Derivation of NS with $j \times B$

The dimensional NS

$$\rho\left\{\frac{d\boldsymbol{u}}{dt}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right\}=-\nabla p+\rho\mu\,\nabla^2\boldsymbol{u}+\boldsymbol{f}+\boldsymbol{j}\times\boldsymbol{B}$$

Where $\boldsymbol{f}$ is some ***dimensional*** force per unit volume.

non-dimensionalizing by

$$\boldsymbol{u}^*=\boldsymbol{u}/U_c \qquad \boldsymbol{B}^*=\boldsymbol{B}/B_c$$

$$t^*=\frac{t}{t_c}\quad , \qquad t_c=\frac{L_c}{U_c} \qquad \nabla^*=L_c\nabla \quad , \quad \rho=\rho_c$$

$$p^*=p/\rho_c U_c^2$$

$$\boldsymbol{j}^*=\frac{\boldsymbol{j}}{\sigma_c U_c B_c}$$

Substituting these expressions in we have

$$\rho_c\left\{\frac{U_c^2}{L_c}\frac{d\boldsymbol{u}^*}{dt}+\frac{U_c^2}{L_c}\boldsymbol{u}^*\cdot\nabla^*\boldsymbol{u}^*\right\}=-\frac{U_c^2}{L_c}\nabla^*p^*\rho_c+\mu\frac{U_c}{L_c^2}\,\nabla^{*\boldsymbol{2}}\boldsymbol{u}^*+\boldsymbol{f}+\sigma_c U_c B_c B_c\,\boldsymbol{j}^*\times\boldsymbol{B}^*$$

Or

$$\frac{U_c^2}{L_c}\frac{d\boldsymbol{u}^*}{dt}+\frac{U_c^2}{L_c}\boldsymbol{u}^*\cdot\nabla^*\boldsymbol{u}^*=-\frac{U_c^2}{L_c}\nabla^*p^*+\nu\frac{U_c}{L_c^2}\,\nabla^{*\boldsymbol{2}}\boldsymbol{u}^*+\frac{\boldsymbol{f}}{\rho_c}+\frac{1}{\rho_c}\sigma_c U_c B_c^2\,\boldsymbol{j}^*\times\boldsymbol{B}^*$$

Making this equation dimensionless yields

$$\frac{d\boldsymbol{u}^*}{dt}+\boldsymbol{u}^*\cdot\nabla^*\boldsymbol{u}^*=-\nabla^*p^*+\frac{\nu}{\underbrace{L_c U_c}_{1/Re}}\,\nabla^{*\boldsymbol{2}}\boldsymbol{u}^*+\frac{L_c}{\rho_c U_c^2}\boldsymbol{f}+\frac{1}{\rho_c U_c}\sigma_c L_c B_c^2\,\boldsymbol{j}^*\times\boldsymbol{B}^*$$

Collecting terms on the $j \times B$ force yields

$$\frac{d\boldsymbol{u}^*}{dt} + \boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* = -\nabla^* p^* + \frac{1}{Re_m} \nabla^{*\mathbf{2}} \boldsymbol{u}^* + \frac{L_c}{\rho_c U_c^2} \boldsymbol{f} + \underbrace{\frac{\nu_c}{L_c U_c}}_{1/Re} \underbrace{L_c^2 B_c^2 \frac{\sigma_c}{\rho_c \nu_c}}_{Ha^2} \boldsymbol{j}^* \times \boldsymbol{B}^*$$

Therefore we have

$$\frac{d\boldsymbol{u}^*}{dt} + \boldsymbol{u}^* \cdot \nabla^* \boldsymbol{u}^* = -\nabla^* p^* + \frac{1}{Re} \nabla^{*\mathbf{2}} \boldsymbol{u}^* + \frac{L_c}{\rho_c U_c^2} \boldsymbol{f} + \frac{Ha^2}{Re} \boldsymbol{j}^* \times \boldsymbol{B}^*$$

Where the subscript variables are the prescribed "characteristic" values. These values are non-changing without loss of generality. Substituting and removing the asterisks yields

$$\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{f} + N \boldsymbol{j} \times \boldsymbol{B}$$

Where

$$Re = \frac{U_c L_c}{\nu}$$

$$N = \frac{\sigma_c L_c B_c^2}{\rho_c U_c} = \frac{Ha^2}{Re}$$

$$Ha = B_c L_c \sqrt{\frac{\sigma_l}{\rho_c \nu}} \qquad L_c = L_{||}$$

We may write the $\boldsymbol{j} \times \boldsymbol{B}$ term as

$$\boldsymbol{j} \times \boldsymbol{B} = \epsilon_{ijk} j_j B_k = \epsilon_{ijk} \left(\epsilon_{jmn} \partial_m \frac{B_n}{\mu} \right) B_k$$

$$= \epsilon_{ijk} \epsilon_{jmn} \left(\partial_m \frac{B_n}{\mu} \right) B_k = \epsilon_{jki} \epsilon_{jmn} \left(\partial_m \frac{B_n}{\mu} \right) B_k$$

$$= (\delta_{km} \delta_{in} - \delta_{kn} \delta_{im}) \left(\partial_m \frac{B_n}{\mu} \right) B_k$$

$$= \left(\partial_k \frac{B_i}{\mu}\right) B_k - \left(\partial_i \frac{B_k}{\mu}\right) B_k$$

$$\boxed{\boldsymbol{j} \times \boldsymbol{B} = B_k \left(\partial_k \frac{B_i}{\mu} - \partial_i \frac{B_k}{\mu}\right)}$$

Writing this in expanded form, we have

$$(j \times B)_i = B_x \left(\partial_x \frac{B_i}{\mu} - \partial_i \frac{B_x}{\mu}\right) + B_y \left(\partial_y \frac{B_i}{\mu} - \partial_i \frac{B_y}{\mu}\right) + B_z \left(\partial_z \frac{B_i}{\mu} - \partial_i \frac{B_z}{\mu}\right)$$

Note that the diagonal is zero.

Conservative form

Attempting to put this into a conservative form, we have

$$B_k \left(\partial_k \frac{B_i}{\mu} - \partial_i \frac{B_k}{\mu}\right) = B_k \partial_k \frac{B_i}{\mu} - B_k \partial_i \frac{B_k}{\mu}$$

Note that we may use the product rule to write

$$\partial_k \left(\frac{B_i}{\mu} B_k\right) = B_k \partial_k \frac{B_i}{\mu} + \frac{B_i}{\mu} \partial_k B_k$$

And

$$\partial_i \left(B_k \frac{B_k}{\mu}\right) = B_k \partial_i \frac{B_k}{\mu} + \frac{B_k}{\mu} \partial_i B_k$$