Steps Towards Multi-Material Domain

1) Low Re_m, Poisson (Gauss-Seidel)

$$\frac{\partial^2 B_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(u_j B_i^0 - u_i B_j^0\right)$$

2) Low Re_m, Pseudo time step (uniform σ, assume $\nabla \cdot B = 0$)

$$\frac{\partial B_i}{\partial s} = -\frac{\partial}{\partial x_j}\left(u_j B_i^0 - u_i B_j^0\right) - \frac{\partial^2 B_i}{\partial x_j^2}$$

3) Low Re_m, Pseudo time step (uniform σ, do NOT assume $\nabla \cdot B = 0$)

$$\frac{\partial B_i}{\partial s} = -\frac{\partial}{\partial x_j}\left(u_j B_i^0 - u_i B_j^0\right) - \left\{\frac{\partial^2 B_j}{\partial x_j x_i} - \frac{\partial^2 B_i}{\partial x_j x_j}\right\}$$

4) Low Re_m, Pseudo time step (non-uniform σ, do NOT assume $\nabla \cdot B = 0$)

$$\frac{\partial B_i}{\partial s} = -\frac{\partial}{\partial x_j}\left(u_j B_i^0 - u_i B_j^0\right) - \left\{\frac{\partial}{\partial x_j}\left(\frac{1}{\sigma}\frac{\partial B_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(\frac{1}{\sigma}\frac{\partial B_i}{\partial x_j}\right)\right\}$$

First two steps

Conclusion: div(B) behaves differently for these two methods

Second two steps

We must write an operator to compute

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\sigma}\frac{\partial B_j}{\partial x_i}\right) \quad \text{and} \quad \frac{\partial}{\partial x_j}\left(\frac{1}{\sigma}\frac{\partial B_i}{\partial x_j}\right)$$

Or, in general

$$\left\{\frac{\partial}{\partial x_j}\left(\frac{1}{\sigma_{cc}}\frac{\partial B_{k,cc}}{\partial x_i}\right)\right\}_{cc}$$

Notation

$CD2(\phi, x_j, 1) = \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x_j}$: 1st derivative, 2nd order central difference along x_j

$CD2(\phi, x_j, 2) = \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x_j^2}$: 2nd derivative, 2nd order central difference along x_j

$OD1(\phi, x_j) = \frac{\phi_{j+1} - \phi_j}{\Delta x_j}$ 1st order one-sided difference along x_j **

$$OD1(OD1(\phi, x_j), x_j) = CD2(\phi, x_j, 2)$$

Operator 1

$$\left\{\frac{\partial}{\partial x_j}\frac{1}{\sigma_{cc}}\frac{\partial B_{k,cc}}{\partial x_i}\right\}_{cc} = \left\{CD2\left(\frac{1}{\sigma_{cc}}\left(\left\{CD2\left(B_{k,cc}, x_i\right)\right\}_{cc}\right), x_j\right)\right\}_{cc}$$

$$= \frac{\frac{1}{\sigma_{cc,j+1}}\left\{\frac{B_{k,cc,i+1} - B_{k,cc,i-1}}{2\Delta x_i}\right\}_{j+1} - \frac{1}{\sigma_{cc,j-1}}\left\{\frac{B_{k,cc,i+1} - B_{k,cc,i-1}}{2\Delta x_i}\right\}_{j-1}}{2\Delta x_j}$$

If $i \neq j$, this provides a 2nd order accurate stencil for the mixed derivatives. But if σ is uniform and $i = j$, this is inconsistent with the 2nd derivative, 2nd order accurate operator:

$$\frac{1}{\sigma_{cc}}\left\{OD1\left(\left\{OD1\left(B_{k,cc}, x_j\right)\right\}_{face}, x_j\right)\right\}_{cc} \neq \frac{1}{\sigma_{cc}}\left\{CD2\left(\left\{CD2\left(B_{k,cc}, x_j\right)\right\}_{cc}, x_j\right)\right\}_{cc}$$

This resulted in oscillations in the magnetic field

Operator 2

$$\left\{\frac{\partial}{\partial x_j}\frac{1}{\sigma_{cc}}\frac{\partial B_{k,cc}}{\partial x_i}\right\}_{cc} = \begin{cases} \left\{CD2\left(\frac{1}{\sigma_{cc}}\left(\left\{CD2\left(B_{k,cc}, x_i\right)\right\}_{cc}\right), x_j\right)\right\}_{cc} & \text{, if} \quad i \neq j \\ \left\{OD1\left(\frac{1}{\sigma_{face}}\left(\left\{OD1\left(B_{k,cc}, x_i\right)\right\}_{face}\right), x_j\right)\right\}_{cc} & \text{, if} \quad i = j \end{cases}$$

But this results in a significantly different magnetic field distribution compared with the Poisson result. Also, I don't like the fact that the operator is directional. I suspect that this will certainly result in divergence errors.