MOONS 3D NS Solver - Explicit Euler

Governing Equations

The semi-discrete, non-dimensional form of the incompressible Navier-Stokes and mass conservation equations are:

$$\frac{d\vec{u}}{dt} + \vec{u} \cdot \nabla\vec{u} = -\nabla p + \nu\, \nabla^2\vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

Now, introducing

$$q = \{u, v, w\}$$

Our equations are:

$$\frac{dq}{dt} = -F(q) - Gp + \nu L_\varepsilon q + bc_1$$

$$Dq = 0$$

Where

$$F(q) = u\nabla \cdot \vec{\mathrm{u}} = (H)_x + (H)_y + (H)_z$$

In order to define the boundary conditions 1 and 2, we will split our solutions into interior and boundary values as follows:

Temporal discretization

We may now use explicit Euler for the non-linear term and implicit for the pressure term results in:

$$\frac{q^{n+1} - q^n}{\Delta t} = -Gp^{n+1} - F(q^n) + \nu L q^n$$

$$Dq^{n+1} = 0$$

Moving all later time steps to the LHS and prior time steps to the RHS, and multiplying by Δt, we have:

$$q^{n+1} = g^n - \Delta t G p^{n+1}$$

Where

$$g^n = q^n + \Delta t\{-F(q^n) + \nu L q^n\}$$

Correction Step

The pressure is unknown at time level $n + 1$, so we use the projection scheme by Chorin. We start by taking the divergence of

$$q^{n+1} = g^n - \Delta t G p^{n+1}$$

to get

$$\Delta t \, L p^{n+1} = D g^n$$

Now, solve Laplace's equation for p^{n+1}:

$$L p^{n+1} = \frac{1}{\Delta t} D g^n$$

First time step

The first time step requires using a self-starting method. Explicit Euler was chosen in this case. The equation for explicit Euler simplifies to no correction:

$$q^{n+1} = g^n$$