Theorem 2.2 ([ST1], Theorem A'). *Let $D_i(H^*(G/H;\mathbb{Q}))$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-derivations of $H^*(G/H;\mathbb{Q})$ which decreases the degree by $i > 0$ where G is a connected, compact Lie group and H is a closed subgroup of maximal rank. Then, for all i,*

$$D_i(H^*(G/H;\mathbb{Q})) = 0.$$

2.2. **Graded endomorphisms on $\mathbf{H}^*_{\mathbb{C}\mathbf{G}}$.** It was conjectured in [O] that any graded endomorphism ϕ of the cohomology algebra $H^*_{\mathbb{C}G}$ is an *Adams* map when $k < n-k$; that is, there exists a rational λ such that $\phi(c_i) = \lambda^i c_i$, for all $i \in I$. Glover and Homer (see [GH1]) and Hoffman (see [Ho1]) proved the conjecture under the following hypothesis respectively:

$$\text{Either } k \leq 3 \text{ and } n > 2k, \text{ or } k > 3 \text{ and } n > 2k^2 - 1. \tag{3}$$

$$\text{The graded endomorphism } \varphi \text{ of } H^*_{\mathbb{C}G} \text{ satisfies } \varphi(c_1) = \lambda c_1, \lambda \neq 0. \tag{4}$$

Let us recall those results proved in [GH1, Ho1] that will be used in the rest of this paper.

Theorem 2.3 ([GH1], Theorem 1, [Ho1], Theorem 1.1). *(i) Assume that the hypothesis (3) is satisfied. Then for every graded endomorphism φ on $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$, there exists a rational λ such that*

$$\varphi(c_i) = \lambda^i c_i, \quad \forall i \in I.$$

(ii) Assume that the hypothesis (4) is satisfied. Then, we have

$$\varphi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n-k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H^*(\mathbb{C}G_{n,k};\mathbb{Q})$.

2.3. **Generalized Dold spaces.** In [Do], the author introduced the notion of *classical Dold manifolds* $P(m,n) := \mathbb{S}^m \times \mathbb{C}P^n/\sim$ where $(s,L) \sim (-s,\bar{L})$, where the involution $L \mapsto \bar{L}$ on $\mathbb{C}G_{n,k}$ is induced from the standard conjugation on $\mathbb{C}^n$, to construct generators in odd dimensions for René Thom's unoriented cobordism ring.

In [NS, MS1], the authors generalized the notion of classical Dold manifolds by replacing the sphere $\mathbb{S}^m$ with an arbitrary topological space S equipped with a free involution α, analogous to the antipodal map on $\mathbb{S}^m$, and $\mathbb{C}P^n$ with an arbitrary topological space X with an involution $\sigma : X \to X$ having a nonempty fixed-point set, analogously to complex conjugation on $\mathbb{C}P^n$. Then the quotient space

$$P(S,\alpha,X,\sigma) := S \times X/\sim, \quad \text{where } (s,x) \sim (\alpha(s),\sigma(x)), \tag{5}$$

is called *generalized Dold space* (in short GDS), often denoted simply as $P(S,X)$. Moreover, the quotient map $S \times X \to P(S,X)$ is a double covering map.

Let us fix a notation Y for $S/\sim_\alpha$, where $s \sim_\alpha \alpha(s), \forall s \in S$. Then, a GDS $P(S,X)$ is the total space of a fiber bundle $X \hookrightarrow P(S,X) \twoheadrightarrow Y$, where the fiber bundle projection is

$$p : P(S,X) \twoheadrightarrow Y, \quad [s,x] \mapsto [s]. \tag{6}$$

Choosing a fixed-point of σ, say $x_0 \in \mathrm{Fix}(\sigma) \neq \emptyset$, we can construct a section of the fiber bundle

$$s : Y \hookrightarrow P(S,X), \quad [s] \mapsto [s,x_0]. \tag{7}$$

(1) Either $\phi(u) = au$ for some $a \in \mathbb{Q}$, or $\phi(u) \in H^*_{\mathbb{C}G} \subseteq H^*_\times$ with $\phi(u)^2 = 0$ in $H^*_\times$.

(2) There exists $\lambda \in \mathbb{Q}\backslash\{0\}$ such that

$$\phi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \textit{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \textit{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \textit{if } k = n-k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H^*_{\mathbb{C}G}$.

Proof. From equation (8) and (9), we have $H^*_\times \cong \mathcal{R}/\mathcal{I} \cong H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G}$, where $\mathcal{R} := \mathbb{Q}[u, c_1, \ldots, c_k]$ and $\mathcal{I} := \langle u^2, h_{n-k+1}, \ldots, h_n\rangle$.

Let $p_1 : H^*_\times = H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G} \to H^*_{\mathbb{C}G}$ be the projection onto the first summand and $i_1 : H^*_{\mathbb{C}G} \hookrightarrow H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G}$ be the inclusion into the first summand. The composite $\phi_1 := p_1 \circ \phi \circ i_1$ is a degree-preserving endomorphism of $H^*_{\mathbb{C}G}$. We have the following diagram:

$$\begin{tikzcd} H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G} \arrow[r, "\phi"] & H^*_{\mathbb{C}G} \oplus uH^*_{\mathbb{C}G} \arrow[d, "p_1"] \\ H^*_{\mathbb{C}G} \arrow[u, hook, "i_1"] \arrow[r, "\phi_1"] & H^*_{\mathbb{C}G} \end{tikzcd} \tag{11}$$

Thus, for $x \in H^*_{\mathbb{C}G} \subset H^*_\times$, one can write $\phi(x) = \phi_1(x) + uP_x$ for some $P_x \in H^*_{\mathbb{C}G} \subset H^*_\times$ because the kernal of p_1, $\ker(p_1) = uH^*_{\mathbb{C}G}$. This implies that

$$\phi(c_i) = \phi_1(c_i) + uP_{c_i}, \forall i \in I. \tag{12}$$

For simplicity, denote P_{c_i} by $P_i \in H^{2i-m}_{\mathbb{C}G}$ which is a polynomial in $c_1, \ldots, c_k$ of degree $2i - m$ as $\deg c_i = 2i$ and $\deg u = m$.

Since $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$, that implies $\phi(c_1)$ is of the form $\lambda c_1 + \mu u$, $\lambda, \mu \in \mathbb{Q}$, $\lambda \neq 0$. Then we have $\phi_1(c_1) = \lambda c_1$, $\lambda \neq 0$ on $H^*_{\mathbb{C}G}$. By Theorem 2.3 part *(ii)*, we have,

$$\phi_1(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n-k, \end{cases} \tag{13}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in H^*_{CG}. Using the observations given above it is convenient to prove part *(2)* first.

proof of part (2): Using (12) and (13), it is sufficient to prove that $P_i = 0$, $\forall i \in I$. By (13), we have that ϕ_1 is an automorphism of $H^*_{\mathbb{C}G}$. Using the invertibility of ϕ_1 and (12), let $D : H^*_{\mathbb{C}G} \to H^*_{\mathbb{C}G}$ be defined by

$$D(x) = P_{\phi_1^{-1}(x)}, \forall x \in H^*_{\mathbb{C}G}.$$

Equivalently, we have $D(\phi_1(x)) = P_x$. Now, we prove that D is a $\mathbb{Q}$-linear transformation and satisfies the Leibniz rule.

$$
\begin{aligned}
uP_{tx} &= \phi(tx) - \phi_1(tx) = t(\phi(x) - \phi_1(x)) = utP_x,\ \forall t \in \mathbb{Q},\\
uP_{x+y} &= \phi(x+y) - \phi_1(x+y) = \phi(x) - \phi_1(x) + \phi(y) - \phi_1(y)\\
&= u(P_x + P_y),\\
uP_{xy} &= \phi(xy) - \phi_1(xy) = \phi(x)\phi(y) - \phi_1(x)\phi_1(y)\\
&= (\phi_1(x) + uP_x)(\phi_1(y) + uP_y) - \phi_1(x)\phi_1(y)\\
&= u(P_x\phi_1(y) + \phi_1(x)P_y).
\end{aligned} \tag{14}
$$

Using (9) and (14), we get

$$
\begin{aligned}
&D(t\phi_1(x)) = tD(\phi_1(x)), \quad D(\phi_1(x) + \phi_1(y)) = D(\phi_1(x)) + D(\phi_1(y)),\\
&D(\phi_1(x)\phi_1(y)) = D(\phi_1(x))\phi_1(y) + \phi_1(x)D(\phi_1(y)).
\end{aligned}
$$

This proves that D is a derivation. For $x \in H^i_{\mathbb{C}G}$, we have $D(x) \in H^{i-m}_{\mathbb{C}G}$ which implies that the derivation D decreases the degree by $\deg(u) = m > 0$. By (2) and Theorem 2.2, we get that D is a zero derivation. In particular

$$D(\phi_1(c_i)) = P_i = 0,\ \forall i \in I.$$

proof of part (1): Since ϕ is a graded endomorphism on $H^*_\times$, therefore

$$\phi(u) = au + P,\ a \in \mathbb{Q},\ \text{satisfying } (au + P)^2 = 0,$$

where P is a homogeneous polynomial in $c_1, \ldots, c_k$ of degree m. We have $P^2 + 2auP = 0$ in $H^*_\times$. Using (9), we get that $2aP = 0$ in $H^*_\times = \mathcal{R}/\mathcal{I}$. Hence, either $a = 0$ or $P \in \mathcal{I}$. □

Remark 3.2. *Theorem 3.1 classifies all graded endomorphisms ϕ of $H^*_\times$ whose image is nonzero in $H^2_{\mathbb{C}G}$ if $n > 2$. In fact, $n > 2$ implies $c_1^2 \neq 0$ and $\phi(u) \neq ac_1$, $a \in \mathbb{Q}\backslash\{0\}$ as $\phi(u)^2 = 0$. Therefore, the only remaining possibility is $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$.*

On the other hand, when $n = 2$, $\mathbb{C}G_{n,k}$ is either a point or $\mathbb{S}^2$ and the classification of graded endomorphisms of $H^*_\times$ is easy.

3.2. In Theorem 3.1, we assume that $\phi(c_1) \neq \mu u$. Let us try to look at the other case where $\phi(c_1) = \mu u$. To address this, we use part (i) of Theorem 2.3 which leads to the following proposition.

Proposition 3.3. *Assume that hypothesis* (3) *is satisfied. Let ϕ be a graded endomorphism such that $\phi(c_1) = \mu u$, $\mu \in \mathbb{Q}$ in $H^*_\times$. Then*

1. *Either $\phi(u) = au$ for some $a \in \mathbb{Q}$, or $\phi(u) \in H^*_{\mathbb{C}G} \subseteq H^*_\times$ with $\phi(u)^2 = 0$ in $H^*_\times$.*
2. *$\phi(c_i) = uP_i$, $\forall i > 1$, where $P_i \in H^{2i-m}_{\mathbb{C}G} \subseteq H^*_\times$.*

Proof. (1): The proof of part *(1)* is exactly the same as the proof of part *(1)* of Theorem 3.1. Therefore, we omit the details.

(2): Using (11), we have that the map ϕ_1 is a graded endomorphism on $H^*_{\mathbb{C}G}$ such that $\phi_1(c_1) = 0$. By Theorem 2.3, $\phi_1(c_i) = 0$, $\forall i \in I$, then by (12), we get $\phi(c_i) = uP_i$ for some $P_i \in H^*_{\mathbb{C}G}$, with $\deg(P_i) = 2i - m$. □

Remark 3.4. *In Theorem 3.1 and Proposition 3.3, if we assume $2m \leq n - k$ then $\phi(u) = 0$ whenever $\phi(u) \in H^*_{\mathbb{C}G}$. This is because $H^*_{\mathbb{C}G}$ has no nontrivial relations up to degree $2(n-k)$ and $u^2 = 0$ implies that $\phi(u)^2 = 0$ forcing $\phi(u) = 0$.*

A graded endomorphism of $H^*_{\mathbb{C}G}$ that vanishes on $H^2_{\mathbb{C}G}$ is expected to be trivial, in view of Hoffman's conjecture [Ho1]. However, unlike the case of the complex Grassmannian, there exist many non-trivial graded endomorphisms of $H^*_\times$ that vanish on $H^2_{\mathbb{C}G}$. The following proposition provides such examples when m is even and $1 \le m \le 2k$.

Proposition 3.5. *For each $i \in I$, choose $P_i \in H^{2i-m}_{\mathbb{C}G} \subseteq H^*_\times$ and either $Q = au$, $a \in \mathbb{Q}$, or $Q \in H^*_{\mathbb{C}G} \subseteq H^*_\times$ with $Q^2 = 0$ in $H^*_\times$. Then there exist a graded endomorphism ϕ on $H^*_\times$ such that*

$$\phi(c_i) = uP_i,\ \forall i \in I,\ \text{and} \quad \phi(u) = Q.$$

Proof. Define ϕ on $H^*_\times = \mathcal{R}/\mathcal{I}$ by $\phi(c_i) = uP_i$, $\forall i \in I$, and $\phi(u) = Q$. It is sufficient to prove that ϕ is well defined, that is, $\mathcal{I} \subseteq \ker(\phi)$. Observe that $u^2 = 0$ in $H^*_\times$ which implies that

$$\phi(c_ic_j) = \phi(c_i)\phi(c_j) = uP_i \cdot uP_j = u^2 P_iP_j = 0. \tag{15}$$

Using (15) and $\phi(u^2) = Q^2 = 0$, we have $\mathcal{I} \subseteq \langle u^2, c_ic_j \mid i,j \in I\rangle \subseteq \ker(\phi)$. □

3.3. In this subsection, we derive some immediate applications of Theorem 3.1.

Corollary 3.6. *Let us consider $X = \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}$ and denote by u_j the generator of $H^{2m_j}(\mathbb{S}^{2m_j};\mathbb{Q})$ corresponding to the fundamental class of $\mathbb{S}^{2m_j}$ for all $1 \le j \le r$. Define*

$$H^*_{\mathbf{m},\mathbb{C}G} := H^*(\mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k};\mathbb{Q}) \ \cong\ H^*_{\mathbb{C}G}[u_1,\dots,u_r]/\langle u_1^2,\dots,u_r^2\rangle,$$

where $\mathbf{m} = (m_1,\dots,m_r)$. Suppose $\phi : H^*_{\mathbf{m},\mathbb{C}G} \to H^*_{\mathbf{m},\mathbb{C}G}$ is a graded endomorphism satisfying $\phi(c_1) = \lambda c_1$, $\lambda \neq 0$. Then

$$\phi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n-k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n-k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H^*_{\mathbb{C}G}$.

Proof. The proof of this corollary is similar to the proof of part *2* of Theorem 3.1. Apply induction on r and replace $\mathbb{C}G_{n,k}$ with $\hat{X} := \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_{i-1}} \times \mathbb{S}^{2m_{i+1}} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}$, and the sphere $\mathbb{S}^m$ with $\mathbb{S}^{2m_i}$ in Theorem 3.1. Since

$$\mathbb{S}^{2m_j} = SO(2m_j+1)/SO(2m_j) \tag{16}$$

where the orthogonal groups $SO(2m_j+1)$ and $SO(2m_j)$ have the same rank m_j. Using (16) and (2), $\hat{X}$ satisfies the hypothesis of Theorem 2.2. Therefore, every $\mathbb{Q}$-linear derivation of $H^*(\hat{X};\mathbb{Q})$ that decreases the degree by $2m_i$ is trivial. □

Let us turn our attention to the generalized Dold spaces $P(m,n,k)$ defined in Subsection 2.4. The following remark helps us to describe endomorphisms of $H^*(P(m,n,k);\mathbb{Q})$ induced by continuous functions on $P(m,n,k)$. These observations will be used in Section 4.

Remark 3.7. *For a continuous map f on $P(m,n,k)$, we have*

$$f_* \circ \pi_*\big(\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})\big) \subseteq \pi_*\big(\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})\big), \tag{17}$$

where $\pi_1(X)$ denotes the fundamental group of a topological space X. Hence, the composite $f \circ \pi$ admits a lift $\tilde{f}$ on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ for the double covering $\pi : \mathbb{S}^m \times \mathbb{C}G_{n,k} \to P(m,n,k)$.

Using Remark 3.7, we get the following commutative diagram,

$$\begin{array}{ccc} H^*(P(m,n,k);\mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \tilde{f}^*} \\ H^*(P(m,n,k);\mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q}). \end{array} \tag{18}$$

where π^* is an injective map. Using Theorem 2.4 and (18) we obtain the following two corollaries.

Corollary 3.8. *Let f^* be an endomorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ satisfying $f^*(c_1^2) \neq 0$. Then f^* is the restriction of a graded endomorphism $\tilde{f}^*$ on $H^*_\times$ satisfying $\tilde{f}^*(c_1) = \lambda c_1, \lambda \neq 0$, to the fixed subring $\mathrm{Fix}(\theta^*)$ of $H^*_\times$ where $\theta = \alpha \times \sigma$.*

Corollary 3.9. *Let f^* be an endomorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ satisfying $f^*(c_1^2) = 0$ and $n > 2$. Then f^* is the restriction of a graded endomorphism $\tilde{f}^*$ on $H^*_\times$ satisfying $\tilde{f}^*(c_1) = au, a \in \mathbb{Q}$, to the fixed subring $\mathrm{Fix}(\theta^*)$ of $H^*_\times$ where $\theta = \alpha \times \sigma$.*

Using Theorem 3.1 in Corollary 3.8, and Proposition 3.3 in Corollary 3.9 along with hypothesis (3), we can determine f^*.

Moreover, there exist graded endomorphisms of $H^*(P(m,n,k))$ that are not induced by any continuous self-map of $P(m,n,k)$, and cannot be realized as restrictions of graded endomorphisms of $H^*_\times$. Let us see an example of such graded endomorphism.

Example 3.10. *If m odd, $n > 2$ and $k = 1$, then $P(m,n,1)$ is fibered by the complex projective space $\mathbb{C}P^{n-1}$ over the real projective space $\mathbb{R}P^m$. In this case, $H^*_\times \cong \mathbb{Q}[u,c_1]/\langle u^2, c_1^n\rangle$ and using (10) and Theorem 2.4, the rational cohomology ring*

$$H^*(P(m,n,1);\mathbb{Q}) \cong \mathbb{Q}[u,b]/\langle u^2, b^{\lfloor (n+1)/2 \rfloor}\rangle,$$

where u is a generator of $H^m(\mathbb{R}P^m;\mathbb{Q})$ and b restricts to $c_1^2 \in H^2(\mathbb{C}P^{n-1};\mathbb{Q})$ under the fiber inclusion.

Consider the endomorphism

$$\phi\colon H^*(P(m,n,1);\mathbb{Q}) \to H^*(P(m,n,1);\mathbb{Q}), \quad \text{defined by} \quad u \mapsto u, b \mapsto -b.$$

Then ϕ is a well-defined graded endomorphism but it cannot be a restriction of a graded endomorphism of $H^*_\times$ because any such map induces $c_1^2 \mapsto \lambda^2 c_1^2$ for some $\lambda \in \mathbb{Q}$, and $\lambda^2 \neq -1$.

The following corollary helps us to understand the relationship between the automorphisms of $H^*(P(m,n,k))$ with the automorphisms of $H^*_\times$.

Corollary 3.11. *Let f^* be an automorphism of $H^*(P(m,n,k);\mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ and assume that $n > 2$. Then $\tilde{f}^*$ is an automorphism of $H^*_\times$, where $\tilde{f}$ is as in Remark 3.7.*
Moreover there exist $\lambda, \mu \in \mathbb{Q}\backslash\{0\}$ such that $\tilde{f}^*(u) = \mu u$ and $\tilde{f}^*(c_i)$ is of the form given in (2) of Theorem 3.1.

The induced diagram in cohomology implies the following commutative diagram.

$$\begin{array}{ccc} \prod_{i=1}^m u_i & \xmapsto{\tilde{f}^*} & \prod_{i=1}^m P_i(u_1,\ldots,u_m) \\ \Big\uparrow{\scriptstyle (q\times \mathrm{id})^*} & & \Big\uparrow{\scriptstyle (q\times \mathrm{id})^*} \\ u & \xmapsto{\quad f^* \quad} & f^*(u) \end{array}$$

This implies that $f^*(u)$ does not contain any nonzero element from $H^*(\mathbb{C}G_{n,k};\mathbb{Z})$. Thus, $f^*(u) = \mu u$ for some $\mu \in \mathbb{Z}$. $\square$

4. Coincidence theory of $P(m,n,k)$

In this section, we study the *coincidence theory* of generalized Dold spaces $P(m,n,k)$ defined in Subsection 2.4. We establish the necessary conditions for a generalized Dold space $P(S,X)$ defined in (5) to satisfy the coincidence property.

4.1. Let us recall certain definitions that will be required in the rest of this section.

Definition 4.1. *Let (X,g) be a pair, where g is a continuous map on a topological space X. The pair (X,g) is said to have the **coincidence property** (in short, CP) if, for every continuous map $f : X \to X$, there exists a point $x \in X$ such that $f(x) = g(x)$.*

If we consider g to be the identity map on X, then the notion of coincidence reduces to that of a fixed point, resulting in the following definition.

Definition 4.2. *A topological space X is said to have **fixed-point property** (FPP) if every continuous map $f : X \to X$ admits a fixed-point; that is, there exists $x \in X$ such that $f(x) = x$.*

The following proposition provides a criteria in terms of the fiber X and the base space $Y := S/{\sim_\alpha}$, allowing one to infer the coincidence properties of the total space $P(S,X)$.

Proposition 4.3. *Let $(P(S,X),g)$ be a pair, where g is a continuous map on the generalized Dold space $P(S,X)$. Then $(P(S,X),g)$ does not have the CP if one of the following hold:*

(1) The continuous map g is a fiber bundle map and the pair $(Y, p\circ g\circ s)$ does not have the CP, where $Y = S/{\sim_\alpha}$ and s denotes a section of the X-bundle projection p defined in (7) and (6).

(2) There exists a σ-equivariant map f (i.e. $f\circ\sigma = \sigma\circ f$) on X and a $\alpha\times\sigma$-equivariant map $\tilde{g}$ on $S\times X$ inducing g such that $\mathrm{id}_S \times f$ coincides with neither $\tilde{g}$ nor $(\alpha\times\sigma)\circ\tilde{g}$.

Proof. (1) Suppose that the pair $(Y, p\circ g\circ s)$ does not have the CP. Then there exists a continuous map $f : Y \to Y$ such that

$$f(x) \neq p\circ g\circ s(x),\ \forall x \in Y. \tag{19}$$

We are given that g is a fiber bundle map, which implies that there exist $g_1 : Y \to Y$, satisfying $p\circ g = g_1\circ p$. Consider $p\circ g\circ s = g_1\circ p\circ s = g_1$. Thus, $p\circ g = g_1\circ p$ implies

$$p\circ g(x) = p\circ g\circ s\circ p(x),\ \forall x \in P(S,X).$$

denote the Euler-Poincaré characteristic of $\mathbb{R}G_{n,k}$ and be defined by

$$\chi(X) := \sum_{i\geq 0} \dim H^i(\mathbb{R}G_{n,k};\mathbb{Z}_2)$$

where $\mathbb{R}G_{n,k}$ denotes the Grassmannian of real k-planes in $\mathbb{R}^n$. Now we observe that $\sum_{i=0}^{d} d_{2i}(-1)^i = \chi(\mathbb{R}G_{n,k})$ where $d_{2i} = \dim H^{2i}(\mathbb{C}G_{n,k};\mathbb{Q}) = \dim H^i(\mathbb{R}G_{n,k};\mathbb{Z}_2)$. It is a well known fact that $\chi(\mathbb{R}G_{n,k}) \neq 0$ if $k(n-k)$ is even.

Let us move to the other case where $\lambda \in \mathbb{Q}\backslash\mathbb{Z}$. Suppose $\sum_{i=0}^{d} d_{2i}\lambda^i = 0$ for some $\lambda = \frac{p}{q}$ where p and q are coprime integers. Since $d_0 = d_d = 1$, using the rational root theorem $p|1$ and $q|1$. Hence, $\lambda = \pm 1$, which is a contradiction. Therefore, we conclude that $\sum_{i=0}^{d} d_{2i}\lambda^i \neq 0$ for all $\lambda \in \mathbb{Q}$. □

Denote the i-th homology groups $H_i(\mathbb{C}G_{n,k};\mathbb{Q})$, $H_i(\mathbb{S}^m;\mathbb{Q})$ and $H_i(\mathbb{S}^m \times \mathbb{C}G_{n,k};\mathbb{Q})$, by $H_i^{\mathbb{C}G}$, $H_i^{\mathbb{S}}$ and $H_i^{\times}$, respectively. Let d denote the complex dimension of $\mathbb{C}G_{n,k}$, given by $d = k(n-k)$. Then we have the following proposition.

Proposition 4.6. *Consider a complex Grassmannian $\mathbb{C}G_{n,k}$ such that the hypothesis (3) is satisfied and $k(n-k)$ is even. Let g be a continuous map on $\mathbb{C}G_{n,k}$ with nonzero Brouwer degree. Then the pair $(\mathbb{C}G_{n,k}, g)$ has the coincidence property.*

Proof. Self-maps with nonzero Brouwer degree induces automorphisms in the rational cohomology algebra. Using Theorem 2.3 part *(i)*, there exist a nonzero rational λ such that $g^*(c_i) = \lambda^i c_i, \forall i \in I$. Let f be a continuous map on $\mathbb{C}G_{n,k}$ and using Theorem 2.3 part *(i)*, there exists $\mu \in \mathbb{Q}$ such that

$$f^*(c_i) = \mu^i c_i, \forall i \in I.$$

Then by the Universal Coefficient Theorem, $\mathrm{Hom}_{\mathbb{Q}}(H_i^{\mathbb{C}G};\mathbb{Q}) \cong H^i_{\mathbb{C}G}$ non-canonically which implies that

$$\varphi \circ f_* = f^*(\varphi),\ \forall \varphi \in \mathrm{Hom}_{\mathbb{Q}}\big(H_{2i}^{\mathbb{C}G},\mathbb{Q}\big) \cong H^{2i}_{\mathbb{C}G}.$$
$$\varphi(f_*(x)) = (f^*(\varphi))(x) = \mu^i\varphi(x) = \varphi(\mu^i x),\ \forall x \in H_{2i}^{\mathbb{C}G}.$$

The last equation implies that $f_*(x) = \mu^i x$, $\forall x \in H_{2i}^{\mathbb{C}G}$. Now observe that $D \circ g^* \circ D^{-1} \circ f_* : H_{2i}^{\mathbb{C}G} \to H_{2i}^{\mathbb{C}G}$ is given by

$$D\circ g^*\circ D^{-1}\circ f_*(x) = D\circ g^*\circ D^{-1}(\mu^i x) = \mu^i D\circ g^*(D^{-1}x) = \mu^i D(\lambda^{d-i}D^{-1}x) = \mu^i\lambda^{d-i}x.$$

Thus for $x \in H_{2i}^{\mathbb{C}G}$, the Lefschetz coincidence number is given by

$$\begin{aligned} L(f,g) &= \textstyle\sum_{i=0}^{d}(-1)^{2i}\mathrm{tr}(D\circ g^*\circ D^{-1}\circ f_*(x)) \\ &= \textstyle\sum_{i=0}^{d} d_{2i}\mu^i\lambda^{d-i} \\ &= \lambda^d \textstyle\sum_{i=0}^{d} d_{2i}(\mu/\lambda)^i \neq 0 \quad (\because \lambda \neq 0) \end{aligned}$$

where d_{2i} denotes $\dim_{\mathbb{Q}} H^{2i}_{\mathbb{C}G}$ and the last equation holds by using Lemma 4.5. Therefore, using Theorem 4.4 the pair $(\mathbb{C}G_{n,k}, g)$ has the coincidence property. □

4.3. Denote by $H_*^{\times} = \bigoplus_{i\geq 0} H_i^{\times}$, $H_*^{\mathbb{C}G} = \bigoplus_{i\geq 0} H_i^{\mathbb{C}G}$, $H_*^{\mathbb{S}} = \bigoplus_{i\geq 0} H_i^{\mathbb{S}}$ and ϑ the fundamental class $[\mathbb{S}^m] \in H_m^{\mathbb{S}}$. Let $\{v_q\}$ be a homogeneous basis of $H_*^{\mathbb{C}G}$, and let $\{\delta_{v_q}\}$ denote the corresponding dual basis of $\mathrm{Hom}(H_*^{\mathbb{C}G},\mathbb{Q}) \cong H^*_{\mathbb{C}G}$, such that $\delta_{v_q}(v_p) = \delta_{qp}$ where δ_{qp} is the Kronecker delta function. Without loss of generality, assume that $1 = v_0 \in \{v_i\}$ represents the generator of $H_0^{\mathbb{C}G} \cong \mathbb{Q}$.

Using similar calculations given above, it is easy to show that

$$\delta_{v_p} \circ \tilde{f}_*(v_q) = \lambda^i \delta_{pq}, \forall v_q \in H_{2i}^{\mathbb{C}G}, \quad \delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(v_q) = 0, \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

Therefore, $\tilde{f}_*(v_q) = \lambda^i v_q, \forall v_q \in H_{2i}^{\mathbb{C}G}$.

If $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$. Again using $H^*_{\mathbb{C}G} \cong \mathrm{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have

$$\tilde{f}^*(\delta_{v_p}) = \lambda^i \delta_{v_p}, \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta \otimes v_p}) \in H^*_{\mathbb{C}G}, \forall v_p \in H_{2i}^{\mathbb{C}G}. \tag{26}$$

By (23) and (26), we get $\delta_{v_p} \circ \tilde{f}_*(v_q) = \lambda^i \delta_{pq}, \forall v_q \in H_{2i}^{\mathbb{C}G}$, which implies that $\tilde{f}_*(x) = \lambda^i x + \vartheta \otimes y$, for some $y \in H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}$. Note that $\tilde{f}^*(\delta_{\vartheta \otimes v_p}) \in H^*_{\mathbb{C}G}$ and equal to some $\sum a_j \delta_{v_j}$. Then

$$\delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{\vartheta \otimes v_p})(\vartheta \otimes v_q) = \sum a_j \delta_{v_j}(\vartheta \otimes v_q) = 0. \tag{27}$$

Hence, $\tilde{f}_*(\vartheta \otimes v_q) \in H_*^{\mathbb{C}G}$ for all $\vartheta \otimes v_q \in \vartheta \otimes H_*^{\mathbb{C}G}$. □

Lemma 4.8. *Assume that the hypothesis* (3) *is satisfied. Let f be a continuous function on $P(m,n,k)$ and $\tilde{f}$ be the lift defined in Remark 3.7 such that $\tilde{f}^*(c_1) = au$, $a \in \mathbb{Q}$. Then the induced map $\tilde{f}_*$ on $H_*^{\times}$ is of the following form.*

(1) $\tilde{f}_*(x) \in \vartheta \otimes H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}, \forall i > 0$.
(2) $\tilde{f}_*(\vartheta \otimes 1) = \mu(\vartheta \otimes 1) + y,\ y \in H_m^{\mathbb{C}G}$,
$\tilde{f}_*(\vartheta \otimes x) \in H_{2i+m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}, i > 0$ *if* $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$

Proof. Using Proposition 3.3, we have $\tilde{f}^*(c_i) = uP_i$, for some $P_i \in H^*_{\mathbb{C}G}$ and either $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ or $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$.
Let us consider the first case where $\tilde{f}^*(u) = \mu u$. Using $H^*_{\mathbb{C}G} \cong \mathrm{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have for $i > 0$

$$\tilde{f}^*(\delta_{v_p}) = \sum a_{jp} \delta_{\vartheta \otimes v_j}, \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta \otimes 1}) = \mu \delta_{\vartheta \otimes 1}, \quad \tilde{f}^*(\delta_{\vartheta \otimes v_p}) = 0, \forall v_p \in H_{2i}^{\mathbb{C}G}. \tag{28}$$

Using (23), (28) and similar calculations given in the proof of Lemma 4.7, we have for $v_p \neq 1$

$$\delta_{v_p} \circ \tilde{f}_*(v_q) = 0, \forall v_q \in H_{2i}^{\mathbb{C}G}, \quad \delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = 0, \forall v_q \in H^*_{2i}$$

that concludes the result.

When $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$ then also we have $\tilde{f}^*(\delta_{v_p}) = \sum a_{jp} \delta_{\vartheta \otimes v_j}, \forall v_p \in H_{2i}^{\mathbb{C}G}, \forall i > 0$ which implies that $\delta_{v_p} \circ \tilde{f}_*(v_q) = 0, \forall v_q \in H_{2i}^{\mathbb{C}G}, \forall i > 0$. □

4.4. The following theorems provide a criteria for the existence of coincidence points between a pair of continuous functions on $P(m,n,k)$.

Theorem 4.9. *Let $P(m,n,k)$ be a generalized Dold manifold with $k < n-k$ and $k(n-k)$ even. Let f and g be two continuous maps on $P(m,n,k)$ and $\tilde{f}, \tilde{g}$ be their lifts as defined in Remark 3.7 such that*

(1) g^* *is an automorphism of* $H^*(P(m,n,k);\mathbb{Q})$.
(2) $\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$.
(3) $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ *if m is odd.*

where s denotes a section of the X-bundle projection p defined in (7) *and* (6)*. Then, there is a point of coincidence of f and g.*

Proof. Using Corollary 3.11, we have $\tilde{g}^*$ is an automorphism on $H^*_\times$ given by $\tilde{g}^*(c_i) = \lambda_1^i c_i$, and $\tilde{g}^*(u) = \mu_1 u$ for some $\lambda_1, \mu_1 \in \mathbb{Q}\backslash\{0\}$ if $k < n-k$.

Using Lemma 4.7, there exist $\lambda \in \mathbb{Q}\backslash\{0\}$ and $\mu \in \mathbb{Q}$ such that $\tilde{f}_*$ is of the following form,

$$\begin{aligned} \tilde{f}_*(x) &= \lambda^i x + \vartheta \otimes y, \text{ for some } y \in H^{\mathbb{C}G}_{2i-m}, \forall x \in H^{\mathbb{C}G}_{2i} \\ \tilde{f}_*(\vartheta \otimes x) = \mu\lambda^i(\vartheta \otimes x), \text{ or } \tilde{f}_*(\vartheta \otimes x) &= z, \text{ for some } z \in H^{\mathbb{C}G}_{2i+m}, \forall x \in H^{\mathbb{C}G}_{2i} \end{aligned} \tag{29}$$

To prove that f has a point of coincidence with g, it is sufficient to prove that either $\tilde{f}$ or the composition $\theta \circ \tilde{f}$ has a point of coincidence with g where $\theta = \alpha \times \sigma$ defined in Section 2.4. By Theorem 4.4, we need to compute $L(\tilde{f}, \tilde{g})$ and $L(\theta \circ \tilde{f}, \tilde{g})$.

For $x \in H^{\mathbb{C}G}_{2i}$, we have

$$\begin{aligned} D\tilde{g}^*D^{-1}\tilde{f}_*(x) &= \mu_1\lambda^i\lambda_1^{d-i}x + \vartheta \otimes y' \text{ for some } y' \in H^{\mathbb{C}G}_{2i-m} \\ D\tilde{g}^*D^{-1}\tilde{f}_*(\vartheta \otimes x) &= \mu\lambda^i\lambda_1^{d-i}(\vartheta \otimes x) + z' \text{ for some } z' \in H^{\mathbb{C}G}_{2i+m}. \end{aligned}$$

where $z^{'} = 0$ or $\mu = 0$ depending on the image of $\tilde{f}_*(\vartheta \otimes x)$. Recall that d_{2i} denote the dimension $\dim H^{2i}_{\mathbb{C}G}$. The Lefschetz number $L(\tilde{f}, \tilde{g})$ is

$$L(\tilde{f}, \tilde{g}) = (\mu_1 + \mu) \sum_{i=0}^{k(n-k)} d_{2i}\lambda^i\lambda_1^{d-i}.$$

Using the Lemma 4.5 and the fact that $\lambda_1 \neq 0$, the sum

$$\sum_{i=0}^{k(n-k)} d_{2i}\lambda^i\lambda_1^{d-i} = \lambda_1^d \sum_{i=0}^{k(n-k)} d_{2i}(\lambda/\lambda_1)^i \neq 0,$$

Since $\tilde{f} \circ \theta = \theta \circ \tilde{f}$, it follows that

$$(\theta \circ \tilde{f})^*(c_i) = (-1)^i \tilde{f}^*(c_i), \forall i \in I, \quad (\theta \circ \tilde{f})^*(u) = \begin{cases} -\tilde{f}^*(u), & \text{if } m \text{ is even,} \\ \tilde{f}^*(u), & \text{if } m \text{ is odd.} \end{cases}$$

If m is even, then

$$\begin{aligned} D\tilde{g}^*D^{-1}(\theta \circ \tilde{f})_*(x) &= \mu_1(-\lambda)^i\lambda_1^{d-i}x + \vartheta \otimes y'' \text{ for some } y'' \in H^{\mathbb{C}G}_{2i-m} \\ D\tilde{g}^*D^{-1}(\theta \circ \tilde{f})_*(\vartheta \otimes x) &= -\mu(-\lambda)^i\lambda_1^{d-i}\vartheta \otimes x + z'' \text{ for some } z'' \in H^{\mathbb{C}G}_{2i+m}. \end{aligned}$$

Thus, the Lefschetz number is

$$L(\theta \circ \tilde{f}, \tilde{g}) = (\mu_1 - \mu) \sum_{i=0}^{k(n-k)} d_{2i}(-\lambda)^i\lambda_1^{d-i}.$$

Also, using $\mu_1 \neq 0$ and Lemma 4.5 it follows that that either $L(\tilde{f}, \tilde{g})$ or $L(\theta \circ \tilde{f}, \tilde{g})$ is nonzero.

If m is odd, $L(\theta \circ \tilde{f}, \tilde{g}) = (\mu_1 + \mu)\sum_{i=0}^{k(n-k)} d_{2i}(-\lambda)^i\lambda_1^{d-i}$. Using Lemma 4.5 and $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ that is $\mu_1 \neq -\mu$, we have both $L(\tilde{f}, \tilde{g})$ and $L(\theta \circ \tilde{f}, \tilde{g})$ are nonzero. This ensures that there exist a point of conincidence between f and g. □

Theorem 4.10. *Let $P(m, n, k)$ be a generalized Dold manifold with $k(n-k)$ even and assume that the hypothesis (3) is satisfied. Let g and f are two continuous maps on $P(m, n, k)$ and $\tilde{g}, \tilde{f}$ be their lifts as defined in Remark 3.7 such that*

(1) g^* is an automorphism of $H^*(P(m, n, k); \mathbb{Q})$.

(2) $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ *if* $\tilde{f}^*(H^*_{\mathbb{C}G}) \nsubseteq H^*_{\mathbb{C}G}$ *and* m *is even.*
(3) $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ *if* m *is odd.*

s denotes a section of the X-bundle projection p defined in (7) and (6). Then, there is a point of coincidence of f and g.

Proof. If $\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$ then we have the result by Theorem 4.9.
Let us consider the other case when $\tilde{f}^*(c_1) = au$, $a \in \mathbb{Q}$, using Theorem 3.3 we have $\tilde{f}^*(c_i) = uP_i$, for some $P_i \in H^{2i-m}_{\mathbb{C}G}$.

If $P_i \neq 0$ for some i in I then $\tilde{f}^*(H^*_{\mathbb{C}G}) \nsubseteq H^*_{\mathbb{C}G}$. Since $\tilde{f}^*$ is graded and by *(2)* we have $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$. Using Lemma 4.8, $\tilde{f}_*$ is of the following form,

$$\begin{aligned} \tilde{f}_*(x) &= \vartheta \otimes y, \text{ for some } y \in H^{\mathbb{C}G}_{2i-m}, \forall x \in H^{\mathbb{C}G}_{2i}, i > 0 \\ \tilde{f}_*(\vartheta \otimes x) &= \mu(\vartheta \otimes x) + z, \text{ for some } z \in H^{\mathbb{C}G}_{2i+m}, \forall x \in H^{\mathbb{C}G}_{2i} \end{aligned} \tag{30}$$

where $\mu = 0$ if $i > 0$. By Corollary 3.11, we have $\tilde{g}^*$ is an automorphism on $H^*_\times$ given by $\tilde{g}^*(c_i) = \lambda_1^i c_i$, and $\tilde{g}^*(u) = \mu_1 u$ for some $\lambda_1, \mu_1 \in \mathbb{Q}\backslash\{0\}$. Using Theorem 4.4 and the similar calculations as done in the proof of Theorem 4.9, we get

$$L(\tilde{f}, \tilde{g}) = (\mu_1 + \mu)d_0\lambda_1^d, \quad L(\theta \circ \tilde{f}, \tilde{g}) = \begin{cases} (\mu_1 - \mu)d_0\lambda_1^d, & \text{if } m \text{ is even,} \\ (\mu_1 + \mu)d_0\lambda_1^d, & \text{if } m \text{ is odd.} \end{cases}$$

Using $\lambda_1 \neq 0$ and $\mu_1 \neq 0$, either $L(\tilde{f}, \tilde{g})$ or $L(\theta \circ \tilde{f}, \tilde{g})$ is non zero if m is even. Using $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ i.e. $\mu_1 \neq -\mu$ we have $L(\tilde{f}, \tilde{g}) = L(\theta \circ \tilde{f}, \tilde{g}) \neq 0$. Hence, we get the result.

Let us consider the case when $P_i = 0, \forall i \in I$, if $\tilde{f}^*(u) = \mu u, \mu \in \mathbb{Q}$ then the proof remains exactly the same as given above. We need to focus on the case when $\tilde{f}^*(u) \in H^*_{\mathbb{C}G}$. Using Lemma 4.8 and (27), we have

$$\tilde{f}_*(x) = \vartheta \otimes y, \text{ for some } y \in H^{\mathbb{C}G}_{2i-m}, \forall x \in H^{\mathbb{C}G}_{2i}, i > 0, \quad \tilde{f}_*(\vartheta \otimes x) \in H^{\mathbb{C}G}_*, \forall x \in H^{\mathbb{C}G}_*.$$

This is exactly the same if we take $\mu = 0$ in (30). The rest of the calculations also remains the same and we get the result. □

Remark 4.11. *There are many situations when the map f satisfies the required hypothesis (2) considered in Theorem 4.9 or Theorem 4.10. Some of them are as follows:*

(1) *The lift $\tilde{f}$ stabilizes a copy of Grassmannian, i.e., $\tilde{f}(\{x_0\} \times \mathbb{C}G_{n,k}) \subseteq \{x_0\} \times \mathbb{C}G_{n,k}$ for some $x_0 \in \mathbb{S}^m$.*
(2) *The map $p_1 \circ \tilde{f}^* \circ i_1 : H^*_{\mathbb{C}G} \to H^*_{\mathbb{C}G}$ is an automorphism, equivalently, $f^*(c_1^2) = \lambda^2 c_1^2$, $\lambda \in \mathbb{Q}\backslash\{0\}$, where p_1 and i_1 are defined in (11).*
(3) *The map $p_2 \circ \tilde{f} \circ i_1 : \mathbb{S}^m \to \mathbb{C}G_{n,k}$ is rationally null homotopic, where p_2 is the projection onto the second summand and i_1 is the inclusion into the first summand.*

Under the assumption $m > 2k$, any continuous map f on the generalized Dold space $P(m, n, k)$, the lift $\tilde{f}$ (from Remark 3.7) satisfies $\tilde{f}^*(c_i) = \lambda^i c_i$ for all $i \in I$. Hence condition *(2)* of Theorem 4.10 may be omitted, and one obtains the following consequence.

Corollary 4.12. *Let $P(m,n,k)$ be a generalized Dold space with m and $k(n-k)$ both even. Assume $m > 2k$, and the hypothesis* (3) *is satisfied. Then, for any continuous function g on $P(m,n,k)$ that induces an automorphism on $H^*(P(m,n,k);\mathbb{Q})$, the pair $(P(m,n,k), g)$ has the coincidence property.*
In particular, for $g = \mathrm{id}$, the space $P(m,n,k)$ has the fixed-point property.

In Theorem 4.10, the first assumption that g^* is an automorphism of $H^*(P(m,n,k);\mathbb{Q})$ can be relaxed by assuming μ is nonzero, which leads to the following proposition.

Proposition 4.13. *Let $P(m,n,k)$ be a generalized Dold manifold with $k(n-k)$ even and assume that the hypothesis* (3) *is satisfied. Let g and f are two continuous maps on $P(m,n,k)$ and $\tilde{g}, \tilde{f}$ be their lifts as defined in Remark 3.7 such that*

(1) $\tilde{g}^*(H^*_{\mathbb{C}G}) = H^*_{\mathbb{C}G}$.
(2) $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}\backslash\{0\}$

Then, there is a point of coincidence of f and g.

The proof of the above proposition is similar to the proof of Theorem 4.10. Therefore, we omit the details.

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