2D Douglas ADI

Consider the equation

$$\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} - f(x)$$

Splitting this we have

$$\left(I - \frac{\Delta t}{2}\alpha_x \Delta_x\right)u^* = \left(I + \frac{\Delta t}{2}\alpha_x \Delta_x + \Delta t \alpha_y \Delta_y\right)u_k - \Delta t F$$

$$\left(I - \frac{\Delta t}{2}\alpha_y \Delta_y\right)u_{k+1} = u^* - \frac{\Delta t}{2}\alpha_y \Delta_y u_k$$

In Douglas' paper, this was written as (in equation 2.5)

$$\frac{1}{2}\Delta_x^2(w_{n+1}^* + w_n) + \Delta_y^2 w_n = \frac{w_{n+1}^* - w_n}{\Delta t}$$

$$\frac{1}{2}\Delta_x^2(w_{n+1}^* + w_n) + \frac{1}{2}\Delta_y^2(w_{n+1} + w_n) = \frac{w_{n+1} - w_n}{\Delta t}$$

Multiplying by Δt, we have

$$\frac{1}{2}\Delta t \Delta_x^2(w_{n+1}^* + w_n) + \Delta t \Delta_y^2 w_n = w_{n+1}^* - w_n$$

$$\frac{1}{2}\Delta t \Delta_x^2(w_{n+1}^* + w_n) + \frac{1}{2}\Delta t \Delta_y^2(w_{n+1} + w_n) = w_{n+1} - w_n$$

Attempting to solve for w_{n+1}^* and w_{n+1}, we have

$$\left(I - \frac{1}{2}\Delta t \Delta_x^2\right)w_{n+1}^* = \frac{1}{2}\Delta t \Delta_x^2 w_n + \Delta t \Delta_y^2 w_n + w_n$$

$$\left(I - \frac{1}{2}\Delta t \Delta_y^2\right)w_{n+1} = \frac{1}{2}\Delta t \Delta_x^2(w_{n+1}^* + w_n) + \frac{1}{2}\Delta t \Delta_y^2 w_n + w_n$$

Or, finally

$$\left(I - \frac{1}{2}\Delta t \Delta_x^2\right)w_{n+1}^* = \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \Delta t \Delta_y^2\right)w_n$$

$$\left(I - \frac{1}{2}\Delta t \Delta_y^2\right)w_{n+1} = \frac{1}{2}\Delta t \Delta_x^2 w_{n+1}^* + \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \frac{1}{2}\Delta t \Delta_y^2\right)w_n$$

Combining

$$\left(I - \frac{1}{2}\Delta t \Delta_x^2\right)w_{n+1}^* = \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \Delta t \Delta_y^2\right)w_n$$

From above to the second equation, we have

$$\left(I - \frac{1}{2}\Delta t \Delta_x^2\right) w_{n+1}^* = \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \Delta t \Delta_y^2\right) w_n$$

$$\left(I - \frac{1}{2}\Delta t \Delta_y^2\right) w_{n+1} = \frac{1}{2}\Delta t \Delta_x^2 w_{n+1}^* + \left(I - \frac{1}{2}\Delta t \Delta_x^2\right) w_{n+1}^* - \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \Delta t \Delta_y^2\right) w_n + \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \frac{1}{2}\Delta t \Delta_y^2\right) w_n$$

We finally have

$$\left(I - \frac{1}{2}\Delta t \Delta_x^2\right) w_{n+1}^* = \left(I + \frac{1}{2}\Delta t \Delta_x^2 + \Delta t \Delta_y^2\right) w_n$$

$$\left(I - \frac{1}{2}\Delta t \Delta_y^2\right) w_{n+1} = w_{n+1}^* - \frac{1}{2}\Delta t \Delta_y^2 w_n$$

This is exactly the same as the form we used in class.

This is written in the last equation (2.7) as

$$\left(I - \frac{\Delta t}{2}\Delta_y^2\right) w_{n+1} = w_{n+1}^* - \frac{\Delta t}{2}\Delta_y^2 w_n$$

3D Douglas ADI

Consider the equation

$$\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} + \alpha_z \frac{\partial^2 u}{\partial z^2} - f(x)$$

Splitting this operator into several steps, we have

$$\frac{1}{2}\alpha_x \Delta_x (u_{n+1}^* + u_n) + \alpha_y \Delta_y u + \alpha_z \Delta_z u - f = \frac{u_{n+1}^* - u_n}{\Delta t}$$

$$\frac{1}{2}\alpha_x \Delta_x (u_{n+1}^* + u_n) + \frac{1}{2}\alpha_y \Delta_y (u_{n+1}^{**} + u_n) + \alpha_z \Delta_z u - f = \frac{u_{n+1}^{**} - u_n}{\Delta t}$$

$$\frac{1}{2}\alpha_x \Delta_x (u_{n+1}^* + u_n) + \frac{1}{2}\alpha_y \Delta_y (u_{n+1}^{**} + u_n) + \frac{1}{2}\alpha_z \Delta_z (u_{n+1} + u_n) - f = \frac{u_{n+1} - u_n}{\Delta t}$$

After rearranging, we have

$$\left(I - \frac{\Delta t}{2}\alpha_x \Delta_x\right) u_{n+1}^* = \left(I + \frac{\Delta t}{2}\alpha_x \Delta_x + \Delta t \alpha_y \Delta_y + \Delta t \alpha_z \Delta_z\right) u_n - \Delta t f$$

$$\left(I - \frac{\Delta t}{2}\alpha_y \Delta_y\right) u_{n+1}^{**} = u_{n+1}^* - \frac{\Delta t}{2}\alpha_y \Delta_y u_n$$

$$\left(I - \frac{\Delta t}{2}\alpha_z \Delta_z\right) u_{n+1}^{**} = u_{n+1}^{**} - \frac{\Delta t}{2}\alpha_z \Delta_z u_n$$

This is the form of the 3D ADI.

Time step selection

In class, we prescribed a time step of

$$\Delta t_j = \frac{4\left(2^j h_0\right)^2}{\alpha \pi^2}$$

For the problem

$$\alpha \nabla^2 u = f$$

There is an issue with this time step selection. Note that if $\alpha = 2$ and $f = cos(2\pi x)$

$$\Delta t_j = \frac{4\left(2^j h_0\right)^2}{2\pi^2}$$

But if $\alpha = 1$ and $f = \frac{1}{2}\cos(2\pi x)$, we have exactly the same PDE, but

$$\Delta t_j = \frac{4\left(2^j h_0\right)^2}{\pi^2}$$

In order to rectify the same result. Let's choose the timestep

$$\Delta t_j^* = \frac{A(f)}{\underbrace{\alpha}_{k}} \Delta t_j$$

Where $A(f)$ is the amplitude of f. For the case previously stated, note that $k = \frac{1}{2}$. Let's verify that this workaround will produce the same result. Suppose $\alpha = 1$ and $f = \frac{1}{2}\cos(2\pi x)$. Then we have

$$A(f) = \frac{1}{2} \quad , k = \frac{1/2}{1} = \frac{1}{2}$$

Although crude, this seems to be a more appropriate choice for the time step.