CT method stencil (uniform grid)

Computation of electric field

The first step is to compute the electric field:

$$E_{edge} = \frac{j_{edge}}{\sigma_{edge}} - interp(u \times B)_{cc \to edge}$$

This electric field is located at an edge.

Let the index i, j denote the cell center. Let cell faces be located at half positions appropriately. Looking at the electric field in 2D, the z component of the electric field would be

$$E_z = \frac{1}{\sigma_{edge}} \{\partial_x B_y - \partial_y B_x\}_{edge} - interp(uB_y - vB_x)_{cc \to edge}$$

Using indexes, in 2D for a uniform grid, current term may be written as

$$j_{z,i+\frac{1}{2},j+\frac{1}{2}} = \frac{4}{\sigma_{i,j} + \sigma_{i+1,j} + \sigma_{i,j+1} + \sigma_{i+1,j+1}} \left\{\partial_x B_{y\,i,j+\frac{1}{2}} - \partial_y B_{x\,i+\frac{1}{2},j}\right\}_{i+\frac{1}{2},j+\frac{1}{2}}$$

The derivatives inside the current term must be collocated to subtract, so linearly interpolate them:

$$j_{z,i+\frac{1}{2},j+\frac{1}{2}} = \frac{4}{\sigma_{i+\frac{1}{2},j+\frac{1}{2}}} \left\{ \frac{\frac{B_{y,j} + B_{y,j+1}}{2}|_{i+1} - \frac{B_{y,j} + B_{y,j+1}}{2}|_{i}}{\Delta x} - \frac{\frac{B_{x,i} + B_{x,i+1}}{2}|_{j+1} - \frac{B_{x,i} + B_{x,i+1}}{2}|_{j}}{\Delta y} \right\}_{i+\frac{1}{2},j+\frac{1}{2}}$$

Where

$$\sigma_{i+\frac{1}{2},j+\frac{1}{2}} = \sigma_{i,j} + \sigma_{i+1,j} + \sigma_{i,j+1} + \sigma_{i+1,j+1}$$

Now looking at the $u \times B$ term, we have

$$interp(u \times B)_{cc \to edge} = \frac{(uB_y - vB_x)_{i,j} + (uB_y - vB_x)_{i+1,j} + (uB_y - vB_x)_{i,j+1} + (uB_y - vB_x)_{i+1,j+1}}{4}$$

Finally, we have

$$E_{z,i+\frac{1}{2},j+\frac{1}{2}} = \frac{4}{\sigma_{i+\frac{1}{2},j+\frac{1}{2}}} \left\{ \frac{\frac{B_{y,j} + B_{y,j+1}}{2}|_{i+1} - \frac{B_{y,j} + B_{y,j+1}}{2}|_{i}}{\Delta x} - \frac{\frac{B_{x,i} + B_{x,i+1}}{2}|_{j+1} - \frac{B_{x,i} + B_{x,i+1}}{2}|_{j}}{\Delta y} \right\}_{i+\frac{1}{2},j+\frac{1}{2}}$$

$$- \frac{(uB_y - vB_x)_{i,j} + (uB_y - vB_x)_{i+1,j} + (uB_y - vB_x)_{i,j+1} + (uB_y - vB_x)_{i+1,j+1}}{4}$$

Where

$$\sigma_{i+\frac{1}{2},j+\frac{1}{2}} = \sigma_{i,j} + \sigma_{i+1,j} + \sigma_{i,j+1} + \sigma_{i+1,j+1}$$

Note:
Although the interpolation of the current seems to be only first order accurate, looking at the stencil for the current, it is clear that 2nd order accuracy is achieved due to symmetry (the current is computed using a symmetric stencil about the cell edge).

Curl of Electric field

Once the electric field components are computed, take the curl of the electric field (which lives on the cell edge) and update B from Faradays Law. This result will live on the cell face:

$$\frac{\partial B}{\partial t} = -\nabla \times E$$

$$b^{n+1}_{x,i+\frac{1}{2},j} = b^{n}_{x,i+\frac{1}{2},j} - \Delta t \frac{\left(E_{z,j+\frac{1}{2}} - E_{z,j-\frac{1}{2}}\right)_{i+\frac{1}{2}}}{\Delta y}$$

$$b^{n+1}_{y,i,j+\frac{1}{2}} = b^{n}_{y,i,j+\frac{1}{2}} - \Delta t \frac{\left(E_{z,i+\frac{1}{2}} - E_{z,i-\frac{1}{2}}\right)_{j+\frac{1}{2}}}{\Delta x}$$

Note that the lower case b indicates that the data lives on the cell face. This equation can be interpolated (for both time steps) to the cell center:

$$B^{n+1}_{x,i,j} = B^{n}_{x,i,j} - \Delta t \left\{ \frac{\frac{\left(E_{z,j+\frac{1}{2}} - E_{z,j-\frac{1}{2}}\right)_{i+\frac{1}{2}}}{\Delta y} + \frac{\left(E_{z,j+\frac{1}{2}} - E_{z,j-\frac{1}{2}}\right)_{i-\frac{1}{2}}}{\Delta y}}{2} \right\}$$

$$B^{n+1}_{y,i,j} = B^{n}_{y,i,j} - \Delta t \left\{ \frac{\frac{\left(E_{z,i+\frac{1}{2}} - E_{z,i-\frac{1}{2}}\right)_{j+\frac{1}{2}}}{\Delta x} + \frac{\left(E_{z,i+\frac{1}{2}} - E_{z,i-\frac{1}{2}}\right)_{j-\frac{1}{2}}}{\Delta x}}{2} \right\}$$

Note that the divergence of the magnetic field must be computed from the magnetic field before interpolating because this method enforces

$$(\nabla \cdot b)_{i,j} = \frac{\left(b_{x,i+\frac{1}{2}} - b_{x,i-\frac{1}{2}}\right)_{j}}{\Delta x} + \frac{\left(b_{y,j+\frac{1}{2}} - b_{y,j-\frac{1}{2}}\right)_{i}}{\Delta y} = 0$$