- We say a bijection $\alpha : \pi_0(T_{1,1}) \to \pi_0(T_{2,1})$ is *order-preserving* if it coincides with the bijection obtained by recording the orders of appearance of the components of $T_{i,1}$ along T_i from a_i to b_i.

Remark 8.11. Using the above notation, note that an order-preserving bijection $\alpha : \pi_0(T_{1,1}) \to \pi_0(T_{2,1})$ automatically induces an order-preserving bijection $\beta : \pi_0(T_{1,2}) \to \pi_0(T_{2,2})$. Moreover, in this paper, we will only be working with intervals between a pair of points, which are usually denoted by (some form of) a, b. We will always take order-preserving to mean as one moves from the a-side to the b-side of the given intervals.

We also set one last bit of notation: Given a stable tree $T = T_e \cup T_c$ for an ϵ-setup $(\mathcal{Y}; \{a,b\})$, if $y \in \mathcal{Y} \cup \{a,b\}$, we let C_y denote the cluster containing y, and $\mu(C_y)$ the corresponding component of T^c_c.

The following definition contains the stability properties we want:

Definition 8.12 (Stable decomposition)**.** Let $\mathcal{Z}$ be δ-hyperbolic and geodesic.

(1) Given $N > 0$, two ϵ-setups $(a,b;\mathcal{Y})$ and $(a',b';\mathcal{Y}')$ are (N,ϵ)-*admissible* if
  (a) $d_{\mathcal{Z}}(a,a'), d_{\mathcal{Z}}(b,b') \le \epsilon$,
  (b) $\mathcal{Y}, \mathcal{Y}' \subset \mathcal{N}_{\epsilon/2}(\lambda(a,b)) \cap \mathcal{N}_{\epsilon/2}(\lambda(a',b'))$,
  (c) $|\mathcal{Y} \triangle \mathcal{Y}'| < N$.
(2) Given an ϵ-setup $(a,b;\mathcal{Y})$, an *edge decomposition* of its stable interval $T = T_e \cup T_c$ is a collection of subintervals $T_s \subset T_e$.
(3) Given $L > 0$, $\mathcal{Y}_0 \subset \mathcal{Y} \cap \mathcal{Y}'$, and two (N,ϵ)-admissible setups $(a,b;\mathcal{Y})$ and $(a',b';\mathcal{Y}')$, we say that two edge decompositions $T_s \subset T_e$ and $T'_s \subset T'_e$ are $\mathcal{Y}_0$-*stably L-compatible* if
  (a) Each component of T_s and T'_s has positive integer length with endpoints at vertices of T_e and T'_e, respectively.
  (b) There is an order-preserving bijection $\alpha : \pi_0(T_s) \to \pi_0(T'_s)$ between the sets of stable components.
  (c) For each *stable pair* (E,E') identified by α, there exists an order-preserving isometry $i_{E,E'} : E \to E'$.
  (d) For all but at most L pairs of stable components (E,E'), we have $\phi(E) = \phi(E')$ are identical with $\phi(x) = \phi'(i_{E,E'}(x))$ for all $x \in E$.
  (e) For the (at most) L-many remaining stable pairs (E,E'), we have $d_{\mathcal{Z}}(\phi(x), \phi'(i_{E,E'}(x))) < L$ for all $x \in E$.
  (f) The complements $T_e - T_s$ and $T'_e - T'_s$ consist of at most L-many *unstable components* of diameter at most L.
  (g) The induced order-preserving bijection $\beta : \pi_0(T - T_s) \to \pi_0(T' - T'_s)$ satisfies:
    (i) (Endpoints) Let D_a denote the component of $T - T_s$ containing $\mu(C_a)$, and define $D_b, D_{a'}, D_{b'}$ similarly. Then $\beta(D_a) = D_{a'}$ and $\beta(D_b) = D_{b'}$.
    (ii) (Identifying clusters) For any $y \in \mathcal{Y}_0$, let D_y, D'_y denote the components of $T - T_s$ and $T' - T'_s$ containing $\mu(C_y), \mu(C'_y)$, respectively. Then $\beta(D_y) = D'_y$.

Remark 8.13. The above definition is a refinement of the properties contained in [DMS20, Stable Tree Theorem 3.2], where we have set up these stable decompositions to plug nicely into the cubical model machinery described earlier in the paper. How this process works is described below in more detail in Subsection 9.3;

but for now we motivate some parts of the definition. Roughly speaking, the points $\mathcal{Y}$ and $\mathcal{Y}'$ stand in for the relative projection data on each interval, with each component of $T_e - T_s$ and $T'_e - T'_s$ being labeled by some such point. Eventually, we will collapse each of the components of $T_e - T_s$ and $T'_e - T'_s$ to points. Item (3g) says that the common labels $\mathcal{Y} \cap \mathcal{Y}'$ are identified, as well as the endpoint pairs a, a' and b, b'. The rest of the statement now says that the components of T_s and T'_s are in bijection with each other, where identified components are isometric, and the order-preserving properties explain how to glue these components to obtain an isometry between collapsed intervals. Finally, the proximity properties in items (3d) and (3e) are necessary for controlling the maps back into the ambient HHS; see the proof of Theorem 9.9. A more general version can be found in [DMS25, Definition 10.18].

8.5. **The stable interval theorem.** With Definition 8.12 in hand, we can now state the main result of this section, Theorem 8.14 below. It says that the stable intervals for a pair of admissible ϵ-setups admit compatible stable decompositions—that is, decompositions into subintervals which are in bijective correspondence with identified pairs uniformly close, up to ignored a bounded collection of bounded-length subintervals. We first state this technical result, and then prove a corollary which gives the main upshot for our purposes of building stable cubical models.

For a bit of setup, recall the notion of a thickening of an interval from Subsection 4.4. This was a way of taking an interval with a decomposition $T = A \cup B$ into collections of segments, and expanding and combining one collection of the segments. Our stable intervals come with such a decomposition $T = T_e \cup T_c$, and we will always mean a thickening of a stable interval T to be a thickening of T along the components of the cluster forest T_c. Finally, we denote the components of the resulting thickening by $T = \mathbb{T}_e \cup \mathbb{T}_c$, and we note that $\mathbb{T}_e \subset T_e$ and $T_c \subset \mathbb{T}_c$.

Theorem 8.14 (Stable intervals). *Let $\mathcal{Z}$ be δ-hyperbolic and geodesic. For any $\epsilon, N > 0$ and positive integers $r_1, r_2 > 0$, there exist $L_1 = L_1(\delta, \epsilon, N) > 0$ and $L_2 = L_2(\delta, \epsilon, N, r_1, r_2) > 0$ so that the following holds. Suppose $(a, b; \mathcal{Y})$ and $(a', b'; \mathcal{Y}')$ are (N, ϵ)-admissible ϵ-setups and let $T = T_e \cup T_c$ and $T' = T'_e \cup T'_c$ denote their stable intervals. Then*

1. *There exist $(\mathcal{Y} \cap \mathcal{Y}')$-stable L_1-compatible decompositions $T_s \subseteq T_e$ and $T'_s \subseteq T'_e$, and*
2. *There exist $(\mathcal{Y} \cap \mathcal{Y}')$-stable L_2-compatible edge decompositions $\mathbb{T}_s \subseteq \mathbb{T}_e$ and $\mathbb{T}'_s \subseteq \mathbb{T}'_e$ of the (r_1, r_2)-thickenings of T, T' along T_c, T'_c.*
   - *Moreover, we have $\mathbb{T}_s \subseteq T_s$ and $\mathbb{T}'_s \subseteq T'_s$.*

Remark 8.15. In the HHS setting, we will be able to control each of the constants $\delta, \epsilon, N, r_1, r_2$ in terms of the ambient HHS structure. We have written the statement with two conclusions because the first conclusion is a mild reformulation of the original theorem [DMS20, Theorem 3.2], while the second is what we actually need for the cubulation machine discussed in this paper. Notably, the second statement will be an easy consequence of the first.

Before we move onto the proof, we observe the following corollary, whose statement and proof motivate the statement of Theorem 8.14. In particular, this corollary will allow us to verify the interval-wise condition in Proposition 6.12 and give us the cubical isomorphism we need for the Stable Cubulations Theorem 9.9; see Subsection 9.3.