Outline of my CT method

A note about notation: B is located at the cell center, b is located at the cell face and bold faces have been neglected. $cc = cell\ center$

1) Compute the electric field at the cell edge using the appropriate fluxes
$$E_{edge} = \frac{1}{Re_m}\frac{j_{edge}}{\sigma_{edge}} - interp(u_{cc} \times B_{cc})_{cc\rightarrow edge}$$
Where j_{edge} is computed from the 4 surrounding cell centers.
2) Compute the flux of B from the edge curl of the electric field
$$\{\nabla \times E_{edge}\} \in face$$
3) Interpolate the CT method formula in "Tóth, G. The divergence Constraint in Shock-Capturing MHD Codes. J. Comput. Phys. 161, 605–652 (2000)."

$$b^{n+1}_{face} = b^{n}_{face} - \Delta t\{\nabla \times E_{edge}\}_{face}$$

$$interp(b^{n+1})_{face\rightarrow cc} = interp(b^{n})_{face\rightarrow cc} - \Delta t\ interp\left(\{\nabla \times E_{edge}\}_{face}\right)_{face\rightarrow c}$$

$$B^{n+1} = B^{n} - \Delta t\ interp\left(\{\nabla \times E_{edge}\}_{face}\right)_{face\rightarrow c}$$

This enforces

$$\{\nabla \cdot b\}_{cc} = 0$$

Maybe this can be represented in a cell centered B-field, need to look into this though.

Equation

$$\frac{\partial B_i}{\partial s} = -\frac{\partial}{\partial x_j}\left(u_j B_i^0 - u_i B_j^0\right) - \left\{\frac{\partial}{\partial x_j}\left(\frac{1}{\sigma}\frac{\partial B_j}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(\frac{1}{\sigma}\frac{\partial B_i}{\partial x_j}\right)\right\}$$

$$\frac{\partial B_i}{\partial s} = -\{\nabla \times E\}_i$$

$$\frac{\partial B_i}{\partial s} = -\epsilon_{i,j,k}\partial_j\left\{\frac{j}{\sigma} - u \times B^0\right\}_k$$

$$\frac{\partial B_i}{\partial s} = -\left\{\begin{array}{c} \left(\partial_y\left\{\frac{j}{\sigma} - u \times B^0\right\}_z - \partial_z\left\{\frac{j}{\sigma} - u \times B^0\right\}_y\right)\hat{\imath} \\ -\left(\partial_x\left\{\frac{j}{\sigma} - u \times B^0\right\}_z - \partial_z\left\{\frac{j}{\sigma} - u \times B^0\right\}_x\right)\hat{\jmath} \\ +\left(\partial_x\left\{\frac{j}{\sigma} - u \times B^0\right\}_y - \partial_y\left\{\frac{j}{\sigma} - u \times B^0\right\}_x\right)\hat{k} \end{array}\right\}$$

Meanwhile

$$j = \nabla \times B = \left\{ \begin{matrix} \left(\partial_y B_z - \partial_z B_y\right)\hat{\imath} \\ -(\partial_x B_z - \partial_z B_x)\hat{\jmath} \\ +\left(\partial_x B_y - \partial_y B_x\right)\hat{k} \end{matrix} \right\}$$

Therefore we have

$$\frac{\partial B_i}{\partial s} = -\left\{ \begin{matrix} \left(\partial_y \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y\right)\hat{\imath} \\ -\left(\partial_x \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{\jmath} \\ +\left(\partial_x \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y - \partial_y \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{k} \end{matrix} \right\}$$

$$B_i^{n+1} = B_i^n - \Delta s \left\{ \begin{matrix} \left(\partial_y \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y\right)\hat{\imath} \\ -\left(\partial_x \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{\jmath} \\ +\left(\partial_x \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y - \partial_y \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{k} \end{matrix} \right\}$$

The divergence of this at the next time step is:

$$\nabla \cdot b_i^{n+1} = \nabla \cdot b_i^n - \Delta s \nabla \cdot \left\{ \begin{matrix} \left(\partial_y \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y\right)\hat{\imath} \\ -\left(\partial_x \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{\jmath} \\ +\left(\partial_x \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y - \partial_y \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{k} \end{matrix} \right\}$$

$$\nabla \cdot b_i^{n+1} = -\Delta s \left\{ \begin{matrix} \partial_x\left(\partial_y \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y\right)\hat{\imath} \\ -\partial_y\left(\partial_x \left\{\frac{1}{\sigma}\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{\jmath} \\ +\partial_z\left(\partial_x \left\{\frac{-1}{\sigma}\left(\partial_x B_z - \partial_z B_x\right) - u \times B^0\right\}_y - \partial_y \left\{\frac{1}{\sigma}\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{k} \end{matrix} \right\}$$

For uniform properties, we have

$$\nabla \cdot b_i^{n+1} = -\Delta s \left\{ \begin{matrix} \partial_x\left(\partial_y \left\{\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \{-(\partial_x B_z - \partial_z B_x) - u \times B^0\}_y\right)\hat{\imath} \\ -\partial_y\left(\partial_x \left\{\left(\partial_x B_y - \partial_y B_x\right) - u \times B^0\right\}_z - \partial_z \left\{\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{\jmath} \\ +\partial_z\left(\partial_x \{-(\partial_x B_z - \partial_z B_x) - u \times B^0\}_y - \partial_y \left\{\left(\partial_y B_z - \partial_z B_y\right) - u \times B^0\right\}_x\right)\hat{k} \end{matrix} \right\}$$