3D amplification factor - verifying separable generalization

Using the result from the 2D and 3D Douglas ADI,

$$\left(I - \frac{\Delta t}{2}\alpha_x \Delta_x\right) u_{n+1}^{*} = \left(I + \frac{\Delta t}{2}\alpha_x \Delta_x + \Delta t \alpha_y \Delta_y + \Delta t \alpha_z \Delta_z\right) u_n - \Delta t f$$

$$\left(I - \frac{\Delta t}{2}\alpha_y \Delta_y\right) u_{n+1}^{**} = u_{n+1}^{*} - \frac{\Delta t}{2}\alpha_y \Delta_y u_n$$

$$\left(I - \frac{\Delta t}{2}\alpha_z \Delta_z\right) u_{n+1} = u_{n+1}^{**} - \frac{\Delta t}{2}\alpha_z \Delta_z u_n$$

For simplicity, let

$$D_\theta = \frac{\Delta t}{2}\alpha_\theta \Delta_\theta \quad ,\theta = x, y, z$$

Substitute for error and we have

$$(I - D_x) e_{n+1}^{*} = \left(I + D_x + 2D_y + 2D_z\right) e_n$$

$$\left(I - D_y\right) e_{n+1}^{**} = e_{n+1}^{*} - D_y e_n$$

$$(I - D_z) e_{n+1} = e_{n+1}^{**} - D_z e_n$$

Therefore

$$e_{n+1}^{*} = \frac{\left(I + D_x + 2D_y + 2D_z\right)}{(I - D_x)} e_n$$

$$e_{n+1}^{**} = \frac{\left\{\frac{\left(I + D_x + 2D_y + 2D_z\right)}{(I - D_x)} - D_y\right\}}{\left(I - D_y\right)} e_n$$

$$e_{n+1} = \frac{\left\{\frac{\left(I + D_x + 2D_y + 2D_z\right)}{(I - D_x)\left(I - D_y\right)} - \frac{D_y}{\left(I - D_y\right)} - D_z\right\}}{(I - D_z)} e_n$$

Or

$$\boxed{e_{n+1} = \left\{\frac{\left(I + D_x + 2D_y + 2D_z\right)}{(I - D_x)\left(I - D_y\right)(I - D_z)} - \frac{D_y}{\left(I - D_y\right)(I - D_z)} - \frac{D_z}{(I - D_z)}\right\} e_n}$$

Amplification factor

Let

$$G = \frac{e_{n+1}}{e_n}$$

And finding the **least common denominator**, we have

$$G = \frac{I + D_x + 2D_y + 2D_z - (I - D_x)D_y - (I - D_x)(I - D_y)D_z}{(I - D_x)(I - D_y)(I - D_z)}$$

Looking at only the numerator, we have

$$G_{num} = I + D_x + 2D_y + 2D_z - D_y + D_x D_y - \{I - D_y - D_x + D_x D_y\} D_z$$

$$= I + D_x + 2D_y + 2D_z - D_y + D_x D_y - D_z + D_y D_z + D_x D_z - D_x D_y D_z$$

$$= I + (D_x + D_y + D_z) + (D_x D_y + D_y D_z + D_x D_z) - (D_x D_y D_z)$$

Therefore our result looks like

$$\boxed{G = \frac{I + (D_x + D_y + D_z) + (D_x D_y + D_y D_z + D_x D_z) - (D_x D_y D_z)}{(I - D_x)(I - D_y)(I - D_z)}}$$

Desired form

I would like to write the numerator in the form

$$G_{num} = (I + D_x)(I + D_y)(I + D_z)$$

Expanding this out, we have

$$G_{num} = (I + D_x + D_y + D_x D_y)(I + D_z)$$

$$= (I + D_x + D_y + D_x D_y) + (D_z + D_x D_z + D_y D_z + D_x D_y D_z)$$

Or

$$= I + (D_x + D_y + D_z) + (D_x D_y + D_x D_z + D_y D_z) + (D_x D_y D_z)$$

If we assume that the 3rd order term is negligible, then we may write the original G in the form

$$G = \frac{(I + D_x)(I + D_y)(I + D_z)}{(I - D_x)(I - D_y)(I - D_z)} = G_x G_y G_z$$

But strictly speaking, this is not true.