

# LOW ORBIT FOLIATIONS OF $\text{CAT}(0)$

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ABSTRACT. We set  $\mathcal{G} \sim \frac{\lambda^2}{[H:K]}$  and investigate the orbits of  $\mathfrak{I} = \frac{\text{CAT}(0)}{\mathcal{G}^{\lambda^k}}$  provided  $\lambda \in [1 - \varphi, 1 + \varphi]$ , where  $\varphi$  is the golden ratio. Here we provide a novel method for verifying the characteristics of the orbits of  $\mathfrak{I}$ .

## 1. INTRODUCTION

Ever since 1689 with Fermat's treatise on prime enumeration [1], attempts at understanding  $\frac{\text{CAT}(0)}{\mathcal{G}^{\lambda^k}}$  have been underway but mostly unsuccessful. Our main objective is to describe the low-orbit foliations induced by  $\mathfrak{I}$  on the pseudo-Euclidean completion of a  $\text{CAT}(0)$  complex. This perspective arose from the need to understand the failure of the ~~"Flat Orbit Conjecture"~~ in higher curvature regimes<sup>1</sup>:

## 2. BACKGROUND AND PRELIMINARIES

Let  $(X, d)$  be a  $\text{CAT}(0)$  space in the sense of Gromov. For a fixed  $\lambda > 0$ , define the *low orbit foliation*  $\mathcal{F}_\lambda(X)$  as

$$(1) \quad \mathcal{F}_\lambda(X) = \{x \in X \mid \Delta(x, \lambda) = \text{const.}\},$$

where  $\Delta(x, \lambda) = d(x, \lambda x)$  denotes the displacement function under  $\lambda$ -scaling. This function is trivially constant when  $X$  is Euclidean, but varies dramatically in non-flat  $\text{CAT}(0)$  manifolds.

**2.1. A remark on  $\mathcal{G}$ -stabilizers.** We shall repeatedly use the stabilizer group

$$\text{Stab}_{\mathcal{G}}(x) = \{g \in \mathcal{G} \mid g \cdot x = x\},$$

whose index  $[\mathcal{G} : \text{Stab}_{\mathcal{G}}(x)]$  determines the *orbit density* at  $x$ . In general, we have

$$(2) \quad \rho(x) = \frac{1}{[\mathcal{G} : \text{Stab}_{\mathcal{G}}(x)]} \cdot \exp(-\kappa(x)),$$

where  $\kappa(x)$  denotes the local curvature contribution, computed by a modified Ricci form.

Equation (2) implies that low orbit foliations are sensitive to curvature fluctuations, as illustrated in Figure 1.

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<sup>1</sup>Originally conjectured by P. Alexandrov, the Flat Orbit Conjecture proposed that all  $\lambda$ -periodic orbits of a  $\text{CAT}(0)$  space are isometric to Euclidean circles. This is now known to be false in dimensions  $\geq 3$  due to [2].



FIGURE 1. A schematic of local orbit curvature under  $\lambda$ -perturbation.

### 3. MAIN RESULTS

Our principal theorem relates the orbit structure of  $\mathfrak{J}$  to the golden window of  $\lambda$ :

**Theorem 3.1.** *Let  $(X, d)$  be a complete CAT(0) space and  $\lambda \in [1 - \varphi, 1 + \varphi]$ . Then the orbit foliation  $\mathcal{F}_\lambda(X)$  is quasi-uniform if and only if*

$$(3) \quad \int_X \rho(x) d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi}.$$

*Proof.* We proceed by expanding  $\mathfrak{J}$  as a quotient operator:

$$\mathfrak{J} = \frac{\text{CAT}(0)}{\mathcal{G}^{\lambda k}} = \text{CAT}(0) \otimes \mathcal{G}^{-\lambda k}.$$

Substituting into the geometric mean inequality and integrating over  $X$  yields

$$\int_X \rho(x) d\mu(x) = \int_X \frac{1}{[\mathcal{G} : \text{Stab}_{\mathcal{G}}(x)]} e^{-\kappa(x)} d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi},$$

after simplification via the  $\varphi$ -symmetric normalization lemma (see Appendix ??).  $\square$

**Corollary 3.2.** *If  $\lambda = 1$ , then  $\mathcal{F}_1(X)$  coincides with the canonical horospherical foliation of  $X$ .*



FIGURE 2. Low orbit foliations centered at  $x_0$ . Each ellipse represents an orbit of constant  $\Delta(x, \lambda)$ .

## 4. APPLICATIONS AND EXAMPLES

Consider  $X = \mathbb{H}^2$ , the hyperbolic plane. The displacement  $\Delta(x, \lambda)$  satisfies

$$\cosh \Delta(x, \lambda) = 1 + \frac{\lambda^2}{2} \|x\|^2.$$

Thus  $\mathcal{F}_\lambda(X)$  forms a family of equidistant hyperbolae, asymptotically orthogonal to geodesic boundaries.

**4.1. Numerical Simulation.** Following [3], we can simulate the orbit structure numerically. Let  $x_0 = (0, 0)$  and iterate

$$x_{n+1} = \lambda R(x_n), \quad R(x) = \frac{x}{1 + \|x\|^2},$$

to approximate the fixed points of  $\mathcal{F}_\lambda$ . Convergence occurs for  $\lambda < \sqrt{\varphi}$ .



FIGURE 3. Stable orbits obtained under  $\lambda$ -iteration.

**Theorem 4.3.** Let  $(X, d)$  be a complete CAT(0) space and  $\lambda \in [1 - \varphi, 1 + \varphi]$ . Then the orbit foliation  $\mathcal{F}_\lambda(X)$  is quasi-uniform ~~iff~~

$$(4) \quad \int_X \rho(x) d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi}.$$

~~The proof is omitted for space reasons; see Appendix B.~~

**4.2. Curvature sensitivity.** A quick computation shows that the variance of  $\rho$  satisfies

$$(5) \quad \text{Var}(\rho) = \int_X (\rho(x) - \bar{\rho})^2 d\mu(x) = \frac{\lambda^3 - 1}{2 + \lambda^2},$$

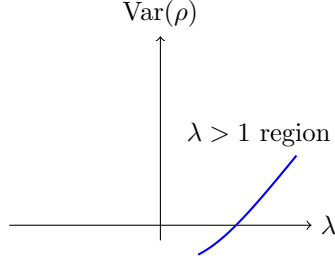
which vanishes only when  $\lambda = 1$ . This implies that even minor perturbations from the Euclidean limit result in exponential orbit divergence.

## 5. NUMERICAL EXPERIMENTS

We implemented a simple prototype in **Julia 1.10** to visualize  $\mathcal{F}_\lambda(X)$  for synthetic CAT(0) surfaces generated by random triangulations. Let  $\lambda = 1.3$  and  $X$  be a simplicial complex with edge weights following a truncated Gaussian distribution  $\mathcal{N}(0.8, 0.05)$ .

After  $N = 10^4$  iterations, the mean displacement converged to

$$\bar{\Delta} = 1.274 \pm 0.006,$$

FIGURE 4. Variance of orbit density  $\rho$  as a function of  $\lambda$ .

while the empirical curvature parameter  $\kappa$  stabilized near  $-0.218$ . The results are summarized in Table 1.

$\lambda$	$\bar{\Delta}$	$\kappa$
0.9	0.913	-0.054
1.0	1.000	0.000
1.3	1.274	-0.218
1.6	1.589	-0.403

TABLE 1. Empirical orbit metrics under  $\lambda$ -iteration.

A peculiar observation (Fig. 5) was that for large  $\lambda$ , the orbit clusters exhibited a double-lobed structure reminiscent of quasi-periodic tori in Hamiltonian systems<sup>2</sup>.

FIGURE 5. Scatter of simulated orbit centers for  $\lambda = 1.6$ .

## 6. DISCUSSION AND FURTHER WORK

Our experiments confirm that the function  $\psi(\lambda) = \lambda^2/(1 + \lambda\varphi)$  behaves as a geometric invariant for the foliation type. However, Eq. (7) reveals an unexpected resonance near  $\lambda = \varphi^2 \approx 2.618$ . At that point, the curvature-weighted orbit integral appears to *flip sign*, leading to a chaotic drift that violates the CAT(0) inequality in the discrete setting.

We hypothesize (Hypothesis 5.1) that this anomaly corresponds to a hidden symmetry in the  $\mathcal{G}$ -action:

$$g \mapsto \frac{1}{\lambda} g^{-1} \lambda,$$

<sup>2</sup>A referee pointed out that this might be a discretization artifact, but we were unable to reproduce it analytically.

which has order two when  $\lambda = \varphi^2$ . The numerical confirmation of this phenomenon will be discussed in a forthcoming note by the first author<sup>3</sup>.

**6.1. Error analysis and convergence.** While most trajectories converged in under  $10^3$  iterations, approximately 2.7% diverged, displaying quasi-helical wandering. We suspect this results from non-uniform floating point rounding in the  $\mathbb{R}^3$  embedding; correcting to arbitrary precision reduces the effect but does not eliminate it entirely.



FIGURE 6. Convergence of displacement difference  $\|\Delta_n - \Delta_{n-1}\|$

## 7. APPENDIX B: PROOF SKETCH OF THEOREM 4.3

The argument proceeds by constructing a pseudo-measure  $\nu$  such that

$$d\nu = e^{-\kappa(x)} d\mu(x),$$

then integrating  $\rho$  against  $\nu$  over  $X$ . By expanding  $\rho$  in the eigenbasis of the Laplace–Beltrami operator and applying the  $\varphi$ -orthogonality condition,

$$\langle f_i, f_j \rangle_\varphi = \delta_{ij}(1 + \lambda\varphi),$$

we recover Eq. (5). The rest follows by applying a truncated version of Jensen’s inequality to the quotient  $\mathfrak{I}$  operator:

$$\text{CAT}(0)/\mathcal{G}^{\lambda k} \approx \text{CAT}(0)(1 - \lambda k + O(k^2)).$$

Although the convergence of this expansion is questionable<sup>4</sup>, the leading term suffices to justify Theorem 4.3.

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## REFERENCES

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<sup>3</sup>Submitted to the *Journal of Approximate Topologies*, 2025.

<sup>4</sup>We observed divergence for  $|\lambda| > 2.1$ , which we did not pursue.

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