Peter's Formulation (variable properties)

Dimensional induction equation

Maxwell's equations (while neglecting the displacement current):

$$\boldsymbol{j} = \nabla \times \frac{\boldsymbol{B}}{\mu_m} \quad , \quad \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \quad , \quad \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B})$$

$$\nabla \cdot \boldsymbol{j} = 0 \quad , \quad \nabla \cdot \boldsymbol{B} = 0 \quad , \quad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_m}$$

Solving for the electric field in terms of the current in Ohm's law yields

$$\boldsymbol{E} = \frac{\boldsymbol{j}}{\sigma} - \boldsymbol{V} \times \boldsymbol{B}$$

Plugging this into Faraday's Law yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left\{\frac{\boldsymbol{j}}{\sigma} - \boldsymbol{V} \times \boldsymbol{B}\right\}$$

Distributing we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) - \nabla \times \frac{\boldsymbol{j}}{\sigma}$$

Applying Ampere's Law to the current yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) - \nabla \times \left\{\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu_m}\right\}$$

Using the vector identity

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$$

On the $\nabla \times (\boldsymbol{V} \times \boldsymbol{B})$ term, using $\nabla \cdot \boldsymbol{V} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$ we have:

$$\frac{\partial \boldsymbol{B}}{\partial t} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{V} - (\boldsymbol{V} \cdot \nabla)\boldsymbol{B} - \nabla \times \left\{\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu_m}\right\}$$

Or, putting this all on one side:

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{V} + \nabla \times \left\{\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu_m}\right\} = 0$$

Noting that

$$\partial_j V_i B_j = \underbrace{V_i \partial_j B_j}_{0} + B_j \partial_j V_i = B_j \partial_j V_i$$

$$\partial_j V_j B_i = V_j \partial_j B_i + \underbrace{B_i \partial_j V_j}_{0} = V_j \partial_j B_i$$

Again using $\nabla \cdot \boldsymbol{V} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$, implies that

$$(\boldsymbol{V}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{V}=V_j\partial_j B_i-B_j\partial_j V_i=\partial_j V_j B_i-\partial_j V_i B_j=\partial_j(V_j B_i-V_i B_j)$$

In vector notation, we may write our equation as

$$\frac{\partial\boldsymbol{B}}{\partial t}+\nabla\cdot(\boldsymbol{VB}-\boldsymbol{BV})+\nabla\times\left\{\frac{1}{\sigma}\nabla\times\boldsymbol{H}\right\}=0$$

Or, more explicitly, in mixed vector-index notation

$$\frac{\partial B_i}{\partial t}+\partial_j(V_j B_i-V_i B_j)+\nabla\times\left\{\frac{1}{\sigma}\nabla\times\boldsymbol{H}\right\}=0$$

And the last term may be written as

$$\nabla\times\left\{\frac{1}{\sigma}\nabla\times\frac{\boldsymbol{B}}{\mu_m}\right\}=\varepsilon_{ijk}\,\partial_j\left(\frac{1}{\sigma}\varepsilon_{kmn}\,\partial_m H_n\right)=\varepsilon_{ijk}\,\varepsilon_{kmn}\,\partial_j\left(\frac{1}{\sigma}\partial_m H_n\right)$$

So we may write this as

$$\frac{\partial B_i}{\partial t}+\partial_j(V_j B_i-V_i B_j)+\varepsilon_{ijk}\,\varepsilon_{kmn}\,\partial_j\left(\frac{1}{\sigma}\partial_m H_n\right)=0$$

Using the kronecker delta identity, the last term is

$$\varepsilon_{ijk}\,\varepsilon_{kmn}=\varepsilon_{kij}\,\varepsilon_{kmn}=(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\partial_j\left(\frac{1}{\sigma}\partial_m H_n\right)$$

$$=\partial_j\left(\frac{1}{\sigma}\partial_i H_j\right)-\partial_j\left(\frac{1}{\sigma}\partial_j H_i\right)=\partial_j\left(\frac{1}{\sigma}\{\partial_i H_j-\partial_j H_i\}\right)$$

So we may write

$$\frac{\partial B_i}{\partial t}+\partial_j(V_j B_i-V_i B_j)+\partial_j\left(\frac{1}{\sigma}\{\partial_i H_j-\partial_j H_i\}\right)=0$$

So we finally have a dimensional *conservative* Finite Difference Method (FDM) formulation that is prepared to be integrated for a Finite Volume Method (FVM) formulation:

$$\boxed{\frac{\partial B_i}{\partial t}+\partial_j(V_j B_i-V_i B_j)+\partial_j\left(\frac{1}{\sigma}\left\{\partial_i\frac{B_j}{\mu}-\partial_j\frac{B_i}{\mu}\right\}\right)=0}$$

Non-dimensionalizing

Introducing the following reference values to non-dimensionalize by

$$\boldsymbol{B}^*=\frac{\boldsymbol{B}}{B_c}\qquad\sigma^*=\frac{\sigma}{\sigma_c}\qquad\mu^*=\frac{\mu}{\mu_c}\qquad\boldsymbol{V}^*=\frac{\boldsymbol{V}}{V_c}\qquad\nabla^*=L_c\nabla\qquad t^*=\frac{t}{t_c}\qquad t_c=L_c/V_c$$

Substituting the magnetic field and canceling the reference factor we have

$$\frac{\partial B_i^*}{\partial t}+\partial_j(V_j B_i^*-V_i B_j^*)+\partial_j\left(\frac{1}{\sigma}\left\{\partial_i\frac{B_j^*}{\mu}-\partial_j\frac{B_i^*}{\mu}\right\}\right)=0$$

Now substituting the velocity and time we have

$$\frac{V_c}{L_c}\frac{\partial B_i^*}{\partial t^*} + \frac{V_c}{L_c}\partial_j\left(V_j^* B_i^* - V_i^* B_j^*\right) + \frac{1}{L_c^2 \sigma_c \mu_c}\partial_j\left(\frac{1}{\sigma^*}\left\{\partial_i \frac{B_j^*}{\mu^*} - \partial_j \frac{B_i^*}{\mu^*}\right\}\right) = 0$$

Multiplying out we have

$$\frac{\partial B_i^*}{\partial t^*} + \partial_j\left(V_j^* B_i^* - V_i^* B_j^*\right) + \frac{1}{L_c V_c \sigma_c \mu_c}\partial_j\left(\frac{1}{\sigma^*}\left\{\partial_i \frac{B_j^*}{\mu^*} - \partial_j \frac{B_i^*}{\mu^*}\right\}\right) = 0$$

Removing the asterisks we have

$$\boxed{\frac{\partial B_i}{\partial t} + \partial_j\left(V_j B_i - V_i B_j\right) + \frac{1}{Re_m}\partial_j\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_j}{\mu} - \partial_j \frac{B_i}{\mu}\right\}\right) = 0}$$

Introducing the characteristic magnetic Reynolds number

$$Re_m = \frac{V_c L_c}{(\mu_c \sigma_c)^{-1}} = \mu_c \sigma_c V_c L_c$$

Expanded form

Expanding the dummy index j gives us the equation for the ith component:

$$\frac{\partial B_i}{\partial t} + \partial_x\left(V_x B_i - V_i B_x\right) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z\left(V_z B_i - V_i B_z\right) +$$

$$\frac{1}{Re_m}\partial_x\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_x}{\mu} - \partial_x \frac{B_i}{\mu}\right\}\right) + \frac{1}{Re_m}\partial_y\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_y}{\mu} - \partial_y \frac{B_i}{\mu}\right\}\right) + \frac{1}{Re_m}\partial_z\left(\frac{1}{\sigma}\left\{\partial_i \frac{B_z}{\mu} - \partial_z \frac{B_i}{\mu}\right\}\right) = source\ terms$$

Expanded form for uniform properties

Expanding the dummy index j gives us the equation for the ith component:

$$\frac{\partial B_i}{\partial t} + \partial_x\left(V_x B_i - V_i B_x\right) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z\left(V_z B_i - V_i B_z\right) +$$

$$\frac{1}{Re_m}\partial_x\left(\left\{\partial_i B_x - \partial_x B_i\right\}\right) + \frac{1}{Re_m}\partial_y\left(\left\{\partial_i B_y - \partial_y B_i\right\}\right) + \frac{1}{Re_m}\partial_z\left(\left\{\partial_i B_z - \partial_z B_i\right\}\right) = source\ terms$$

Or

$$\frac{\partial B_i}{\partial t} + \partial_x\left(V_x B_i - V_i B_x\right) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z\left(V_z B_i - V_i B_z\right) +$$

$$\frac{1}{Re_m}\left\{\partial_x \partial_i B_x - \partial_x \partial_x B_i\right\} + \frac{1}{Re_m}\left\{\partial_y \partial_i B_y - \partial_y \partial_y B_i\right\} + \frac{1}{Re_m}\left\{\partial_z \partial_i B_z - \partial_z \partial_z B_i\right\} = source\ terms$$

Expanded form for uniform properties (weak form in B)

$$\frac{\partial B_i}{\partial t} + \partial_x\left(V_x B_i - V_i B_x\right) + \partial_y\left(V_y B_i - V_i B_y\right) + \partial_z\left(V_z B_i - V_i B_z\right) - \frac{1}{Re_m}\underbrace{\left\{\partial_x \partial_x B_i + \partial_y \partial_y B_i + \partial_z \partial_z B_i\right\}}_{\nabla^2 B_i} = source\ terms$$

For the low Re_m approximation, we have

$$\frac{\partial B_i}{\partial t} + \frac{1}{Re_m}\underbrace{\left\{-\left(\partial_x \partial_x B_i + \partial_y \partial_y B_i + \partial_z \partial_z B_i\right)\right\}}_{-\nabla^2 B_i} = \partial_j\left(V_j B_{i0} - V_i B_{j0}\right)$$

Finite Volume Method (FVM) form

Integrating this equation over the control volume yields

$$\int_{\Omega} \frac{\partial B_i}{\partial t} d\Omega + \int_{\Omega} \partial_j \left(V_j B_i - V_i B_j\right) d\Omega + \int_{\Omega} \partial_j \left(\frac{1}{\sigma}\left\{\partial_i \frac{B_j}{\mu} - \partial_j \frac{B_i}{\mu}\right\}\right) d\Omega = 0$$

Using RTT on the first term and Gauss' divergence theorem on the second and third terms yields

$$\int_{\Omega} \frac{\partial B_i}{\partial t} d\Omega + \int_{\Omega} n_j \left(V_j B_i - V_i B_j\right) dS + \int_{\Omega} n_j \left(\frac{1}{\sigma}\left\{\partial_i \frac{B_j}{\mu} - \partial_j \frac{B_i}{\mu}\right\}\right) d\Omega = 0$$

Let

$$F_i = n_j \left(V_j B_i - V_i B_j\right)$$

$$G_i = -n_j \left(\frac{1}{\sigma}\left\{\partial_i \frac{B_j}{\mu} - \partial_j \frac{B_i}{\mu}\right\}\right)$$

And then we have

$$\int_{\Omega} \frac{\partial B_i}{\partial t} d\Omega + \int_{\partial\Omega} F_i \, dS + \int_{\partial\Omega} -G_i dS = 0$$

Or

$$\frac{\partial}{\partial t}\int_{\Omega} B_i \, d\Omega + \int_{\partial\Omega} F_i \, dS = \int_{\partial\Omega} G_i dS$$

Integrating this, we may solve this system using Finite Volume Method (FVM):

$$\frac{\left(B_i^{n+1} - B_i^{n}\right)}{\Delta t}\Omega = \sum_{faces} (G_i - F_i)\Delta S$$

Or

$$B_i^{n+1} = B_i^{n} + \frac{\Delta t}{\Omega} \sum_{faces} (G_i - F_i)\Delta S$$