B - formulation

$$\begin{cases} \dfrac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} \\[2ex] \vec{J} = \sigma(\vec{E} + \vec{U}\times\vec{B}) \\[2ex] \nabla\times\vec{H} = \vec{J} + \cancelto{0}{\dfrac{\partial \vec{D}}{\partial t}} \\[2ex] \nabla\cdot\vec{B} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \nabla\cdot\vec{U} = 0 \quad \text{(incompressible)} \\[2ex] \vec{H} = \dfrac{\vec{B}}{\mu_m} \end{cases}$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = -\nabla\times\left(\frac{\vec{J}}{\sigma} - \vec{U}\times\vec{B}\right) = -\nabla\times\left(\frac{1}{\sigma}\nabla\times\left(\frac{\vec{B}}{\mu_m}\right)\right) + \nabla\times(\vec{U}\times\vec{B})$$

$$\text{and} \quad \nabla\times(\vec{U}\times\vec{B}) = \vec{B}\cdot\nabla\vec{U} - \vec{U}\cdot\nabla\vec{B} + \vec{U}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{U})$$

$$-\nabla\cdot(\vec{U}\vec{B} - \vec{B}\vec{U}) = \nabla\cdot(\vec{B}\vec{U}) - \nabla\cdot(\vec{U}\vec{B})$$

$$= (\nabla\cdot\vec{B})\vec{U} + \vec{B}\cdot\nabla\vec{U} - (\nabla\cdot\vec{U})\vec{B} - \vec{U}\cdot\nabla\vec{B} = \nabla\times(\vec{U}\times\vec{B})$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} + \vec{U}\cdot\nabla\vec{B} + \nabla\times\left(\frac{1}{\sigma}\nabla\times\left(\frac{\vec{B}}{\mu_m}\right)\right) = \vec{B}\cdot\nabla\vec{U}}$$

viscous term

$$\text{or:} \quad \frac{\partial \vec{B}}{\partial t} + \nabla\cdot\underline{(\vec{U}\vec{B} - \vec{B}\vec{U})} + \nabla\times\left(\frac{1}{\sigma}\nabla\times\left(\frac{\vec{B}}{\mu_m}\right)\right) = 0$$

$$\text{let:} \quad \vec{F} = \underbrace{\underset{\sim}{n}\cdot(\vec{U}\vec{B} - \vec{B}\vec{U})}_{\text{inviscous term}} = \begin{bmatrix} (U_y B_x - U_x B_y) n_y + (U_z B_x - U_x B_z) n_z \\ (U_z B_y - U_y B_z) n_z + (U_x B_y - U_y B_x) n_x \\ (U_x B_z - U_z B_x) n_x + (U_y B_z - U_z B_y) n_y \end{bmatrix}$$

$$\vec{G} = \frac{1}{\sigma}(\underset{\sim}{n}\cdot\nabla\vec{H} - \nabla\vec{H}\cdot\underset{\sim}{n}) = \frac{1}{\sigma}\underset{\sim}{n}\cdot(\nabla\vec{H} - (\nabla\vec{H})^T) \quad \text{— viscous term}$$

$$= \frac{1}{\sigma}\begin{bmatrix} \left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right) n_y + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) n_z \\[1ex] \left(\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y}\right) n_z + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) n_x \\[1ex] \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) n_x + \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) n_y \end{bmatrix}$$

$$\int_{\Omega} \nabla\times\left(\frac{1}{\sigma}\nabla\times\left(\frac{\vec{B}}{\mu_m}\right)\right)d\Omega = \oint_{\partial\Omega} \underset{\sim}{n}\times\left[\frac{1}{\sigma}\nabla\times\left(\frac{\vec{B}}{\mu_m}\right)\right]ds$$

$$= \oint_{\partial\Omega} \frac{1}{\sigma}\,\underset{\sim}{n}\times(\nabla\times\vec{H})\,ds = \oint_{\partial\Omega}\frac{1}{\sigma}\left(\nabla\vec{H}\cdot\underset{\sim}{n} - \underset{\sim}{n}\cdot\nabla\vec{H}\right)ds$$

$$\Rightarrow \int_{\Omega}\frac{\partial\vec{B}}{\partial t}\,d\Omega + \oint_{\partial\Omega}\vec{F}\,ds = \oint_{\partial\Omega}\vec{G}\,ds$$

$$\vec{J} = \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \nabla\times\vec{H} = \begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \frac{1}{\mu_m}\begin{bmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \end{bmatrix}$$

$$\Rightarrow \vec{G} = \frac{1}{\sigma}\begin{bmatrix} -J_z n_y + J_y n_z \\ -J_x n_z + J_z n_x \\ -J_y n_x + J_x n_y \end{bmatrix} = \frac{1}{\sigma}\begin{bmatrix} 0 & -J_z & J_y \\ J_z & 0 & -J_x \\ -J_y & J_x & 0 \end{bmatrix}\cdot\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$$\Rightarrow \frac{\Omega}{\Delta t}\left(\vec{B}^{n+1} - \vec{B}^{n}\right) = \sum_{faces}(\vec{G} - \vec{F})\,\Delta S$$

$$\vec{B}^{n+1} = \vec{B}^{n} + \frac{\Delta t}{\Omega}\sum_{faces}(\vec{G} - \vec{F})\,\Delta S$$

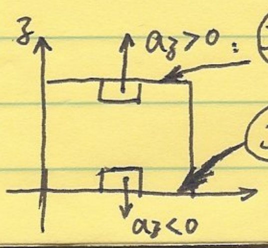
$$\vec{J} = \frac{\int_{\Omega}\vec{J}\,d\Omega}{\Omega} = \frac{\int_{\Omega}\frac{1}{\mu_m}(\nabla\times\vec{B})\,d\Omega}{\Omega} = \frac{1}{\mu_m\Omega}\int_{\Omega}\nabla\times\vec{B}\,d\Omega = \frac{1}{\mu_m\Omega}\oint_{\partial\Omega}\vec{n}\times\vec{B}\,ds$$

$$= \frac{1}{\mu_m\Omega}\sum_{faces}\vec{n}\times\vec{B}_f\,\Delta S = \frac{1}{\mu_m\Omega}\sum_{faces}\begin{bmatrix} n_y B_z - n_z B_y \\ n_z B_x - n_x B_z \\ n_x B_y - n_y B_x \end{bmatrix}\Delta S$$

μ_m : magnetic permeability $\quad \mu_r = \frac{\mu_m}{\mu_0}$ (relative per...)

$\mu_0 \rightarrow (4\pi e{-7})$ H/m

BC: $\vec{J}\cdot\underset{\sim}{n} = 0 \quad\Rightarrow\quad \begin{cases} J_y = \frac{1}{\mu_m}\frac{\partial B_x}{\partial z} \\ J_z = -\frac{1}{\mu_m}\frac{\partial B_x}{\partial y} \end{cases}$



$a_z>0$: $J_y = \frac{1}{\mu_m}\frac{0-B_x}{d_1}$

$J_y = \frac{1}{\mu_m}\frac{B_x-0}{d_1}$