MOONS - Derivation of Conservative FD Formulation

Dimensional induction equation

Maxwell's equations (while neglecting the displacement current):

$$\boldsymbol{j} = \nabla \times \frac{\boldsymbol{B}}{\mu_m} \quad , \quad \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \quad , \quad \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B})$$

$$\nabla \cdot \boldsymbol{j} = 0 \quad , \quad \nabla \cdot \boldsymbol{B} = 0 \quad , \quad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_m}$$

Solving for the electric field in terms of the current in Ohm's law yields

$$\boldsymbol{E} = \frac{\boldsymbol{j}}{\sigma} - \boldsymbol{V} \times \boldsymbol{B}$$

Plugging this into Faraday's Law yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left\{\frac{\boldsymbol{j}}{\sigma} - \boldsymbol{V} \times \boldsymbol{B}\right\}$$

Distributing we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) - \nabla \times \frac{\boldsymbol{j}}{\sigma}$$

Applying Ampere's Law to the current yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) - \nabla \times \left\{\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu_m}\right\}$$

Using the vector identity

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$$

And noting that

$$\partial_j V_i B_j = \underbrace{V_i \partial_j B_j}_{0} + B_j \partial_j V_i = B_j \partial_j V_i$$

$$\partial_j V_j B_i = V_j \partial_j B_i + \underbrace{B_i \partial_j V_j}_{0} = V_j \partial_j B_i$$

We can write our advective term as

$$\nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = \boldsymbol{V}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{V}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{V} - (\boldsymbol{V} \cdot \nabla)\boldsymbol{B} = \partial_j V_i B_j - \partial_j V_j B_i = -\partial_j (V_j B_i - V_i B_j)$$

Moving everything to the RHS, we may write our equation in vector form as

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{V}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{V}) + \nabla \times \left\{\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu_m}\right\} = 0$$

Or, more explicitly, in mixed vector-index notation

$$\frac{\partial B_i}{\partial t} + \partial_j (V_j B_i - V_i B_j) + \nabla \times \left\{ \frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu_m} \right\} = 0$$

Using the kronecker delta identity, $\varepsilon_{ijk}\, \varepsilon_{kmn} = \varepsilon_{kij}\, \varepsilon_{kmn} = (\delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm})$, the last term is

$$\nabla \times \left\{ \frac{1}{\sigma} \nabla \times \boldsymbol{H} \right\} = \varepsilon_{ijk}\, \partial_j \left(\frac{1}{\sigma} \varepsilon_{kmn}\, \partial_m H_n \right) = \varepsilon_{ijk}\, \varepsilon_{kmn}\, \partial_j \left(\frac{1}{\sigma} \partial_m H_n \right) = (\delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm}) \partial_j \left(\frac{1}{\sigma} \partial_m H_n \right)$$

$$= \partial_j \left(\frac{1}{\sigma} \partial_i H_j \right) - \partial_j \left(\frac{1}{\sigma} \partial_j H_i \right) = \partial_j \left(\frac{1}{\sigma} \{ \partial_i H_j - \partial_j H_i \} \right)$$

So we may write

$$\frac{\partial B_i}{\partial t} + \partial_j (V_j B_i - V_i B_j) + \partial_j \left(\frac{1}{\sigma} \{ \partial_i H_j - \partial_j H_i \} \right) = 0$$

So we finally have a dimensional *conservative* Finite Difference Method (FDM) formulation that is prepared to be integrated for a Finite Volume Method (FVM) formulation:

$$\boxed{\frac{\partial B_i}{\partial t} + \partial_j (V_j B_i - V_i B_j) + \partial_j \left(\frac{1}{\sigma} \left\{ \partial_i \frac{B_j}{\mu} - \partial_j \frac{B_i}{\mu} \right\} \right) = 0}$$

Non-dimensionalizing

Introducing the following reference values to non-dimensionalize by

$$\boldsymbol{B}^* = \frac{\boldsymbol{B}}{B_c} \qquad \sigma^* = \frac{\sigma}{\sigma_c} \qquad \mu^* = \frac{\mu}{\mu_c} \qquad \boldsymbol{V}^* = \frac{\boldsymbol{V}}{V_c} \qquad \nabla^* = L_c \nabla \qquad t^* = \frac{t}{t_c} \qquad t_c = L_c / V_c$$

Substituting the magnetic field and canceling the reference factor we have

$$\frac{\partial B_i^*}{\partial t} + \partial_j (V_j B_i^* - V_i B_j^*) + \partial_j \left(\frac{1}{\sigma} \left\{ \partial_i \frac{B_j^*}{\mu} - \partial_j \frac{B_i^*}{\mu} \right\} \right) = 0$$

Now substituting the velocity and time we have

$$\frac{V_c}{L_c} \frac{\partial B_i^*}{\partial t^*} + \frac{V_c}{L_c} \partial_j (V_j^* B_i^* - V_i^* B_j^*) + \frac{1}{L_c^2 \sigma_c \mu_c} \partial_j \left(\frac{1}{\sigma^*} \left\{ \partial_i \frac{B_j^*}{\mu^*} - \partial_j \frac{B_i^*}{\mu^*} \right\} \right) = 0$$

Multiplying out we have

$$\frac{\partial B_i^*}{\partial t^*} + \partial_j (V_j^* B_i^* - V_i^* B_j^*) + \frac{1}{L_c V_c \sigma_c \mu_c} \partial_j \left(\frac{1}{\sigma^*} \left\{ \partial_i \frac{B_j^*}{\mu^*} - \partial_j \frac{B_i^*}{\mu^*} \right\} \right) = 0$$

Removing the asterisks we have

$$\boxed{\frac{\partial B_i}{\partial t} + \partial_j (V_j B_i - V_i B_j) + \frac{1}{Re_m} \partial_j \left(\frac{1}{\sigma} \left\{ \partial_i \frac{B_j}{\mu} - \partial_j \frac{B_i}{\mu} \right\} \right) = 0}$$

Introducing the characteristic magnetic Reynolds number

$$Re_m = \frac{V_c L_c}{(\mu_c \sigma_c)^{-1}} = \mu_c \sigma_c V_c L_c$$

Expanded form

Expanding the dummy index j gives us the equation for the ith component:

$$\frac{\partial B_i}{\partial t} + \partial_x(V_x B_i - V_i B_x) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z(V_z B_i - V_i B_z) +$$

$$\frac{1}{Re_m}\partial_x\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_x}{\mu} - \partial_x \frac{B_i}{\mu}\right\}\right) + \frac{1}{Re_m}\partial_y\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_y}{\mu} - \partial_y \frac{B_i}{\mu}\right\}\right) + \frac{1}{Re_m}\partial_z\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_z}{\mu} - \partial_z \frac{B_i}{\mu}\right\}\right) = source\ terms$$

Expanded form for uniform properties

Expanding the dummy index j gives us the equation for the ith component:

$$\frac{\partial B_i}{\partial t} + \partial_x(V_x B_i - V_i B_x) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z(V_z B_i - V_i B_z) +$$

$$\frac{1}{Re_m}\partial_x(\{\partial_i B_x - \partial_x B_i\}) + \frac{1}{Re_m}\partial_y\left(\left\{\partial_i B_y - \partial_y B_i\right\}\right) + \frac{1}{Re_m}\partial_z(\{\partial_i B_z - \partial_z B_i\}) = source\ terms$$

Or

$$\frac{\partial B_i}{\partial t} + \partial_x(V_x B_i - V_i B_x) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z(V_z B_i - V_i B_z) +$$

$$\frac{1}{Re_m}\{\partial_x \partial_i B_x - \partial_x \partial_x B_i\} + \frac{1}{Re_m}\left\{\partial_y \partial_i B_y - \partial_y \partial_y B_i\right\} + \frac{1}{Re_m}\{\partial_z \partial_i B_z - \partial_z \partial_z B_i\} = source\ terms$$