MOONS - Successive Over-Relaxation (SOR)

Equations

The finite difference equation to solve begins with the Poisson equation:

$$\nabla \cdot (\sigma \nabla u) = f$$

Non-uniform grid for cell based data

Using the definition

$$\Delta h_{ci} = h_{ci} - h_{ci-1}$$

$$\Delta h_{ni} = h_{ni} - h_{ni-1}$$

For a non-uniform stencil, we have

$$\frac{\sigma_{i+\frac{1}{2}}\left(\frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta x_{ci+1}}\right) - \sigma_{i-\frac{1}{2}}\left(\frac{u_{i,j,k} - u_{i-1,j,k}}{\Delta x_{ci}}\right)}{\Delta x_{ni+1}} + \frac{\sigma_{j+\frac{1}{2}}\left(\frac{u_{i,j+1,k} - u_{i,j,k}}{\Delta y_{cj+1}}\right) - \sigma_{j-\frac{1}{2}}\left(\frac{u_{i,j,k} - u_{i,j-1,k}}{\Delta y_{cj}}\right)}{\Delta y_{nj+1}} + \frac{\sigma_{k+\frac{1}{2}}\left(\frac{u_{i,j,k+1} - u_{i,j,k}}{\Delta z_{ck+1}}\right) - \sigma_{k-\frac{1}{2}}\left(\frac{u_{i+1,j,k} - u_{i,j,k-1}}{\Delta z_{ck}}\right)}{\Delta z_{nk+1}} = f_{ijk}$$

Collecting terms we have

$$u_{i,j,k}\left(-\frac{1}{\Delta x_{ni+1}}\left(\frac{\sigma_{i+\frac{1}{2}}}{\Delta x_{ci+1}} + \frac{\sigma_{i-\frac{1}{2}}}{\Delta x_{ci}}\right) - \frac{1}{\Delta y_{nj+1}}\left(\frac{\sigma_{j+\frac{1}{2}}}{\Delta y_{cj+1}} + \frac{\sigma_{i-\frac{1}{2}}}{\Delta y_{cj}}\right) - \frac{1}{\Delta z_{nk+1}}\left(\frac{\sigma_{k+\frac{1}{2}}}{\Delta z_{ck+1}} + \frac{\sigma_{k-\frac{1}{2}}}{\Delta z_{ck}}\right)\right) +$$

$$u_{i+1,j,k}\left(\frac{\sigma_{i+\frac{1}{2}}}{\Delta x_{ci+1}\Delta x_{ni+1}}\right) + u_{i-1,j,k}\left(\frac{\sigma_{i-\frac{1}{2}}}{\Delta x_{ci}\Delta x_{ni+1}}\right) + u_{i,j+1,k}\left(\frac{\sigma_{j+\frac{1}{2}}}{\Delta y_{cj+1}\Delta y_{nj+1}}\right) + u_{i,j-1,k}\left(\frac{\sigma_{j-\frac{1}{2}}}{\Delta y_{cj}\Delta y_{nj+1}}\right) + \cdots + u_{i,j,k+1} + u_{i,j,k-1} = f_{ijk}$$

Solving for u_{ijk} we have

$$u_{ijk} = \frac{u_{i+1,j,k}\left(\frac{\sigma_{i+\frac{1}{2}}}{\Delta x_{ci+1}\Delta x_{ni+1}}\right) + u_{i-1,j,k}\left(\frac{\sigma_{i-\frac{1}{2}}}{\Delta x_{ci}\Delta x_{ni+1}}\right) + u_{i,j+1,k}\left(\frac{\sigma_{j+\frac{1}{2}}}{\Delta y_{cj+1}\Delta y_{nj+1}}\right) + u_{i,j-1,k}\left(\frac{\sigma_{j-\frac{1}{2}}}{\Delta y_{cj}\Delta y_{nj+1}}\right) + u_{i,j,k+1}\left(\frac{\sigma_{k+\frac{1}{2}}}{\Delta z_{ck+1}\Delta z_{nk+1}}\right) + u_{i,j,k-1}\left(\frac{\sigma_{k-\frac{1}{2}}}{\Delta z_{ck}\Delta z_{nk+1}}\right) - f_{ijk}}{\frac{1}{\Delta x_{ni+1}}\left(\frac{\sigma_{i+\frac{1}{2}}}{\Delta x_{ci+1}} + \frac{\sigma_{i-\frac{1}{2}}}{\Delta x_{ci}}\right) + \frac{1}{\Delta y_{nj+1}}\left(\frac{\sigma_{j+\frac{1}{2}}}{\Delta y_{cj+1}} + \frac{\sigma_{i-\frac{1}{2}}}{\Delta y_{cj}}\right) + \frac{1}{\Delta z_{nk+1}}\left(\frac{\sigma_{k+\frac{1}{2}}}{\Delta z_{ck+1}} + \frac{\sigma_{k-\frac{1}{2}}}{\Delta z_{ck}}\right)}$$

In MOONS, this is written as

$$u_{ijk} = \frac{u_{i+1,j,k}\left(\frac{\sigma_{i+\frac{1}{2}}}{\Delta x_{ci+1}\Delta x_{ni+1}}\right) + u_{i-1,j,k}\left(\frac{\sigma_{i-\frac{1}{2}}}{\Delta x_{ci}\Delta x_{ni+1}}\right) + u_{i,j+1,k}\left(\frac{\sigma_{j+\frac{1}{2}}}{\Delta y_{cj+1}\Delta y_{nj+1}}\right) + u_{i,j-1,k}\left(\frac{\sigma_{j-\frac{1}{2}}}{\Delta y_{cj}\Delta y_{nj+1}}\right) + u_{i,j,k+1}\left(\frac{\sigma_{k+\frac{1}{2}}}{\Delta z_{ck+1}\Delta z_{nk+1}}\right) + u_{i,j,k-1}\left(\frac{\sigma_{k-\frac{1}{2}}}{\Delta z_{ck}\Delta z_{nk+1}}\right) - f_{ijk}}{r}$$

Where

$$r = \frac{1}{\Delta x_{ni+1}}\left(\frac{\sigma_{i+\frac{1}{2}}}{\Delta x_{ci+1}} + \frac{\sigma_{i-\frac{1}{2}}}{\Delta x_{ci}}\right) + \frac{1}{\Delta y_{nj+1}}\left(\frac{\sigma_{j+\frac{1}{2}}}{\Delta y_{cj+1}} + \frac{\sigma_{i-\frac{1}{2}}}{\Delta y_{cj}}\right) + \frac{1}{\Delta z_{nk+1}}\left(\frac{\sigma_{k+\frac{1}{2}}}{\Delta z_{ck+1}} + \frac{\sigma_{k-\frac{1}{2}}}{\Delta z_{ck}}\right)$$

Non-uniform grid for node based data

Using the definition

$$\Delta h_{ci} = h_{ci} - h_{ci-1}$$

$$\Delta h_{ni} = h_{ni} - h_{ni-1}$$

For a non-uniform stencil, we have

$$\frac{\left(\frac{u_{i+1,j,k} - u_{i,j,k}}{\Delta x_{ni+1}}\right) - \left(\frac{u_{i,j,k} - u_{i-1,j,k}}{\Delta x_{ni}}\right)}{\Delta x_{ci+1}} + \cdots = f_{ijk}$$

Collecting terms

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Solving for u_{ijk} we have

$$u_{ijk} = \frac{u_{i+1,j,k}\left(\frac{1}{\Delta x_{ci+1}\Delta x_{ni+1}}\right) + u_{i-1,j,k}\left(\frac{1}{\Delta x_{ci}\Delta x_{ni+1}}\right) + u_{i,j+1,k}\left(\frac{1}{\Delta y_{cj+1}\Delta y_{nj+1}}\right) + u_{i,j-1,k}\left(\frac{1}{\Delta y_{cj}\Delta y_{nj+1}}\right) + u_{i,j,k+1}\left(\frac{1}{\Delta z_{ck+1}\Delta z_{nk+1}}\right) + u_{i,j,k-1}\left(\frac{1}{\Delta z_{ck}\Delta z_{nk+1}}\right) - f_{ijk}}{\left(\frac{1}{\Delta x_{ci+1}}\left(\frac{1}{\Delta x_{ni+1}} + \frac{1}{\Delta x_{ni}}\right) + \frac{1}{\Delta y_{cj+1}}\left(\frac{1}{\Delta y_{nj+1}} + \frac{1}{\Delta y_{nj}}\right) + \frac{1}{\Delta z_{ck+1}}\left(\frac{1}{\Delta z_{nk+1}} + \frac{1}{\Delta z_{nk}}\right)\right)}$$

SOR Parameter

The SOR Poisson solver was developed and the following optimal parameter was used:

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \left\{\frac{\left(\cos\left(\frac{\pi}{N_x}\right) + \cos\left(\frac{\pi}{N_x}\right) + \cos\left(\frac{\pi}{N_z}\right)\right)}{3}\right\}^2}}$$

References

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