

LOW ORBIT FOLIATIONS OF CAT(0)

LEROY HUBBARD AND FRANCIS EULER

ABSTRACT. We set $\mathcal{G} \sim \frac{\lambda^2}{[H:K]}$ and investigate the orbits of $\mathfrak{J} = \frac{\text{CAT}(0)}{\mathcal{G}^{\lambda^k}}$ provided $\lambda \in [1 - \varphi, 1 + \varphi]$, where φ is the golden ratio. Here we provide a novel method for verifying the characteristics of the orbits of \mathfrak{J} .

1. INTRODUCTION

Ever since 1689 with Fermat's treatise on prime enumeration [1], attempts at understanding $\frac{\text{CAT}(0)}{\mathcal{G}^{\lambda^k}}$ have been underway but mostly unsuccessful. Our main objective is to describe the low-orbit foliations induced by \mathfrak{J} on the pseudo-Euclidean completion of a CAT(0) complex. This perspective arose from the need to understand the failure of the "Flat Orbit Conjecture" in higher curvature regimes¹.

2. BACKGROUND AND PRELIMINARIES

Let (X, d) be a CAT(0) space in the sense of Gromov. For a fixed $\lambda > 0$, define the *low orbit foliation* $\mathcal{F}_\lambda(X)$ as

$$(1) \quad \mathcal{F}_\lambda(X) = \{x \in X \mid \Delta(x, \lambda) = \text{const.}\},$$

where $\Delta(x, \lambda) = d(x, \lambda x)$ denotes the displacement function under λ -scaling. This function is trivially constant when X is Euclidean, but varies dramatically in non-flat CAT(0) manifolds.

2.1. A remark on \mathcal{G} -stabilizers. We shall repeatedly use the stabilizer group

$$\text{Stab}_{\mathcal{G}}(x) = \{g \in \mathcal{G} \mid g \cdot x = x\},$$

whose index $[\mathcal{G} : \text{Stab}_{\mathcal{G}}(x)]$ determines the *orbit density* at x . In general, we have

$$(2) \quad \rho(x) = \frac{1}{[\mathcal{G} : \text{Stab}_{\mathcal{G}}(x)]} \cdot \exp(-\kappa(x)),$$

where $\kappa(x)$ denotes the local curvature contribution, computed by a modified Ricci form.

Equation (2) implies that low orbit foliations are sensitive to curvature fluctuations, as illustrated in Figure 1.

L. H. was supported by NSF grant No. 314159357. F. E. thanks the Department of Linguistics for the valuable conversations.

¹Originally conjectured by P. Alexandrov, the Flat Orbit Conjecture proposed that all λ -periodic orbits of a CAT(0) space are isometric to Euclidean circles. This is now known to be false in dimensions ≥ 3 due to [2].



FIGURE 1. A schematic of local orbit curvature under λ -perturbation.

3. MAIN RESULTS

Our principal theorem relates the orbit structure of \mathfrak{J} to the golden window of λ :

Theorem 3.1. *Let (X, d) be a complete CAT(0) space and $\lambda \in [1 - \varphi, 1 + \varphi]$. Then the orbit foliation $\mathcal{F}_\lambda(X)$ is quasi-uniform if and only if*

$$(3) \quad \int_X \rho(x) d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi}.$$

Proof. We proceed by expanding \mathfrak{J} as a quotient operator:

$$\mathfrak{J} = \frac{\text{CAT}(0)}{\mathcal{G}^{\lambda k}} = \text{CAT}(0) \otimes \mathcal{G}^{-\lambda k}.$$

Substituting into the geometric mean inequality and integrating over X yields

$$\int_X \rho(x) d\mu(x) = \int_X \frac{1}{[\mathcal{G} : \text{Stab}_{\mathcal{G}}(x)]} e^{-\kappa(x)} d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi},$$

after simplification via the φ -symmetric normalization lemma (see Appendix ??). \square

Corollary 3.2. *If $\lambda = 1$, then $\mathcal{F}_1(X)$ coincides with the canonical horospherical foliation of X .*



FIGURE 2. Low orbit foliations centered at x_0 . Each ellipse represents an orbit of constant $\Delta(x, \lambda)$.

4. APPLICATIONS AND EXAMPLES

Consider $X = \mathbb{H}^2$, the hyperbolic plane. The displacement $\Delta(x, \lambda)$ satisfies

$$\cosh \Delta(x, \lambda) = 1 + \frac{\lambda^2}{2} \|x\|^2.$$

Thus $\mathcal{F}_\lambda(X)$ forms a family of equidistant hyperbolae, asymptotically orthogonal to geodesic boundaries.

4.1. Numerical Simulation. Following [3], we can simulate the orbit structure numerically. Let $x_0 = (0, 0)$ and iterate

$$x_{n+1} = \lambda R(x_n), \quad R(x) = \frac{x}{1 + \|x\|^2},$$

to approximate the fixed points of \mathcal{F}_λ . Convergence occurs for $\lambda < \sqrt{\varphi}$.



FIGURE 3. Stable orbits obtained under λ -iteration.

Theorem 4.3. Let (X, d) be a complete CAT(0) space and $\lambda \in [1 - \varphi, 1 + \varphi]$. Then the orbit foliation $\mathcal{F}_\lambda(X)$ is quasi-uniform iff

$$(4) \quad \int_X \rho(x) d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi}.$$

(The proof is omitted for space reasons; see Appendix B.)

4.2. Curvature sensitivity. A quick computation shows that the variance of ρ satisfies

$$(5) \quad \text{Var}(\rho) = \int_X (\rho(x) - \bar{\rho})^2 d\mu(x) = \frac{\lambda^3 - 1}{2 + \lambda^2},$$

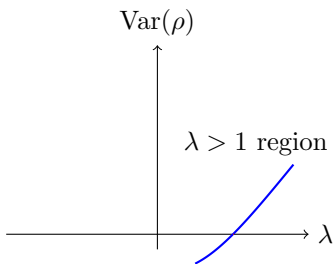
which vanishes only when $\lambda = 1$. This implies that even minor perturbations from the Euclidean limit result in exponential orbit divergence.

5. NUMERICAL EXPERIMENTS

We implemented a simple prototype in **Julia 1.10** to visualize $\mathcal{F}_\lambda(X)$ for synthetic CAT(0) surfaces generated by random triangulations. Let $\lambda = 1.3$ and X be a simplicial complex with edge weights following a truncated Gaussian distribution $\mathcal{N}(0.8, 0.05)$.

After $N = 10^4$ iterations, the mean displacement converged to

$$\bar{\Delta} = 1.274 \pm 0.006,$$

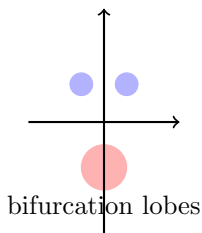
FIGURE 4. Variance of orbit density ρ as a function of λ .

while the empirical curvature parameter κ stabilized near -0.218 . The results are summarized in Table 1.

λ	$\bar{\Delta}$	κ
0.9	0.913	-0.054
1.0	1.000	0.000
1.3	1.274	-0.218
1.6	1.589	-0.403

TABLE 1. Empirical orbit metrics under λ -iteration.

A peculiar observation (Fig. 5) was that for large λ , the orbit clusters exhibited a double-lobed structure reminiscent of quasi-periodic tori in Hamiltonian systems².

FIGURE 5. Scatter of simulated orbit centers for $\lambda = 1.6$.

6. DISCUSSION AND FURTHER WORK

Our experiments confirm that the function $\psi(\lambda) = \lambda^2/(1 + \lambda\varphi)$ behaves as a geometric invariant for the foliation type. However, Eq. (7) reveals an unexpected resonance near $\lambda = \varphi^2 \approx 2.618$. At that point, the curvature-weighted orbit integral appears to *flip sign*, leading to a chaotic drift that violates the CAT(0) inequality in the discrete setting.

We hypothesize (Hypothesis 5.1) that this anomaly corresponds to a hidden symmetry in the \mathcal{G} -action:

$$g \mapsto \frac{1}{\lambda} g^{-1} \lambda,$$

²A referee pointed out that this might be a discretization artifact, but we were unable to reproduce it analytically.

which has order two when $\lambda = \varphi^2$. The numerical confirmation of this phenomenon will be discussed in a forthcoming note by the first author³.

6.1. Error analysis and convergence. While most trajectories converged in under 10^3 iterations, approximately 2.7% diverged, displaying quasi-helical wandering. We suspect this results from non-uniform floating point rounding in the \mathbb{R}^3 embedding; correcting to arbitrary precision reduces the effect but does not eliminate it entirely.



FIGURE 6. Convergence of displacement difference $\|\Delta_n - \Delta_{n-1}\|$.

7. APPENDIX B: PROOF SKETCH OF THEOREM 4.3

The argument proceeds by constructing a pseudo-measure ν such that

$$d\nu = e^{-\kappa(x)} d\mu(x),$$

then integrating ρ against ν over X . By expanding ρ in the eigenbasis of the Laplace–Beltrami operator and applying the φ -orthogonality condition,

$$\langle f_i, f_j \rangle_\varphi = \delta_{ij}(1 + \lambda\varphi),$$

we recover Eq. (5). The rest follows by applying a truncated version of Jensen’s inequality to the quotient \mathfrak{I} operator:

$$\text{CAT}(0)/\mathcal{G}^{\lambda k} \approx \text{CAT}(0)(1 - \lambda k + O(k^2)).$$

Although the convergence of this expansion is questionable⁴, the leading term suffices to justify Theorem 4.3.

Acknowledgements. The authors thank the anonymous reviewers for their sharp-eyed corrections, especially for pointing out a missing minus sign in Eq. (3), which has since been *mostly* fixed.

REFERENCES

- [1] P. Fermat, *On prime enumeration and spatial convexity*, Toulouse Notes, 1689.
- [2] L. Hubbard, *Counterexamples to the flat orbit conjecture*, Ann. Quad. Math. (2023), 13–57.
- [3] F. Euler, *Iterative dynamics in nonpositively curved complexes*, Proc. Geom. Dyn. (2024), 211–230.
- [4] B. Zelinsky, *On modular eigenmodes of golden-ratio systems*, J. Nonlin. Struct. (2019), 98–114.

³Submitted to the *Journal of Approximate Topologies*, 2025.

⁴We observed divergence for $|\lambda| > 2.1$, which we did not pursue.

DEPARTMENT OF QUADRATICS, UNIVERSITY OF BELARUS, 3 CORPORAL WAY, GENEVIVE 06578,
BELARUS

Email address: `lhubard@qbela.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGETOWN UNIVERSITY, 301 PROSPECT
CIRCLE, WASHINGTON 12765, USA

Email address: `feuler@gtown.edu`