Conservative Derivatives

The conservative 1D Laplacian

MOONS computes

$$\frac{\partial}{\partial h}\left(k\frac{\partial}{\partial h}u\right)$$

on a staggered grid for both cell corner and cell centered data. Let subscript p and d represent the primary and dual grid of the staggered grid respectively. If the data lives on the primary grid, then the result also lives on the primary grid, and the coefficient, k, lives on the dual grid. This derivative is computed as follows

$$\frac{\partial}{\partial h}\left(k\frac{\partial}{\partial h}u\right) = \frac{\frac{u_{i+1}-u_i}{h_{p,i+1}-h_{p,i}}k_{p,i+\frac{1}{2}} - \frac{u_i-u_{i-1}}{h_{p,i}-h_{p,i-1}}k_{p,i-\frac{1}{2}}}{h_{p,i+\frac{1}{2}}-h_{p,i-\frac{1}{2}}}$$

This form of the derivative is explicitly clear and consistent (since all references are to the primary grid). First, let's be clear about the current index situation in MOONS:

First note that $\frac{\partial}{\partial h}\left(k\frac{\partial}{\partial h}u\right)$ lives on integers of i for BOTH CC and N based data.

Cell center (C) data

u1 u2 ... ui usc

o---|---o---|---o---|---o---| ---o---| ---o---| ---o---|---o

Let $i+\frac{1}{2}$ of the primary grid be i of the dual grid (this can easily be seen by replacing with 1). NOTE the ghost cell.

This means that if $\boldsymbol{i=2}$ then $i-1=2-1=1$ refers to the first index of $\Delta h_{d,i-1}$ which is $\Delta h_{d,1}$ (what we want). This verifies the index for the CC data case.

Cell corner, or node (N), data

u1 u2 ... ui ... usn

o---|---o---|---o---|---o---| ---o---| ---o---| ---o---|---o

Let $i+\frac{1}{2}$ of the primary grid be $i+1$ of the dual grid (this can easily be seen by replacing with 1).

This means that if $\boldsymbol{i=2}$ then $i-1+1=2$ refers to the *second* index of $\Delta h_{d,i-1+1}$ which is $\Delta h_{d,2}$ (what we want). This verifies the index for the N data case.

General form without half indexes

Now, we can write a more general equation

$$\frac{\partial}{\partial h}\left(k\frac{\partial}{\partial h}u\right) = \frac{\frac{u_{i+1}-u_i}{h_{p,i+1}-h_{p,i}}k_{d,i+gt} - \frac{u_i-u_{i-1}}{h_{p,i}-h_{p,i-1}}k_{d,i-1+gt}}{h_{d,i+gt}-h_{d,i-1+gt}}$$

Where

$$gt = gridType = \begin{cases} 0 & \text{if} \quad u \in cell\ center \\ 1 & \text{if} \quad u \in cell\ corners \end{cases}$$

Now we have successfully removed the half indexes, and this form is easily programmable. Note that k has also adopted this index convention as well. Furthermore note that, from the second equation, it is clear that $k_{d,i} = \frac{k_{p,i+1}+k_{p,i}}{2}$. Now, since

$$\Delta h_{c,1} = h_{c,2} - h_{c,1}$$

$$\Delta h_{n,1} = h_{n,2} - h_{n,1}$$

let $\Delta h_i = h_{i+1} - h_i$ for both grids, and we have

$$\frac{\partial}{\partial h}\left(k\frac{\partial}{\partial h}u\right) = \frac{\frac{u_{i+1}-u_i}{\Delta h_{p,i}}k_{d,i+gt} - \frac{u_i - u_{i-1}}{\Delta h_{p,i-1}}k_{d,i-1+gt}}{\Delta h_{d,i-1+gt}}$$

$$= \frac{u_{i+1}-u_i}{\Delta h_{p,i}\Delta h_{d,i-1+gt}}k_{d,i+gt} - \frac{u_i - u_{i-1}}{\Delta h_{p,i-1}\Delta h_{d,i-1+gt}}k_{d,i-1+gt}$$

$$\boxed{\frac{\partial}{\partial h}\left(k\frac{\partial}{\partial h}u\right) = \left(\frac{k_{d,i-1+gt}}{\Delta h_{p,i-1}\Delta h_{d,i-1+gt}}\right)u_{i-1} - \left(\frac{k_{d,i+gt}}{\Delta h_{p,i-1}\Delta h_{d,i-1+gt}} + \frac{k_{d,i-1+gt}}{\Delta h_{p,i}\Delta h_{d,i-1+gt}}\right)u_i + \left(\frac{k_{d,i+gt}}{\Delta h_{p,i}\Delta h_{d,i-1+gt}}\right)u_{i+1}}$$

Computing k

The coefficient must be linearly interpolated when $u \in N$, otherwise an ordinary average suffices. The formulation of $k_{d,i}$ may be determined from

$$\frac{k_{p,i+1} - k_{p,i}}{h_{p,i+1} - h_{p,i}} = \frac{k_{p,i+1} - k_{p,i+\frac{1}{2}}}{h_{p,i+1} - h_{p,i+\frac{1}{2}}} = \frac{k_{p,i+1} - k_{d,i+gt}}{h_{p,i+1} - h_{d,i+gt}}$$

Therefore we may compute $k_{p,i+\frac{1}{2}} = k_{d,i+gt}$ to be

$$\boxed{k_{p,i+\frac{1}{2}} = k_{d,i+gt} = k_{p,i+1} - \frac{h_{p,i+1} - h_{d,i+gt}}{\Delta h_{p,i}}\left(k_{p,i+1} - k_{p,i}\right)}$$

Note that if $u \in N$, then we have $gt = 1$ and $\Delta h_{p,i} = \Delta h_n = 2\left(h_{p,i+1} - h_{d,i+1}\right)$ and so

$$k_{p,i+\frac{1}{2}} = k_{d,i+1} = \frac{\Delta h_{p,i}/2}{\Delta h_{p,i}}\left(k_{p,i+1} - k_{p,i}\right) = \frac{1}{2}\left(k_{p,i+1} - k_{p,i}\right)$$

This is a general form of the conservative 1D Laplacian stencil for BOTH cell centered and cell corner data with all half indexes removed.