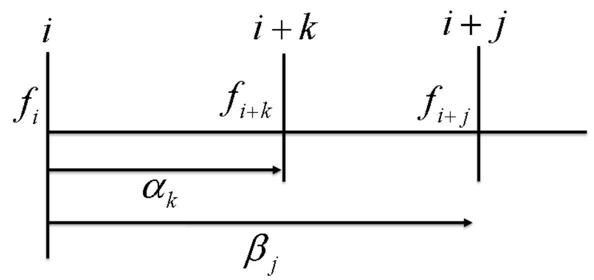
Non-uniform grid stencils results

This document contains the results and examples of the 1st and 2nd derivative of a function using function values at locations x_i, α_k and β_j with respective indexes. Below are diagrams of the possible stencils that may be made from the results of this document. The following analysis is valid for non-uniform grids.

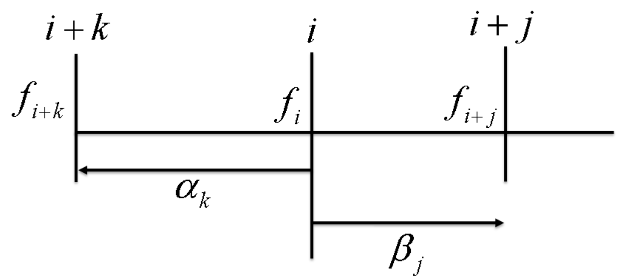
Stencil Possibility 1



k, j may be positive or negative

α_k, β_j may be positive or negative

Stencil Possibility 2



k, j may be positive or negative

α_k, β_j may be positive or negative

Results

These results were obtained by solving the following system of equations

$$f_{i+k} = f_i + \alpha_k f_i' + \frac{1}{2}\alpha_k{}^2 f_i'' + O(\alpha_k^3) \quad , \quad f_{i+j} = f_i + \beta_j f_i' + \frac{1}{2}\beta_j{}^2 f_i'' + O(\beta_j{}^3)$$

$$f_{i+k} = f_i + \alpha_k f_i' + \frac{1}{2}\alpha_k{}^2 f_i'' + \frac{1}{3!}\alpha_k{}^3 f_i''' + O(\alpha_k^4) \quad , \quad f_{i+j} = f_i + \beta_j f_i' + \frac{1}{2}\beta_j{}^2 f_i'' + \frac{1}{3!}\beta_j{}^3 f_i''' + O(\beta_j{}^4)$$

1st derivative

$$\boxed{f_i' = \frac{1}{(\beta_j - \alpha_k)}\left\{f_i\left(\frac{\alpha_k}{\beta_j} - \frac{\beta_j}{\alpha_k}\right) + \frac{\beta_j}{\alpha_k} f_{i+k} - \frac{\alpha_k}{\beta_j} f_{i+j}\right\} + O(\alpha_k \beta_j)} \qquad \alpha_k \neq \beta_j$$

Solving for f_i for BCs

$$\boxed{f_i = \frac{\left(f_i'(\beta_j - \alpha_k) - \frac{\beta_j}{\alpha_k} f_{i+k} + \frac{\alpha_k}{\beta_j} f_{i+j}\right)}{\left(\frac{\alpha_k}{\beta_j} - \frac{\beta_j}{\alpha_k}\right)} + O(\alpha_k \beta_j)} \qquad \alpha_k \neq \beta_j$$

2nd derivative

$$\boxed{f_i'' = \frac{2f_i}{\alpha_k \beta_j} + \frac{2f_{i+k}}{\alpha_k{}^2 - \alpha_k \beta_j} + \frac{2f_{i+j}}{\beta_j{}^2 - \beta_j \alpha_k} + O\left(-\beta_j \alpha_k - (\beta_j + \alpha_k)\right)} \qquad \alpha_k \neq \beta_j$$

Examples - 1st derivative

Example 1 (central difference)

Let

$$\alpha_k = -\Delta x \qquad k = -1$$

$$\beta_j = \Delta x \qquad j = 1$$

This results in

$$\frac{\beta_j}{\alpha_k} = -1 \quad , \quad \beta_j - \alpha_k = 2\Delta x$$

$$f_i^{'} = \frac{1}{2\Delta x}\{f_i(-1+1) - f_{i-1} + f_{i+1}\}$$

$$= \frac{1}{2\Delta x}(f_{i-1} + f_{i+1})$$

The order of accuracy should be

$$O(\alpha_k \beta_j) = -O(\Delta x \Delta x) = O(\Delta x^2)$$

Which is the 2nd order accurate, first derivative central differencing equation. Note that the indexes k and j should be -1 and 1 respectively.

Example 2 (forward difference)

Let

$$\alpha_k = \Delta x \qquad k = 1$$

$$\beta_j = 2\Delta x \qquad j = 2$$

This results in

$$\frac{\beta_j}{\alpha_k} = 2 \quad , \quad \beta_j - \alpha_k = \Delta x$$

$$f_i^{'} = \frac{1}{\Delta x}\{f_i\left(\frac{1}{2} - 2\right) + 2f_{i+1} + \frac{1}{2}f_{i+2}\}$$

$$= \frac{1}{2\Delta x}\{-3f_i + 4f_{i+1} - f_{i+2}\}$$

The order of accuracy should be

$$O(\alpha_k \beta_j) = O(\Delta x 2\Delta x) = O(\Delta x^2)$$

Which is the 2nd order accurate, first derivative forward differencing equation.

Examples - 2nd derivatives

Example 1 (central difference)

Let

$$\alpha_k = -\Delta x \qquad k = -1$$

$$\beta_j = \Delta x \qquad j = 1$$

This results in

$$\frac{\beta_j}{\alpha_k} = -1 \quad , \quad \beta_j - \alpha_k = 2\Delta x$$

$$f_i'' = 2f_i \frac{1}{-\Delta x^2} - 2f_{i+k} \frac{1}{(-\Delta x)^2\{-2\}} + 2f_{i+j} \frac{1}{\Delta x(-\Delta x)\{-2\}}$$

$$= \frac{-2f_i + f_{i+k} + f_{i+j}}{\Delta x^2}$$

Which is the 2nd order central difference scheme.

Example 2 (forward difference)

Let

$$\alpha_k = \Delta x \qquad k = 1$$

$$\beta_j = 2\Delta x \qquad j = 2$$

This results in

$$\frac{\beta_j}{\alpha_k} = 2 \quad , \quad \beta_j - \alpha_k = \Delta x$$

$$f_i'' = \frac{2f_i}{\Delta x 2\Delta x} + \frac{2f_{i+k}}{\{\Delta x^2 - 2\Delta x^2\}} + \frac{2f_{i+j}}{\{(2\Delta x)^2 - 2\Delta x^2\}}$$

$$f_i'' = \frac{f_i}{\Delta x^2} - \frac{2f_{i+k}}{\Delta x^2} + \frac{f_{i+j}}{\Delta x^2}$$

$$f_i'' = \frac{f_i - 2f_{i+k} + f_{i+j}}{\Delta x^2}$$

Which is the 2nd order forward difference scheme.

Example 3 (non-uniform grid)

The diffusive terms may be written as

$$f'' = \frac{1}{\Delta x_{i+\frac{1}{2}}} \left(\frac{f_{i+1} - f_i}{\Delta x_{i+1}} - \frac{f_i - f_{i-1}}{\Delta x_i} \right)$$

$$= \frac{2}{\Delta x_{i+1} + \Delta x_i}\left(\frac{f_{i+1} - f_i}{\Delta x_{i+1}} - \frac{f_i - f_{i-1}}{\Delta x_i}\right)$$

Factoring terms

$$= f_i\left(\frac{-2}{(\Delta x_{i+1} + \Delta x_i)}\left(\frac{1}{\Delta x_{i+1}} + \frac{1}{\Delta x_i}\right)\right) + f_{i-1}\left(\frac{2}{\Delta x_i(\Delta x_{i+1} + \Delta x_i)}\right) + f_{i+1}\left(\frac{2}{\Delta x_{i+1}(\Delta x_{i+1} + \Delta x_i)}\right)$$

First term

$$\frac{-2}{(\Delta x_{i+1} + \Delta x_i)}\left(\frac{1}{\Delta x_{i+1}} + \frac{1}{\Delta x_i}\right) = \frac{-2}{\Delta x_i \Delta x_{i+1}(\Delta x_{i+1} + \Delta x_i)}(\Delta x_i + \Delta x_{i+1}) = \frac{-2}{\Delta x_i \Delta x_{i+1}}$$

Returning

$$f'' = 2f_i \frac{1}{-\Delta x_i \Delta x_{i+1}} + 2f_{i-1}\left(\frac{1}{\Delta x_i(\Delta x_{i+1} + \Delta x_i)}\right) + 2f_{i+1}\left(\frac{1}{\Delta x_{i+1}(\Delta x_{i+1} + \Delta x_i)}\right)$$

Let

$$\alpha_k = -\Delta x_i$$

$$k = -1$$

$$\beta_j = \Delta x_{i+1}$$

$$j = 1$$

Substituting

$$f'' = 2f_i \frac{1}{\alpha_k \beta_j} + 2f_{i-1}\left(\frac{1}{-\alpha_k(\beta_j - \alpha_k)}\right) + 2f_{i+1}\left(\frac{1}{\beta_j(\beta_j - \alpha_k)}\right)$$

$$= \frac{2f_i}{\alpha_k \beta_j} + \frac{2f_{i-1}}{\alpha_k^2 - \alpha_k \beta_j} + \frac{2f_{i+1}}{\beta_j^2 - \beta_j \alpha_k}$$

This is the same form of the equation that I've derived.