2D and 3D Douglas ADI

2D Douglas ADI

Consider the equation

$$\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} - f(x)$$

Splitting this we have

$$\left(I - \frac{\Delta t}{2} \alpha_x \Delta_x\right) u^* = \left(I + \frac{\Delta t}{2} \alpha_x \Delta_x + \Delta t \alpha_y \Delta_y\right) u_k - \Delta t F$$

$$\left(I - \frac{\Delta t}{2} \alpha_y \Delta_y\right) u_{k+1} = u^* - \frac{\Delta t}{2} \alpha_y \Delta_y u_k$$

In Douglas' paper, this was written as (in equation 2.5)

$$\frac{1}{2} \Delta_x^2 (w_{n+1}^* + w_n) + \Delta_y^2 w_n = \frac{w_{n+1}^* - w_n}{\Delta t}$$

$$\frac{1}{2} \Delta_x^2 (w_{n+1}^* + w_n) + \frac{1}{2} \Delta_y^2 (w_{n+1} + w_n) = \frac{w_{n+1} - w_n}{\Delta t}$$

Multiplying by Δt, we have

$$\frac{1}{2} \Delta t \Delta_x^2 (w_{n+1}^* + w_n) + \Delta t \Delta_y^2 w_n = w_{n+1}^* - w_n$$

$$\frac{1}{2} \Delta t \Delta_x^2 (w_{n+1}^* + w_n) + \frac{1}{2} \Delta t \Delta_y^2 (w_{n+1} + w_n) = w_{n+1} - w_n$$

Attempting to solve for w_{n+1}^* and w_{n+1}, we have

$$\left(I - \frac{1}{2} \Delta t \Delta_x^2\right) w_{n+1}^* = \frac{1}{2} \Delta t \Delta_x^2 w_n + \Delta t \Delta_y^2 w_n + w_n$$

$$\left(I - \frac{1}{2} \Delta t \Delta_y^2\right) w_{n+1} = \frac{1}{2} \Delta t \Delta_x^2 (w_{n+1}^* + w_n) + \frac{1}{2} \Delta t \Delta_y^2 w_n + w_n$$

Or, finally

$$\left(I - \frac{1}{2} \Delta t \Delta_x^2\right) w_{n+1}^* = \left(I + \frac{1}{2} \Delta t \Delta_x^2 + \Delta t \Delta_y^2\right) w_n$$

$$\left(I - \frac{1}{2} \Delta t \Delta_y^2\right) w_{n+1} = \frac{1}{2} \Delta t \Delta_x^2 w_{n+1}^* + \left(I + \frac{1}{2} \Delta t \Delta_x^2 + \frac{1}{2} \Delta t \Delta_y^2\right) w_n$$

Combining

$$\left(I - \frac{1}{2} \Delta t \Delta_x^2\right) w_{n+1}^* = \left(I + \frac{1}{2} \Delta t \Delta_x^2 + \Delta t \Delta_y^2\right) w_n$$

From above to the second equation, we have

$$\left(I-\frac{1}{2}\Delta t\Delta_x^2\right)w_{n+1}^* = \left(I+\frac{1}{2}\Delta t\Delta_x^2+\Delta t\Delta_y^2\right)w_n$$

$$\left(I-\frac{1}{2}\Delta t\Delta_y^2\right)w_{n+1} = \frac{1}{2}\Delta t\Delta_x^2 w_{n+1}^* + \left(I-\frac{1}{2}\Delta t\Delta_x^2\right)w_{n+1}^* - \left(I+\frac{1}{2}\Delta t\Delta_x^2+\Delta t\Delta_y^2\right)w_n + \left(I+\frac{1}{2}\Delta t\Delta_x^2+\frac{1}{2}\Delta t\Delta_y^2\right)w_n$$

We finally have

$$\left(I-\frac{1}{2}\Delta t\Delta_x^2\right)w_{n+1}^* = \left(I+\frac{1}{2}\Delta t\Delta_x^2+\Delta t\Delta_y^2\right)w_n$$

$$\left(I-\frac{1}{2}\Delta t\Delta_y^2\right)w_{n+1} = w_{n+1}^* - \frac{1}{2}\Delta t\Delta_y^2 w_n$$

This is exactly the same as the form we used in class.

This is written in the last equation (2.7) as

$$\left(I-\frac{\Delta t}{2}\Delta_y^2\right)w_{n+1} = w_{n+1}^* - \frac{\Delta t}{2}\Delta_y^2 w_n$$

3D Douglas ADI

Consider the equation

$$\frac{\partial u}{\partial t} = \alpha_x\frac{\partial^2 u}{\partial x^2} + \alpha_y\frac{\partial^2 u}{\partial y^2} + \alpha_z\frac{\partial^2 u}{\partial z^2} - f(x)$$

Splitting this operator into several steps, we have

$$\frac{1}{2}\alpha_x\Delta_x(u_{n+1}^*+u_n) + \alpha_y\Delta_y u + \alpha_z\Delta_z u - f = \frac{u_{n+1}^*-u_n}{\Delta t}$$

$$\frac{1}{2}\alpha_x\Delta_x(u_{n+1}^*+u_n) + \frac{1}{2}\alpha_y\Delta_y(u_{n+1}^{**}+u_n) + \alpha_z\Delta_z u - f = \frac{u_{n+1}^{**}-u_n}{\Delta t}$$

$$\frac{1}{2}\alpha_x\Delta_x(u_{n+1}^*+u_n) + \frac{1}{2}\alpha_y\Delta_y(u_{n+1}^{**}+u_n) + \frac{1}{2}\alpha_z\Delta_z(u_{n+1}+u_n) - f = \frac{u_{n+1}-u_n}{\Delta t}$$

After rearranging, we have

$$\left(I-\frac{\Delta t}{2}\alpha_x\Delta_x\right)u_{n+1}^* = \left(I+\frac{\Delta t}{2}\alpha_x\Delta_x+\Delta t\alpha_y\Delta_y+\Delta t\alpha_z\Delta_z\right)u_n - \Delta t f$$

$$\left(I-\frac{\Delta t}{2}\alpha_y\Delta_y\right)u_{n+1}^{**} = u_{n+1}^* - \frac{\Delta t}{2}\alpha_y\Delta_y u_n$$

$$\left(I-\frac{\Delta t}{2}\alpha_z\Delta_z\right)u_{n+1} = u_{n+1}^{**} - \frac{\Delta t}{2}\alpha_z\Delta_z u_n$$

This is the form of the 3D ADI.

3D Douglas ADI - for analysis

Without loss of generality, we may replace $\alpha_j \Delta_j$ with simply Δ_j. For analysis, it is useful to write the 3D ADI equation in the form where u^{*}_{n+1} and u^{**}_{n+1} are eliminated. The result we wish to reach is that of eq 4.1 in the Douglas paper. Combining the three above equations, we have

$$\left(I-\frac{\Delta t}{2}\Delta_x\right)\left\{\left(I-\frac{\Delta t}{2}\Delta_y\right)\left[\left(I-\frac{\Delta t}{2}\Delta_z\right)u_{n+1}+\frac{\Delta t}{2}\Delta_z u_n\right]+\frac{\Delta t}{2}\Delta_y u_n\right\}=\left(I+\frac{\Delta t}{2}\Delta_x+\Delta t\Delta_y+\Delta t\Delta_z\right)u_n-\Delta t f$$

Distributing

$$\left(I-\frac{\Delta t}{2}\Delta_x\right)\left\{\left(I-\frac{\Delta t}{2}\Delta_y\right)\left(I-\frac{\Delta t}{2}\Delta_z\right)u_{n+1}+\left(I-\frac{\Delta t}{2}\Delta_y\right)\frac{\Delta t}{2}\Delta_z u_n+\frac{\Delta t}{2}\Delta_y u_n\right\}=\left(I+\frac{\Delta t}{2}\Delta_x+\Delta t\Delta_y+\Delta t\Delta_z\right)u_n-\Delta t f$$

again

$$\left(I-\frac{\Delta t}{2}\Delta_x\right)\left(I-\frac{\Delta t}{2}\Delta_y\right)\left(I-\frac{\Delta t}{2}\Delta_z\right)u_{n+1}+\left(I-\frac{\Delta t}{2}\Delta_x\right)\left(I-\frac{\Delta t}{2}\Delta_y\right)\frac{\Delta t}{2}\Delta_z u_n+\left(I-\frac{\Delta t}{2}\Delta_x\right)\frac{\Delta t}{2}\Delta_y u_n$$
$$=\left(I+\frac{\Delta t}{2}\Delta_x+\Delta t\Delta_y+\Delta t\Delta_z\right)u_n-\Delta t f$$

Multiplying the first term out (using dummy variables), we have

$$(1-A)(1-B)(1-C)=(1-C)(1-A-B+AB)=1-A-B+AB-C+AC+BC-ABC$$
$$=1-(A+B+C)+AB+AC+BC-ABC$$

This means we have

$$\left(I-\frac{\Delta t}{2}\Delta_x\right)\left(I-\frac{\Delta t}{2}\Delta_y\right)\left(I-\frac{\Delta t}{2}\Delta_z\right)u_{n+1}=u_{n+1}-\frac{\Delta t}{2}(\Delta_x+\Delta_y+\Delta_z)u_{n+1}+\left(\frac{\Delta t}{2}\right)^2(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}$$
$$-\left(\frac{\Delta t}{2}\right)^3\Delta_x\Delta_y\Delta_z u_{n+1}$$

Next, the second term

$$(1-A)(1-B)=1-A-B+AB=1-(A+B)+AB$$

which yields

$$\left(I-\frac{\Delta t}{2}\Delta_x\right)\left(I-\frac{\Delta t}{2}\Delta_y\right)\frac{\Delta t}{2}\Delta_z u_n=\frac{\Delta t}{2}\Delta_z u_n-\left(\frac{\Delta t}{2}\right)^2(\Delta_x+\Delta_y)\Delta_z u_n+\left(\frac{\Delta t}{2}\right)^3\Delta_x\Delta_y\Delta_z u_n$$

Next, the third term we have

$$\left(I-\frac{\Delta t}{2}\Delta_x\right)\frac{\Delta t}{2}\Delta_y u_n=\frac{\Delta t}{2}\Delta_y u_n-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y u_n$$

The first term on the RHS, we can write as

$$\left(I+\frac{\Delta t}{2}\Delta_x+\Delta t\Delta_y+\Delta t\Delta_z\right)u_n=u_n+\frac{\Delta t}{2}(\Delta_x+2\Delta_y+2\Delta_z)u_n$$

Putting this all together, we have

$$u_{n+1}-\frac{\Delta t}{2}(\Delta_x+\Delta_y+\Delta_z)u_{n+1}+\left(\frac{\Delta t}{2}\right)^2(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}-\left(\frac{\Delta t}{2}\right)^3\Delta_x\Delta_y\Delta_z u_{n+1}+\frac{\Delta t}{2}\Delta_z u_n$$
$$-\left(\frac{\Delta t}{2}\right)^2(\Delta_x+\Delta_y)\Delta_z u_n+\left(\frac{\Delta t}{2}\right)^3\Delta_x\Delta_y\Delta_z u_n+\frac{\Delta t}{2}\Delta_y u_n-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y u_n$$
$$=u_n+\frac{\Delta t}{2}(\Delta_x+2\Delta_y+2\Delta_z)u_n-\Delta t f$$

Multiplying by $\frac{2}{\Delta t}$ we have

$$2\frac{u_{n+1}}{\Delta t}-(\Delta_x+\Delta_y+\Delta_z)u_{n+1}+\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z u_{n+1}+\Delta_z u_n-\frac{\Delta t}{2}(\Delta_x+\Delta_y)\Delta_z u_n$$
$$+\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z u_n+\Delta_y u_n-\frac{\Delta t}{2}\Delta_x\Delta_y u_n=2\frac{u_n}{\Delta t}+(\Delta_x+2\Delta_y+2\Delta_z)u_n-2f$$

Rearranging, we have

$$2\frac{u_{n+1}-u_n}{\Delta t}+\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z u_{n+1}-\frac{\Delta t}{2}(\Delta_x+\Delta_y)\Delta_z u_n+\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z u_n-\frac{\Delta t}{2}\Delta_x\Delta_y u_n$$
$$=(\Delta_x+\Delta_y+\Delta_z)(u_{n+1}+u_n)-2f$$

Flipping sides (not changing signs), we have

$$(\Delta_x+\Delta_y+\Delta_z)(u_{n+1}+u_n)$$
$$=2\frac{u_{n+1}-u_n}{\Delta t}+\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z u_{n+1}-\frac{\Delta t}{2}(\Delta_x+\Delta_y)\Delta_z u_n$$
$$+\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z u_n-\frac{\Delta t}{2}\Delta_x\Delta_y u_n+2f$$

Rearranging again we have

$$(\Delta_x+\Delta_y+\Delta_z)(u_{n+1}+u_n)$$
$$=2\frac{u_{n+1}-u_n}{\Delta t}+\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z(u_{n+1}-u_n)-\frac{\Delta t}{2}(\Delta_x+\Delta_y)\Delta_z u_n$$
$$-\frac{\Delta t}{2}\Delta_x\Delta_y u_n+2f$$

Multiplying some terms out we have

$$-\frac{\Delta t}{2}(\Delta_x+\Delta_y)\Delta_z u_n-\frac{\Delta t}{2}\Delta_x\Delta_y u_n=-\frac{\Delta t}{2}\Delta_x\Delta_z u_n-\frac{\Delta t}{2}\Delta_y\Delta_z u_n-\frac{\Delta t}{2}\Delta_x\Delta_y u_n=-\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_n$$

Therefore we have

$$(\Delta_x+\Delta_y+\Delta_z)(u_{n+1}+u_n)$$
$$=2\frac{u_{n+1}-u_n}{\Delta t}+\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_{n+1}-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z(u_{n+1}-u_n)$$
$$-\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)u_n+2f$$

Combining we have, finally

$$(\Delta_x+\Delta_y+\Delta_z)(u_{n+1}+u_n)=2\frac{u_{n+1}-u_n}{\Delta t}+\frac{\Delta t}{2}(\Delta_x\Delta_y+\Delta_x\Delta_z+\Delta_y\Delta_z)(u_{n+1}-u_n)-\left(\frac{\Delta t}{2}\right)^2\Delta_x\Delta_y\Delta_z(u_{n+1}-u_n)+2f$$