Higher Order Interpolation Formulas

3rd order cell based extrapolation

Backward Extrapolation

Approximating several function values around f_i we have

$$p_i = f_i + (h_{ci} - h_{ni})f_i^{'} + \frac{(h_{ci} - h_{ni})^2}{2} f_i^{''} + O(\Delta h^3)$$

$$p_{i+1} = f_i + (h_{ci+1} - h_{ni})f_i^{'} + \frac{(h_{ci+1} - h_{ni})^2}{2} f_i^{''} + O(\Delta h^3)$$

$$f_{i+1} = f_i + (h_{ni+1} - h_{ni})f_i^{'} + \frac{(h_{ni+1} - h_{ni})^2}{2} f_i^{''} + O(\Delta h^3)$$

We may rewrite this as

$$p_i = f_i + \alpha_1 f_i^{'} + \frac{\alpha_1^2}{2} f_i^{''}$$

$$p_{i+1} = f_i + \alpha_2 f_i^{'} + \frac{\alpha_2^2}{2} f_i^{''}$$

$$f_{i+1} = f_i + \alpha_3 f_i^{'} + \frac{\alpha_3^2}{2} f_i^{''}$$

Or

$$-f_i = -p_i + \alpha_1 f_i^{'} + \frac{\alpha_1^2}{2} f_i^{''}$$

$$p_{i+1} - f_i = \alpha_2 f_i^{'} + \frac{\alpha_2^2}{2} f_i^{''}$$

$$f_{i+1} - f_i = \alpha_3 f_i^{'} + \frac{\alpha_3^2}{2} f_i^{''}$$

In matrix form we may write

$$Ax = b$$

Where

$$A = \begin{bmatrix} 1 & \alpha_1 & \frac{\alpha_1^2}{2} \\ 0 & \alpha_2 & \frac{\alpha_2^2}{2} \\ 0 & \alpha_3 & \frac{\alpha_3^2}{2} \end{bmatrix}, \quad x = \begin{bmatrix} -p_i \\ f_i{}' \\ f_i{}'' \end{bmatrix}, \quad b = \begin{bmatrix} -f_i \\ p_{i+1} - f_i \\ f_{i+1} - f_i \end{bmatrix}$$

The only solution we seek is for p_i which, by Cramer's rule is

$$p_i = -\frac{\begin{vmatrix} -f_i & \alpha_1 & \frac{\alpha_1^2}{2} \\ p_{i+1}-f_i & \alpha_2 & \frac{\alpha_2^2}{2} \\ f_{i+1}-f_i & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}}{\det(A)} = -\frac{\begin{vmatrix} -f_i & \alpha_1 & \frac{\alpha_1^2}{2} \\ p_{i+1}-f_i & \alpha_2 & \frac{\alpha_2^2}{2} \\ f_{i+1}-f_i & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}}{\begin{vmatrix} 1 & \alpha_1 & \frac{\alpha_1^2}{2} \\ 0 & \alpha_2 & \frac{\alpha_2^2}{2} \\ 0 & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}}$$

Let's tackle these separately

$$\det(A) = \left(\alpha_2 \frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right) - \alpha_1(0) + \frac{\alpha_1^2}{2}(0)$$

The numerator is

$$-\begin{vmatrix} -f_i & \alpha_1 & \frac{\alpha_1^2}{2} \\ p_{i+1}-f_i & \alpha_2 & \frac{\alpha_2^2}{2} \\ f_{i+1}-f_i & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}$$

$$= -\left\{-f_i\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right) - \alpha_1\left([p_{i+1}-f_i]\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}[f_{i+1}-f_i]\right) + \frac{\alpha_1^2}{2}([p_{i+1}-f_i]\alpha_3 - \alpha_2[f_{i+1}-f_i])\right\}$$

$$= f_i\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right) + \alpha_1\left([p_{i+1}-f_i]\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}[f_{i+1}-f_i]\right) - \frac{\alpha_1^2}{2}([p_{i+1}-f_i]\alpha_3 - \alpha_2[f_{i+1}-f_i])$$

Factoring terms we have

$$= f_i\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3 + \alpha_1\left(-\frac{\alpha_3^2}{2} + \frac{\alpha_2^2}{2}\right) - \frac{\alpha_1^2}{2}(-\alpha_3+\alpha_2)\right) + p_{i+1}\left(\alpha_1\frac{\alpha_3^2}{2} - \frac{\alpha_1^2}{2}\alpha_3\right) + f_{i+1}\left(\alpha_1\left(-\frac{\alpha_2^2}{2}\right) + \frac{\alpha_1^2}{2}\alpha_2\right)$$

Therefore, p_i is

$$p_i = \frac{f_i\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3 + \alpha_1\left(-\frac{\alpha_3^2}{2} + \frac{\alpha_2^2}{2}\right) - \frac{\alpha_1^2}{2}(-\alpha_3+\alpha_2)\right) + p_{i+1}\left(\alpha_1\frac{\alpha_3^2}{2} - \frac{\alpha_1^2}{2}\alpha_3\right) + f_{i+1}\left(\alpha_1\left(-\frac{\alpha_2^2}{2}\right) + \frac{\alpha_1^2}{2}\alpha_2\right)}{\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right)}$$

Or

$$p_i = f_i\frac{\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3 + \alpha_1\left(-\frac{\alpha_3^2}{2} + \frac{\alpha_2^2}{2}\right) - \frac{\alpha_1^2}{2}(-\alpha_3+\alpha_2)\right)}{\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right)} + p_{i+1}\frac{\left(\alpha_1\frac{\alpha_3^2}{2} - \frac{\alpha_1^2}{2}\alpha_3\right)}{\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right)} + f_{i+1}\frac{\left(\alpha_1\left(-\frac{\alpha_2^2}{2}\right) + \frac{\alpha_1^2}{2}\alpha_2\right)}{\left(\alpha_2\frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2}\alpha_3\right)}$$

Factoring out the 1/2 we have

$$p_i = f_i \frac{\left(\alpha_2\alpha_3^2 - \alpha_2^2\alpha_3 + \alpha_1(-\alpha_3^2 + \alpha_2^2) - \alpha_1^2(-\alpha_3 + \alpha_2)\right)}{(\alpha_2\alpha_3^2 - \alpha_2^2\alpha_3)} + p_{i+1}\frac{(\alpha_1\alpha_3^2 - \alpha_1^2\alpha_3)}{(\alpha_2\alpha_3^2 - \alpha_2^2\alpha_3)} + f_{i+1}\frac{(\alpha_1(-\alpha_2^2) + \alpha_1^2\alpha_2)}{(\alpha_2\alpha_3^2 - \alpha_2^2\alpha_3)}$$

Factoring some more

$$p_i = f_i \frac{(\alpha_2\alpha_3^2 - \alpha_2^2\alpha_3 - \alpha_3^2\alpha_1 + \alpha_2^2\alpha_1 + \alpha_3\alpha_1^2 - \alpha_2\alpha_1^2)}{\alpha_2\alpha_3(\alpha_3 - \alpha_2)} + p_{i+1}\frac{\alpha_1\alpha_3(\alpha_3 - \alpha_1)}{\alpha_2\alpha_3(\alpha_3 - \alpha_2)} + f_{i+1}\frac{\alpha_1\alpha_2(\alpha_1 - \alpha_2)}{\alpha_2\alpha_3(\alpha_3 - \alpha_2)}$$

Slightly simplifying we have

Result

$$p_i = f_i \frac{\alpha_2\alpha_3^2 - \alpha_2^2\alpha_3 - \alpha_3^2\alpha_1 + \alpha_2^2\alpha_1 + \alpha_3\alpha_1^2 - \alpha_2\alpha_1^2}{\alpha_2\alpha_3(\alpha_3 - \alpha_2)} + p_{i+1}\frac{\alpha_1(\alpha_3 - \alpha_1)}{\alpha_2(\alpha_3 - \alpha_2)} + f_{i+1}\frac{\alpha_1(\alpha_1 - \alpha_2)}{\alpha_3(\alpha_3 - \alpha_2)}$$

Or

$$p_i = f_i \frac{\left(\alpha_3 - \alpha_2 - \frac{\alpha_3\alpha_1}{\alpha_2} + \frac{\alpha_2\alpha_1}{\alpha_3} + \frac{\alpha_1^2}{\alpha_2} - \frac{\alpha_1^2}{\alpha_3}\right)}{(\alpha_3 - \alpha_2)} + p_{i+1}\frac{\alpha_1(\alpha_3 - \alpha_1)}{\alpha_2(\alpha_3 - \alpha_2)} + f_{i+1}\frac{\alpha_1(\alpha_1 - \alpha_2)}{\alpha_3(\alpha_3 - \alpha_2)}$$

Forward Extrapolation

This is very similar to the backward extrapolation, the only difference here is that we are now looking for a different function location and we approximate closer to the end point. Approximating several function values around f_i we have

$$p_{i+1} = f_{i+1} + (h_{ci+1} - h_{ni+1})f_{i+1}^{'} + \frac{(h_{ci+1} - h_{ni+1})^2}{2}f_{i+1}^{''} + O(\Delta h^3)$$

$$p_i = f_{i+1} + (h_{ci} - h_{n+1i})f_{i+1}^{'} + \frac{(h_{ci} - h_{n+1i})^2}{2}f_{i+1}^{''} + O(\Delta h^3)$$

$$f_i = f_{i+1} + (h_{ni} - h_{ni+1})f_{i+1}^{'} + \frac{(h_{ni} - h_{ni+1})^2}{2}f_{i+1}^{''} + O(\Delta h^3)$$

We may rewrite this as

$$p_{i+1} = f_{i+1} + \alpha_1 f_{i+1}^{'} + \frac{\alpha_1^2}{2}f_{i+1}^{''}$$

$$p_i = f_{i+1} + \alpha_2 f_{i+1}^{'} + \frac{\alpha_2^2}{2}f_{i+1}^{''}$$

$$f_i = f_{i+1} + \alpha_3 f_{i+1}^{'} + \frac{\alpha_3^2}{2}f_{i+1}^{''}$$

Or

$$-f_{i+1} = -p_{i+1} + \alpha_1 f_{i+1}^{'} + \frac{\alpha_1^2}{2}f_{i+1}^{''}$$

$$p_i - f_{i+1} = \alpha_2 f_{i+1}^{'} + \frac{\alpha_2^2}{2} f_{i+1}^{''}$$

$$f_i - f_{i+1} = \alpha_3 f_{i+1}^{'} + \frac{\alpha_3^2}{2} f_{i+1}^{''}$$

In matrix form we may write

$$Ax = b$$

Where

$$A = \begin{bmatrix} 1 & \alpha_1 & \frac{\alpha_1^2}{2} \\ 0 & \alpha_2 & \frac{\alpha_2^2}{2} \\ 0 & \alpha_3 & \frac{\alpha_3^2}{2} \end{bmatrix}, \qquad x = \begin{bmatrix} -p_{i+1} \\ f_{i+1}^{'} \\ f_{i+1}^{''} \end{bmatrix}, \quad b = \begin{bmatrix} -f_{i+1} \\ p_i - f_{i+1} \\ f_i - f_{i+1} \end{bmatrix}$$

The only solution we seek is for p_{i+1} which, by Cramer's rule is

$$p_{i+1} = -\frac{\begin{vmatrix} -f_{i+1} & \alpha_1 & \frac{\alpha_1^2}{2} \\ p_i - f_{i+1} & \alpha_2 & \frac{\alpha_2^2}{2} \\ f_i - f_{i+1} & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}}{\det(A)} = -\frac{\begin{vmatrix} -f_{i+1} & \alpha_1 & \frac{\alpha_1^2}{2} \\ p_i - f_{i+1} & \alpha_2 & \frac{\alpha_2^2}{2} \\ f_i - f_{i+1} & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}}{\begin{bmatrix} 1 & \alpha_1 & \frac{\alpha_1^2}{2} \\ 0 & \alpha_2 & \frac{\alpha_2^2}{2} \\ 0 & \alpha_3 & \frac{\alpha_3^2}{2} \end{bmatrix}},$$

Let's tackle these separately. The A matrix is the same, so we have

$$\det(A) = \left(\alpha_2 \frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2} \alpha_3\right) - \alpha_1 (0) + \frac{\alpha_1^2}{2} (0)$$

The numerator is

$$-\begin{vmatrix} -f_{i+1} & \alpha_1 & \frac{\alpha_1^2}{2} \\ p_i - f_{i+1} & \alpha_2 & \frac{\alpha_2^2}{2} \\ f_i - f_{i+1} & \alpha_3 & \frac{\alpha_3^2}{2} \end{vmatrix}$$

$$= -\left\{(-f_{i+1}) \left(\alpha_2 \frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2} \alpha_3\right) - (\alpha_1) \left((p_i - f_{i+1}) \frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2} (f_i - f_{i+1})\right) + \left(\frac{\alpha_1^2}{2}\right) ((p_i - f_{i+1})\alpha_3 - \alpha_2 (f_i - f_{i+1}))\right\}$$

$$= f_{i+1} \left(\alpha_2 \frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2} \alpha_3\right) + \alpha_1 \left((p_i - f_{i+1}) \frac{\alpha_3^2}{2} - \frac{\alpha_2^2}{2} (f_i - f_{i+1})\right) + \frac{\alpha_1^2}{2} (\alpha_2 (f_i - f_{i+1}) - (p_i - f_{i+1})\alpha_3)$$

Factoring terms we have

$$= f_{i+1}\left(\alpha_2\frac{\alpha_3^2}{2}-\frac{\alpha_2^2}{2}\alpha_3-\alpha_1\frac{\alpha_3^2}{2}+\alpha_1\frac{\alpha_2^2}{2}-\frac{\alpha_1^2}{2}\alpha_2+\frac{\alpha_1^2}{2}\alpha_3\right)+p_i\left(\alpha_1\frac{\alpha_3^2}{2}-\frac{\alpha_1^2}{2}\alpha_3\right)+f_i\left(-\alpha_1\frac{\alpha_2^2}{2}+\frac{\alpha_1^2}{2}\alpha_2\right)$$

Therefore, p_{i+1} is

$$p_{i+1}=\frac{f_{i+1}\left(\alpha_2\frac{\alpha_3^2}{2}-\frac{\alpha_2^2}{2}\alpha_3-\alpha_1\frac{\alpha_3^2}{2}+\alpha_1\frac{\alpha_2^2}{2}-\frac{\alpha_1^2}{2}\alpha_2+\frac{\alpha_1^2}{2}\alpha_3\right)+p_i\left(\alpha_1\frac{\alpha_3^2}{2}-\frac{\alpha_1^2}{2}\alpha_3\right)+f_i\left(-\alpha_1\frac{\alpha_2^2}{2}+\frac{\alpha_1^2}{2}\alpha_2\right)}{\alpha_2\frac{\alpha_3^2}{2}-\frac{\alpha_2^2}{2}\alpha_3}$$

Dividing by 2 we have

$$p_{i+1}=\frac{f_{i+1}(\alpha_2\alpha_3^2-\alpha_2^2\alpha_3-\alpha_1\alpha_3^2+\alpha_1\alpha_2^2-\alpha_1^2\alpha_2+\alpha_1^2\alpha_3)+p_i(\alpha_1\alpha_3^2-\alpha_1^2\alpha_3)+f_i(-\alpha_1\alpha_2^2+\alpha_1^2\alpha_2)}{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3}$$

Splitting, and slightly rearranging this, we have

$$p_{i+1}=f_{i+1}\frac{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3-\alpha_1\alpha_3^2+\alpha_1\alpha_2^2-\alpha_1^2\alpha_2+\alpha_1^2\alpha_3}{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3}+p_i\frac{\alpha_1\alpha_3^2-\alpha_1^2\alpha_3}{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3}+f_i\frac{\alpha_1^2\alpha_2-\alpha_1\alpha_2^2}{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3}$$

Factoring more

$$p_{i+1}=f_{i+1}\frac{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3-\alpha_1\alpha_3^2+\alpha_1\alpha_2^2-\alpha_1^2\alpha_2+\alpha_1^2\alpha_3}{\alpha_2\alpha_3(\alpha_3-\alpha_2)}+p_i\frac{\alpha_1\alpha_3(\alpha_3-\alpha_1)}{\alpha_2\alpha_3(\alpha_3-\alpha_2)}+f_i\frac{\alpha_1\alpha_2(\alpha_1-\alpha_2)}{\alpha_2\alpha_3(\alpha_3-\alpha_2)}$$

Simplifying

Result

$$p_{i+1}=f_{i+1}\frac{\alpha_2\alpha_3^2-\alpha_2^2\alpha_3-\alpha_1\alpha_3^2+\alpha_1\alpha_2^2-\alpha_1^2\alpha_2+\alpha_1^2\alpha_3}{\alpha_2\alpha_3(\alpha_3-\alpha_2)}+p_i\frac{\alpha_1(\alpha_3-\alpha_1)}{\alpha_2(\alpha_3-\alpha_2)}+f_i\frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha_3(\alpha_3-\alpha_2)}$$

Or

$$p_{i+1}=f_{i+1}\frac{\alpha_3-\alpha_2-\frac{\alpha_1\alpha_3}{\alpha_2}+\frac{\alpha_1\alpha_2}{\alpha_3}-\frac{\alpha_1^2}{\alpha_3}+\frac{\alpha_1^2}{\alpha_2}}{(\alpha_3-\alpha_2)}+p_i\frac{\alpha_1(\alpha_3-\alpha_1)}{\alpha_2(\alpha_3-\alpha_2)}+f_i\frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha_3(\alpha_3-\alpha_2)}$$