COHOMOGENEITY ONE Spin(7) METRICS WITH GENERIC ALOFF–WALLACH SPACES AS PRINCIPAL ORBITS

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ABSTRACT. This paper establishes the existence of forward complete cohomogeneity one Spin(7) metrics with generic Aloff–Wallach spaces $N_{k,l}$ as principal orbits and $\mathbb{CP}^2$ as the singular orbit, building on Reidegeld's analysis of the initial value problem. We construct three continuous one-parameter families of non-compact Spin(7) metrics. Each family contains a limiting asymptotically conical (AC) metric, while the other metrics in the families are asymptotically locally conical (ALC). Moreover, two of the AC metrics share the same asymptotic cone, exhibiting a geometric transition phenomenon analogous to that found by Lehmann in the exceptional case.

1. Introduction

Metrics with Spin(7) holonomy are of great interest in both differential geometry and theoretical physics. The first example was constructed in [Bry87] on the cone over the Berger space $SO(5)/SO(3)$. The first complete example was constructed in [BS89], defined on the spinor bundle over $\mathbb{S}^4$. The first compact example was given in [Joy99]. The compact examples rely on resolving orbifold singularities followed by delicate analytic perturbation, whereas most non-compact examples are obtained by exploiting symmetry, which reduces the Spin(7) equations to a system of ODEs. These examples are of cohomogeneity one.

In this paper, we follow the latter approach to seek further examples of Spin(7) metrics, where the geometry is determined by the choice of principal orbit. Among the possible homogeneous 7-manifolds, Aloff–Wallach spaces stand out as ideal candidates for constructing new Spin(7) metrics. An Aloff–Wallach space is a homogeneous space $N_{k,l} := SU(3)/U(1)_{k,l}$, where $U(1)_{k,l}$ is embedded in $SU(3)$ as

$$\operatorname{diag}\left(e^{k\sqrt{-1}t}, e^{l\sqrt{-1}t}, e^{-(k+l)\sqrt{-1}t}\right).$$

Without loss of generality, we set (k,l) to be non-negative and coprime. Due to the geometric differences, an Aloff–Wallach space is *exceptional* if $kl(k-l)=0$ and *generic* if otherwise. Aloff–Wallach spaces possess remarkably rich topological and geometric structures. Setting $\Delta = k^2+kl+l^2$, we have

$$H^4(N_{k,l},\mathbb{Z}) = \mathbb{Z}/\Delta\mathbb{Z},$$

2020 Mathematics Subject Classification: 53C25 (primary).

Keywords: Spin(7) metric, cohomogeneity one metric.

The author is supported by the NSFC (No. 12301078), the Natural Science Foundation of Jiangsu Province (BK-20220282), and the XJTLU Research Development Funding (RDF-21-02-083).

which shows that there exist infinitely many homotopy types among them. Moreover, there are known examples of homeomorphic but non-diffeomorphic Aloff–Wallach spaces [KS91]. An Aloff–Wallach space can be regarded as a principal $\mathbb{S}^1$-bundle over $SU(3)/T^2$, whose bundle structure depends on the chosen embedding of $U(1)_{k,l} \subset SU(3)$. Alternatively, each $N_{k,l}$ can also be viewed as lens space bundles

$$\mathbb{S}^3/\mathbb{Z}_i \hookrightarrow N_{k,l} \to \mathbb{CP}^2, \quad i \in \{k+l, l, k\},$$

giving rise to three topologically distinct cohomogeneity one orbifold $M^i_{k,l}$, each as an $\mathbb{R}^4/\mathbb{Z}_i$-bundle over $\mathbb{CP}^2$. By [Wan82, CR84, PP84, KV93], each generic Aloff–Wallach space admits two non-isometric $SU(3)$-invariant Einstein metrics. The Euclidean metric cones over these homogeneous Einstein metrics are Spin(7) [Bär93]. This naturally motivates the study of cohomogeneity one Spin(7) metrics on $M^i_{k,l}$.

For the exceptional cases, the picture is largely understood. For $N_{1,1}$, the orbifold $M^2_{1,1}$ admits the Bryant–Salamon metric in [BS89], which belongs to the $\mathbb{B}_8$ family introduced numerically in [CGLP02] and later proved to exist in [Baz07, Baz08]. The space $M^1_{1,1}$ is the manifold $T^*\mathbb{CP}^2$, and it carries the Calabi metric [Cal79], which is HyperKähler. The metric appears as the limiting AC metric of the one-parameter family of ALC Spin(7) metrics constructed in [Chi22]. For $N_{1,0}$, it was conjectured in [CGLP02] and [GST03] that there exist two topologically distinct resolutions of the same Spin(7) cone. The conjecture was later confirmed in [Leh22], where two topologically different continuous families of ALC Spin(7) metrics were constructed: one has singular orbit $\mathbb{S}^5$, and the other has $\mathbb{CP}^2$. Each family contains an AC limiting metric, and the two AC metrics share the same asymptotic cone based on the unique $SU(3) \times U(1)$-invariant nearly parallel G_2 structure on $N_{1,0}$.

In contrast, the situation for generic Aloff–Wallach spaces remains less understood. By Reidegeld's local analysis in [Rei11], a singular orbit as $\mathbb{S}^5$ can only occur in the $N_{1,0}$ case. The main interest lies in forward complete examples where $N_{k,l}$ collapses to $\mathbb{CP}^2$. Explicit isolated examples were obtained in [CGLP02, KY02]. The generic case was further investigated in [Chi22], where we proved the existence of a continuous one-parameter family of ALC Spin(7) metrics on $M^{k+l}_{k,l}$ and $M^k_{k,l}$ with an artificial assumption $k > l$.

The present paper serves as a sequel to [Chi22] in two aspects. Firstly, we reconstruct an invariant set that applies to all coprime pairs (k, l). Being derived from comparing the metric components of the $\mathbb{S}^1$-fiber and the $SU(3)/T^2$ base, the new invariant set is more geometrically motivated. Secondly, we construct another invariant set, where the $\mathbb{S}^1$-fiber blows up. The limiting integral curves that stay between these two invariant sets represent AC Spin(7) metrics, which are analogous to the Lehmann's AC metrics in the $N_{1,0}$ case.

Theorem 1.1. *For any coprime pair (k, l), on each $M^i_{k,l}$ with $i \in \{k+l, l, k\}$, there exists a continuous one-parameter family of forward complete* Spin(7) *metrics*

$$\left\{\gamma^i_\theta \mid \theta \in (0, \theta_i]\right\}.$$

For $\theta \in (0, \theta_i)$, the metric γ^i_θ is asymptotically locally conical (ALC), with its ALC limit modeled on an $\mathbb{S}^1$-bundle over the G_2 cone on the nearly Kähler $SU(3)/T^2$. For the endpoint θ_i, the limiting metric $\gamma^i_{\theta_i}$ is asymptotically conical (AC). The AC limits of $\gamma^k_{\theta_k}$ and $\gamma^l_{\theta_l}$ are modeled on a homogeneous Einstein metric on $N_{k,l}$, *whereas the AC limit of $\gamma^{k+l}_{\theta_{k+l}}$ is modeled on another homogeneous Einstein metric on $N_{k,l}$.*

The spirit of Lehmann's metrics is reflected in the main theorem above. For each coprime pair (k,l), the orbifolds $M^k_{k,l}$ and $M^l_{k,l}$ are topologically distinct, both carrying AC Spin(7) metrics that share the same asymptotic cone. In particular, if $1 \in \{k,l\}$, one of the resolutions of the Spin(7) cone on $N_{k,l}$ yields a smooth manifold $M^1_{k,l}$, whereas the other still has an orbifold singularity.

This paper is organized as follows. In Section 2, we briefly recall the setup and main equations from our previous work [Chi22]. Section 3 reviews the local analysis developed therein. These two sections are included mainly for completeness, and readers familiar with the subject may safely skip them. In Section 4, we construct invariant sets that lead to the existence of one-parameter families of ALC Spin(7) metrics on each $M^i_{k,l}$. In Section 5, we construct another invariant set, from which the existence of AC Spin(7) metrics on $M^i_{k,l}$ follows.

2. The Spin(7) holonomy cohomogeneity one system

The cohomogeneity one Ricci-flat equations derived in [Rei11] were reformulated in [Chi22] as a dynamical system in the variables

$$(X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3, Z_4)$$

constrained to an algebraic surface. Here the variables X_j represent the normalized principal curvatures, while the variables Z_j correspond normalized metric components. In the following, we give a brief presentation of the dynamical system and the definitions of (X_j, Z_j). For full details and derivations, the reader is referred to [Rei11, Chi22].

For an Aloff–Wallach space $N_{k,l}$, we fix a basis for $\mathfrak{su}(3)$ as in [Rei11, (4.3)]. The isotropy representation $\mathfrak{su}(3)/\mathfrak{u}(1)_{k,l}$ is decomposed as

$$\mathfrak{su}(3)/\mathfrak{u}(1)_{k,l} = \mathbf{2}_{k-l} \oplus \mathbf{2}_{2k+l} \oplus \mathbf{2}_{k+2l} \oplus \mathbf{1}, \tag{2.1}$$

where each subscript denotes the corresponding $\mathfrak{u}(1)_{k,l}$-weight. With our convention that k and l are non-negative coprime integers, the above four irreducible summands are pairwise inequivalent if and only if

$$(k,l) \notin \{(1,0),(0,1),(1,1)\},$$

which is the generic case for Aloff–Wallach spaces. By the $SU(3)$-action, an invariant metric $g_{N_{k,l}}$ on the principal orbit is determined by a positive definite symmetric bilinear form on $\mathfrak{su}(3)/\mathfrak{u}(1)_{k,l}$. Hence, the matrix representation of $g_{N_{k,l}}$ takes the form

$$\begin{bmatrix} a^2 & 0 & & & & & & \\ 0 & a^2 & & & & & & \\ & & b^2 & 0 & & & & \\ & & 0 & b^2 & & & & \\ & & & & c^2 & 0 & & \\ & & & & 0 & c^2 & & \\ & & & & & & f^2 \end{bmatrix}, \tag{2.2}$$

where a^2, b^2, c^2, and f^2 respectively correspond to irreducible summands in (2.1). Consider a cohomogeneity one metric of the form

$$g = dt^2 + g_{N_{k,l}}(t),$$

where components in (2.2) are now functions of t. We follow [EW00, Rei11] to obtain the cohomogeneity one Ricci-flat system:

(2.3)
$$
\begin{aligned}
\frac{\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2&=-\left(2\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}+\frac{\dot{f}}{f}\right)\frac{\dot{a}}{a}+\frac{6}{a^2}+\frac{a^2}{b^2c^2}-\frac{b^2}{a^2c^2}-\frac{c^2}{a^2b^2}-\frac{1}{2}\frac{(k+l)^2}{\Delta^2}\frac{f^2}{a^4},\\
\frac{\ddot{b}}{b}-\left(\frac{\dot{b}}{b}\right)^2&=-\left(2\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}+\frac{\dot{f}}{f}\right)\frac{\dot{b}}{b}+\frac{6}{b^2}+\frac{b^2}{a^2c^2}-\frac{c^2}{a^2b^2}-\frac{a^2}{b^2c^2}-\frac{1}{2}\frac{l^2}{\Delta^2}\frac{f^2}{b^4},\\
\frac{\ddot{c}}{c}-\left(\frac{\dot{c}}{c}\right)^2&=-\left(2\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}+\frac{\dot{f}}{f}\right)\frac{\dot{c}}{c}+\frac{6}{c^2}+\frac{c^2}{a^2b^2}-\frac{a^2}{b^2c^2}-\frac{b^2}{a^2c^2}-\frac{1}{2}\frac{k^2}{\Delta^2}\frac{f^2}{c^4},\\
\frac{\ddot{f}}{f}-\left(\frac{\dot{f}}{f}\right)^2&=-\left(2\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}+\frac{\dot{f}}{f}\right)\frac{\dot{f}}{f}+\frac{1}{2}\frac{(k+l)^2}{\Delta^2}\frac{f^2}{a^4}+\frac{1}{2}\frac{l^2}{\Delta^2}\frac{f^2}{b^4}+\frac{1}{2}\frac{k^2}{\Delta^2}\frac{f^2}{c^4},
\end{aligned}
$$

with a conservation law

(2.4)
$$
\begin{aligned}
&\left(2\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}+2\frac{\dot{c}}{c}+\frac{\dot{f}}{f}\right)^2-2\left(\frac{\dot{a}}{a}\right)^2-2\left(\frac{\dot{b}}{b}\right)^2-2\left(\frac{\dot{c}}{c}\right)^2-\left(\frac{\dot{f}}{f}\right)^2\\
&=12\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-2\left(\frac{a^2}{b^2c^2}+\frac{b^2}{a^2c^2}+\frac{c^2}{a^2b^2}\right)-\frac{1}{2}\frac{(k+l)^2}{\Delta^2}\frac{f^2}{a^4}-\frac{1}{2}\frac{l^2}{\Delta^2}\frac{f^2}{b^4}-\frac{1}{2}\frac{k^2}{\Delta^2}\frac{f^2}{c^4}.
\end{aligned}
$$

Singular orbits of $M_{k,l}^{k+l}$, $M_{k,l}^{l}$ and $M_{k,l}^{k}$ are respectively generated by $\mathbf{2}^{2k+l}\oplus\mathbf{2}^{k+2l}$, $\mathbf{2}^{k-l}\oplus\mathbf{2}^{k+2l}$ and $\mathbf{2}^{k-l}\oplus\mathbf{2}^{2k+l}$. By [EW00, Rei11], the corresponding initial conditions are

$$
\begin{aligned}
&\lim_{t\to 0}(a,b,c,f,\dot{a},\dot{b},\dot{c},\dot{f})=\left(0,a_0,a_0,0,1,0,0,\frac{2\Delta}{k+l}\right),\\
&\lim_{t\to 0}(a,b,c,f,\dot{a},\dot{b},\dot{c},\dot{f})=\left(b_0,0,b_0,0,0,1,0,\frac{2\Delta}{l}\right),\\
&\lim_{t\to 0}(a,b,c,f,\dot{a},\dot{b},\dot{c},\dot{f})=\left(c_0,c_0,0,0,0,0,1,\frac{2\Delta}{k}\right),\\
&a_0,b_0,c_0>0.
\end{aligned}
\tag{2.5}
$$

The Spin(7) equations derived in [Rei11] are

$$
\begin{aligned}
\frac{\dot{a}}{a}&=\frac{b}{ac}+\frac{c}{ab}-\frac{a}{bc}-\frac{k+l}{2\Delta}\frac{f}{a^2},\\
\frac{\dot{b}}{b}&=\frac{c}{ab}+\frac{a}{bc}-\frac{b}{ac}+\frac{l}{2\Delta}\frac{f}{b^2},\\
\frac{\dot{c}}{c}&=\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}+\frac{k}{2\Delta}\frac{f}{c^2},\\
\frac{\dot{f}}{f}&=\frac{k+l}{2\Delta}\frac{f}{a^2}-\frac{l}{2\Delta}\frac{f}{b^2}-\frac{k}{2\Delta}\frac{f}{c^2}.
\end{aligned}
\tag{2.6}
$$

Changing the sign of f in (2.6) yields the Spin(7) condition with the opposite chirality:

$$\begin{aligned}\frac{\dot a}{a}&=\frac{b}{ac}+\frac{c}{ab}-\frac{a}{bc}+\frac{k+l}{2\Delta}\frac{f}{a^2},\\ \frac{\dot b}{b}&=\frac{c}{ab}+\frac{a}{bc}-\frac{b}{ac}-\frac{l}{2\Delta}\frac{f}{b^2},\\ \frac{\dot c}{c}&=\frac{a}{bc}+\frac{b}{ac}-\frac{c}{ab}-\frac{k}{2\Delta}\frac{f}{c^2},\\ \frac{\dot f}{f}&=-\frac{k+l}{2\Delta}\frac{f}{a^2}+\frac{l}{2\Delta}\frac{f}{b^2}+\frac{k}{2\Delta}\frac{f}{c^2}.\end{aligned}\tag{2.7}$$

We follow the change of coordinates in [Chi22], which transforms the Ricci-flat system to a dynamical system on an algebraic surface. Let L be the shape operator of $N_{k,l}$ in $M^i_{k,l}$. Normalize the orbit space by $d\eta=\mathrm{tr}Ldt$. Define functions

$$\begin{aligned}&X_1=\frac{\frac{\dot a}{a}}{\mathrm{tr}L},\quad X_2=\frac{\frac{\dot b}{b}}{\mathrm{tr}L},\quad X_3=\frac{\frac{\dot c}{c}}{\mathrm{tr}L},\quad X_4=\frac{\frac{\dot f}{f}}{\mathrm{tr}L},\\ &Z_1=\frac{\frac{a}{bc}}{\mathrm{tr}L},\quad Z_2=\frac{\frac{b}{ac}}{\mathrm{tr}L},\quad Z_3=\frac{\frac{c}{ab}}{\mathrm{tr}L},\quad Z_4=f\mathrm{tr}L.\end{aligned}\tag{2.8}$$

Let $'$ denote the derivative with respect to η. The original dynamical system (2.3) is transformed to

$$\begin{bmatrix}X_1\\X_2\\X_3\\X_4\\Z_1\\Z_2\\Z_3\\Z_4\end{bmatrix}'=\begin{bmatrix}X_1(G-1)+R_1\\X_2(G-1)+R_2\\X_3(G-1)+R_3\\X_4(G-1)+R_4\\Z_1(G+X_1-X_2-X_3)\\Z_2(G+X_2-X_3-X_1)\\Z_3(G+X_3-X_1-X_2)\\Z_4(-G+X_4)\end{bmatrix},\tag{2.9}$$

where

$$\begin{aligned}G&=2X_1^2+2X_2^2+2X_3^2+X_4^2,\\ R_1&=6Z_2Z_3+Z_1^2-Z_2^2-Z_3^2-\frac{1}{2}\frac{(k+l)^2}{\Delta^2}Z_2^2Z_3^2Z_4^2,\\ R_2&=6Z_1Z_3+Z_2^2-Z_3^2-Z_1^2-\frac{1}{2}\frac{l^2}{\Delta^2}Z_1^2Z_3^2Z_4^2,\\ R_3&=6Z_1Z_2+Z_3^2-Z_1^2-Z_2^2-\frac{1}{2}\frac{k^2}{\Delta^2}Z_1^2Z_2^2Z_4^2,\\ R_4&=\frac{1}{2}\frac{(k+l)^2}{\Delta^2}Z_2^2Z_3^2Z_4^2+\frac{1}{2}\frac{l^2}{\Delta^2}Z_1^2Z_3^2Z_4^2+\frac{1}{2}\frac{k^2}{\Delta^2}Z_1^2Z_2^2Z_4^2.\end{aligned}\tag{2.10}$$

The conservation law (2.4) becomes

$$G-1+2R_1+2R_2+2R_3+R_4=0.\tag{2.11}$$

Since $\left(\frac{1}{\mathrm{tr}(L)}\right)'=\frac{1}{\mathrm{tr}(L)}G$, the quantity $\frac{1}{\mathrm{tr}(L)}$ can be treated as a function of η by

$$\frac{1}{\mathrm{tr}(L)}=\exp\left(\int_{\eta^*}^{\eta}Gd\tilde\eta+C\right).$$

To recover the original coordinates, we simply compute

$$t=\int_{\eta^*}^{\eta}\frac{1}{\mathrm{tr}(L)}d\tilde{\eta}=\int_{\eta^*}^{\eta}\exp\left(\int_{\eta^{**}}^{\eta^*}Gd\tilde{\tilde{\eta}}+C\right)d\tilde{\eta}+t_0$$

and

$$a=\frac{1}{\mathrm{tr}(L)}\frac{1}{\sqrt{Z_2Z_3}},\quad b=\frac{1}{\mathrm{tr}(L)}\frac{1}{\sqrt{Z_1Z_3}},\quad c=\frac{1}{\mathrm{tr}(L)}\frac{1}{\sqrt{Z_1Z_2}},\quad f=\frac{Z_4}{\mathrm{tr}(L)}.$$

Although our main focus is on constructing Spin(7) metrics, it is more convenient to begin with the full Ricci-flat system, where several key estimates are transparent. From the definition of X_j in (2.8), one expects the equality

$$2X_1+2X_2+2X_3+X_4=1 \tag{2.12}$$

be preserved by the new dynamical system. Indeed, since

(2.13)
$$\begin{aligned}(2X_1+2X_2+2X_3+X_4)'&=(2X_1+2X_2+2X_3+X_4)(G-1)+2R_1+2R_2+2R_3+R_4\\&=(2X_1+2X_2+2X_3+X_4)(G-1)+1-G\quad\text{by (2.11)}\\&=(2X_1+2X_2+2X_3+X_4-1)(G-1)\end{aligned},$$

the set

$$\{2X_1+2X_2+2X_3+X_4=1\}$$

is invariant. We can also assume each Z_j to be non-negative since the set $\{Z_j=0\}$ is invariant by (2.9). A straightforward observation gives the following proposition.

Proposition 2.1. The set $\{X_4\geq 0\}$ is invariant.

Proof. We have

$$X_4'|_{X_4=0}=R_4=\frac{1}{2}\frac{(k+l)^2}{\Delta^2}Z_2^2Z_3^2Z_4^2+\frac{1}{2}\frac{l^2}{\Delta^2}Z_1^2Z_3^2Z_4^2+\frac{1}{2}\frac{k^2}{\Delta^2}Z_1^2Z_2^2Z_4^2\geq 0. \tag{2.14}$$

If a non-transversal intersection emerges on an integral curve, it is necessary that the integral curve is in the invariant set $\{Z_j=0\}$ for some $j\in\{1,2,3,4\}$. Hence, the derivative X_4' vanishes identically. The possibility of non-transversal intersections is excluded. □

By our discussion above, a cohomogeneity one Ricci-flat metric with a generic $N_{k,l}$ as the principal orbit is represented by an integral curve to (2.9) in the following invariant subset of $\mathbb{R}^8$:

(2.15)
$$\begin{aligned}\mathcal{C}_{RF}:=&\{G-1+2R_1+2R_2+2R_3+R_4=0\}\cap\{2X_1+2X_2+2X_3+X_4=1\}\\&\cap\{Z_1,Z_2,Z_3,Z_4\geq 0\}\cap\{X_4\geq 0\},\end{aligned}$$

a 6-dimensional algebraic surface with boundaries. By [BDW15, Lemma 5.1], we have $\lim_{\eta\to\infty}t=\infty$. Therefore, if an integral curve is defined on $\mathbb{R}$, the corresponding Ricci-flat metric is forward complete.

The Spin(7) equations (2.6)–(2.7) are first-order subsystems of the second-order system (2.3). In the new coordinates, they appear as invariant algebraic surfaces and further reduce the dimension of the phase space. Specifically, equations

(2.6)–(2.7) become

$$\begin{aligned}\mathcal{S}_1^{\pm}: X_1&=Z_2+Z_3-Z_1\mp\frac{k+l}{2\Delta}Z_2Z_3Z_4,\\ \mathcal{S}_2^{\pm}: X_2&=Z_3+Z_1-Z_2\pm\frac{l}{2\Delta}Z_1Z_3Z_4,\\ \mathcal{S}_3^{\pm}: X_3&=Z_1+Z_2-Z_3\pm\frac{k}{2\Delta}Z_1Z_2Z_4,\\ \mathcal{S}_4^{\pm}: X_4&=\pm\frac{k+l}{2\Delta}Z_2Z_3Z_4\mp\frac{l}{2\Delta}Z_1Z_3Z_4\mp\frac{k}{2\Delta}Z_1Z_2Z_4.\end{aligned}\tag{2.16}$$

Substituting each X_j in (2.11) using (2.16), we obtain

$$\left(2(Z_1+Z_2+Z_3)\mp\frac{k+l}{2\Delta}Z_2Z_3Z_4\pm\frac{l}{2\Delta}Z_1Z_3Z_4\pm\frac{k}{2\Delta}Z_1Z_2Z_4\right)^2=1.$$

On the other hand, substituting each X_j in (2.12) yields

$$2(Z_1+Z_2+Z_3)\mp\frac{k+l}{2\Delta}Z_2Z_3Z_4\pm\frac{l}{2\Delta}Z_1Z_3Z_4\pm\frac{k}{2\Delta}Z_1Z_2Z_4=1,\tag{2.17}$$

or equivalently

$$2(Z_1+Z_2+Z_3)-X_4=1.\tag{2.18}$$

By [Chi22, Proposition 2.2], the sets

$$\mathcal{C}_{\mathrm{Spin}(7)}^{\pm}=\mathcal{C}_{RF}\cap\left(\bigcap_{i=1}^{4}\mathcal{S}_i^{\pm}\right)\cap\{2(Z_1+Z_2+Z_3)-X_4=1\}\tag{2.19}$$

are invariant. A cohomogeneity one Spin(7) metric with a generic $N_{k,l}$ as the principal orbit is represented by an integral curve to (2.9) restricted to $(\mathcal{C}_{\mathrm{Spin}(7)}^{+}\cup\mathcal{C}_{\mathrm{Spin}(7)}^{-})\subset\mathcal{C}_{RF}$, a 3-dimensional algebraic surface in $\mathbb{R}^8$ with boundaries.

It is apparent that

$$\mathcal{C}_{G_2}=\mathcal{C}_{\mathrm{Spin}(7)}^{+}\cap\mathcal{C}_{\mathrm{Spin}(7)}^{-}\cap\{X_4=0\}\cap\{Z_4=0\}\tag{2.20}$$

is a 2-dimensional invariant subset. System (2.9) restricted to $\mathcal{C}_{G_2}$ is essentially the one for cohomogeneity one G_2 metrics with $\mathbb{CP}^2$ as the singular orbit and $SU(3)/T^2$ as the principal orbit. For a forward complete ALC Spin(7) metric, components a,b and c in (2.2) grow linearly and f converges to a constant as $t\to\infty$. The corresponding integral curve converges to the invariant set $\mathcal{C}_{G_2}$ as $\eta\to\infty$.

3. Local Existence

In this section, we study the critical points of the Spin(7) system. These critical points encode the initial conditions (2.5) and the AC/ALC asymptotics. Linearizations at the initial condition critical points yield the local existence of Spin(7) metrics near the tubular neighbourhood of $\mathbb{CP}^2$, recovering the result of [Rei11].

The following is the complete list of critical points of (2.9) on $\mathcal{C}_{\mathrm{Spin}(7)}^{\pm}$. We refer the reader to [Chi22] for the complete list of critical points of (2.9) on $\mathcal{C}_{RF}$.

I $P_0^{k+l}:=\left(\frac{1}{3},0,0,\frac{1}{3},0,\frac{1}{3},\frac{1}{3},\frac{6\Delta}{k+l}\right)\in\mathcal{C}_{\mathrm{Spin}}^{+}$;
$P_0^{l}:=\left(0,\frac{1}{3},0,\frac{1}{3},\frac{1}{3},0,\frac{1}{3},\frac{6\Delta}{l}\right)$, $\quad P_0^{k}:=\left(0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0,\frac{6\Delta}{k}\right)\in\mathcal{C}_{\mathrm{Spin}}^{-}$.

These critical points represent the initial conditions (2.5) in the new coordinate. Integral curves that emanate from these points represent Ricci-flat metrics that are defined on the tubular neighborhood around $\mathbb{CP}^2$ in $M_{k,l}^{k+l}$, $M_{k,l}^{l}$ and $M_{k,l}^{k}$, respectively.

II (a) $P_{ALC} := \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0\right) \in \mathcal{C}_{G_2}$

If an integral curve converges to P_{ALC}, the corresponding metric has ALC asymptotics, with its end modeled on an $\mathbb{S}^1$-bundle over the cone on the homogeneous nearly Kähler metric $SU(3)/T^2$.

III $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{5}}{2}, 0, 0, 0\right)$, $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{\sqrt{5}}{2}, 0, 0\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{\sqrt{5}}{2}, 0\right) \in \mathcal{C}_{G_2}$.

These critical points are sources in the subsystem on $\mathcal{C}_{G_2}$. Singular G_2 metrics in [CS02] are represented by integral curves that emanate from these points.

IV $Q_0^{k+l} = \left(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{4}, \frac{1}{4}, 0\right)$, $Q_0^{l} = \left(0, \frac{1}{2}, 0, 0, \frac{1}{4}, 0, \frac{1}{4}, 0\right)$, $Q_0^{k} = \left(0, 0, \frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}, 0, 0\right) \in \mathcal{C}_{G_2}$.

These critical points are saddles in the subsystem on $\mathcal{C}_{G_2}$. The smooth G_2 metrics that collapse to $\mathbb{CP}^2$ in [BS89, GPP90] are represented by integral curves that emanate from these points.

V $\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, z_1, z_2, z_3, z_4\right)$, where $R_i(z_1, z_2, z_3, z_4) = \frac{6}{49}$ for each i.

There are exactly two critical points of this type, corresponding to the homogeneous Einstein metrics on $N_{k,l}$ [Wan82, CR84, PP84, KV93]. Viewed as trivial integral curves of (2.9), they represent Ricci-flat metric cones over these homogeneous Einstein metrics, which are in fact Spin(7) [Bär93]. Below we recover the existence of these critical points and show that the associated Spin(7) cones have opposite chirality.

Proposition 3.1. Each of $\mathcal{C}^{\pm}_{\mathrm{Spin}(7)}$ admits exactly one critical point of Type V, respectively denoted as $P^{\pm}_{AC}$.

Proof. By the aforementioned works, there are exactly two critical points of Type V in $\mathcal{C}^{+}_{\mathrm{Spin}(7)} \cup \mathcal{C}^{-}_{\mathrm{Spin}(7)}$. We show that each of $\mathcal{C}^{\pm}_{\mathrm{Spin}(7)}$ contains one.

By (2.18), Type V critical points on $\mathcal{C}^{\pm}_{\mathrm{Spin}(7)}$ are on the hypersurface $\{Z_1 + Z_2 + Z_3 = \frac{4}{7}\}$. Substituting $X_j = \frac{1}{7} = \frac{1}{4}(Z_1 + Z_2 + Z_3)$ in $\mathcal{S}_j^{\pm}$ for $j \in \{1, 2, 3\}$, we have

$$\begin{aligned} &\frac{1}{Z_2 Z_3}(3Z_2 + 3Z_3 - 5Z_1) \mp \frac{k+l}{2\Delta} Z_4 = 0, \\ &\frac{1}{Z_3 Z_1}(3Z_3 + 3Z_1 - 5Z_2) \pm \frac{l}{2\Delta} Z_4 = 0, \\ &\frac{1}{Z_1 Z_2}(3Z_1 + 3Z_2 - 5Z_3) \pm \frac{k}{2\Delta} Z_4 = 0. \end{aligned} \tag{3.1}$$

Summing the above three equations, we have

$$5(Z_1^2 + Z_2^2 + Z_3^2) = 6(Z_2 Z_3 + Z_3 Z_1 + Z_1 Z_2),$$

while the last two equations above yield

$$k Z_2 (3Z_3 + 3Z_1 - 5Z_2) = l Z_3 (3Z_1 + 3Z_2 - 5Z_3).$$

Therefore, Type V critical points are realized as intersections between the the circle

$$\left\{ Z_1 + Z_2 + Z_3 = \frac{4}{7} \right\} \cap \{5(Z_1^2 + Z_2^2 + Z_3^2) = 6(Z_2 Z_3 + Z_3 Z_1 + Z_1 Z_2)\}.$$

and the hyperbola

$$\left\{Z_1 + Z_2 + Z_3 = \frac{4}{7}\right\} \cap \{kZ_2(3Z_3 + 3Z_1 - 5Z_2) = lZ_3(3Z_1 + 3Z_2 - 5Z_3)\}.$$

Consider $\alpha = \frac{Z_2}{Z_1}$, $\beta = \frac{Z_3}{Z_1}$, it suffices to solve the equations $L_1 = L_2 = 0$, where

$$\begin{aligned} L_1 &= 5(1 + \alpha^2 + \beta^2) - (6\alpha\beta + 6\beta + 6\alpha) = (\alpha + \beta - 3)^2 + 4(\alpha - \beta)^2 - 4 \\ L_2 &= k\alpha(3 + 3\beta - 5\alpha) - l\beta(3 + 3\alpha - 5\beta). \end{aligned} \tag{3.2}$$

The level curve $L_1 = 0$ is an ellipse in the (α, β)-space that passes through

$$\left(\frac{2}{5}, 1\right), \quad \left(1, \frac{2}{5}\right), \quad \left(2, \frac{13}{5}\right), \quad \left(\frac{13}{5}, 2\right),$$

while the level curve $L_2 = 0$ is a hyperbola. Since

$$L_2\left(\frac{2}{5}, 1\right) = k\frac{8}{5} + l\frac{4}{5}, \quad L_2\left(1, \frac{2}{5}\right) = -k\frac{4}{5} - l\frac{8}{5},$$

$$L_2\left(2, \frac{13}{5}\right) = k\frac{58}{5} + l\frac{52}{5}, \quad L_2\left(\frac{13}{5}, 2\right) = -k\frac{52}{5} - l\frac{58}{5},$$

there exists an intersection point on the elliptical arc

$$\{L_1 = 0\} \cap \left\{\alpha + \beta < \frac{7}{5}\right\} = \{L_1 = 0\} \cap \left\{\alpha + \beta < \frac{7}{5}\right\} \cap \left\{-\frac{3}{5} < \alpha - \beta < \frac{3}{5}\right\},$$

another on

$$\{L_1 = 0\} \cap \left\{\alpha + \beta > \frac{23}{5}\right\} = \{L_1 = 0\} \cap \left\{\alpha + \beta > \frac{23}{5}\right\} \cap \left\{-\frac{3}{5} < \alpha - \beta < \frac{3}{5}\right\}.$$

For a Type V critical point, we have

$$\frac{1}{7} = X_4 = \pm\left(\frac{k + l}{2\Delta} Z_2 Z_3 - \frac{l}{2\Delta} Z_1 Z_3 - \frac{k}{2\Delta} Z_1 Z_2\right) Z_4.$$

For the first intersection point, we have $\alpha, \beta < 1$, equivalently $Z_2, Z_3 < Z_1$. This makes the expression in parentheses negative and thus the point lies in $\mathcal{C}^-_{\text{Spin}(7)}$ in order for $Z_4 > 0$. For the second intersection point, we have $\alpha, \beta > 2$, equivalently $Z_2, Z_3 > 2Z_1$. The expression in parentheses is positive, so the intersection point must lie in $\mathcal{C}^+_{\text{Spin}(7)}$ for $Z_4 > 0$. $\square$

Each Type I critical point P_0^i is hyperbolic with a single positive eigenvalue $\frac{2}{3}$, and it has two unstable eigenvectors v_1 and v_2 that are tangent to $\mathcal{C}^\pm_{\text{Spin}(7)}$. Below we list v_1 and v_2 for each P_0^i.

$$
\begin{array}{c|c|c|c}
 & P_0^{k+l} & P_0^l & P_0^k \\
\hline
v_1 & \begin{bmatrix} 2\\0\\0\\-4\\0\\-1\\-1\\-36\frac{\Delta}{k+l} \end{bmatrix} & \begin{bmatrix} 0\\2\\0\\-4\\-1\\0\\-1\\-36\frac{\Delta}{l} \end{bmatrix} & \begin{bmatrix} 0\\0\\2\\-4\\-1\\-1\\0\\-36\frac{\Delta}{k} \end{bmatrix} \\
v_2 & \begin{bmatrix} -3(k+l)\\4k+5l\\5k+4l\\-12(k+l)\\3(k+l)\\-5k-4l\\-4k-5l\\0 \end{bmatrix} & \begin{bmatrix} 5l+k\\-3l\\4l-k\\-12l\\-4l+k\\3l\\-5l-k\\0 \end{bmatrix} & \begin{bmatrix} 5k+l\\4k-l\\-3k\\-12k\\-4k+l\\-5k-l\\3k\\0 \end{bmatrix}
\end{array}.
$$

By the Hartman–Grobman Theorem, there is a one-to-one correspondence between linearized solutions and the corresponding integral curves emanating from P_0^i. By the unstable version of [CL55, Theorem 4.5], it is unambiguous to denote the integral curve emanating from P_0^i by γ_θ^i, where

$$
\gamma_\theta^i = P_0^i + s_1 e^{\frac{2\eta}{3}} v_1 + s_2 e^{\frac{2\eta}{3}} v_2 + O(e^{\left(\frac{2}{3}+\epsilon\right)\eta}), \quad (s_1, s_2) = (\cos(\theta), \sin(\theta)). \tag{3.3}
$$

The normalization $s_1^2+s_2^2 = 1$ removes the scaling redundancy, and we set $\theta \in [0,\pi]$ so that each Z_j is non-negative. We hence obtain

$$
\{\gamma_\theta^i \mid \theta \in [0,\pi]\},
$$

a continuous one-parameter family of Spin(7) metrics defined on a tubular neighbourhood of $\mathbb{CP}^2$ in $M_{k,l}^i$. All integral curves in the interior represent non-degenerate metrics, since each $Z_j > 0$. The two integral curves at the boundaries $\theta = 0, \pi$ represent degenerate metrics, as some Z_j vanishes identically. Specifically, for each i, integral curves γ_0^i and γ_π^i lie on the following algebraic curves $\mathcal{W}_i$.

$$
\begin{aligned}
\mathcal{W}_{k+l} &:= \mathcal{C}^+_{\mathrm{Spin}(7)} \cap \{X_2 = X_3 = 0\} \cap \{X_1 = 1 - 2Z_2\} \cap \{X_4 = 4Z_2 - 1\} \\
&\quad \cap \{Z_2 = Z_3\} \cap \{Z_1 = 0\} \cap \left\{ Z_4 = \frac{2\Delta}{k+l} \frac{4Z_2 - 1}{Z_2^2} \right\}, \\
\mathcal{W}_l &:= \mathcal{C}^-_{\mathrm{Spin}(7)} \cap \{X_1 = X_3 = 0\} \cap \{X_2 = 1 - 2Z_3\} \cap \{X_4 = 4Z_3 - 1\} \\
&\quad \cap \{Z_3 = Z_1\} \cap \{Z_2 = 0\} \cap \left\{ Z_4 = \frac{2\Delta}{l} \frac{4Z_3 - 1}{Z_3^2} \right\}, \\
\mathcal{W}_k &:= \mathcal{C}^-_{\mathrm{Spin}(7)} \cap \{X_1 = X_2 = 0\} \cap \{X_3 = 1 - 2Z_1\} \cap \{X_4 = 4Z_1 - 1\} \\
&\quad \cap \{Z_1 = Z_2\} \cap \{Z_3 = 0\} \cap \left\{ Z_4 = \frac{2\Delta}{k} \frac{4Z_1 - 1}{Z_1^2} \right\}.
\end{aligned} \tag{3.4}
$$

Along γ_0^i, we have $Z_4 \in (0, \frac{6\Delta}{i})$, and the integral curve joins P_0^i and Q_0^i. Along γ_π^i, we have $Z_4 > \frac{6\Delta}{i}$.

Geometrically, the parameter θ governs the initial differences among principal curvatures. For example, for γ_θ^{k+l}, we have

(3.5)
$$\begin{aligned}\cot(\theta) = \frac{s_1}{s_2} = \lim_{\eta \to -\infty} \frac{k+l}{2} \frac{X_1 - X_2 - X_3 - X_4}{Z_1} &= \lim_{t \to 0} \frac{k+l}{2} \frac{bc}{a} \left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b} - \frac{\dot{c}}{c} - \frac{\dot{f}}{f} \right) \\ = -\frac{3(k+l)}{16\Delta}\left(12\Delta + q\right), \end{aligned}$$

where q is the free third-order parameter appearing in the power series [Rei11, (7.1)].

4. Invariant sets for ALC metrics

The compact invariant sets in [Chi22] are constructed using the quantity $Z_4 = f\mathrm{tr}(L)$. If $k > l$, the condition

$$f\mathrm{tr}(L) \le \lim_{t \to 0} \left(f\mathrm{tr}(L)\right)(\gamma_\theta^i)$$

defines a compact invariant set in the (X_j, Z_j)-space for $i \in \{k+l, k\}$, which helps prove the forward completeness for metrics on $M_{k,l}^{k+l}$ and $M_{k,l}^k$. The set fails to be compact if $i = l$. This is the essential limitation of the old construction.

In this section, we introduce a new inequality,

$$(k+l)\frac{f}{a} + l\frac{f}{b} + k\frac{f}{c} \le 2\Delta. \tag{4.1}$$

This condition is more geometrically natural, as it compares the $\mathbb{S}^1$-fiber in $N_{k,l}$ with the other metric components. The inequality admits dihedral symmetry among pairs $(a, k+l)$, (b, l), and (c, k). The associated invariant sets bound Z_1, Z_2, Z_3 simultaneously. This helps prove that a γ_θ^i with a sufficiently small $\theta \ge 0$ remains in a compact subset for *all three cases* $i \in \{k+l, l, k\}$. In Section 4.1 we show that (4.1) defines an invariant set inside $\mathcal{C}^+_{\mathrm{Spin}(7)}$ for γ_θ^{k+l}. In Section 4.2 we prove the analogous statement in $\mathcal{C}^-_{\mathrm{Spin}(7)}$ for γ_θ^l and γ_θ^k.

4.1. $\mathcal{C}^+_{\mathrm{Spin}(7)}$.

Define

$$\begin{aligned} \mathcal{D}^+ := \mathcal{C}^+_{\mathrm{Spin}(7)} &\cap \{Z_1 \le Z_2\} \cap \{Z_1 \le Z_3\} \\ &\cap \left\{ \left(k\sqrt{Z_1 Z_2} + l\sqrt{Z_1 Z_3} + (k+l)\sqrt{Z_2 Z_3}\right) Z_4 \le 2\Delta \right\}. \end{aligned} \tag{4.2}$$

Lemma 4.1. *The set $\mathcal{D}^+$ is invariant.*

Proof. By (2.9) and (2.16), we have

$$\begin{aligned} (Z_2 - Z_1)'|_{Z_2 - Z_1 = 0} &= 2Z_1(X_2 - X_1) \\ &= 2Z_1\left(\frac{l}{2\Delta} Z_1 Z_3 Z_4 + \frac{k+l}{2\Delta} Z_2 Z_3 Z_4\right), \\ &\ge 0 \end{aligned} \tag{4.3}$$

$$(4.4)\quad \begin{aligned}(Z_3-Z_1)'|_{Z_3-Z_1=0} &= 2Z_1(X_3-X_1)\\ &= 2Z_1\left(\frac{k}{2\Delta}Z_1Z_2Z_4+\frac{k+l}{2\Delta}Z_2Z_3Z_4\right).\\ &\geq 0\end{aligned}$$

It suffices to show that an integral curve in $\mathcal{D}^+$ does not escape through the boundary $\mathcal{D}^+\cap\{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta\}$.

We have

(4.5)

$$\begin{aligned}&\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)'\Big|_{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta}\\ &= k\sqrt{Z_1Z_2}Z_4(X_4-X_3)+l\sqrt{Z_1Z_3}Z_4(X_4-X_2)+(k+l)\sqrt{Z_2Z_3}Z_4(X_4-X_1)\\ &= Z_4\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)X_4\\ &\quad + Z_4(k\sqrt{Z_1Z_2}(-X_3)+l\sqrt{Z_1Z_3}(-X_2)+(k+l)\sqrt{Z_2Z_3}(-X_1))\\ &= Z_4\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)\left(\frac{k+l}{2\Delta}Z_2Z_3Z_4-\frac{l}{2\Delta}Z_1Z_3Z_4-\frac{k}{2\Delta}Z_1Z_2Z_4\right)\\ &\quad + k\sqrt{Z_1Z_2}Z_4\left(-\frac{k}{2\Delta}Z_1Z_2Z_4+Z_3-Z_1-Z_2\right)\\ &\quad + l\sqrt{Z_1Z_3}Z_4\left(-\frac{l}{2\Delta}Z_1Z_3Z_4+Z_2-Z_3-Z_1\right)\\ &\quad + (k+l)\sqrt{Z_2Z_3}Z_4\left(\frac{k+l}{2\Delta}Z_2Z_3Z_4+Z_1-Z_2-Z_3\right).\end{aligned}$$

Let

$$(4.6)\qquad Z_1Z_4=\zeta,\quad \sqrt{\frac{Z_2}{Z_1}}=\alpha,\quad \sqrt{\frac{Z_3}{Z_1}}=\beta.$$

The above computation becomes

(4.7)

$$\begin{aligned}&\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)'\Big|_{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta}\\ &= Z_1\zeta^2(k\alpha+l\beta+(k+l)\alpha\beta)\left(\frac{k+l}{2\Delta}\alpha^2\beta^2-\frac{l}{2\Delta}\beta^2-\frac{k}{2\Delta}\alpha^2\right)\\ &\quad + Z_1\zeta^2\left(\frac{(k+l)^2}{2\Delta}\alpha^3\beta^3-\frac{l^2}{2\Delta}\beta^3-\frac{k^2}{2\Delta}\alpha^3\right)\\ &\quad + Z_1\zeta(k\alpha(\beta^2-1-\alpha^2)+l\beta(\alpha^2-1-\beta^2)+(k+l)\alpha\beta(1-\alpha^2-\beta^2)).\end{aligned}$$

With $\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta$, we have

$$(4.8)\qquad \zeta=\frac{2\Delta}{k\alpha+l\beta+(k+l)\alpha\beta}.$$

The equation (4.7) becomes

(4.9)
$$\begin{aligned}
&\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)'\Big|_{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta}\\
&=\frac{Z_1\zeta}{k\alpha+l\beta+(k+l)\alpha\beta}(-\beta^2\Xi_0l^2-\alpha\beta\Xi_1kl-\alpha^2\Xi_2k^2),
\end{aligned}$$

where

$$\begin{aligned}
\Xi_0&=(\alpha+1)^2\beta^2+(1-\alpha)(2\alpha^2+3\alpha+2)\beta+(\alpha^2-1)^2,\\
\Xi_1&=2\alpha\beta(\alpha-\beta)^2+(\alpha+\beta)(2\alpha^2-3\alpha\beta+2\beta^2)+(\alpha-\beta)^2+\alpha+\beta+2,\\
\Xi_2&=(\beta+1)^2\alpha^2+(1-\beta)(2\beta^2+3\beta+2)\alpha+(\beta^2-1)^2.
\end{aligned}\tag{4.10}$$

Since $Z_2,Z_3\geq Z_1$ in $\mathcal{D}^+$, we consider each Ξ_j for $(\alpha,\beta)\in[1,\infty)\times[1,\infty)$. Since the discriminant of Ξ_0 (as a quadratic function of β) is

$$\delta_\beta(\Xi_0)=-\alpha(\alpha-1)^2(4\alpha^2+7\alpha+4)\leq 0,$$

the function $\Xi_0\geq 0$ and vanishes only if $(\alpha,\beta)=(1,0)$. By the transformation $\alpha\leftrightarrow\beta$, the function Ξ_2 is also non-negative and vanishes only if $(\alpha,\beta)=(0,1)$. The function Ξ_1 is apparently positive. Therefore, the derivative (4.5) is negative if $Z_1\neq 0$.

If an integral curve leaves $\mathcal{D}^+$ non-transversally through

$$\{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta\}\cap\{Z_1=0\},$$

we have

(4.11)
$$\begin{aligned}
&\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)'\Big|_{\{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta\}\cap\{Z_1=0\}}\\
&=(k+l)\sqrt{Z_2Z_3}Z_4(X_4-X_1)\\
&=2\Delta\left(\frac{k+l}{\Delta}Z_2Z_3Z_4-Z_2-Z_3\right)\\
&=2\Delta\left(2\sqrt{Z_2Z_3}-Z_2-Z_3\right)\\
&\leq 0.
\end{aligned}$$

Therefore, the non-transversal intersection satisfies

$$(k+l)\sqrt{Z_2Z_3}Z_4=2\Delta,\quad Z_1=0,\quad Z_2=Z_3.$$

By (2.16) and (2.17), the intersection point is P_0^{k+l}, a contradiction. Hence, the set $\mathcal{D}^+$ is invariant. □

4.2. $\mathcal{C}^-_{\mathrm{Spin}(7)}$. Define

$$\begin{aligned}
\mathcal{D}^-:=\;&\mathcal{C}^-_{\mathrm{Spin}(7)}\cap\{Z_1\geq Z_2\}\cap\{Z_1\geq Z_3\}\\
&\cap\left\{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\leq 2\Delta\right\}.
\end{aligned}\tag{4.12}$$

Lemma 4.2. *The set $\mathcal{D}^-$ is invariant.*

Proof. By (2.9) and (2.16), we have

$$
\begin{aligned}
(Z_1-Z_2)'|_{Z_1-Z_2=0} &= 2Z_1(X_1-X_2)\\
&= 2Z_1\left(\frac{k+l}{2\Delta}Z_2Z_3Z_4+\frac{l}{2\Delta}Z_1Z_3Z_4\right),\\
&\geq 0
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
(Z_1-Z_3)'|_{Z_1-Z_3=0} &= 2Z_1(X_1-X_3)\\
&= 2Z_1\left(\frac{k+l}{2\Delta}Z_2Z_3Z_4+\frac{k}{2\Delta}Z_1Z_2Z_4\right).\\
&\geq 0
\end{aligned}
\tag{4.14}
$$

For $\mathcal{D}^-\cap\{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta\}$, we have

(4.15)

$$
\begin{aligned}
&\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)'\Big|_{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta}\\
&= k\sqrt{Z_1Z_2}Z_4(X_4-X_3)+l\sqrt{Z_1Z_3}Z_4(X_4-X_2)+(k+l)\sqrt{Z_2Z_3}Z_4(X_4-X_1)\\
&= Z_4(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3})\left(-\frac{k+l}{2\Delta}Z_2Z_3Z_4+\frac{l}{2\Delta}Z_1Z_3Z_4+\frac{k}{2\Delta}Z_1Z_2Z_4\right)\\
&\quad + k\sqrt{Z_1Z_2}Z_4\left(\frac{k}{2\Delta}Z_1Z_2Z_4+Z_3-Z_1-Z_2\right)\\
&\quad + l\sqrt{Z_1Z_3}Z_4\left(\frac{l}{2\Delta}Z_1Z_3Z_4+Z_2-Z_3-Z_1\right)\\
&\quad + (k+l)\sqrt{Z_2Z_3}Z_4\left(-\frac{k+l}{2\Delta}Z_2Z_3Z_4+Z_1-Z_2-Z_3\right).\\
&= Z_1\zeta^2(k\alpha+l\beta+(k+l)\alpha\beta)\left(-\frac{k+l}{2\Delta}\alpha^2\beta^2+\frac{l}{2\Delta}\beta^2+\frac{k}{2\Delta}\alpha^2\right)\\
&\quad + Z_1\zeta^2\left(-\frac{(k+l)^2}{2\Delta}\alpha^3\beta^3+\frac{l^2}{2\Delta}\beta^3+\frac{k^2}{2\Delta}\alpha^3\right)\\
&\quad + Z_1\zeta(k\alpha(\beta^2-1-\alpha^2)+l\beta(\alpha^2-1-\beta^2)+(k+l)\alpha\beta(1-\alpha^2-\beta^2))
\end{aligned}
$$

where (ζ,α,β) are as in (4.6). Again by (4.8), the equation (4.15) becomes

(4.16)

$$
\begin{aligned}
&\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)'\Big|_{\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4=2\Delta}\\
&= \frac{Z_1\zeta}{k\alpha+l\beta+(k+l)\alpha\beta}(-\beta^2\Theta_0l^2-\alpha\beta\Theta_1kl-\alpha^2\Theta_2k^2),
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta_0 &= (\alpha+1)^2\beta^2+(\alpha-1)(2\alpha^2+3\alpha+2)\beta+(\alpha^2-1)^2\\
\Theta_1 &= 2\alpha\beta(\alpha+\beta)^2+(\alpha+\beta)(2\alpha^2-\alpha\beta+2\beta^2)-(\alpha+\beta)^2-(\alpha+\beta)+2\\
\Theta_2 &= (\beta+1)^2\alpha^2+(\beta-1)(2\beta^2+3\beta+2)\alpha+(\beta^2-1)^2
\end{aligned}
\tag{4.17}
$$

Since $Z_1 \geq Z_2, Z_3$ in $\mathcal{D}^-$, we consider each Θ_j for $(\alpha, \beta) \in [0,1] \times [0,1]$. Note that

$$\delta_\beta(\Theta_0) = \delta_\beta(\Xi_0) \leq 0.$$

Thus the function $\Theta_0 \geq 0$ and vanishes only if $(\alpha, \beta) \in \{(1,0),(0,1)\}$. By the transformation $\alpha \leftrightarrow \beta$, the function Θ_2 has the same property. We show that $\Theta_1 > 0$ in the following.

Since

$$\Theta_1(0,\beta) = (\beta+1)(2\beta^2 - 3\beta + 2) > 0, \quad \Theta_1(\alpha, 0) = (\alpha+1)(2\alpha^2 - 3\alpha + 2) > 0,$$

it suffices to show that the minimum of Θ_1 in $[0, +\infty) \times [0, +\infty)$ is positive. Since

$$\frac{\partial \Theta_1}{\partial \alpha} - \frac{\partial \Theta_1}{\partial \beta} = (\beta - \alpha)(\beta + \alpha)(2\beta + 2\alpha - 5),$$

the minimum of Θ_2 in the interior (if it exists) satisfies $2\beta + 2\alpha - 5$ or $\alpha = \beta$. Since $\Theta_1\left(\alpha, \frac{5-2\alpha}{2}\right) = \frac{49}{2} > 0$, and

$$\Theta_1(\alpha,\alpha) = 8\alpha^4 + 6\alpha^3 - 4\alpha^2 - 2\alpha + 2 = 3\alpha^4 + 2\alpha^3 + (\alpha^4 - \alpha^2 + 1) + (2\alpha^2 + \alpha - 1)^2 > 0,$$

the function Θ_1 is positive on $[0, +\infty) \times [0, +\infty)$. Since $Z_1 \geq Z_2, Z_3$ in $\mathcal{D}^-$, the vanishing of Z_1 forces Z_2 and Z_3 to vanish. Therefore, the variable Z_1 does not vanish and one of α, β must be positive on the boundary

$$\mathcal{D}^- \cap \{\left(k\sqrt{Z_1 Z_2} + l\sqrt{Z_1 Z_3} + (k+l)\sqrt{Z_2 Z_3}\right) Z_4 = 2\Delta\}.$$

Therefore, the polynomial $-\beta^2\Theta_0 l^2 - \alpha\beta\Theta_1 kl - \alpha^2\Theta_2 k^2 \leq 0$ and only vanishes at $(1,0)$ and $(0,1)$. Equivalently, the derivative (4.16) is non-positive and only vanishes at critical points P_0^k and P_0^l. The proof is complete. $\square$

Proposition 4.3. Each Z_j with $j \in \{1,2,3\}$ is bounded in $\mathcal{D}^\pm$.

Proof. If $k\sqrt{Z_1 Z_2} + l\sqrt{Z_1 Z_3} + (k+l)\sqrt{Z_2 Z_3} = 0$, at least two of Z_1, Z_2, Z_3 vanish. Hence, the variable X_4 vanishes by (2.16). The equation (2.18) implies

$$2(Z_1 + Z_2 + Z_3) = 1 + X_4 = 1.$$

Therefore, each Z_j with $j \in \{1,2,3\}$ is bounded in this case.

Now consider the case $k\sqrt{Z_1 Z_2} + l\sqrt{Z_1 Z_3} + (k+l)\sqrt{Z_2 Z_3} \neq 0$. By (2.17) and the definition of $\mathcal{D}^+$, we have

$$\begin{aligned}
2(Z_1 + Z_2 + Z_3) &= 1 + \left(\frac{l}{2\Delta}(Z_2 - Z_1)Z_3 + \frac{k}{2\Delta}(Z_3 - Z_1)Z_2\right) Z_4 \\
&\leq 1 + \frac{l(Z_2 - Z_1)Z_3 + k(Z_3 - Z_1)Z_2}{k\sqrt{Z_1 Z_2} + l\sqrt{Z_1 Z_3} + (k+l)\sqrt{Z_2 Z_3}} \\
&\leq 1 + \frac{(k+l)Z_2 Z_3}{(k+l)\sqrt{Z_2 Z_3}} \\
&= 1 + \sqrt{Z_2 Z_3}.
\end{aligned} \tag{4.18}$$

Therefore, we have $2Z_1 + \frac{1}{2}(\sqrt{Z_2} - \sqrt{Z_3})^2 + \frac{3}{2}(Z_2 + Z_3) \leq 1$. Thus Z_1, Z_2, Z_3 are all bounded.

On the other hand, by (2.17) and the definition of $\mathcal{D}^-$, we have

$$
\begin{aligned}
2(Z_1+Z_2+Z_3) &= 1+\left(\frac{l}{2\Delta}(Z_1-Z_2)Z_3+\frac{k}{2\Delta}(Z_1-Z_3)Z_2\right)Z_4\\
&\leq 1+\frac{l(Z_1-Z_2)Z_3+k(Z_1-Z_3)Z_2}{k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}}\\
&\leq 1+\frac{lZ_1Z_3+kZ_1Z_2}{k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}}\\
&\leq 1+\frac{lZ_1Z_3+kZ_1Z_2}{kZ_2+lZ_3}\quad \text{since } Z_1\geq Z_2,Z_3 \text{ in } \mathcal{D}^-\\
&= 1+Z_1.
\end{aligned}
\tag{4.19}
$$

Therefore, we have $Z_1+2Z_2+2Z_3\leq 1$, and each of Z_1, Z_2, Z_3 is bounded. □

4.3. **Existence of ALC metrics.**

Proposition 4.4. If a γ^i_θ with $\theta\in(0,\pi)$ enters $\mathcal{D}^\pm$, the integral curve is defined on $\mathbb{R}$.

Proof. By (2.12) and the Cauchy–Schwarz inequality we have $G\geq\frac{1}{7}$. For $\theta\in(0,\pi)$ the quantity $\frac{Z_4}{Z_1^2Z_2^2Z_3^2}$ is positive once γ^i_θ leaves P^i_0. A direct computation yields

$$
\left(\frac{Z_4}{Z_1^2Z_2^2Z_3^2}\right)'=\frac{Z_4}{Z_1^2Z_2^2Z_3^2}(1-7G)\leq 0,
\tag{4.20}
$$

so $\frac{Z_4}{Z_1^2Z_2^2Z_3^2}$ is non-increasing along the integral curve. Let μ denote its value at some η_*. Then for all $\eta>\eta_*$ we have $Z_4\leq\mu Z_1^2Z_2^2Z_3^2$.

As the integral curve enters $\mathcal{D}^\pm$, each of Z_1, Z_2, Z_3 is bounded by Proposition 4.3. Hence, the function Z_4 is also bounded along the integral curve, and so is each X_j by (2.16). Subsequently, the integral curve stays in a compact subset of $\mathcal{D}^\pm$ by Lemma 4.1-4.2. Thus the integral curve is defined on $\mathbb{R}$. □

The critical point P^{k+l}_0 is on the boundary of $\mathcal{D}^+$, while P^k_0 and P^l_0 are on the boundary of $\mathcal{D}^-$. Although the inequality (4.1) fails to hold initially for $\theta\in(0,\pi)$, we show that for a $\theta>0$ sufficiently small, the integral curve γ^i_θ eventually satisfies the inequality and consequently enters the corresponding invariant set at finite time. This provides a unified construction for all three families of forward complete Spin(7) metrics.

Lemma 4.5. *For each $i\in\{k+l,l,k\}$, there exists a sufficiently small $\theta_*>0$ such that each γ^i_θ eventually enters $\mathcal{D}^\pm$ if $\theta\in[0,\theta_*)$.*

Proof. Since $Z_1=0$ and $Z_2=Z_3=\frac{1}{3}$ at P^{k+l}_0, the first two defining inequalities in (4.2) are automatically satisfied by γ^{k+l}_θ for any $\theta\in[0,\pi]$. Recall that $Z_2=0$ and $Z_1=Z_3=\frac{1}{3}$ at P^l_0. By (3.3), it is clear that $Z_1\geq Z_3$ initially along γ^l_θ. Therefore, the first two defining inequalities in (4.12) are satisfied by γ^l_θ for any $\theta\in[0,\pi]$. By a similar argument, the same conclusion holds for γ^k_θ for any $\theta\in[0,\pi]$.

It boils down to investigating the function $\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4$. Fix an $\eta_\bullet\in\mathbb{R}$ and let $p^i_\bullet(\theta)=\gamma^i_\theta(\eta_\bullet)$. As the integral curve γ^i_0 is contained in the invariant set $\mathcal{W}_i$ that joins P^i_0 and Q^i_0, it is clear that

$$
\left(\left(k\sqrt{Z_1Z_2}+l\sqrt{Z_1Z_3}+(k+l)\sqrt{Z_2Z_3}\right)Z_4\right)(p^i_\bullet(0))<2\Delta.
$$

Since the composite function is continuous, there exists a sufficiently small $\theta_* > 0$ such that

$$\left(\left(k\sqrt{Z_1Z_2} + l\sqrt{Z_1Z_3} + (k+l)\sqrt{Z_2Z_3}\right) Z_4\right)(p^i_\bullet(\theta)) < 2\Delta$$

if $\theta \in [0, \theta_*)$. Therefore, the point $p^i_\bullet(\theta)$ is contained in the interior of $\mathcal{D}^\pm$ if $\theta \in (0, \theta_*)$, meaning that the corresponding γ^i_θ enters $\mathcal{D}^\pm$ eventually. The proof is complete. □

Combining the results above, each γ^i_θ with $\theta \in (0, \theta_*)$ is defined on $\mathbb{R}$. Thus we obtain a continuous one-parameter family of forward complete Spin(7) metrics on each $M^i_{k,l}$. We show below that these metrics are ALC.

Proposition 4.6. Critical points $P^\pm_{AC}$ are not in the closed set $\mathcal{D}^+ \cup \mathcal{D}^-$.

Proof. We first show that for any distinct $m, n \in \{1, 2, 3\}$, the inequality $\sqrt{Z_mZ_n} \leq \frac{1}{7}$ holds at $P^\pm_{AC}$.

In the proof of Proposition 3.1, it is known that $P^\pm_{AC}$ are on the circle

$$\left\{Z_1 + Z_2 + Z_3 = \frac{4}{7}\right\} \cap \{5(Z_1^2 + Z_2^2 + Z_3^2) = 6(Z_2Z_3 + Z_3Z_1 + Z_1Z_2)\}. \tag{4.21}$$

By symmetry, it suffices to prove that $\sqrt{Z_2Z_3} \leq \frac{1}{7}$. Eliminating Z_1 from (4.21), we have

$$0 = 16\left(Z_2 + Z_3 - \frac{2}{7}\right)^2 + \frac{16}{49} - 16Z_2Z_3 \geq \frac{16}{49} - 16Z_2Z_3. \tag{4.22}$$

Hence $\sqrt{Z_2Z_3} \leq \frac{1}{7}$.

For P^+_{AC}, we have $\frac{1}{7} = \frac{1}{2\Delta}((k+l)Z_2Z_3 - lZ_1Z_3 - kZ_1Z_2)Z_4$ by (2.16). Therefore,

$$\begin{aligned} &((k+l)\sqrt{Z_2Z_3} + l\sqrt{Z_1Z_3} + k\sqrt{Z_1Z_2})Z_4 \\ &= \frac{2\Delta}{7}\frac{(k+l)\sqrt{Z_2Z_3} + l\sqrt{Z_1Z_3} + k\sqrt{Z_1Z_2}}{(k+l)Z_2Z_3 - lZ_1Z_3 - kZ_1Z_2} \\ &> \frac{2\Delta}{7}\frac{1}{\sqrt{Z_2Z_3}} \quad \text{each } Z_j \text{ is positive at } P^+_{AC} \\ &\geq 2\Delta \quad \text{by } \sqrt{Z_2Z_3} \leq \frac{1}{7} \end{aligned} \tag{4.23}$$

at P^+_{AC}. Hence $P^+_{AC} \notin \mathcal{D}^+ \cup \mathcal{D}^-$.

For P^-_{AC}, we have $\frac{1}{7} = \frac{1}{2\Delta}(-(k+l)Z_2Z_3 + lZ_1Z_3 + kZ_1Z_2)Z_4$ by (2.16). Therefore,

$$\begin{aligned} &((k+l)\sqrt{Z_2Z_3} + l\sqrt{Z_1Z_3} + k\sqrt{Z_1Z_2})Z_4 \\ &= \frac{2\Delta}{7}\frac{(k+l)\sqrt{Z_2Z_3} + l\sqrt{Z_1Z_3} + k\sqrt{Z_1Z_2}}{-(k+l)Z_2Z_3 + lZ_1Z_3 + kZ_1Z_2} \\ &> \frac{2\Delta}{7}\frac{l\sqrt{Z_1Z_3} + k\sqrt{Z_1Z_2}}{lZ_1Z_3 + kZ_1Z_2} \quad \text{each } Z_j \text{ is positive at } P^-_{AC} \\ &\geq 2\Delta \quad \text{by } \sqrt{Z_1Z_2}, \sqrt{Z_1Z_3} \leq \frac{1}{7} \end{aligned} \tag{4.24}$$

at P^-_{AC}. Thus $P^-_{AC} \notin \mathcal{D}^+ \cup \mathcal{D}^-$. □

Lemma 4.7. *If a γ^i_θ with $\theta \in (0, \pi)$ enters $\mathcal{D}^\pm$, the integral curve converges to P_{ALC}.*

Proof. Recall that for $\theta \in (0,\pi)$, the quantity $\frac{Z_4}{Z_1^2 Z_2^2 Z_3^2}$ is non-increasing along γ_θ^i by (4.20). Since the integral curve is confined in a compact subset and is defined on $\mathbb{R}$ by Proposition 4.4, the limit $\lim_{\eta\to\infty} \frac{Z_4}{Z_1^2 Z_2^2 Z_3^2}$ exists.

If $\lim_{\eta\to\infty} \frac{Z_4}{Z_1^2 Z_2^2 Z_3^2} \neq 0$, the ω-limit set is a subset of $\{G = \frac{1}{7}\}$ by (4.20). Cauchy–Schwarz inequality implies that the ω-limit set is contained in $\{X_1 = X_2 = X_3 = X_4 = \frac{1}{7}\}$. Since the ω-limit set is invariant, it must be contained in the finite set $\{P_{AC}^+, P_{AC}^-\}$, meaning the integral curve converges to one of the critical points. Since $P_{AC}^\pm$ are not in $\mathcal{D}^+ \cup \mathcal{D}^-$ and each $\mathcal{D}^\pm$ is invariant, this leads to a contradiction.

If $\lim_{\eta\to\infty} \frac{Z_4}{Z_1^2 Z_2^2 Z_3^2} = 0$, the ω-limit set is contained in $\{Z_4 = 0\}$, since the other Z_j are bounded above. By (2.16), the ω-limit set is contained in $\mathcal{C}_{G_2}$. By rewriting the G_2 equations from [CS02] in our coordinates, the integral curves on $\mathcal{C}_{G_2}$ are invariant algebraic curves given by

$$\mathcal{C}_{G_2} \cap \{\cos(\xi) Z_3(Z_2 - Z_1) - \sin(\xi) Z_2(Z_3 - Z_1) = 0\}, \quad \xi \in [0,\pi).$$

For each fixed ξ, the algebraic curve consists of two integral curves, both of which converge to P_{ALC}. Hence, the critical point P_{ALC} belongs to the ω-limit set. Since P_{ALC} is a sink by [Chi22, (3.5)], the integral curve converges to P_{ALC}. $\square$

5. Invariant set where the circle fiber blows up

In this section, we construct invariant sets where integral curves do not converge to any of the critical points of the Spin(7) system. Geometrically, integral curves entering these sets correspond to metrics whose principal curvature of the $\mathbb{S}^1$-fiber in $N_{k,l}$ dominates the remaining principal curvatures beyond a fixed threshold. Once this threshold is exceeded, the $\mathbb{S}^1$-fiber expands too rapidly for the metric to exhibit either ALC or AC asymptotics.

Lemma 5.1. *Define*

$$\mathcal{B}^\pm := \mathcal{C}^\pm_{\mathrm{Spin}(7)} \cap \{X_4 - X_1 - X_2 - X_3 \geq 0\}.$$

The sets $\mathcal{B}^\pm$ are invariant.

Proof. Consider

$$\begin{aligned}
&(X_4 - X_1 - X_2 - X_3)'|_{X_4 - X_1 - X_2 - X_3 = 0} \\
&= (X_4 - X_1 - X_2 - X_3)(G-1) + R_4 - R_1 - R_2 - R_3 \\
&= \left(\frac{(k+l)^2}{\Delta^2} Z_2^2 Z_3^2 + \frac{l^2}{\Delta^2} Z_1^2 Z_3^2 + \frac{k^2}{\Delta^2} Z_1^2 Z_2^2\right) Z_4^2 \\
&\quad + Z_1^2 + Z_2^2 + Z_3^2 - 6Z_1Z_2 - 6Z_2Z_3 - 6Z_1Z_3
\end{aligned} \tag{5.1}$$

By (2.12) and (2.18), the equation $X_4 - X_1 - X_2 - X_3 = 0$ is equivalent to $Z_1 + Z_2 + Z_3 = \frac{2}{3}$. By (2.17), we have

$$Z_4^2 = \left(\frac{2(Z_1+Z_2+Z_3) - 1}{\frac{k+l}{2\Delta} Z_2 Z_3 - \frac{l}{2\Delta} Z_1 Z_3 - \frac{k}{2\Delta} Z_1 Z_2}\right)^2 = \left(\frac{1}{2} \frac{Z_1 + Z_2 + Z_3}{\frac{k+l}{2\Delta} Z_2 Z_3 - \frac{l}{2\Delta} Z_1 Z_3 - \frac{k}{2\Delta} Z_1 Z_2}\right)^2.$$

The computation (5.1) becomes

(5.2)
$$
\begin{aligned}
&(X_4-X_1-X_2-X_3)'|_{X_4-X_1-X_2-X_3=0}\\
&=\left(\frac{(k+l)^2}{\Delta^2}Z_2^2Z_3^2+\frac{l^2}{\Delta^2}Z_1^2Z_3^2+\frac{k^2}{\Delta^2}Z_1^2Z_2^2\right)\left(\frac{1}{2}\frac{Z_1+Z_2+Z_3}{\frac{k+l}{2\Delta}Z_2Z_3-\frac{l}{2\Delta}Z_1Z_3-\frac{k}{2\Delta}Z_1Z_2}\right)^2\\
&\quad+Z_1^2+Z_2^2+Z_3^2-6Z_1Z_2-6Z_2Z_3-6Z_1Z_3\\
&=\frac{2\Phi}{((k+l)Z_2Z_3-lZ_1Z_3-kZ_1Z_2)^2},
\end{aligned}
$$

where

(5.3)
$$
\begin{aligned}
\Phi &= \Phi_2l^2+\Phi_1kl+\Phi_0k^2\\
\Phi_2 &= Z_1^4Z_3^2-3Z_1^3Z_2Z_3^2-2Z_1^3Z_3^3+8Z_1^2Z_2^2Z_3^2+4Z_1^2Z_2Z_3^3+Z_1^2Z_3^4-3Z_1Z_2^3Z_3^2\\
&\quad+4Z_1Z_2^2Z_3^3-Z_1Z_2Z_3^4+Z_2^4Z_3^2-2Z_2^3Z_3^3+Z_2^2Z_3^4,\\
\Phi_1 &= Z_1^4Z_2Z_3-7Z_1^3Z_2^2Z_3-7Z_1^3Z_2Z_3^2+7Z_1^2Z_2^3Z_3+8Z_1^2Z_2^2Z_3^2\\
&\quad+7Z_1^2Z_2Z_3^3-Z_1\,Z_2^4Z_3+Z_1Z_2^3Z_3^2+Z_1Z_2^2Z_3^3-Z_1Z_2Z_3^4+2Z_2^4Z_3^2-4Z_2^3Z_3^3+2Z_2^2Z_3^4,\\
\Phi_0 &= Z_1^4Z_2^2-2Z_1^3Z_2^3-3Z_1^3Z_2^2Z_3+Z_1^2Z_2^4+4Z_1^2Z_2^3Z_3+8Z_1^2Z_2^2Z_3^2-Z_1Z_2^4Z_3\\
&\quad+4Z_1Z_2^3Z_3^2-3Z_1Z_2^2Z_3^3+Z_2^4Z_3^2-2Z_2^3Z_3^3+Z_2^2Z_3^4.
\end{aligned}
$$

We show that $\Phi_2\geq 0$ and its zero set in $\{X_4-X_1-X_2-X_3=0\}$ (equivalently $\{Z_1+Z_2+Z_3=\frac{2}{3}\}$) is $\{Z_3=0\}\cup\{P_0^{k+l},P_0^l\}$. Define $a=\frac{Z_1}{Z_3}$ and $b=\frac{Z_2}{Z_3}$. Rewrite $\Phi_2=Z_3^6\phi_2$, where

(5.4) $$\phi_2(a,b)=a^4-3a^3b-2a^3+8a^2b^2+4a^2b+a^2-3ab^3+4ab^2-ab+b^4-2b^3+b^2.$$

Since

(5.5)
$$
\begin{aligned}
\frac{\partial\phi_2}{\partial a}+\frac{\partial\phi_2}{\partial b}&=a(a-1)^2+b(b-1)^2+ab(7a+7b+16)\geq 0,\\
\frac{\partial\phi_2}{\partial a}-\frac{\partial\phi_2}{\partial b}&=(a-b)(7a^2-18ab+7b^2-10a-10b+3),
\end{aligned}
$$

the critical points of ϕ_2 in $[0,\infty)\times[0,\infty)$ are $(0,0)$, $(1,0)$, or $(0,1)$. The Hessian of ϕ_2 at these points are

$$
\mathrm{Hess}\phi_2(0,0)=\begin{bmatrix}2&-1\\-1&2\end{bmatrix},\quad \mathrm{Hess}\phi_2(1,0)=\begin{bmatrix}2&-2\\-2&26\end{bmatrix},\quad \mathrm{Hess}\phi_2(0,1)=\begin{bmatrix}26&-2\\-2&2\end{bmatrix}.
$$

Hence, these points are local minima. Since ϕ_2 vanishes at these point, the function $\phi_2\geq 0$ on $[0,\infty)\times[0,\infty)$ and it only vanishes at $(0,0)$, $(1,0)$, and $(0,1)$.

By (2.12), the equation $X_4-X_1-X_2-X_3=0$ is equivalent to $X_4=\frac{1}{3}$. By (2.16), at most one of Z_1, Z_2, Z_3 vanishes. Threfore, we can ignore $(0,0)$. Correspondingly, the zero set of $\Phi_2=Z_3^6\phi_2$ zero set in $\{X_4-X_1-X_2-X_3=0\}$ is $\{Z_3=0\}\cup\{P_0^l,P_0^{k+l}\}$. By the symmetry $(l,Z_2)\leftrightarrow(k,Z_3)$, the function Φ_0 is non-negative and its zero set is $\{Z_2=0\}\cup\{P_0^k,P_0^{k+l}\}$. Since

(5.6)
$$\begin{aligned}\delta&=\Phi_1^2-4\Phi_0\Phi_2\\&=-3Z_1^2Z_2^2Z_3^2\left(Z_1+Z_2+Z_3\right)^2\left(Z_1^2+Z_2^2+Z_3^2-2Z_2Z_3-2Z_1Z_3-2Z_1Z_2\right)^2\\&\leq 0,\end{aligned}$$

the derivative (5.2) is non-negative. If a non-transversal intersection occurs in $\{Z_1+Z_2+Z_3=\frac{2}{3}\}$, it is necessary that $\delta=0$ at the intersection point. If $Z_1Z_2Z_3=0$, we obtain one of P_0^i from $\Phi=0$. Otherwise, we have

$$Z_1^2+Z_2^2+Z_3^2=2Z_2Z_3+2Z_1Z_3+2Z_1Z_2.$$

With $Z_1^2+Z_2^2+Z_3^2+2Z_2Z_3+2Z_1Z_3+2Z_1Z_2=(Z_1+Z_2+Z_3)^2=\frac{4}{9}$, it is clear that

$$Z_1^2+Z_2^2+Z_3^2=2Z_2Z_3+2Z_1Z_3+2Z_1Z_2=\frac{2}{9}.$$

Therefore, at the non-transversal intersection point, the derivative (5.2) becomes

(5.7)
$$\begin{aligned}&(X_4-X_1-X_2-X_3)'|_{X_4-X_1-X_2-X_3=0}\\&=(X_4-X_1-X_2-X_3)(G-1)+R_4-R_1-R_2-R_3\\&=\left(\frac{(k+l)^2}{\Delta^2}Z_2^2Z_3^2+\frac{l^2}{\Delta^2}Z_1^2Z_3^2+\frac{k^2}{\Delta^2}Z_1^2Z_2^2\right)\left(\frac{1}{2}\frac{Z_1+Z_2+Z_3}{\frac{k+l}{2\Delta}Z_2Z_3-\frac{l}{2\Delta}Z_1Z_3-\frac{k}{2\Delta}Z_1Z_2}\right)^2\\&\quad+Z_1^2+Z_2^2+Z_3^2-6Z_1Z_2-6Z_2Z_3-6Z_1Z_3\\&=\left(\frac{(k+l)^2}{\Delta^2}Z_2^2Z_3^2+\frac{l^2}{\Delta^2}Z_1^2Z_3^2+\frac{k^2}{\Delta^2}Z_1^2Z_2^2\right)\left(\frac{1}{3}\frac{1}{\frac{k+l}{2\Delta}Z_2Z_3-\frac{l}{2\Delta}Z_1Z_3-\frac{k}{2\Delta}Z_1Z_2}\right)^2-\frac{4}{9}\\&=\frac{8}{9}\frac{Z_1Z_2Z_3}{\left((k+l)Z_2Z_3-lZ_1Z_3-kZ_1Z_2\right)^2}\left((k+l)lZ_3+(k+l)kZ_2-lkZ_1\right).\end{aligned}$$

The non-transversal intersection point in $\{Z_1+Z_2+Z_3=\frac{2}{3}\}$ with $Z_1Z_2Z_3\neq 0$ is hence characterized by the following set of equations

$$\begin{aligned}Z_1+Z_2+Z_3&=\frac{2}{3},\\Z_1^2+Z_2^2+Z_3^2&=2Z_2Z_3+2Z_1Z_3+2Z_1Z_2,\\(k+l)lZ_3+(k+l)kZ_2-lkZ_1&=0,\end{aligned}\tag{5.8}$$

whose only solution is $P_*=\left(-\frac{kl}{3\Delta},\frac{(k+l)k}{3\Delta},\frac{(k+l)l}{3\Delta},\frac{1}{3},\frac{(k+l)^2}{3\Delta},\frac{l^2}{3\Delta},\frac{k^2}{3\Delta},\frac{6\Delta^2}{kl(k+l)}\right)\in\mathcal{C}^-_{\mathrm{Spin}(7)}$. By continuous dependence, the integral curve that starts at P_* also enters the interior of $\mathcal{B}^-$. Therefore, both sets $\mathcal{B}^\pm$ are invariant. □

Recall that each γ_π^i lies on the algebraic curve $\mathcal{W}_i$ in (3.4), where one of the Z_j vanishes identically while the other two remain greater than $\frac{1}{3}$. Hence, these integral curves lie in $\mathcal{B}^+\cup\mathcal{B}^-$. By (3.3) and Lemma 5.1, each γ_θ^i stays in $\mathcal{B}^+\cup\mathcal{B}^-$ initially whenever $\theta\in\left(\pi-\arctan\left(\frac{1}{3i}\right),\pi\right]$.

For each γ_θ^i, define

$$\theta_i:=\sup\{\theta\in[0,\pi]\mid\gamma_\theta^i\text{ eventually enters }\mathcal{D}^+\cup\mathcal{D}^-\}.$$

Lemma 5.2. *The integral curve $\gamma^{k+l}_{\theta_{k+l}}$ converges to P^+_{AC}. The integral curves $\gamma^l_{\theta_l}$ and $\gamma^k_{\theta_k}$ converge to P^-_{AC}.*

Proof. By Lemma 4.5, we have $\theta_i \geq \theta_i^* > 0$. By our discussion above, we also have $\theta_i \leq \pi - \arctan\left(\frac{1}{3i}\right) < \pi$. Hence $\theta_i \in (0, \pi)$.

Suppose a $\gamma^i_{\theta_i}$ eventually enters $\mathcal{D}^+ \cup \mathcal{D}^-$ in finite time. By continuous dependence, the integral curve $\gamma^i_{\theta_i+\epsilon}$ eventually enters $\mathcal{D}^+ \cup \mathcal{D}^-$ in finite time for a sufficiently small $\epsilon > 0$. Since $\mathcal{D}^+ \cup \mathcal{D}^-$ is invariant by Lemma 4.1-4.2, this contradicts the definition of θ_i.

On the other hand, suppose $\gamma^i_{\theta_i}$ eventually enters $\mathcal{B}^+ \cup \mathcal{B}^-$ in finite time. By continuous dependence, the integral curve $\gamma^i_{\theta_i-\epsilon}$ eventually enters $\mathcal{B}^+ \cup \mathcal{B}^-$ in finite time for a sufficiently small $\epsilon > 0$. As $\mathcal{B}^+ \cup \mathcal{B}^-$ is invariant by Lemma 5.1, this also contradicts the definition of θ_i. Hence, each $\gamma^i_{\theta_i}$ stays in the region between $\mathcal{D}^+ \cup \mathcal{D}^-$ and $\mathcal{B}^+ \cup \mathcal{B}^-$. In particular, the integral curve $\gamma^{k+l}_{\theta_{k+l}}$ remains in

$$\mathcal{A}^+ := \mathcal{C}^+_{\mathrm{Spin}(7)} \cap \left\{ \left(k\sqrt{Z_1Z_2} + l\sqrt{Z_1Z_3} + (k+l)\sqrt{Z_2Z_3}\right) Z_4 \geq 2\Delta \right\} \cap \left\{ Z_1 + Z_2 + Z_3 \leq \frac{2}{3} \right\}.$$

By the boundedness of Z_j for $j \in \{1,2,3\}$ and (4.20), we know that $\gamma^{k+l}_{\theta_{k+l}}$ is in a compact subset of $\mathcal{A}^+$. Hence, the integral curve is defined on $\mathbb{R}$. Furthermore, by (4.20) and the first defining inequality of $\mathcal{A}^+$, the function $\frac{Z_4}{Z_1^2Z_2^2Z_3^2}$ decreases to some positive limit. Hence $G \to \frac{1}{7}$ along $\gamma^{k+l}_{\theta_{k+l}}$. Since the ω-limit set is invariant, the ω-limit set of $\gamma^{k+l}_{\theta_{k+l}}$ is contained in $\{P^+_{AC}, P^-_{AC}\}$. By Proposition 3.1 and Proposition 4.6, the only critical points in $\mathcal{A}^+$ are P^{k+l}_0 and P^+_{AC}. Hence, the integral curve converges to P^+_{AC}.

Analogously, the integral curves $\gamma^l_{\theta_l}$ and $\gamma^k_{\theta_k}$ remain in

$$\mathcal{A}^- := \mathcal{C}^-_{\mathrm{Spin}(7)} \cap \left\{ \left(k\sqrt{Z_1Z_2} + l\sqrt{Z_1Z_3} + (k+l)\sqrt{Z_2Z_3}\right) Z_4 \geq 2\Delta \right\} \cap \left\{ Z_1 + Z_2 + Z_3 \leq \frac{2}{3} \right\},$$

and their ω-limit sets lie in $\{P^+_{AC}, P^-_{AC}\}$. By Proposition 3.1 and Proposition 4.6, the only critical points in $\mathcal{A}^-$ are P^l_0, P^k_0, and P^-_{AC}. Therefore, both integral curves converge to P^-_{AC}. $\square$

Theorem 1.1 is established by Lemma 4.5, Lemma 4.7 and Lemma 5.2.

We conclude by mentioning conically singular Spin(7) metrics, whose principal orbit collapses to a point as $t \to 0$. Near the singular end these metrics are asymptotic to the Euclidean cone over the homogeneous Einstein $N_{k,l}$, while at infinity they exhibit ALC asymptotics. These metrics are represented by integral curves that emanate from $P^\pm_{AC}$. Such examples are already known in exceptional cases: for $N_{1,1}$, this includes the $\mathbb{A}_8$ metric in [CGLP02] and the Γ_0 metric in [Chi21, Theorem 1.3]. For $N_{1,0}$, the conically singular metric was constructed in [Leh22]. It is natural to expect that similar solutions should exist for general coprime pairs (k,l). The main obstacle toward a uniform proof appears to be computational: the coordinates of $P^\pm_{AC}$ are determined by quartic equations. It would be interesting to understand whether these computational difficulties can be bypassed, perhaps by a more conceptual approach.

Acknowledgements

The author thanks McKenzie Wang and Lorenzo Foscolo for their valuable discussions and helpful comments.

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