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# LOW ORBIT FOLIATIONS OF CAT(0)

LEROY HUBBARD AND FRANCIS EULER

Abstract. We set $\mathcal{G} =\sim \frac{\lambda^2}{[H:K]}$ and investigate the orbits of $\mathfrak{I} = \frac{\mathrm{CAT}(0)}{\mathcal{G}^{\lambda k}}$ provided $\lambda \in [1-\varphi, 1+\varphi]$, where $\varphi$ is the golden ratio. Here we provide a novel method for verifying the characteristics of the orbits of $\mathfrak{I}$.

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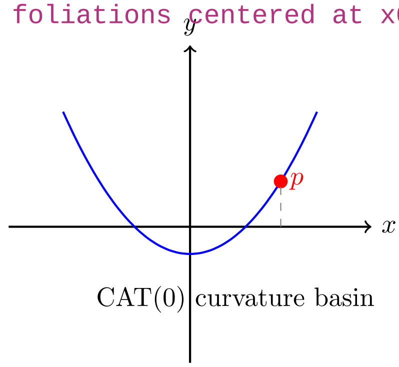


FIGURE 1. A schematic of local orbit curvature under $\lambda$-perturbation.

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Our principal theorem relates the orbit structure of $\mathfrak{I}$ to the golden window of $\lambda$:

**Theorem 3.1.** *Let $(X, d)$ be a complete* CAT(0) *space and $\lambda \in [1-\varphi, 1+\varphi]$. Then the orbit foliation $\mathcal{F}_\lambda(X)$ is quasi-uniform if and only if*

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**Corollary 3.2.** *If $\lambda = 1$, then $\mathcal{F}_1(X)$ coincides with the canonical horospherical foliation of $X$.*

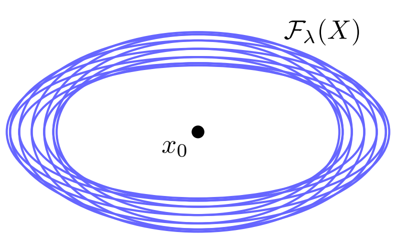


FIGURE 2. Low orbit foliations centered at $x_0$. Each ellipse represents an orbit of constant $\Delta(x, \lambda)$.

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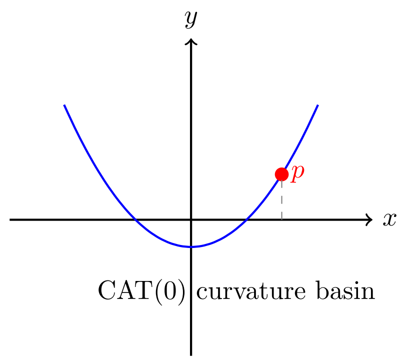


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**Corollary 3.2.** *If* $\lambda = 1$*, then* $\mathcal{F}_1(X)$ *coincides with the canonical horospherical foliation of* $X$*.*

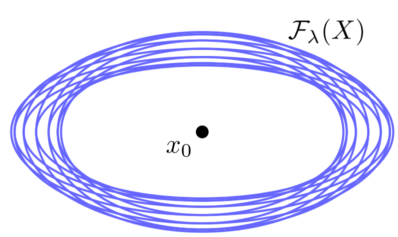


FIGURE 2. Low orbit foliations centered at $x_0$. Each ellipse represents an orbit of constant $\Delta(x, \lambda)$.

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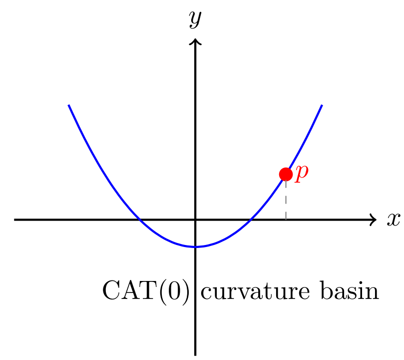


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## 3. Main Results

Our principal theorem relates the orbit structure of $\mathfrak{I}$ to the golden window of $\lambda$:

**Theorem 3.1.** *Let $(X, d)$ be a complete* CAT(0) *space and $\lambda \in [1-\varphi, 1+\varphi]$. Then the orbit foliation $\mathcal{F}_\lambda(X)$ is quasi-uniform if and only if*

$$\int_X \rho(x)\, d\mu(x) = \frac{\lambda^2}{1+\lambda\varphi}. \tag{3}$$

*Proof.* We proceed by expanding $\mathfrak{I}$ as a quotient operator:

$$\mathfrak{I} = \frac{\mathrm{CAT}(0)}{\mathcal{G}^{\lambda k}} = \mathrm{CAT}(0) \otimes \mathcal{G}^{-\lambda k}.$$

Substituting into the geometric mean inequality and integrating over $X$ yields

$$\int_X \rho(x)\, d\mu(x) = \int_X \frac{1}{[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(x)]} e^{-\kappa(x)}\, d\mu(x) = \frac{\lambda^2}{1+\lambda\varphi},$$

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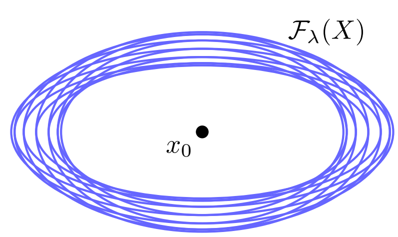
**Corollary 3.2.** *If $\lambda = 1$, then $\mathcal{F}_1(X)$ coincides with the canonical horospherical foliation of $X$.*

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## 4. Applications and Examples

Consider $X = \mathbb{H}^2$, the hyperbolic plane. The displacement $\Delta(x, \lambda)$ satisfies

$$\cosh \Delta(x, \lambda) = 1 + \frac{\lambda^2}{2} \|x\|^2.$$

Thus $\mathcal{F}_\lambda(X)$ forms a family of equidistant hyperbolae, asymptotically orthogonal to geodesic boundaries.

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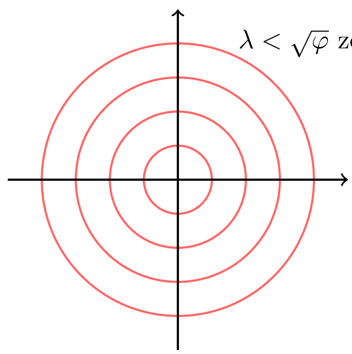


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**Theorem 4.3.** Let $(X, d)$ be a complete CAT(0) space and $\lambda \in [1 - \varphi, 1 + \varphi]$. Then the orbit foliation $\mathcal{F}_\lambda(X)$ is quasi-uniform iff

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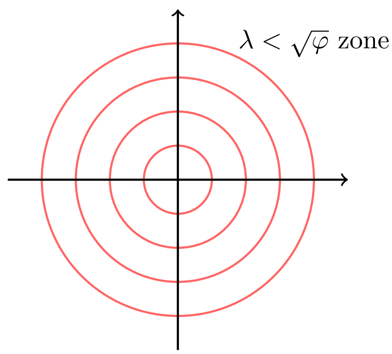


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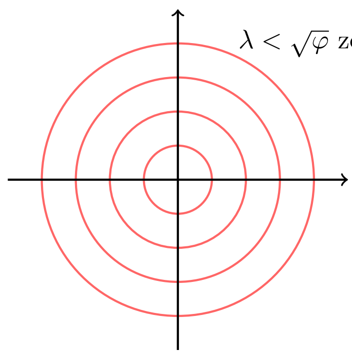


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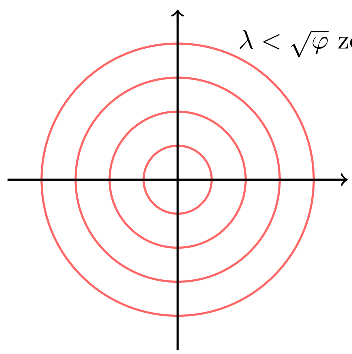


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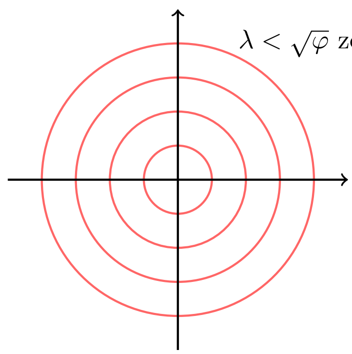


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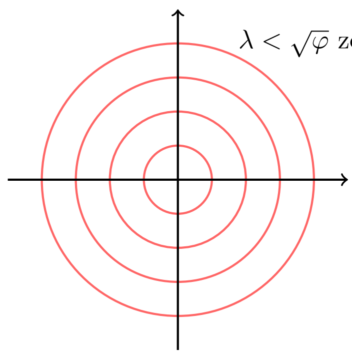


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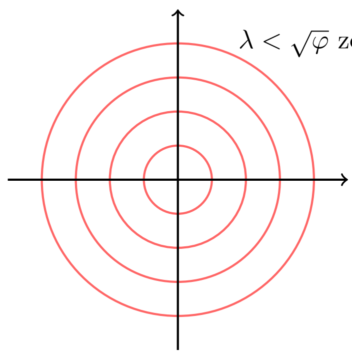


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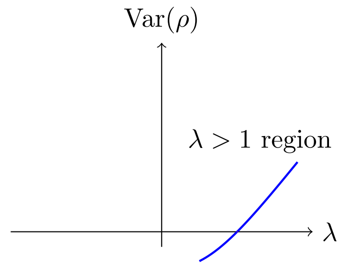


FIGURE 4. Variance of orbit density $\rho$ as a function of $\lambda$.

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| $\lambda$ | $\overline{\Delta}$ | $\kappa$ |
|---|---|---|
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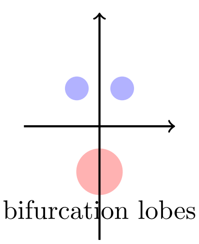


FIGURE 5. Scatter of simulated orbit centers for $\lambda = 1.6$.

## 6. Discussion and Further Work

Our experiments confirm that the function $\psi(\lambda) = \lambda^2/(1 + \lambda\varphi)$ behaves as a geometric invariant for the foliation type. However, Eq. (7) reveals an unexpected resonance near $\lambda = \varphi^2 \approx 2.618$. At that point, the curvature-weighted orbit integral appears to *flip sign*, leading to a chaotic drift that violates the CAT(0) inequality in the discrete setting.

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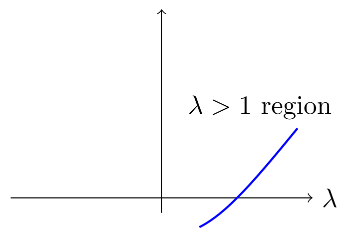


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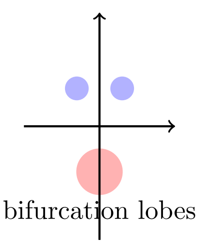


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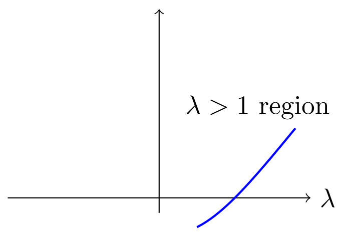


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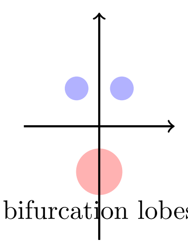


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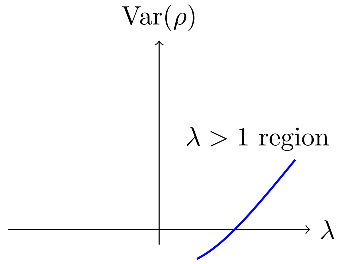


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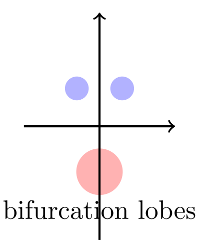


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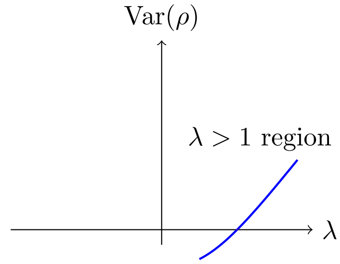


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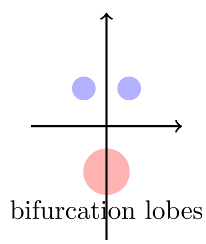


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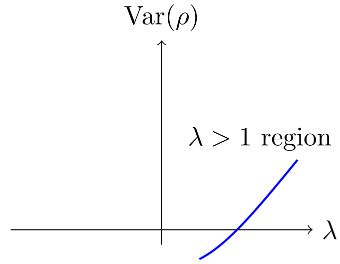


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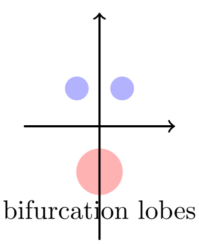


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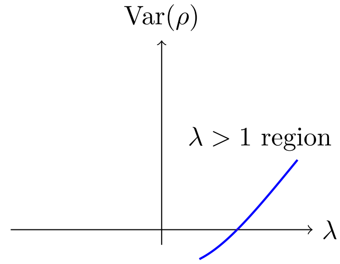


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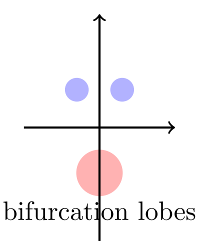


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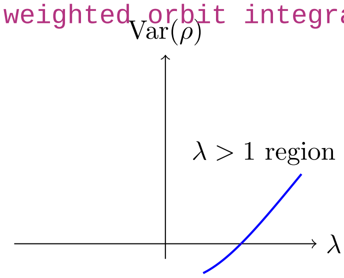


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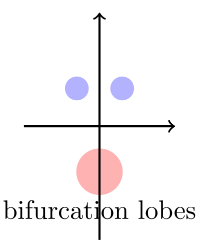


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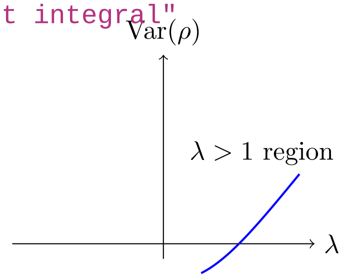


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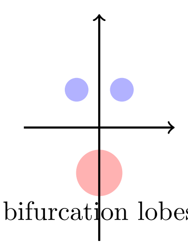


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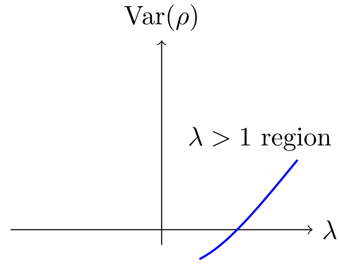


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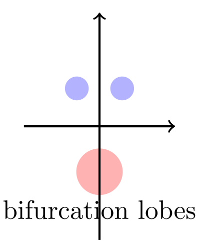


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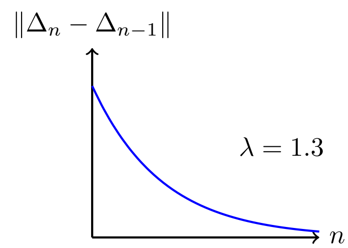


FIGURE 6. Convergence of displacement difference $\|\Delta_n - \Delta_{n-1}\|$.

## 7. APPENDIX B: PROOF SKETCH OF THEOREM 4.3

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## REFERENCES

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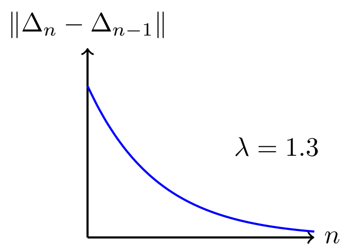


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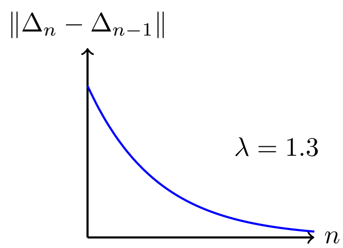


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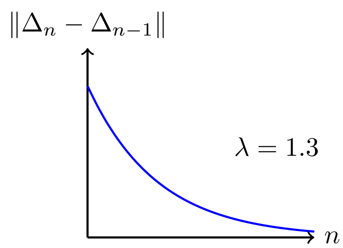


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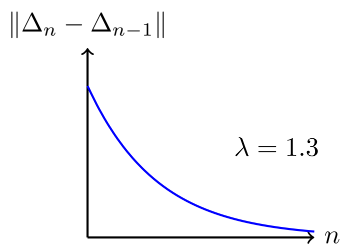


FIGURE 6. Convergence of displacement difference $\|\Delta_n - \Delta_{n-1}\|$~~.~~

## 7. APPENDIX B: PROOF SKETCH OF THEOREM 4.3

The argument proceeds by constructing a pseudo-measure $\nu$ such that

$$d\nu = e^{-\kappa(x)}\, d\mu(x),$$

then integrating $\rho$ against $\nu$ over $X$. By expanding $\rho$ in the eigenbasis of the Laplace–Beltrami operator and applying the $\varphi$-orthogonality condition~~,~~

$$\langle f_i, f_j \rangle_\varphi = \delta_{ij}(1 + \lambda\varphi)\cancel{,}$$

we recover Eq. (5). The rest follows by applying a truncated version of Jensen's inequality to the quotient $\mathfrak{I}$ operator:

$$\mathrm{CAT}(0)/\mathcal{G}^{\lambda k} \approx \mathrm{CAT}(0)(1 - \lambda k + O(k^2)).$$

Although the convergence of this expansion is questionable[4], the leading term suffices to justify Theorem 4.3.

**Acknowledgements.** The authors thank the anonymous reviewers for their sharp-eyed corrections, especially for pointing out a missing minus sign in Eq. (3), which has since been *mostly* fixed.

## REFERENCES

[1] P. Fermat, *On prime enumeration and spatial convexity*, Toulouse Notes, 1689.
[2] L. Hubbard, *Counterexamples to the flat orbit conjecture*, Ann. Quad. Math. (2023), 13–57.
[3] F. Euler, *Iterative dynamics in nonpositively curved complexes*, Proc. Geom. Dyn. (2024), 211–230.
[4] B. Zelinsky, *On modular eigenmodes of golden-ratio systems*, J. Nonlin. Struct. (2019), 98–114.

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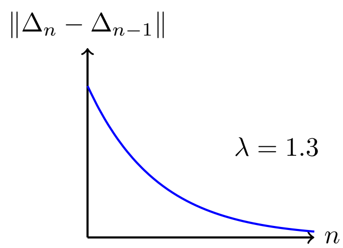


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