Magnetohydrodynamic Object-Oriented Numerical Simulation (MOONS)

Derivations

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1 Navier-Stokes Equation

1.1 Dimensionless NS with $j \times B$ Force

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \bullet \nabla \boldsymbol{u} \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{f} + \boldsymbol{j} \times \boldsymbol{B} \tag{1}$$

Where $\boldsymbol{f}$ is some dimensional force per unit volume. Non-dimensionalizing by

$$\boldsymbol{u}^* = \boldsymbol{u}/U_c \qquad t^* = t/t_c \qquad t_c = a_c/U_c \qquad \nabla^* = a_c \nabla \qquad \rho = \rho_c \tag{2}$$

$$p^* = p/\rho_c U_c^2 \qquad \boldsymbol{j}^* = \boldsymbol{j}/\sigma_c U_c B_c \qquad \boldsymbol{B}^* = \boldsymbol{B}/B_c \tag{3}$$

Yields

$$\rho_c \left(\frac{U_c^2}{a_c} \frac{\partial \boldsymbol{u}^*}{\partial t} + \frac{U_c^2}{a_c} \boldsymbol{u}^* \bullet \nabla^* \boldsymbol{u}^* \right) = -\rho_c \frac{U_c^2}{a_c} \nabla^* p^* + \mu \frac{U_c}{a_c^2} \nabla^{*2} \boldsymbol{u}^* + \boldsymbol{f} + U_c B_c^2 \sigma_c \boldsymbol{j}^* \times \boldsymbol{B}^* \tag{4}$$

Dividing by $\frac{\rho_c U_c^2}{a_c}$ yields

$$\frac{\partial \boldsymbol{u}^*}{\partial t} + \boldsymbol{u}^* \bullet \nabla^* \boldsymbol{u}^* = -\nabla^* p^* + \frac{\nu_c}{U_c a_c} \nabla^{*2} \boldsymbol{u}^* + \frac{a_c}{\rho_c U_c^2} \boldsymbol{f} + \frac{a_c B_c^2 \sigma_c}{\rho_c U_c} \boldsymbol{j}^* \times \boldsymbol{B}^* \tag{5}$$

Re-ordering, and removing the asterisk, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{U_c^2}{a_c} \boldsymbol{u} \bullet \nabla \boldsymbol{u} = -\nabla p + \frac{\nu_c}{U_c a_c} \nabla^2 \boldsymbol{u} + \frac{a_c}{\rho_c U_c^2} \boldsymbol{f} + \frac{\mu_c}{\rho_c U_c a_c} \frac{a_c^2}{b_c^2} b_c^2 B_c^2 \frac{\sigma_c}{\rho_c \mu_c} \boldsymbol{j} \times \boldsymbol{B} \tag{6}$$

Finally, we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \bullet \nabla \boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \frac{a_c}{\rho_c U_c^2}\boldsymbol{f} + \frac{Ha^2}{Re}\left(\frac{a_c}{b_c}\right)^2 \boldsymbol{j} \times \boldsymbol{B} \tag{7}$$

Where

$$Re = \frac{U_c a_c}{\nu_c} \qquad\qquad Ha = B_c b_c \sqrt{\frac{\sigma_c}{\mu_c}} \tag{8}$$

Using the kronecker delta relation, we may write this as

$$\frac{\partial u_i}{\partial t} + u_j \partial_j u_i = -\partial_i p + \frac{1}{Re}\partial_j \partial_j u_i + \frac{a_c}{\rho_c U_c^2} f_i + \frac{Ha^2}{Re}\left(\frac{a_c}{b_c}\right)^2 \boldsymbol{j} \times \boldsymbol{B} \tag{9}$$

We may write the $\boldsymbol{j} \times \boldsymbol{B}$ force as

$$\boldsymbol{j} \times \boldsymbol{B} = \epsilon_i jk j_j B_k = \epsilon_i jk \left(\epsilon_j mn \partial_m \frac{B_n}{\mu} B_k\right) \tag{10}$$

$$= \epsilon_i jk \epsilon_j mn \left(\partial_m \frac{B_n}{\mu} B_k\right) = \epsilon_j ki \epsilon_j mn \left(\partial_m \frac{B_n}{\mu} B_k\right) \tag{11}$$

$$= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) \left(\partial_m \frac{B_n}{\mu} B_k\right) \tag{12}$$

$$= \left(\partial_k \frac{B_i}{\mu} B_k\right) - \left(\partial_i \frac{B_k}{\mu} B_k\right) \tag{13}$$

$$\boldsymbol{j} \times \boldsymbol{B} = B_k \left(\partial_k \frac{B_i}{\mu} - \partial_i \frac{B_k}{\mu}\right) \tag{14}$$

1.2 Time discretization

MOONS implements the same time stepping procedure described in [1]. Below the equations are written for this method with minimal explanation, since it can be found in the reference. Also, a mixed index-vector notation has been used.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\partial_j (u_j^n u_i^n) - \partial_i p^{n+1} + \frac{1}{Re}\partial_j \partial_j u_i^n + \frac{Ha^2}{Re}\left(\frac{a_c}{b_c}\right)^2 \boldsymbol{j^n} \times \boldsymbol{B^n} \tag{15}$$

Estimating for the velocity at the next time step, we have

$$u_i^* = u_i^n + \Delta t \left[-\partial_j (u_j^n u_i^n) + \frac{1}{Re}\partial_j \partial_j u_i^n + \frac{Ha^2}{Re}\left(\frac{a_c}{b_c}\right)^2 \boldsymbol{j^n} \times \boldsymbol{B^n}\right] \tag{16}$$

And the pressure-correction step

$$u_i^{n+1} = u_i^* - \partial_i p^{n+1} \tag{17}$$

Where the implicit pressure is solved in

$$\partial_j \partial_j p^{n+1} = \frac{1}{\Delta t} \partial_i u_i^n \tag{18}$$

2 Induction Equation

2.1 Maxwells equations

Maxwell's equations are

$$\boldsymbol{j} = \nabla \times \frac{\boldsymbol{B}}{\mu} \qquad \frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \qquad \boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{E}) \tag{19}$$

$$\nabla \bullet \boldsymbol{j} = \boldsymbol{0} \qquad \nabla \bullet \boldsymbol{B} = \boldsymbol{0} \qquad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} \tag{20}$$

Solving for the electric field of the current in Ohm's law yields

$$\boldsymbol{E} = \frac{\boldsymbol{j}}{\sigma} - \boldsymbol{u} \times \boldsymbol{B} \tag{21}$$

Plugging this into Faraday's law yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left[\frac{\boldsymbol{j}}{\sigma} - \boldsymbol{u} \times \boldsymbol{B}\right] \tag{22}$$

Distributing we have the induction equation

2.2 Induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times \left[\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu}\right] \tag{23}$$

Non-dimensionalizing this by

$$\nabla^* = b_c \nabla, \qquad t^* = t/(a_c/U_c), \qquad B_i^* = B_i/B_c, \qquad \sigma^* = \sigma/\sigma_c$$

$$\mu^* = \mu/\mu_c, \qquad E_i^* = E_i/(U_c B_c), \qquad j_i^* = \frac{j}{B_i/(\mu_c b_c)}$$

Yields

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times \left[\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu}\right] \tag{24}$$

Using the vector identity

$$\nabla \times (A \times B) = A(\nabla \bullet B) - B(\nabla \bullet A) + (B \bullet \nabla)A - (A \bullet \nabla)B \tag{25}$$

And noting that (index notation helps here)

$$\partial_j(u_i B_j) = u_i \underbrace{\partial_j B_j}_{=0} + B_j \partial_j u_i = B_j \partial_j u_i \tag{26}$$

$$\partial_j(u_j B_i) = u_j \partial_j B_i + B_i \underbrace{\partial_j u_j}_{=0} = u_j \partial_j B_i \tag{27}$$

we can write our advective term as

$$\nabla \times (u \times B) = u(\nabla \bullet B) - B(\nabla \bullet u) + (B \bullet \nabla)u - (u \bullet \nabla)B = \partial_j(u_i B_j) - \partial_j(u_j B_i) = -\partial_j(u_j B_i - u_i B_j) \tag{28}$$

Moving everything to the RHS, we may write our equation in vector form as

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times \left[\frac{1}{\sigma} \nabla \times \frac{\boldsymbol{B}}{\mu} \right] \tag{29}$$

Using the kronecker delta identity, the diffusion term may be written as

$$\nabla \tag{30}$$

$$= \tag{31}$$

Finally, we may write our full induction equation as

$$\frac{\partial B_i}{\partial t} = \tag{32}$$

This is the full induction equation written in a conservative finite difference form. This is the form that is implemented in MOONS.

References

[1] Michael Griebel, Thomas Dornseifer, and Tilman Neunhoeffer. *Numerical simulation in fluid dynamics: a practical introduction*, volume 3. Siam, 1997.