition: If n^2 is even, then n is even.

Answer: Contrapositive: if n is odd then n^2 is odd (shown above). Therefore original holds.

54. Prove by contradiction: sqrt(2) is irrational.

Answer: Assume sqrt(2)=p/q in lowest terms. Then 2q^2=p^2 so p even, write p=2k, substitute gives q even, c

55. Use direct proof to show that if n is a multiple of 6, then n is even.

Answer: If n is multiple of 6, n=6k = 2(3k), hence even.

56. Prove by induction that 1 + 2 + ... + n = n(n+1)/2.

Answer: Base n=1 holds. Assume true for n, then add n+1 to get (n+1)(n+2)/2, so holds for n+1.

57. Prove by induction that 2^n >= n + 1 for all n >= 0.

Answer: Base n=0: 1 >=1. Assume 2^n >= n+1. Then 2^{n+1} = 2*2^n >= 2(n+1) >= n+2, so inequality holds.

58. Prove by induction that n! > 2^n for n >= 4.

Answer: Base n=4: 24 >16. Assume k! > 2^k. Then (k+1)! = (k+1)k! > (k+1)2^k. For k>=4, k+1 >= 5 so product

59. Prove by induction that 3 divides (4^n - 1) for all n >= 1.

Answer: Base n=1: 3 divides 3. Assume 4^k -1 = 3m. Then 4^{k+1}-1 = 4(4^k-1)+3 = 3(4m+1) divisible by 3.

60. Prove by induction that 5^n - 1 is divisible by 4 for all n >= 1.

Answer: Base n=1: 5-1=4. Assume 5^k-1=4m. Then 5^{k+1}-1 =5(5^k-1)+4 = 4(5m+1), divisible by 4.

61. Prove that A subseteq B iff A intersect B = A.

Answer: If A subset B then intersection is A. Conversely if intersection equals A then all elements of A ar

62. Show that (A union B)' = A' intersect B'.

Answer: Use element argument: x not in A union B iff x not in A and x not in B, so in A' intersect B'.

63. Prove that (A intersect B) union (A intersect C) = A intersect (B union C).

Answer: Distribute intersection over union; element-wise equivalence shows equality.

64. Prove that the intersection of two equivalence relations is an equivalence relation.

Answer: Intersection of reflexive relations is reflexive; same for symmetric and transitive. Hence intersec

65. Prove that the sum of two rational numbers is rational.

Answer: If r1 = p/q and r2 = s/t then r1+r2 = (pt+qs)/qt which is rational.

66. Prove that the sum of a rational and an irrational number is irrational.

Answer: If r rational and x irrational and r+x were rational then x=(r+x)-r would be rational, contradictio

67. Prove that there are infinitely many prime numbers.

Answer: Euclid's proof: assume p1,...,pn are all primes. Consider N = p1*...*pn +1. N has a prime divisor n

68. Prove that every integer greater than 1 is divisible by a prime number.

Answer: If n is prime done. Otherwise n has a factor a with 1<a<n. Repeat factorization until a prime facto
69. Prove that the square of any integer is either of the form 4k or 4k+1.
Answer: If n even n=2k then n^2=4k^2=4k'. If n odd n=2k+1 then n^2=4k^2+4k+1=4k'+1.

70. Prove that the product of two odd integers is odd.

Answer: Let a=2m+1, b=2n+1. Then ab=4mn+2m+2n+1=2(2mn+m+n)+1 which is odd.

71. Prove that if A subseteq B and B subseteq C, then A subseteq C.

Answer: If x in A then x in B then x in C, so A subset C.

72. Prove that P -> Q is equivalent to not P or Q.

Answer: By truth table equivalence holds.

73. Show that (P or Q) and (not P or R) implies (Q or R).

Answer: Reason by cases: if P true then (not P or R) gives R true so Q or R holds. If P false then (P or Q)

74. Prove that if a relation is both symmetric and transitive, and if it is reflexive on some element, then

Answer: Let (a,a) in R and (a,b) in R. By symmetry (b,a) in R. By transitivity (b,b) in R. So b is reflexiv

75. Prove that if f: A -> B and g: B -> C are injective, then g o f is injective.

Answer: Same proof as earlier: injectivity preserved under composition.

76. Prove that if f and g are surjective, then g o f is surjective.

Answer: Given c in C, pick b with g(b)=c, pick a with f(a)=b, then g(f(a))=c.

77. Prove that every finite partially ordered set has at least one minimal element.

Answer: If no minimal element exists, for each element there is a smaller one producing infinite descent wh

78. Prove that an equivalence relation partitions a set into mutually disjoint subsets.

Answer: Equivalence classes are disjoint or identical and their union is the whole set; hence a partition.

79. Prove that the power set of a finite set with n elements has 2^n elements.

Answer: Each element may be in or out of a subset (2 choices) so total 2^n. Inductive proof also possible.

80. Prove that for all integers a, b, c, if a divides b and b divides c, then a divides c.

Answer: If b = a k and c = b l then c = a (k l) so a divides c.