RATIONAL COHOMOLOGY ENDOMORPHISMS OF PRODUCT OF SPHERE WITH GRASSMANNIAN AND COINCIDENCE THEORY

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Abstract. We classified graded endomorphisms of the rational cohomology algebra of the product of a sphere and a complex Grassmannian, whose images are nonzero in the second cohomology of the Grassmannian.

We also derive necessary conditions for the generalized Dold spaces to satisfy the coincidence property, in particular the fixed-point property. As an application of our results, we obtain several sufficient conditions for the existence of a point of coincidence between a pair of continuous functions on certain generalized Dold spaces.

1. Introduction

The classification of endomorphisms of the rational cohomology algebra of formal spaces was greatly motivated by Sullivan's theory where it was proved that rational homotopy class of self-maps are completely determined by the induced graded endomorphisms of their rational cohomology algebras.

In [B], the authors developed the foundational work by classifying automorphisms of the rational cohomology algebra of complex Grassmannian. Their results were generalized in [Ho1], where the author classified graded endomorphisms of the rational cohomology algebra of complex Grassmannian which are nonzero on dimension 2. Further, he conjectured that every graded endomorphism vanishing on dimension two is necessarily trivial. This conjecture was proved in [GH1] for several cases.

The cohomology endomorphisms are also studied for a variety of homogeneous spaces G/H, where G is a compact connected Lie group and H is a closed subgroup of maximal rank. This is a topic of interest since past fifty years and are studied in several papers [ST2, BH, HH, P, Du, DF, DZ, L, GS].

However, the behavior of cohomology endomorphisms for the product spaces is comparatively less explored, and this provides a direction for study. We are mainly interested in the product of a sphere with complex Grassmannian $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ because the sphere $\mathbb{S}^m$ has a simple, singly generated cohomology algebra, while the Grassmannian $\mathbb{C}G_{n,k}$ (of k-planes in $\mathbb{C}^n$) carries the rich structure arising from Schubert calculus. We have classified graded endomorphisms of the rational cohomology algebra $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$, whose image has a nonzero component in $H^2(\mathbb{C}G_{n,k},\mathbb{Q})$. As an application we obtain useful results in coincidence theory and in particular, fixed-point theory.

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The rational cohomology algebra of the product

$$H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}) \cong H^*(\mathbb{S}^m,\mathbb{Q}) \otimes H^*(\mathbb{C}G_{n,k};\mathbb{Q})$$

is generated by $u, c_1, c_2, \ldots, c_k$, where $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$ (resp. $H^*(\mathbb{S}^m;\mathbb{Q})$) is generated by certain Chern classes $c_1, \ldots, c_k$ (resp. u). Now, we are ready to state one of the main results of our paper which proves that the rigidity of $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$ persists in $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$ even in the presence of spherical cohomology classes.

Theorem 1.1. *Let ϕ be a graded endomorphism of $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$ satisfying $\phi(c_1) \neq au$, $a \in \mathbb{Q}$. Then there exists a nonzero rational λ such that the following holds.*

(1) If $k < n-k$,

$$\phi(c_i) = \lambda^i c_i, \forall i \in \{1, 2, \ldots, k\}$$

If $k = n-k$, there is an additional possibility of ϕ that it is induced by the homeomorphism

$$\mathbb{C}G_{2k,k} \longrightarrow \mathbb{C}G_{2k,k}, \quad L \longmapsto L^{\perp},$$

where $L^{\perp}$ denotes the orthogonal complement of the k-plane L in $\mathbb{C}^{2k}$.

(2) The image of $H^*(\mathbb{S}^m;\mathbb{Q})$ under ϕ lies in $H^*(\mathbb{S}^m;\mathbb{Q})$ or in $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$ i.e.

$$\phi(u) = \mu u,\ \mu \in \mathbb{Q},\ \textit{ or } \phi(u) \in H^*(\mathbb{C}G_{n,k};\mathbb{Q}),\ \textit{ with } (\phi(u))^2 = 0.$$

Unlike the case of the complex Grassmannian, we cannot expect a graded endomorphism of $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$ to be trivial merely because it vanishes in $H^2(\mathbb{C}G_{n,k};\mathbb{Q})$. In fact, we proved that for any choice of $P_i \in H^{2i-m}(\mathbb{C}G_{n,k};\mathbb{Q})$ and $Q \in \mathbb{Q}u \cup H^*(\mathbb{C}G_{n,k};\mathbb{Q})$ with $Q^2 = 0$, there exist a graded endomorphism ϕ on $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$ such that $\phi(c_i) = uP_i$, $\forall i$ and $\phi(u) = Q$. We also proved that if a continuous function on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ stabilizes a copy of Grassmannian then the induced cohomology endomorphism stabilizes the subalgebra $H^*(\mathbb{S}^m;\mathbb{Q})$.

Our study is also motivated by the theory of generalized Dold spaces because the product space $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ is a double cover of certain generalized Dold spaces (GDS). The classical Dold manifolds were introduced in [Do] to construct odd-dimensional generators for Thom's unoriented cobordism ring. In this paper, we are interested in the GDS introduced in [NS] and defined as

$$P(m,n,k) := \mathbb{S}^m \times \mathbb{C}G_{n,k}/\!\sim, \text{ where } (s, L) \sim (-s, \bar{L}).$$

As an application of Theorem 1.1, we describe endomorphisms of $H^*(P(m,n,k);\mathbb{Q})$ induced by continuous functions on $P(m,n,k)$. Using this description, we prove that every automorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function on $P(m,n,k)$ lifts to an automorphism of $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$ if $n > 2$.

Our broader aim is to apply Theorem 1.1 to obtain results in coincidence theory. Coincidence theory has been extensively studied in [Ho2, GH2, W]. A pair (X, g) where g is a continuous function on X is said to have the coincidence property if g has a point of coincidence with every continuous function on X. In particular, if g is the identity map, then the coincidence property is same as the fixed-point property of X.

We have generalized Theorem 2 of [GH1] to the setting of coincidence theory and proved that the pair $(\mathbb{C}G_{n,k}, g)$ satisfies the coincidence property if $k(n-k)$ is even and g has nonzero Brouwer degree. To conclude, using the Lefschetz Coincidence Theorem and Theorem 1.1, we obtained multiple situations when two continuous functions on the generalized Dold space $P(m,n,k)$ are guaranteed to have a point of coincidence, and found certain pairs $(P(m,n,k), g)$ satisfying the coincidence property.

The paper is organized as follows:
In Section 2 we develop the necessary background and recall some relevant results. Section 3 is devoted to the study of graded endomorphisms of the rational cohomology algebra of $\mathbb{S}^m \times \mathbb{C}G_{n,k}$, from which we extract several consequences. These are applied in Section 4 to obtain the coincidence-theoretic results.

2. Preliminaries

In this section, we discuss some preliminaries and recall some results that will be required to proceed with our study.

2.1. **Cohomology of complex Grassmannians.** Let $\mathbb{C}G_{n,k}$ denote the complex Grassmannian consisting of complex k-planes in $\mathbb{C}^n$. Let $\gamma_{n,k}$ and $\beta_{n,k}$ denote the canonical complex k-plane and $(n-k)$-plane bundles, respectively, over $\mathbb{C}G_{n,k}$. Let the total Chern classes of the vector bundles $\gamma_{n,k}$ and $\beta_{n,k}$ be denoted by $c(\gamma_{n,k}) = c$ and $c(\beta_{n,k}) = \bar{c}$, respectively. Thus,

$$c = 1 + c_1 + c_2 + \cdots + c_k, \quad \bar{c} = 1 + \bar{c}_1 + \bar{c}_2 + \cdots + \bar{c}_{n-k},$$

where c_i and $\bar{c}_i$ denote the i-th Chern classes of $\gamma_{n,k}$ and $\beta_{n,k}$, respectively. Since $\gamma_{n,k} \oplus \beta_{n,k} \cong \varepsilon^n_{\mathbb{C}}$, it follows that $c \cdot \bar{c} = 1$. The cohomology ring of the complex Grassmannian is well known and given by

$$H^*_{\mathbb{C}G} := H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \ldots, c_k, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{n-k}]/\langle h_r : 1 \leq r \leq n\rangle,$$

where the relations h_r for $r = 1, 2, \ldots, n$ are induced from the homogeneous parts of the equation $c \cdot \bar{c} = 1$ and given by

$$h_r := \sum_{i+j=r} c_i \bar{c}_j.$$

Using the relations $h_r, r = 1, 2, ..., n-k$, the generators $\bar{c}_i$ for $i = 1, 2, \ldots, n-k$ can be expressed inductively in terms of c_i for $i = 1, 2, \ldots, k$. Consequently, the relations h_r for $r = n-k+1, \ldots, n$ become homogeneous polynomials in c_i of degree $2r$, where the degree of each c_i is $2i$. Then the cohomology ring $H^*_{\mathbb{C}G}$ can be rewritten as

$$\mathbb{Q}[c_1, c_2, \ldots, c_k]/\langle h_{n-k+1}, h_{n-k+2}, \ldots, h_n\rangle. \tag{1}$$

Since there are no relations among the generators c_i for $i = 1, 2, \ldots, k$ up to degree $2(n-k)$, the set of all monomials of degree $2r$ in terms of $c_1, c_2, \ldots, c_k$ forms a $\mathbb{Q}$-basis of $H^{2r}(\mathbb{C}G_{n,k}; \mathbb{Q})$ for $r \leq n-k$.

From now on, we denote the indexing set $\{1, 2, \ldots, k\}$ by I.

Remark 2.1. *We can assume $k \leq n-k$ for $\mathbb{C}G_{n,k}$ as $\mathbb{C}G_{n,k}$ is homeomorphic to $\mathbb{C}G_{n,n-k}$ by using orthogonal complementation.*

The complex Grassmannian $\mathbb{C}G_{n,k}$ is a homogeneous space and can be represented as the quotient of the unitary group $U(n)$ by the stabilizer subgroup $U(k) \times U(n-k)$ that is

$$\mathbb{C}G_{n,k} = U(n)/(U(k) \times U(n-k)). \tag{2}$$

Now we recall a result given in [ST1].

Theorem 2.2 ([ST1], Theorem A')**.** *Let $D_i(H^*(G/H;\mathbb{Q}))$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-derivations of $H^*(G/H;\mathbb{Q})$ which decreases the degree by $i>0$ where G is a connected, compact Lie group and H is a closed subgroup of maximal rank. Then, for all i,*

$$D_i(H^*(G/H;\mathbb{Q}))=0.$$

2.2. **Graded endomorphisms on $\mathbf{H}^*_{\mathbb{C}\mathbf{G}}$.** It was conjectured in [O] that any graded endomorphism ϕ of the cohomology algebra $H^*_{\mathbb{C}G}$ is an *Adams* map when $k<n-k$; that is, there exists a rational λ such that $\phi(c_i)=\lambda^i c_i$, for all $i\in I$. Glover and Homer (see [GH1]) and Hoffman (see [Ho1]) proved the conjecture under the following hypothesis respectively:

$$\text{Either } k\le 3 \text{ and } n>2k, \text{ or } k>3 \text{ and } n>2k^2-1. \tag{3}$$

$$\text{The graded endomorphism } \varphi \text{ of } H^*_{\mathbb{C}G} \text{ satisfies } \varphi(c_1)=\lambda c_1, \lambda\neq 0. \tag{4}$$

Let us recall those results proved in [GH1, Ho1] that will be used in the rest of this paper.

Theorem 2.3 ([GH1], Theorem 1, [Ho1], Theorem 1.1)**.** *(i) Assume that the hypothesis (3) is satisfied. Then for every graded endomorphism φ on $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$, there exists a rational λ such that*

$$\varphi(c_i)=\lambda^i c_i,\quad \forall i\in I.$$

(ii) Assume that the hypothesis (4) is satisfied. Then, we have

$$\varphi(c_i)=\begin{cases}\lambda^i c_i, \forall i\in I & \textit{if } k<n-k,\\ \lambda^i c_i, \forall i\in I \quad \textit{or} \quad (-\lambda)^i(c^{-1})_i, \forall i\in I & \textit{if } k=n-k,\end{cases}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c=1+c_1+\cdots+c_k$ in $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$.

2.3. **Generalized Dold spaces.** In [Do], the author introduced the notion of *classical Dold manifolds* $P(m,n):=\mathbb{S}^m\times\mathbb{C}P^n/\sim$ where $(s,L)\sim(-s,\bar{L})$, where the involution $L\mapsto\bar{L}$ on $\mathbb{C}G_{n,k}$ is induced from the standard conjugation on $\mathbb{C}^n$, to construct generators in odd dimensions for René Thom's unoriented cobordism ring.

In [NS, MS1], the authors generalized the notion of classical Dold manifolds by replacing the sphere $\mathbb{S}^m$ with an arbitrary topological space S equipped with a free involution α, analogous to the antipodal map on $\mathbb{S}^m$, and $\mathbb{C}P^n$ with an arbitrary topological space X with an involution $\sigma: X\to X$ having a nonempty fixed-point set, analogously to complex conjugation on $\mathbb{C}P^n$. Then the quotient space

$$P(S,\alpha,X,\sigma):=S\times X/\sim,\quad \text{where } (s,x)\sim(\alpha(s),\sigma(x)), \tag{5}$$

is called *generalized Dold space* (in short GDS), often denoted simply as $P(S,X)$. Moreover, the quotient map $S\times X\to P(S,X)$ is a double covering map.

Let us fix a notation Y for $S/\sim_\alpha$, where $s\sim_\alpha \alpha(s), \forall s\in S$. Then, a GDS $P(S,X)$ is the total space of a fiber bundle $X\hookrightarrow P(S,X)\twoheadrightarrow Y$, where the fiber bundle projection is

$$p: P(S,X)\twoheadrightarrow Y,\quad [s,x]\mapsto[s]. \tag{6}$$

Choosing a fixed-point of σ, say $x_0\in \mathrm{Fix}(\sigma)\neq\emptyset$, we can construct a section of the fiber bundle

$$s: Y\hookrightarrow P(S,X),\quad [s]\mapsto[s,x_0]. \tag{7}$$

In fact, we have an embedding $Y \times \mathrm{Fix}(\sigma) \hookrightarrow P(S, X)$, where $\mathrm{Fix}(\sigma) \subseteq X$ has the subspace topology induced from X.

2.4. **Rational cohomology of $\mathbf{P}(\mathbb{S}^{\mathbf{m}}, \mathbb{C}\mathbf{G}_{\mathbf{n},\mathbf{k}})$.** The GDS $P(\mathbb{S}^m, \mathbb{C}G_{n,k})$ is defined as

$$\mathbb{S}^m \times \mathbb{C}G_{n,k}/\sim, \text{ where } (s, L) \sim (-s, \bar{L}),$$

for which, $\mathbb{S}^m$ is equipped with the free action generated by the antipodal map α and the involution $\sigma : L \mapsto \bar{L}$ on $\mathbb{C}G_{n,k}$ is induced from the standard complex conjugation on $\mathbb{C}^n$. We denote $P(\mathbb{S}^m, \mathbb{C}G_{n,k})$ simply by $P(m, n, k)$. By the Künneth formula, we have

(8)

$$H^*_\times := H^*(\mathbb{S}^m \times \mathbb{C}\mathbb{G}_{n,k};\mathbb{Q}) \cong H^*(\mathbb{S}^m;\mathbb{Q}) \otimes H^*(\mathbb{C}\mathbb{G}_{n,k};\mathbb{Q}) \cong \frac{\mathbb{Q}[u, c_1, \ldots, c_k]}{\langle u^2, h_{n-k+1}, \ldots, h_n\rangle}$$

where $u \in H^m(\mathbb{S}^m;\mathbb{Q})$ denotes the generator corresponding to the fundamental class of $\mathbb{S}^m$. Note that

$$H^*_\times \cong H^*_{\mathbb{C}G}[u]/\langle u^2\rangle \cong H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G}, \tag{9}$$

where the latter isomorphism is a $\mathbb{Q}$-module isomorphism. We have that $H^*_{\mathbb{C}G}$ is a subring of $H^*_\times$. The product involution $\theta := \alpha \times \sigma$ on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ induces an involution θ^* on $H^*_\times$ given by

$$\theta^*(c_i) = (-1)^i c_i, i \in I, \quad \theta^*(u) = \begin{cases} u, & \text{if } m \text{ is odd,} \\ -u, & \text{if } m \text{ is even.} \end{cases} \tag{10}$$

The cohomology ring $H^*(P(m, n, k);\mathbb{Q})$ was computed in [MS2] and the following result was proved.

Theorem 2.4 ([MS2, Theorem 3.13]). *The cohomology algebra $H^*(P(m, n, k);\mathbb{Q})$ is isomorphic to the subalgebra $\mathrm{Fix}(\theta^*) \subseteq H^*(\mathbb{S}^m \times \mathbb{C}\mathbb{G}_{n,k};\mathbb{Q})$, generated by the following elements:*

$$u\,c_{2p-1}, \quad c_{2j}, \quad c_{2p-1}\,c_{2q-1},\ \forall 2p-1, 2q-1, 2j \in I,\ \textit{if } m \textit{ is even};$$
$$u, \quad c_{2j}, \quad c_{2p-1}\,c_{2q-1},\ \forall 2p-1, 2q-1, 2j \in I,\ \textit{if } m \textit{ is odd.}$$

A description of the cohomology algebra $H^*(P(m, n, k);\mathbb{Q})$, as a quotient of a polynomial algebra, can be deduced as a particular case in Theorem 3.14 of [MS2].

3. Graded endomorphisms of $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$

In this section, we classify graded endomorphisms of the rational cohomology algebra $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$ whose images are nonzero in $H^2(\mathbb{C}G_{n,k};\mathbb{Q})$. Our approach relies on the study of graded endomorphisms of $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$ from [GH1] and [Ho1]. Assume $m > 0$ for the rest of this paper.

3.1. The cohomology ring of the complex Grassmannian $\mathbb{C}G_{n,k}$ is generated by the Chern classes $c_i, \forall i \in I$ as given in (1). In (8), we see that the cohomology ring of $S^m \times \mathbb{C}G_{n,k}$ is generated by $u, c_i, \forall i \in I$. Therefore, it is sufficient to describe the images of the generators to classify graded endomorphisms of $H^*_\times$. The following is the main result of this section.

Theorem 3.1. *Let ϕ be a graded endomorphism of $H^*_\times$ satisfying $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$. Then the following holds,* *(1) Either* $\phi(u) = au$ *for some* $a \in \mathbb{Q}$*, or* $\phi(u) \in H^*_{\mathbb{C}G} \subseteq H^*_{\times}$ *with* $\phi(u)^2 = 0$ *in* $H^*_{\times}$*.*

(2) There exists $\lambda \in \mathbb{Q}\backslash\{0\}$ *such that*

$$\phi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \textit{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \textit{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \textit{if } k = n-k, \end{cases}$$

where $(c^{-1})_i$ *is the* $2i$*-dimensional part of the inverse of* $c = 1 + c_1 + \cdots + c_k$ *in* $H^*_{\mathbb{C}G}$*.*

Proof. From equation (8) and (9), we have $H^*_{\times} \cong \mathcal{R}/\mathcal{I} \cong H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G}$, where $\mathcal{R} := \mathbb{Q}[u, c_1, \ldots, c_k]$ and $\mathcal{I} := \langle u^2, h_{n-k+1}, \ldots, h_n\rangle$.

Let $p_1 : H^*_{\times} = H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G} \to H^*_{\mathbb{C}G}$ be the projection onto the first summand and $i_1 : H^*_{\mathbb{C}G} \hookrightarrow H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G}$ be the inclusion into the first summand. The composite $\phi_1 := p_1 \circ \phi \circ i_1$ is a degree-preserving endomorphism of $H^*_{\mathbb{C}G}$. We have the following diagram:

$$\begin{tikzcd} H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G} \arrow[r, "\phi"] & H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G} \arrow[d, two heads, "p_1"] \\ H^*_{\mathbb{C}G} \arrow[u, hook, "i_1"] \arrow[r, "\phi_1"] & H^*_{\mathbb{C}G} \end{tikzcd} \tag{11}$$

Thus, for $x \in H^*_{\mathbb{C}G} \subset H^*_{\times}$, one can write $\phi(x) = \phi_1(x) + uP_x$ for some $P_x \in H^*_{\mathbb{C}G} \subset H^*_{\times}$ because the kernal of p_1, $\ker(p_1) = uH^*_{\mathbb{C}G}$. This implies that

$$\phi(c_i) = \phi_1(c_i) + uP_{c_i}, \forall i \in I. \tag{12}$$

For simplicity, denote P_{c_i} by $P_i \in H^{2i-m}_{\mathbb{C}G}$ which is a polynomial in $c_1, \ldots, c_k$ of degree $2i - m$ as $\deg c_i = 2i$ and $\deg u = m$.

Since $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$, that implies $\phi(c_1)$ is of the form $\lambda c_1 + \mu u$, $\lambda, \mu \in \mathbb{Q}$, $\lambda \neq 0$. Then we have $\phi_1(c_1) = \lambda c_1$, $\lambda \neq 0$ on $H^*_{\mathbb{C}G}$. By Theorem 2.3 part *(ii)*, we have,

$$\phi_1(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n-k, \end{cases} \tag{13}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in H^*_{CG}. Using the observations given above it is convenient to prove part *(2)* first.

proof of part (2): Using (12) and (13), it is sufficient to prove that $P_i = 0$, $\forall i \in I$. By (13), we have that ϕ_1 is an automorphism of $H^*_{\mathbb{C}G}$. Using the invertibility of ϕ_1 and (12), let $D : H^*_{\mathbb{C}G} \to H^*_{\mathbb{C}G}$ be defined by

$$D(x) = P_{\phi_1^{-1}(x)}, \forall x \in H^*_{\mathbb{C}G}.$$

Equivalently, we have $D(\phi_1(x)) = P_x$. Now, we prove that D is a $\mathbb{Q}$-linear transformation and satisfies the Leibniz rule.

$$\begin{aligned}
uP_{tx} &= \phi(tx) - \phi_1(tx) = t(\phi(x) - \phi_1(x)) = utP_x,\ \forall t \in \mathbb{Q},\\
uP_{x+y} &= \phi(x+y) - \phi_1(x+y) = \phi(x) - \phi_1(x) + \phi(y) - \phi_1(y)\\
&= u(P_x + P_y),\\
uP_{xy} &= \phi(xy) - \phi_1(xy) = \phi(x)\phi(y) - \phi_1(x)\phi_1(y)\\
&= (\phi_1(x) + uP_x)(\phi_1(y) + uP_y) - \phi_1(x)\phi_1(y)\\
&= u(P_x\phi_1(y) + \phi_1(x)P_y).
\end{aligned} \tag{14}$$

Using (9) and (14), we get

$$\begin{aligned}
&D(t\phi_1(x)) = tD(\phi_1(x)), \quad D(\phi_1(x) + \phi_1(y)) = D(\phi_1(x)) + D(\phi_1(y)),\\
&D(\phi_1(x)\phi_1(y)) = D(\phi_1(x))\phi_1(y) + \phi_1(x)D(\phi_1(y)).
\end{aligned}$$

This proves that D is a derivation. For $x \in H^i_{\mathbb{C}G}$, we have $D(x) \in H^{i-m}_{\mathbb{C}G}$ which implies that the derivation D decreases the degree by $\deg(u) = m > 0$. By (2) and Theorem 2.2, we get that D is a zero derivation. In particular

$$D(\phi_1(c_i)) = P_i = 0,\ \forall i \in I.$$

proof of part (1): Since ϕ is a graded endomorphism on $H^*_\times$, therefore

$$\phi(u) = au + P,\ a \in \mathbb{Q},\ \text{satisfying } (au+P)^2 = 0,$$

where P is a homogeneous polynomial in $c_1, \ldots, c_k$ of degree m. We have $P^2 + 2auP = 0$ in $H^*_\times$. Using (9), we get that $2aP = 0$ in $H^*_\times = \mathcal{R}/\mathcal{I}$. Hence, either $a = 0$ or $P \in \mathcal{I}$. □

Remark 3.2. *Theorem 3.1 classifies all graded endomorphisms ϕ of $H^*_\times$ whose image is nonzero in $H^2_{\mathbb{C}G}$ if $n > 2$. In fact, $n > 2$ implies $c_1^2 \neq 0$ and $\phi(u) \neq ac_1$, $a \in \mathbb{Q}\backslash\{0\}$ as $\phi(u)^2 = 0$. Therefore, the only remaining possibility is $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$.*

On the other hand, when $n = 2$, $\mathbb{C}G_{n,k}$ is either a point or $\mathbb{S}^2$ and the classification of graded endomorphisms of $H^*_\times$ is easy.

3.2. In Theorem 3.1, we assume that $\phi(c_1) \neq \mu u$. Let us try to look at the other case where $\phi(c_1) = \mu u$. To address this, we use part (i) of Theorem 2.3 which leads to the following proposition.

Proposition 3.3. *Assume that hypothesis (3) is satisfied. Let ϕ be a graded endomorphism such that $\phi(c_1) = \mu u$, $\mu \in \mathbb{Q}$ in $H^*_\times$. Then*

(1) Either $\phi(u) = au$ for some $a \in \mathbb{Q}$, or $\phi(u) \in H^*_{\mathbb{C}G} \subseteq H^*_\times$ with $\phi(u)^2 = 0$ in $H^*_\times$.

(2) $\phi(c_i) = uP_i$, $\forall i > 1$, where $P_i \in H^{2i-m}_{\mathbb{C}G} \subseteq H^*_\times$.

Proof. (1): The proof of part *(1)* is exactly the same as the proof of part *(1)* of Theorem 3.1. Therefore, we omit the details.

(2): Using (11), we have that the map ϕ_1 is a graded endomorphism on $H^*_{\mathbb{C}G}$ such that $\phi_1(c_1) = 0$. By Theorem 2.3, $\phi_1(c_i) = 0$, $\forall i \in I$, then by (12), we get $\phi(c_i) = uP_i$ for some $P_i \in H^*_{\mathbb{C}G}$, with $\deg(P_i) = 2i - m$. □

Remark 3.4. *In Theorem 3.1 and Proposition 3.3, if we assume $2m \leq n - k$ then $\phi(u) = 0$ whenever $\phi(u) \in H^*_{\mathbb{C}G}$. This is because $H^*_{\mathbb{C}G}$ has no nontrivial relations up to degree $2(n-k)$ and $u^2 = 0$ implies that $\phi(u)^2 = 0$ forcing $\phi(u) = 0$.*

A graded endomorphism of $H^*_{\mathbb{C}G}$ that vanishes on $H^2_{\mathbb{C}G}$ is expected to be trivial, in view of Hoffman's conjecture [Ho1]. However, unlike the case of the complex Grassmannian, there exist many non-trivial graded endomorphisms of $H^*_\times$ that vanish on $H^2_{\mathbb{C}G}$. The following proposition provides such examples when m is even and $1 \le m \le 2k$.

Proposition 3.5. *For each $i \in I$, choose $P_i \in H^{2i-m}_{\mathbb{C}G} \subseteq H^*_\times$ and either $Q = au$, $a \in \mathbb{Q}$, or $Q \in H^*_{\mathbb{C}G} \subseteq H^*_\times$ with $Q^2 = 0$ in $H^*_\times$. Then there exist a graded endomorphism ϕ on $H^*_\times$ such that*

$$\phi(c_i) = uP_i,\ \forall i \in I,\ \text{and} \quad \phi(u) = Q.$$

Proof. Define ϕ on $H^*_\times = \mathcal{R}/\mathcal{I}$ by $\phi(c_i) = uP_i,\ \forall i \in I$, and $\phi(u) = Q$. It is sufficient to prove that ϕ is well defined, that is, $\mathcal{I} \subseteq \ker(\phi)$. Observe that $u^2 = 0$ in $H^*_\times$ which implies that

$$\phi(c_i c_j) = \phi(c_i)\phi(c_j) = uP_i \cdot uP_j = u^2 P_i P_j = 0. \tag{15}$$

Using (15) and $\phi(u^2) = Q^2 = 0$, we have $\mathcal{I} \subseteq \langle u^2, c_i c_j \mid i, j \in I\rangle \subseteq \ker(\phi)$. ☐

3.3. In this subsection, we derive some immediate applications of Theorem 3.1.

Corollary 3.6. *Let us consider $X = \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}$ and denote by u_j the generator of $H^{2m_j}(\mathbb{S}^{2m_j};\mathbb{Q})$ corresponding to the fundamental class of $\mathbb{S}^{2m_j}$ for all $1 \le j \le r$. Define*

$$H^*_{\mathbf{m},\mathbb{C}G} := H^*(\mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k};\mathbb{Q}) \ \cong\ H^*_{\mathbb{C}G}[u_1, \ldots, u_r]/\langle u_1^2, \ldots, u_r^2\rangle,$$

where $\mathbf{m} = (m_1, \ldots, m_r)$. Suppose $\phi : H^*_{\mathbf{m},\mathbb{C}G} \to H^*_{\mathbf{m},\mathbb{C}G}$ is a graded endomorphism satisfying $\phi(c_1) = \lambda c_1$, $\lambda \neq 0$. Then

$$\phi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n-k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H^*_{\mathbb{C}G}$.

Proof. The proof of this corollary is similar to the proof of part *2* of Theorem 3.1. Apply induction on r and replace $\mathbb{C}G_{n,k}$ with $\hat{X} := \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_{i-1}} \times \mathbb{S}^{2m_{i+1}} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}$, and the sphere $\mathbb{S}^m$ with $\mathbb{S}^{2m_i}$ in Theorem 3.1. Since

$$\mathbb{S}^{2m_j} = SO(2m_j + 1)/SO(2m_j) \tag{16}$$

where the orthogonal groups $SO(2m_j + 1)$ and $SO(2m_j)$ have the same rank m_j. Using (16) and (2), $\hat{X}$ satisfies the hypothesis of Theorem 2.2. Therefore, every $\mathbb{Q}$-linear derivation of $H^*(\hat{X};\mathbb{Q})$ that decreases the degree by $2m_i$ is trivial. ☐

Let us turn our attention to the generalized Dold spaces $P(m, n, k)$ defined in Subsection 2.4. The following remark helps us to describe endomorphisms of $H^*(P(m, n, k);\mathbb{Q})$ induced by continuous functions on $P(m, n, k)$. These observations will be used in Section 4.

Remark 3.7. *For a continuous map f on $P(m, n, k)$, we have*

$$f_* \circ \pi_*\big(\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})\big) \subseteq \pi_*\big(\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})\big), \tag{17}$$

where $\pi_1(X)$ denotes the fundamental group of a topological space X. Hence, the composite $f \circ \pi$ admits a lift $\tilde{f}$ on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ for the double covering $\pi : \mathbb{S}^m \times \mathbb{C}G_{n,k} \to P(m, n, k)$.

Using Remark 3.7, we get the following commutative diagram,

$$\begin{array}{ccc} H^*(P(m,n,k);\mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \tilde{f}^*} \\ H^*(P(m,n,k);\mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}). \end{array} \tag{18}$$

where π^* is an injective map. Using Theorem 2.4 and (18) we obtain the following two corollaries.

Corollary 3.8. *Let f^* be an endomorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ satisfying $f^*(c_1^2) \neq 0$. Then f^* is the restriction of a graded endomorphism $\tilde{f}^*$ on $H^*_\times$ satisfying $\tilde{f}^*(c_1) = \lambda c_1, \lambda \neq 0$, to the fixed subring* Fix(θ^*) *of $H^*_\times$ where $\theta = \alpha \times \sigma$.*

Corollary 3.9. *Let f^* be an endomorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ satisfying $f^*(c_1^2) = 0$ and $n > 2$. Then f^* is the restriction of a graded endomorphism $\tilde{f}^*$ on $H^*_\times$ satisfying $\tilde{f}^*(c_1) = au, a \in \mathbb{Q}$, to the fixed subring* Fix(θ^*) *of $H^*_\times$ where $\theta = \alpha \times \sigma$.*

Using Theorem 3.1 in Corollary 3.8, and Proposition 3.3 in Corollary 3.9 along with hypothesis (3), we can determine f^*.

Moreover, there exist graded endomorphisms of $H^*(P(m,n,k))$ that are not induced by any continuous self-map of $P(m,n,k)$, and cannot be realized as restrictions of graded endomorphisms of $H^*_\times$. Let us see an example of such graded endomorphism.

Example 3.10. *If m odd, $n > 2$ and $k = 1$, then $P(m,n,1)$ is fibered by the complex projective space $\mathbb{C}P^{n-1}$ over the real projective space $\mathbb{R}P^m$. In this case, $H^*_\times \cong \mathbb{Q}[u,c_1]/\langle u^2, c_1^n\rangle$ and using (10) and Theorem 2.4, the rational cohomology ring*

$$H^*(P(m,n,1);\mathbb{Q}) \cong \mathbb{Q}[u,b]/\langle u^2, b^{\lfloor (n+1)/2 \rfloor}\rangle,$$

where u is a generator of $H^m(\mathbb{R}P^m;\mathbb{Q})$ and b restricts to $c_1^2 \in H^2(\mathbb{C}P^{n-1};\mathbb{Q})$ under the fiber inclusion.

Consider the endomorphism

$$\phi\colon H^*(P(m,n,1);\mathbb{Q}) \to H^*(P(m,n,1);\mathbb{Q}), \quad \text{defined by} \quad u \mapsto u, b \mapsto -b.$$

Then ϕ is a well-defined graded endomorphism but it cannot be a restriction of a graded endomorphism of $H^*_\times$ because any such map induces $c_1^2 \mapsto \lambda^2 c_1^2$ for some $\lambda \in \mathbb{Q}$, and $\lambda^2 \neq -1$.

The following corollary helps us to understand the relationship between the automorphisms of $H^*(P(m,n,k))$ with the automorphisms of $H^*_\times$.

Corollary 3.11. *Let f^* be an automorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ and assume that $n > 2$. Then $\tilde{f}^*$ is an automorphism of $H^*_\times$, where $\tilde{f}$ is as in Remark 3.7.*
Moreover there exist $\lambda, \mu \in \mathbb{Q}\backslash\{0\}$ such that $\tilde{f}^*(u) = \mu u$ and $\tilde{f}^*(c_i)$ is of the form given in (2) of Theorem 3.1.

Proof. Using Remark 3.7, we have $\tilde{f}^*$ is a graded endomorphism of $H^*_\times$. When $n > 2$, we have $c_1^2 \neq 0$ in $\mathrm{Fix}(\theta^*) \subseteq H^*_\times$, where $\mathrm{Fix}(\theta^*)$ is the fixed subring under θ^* defined in (10). Since f^* is an automorphism, we have $f^*(c_1^2) \neq 0$. Using Corollary 3.8, there exist $\lambda \in \mathbb{Q}$ such that $\tilde{f}^*(c_1) = \lambda c_1, \lambda \neq 0$.
By Theorem 3.1, $\tilde{f}^*(c_i)$ is of the form given in *(2)* of Theorem 3.1. Also,

$$\tilde{f}^*(u) = \mu u,\ \mu \in \mathbb{Q} \quad \text{ or } \quad \tilde{f}^*(u) = Q$$

where Q is a polynomial of degree m in $H^*_{\mathbb{C}G}$ with $Q^2 = 0$. To conclude the result, we need to prove that $\tilde{f}^*(u) = \mu u$ where $\mu \neq 0$.
Suppose that $\tilde{f}^*(u) = Q$, then the image set $\mathrm{Im}\tilde{f}^* \subseteq H^*_{\mathbb{C}G}$. Using Corollary 3.8, we get

$$\mathrm{Im}f^* \cong \mathrm{Im}\tilde{f}^*|_{Fix(\theta^*)} \subseteq H^*_{\mathbb{C}G}.$$

This is a contradiction to the assumption that f^* is an automorphism because using Theorem 2.4, either u or uc_1 (depending on the parity of m) is in $\mathrm{Im}f^* = \mathrm{Fix}(\theta^*) \cong H^*(P(m,n,k);\mathbb{Q})$. Therefore, $\tilde{f}^*(u) = \mu u, \mu \in \mathbb{Q}$ and $\mu \neq 0$ because f^* is an automorphism. □

3.4. The following theorem provides a criterion for the image of the spherical cohomology class mapped to a scaler multiple of itself under the graded endomorphism on $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Z})$ induced from a continuous map.

Theorem 3.12. *Let f be a continuous map on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ such that it stabilizes a copy of Grassmannian $\{x_0\} \times \mathbb{C}G_{n,k}$ for some $x_0 \in \mathbb{S}^m$. Then the induced endomorphism in cohomology satisfies $f^*(u) = \mu u$ for some $\mu \in \mathbb{Z}$.*

Proof. Let $\mathbb{T}^m$ be the torus $(\mathbb{S}^1)^m$ and $q : \mathbb{T}^m \to \mathbb{S}^m$ be the quotient map that collapses the complement C of an open disk $D \subset \mathbb{T}^m$ to the point x_0 in $\mathbb{S}^m$. Denote p_i the i-th projection map on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ for $i = 1, 2$ and $s : \mathbb{S}^m \setminus \{x_0\} \to D$ is the inverse of the restriction $q|_D$. Since f stabilizes $\{x_0\} \times \mathbb{C}G_{n,k}$, define continuous maps $g : \mathbb{C}G_{n,k} \to \mathbb{C}G_{n,k}$ by $(x_0, g(y)) = f(x_0, y)$ and $\tilde{f} : \mathbb{T}^m \times \mathbb{C}G_{n,k} \to \mathbb{T}^m \times \mathbb{C}G_{n,k}$ by

$$\tilde{f}(x,y) = \begin{cases} \big(s \circ p_1 \circ f(q(x), y),\ p_2 \circ f(q(x), y)\big), & x \in D, \\ \big(x, g(y)\big), & x \in C. \end{cases}$$

Then it is easy to check that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T}^m \times \mathbb{C}G_{n,k} & \xrightarrow{\tilde{f}} & \mathbb{T}^m \times \mathbb{C}G_{n,k} \\ {\scriptstyle q\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle q\times \mathrm{id}} \\ \mathbb{S}^m \times \mathbb{C}G_{n,k} & \xrightarrow{f} & \mathbb{S}^m \times \mathbb{C}G_{n,k} \end{array}$$

Since, the quotient map q has Brouwer degree 1, the induced map on rational cohomology $q^* : H^*(\mathbb{S}^m;\mathbb{Z}) \to H^*(\mathbb{T}^m;\mathbb{Z})$ sends $u \mapsto 1 \cdot u_1 u_2 \dots u_m$ where u_i denote the one dimensional cohomology class corresponding to the fundamental class of the i-th circle factor of $\mathbb{T}^m$ for $i \in \{1, 2, \dots, m\}$ with appropriate orientation. Since $H^{\mathrm{odd}}(\mathbb{C}G_{n,k};\mathbb{Z}) = 0$, the induced map $\tilde{f}^*$ sends each u_i to a polynomial $P_i(u_1, \dots, u_m)$. We slightly abuse notation by using the same symbols for the cohomology classes of $H^*(\mathbb{S}^m;\mathbb{Z})$ and $H^*(\mathbb{C}G_{n,k};\mathbb{Z})$ when viewed in $H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Z})$.

The induced diagram in cohomology implies the following commutative diagram.

$$\begin{tikzcd} \prod_{i=1}^m u_i \arrow[r, mapsto, "\tilde{f}^*"] & \prod_{i=1}^m P_i(u_1,\dots,u_m) \\ u \arrow[r, mapsto, "f^*"] \arrow[u, mapsto, "(q\times \mathrm{id})^*"] & f^*(u) \arrow[u, mapsto, "(q\times \mathrm{id})^*"] \end{tikzcd}$$

This implies that $f^*(u)$ does not contain any nonzero element from $H^*(\mathbb{C}G_{n,k};\mathbb{Z})$. Thus, $f^*(u) = \mu u$ for some $\mu \in \mathbb{Z}$. $\square$

4. Coincidence theory of $P(m,n,k)$

In this section, we study the *coincidence theory* of generalized Dold spaces $P(m,n,k)$ defined in Subsection 2.4. We establish the necessary conditions for a generalized Dold space $P(S,X)$ defined in (5) to satisfy the coincidence property.

4.1. Let us recall certain definitions that will be required in the rest of this section.

Definition 4.1. *Let (X,g) be a pair, where g is a continuous map on a topological space X. The pair (X,g) is said to have the* ***coincidence property*** *(in short, CP) if, for every continuous map $f: X\to X$, there exists a point $x\in X$ such that $f(x)=g(x)$.*

If we consider g to be the identity map on X, then the notion of coincidence reduces to that of a fixed point, resulting in the following definition.

Definition 4.2. *A topological space X is said to have* ***fixed-point property*** *(FPP) if every continuous map $f: X\to X$ admits a fixed-point; that is, there exists $x\in X$ such that $f(x)=x$.*

The following proposition provides a criteria in terms of the fiber X and the base space $Y := S/{\sim_\alpha}$, allowing one to infer the coincidence properties of the total space $P(S,X)$.

Proposition 4.3. *Let $(P(S,X),g)$ be a pair, where g is a continuous map on the generalized Dold space $P(S,X)$. Then $(P(S,X),g)$ does not have the CP if one of the following hold:*

(1) The continuous map g is a fiber bundle map and the pair $(Y, p\circ g\circ s)$ does not have the CP, where $Y = S/{\sim_\alpha}$ and s denotes a section of the X-bundle projection p defined in (7) and (6).

(2) There exists a σ-equivariant map f (i.e. $f\circ\sigma = \sigma\circ f$) on X and a $\alpha\times\sigma$-equivariant map $\tilde{g}$ on $S\times X$ inducing g such that $\mathrm{id}_S\times f$ coincides with neither $\tilde{g}$ nor $(\alpha\times\sigma)\circ\tilde{g}$.

Proof. *(1)* Suppose that the pair $(Y, p\circ g\circ s)$ does not have the CP. Then there exists a continuous map $f: Y\to Y$ such that

$$f(x) \neq p\circ g\circ s(x),\ \forall x\in Y. \tag{19}$$

We are given that g is a fiber bundle map, which implies that there exist $g_1 : Y\to Y$, satisfying $p\circ g = g_1\circ p$. Consider $p\circ g\circ s = g_1\circ p\circ s = g_1$. Thus, $p\circ g = g_1\circ p$ implies

$$p\circ g(x) = p\circ g\circ s\circ p(x),\ \forall x\in P(S,X).$$

Define the map $\phi := s \circ f \circ p$ on $P(S,X)$. We claim that $\phi(y) \neq g(y)$, $\forall y \in P(S,X)$. Suppose there exist $y \in P(S,X)$ such that $\phi(y) = g(y)$, then

$$p \circ g \circ s(p(y)) = p \circ g(y) = p \circ s \circ f \circ p(y) = f(p(y)),$$

which contradicts (19).

(2) Let G denote the group of deck transformations of the double covering $\pi : S \times X \to P(S,X)$, generated by the free involution $\alpha \times \sigma$. The proof then follows from a general observation that if for two G-equivariant maps $\tilde{\phi}, \tilde{\psi}$ on $S \times X$, the maps $\tilde{\phi}$ and $t \cdot \tilde{\psi}$ have no point of coincidence, for any $t \in G$; then the maps they induce on the orbit space $P(S,X)$, namely ϕ, ψ, are also coincidence-free. □

In particular if we take g to be the identity map in Proposition 4.3, we recover Proposition 7.2.1 of [M], which proves that if the base Y does not have the FPP, or there exist a σ-equivarinat map on the fibre X with no fixed point then $P(S,X)$ does not have the FPP. As a consequence, $P(m,n)$ does not have the FPP if either m or n is odd.

4.2. Let us recall a well known result in coincidence theory, the Lefschetz Coincidence Theorem, which will be used to prove results in the rest of this paper.

For a closed oriented manifold M of dimension n, let $[M] \in H^n(M;\mathbb{Q})$ denote a chosen fundamental class. Then we have the Poincaré duality isomorphism $D_M : H^k(M;\mathbb{Q}) \to H_{n-k}(M;\mathbb{Q})$, defined by

$$D_M(\alpha) = [M] \frown \alpha, \ \forall \alpha \in H^k(M;\mathbb{Z}). \tag{20}$$

Theorem 4.4 (Lefschetz Coincidence Theorem)**.** *Let f, g be two continuous maps on a compact, connected, oriented manifold M of dimension n. The Lefschetz coincidence number is defined as*

$$L(f,g) := \sum_{i=0}^{n} (-1)^i \mathrm{tr}\big(D_M \circ g^* \circ D_M^{-1} \circ f_* \ : \ H_i(M;\mathbb{Q}) \longrightarrow H_i(M;\mathbb{Q})\big).$$

If $L(f,g) \neq 0$, then there exists $x \in M$ such that $f(x) = g(x)$.

When $g = \mathrm{id}_M$, the theorem reduces to the Lefschetz Fixed-Point Theorem for M.

To study the coincidence theory of generalized Dold spaces fibred by complex Grassmannians over real projective spaces, it is helpful to first understand the coincidence theory of complex Grassmannians, a topic of independent interest. We now prove the following lemma (cf. Theorem 2, [GH1]) to prove Proposition 4.6.

Lemma 4.5. *Let d_{2i} be the $2i$-th Betti number of a complex Grassmannian $\mathbb{C}G_{n,k}$ with $d = k(n-k)$ even. Then the sum $\sum_{i=0}^{d} d_{2i}\lambda^i \neq 0$, $\forall \lambda \in \mathbb{Q}$.*

Proof. Let us consider the sum $\sum_{i=0}^{d} d_{2i}\lambda^i$, when λ is an integer. Clearly,

$$\sum_{i=0}^{d} d_{2i}\lambda^i \equiv 1 \pmod{\lambda}.$$

Hence, $\sum_{i=0}^{d} d_{2i}\lambda^i \neq 0$, if $\lambda \neq \pm 1$. When $\lambda = 1$, the sum is also positive and therefore nonzero. It remains to consider the case where $\lambda = -1$. Let $\chi(\mathbb{R}G_{n,k})$

denote the Euler-Poincaré characteristic of $\mathbb{R}G_{n,k}$ and be defined by

$$\chi(X) := \sum_{i\geq 0} \dim H^i(\mathbb{R}G_{n,k};\mathbb{Z}_2)$$

where $\mathbb{R}G_{n,k}$ denotes the Grassmannian of real k-planes in $\mathbb{R}^n$. Now we observe that $\sum_{i=0}^{d} d_{2i}(-1)^i = \chi(\mathbb{R}G_{n,k})$ where $d_{2i} = \dim H^{2i}(\mathbb{C}G_{n,k};\mathbb{Q}) = \dim H^i(\mathbb{R}G_{n,k};\mathbb{Z}_2)$. It is a well known fact that $\chi(\mathbb{R}G_{n,k}) \neq 0$ if $k(n-k)$ is even.

Let us move to the other case where $\lambda \in \mathbb{Q}\backslash\mathbb{Z}$. Suppose $\sum_{i=0}^{d} d_{2i}\lambda^i = 0$ for some $\lambda = \frac{p}{q}$ where p and q are coprime integers. Since $d_0 = d_d = 1$, using the rational root theorem $p|1$ and $q|1$. Hence, $\lambda = \pm 1$, which is a contradiction. Therefore, we conclude that $\sum_{i=0}^{d} d_{2i}\lambda^i \neq 0$ for all $\lambda \in \mathbb{Q}$. □

Denote the i-th homology groups $H_i(\mathbb{C}G_{n,k};\mathbb{Q})$, $H_i(\mathbb{S}^m;\mathbb{Q})$ and $H_i(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$, by $H_i^{\mathbb{C}G}$, $H_i^{\mathbb{S}}$ and $H_i^{\times}$, respectively. Let d denote the complex dimension of $\mathbb{C}G_{n,k}$, given by $d = k(n-k)$. Then we have the following proposition.

Proposition 4.6. *Consider a complex Grassmannian $\mathbb{C}G_{n,k}$ such that the hypothesis (3) is satisfied and $k(n-k)$ is even. Let g be a continuous map on $\mathbb{C}G_{n,k}$ with nonzero Brouwer degree. Then the pair $(\mathbb{C}G_{n,k}, g)$ has the coincidence property.*

Proof. Self-maps with nonzero Brouwer degree induces automorphisms in the rational cohomology algebra. Using Theorem 2.3 part *(i)*, there exist a nonzero rational λ such that $g^*(c_i) = \lambda^i c_i, \forall i \in I$. Let f be a continuous map on $\mathbb{C}G_{n,k}$ and using Theorem 2.3 part *(i)*, there exists $\mu \in \mathbb{Q}$ such that

$$f^*(c_i) = \mu^i c_i, \forall i \in I.$$

Then by the Universal Coefficient Theorem, $\mathrm{Hom}_{\mathbb{Q}}(H_i^{\mathbb{C}G};\mathbb{Q}) \cong H^i_{\mathbb{C}G}$ non-canonically which implies that

$$\varphi \circ f_* = f^*(\varphi),\ \forall \varphi \in \mathrm{Hom}_{\mathbb{Q}}\big(H_{2i}^{\mathbb{C}G},\mathbb{Q}\big) \cong H^{2i}_{\mathbb{C}G}.$$
$$\varphi(f_*(x)) = (f^*(\varphi))(x) = \mu^i\varphi(x) = \varphi(\mu^i x),\ \forall x \in H_{2i}^{\mathbb{C}G}.$$

The last equation implies that $f_*(x) = \mu^i x$, $\forall x \in H_{2i}^{\mathbb{C}G}$. Now observe that $D \circ g^* \circ D^{-1} \circ f_* : H_{2i}^{\mathbb{C}G} \to H_{2i}^{\mathbb{C}G}$ is given by

$$D\circ g^*\circ D^{-1}\circ f_*(x) = D\circ g^*\circ D^{-1}(\mu^i x) = \mu^i D\circ g^*(D^{-1}x) = \mu^i D(\lambda^{d-i}D^{-1}x) = \mu^i\lambda^{d-i}x.$$

Thus for $x \in H_{2i}^{\mathbb{C}G}$, the Lefschetz coincidence number is given by

$$\begin{aligned} L(f,g) &= \textstyle\sum_{i=0}^{d}(-1)^{2i}\mathrm{tr}(D \circ g^* \circ D^{-1} \circ f_*(x)) \\ &= \textstyle\sum_{i=0}^{d} d_{2i}\mu^i\lambda^{d-i} \\ &= \lambda^d \textstyle\sum_{i=0}^{d} d_{2i}(\mu/\lambda)^i \neq 0 \quad (\because \lambda \neq 0) \end{aligned}$$

where d_{2i} denotes $\dim_{\mathbb{Q}} H^{2i}_{\mathbb{C}G}$ and the last equation holds by using Lemma 4.5. Therefore, using Theorem 4.4 the pair $(\mathbb{C}G_{n,k}, g)$ has the coincidence property. □

4.3. Denote by $H_*^{\times} = \bigoplus_{i\geq 0} H_i^{\times}$, $H_*^{\mathbb{C}G} = \bigoplus_{i\geq 0} H_i^{\mathbb{C}G}$, $H_*^{\mathbb{S}} = \bigoplus_{i\geq 0} H_i^{\mathbb{S}}$ and ϑ the fundamental class $[\mathbb{S}^m] \in H_m^{\mathbb{S}}$. Let $\{v_q\}$ be a homogeneous basis of $H_*^{\mathbb{C}G}$, and let $\{\delta_{v_q}\}$ denote the corresponding dual basis of $\mathrm{Hom}(H_*^{\mathbb{C}G},\mathbb{Q}) \cong H^*_{\mathbb{C}G}$, such that $\delta_{v_q}(v_p) = \delta_{qp}$ where δ_{qp} is the Kronecker delta function. Without loss of generality, assume that $1 = v_0 \in \{v_i\}$ represents the generator of $H_0^{\mathbb{C}G} \cong \mathbb{Q}$.

Over $\mathbb{Q}$, the Künneth Theorem yields the following decompositions

$$H_i^\times \cong H_i^{\mathbb{C}G} \oplus (\vartheta \otimes H_{i-m}^{\mathbb{C}G}), \qquad H_\times^i \cong H_{\mathbb{C}G}^i \oplus uH_{\mathbb{C}G}^{i-m}, \tag{21}$$

where $u \in H_\times^m \cong \mathrm{Hom}(H_m^\times, \mathbb{Q})$ corresponds to the element $\delta_{\vartheta\otimes 1}$.

Using (21), we can extend the chosen basis $\{v_q\}$ of $H_*^{\mathbb{C}G}$ to $\{v_q\} \cup \{\vartheta \otimes v_q\}$ of $H_*^\times$ such that the corresponding dual basis can also be extended from $\{\delta_{v_q}\}$ of $\mathrm{Hom}(H_*^{\mathbb{C}G};\mathbb{Q})$ to $\{\delta_{v_q}\} \cup \{\delta_{\vartheta\otimes v_q}\}$ of $\mathrm{Hom}(H_*^\times;\mathbb{Q})$ satisfying:

$$\delta_{v_q}(v_p) = \delta_{qp}, \quad \delta_{v_q}(\vartheta \otimes v_p) = 0, \quad \delta_{\vartheta\otimes v_q}(v_p) = 0, \quad \delta_{\vartheta\otimes v_q}(\vartheta \otimes v_p) = \delta_{qp}. \tag{22}$$

Let f be a continuous function on $P(m,n,k)$. Using Remark 3.7 and the Universal Coefficient Theorem, there exist a lift $\tilde{f}$ on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ satisfying

$$\varphi \circ \tilde{f}_* = \tilde{f}^*(\varphi), \, \forall \varphi \in \mathrm{Hom}_{\mathbb{Q}}\big(H_{2i}^{\mathbb{C}G}, \mathbb{Q}\big) \cong H_{\mathbb{C}G}^{2i}. \tag{23}$$

Poincaré duality on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ can be described in terms of the duality on the Grassmannian factor. Let $D_{\mathbb{C}G}\colon H_{\mathbb{C}G}^i \to H_{2d-i}^{\mathbb{C}G}$ be the Poincaré duality isomorphism defined in (20) for $\mathbb{C}G_{n,k}$, where $d = k(n-k)$. The Poincaré duality isomorphism on the product is then determined on the basis elements by

$$D\colon H_\times^j \to H_{m+2d-j}^\times, \quad \delta_{v_i} \mapsto \vartheta \otimes D_{\mathbb{C}G}(\delta_{v_i}) \quad \text{and} \quad \delta_{\vartheta\otimes v_i} \mapsto D_{\mathbb{C}G}(\delta_{v_i}). \tag{24}$$

We are now ready to establish the following lemmas, which will be useful in the sequel.

Lemma 4.7. *Let f be a continuous function on $P(m,n,k)$ and $\tilde{f}$ be the lift defined in Remark 3.7 such that $\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$ and $k < n-k$. Then there exist $\lambda \in \mathbb{Q}\backslash\{0\}$ and $\mu \in \mathbb{Q}$ such that the induced map $\tilde{f}_*$ on $H_*^\times$ is of the following form.*

1. *Either $\tilde{f}_*(\vartheta \otimes x) = \mu\lambda^i(\vartheta \otimes x)$, $\forall x \in H_{2i}^{\mathbb{C}G}$ or $\tilde{f}_*(\vartheta \otimes x) \in H_*^{\mathbb{C}\mathbb{G}}$, $\forall x \in H_*^{\mathbb{C}G}$.*
2. *$\tilde{f}_*(x) = \lambda^i x + \vartheta \otimes y$, for some $y \in H_{2i-m}^{\mathbb{C}G}$, $\forall x \in H_{2i}^{\mathbb{C}G}$.*

Moreover, $y = 0$ in (2) if $\tilde{f}_*(\vartheta \otimes x) = \mu\lambda^i(\vartheta \otimes x)$, $\forall x \in H_{2i}^{\mathbb{C}G}$.

Proof. Using Theorem 3.1, there exist $\lambda \in \mathbb{Q}\backslash\{0\}$ such that $\tilde{f}^*(c_i) = \lambda^i c_i, \forall i \in I$ and either $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ or $\tilde{f}^*(u) \in H_{\mathbb{C}G}^*$. It is sufficient to prove the result for the chosen basis $\{v_q\} \cup \{\vartheta \otimes v_q\}$ of $H_*^\times$.

Let us consider the first case where $\tilde{f}^*(u) = \mu u$. Using $H_{\mathbb{C}G}^* \cong \mathrm{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have

$$\tilde{f}^*(\delta_{v_p}) = \lambda^i \delta_{v_p}, \, \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta\otimes v_p}) = \mu\lambda^i(\delta_{\vartheta\otimes v_p}), \, \forall v_p \in H_{2i}^{\mathbb{C}G}. \tag{25}$$

If m is odd, then the coefficient of any basis element $v_p \in H_*^{\mathbb{C}G}$ in $\tilde{f}_*(\vartheta \otimes v_q)$ and $\vartheta \otimes v_p$ in $\tilde{f}_*(v_q)$ is zero because $\tilde{f}_*$ is a graded map. Let us consider the case where $m = 2s$. By (23) and (25), the coefficient of a basis element $v_p \in H_{2i+m}^{\mathbb{C}G}$ in $\tilde{f}_*(\vartheta\otimes v_q)$ written as a $\mathbb{Q}$-linear combination of the basis elements from $\{v_q\} \cup \{\vartheta \otimes v_q\}$ is the following:

$$\delta_{v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{v_p})(\vartheta \otimes v_q) = \lambda^{i+s}\delta_{v_p}(\vartheta \otimes v_q) = 0, \, \forall v_q \in H_{2i}^{\mathbb{C}G}$$

and the coefficient of a basis element $\vartheta \otimes v_p \in \vartheta \otimes H_{2i}^{\mathbb{C}G}$ in $\tilde{f}_*(\vartheta \otimes v_q)$ is

$$\delta_{\vartheta\otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{\vartheta\otimes v_p})(\vartheta \otimes v_q) = \mu\lambda^i \, \delta_{\vartheta\otimes v_p}(\vartheta \otimes v_q) = \mu\lambda^i\delta_{pq}, \, \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

This implies that

$$\tilde{f}_*(\vartheta \otimes v_q) = \mu\lambda^i(\vartheta \otimes v_q), \, \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

Using similar calculations given above, it is easy to show that

$$\delta_{v_p} \circ \tilde{f}_*(v_q) = \lambda^i \delta_{pq},\ \forall v_q \in H_{2i}^{\mathbb{C}G}, \quad \delta_{\vartheta\otimes v_p} \circ \tilde{f}_*(v_q) = 0,\ \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

Therefore, $\tilde{f}_*(v_q) = \lambda^i v_q,\ \forall v_q \in H_{2i}^{\mathbb{C}G}$.

If $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$. Again using $H^*_{\mathbb{C}G} \cong \mathrm{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have

$$\tilde{f}^*(\delta_{v_p}) = \lambda^i \delta_{v_p},\ \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta\otimes v_p}) \in H^*_{\mathbb{C}G},\ \forall v_p \in H_{2i}^{\mathbb{C}G}. \tag{26}$$

By (23) and (26), we get $\delta_{v_p} \circ \tilde{f}_*(v_q) = \lambda^i \delta_{pq},\ \forall v_q \in H_{2i}^{\mathbb{C}G}$, which implies that $\tilde{f}_*(x) = \lambda^i x + \vartheta \otimes y$, for some $y \in H_{2i-m}^{\mathbb{C}G},\ \forall x \in H_{2i}^{\mathbb{C}G}$. Note that $\tilde{f}^*(\delta_{\vartheta\otimes v_p}) \in H^*_{\mathbb{C}G}$ and equal to some $\sum a_j \delta_{v_j}$. Then

$$\delta_{\vartheta\otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{\vartheta\otimes v_p})(\vartheta \otimes v_q) = \sum a_j \delta_{v_j}(\vartheta \otimes v_q) = 0. \tag{27}$$

Hence, $\tilde{f}_*(\vartheta \otimes v_q) \in H_*^{\mathbb{C}G}$ for all $\vartheta \otimes v_q \in \vartheta \otimes H_*^{\mathbb{C}G}$. □

Lemma 4.8. *Assume that the hypothesis (3) is satisfied. Let f be a continuous function on $P(m,n,k)$ and $\tilde{f}$ be the lift defined in Remark 3.7 such that $\tilde{f}^*(c_1) = au$, $a \in \mathbb{Q}$. Then the induced map $\tilde{f}_*$ on $H_*^{\times}$ is of the following form.*

1. *$\tilde{f}_*(x) \in \vartheta \otimes H_{2i-m}^{\mathbb{C}G},\ \forall x \in H_{2i}^{\mathbb{C}G},\ \forall i > 0$.*
2. *$\tilde{f}_*(\vartheta \otimes 1) = \mu(\vartheta \otimes 1) + y,\ y \in H_m^{\mathbb{C}G}$,*
   $\tilde{f}_*(\vartheta \otimes x) \in H_{2i+m}^{\mathbb{C}G},\ \forall x \in H_{2i}^{\mathbb{C}G}, i > 0$ if $\tilde{f}^*(u) = \mu u,\ \mu \in \mathbb{Q}$

Proof. Using Proposition 3.3, we have $\tilde{f}^*(c_i) = uP_i$, for some $P_i \in H^*_{\mathbb{C}G}$ and either $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ or $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$.
Let us consider the first case where $\tilde{f}^*(u) = \mu u$. Using $H^*_{\mathbb{C}G} \cong \mathrm{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have for $i > 0$

$$\tilde{f}^*(\delta_{v_p}) = \sum a_{jp} \delta_{\vartheta\otimes v_j},\ \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta\otimes 1}) = \mu \delta_{\vartheta\otimes 1}, \quad \tilde{f}^*(\delta_{\vartheta\otimes v_p}) = 0,\ \forall v_p \in H_{2i}^{\mathbb{C}G}. \tag{28}$$

Using (23), (28) and similar calculations given in the proof of Lemma 4.7, we have for $v_p \neq 1$

$$\delta_{v_p} \circ \tilde{f}_*(v_q) = 0,\ \forall v_q \in H_{2i}^{\mathbb{C}G}, \quad \delta_{\vartheta\otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = 0,\ \forall v_q \in H^*_{2i}$$

that concludes the result.

When $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$ then also we have $\tilde{f}^*(\delta_{v_p}) = \sum a_{jp}\delta_{\vartheta\otimes v_j},\ \forall v_p \in H_{2i}^{\mathbb{C}G},\ \forall i > 0$ which implies that $\delta_{v_p} \circ \tilde{f}_*(v_q) = 0,\ \forall v_q \in H_{2i}^{\mathbb{C}G},\ \forall i > 0$. □

4.4. The following theorems provide a criteria for the existence of coincidence points between a pair of continuous functions on $P(m,n,k)$.

Theorem 4.9. *Let $P(m,n,k)$ be a generalized Dold manifold with $k < n-k$ and $k(n-k)$ even. Let f and g be two continuous maps on $P(m,n,k)$ and $\tilde{f}, \tilde{g}$ be their lifts as defined in Remark 3.7 such that*

1. *g^* is an automorphism of $H^*(P(m,n,k);\mathbb{Q})$.*
2. *$\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$.*
3. *$\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ if m is odd.*

where s denotes a section of the X-bundle projection p defined in (7) and (6). Then, there is a point of coincidence of f and g.

Proof. Using Corollary 3.11, we have $\tilde{g}^*$ is an automorphism on $H^*_\times$ given by $\tilde{g}^*(c_i) = \lambda_1^i c_i$, and $\tilde{g}^*(u) = \mu_1 u$ for some $\lambda_1, \mu_1 \in \mathbb{Q}\backslash\{0\}$ if $k < n-k$.

Using Lemma 4.7, there exist $\lambda \in \mathbb{Q}\backslash\{0\}$ and $\mu \in \mathbb{Q}$ such that $\tilde{f}_*$ is of the following form,

$$\begin{aligned} \tilde{f}_*(x) &= \lambda^i x + \vartheta \otimes y, \text{ for some } y \in H^{\mathbb{C}G}_{2i-m}, \forall x \in H^{\mathbb{C}G}_{2i} \\ \tilde{f}_*(\vartheta \otimes x) &= \mu\lambda^i(\vartheta \otimes x), \text{ or } \tilde{f}_*(\vartheta \otimes x) = z, \text{ for some } z \in H^{\mathbb{C}G}_{2i+m}, \forall x \in H^{\mathbb{C}G}_{2i} \end{aligned} \tag{29}$$

To prove that f has a point of coincidence with g, it is sufficient to prove that either $\tilde{f}$ or the composition $\theta \circ \tilde{f}$ has a point of coincidence with g where $\theta = \alpha \times \sigma$ defined in Section 2.4. By Theorem 4.4, we need to compute $L(\tilde{f}, \tilde{g})$ and $L(\theta \circ \tilde{f}, \tilde{g})$.

For $x \in H^{\mathbb{C}G}_{2i}$, we have

$$\begin{aligned} D\tilde{g}^* D^{-1} \tilde{f}_*(x) &= \mu_1 \lambda^i \lambda_1^{d-i} x + \vartheta \otimes y' \text{ for some } y' \in H^{\mathbb{C}G}_{2i-m} \\ D\tilde{g}^* D^{-1} \tilde{f}_*(\vartheta \otimes x) &= \mu\lambda^i \lambda_1^{d-i}(\vartheta \otimes x) + z' \text{ for some } z' \in H^{\mathbb{C}G}_{2i+m}. \end{aligned}$$

where $z^{'} = 0$ or $\mu = 0$ depending on the image of $\tilde{f}_*(\vartheta \otimes x)$. Recall that d_{2i} denote the dimension $\dim H^{2i}_{\mathbb{C}G}$. The Lefschetz number $L(\tilde{f}, \tilde{g})$ is

$$L(\tilde{f}, \tilde{g}) = (\mu_1 + \mu) \sum_{i=0}^{k(n-k)} d_{2i} \lambda^i \lambda_1^{d-i}.$$

Using the Lemma 4.5 and the fact that $\lambda_1 \neq 0$, the sum

$$\sum_{i=0}^{k(n-k)} d_{2i} \lambda^i \lambda_1^{d-i} = \lambda_1^d \sum_{i=0}^{k(n-k)} d_{2i} (\lambda/\lambda_1)^i \neq 0,$$

Since $\tilde{f} \circ \theta = \theta \circ \tilde{f}$, it follows that

$$(\theta \circ \tilde{f})^*(c_i) = (-1)^i \tilde{f}^*(c_i), \forall i \in I, \quad (\theta \circ \tilde{f})^*(u) = \begin{cases} -\tilde{f}^*(u), & \text{if } m \text{ is even,} \\ \tilde{f}^*(u), & \text{if } m \text{ is odd.} \end{cases}$$

If m is even, then

$$\begin{aligned} D\tilde{g}^* D^{-1} (\theta \circ \tilde{f})_*(x) &= \mu_1 (-\lambda)^i \lambda_1^{d-i} x + \vartheta \otimes y'' \text{ for some } y'' \in H^{\mathbb{C}G}_{2i-m} \\ D\tilde{g}^* D^{-1} (\theta \circ \tilde{f})_*(\vartheta \otimes x) &= -\mu(-\lambda)^i \lambda_1^{d-i} \vartheta \otimes x + z'' \text{ for some } z'' \in H^{\mathbb{C}G}_{2i+m}. \end{aligned}$$

Thus, the Lefschetz number is

$$L(\theta \circ \tilde{f}, \tilde{g}) = (\mu_1 - \mu) \sum_{i=0}^{k(n-k)} d_{2i} (-\lambda)^i \lambda_1^{d-i}.$$

Also, using $\mu_1 \neq 0$ and Lemma 4.5 it follows that that either $L(\tilde{f}, \tilde{g})$ or $L(\theta \circ \tilde{f}, \tilde{g})$ is nonzero.

If m is odd, $L(\theta \circ \tilde{f}, \tilde{g}) = (\mu_1 + \mu) \sum_{i=0}^{k(n-k)} d_{2i} (-\lambda)^i \lambda_1^{d-i}$. Using Lemma 4.5 and $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ that is $\mu_1 \neq -\mu$, we have both $L(\tilde{f}, \tilde{g})$ and $L(\theta \circ \tilde{f}, \tilde{g})$ are nonzero. This ensures that there exist a point of conincidence between f and g. ☐

Theorem 4.10. *Let $P(m, n, k)$ be a generalized Dold manifold with $k(n-k)$ even and assume that the hypothesis (3) is satisfied. Let g and f are two continuous maps on $P(m, n, k)$ and $\tilde{g}, \tilde{f}$ be their lifts as defined in Remark 3.7 such that*

(1) g^* is an automorphism of $H^*(P(m, n, k); \mathbb{Q})$.

(2) $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ *if* $\tilde{f}^*(H^*_{\mathbb{C}G}) \nsubseteq H^*_{\mathbb{C}G}$ *and* m *is even.*
(3) $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ *if* m *is odd.*

s denotes a section of the X-bundle projection p defined in (7) *and* (6)*. Then, there is a point of coincidence of f and g.*

Proof. If $\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$ then we have the result by Theorem 4.9.
Let us consider the other case when $\tilde{f}^*(c_1) = au$, $a \in \mathbb{Q}$, using Theorem 3.3 we have $\tilde{f}^*(c_i) = uP_i$, for some $P_i \in H^{2i-m}_{\mathbb{C}G}$.

If $P_i \neq 0$ for some i in I then $\tilde{f}^*(H^*_{\mathbb{C}G}) \nsubseteq H^*_{\mathbb{C}G}$. Since $\tilde{f}^*$ is graded and by *(2)* we have $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$. Using Lemma 4.8, $\tilde{f}_*$ is of the following form,

$$\begin{aligned} \tilde{f}_*(x) &= \vartheta \otimes y, \text{ for some } y \in H^{\mathbb{C}G}_{2i-m}, \forall x \in H^{\mathbb{C}G}_{2i}, i > 0 \\ \tilde{f}_*(\vartheta \otimes x) &= \mu(\vartheta \otimes x) + z, \text{ for some } z \in H^{\mathbb{C}G}_{2i+m}, \forall x \in H^{\mathbb{C}G}_{2i} \end{aligned} \tag{30}$$

where $\mu = 0$ if $i > 0$. By Corollary 3.11, we have $\tilde{g}^*$ is an automorphism on $H^*_\times$ given by $\tilde{g}^*(c_i) = \lambda_1^i c_i$, and $\tilde{g}^*(u) = \mu_1 u$ for some $\lambda_1, \mu_1 \in \mathbb{Q}\backslash\{0\}$. Using Theorem 4.4 and the similar calculations as done in the proof of Theorem 4.9, we get

$$L(\tilde{f}, \tilde{g}) = (\mu_1 + \mu) d_0 \lambda_1^d, \quad L(\theta \circ \tilde{f}, \tilde{g}) = \begin{cases} (\mu_1 - \mu) d_0 \lambda_1^d, & \text{if } m \text{ is even}, \\ (\mu_1 + \mu) d_0 \lambda_1^d, & \text{if } m \text{ is odd}. \end{cases}$$

Using $\lambda_1 \neq 0$ and $\mu_1 \neq 0$, either $L(\tilde{f}, \tilde{g})$ or $L(\theta \circ \tilde{f}, \tilde{g})$ is non zero if m is even. Using $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ i.e. $\mu_1 \neq -\mu$ we have $L(\tilde{f}, \tilde{g}) = L(\theta \circ \tilde{f}, \tilde{g}) \neq 0$. Hence, we get the result.

Let us consider the case when $P_i = 0$, $\forall i \in I$, if $\tilde{f}^*(u) = \mu u, \mu \in \mathbb{Q}$ then the proof remains exactly the same as given above. We need to focus on the case when $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$. Using Lemma 4.8 and (27), we have

$$\tilde{f}_*(x) = \vartheta \otimes y, \text{ for some } y \in H^{\mathbb{C}G}_{2i-m}, \forall x \in H^{\mathbb{C}G}_{2i}, i > 0, \quad \tilde{f}_*(\vartheta \otimes x) \in H^{\mathbb{C}G}_*, \forall x \in H^{\mathbb{C}G}_*.$$

This is exactly the same if we take $\mu = 0$ in (30). The rest of the calculations also remains the same and we get the result. □

Remark 4.11. *There are many situations when the map f satisfies the required hypothesis (2) considered in Theorem 4.9 or Theorem 4.10. Some of them are as follows:*

(1) The lift $\tilde{f}$ stabilizes a copy of Grassmannian, i.e., $\tilde{f}(\{x_0\} \times \mathbb{C}G_{n,k}) \subseteq \{x_0\} \times \mathbb{C}G_{n,k}$ for some $x_0 \in \mathbb{S}^m$.
(2) The map $p_1 \circ \tilde{f}^* \circ i_1 : H^*_{\mathbb{C}G} \to H^*_{\mathbb{C}G}$ is an automorphism, equivalently, $f^*(c_1^2) = \lambda^2 c_1^2$, $\lambda \in \mathbb{Q}\backslash\{0\}$, where p_1 and i_1 are defined in (11).
(3) The map $p_2 \circ \tilde{f} \circ i_1 : \mathbb{S}^m \to \mathbb{C}G_{n,k}$ is rationally null homotopic, where p_2 is the projection onto the second summand and i_1 is the inclusion into the first summand.

Under the assumption $m > 2k$, any continuous map f on the generalized Dold space $P(m, n, k)$, the lift $\tilde{f}$ (from Remark 3.7) satisfies $\tilde{f}^*(c_i) = \lambda^i c_i$ for all $i \in I$. Hence condition *(2)* of Theorem 4.10 may be omitted, and one obtains the following consequence.

Corollary 4.12. *Let $P(m,n,k)$ be a generalized Dold space with m and $k(n-k)$ both even. Assume $m > 2k$, and the hypothesis (3) is satisfied. Then, for any continuous function g on $P(m,n,k)$ that induces an automorphism on $H^*(P(m,n,k);\mathbb{Q})$, the pair $(P(m,n,k),g)$ has the coincidence property.*
In particular, for $g = \mathrm{id}$, the space $P(m,n,k)$ has the fixed-point property.

In Theorem 4.10, the first assumption that g^* is an automorphism of $H^*(P(m,n,k);\mathbb{Q})$ can be relaxed by assuming μ is nonzero, which leads to the following proposition.

Proposition 4.13. *Let $P(m,n,k)$ be a generalized Dold manifold with $k(n-k)$ even and assume that the hypothesis (3) is satisfied. Let g and f are two continuous maps on $P(m,n,k)$ and $\tilde{g},\tilde{f}$ be their lifts as defined in Remark 3.7 such that*

(1) $\tilde{g}^*(H^*_{\mathbb{C}G}) = H^*_{\mathbb{C}G}$.
(2) $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}\backslash\{0\}$

Then, there is a point of coincidence of f and g.

The proof of the above proposition is similar to the proof of Theorem 4.10. Therefore, we omit the details.

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