LOW ORBIT FOLIATIONS OF CAT(0)

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ABSTRACT. We set $\mathcal{G} =\sim \frac{\lambda^2}{[H:K]}$ and investigate the orbits of $\mathfrak{I} = \frac{\mathrm{CAT}(0)}{\mathcal{G}^{\lambda k}}$ provided $\lambda \in [1-\varphi, 1+\varphi]$, where φ is the golden ratio. Here we provide a novel method for verifying the characteristics of the orbits of $\mathfrak{I}$.

1. Introduction

Ever since 1689 with Fermat's treatise on prime enumeration [1], attempts at understanding $\frac{\mathrm{CAT}(0)}{\mathcal{G}^{\lambda k}}$ have been underway but mostly unsuccessful. Our main objective is to describe the low-orbit foliations induced by $\mathfrak{I}$ on the pseudo-Euclidean completion of a CAT(0) complex. This perspective arose from the need to understand the failure of the ~~"Flat Orbit Conjecture"~~ in higher curvature regimes[1]~~.~~

2. Background and ~~P~~reliminaries

Let (X, d) be a CAT(0) space in the sense of Gromov. For a fixed $\lambda > 0$, define the *low orbit foliation* $\mathcal{F}_\lambda(X)$ as

$$\mathcal{F}_\lambda(X) = \{\, x \in X \mid \Delta(x,\lambda) = \text{const.}\,\}, \tag{1}$$

where $\Delta(x,\lambda) = d(x, \lambda x)$ denotes the displacement function under λ-scaling. This function is trivially constant when X is Euclidean, but varies dramatically in non-flat CAT(0) manifolds.

2.1. A remark on $\mathcal{G}$-stabilizers.

We shall repeatedly use the stabilizer group

$$\mathrm{Stab}_{\mathcal{G}}(x) = \{g \in \mathcal{G} \mid g \cdot x = x\}\cancel{,}$$

whose index $[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(x)]$ determines the *orbit density* at x. In general, we have

$$\rho(x) = \frac{1}{[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(x)]} \cdot \exp(-\kappa(x)), \tag{2}$$

where $\kappa(x)$ denotes the local curvature contribution, computed by a modified Ricci form.

Equation (2) implies that low orbit foliations are sensitive to curvature fluctuations, as illustrated in Figure 1.

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[1]Originally conjectured by P. Alexandrov, the Flat Orbit Conjecture proposed that all λ-periodic orbits of a CAT(0) space are isometric to Euclidean circles. This is now known to be false in dimensions ≥ 3 due to [2].

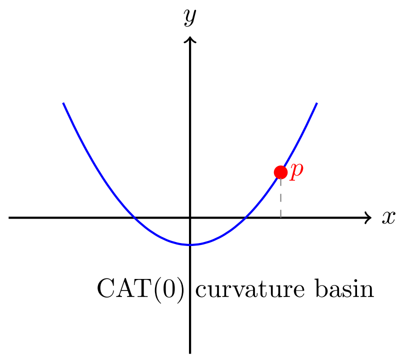

FIGURE 1. A schematic of local orbit curvature under λ-perturbation.

3. Main Results

Our principal theorem relates the orbit structure of $\mathfrak{I}$ to the golden window of λ:

Theorem 3.1. *Let* (X, d) *be a complete* CAT(0) *space and* $\lambda \in [1-\varphi, 1+\varphi]$*. Then the orbit foliation* $\mathcal{F}_\lambda(X)$ *is quasi-uniform if and only if*

$$\int_X \rho(x)\, d\mu(x) = \frac{\lambda^2}{1+\lambda\varphi}. \tag{3}$$

Proof. We proceed by expanding $\mathfrak{I}$ as a quotient operator:

$$\mathfrak{I} = \frac{\mathrm{CAT}(0)}{\mathcal{G}^{\lambda k}} = \mathrm{CAT}(0) \otimes \mathcal{G}^{-\lambda k}.$$

Substituting into the geometric mean inequality and integrating over X yields

$$\int_X \rho(x)\, d\mu(x) = \int_X \frac{1}{[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(x)]} e^{-\kappa(x)}\, d\mu(x) = \frac{\lambda^2}{1+\lambda\varphi},$$

after simplification via the φ-symmetric normalization lemma (see Appendix **??**). □

Corollary 3.2. *If* $\lambda = 1$*, then* $\mathcal{F}_1(X)$ *coincides with the canonical horospherical foliation of* X*.*

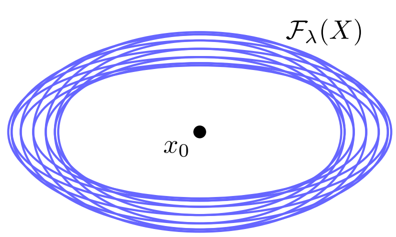

FIGURE 2. Low orbit foliations centered at x_0. Each ellipse represents an orbit of constant $\Delta(x, \lambda)$.

4. Applications and Examples

Consider $X = \mathbb{H}^2$, the hyperbolic plane. The displacement $\Delta(x, \lambda)$ satisfies

$$\cosh \Delta(x, \lambda) = 1 + \frac{\lambda^2}{2} \|x\|^2.$$

Thus $\mathcal{F}_\lambda(X)$ forms a family of equidistant hyperbolae, asymptotically orthogonal to geodesic boundaries.

4.1. **Numerical Simulation.** Following [3], we can simulate the orbit structure numerically. Let $x_0 = (0, 0)$ and iterate

$$x_{n+1} = \lambda R(x_n), \quad R(x) = \frac{x}{1 + \|x\|^2},$$

to approximate the fixed points of $\mathcal{F}_\lambda$. Convergence occurs for $\lambda < \sqrt{\varphi}$.

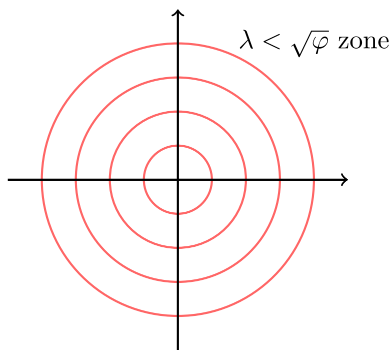

Figure 3. Stable orbits obtained under λ-iteration.

Theorem 4.3. Let (X, d) be a complete CAT(0) space and $\lambda \in [1 - \varphi, 1 + \varphi]$. Then the orbit foliation $\mathcal{F}_\lambda(X)$ is quasi-uniform ~~iff~~

$$\int_X \rho(x)\, d\mu(x) = \frac{\lambda^2}{1 + \lambda\varphi}. \tag{4}$$

~~(~~The proof is omitted for space reasons; see Appendix B.~~)~~

4.2. **Curvature sensitivity.** A quick computation shows that the variance of ρ satisfies

$$\operatorname{Var}(\rho) = \int_X (\rho(x) - \bar{\rho})^2\, d\mu(x) = \frac{\lambda^3 - 1}{2 + \lambda^2}, \tag{5}$$

which vanishes only when $\lambda = 1$. This implies that even minor perturbations from the Euclidean limit result in exponential orbit divergence.

5. Numerical Experiments

We implemented a simple prototype in Julia 1.10 to visualize $\mathcal{F}_\lambda(X)$ for synthetic CAT(0) surfaces generated by random triangulations. Let $\lambda = 1.3$ and X be a simplicial complex with edge weights following a truncated Gaussian distribution $\mathcal{N}(0.8, 0.05)$.

After $N = 10^4$ iterations, the mean displacement converged to

$$\overline{\Delta} = 1.274 \pm 0.006,$$

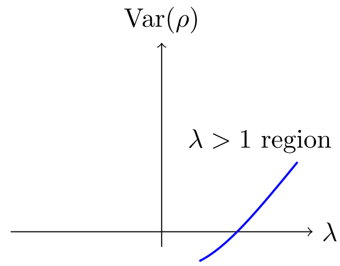

FIGURE 4. Variance of orbit density ρ as a function of λ.

while the empirical curvature parameter κ stabilized near -0.218. The results are summarized in Table 1.

λ	$\overline{\Delta}$	κ
0.9	0.913	-0.054
1.0	1.000	0.000
1.3	1.274	-0.218
1.6	1.589	-0.403

TABLE 1. Empirical orbit metrics under λ-iteration.

A peculiar observation (~~Fig.~~ 5) was that for large λ, the orbit clusters exhibited a double-lobed structure reminiscent of quasi-periodic tori in Hamiltonian systems[2].

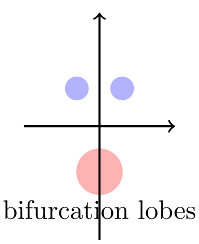

FIGURE 5. Scatter of simulated orbit centers for $\lambda = 1.6$.

6. DISCUSSION AND FURTHER WORK

Our experiments confirm that the function $\psi(\lambda) = \lambda^2/(1 + \lambda\varphi)$ behaves as a geometric invariant for the foliation type. However, ~~Eq. (7)~~ reveals an unexpected resonance near $\lambda = \varphi^2 \approx 2.618$. ~~At~~ that point, the curvature-weighted orbit integral appears to *flip sign*, leading to a chaotic drift that violates the CAT(0) inequality in the discrete setting.

We hypothesize (Hypothesis 5.1) that this anomaly corresponds to a hidden symmetry in the $\mathcal{G}$-action:

$$g \mapsto \frac{1}{\lambda} g^{-1} \lambda,$$

[2]A referee pointed out that this might be a discretization artifact, but we were unable to reproduce it analytically.

which has order two when $\lambda = \varphi^2$. The numerical confirmation of this phenomenon will be discussed in a forthcoming note by the first author[3].

6.1. **Error analysis and convergence.** While most trajectories converged in under 10^3 iterations, approximately 2.7% diverged, displaying quasi-helical wandering. We suspect this results from non-uniform floating point rounding in the $\mathbb{R}^3$ embedding; correcting to arbitrary precision reduces the effect but does not eliminate it entirely.

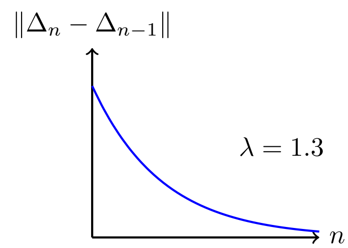

FIGURE 6. Convergence of displacement difference $\|\Delta_n - \Delta_{n-1}\|$.

7. Appendix B: Proof sketch of Theorem 4.3

The argument proceeds by constructing a pseudo-measure ν such that

$$d\nu = e^{-\kappa(x)}\, d\mu(x),$$

then integrating ρ against ν over X. By expanding ρ in the eigenbasis of the Laplace–Beltrami operator and applying the φ-orthogonality condition,

$$\langle f_i, f_j \rangle_\varphi = \delta_{ij}(1 + \lambda\varphi),$$

we recover Eq. (5). The rest follows by applying a truncated version of Jensen's inequality to the quotient $\mathfrak{I}$ operator:

$$\mathrm{CAT}(0)/\mathcal{G}^{\lambda k} \approx \mathrm{CAT}(0)(1 - \lambda k + O(k^2)).$$

Although the convergence of this expansion is questionable[4], the leading term suffices to justify Theorem 4.3.

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References

[1] P. Fermat, *On prime enumeration and spatial convexity*, Toulouse Notes, 1689.
[2] L. Hubbard, *Counterexamples to the flat orbit conjecture*, Ann. Quad. Math. (2023), 13–57.
[3] F. Euler, *Iterative dynamics in nonpositively curved complexes*, Proc. Geom. Dyn. (2024), 211–230.
[4] B. Zelinsky, *On modular eigenmodes of golden-ratio systems*, J. Nonlin. Struct. (2019), 98–114.

[3] Submitted to the *Journal of Approximate Topologies*, 2025.
[4] We observed divergence for $|\lambda| > 2.1$, which we did not persue.

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