

functions on the generalized Dold space $P(m, n, k)$ are guaranteed to have a point of coincidence, and found certain pairs $(P(m, n, k), g)$ satisfying the coincidence property.

The paper is organized as follows:

In Section 2 we develop the necessary background and recall some relevant results. Section 3 is devoted to the study of graded endomorphisms of the rational cohomology algebra of $\mathbb{S}^m \times \mathbb{C}G_{n,k}$, from which we extract several consequences. These are applied in Section 4 to obtain the coincidence-theoretic results.

2. PRELIMINARIES

In this section, we discuss some preliminaries and recall some results that will be required to proceed with our study.

2.1. Cohomology of complex Grassmannians. Let $\mathbb{C}G_{n,k}$ denote the complex Grassmannian consisting of complex k -planes in \mathbb{C}^n . Let $\gamma_{n,k}$ and $\beta_{n,k}$ denote the canonical complex k -plane and $(n-k)$ -plane bundles, respectively, over $\mathbb{C}G_{n,k}$. Let the total Chern classes of the vector bundles $\gamma_{n,k}$ and $\beta_{n,k}$ be denoted by $c(\gamma_{n,k}) = c$ and $c(\beta_{n,k}) = \bar{c}$, respectively. Thus,

$$c = 1 + c_1 + c_2 + \cdots + c_k, \quad \bar{c} = 1 + \bar{c}_1 + \bar{c}_2 + \cdots + \bar{c}_{n-k},$$

where c_i and \bar{c}_i denote the i -th Chern classes of $\gamma_{n,k}$ and $\beta_{n,k}$, respectively. Since $\gamma_{n,k} \oplus \beta_{n,k} \cong \varepsilon_{\mathbb{C}}^n$, it follows that $c \cdot \bar{c} = 1$. The cohomology ring of the complex Grassmannian is well known and given by

$$H_{\mathbb{C}G}^* := H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \dots, c_k, \bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-k}] / \langle h_r : 1 \leq r \leq n \rangle,$$

where the relations h_r for $r = 1, 2, \dots, n$ are induced from the homogeneous parts of the equation $c \cdot \bar{c} = 1$ and given by

$$h_r := \sum_{i+j=r} c_i \bar{c}_j.$$

Using the relations $h_r, r = 1, 2, \dots, n - k$, the generators \bar{c}_i for $i = 1, 2, \dots, n - k$ can be expressed inductively in terms of c_i for $i = 1, 2, \dots, k$. Consequently, the relations h_r for $r = n - k + 1, \dots, n$ become homogeneous polynomials in c_i of degree $2r$, where the degree of each c_i is $2i$. Then the cohomology ring $H_{\mathbb{C}G}^*$ can be rewritten as

$$(1) \quad \mathbb{Q}[c_1, c_2, \dots, c_k] / \langle h_{n-k+1}, h_{n-k+2}, \dots, h_n \rangle.$$

Since there are no relations among the generators c_i for $i = 1, 2, \dots, k$ up to degree $2(n - k)$, the set of all monomials of degree $2r$ in terms of c_1, c_2, \dots, c_k forms a \mathbb{Q} -basis of $H^{2r}(\mathbb{C}G_{n,k}; \mathbb{Q})$ for $r \leq n - k$.

From now on, we denote the indexing set $\{1, 2, \dots, k\}$ by I .

Remark 2.1. We can assume $k \leq n - k$ for $\mathbb{C}G_{n,k}$ as $\mathbb{C}G_{n,k}$ is homeomorphic to $\mathbb{C}G_{n,n-k}$ by using orthogonal complementation.

The complex Grassmannian $\mathbb{C}G_{n,k}$ is a homogeneous space and can be represented as the quotient of the unitary group $U(n)$ by the stabilizer subgroup $U(k) \times U(n - k)$ that is

$$(2) \quad \mathbb{C}G_{n,k} = U(n) / (U(k) \times U(n - k)).$$

Now we recall a result given in [ST1].

Theorem 2.2 ([ST1], Theorem A'). *Let $D_i(H^*(G/H; \mathbb{Q}))$ be the \mathbb{Q} -vector space of \mathbb{Q} -derivations of $H^*(G/H; \mathbb{Q})$ which decreases the degree by $i > 0$ where G is a connected, compact Lie group and H is a closed subgroup of maximal rank. Then, for all i ,*

$$D_i(H^*(G/H; \mathbb{Q})) = 0.$$

2.2. Graded endomorphisms on $H_{\mathbb{C}G}^*$. It was conjectured in [O] that any graded endomorphism ϕ of the cohomology algebra $H_{\mathbb{C}G}^*$ is an Adams map when $k < n - k$; that is, there exists a rational λ such that $\phi(c_i) = \lambda^i c_i$, for all $i \in I$. Glover and Homer (see [GH1]) and Hoffman (see [Ho1]) proved the conjecture under the following hypothesis respectively:

- (3) Either $k \leq 3$ and $n > 2k$, or $k > 3$ and $n > 2k^2 - 1$.
- (4) The graded endomorphism φ of $H_{\mathbb{C}G}^*$ satisfies $\varphi(c_1) = \lambda c_1, \lambda \neq 0$.

Let us recall those results proved in [GH1, Ho1] that will be used in the rest of this paper.

Theorem 2.3 ([GH1], Theorem 1, [Ho1], Theorem 1.1). *(i) Assume that the hypothesis (3) is satisfied. Then for every graded endomorphism φ on $H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$, there exists a rational λ such that*

$$\varphi(c_i) = \lambda^i c_i, \quad \forall i \in I.$$

(ii) Assume that the hypothesis (4) is satisfied. Then, we have

$$\varphi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n - k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n - k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$ -dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$.

2.3. Generalized Dold spaces. In [Do], the author introduced the notion of *classical Dold manifolds* $P(m, n) := \mathbb{S}^m \times \mathbb{C}P^n / \sim$ where $(s, L) \sim (-s, \bar{L})$, where the involution $L \mapsto \bar{L}$ on $\mathbb{C}G_{n,k}$ is induced from the standard conjugation on \mathbb{C}^n , to construct generators in odd dimensions for René Thom's unoriented cobordism ring.

In [NS, MS1], the authors generalized the notion of classical Dold manifolds by replacing the sphere \mathbb{S}^m with an arbitrary topological space S equipped with a free involution α , analogous to the antipodal map on \mathbb{S}^m , and $\mathbb{C}P^n$ with an arbitrary topological space X with an involution $\sigma : X \rightarrow X$ having a nonempty fixed-point set, analogously to complex conjugation on $\mathbb{C}P^n$. Then the quotient space

$$(5) \quad P(S, \alpha, X, \sigma) := S \times X / \sim, \quad \text{where } (s, x) \sim (\alpha(s), \sigma(x)),$$

is called *generalized Dold space* (in short GDS), often denoted simply as $P(S, X)$. Moreover, the quotient map $S \times X \rightarrow P(S, X)$ is a double covering map.

Let us fix a notation Y for S / \sim_α , where $s \sim_\alpha \alpha(s), \forall s \in S$. Then, a GDS $P(S, X)$ is the total space of a fiber bundle $X \hookrightarrow P(S, X) \twoheadrightarrow Y$, where the fiber bundle projection is

$$(6) \quad p : P(S, X) \twoheadrightarrow Y, \quad [s, x] \mapsto [s].$$

Choosing a fixed-point of σ , say $x_0 \in \text{Fix}(\sigma) \neq \emptyset$, we can construct a section of the fiber bundle

$$(7) \quad s : Y \hookrightarrow P(S, X), \quad [s] \mapsto [s, x_0].$$

- (1) Either $\phi(u) = au$ for some $a \in \mathbb{Q}$, or $\phi(u) \in H_{\mathbb{C}G}^* \subseteq H_{\times}^*$ with $\phi(u)^2 = 0$ in H_{\times}^* .
(2) There exists $\lambda \in \mathbb{Q} \setminus \{0\}$ such that

$$\phi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n - k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n - k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$ -dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H_{\mathbb{C}G}^*$.

Proof. From equation (8) and (9), we have $H_{\times}^* \cong \mathcal{R}/\mathcal{I} \cong H_{\mathbb{C}G}^* \oplus uH_{\mathbb{C}G}^*$, where $\mathcal{R} := \mathbb{Q}[u, c_1, \dots, c_k]$ and $\mathcal{I} := \langle u^2, h_{n-k+1}, \dots, h_n \rangle$.

Let $p_1 : H_{\times}^* = H_{\mathbb{C}G}^* \oplus uH_{\mathbb{C}G}^* \rightarrow H_{\mathbb{C}G}^*$ be the projection onto the first summand and $i_1 : H_{\mathbb{C}G}^* \hookrightarrow H_{\mathbb{C}G}^* \oplus uH_{\mathbb{C}G}^*$ be the inclusion into the first summand. The composite $\phi_1 := p_1 \circ \phi \circ i_1$ is a degree-preserving endomorphism of $H_{\mathbb{C}G}^*$. We have the following diagram:

$$(11) \quad \begin{array}{ccc} H_{\mathbb{C}G}^* \oplus uH_{\mathbb{C}G}^* & \xrightarrow{\phi} & H_{\mathbb{C}G}^* \oplus uH_{\mathbb{C}G}^* \\ i_1 \uparrow & & \downarrow p_1 \\ H_{\mathbb{C}G}^* & \xrightarrow{\phi_1} & H_{\mathbb{C}G}^* \end{array}$$

Thus, for $x \in H_{\mathbb{C}G}^* \subset H_{\times}^*$, one can write $\phi(x) = \phi_1(x) + uP_x$ for some $P_x \in H_{\mathbb{C}G}^* \subset H_{\times}^*$ because the kernel of p_1 , $\ker(p_1) = uH_{\mathbb{C}G}^*$. This implies that

$$(12) \quad \phi(c_i) = \phi_1(c_i) + uP_{c_i}, \quad \forall i \in I.$$

For simplicity, denote P_{c_i} by $P_i \in H_{\mathbb{C}G}^{2i-m}$ which is a polynomial in c_1, \dots, c_k of degree $2i - m$ as $\deg c_i = 2i$ and $\deg u = m$.

Since $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$, that implies $\phi(c_1)$ is of the form $\lambda c_1 + \mu u$, $\lambda, \mu \in \mathbb{Q}$, $\lambda \neq 0$. Then we have $\phi_1(c_1) = \lambda c_1$, $\lambda \neq 0$ on $H_{\mathbb{C}G}^*$. By Theorem 2.3 part (ii), we have,

$$(13) \quad \phi_1(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n - k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n - k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$ -dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H_{\mathbb{C}G}^*$. Using the observations given above it is convenient to prove part (2) first.

proof of part (2): Using (12) and (13), it is sufficient to prove that $P_i = 0$, $\forall i \in I$. By (13), we have that ϕ_1 is an automorphism of $H_{\mathbb{C}G}^*$. Using the invertibility of ϕ_1 and (12), let $D : H_{\mathbb{C}G}^* \rightarrow H_{\mathbb{C}G}^*$ be defined by

$$D(x) = P_{\phi_1^{-1}(x)}, \quad \forall x \in H_{\mathbb{C}G}^*.$$

Equivalently, we have $D(\phi_1(x)) = P_x$. Now, we prove that D is a \mathbb{Q} -linear transformation and satisfies the Leibniz rule.

$$\begin{aligned}
 uP_{tx} &= \phi(tx) - \phi_1(tx) = t(\phi(x) - \phi_1(x)) = utP_x, \forall t \in \mathbb{Q}, \\
 uP_{x+y} &= \phi(x+y) - \phi_1(x+y) = \phi(x) - \phi_1(x) + \phi(y) - \phi_1(y) \\
 &\quad = u(P_x + P_y), \\
 (14) \quad uP_{xy} &= \phi(xy) - \phi_1(xy) = \phi(x)\phi(y) - \phi_1(x)\phi_1(y) \\
 &\quad = (\phi_1(x) + uP_x)(\phi_1(y) + uP_y) - \phi_1(x)\phi_1(y) \\
 &\quad = u(P_x\phi_1(y) + \phi_1(x)P_y).
 \end{aligned}$$

Using (9) and (14), we get

$$\begin{aligned}
 D(t\phi_1(x)) &= tD(\phi_1(x)), \quad D(\phi_1(x) + \phi_1(y)) = D(\phi_1(x)) + D(\phi_1(y)), \\
 D(\phi_1(x)\phi_1(y)) &= D(\phi_1(x))\phi_1(y) + \phi_1(x)D(\phi_1(y)).
 \end{aligned}$$

This proves that D is a derivation. For $x \in H_{\mathbb{C}G}^i$, we have $D(x) \in H_{\mathbb{C}G}^{i-m}$ which implies that the derivation D decreases the degree by $\deg(u) = m > 0$. By (2) and Theorem 2.2, we get that D is a zero derivation. In particular

$$D(\phi_1(c_i)) = P_i = 0, \forall i \in I.$$

proof of part (1): Since ϕ is a graded endomorphism on H_{\times}^* , therefore

$$\phi(u) = au + P, a \in \mathbb{Q}, \text{ satisfying } (au + P)^2 = 0,$$

where P is a homogeneous polynomial in c_1, \dots, c_k of degree m . We have $P^2 + 2auP = 0$ in H_{\times}^* . Using (9), we get that $2aP = 0$ in $H_{\times}^* = \mathcal{R}/\mathcal{I}$. Hence, either $a = 0$ or $P \in \mathcal{I}$. \square

Remark 3.2. Theorem 3.1 classifies all graded endomorphisms ϕ of H_{\times}^* whose image is nonzero in $H_{\mathbb{C}G}^2$ if $n > 2$. In fact, $n > 2$ implies $c_1^2 \neq 0$ and $\phi(u) \neq ac_1$, $a \in \mathbb{Q} \setminus \{0\}$ as $\phi(u)^2 = 0$. Therefore, the only remaining possibility is $\phi(c_1) \neq \mu u$, $\mu \in \mathbb{Q}$.

On the other hand, when $n = 2$, $\mathbb{C}G_{n,k}$ is either a point or \mathbb{S}^2 and the classification of graded endomorphisms of H_{\times}^* is easy.

3.2. In Theorem 3.1, we assume that $\phi(c_1) \neq \mu u$. Let us try to look at the other case where $\phi(c_1) = \mu u$. To address this, we use part (i) of Theorem 2.3 which leads to the following proposition.

Proposition 3.3. Assume that hypothesis (3) is satisfied. Let ϕ be a graded endomorphism such that $\phi(c_1) = \mu u$, $\mu \in \mathbb{Q}$ in H_{\times}^* . Then

- (1) Either $\phi(u) = au$ for some $a \in \mathbb{Q}$, or $\phi(u) \in H_{\mathbb{C}G}^* \subseteq H_{\times}^*$ with $\phi(u)^2 = 0$ in H_{\times}^* .
- (2) $\phi(c_i) = uP_i$, $\forall i > 1$, where $P_i \in H_{\mathbb{C}G}^{2i-m} \subseteq H_{\times}^*$.

Proof. (1): The proof of part (1) is exactly the same as the proof of part (1) of Theorem 3.1. Therefore, we omit the details.

(2): Using (11), we have that the map ϕ_1 is a graded endomorphism on $H_{\mathbb{C}G}^*$ such that $\phi_1(c_1) = 0$. By Theorem 2.3, $\phi_1(c_i) = 0$, $\forall i \in I$, then by (12), we get $\phi(c_i) = uP_i$ for some $P_i \in H_{\mathbb{C}G}^*$, with $\deg(P_i) = 2i - m$. \square

Remark 3.4. In Theorem 3.1 and Proposition 3.3, if we assume $2m \leq n - k$ then $\phi(u) = 0$ whenever $\phi(u) \in H_{\mathbb{C}G}^*$. This is because $H_{\mathbb{C}G}^*$ has no nontrivial relations up to degree $2(n - k)$ and $u^2 = 0$ implies that $\phi(u)^2 = 0$ forcing $\phi(u) = 0$.

A graded endomorphism of $H_{\mathbb{C}G}^*$ that vanishes on $H_{\mathbb{C}G}^2$ is expected to be trivial, in view of Hoffman's conjecture [Ho1]. However, unlike the case of the complex Grassmannian, there exist many non-trivial graded endomorphisms of H_{\times}^* that vanish on $H_{\mathbb{C}G}^2$. The following proposition provides such examples when m is even and $1 \leq m \leq 2k$.

Proposition 3.5. *For each $i \in I$, choose $P_i \in H_{\mathbb{C}G}^{2i-m} \subseteq H_{\times}^*$ and either $Q = au$, $a \in \mathbb{Q}$, or $Q \in H_{\mathbb{C}G}^* \subseteq H_{\times}^*$ with $Q^2 = 0$ in H_{\times}^* . Then there exist a graded endomorphism ϕ on H_{\times}^* such that*

$$\phi(c_i) = uP_i, \forall i \in I, \text{ and } \phi(u) = Q.$$

Proof. Define ϕ on $H_{\times}^* = \mathcal{R}/\mathcal{I}$ by $\phi(c_i) = uP_i$, $\forall i \in I$, and $\phi(u) = Q$. It is sufficient to prove that ϕ is well defined, that is, $\mathcal{I} \subseteq \ker(\phi)$. Observe that $u^2 = 0$ in H_{\times}^* which implies that

$$(15) \quad \phi(c_i c_j) = \phi(c_i)\phi(c_j) = uP_i \cdot uP_j = u^2 P_i P_j = 0.$$

Using (15) and $\phi(u^2) = Q^2 = 0$, we have $\mathcal{I} \subseteq \langle u^2, c_i c_j \mid i, j \in I \rangle \subseteq \ker(\phi)$. \square

3.3. In this subsection, we derive some immediate applications of Theorem 3.1.

Corollary 3.6. *Let us consider $X = \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}$ and denote by u_j the generator of $H^{2m_j}(\mathbb{S}^{2m_j}; \mathbb{Q})$ corresponding to the fundamental class of \mathbb{S}^{2m_j} for all $1 \leq j \leq r$. Define*

$$H_{\mathbf{m}, \mathbb{C}G}^* := H^*(\mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}; \mathbb{Q}) \cong H_{\mathbb{C}G}^*[u_1, \dots, u_r]/\langle u_1^2, \dots, u_r^2 \rangle,$$

where $\mathbf{m} = (m_1, \dots, m_r)$. Suppose $\phi : H_{\mathbf{m}, \mathbb{C}G}^* \rightarrow H_{\mathbf{m}, \mathbb{C}G}^*$ is a graded endomorphism satisfying $\phi(c_1) = \lambda c_1$, $\lambda \neq 0$. Then

$$\phi(c_i) = \begin{cases} \lambda^i c_i, \forall i \in I & \text{if } k < n - k, \\ \lambda^i c_i, \forall i \in I \quad \text{or} \quad (-\lambda)^i (c^{-1})_i, \forall i \in I & \text{if } k = n - k, \end{cases}$$

where $(c^{-1})_i$ is the $2i$ -dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H_{\mathbb{C}G}^*$.

Proof. The proof of this corollary is similar to the proof of part 2 of Theorem 3.1. Apply induction on r and replace $\mathbb{C}G_{n,k}$ with $\hat{X} := \mathbb{S}^{2m_1} \times \cdots \times \mathbb{S}^{2m_{i-1}} \times \mathbb{S}^{2m_{i+1}} \times \cdots \times \mathbb{S}^{2m_r} \times \mathbb{C}G_{n,k}$, and the sphere \mathbb{S}^m with \mathbb{S}^{2m_i} in Theorem 3.1. Since

$$(16) \quad \mathbb{S}^{2m_j} = SO(2m_j + 1)/SO(2m_j)$$

where the orthogonal groups $SO(2m_j + 1)$ and $SO(2m_j)$ have the same rank m_j . Using (16) and (2), \hat{X} satisfies the hypothesis of Theorem 2.2. Therefore, every \mathbb{Q} -linear derivation of $H^*(\hat{X}; \mathbb{Q})$ that decreases the degree by $2m_i$ is trivial. \square

Let us turn our attention to the generalized Dold spaces $P(m, n, k)$ defined in Subsection 2.4. The following remark helps us to describe endomorphisms of $H^*(P(m, n, k); \mathbb{Q})$ induced by continuous functions on $P(m, n, k)$. These observations will be used in Section 4.

Remark 3.7. *For a continuous map f on $P(m, n, k)$, we have*

$$(17) \quad f_* \circ \pi_* (\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})) \subseteq \pi_* (\pi_1(\mathbb{S}^m \times \mathbb{C}G_{n,k})),$$

where $\pi_1(X)$ denotes the fundamental group of a topological space X . Hence, the composite $f \circ \pi$ admits a lift \tilde{f} on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ for the double covering $\pi : \mathbb{S}^m \times \mathbb{C}G_{n,k} \rightarrow P(m, n, k)$.

Using Remark 3.7, we get the following commutative diagram,

$$(18) \quad \begin{array}{ccc} H^*(P(m,n,k); \mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k}; \mathbb{Q}) \\ f^* \downarrow & & \downarrow \bar{f}^* \\ H^*(P(m,n,k); \mathbb{Q}) & \xrightarrow{\pi^*} & H^*(\mathbb{S}^m \times \mathbb{C}G_{n,k}; \mathbb{Q}). \end{array}$$

where π^* is an injective map. Using Theorem 2.4 and (18) we obtain the following two corollaries.

Corollary 3.8. *Let f^* be an endomorphism of $H^*(P(m,n,k); \mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ satisfying $f^*(c_1^2) \neq 0$. Then f^* is the restriction of a graded endomorphism \tilde{f}^* on H_\times^* satisfying $\tilde{f}^*(c_1) = \lambda c_1$, $\lambda \neq 0$, to the fixed subring $\text{Fix}(\theta^*)$ of H_\times^* where $\theta = \alpha \times \sigma$.*

Corollary 3.9. *Let f^* be an endomorphism of $H^*(P(m,n,k); \mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ satisfying $f^*(c_1^2) = 0$ and $n > 2$. Then f^* is the restriction of a graded endomorphism \tilde{f}^* on H_\times^* satisfying $\tilde{f}^*(c_1) = au$, $a \in \mathbb{Q}$, to the fixed subring $\text{Fix}(\theta^*)$ of H_\times^* where $\theta = \alpha \times \sigma$.*

Using Theorem 3.1 in Corollary 3.8, and Proposition 3.3 in Corollary 3.9 along with hypothesis (3), we can determine f^* .

Moreover, there exist graded endomorphisms of $H^*(P(m,n,k))$ that are not induced by any continuous self-map of $P(m,n,k)$, and cannot be realized as restrictions of graded endomorphisms of H_\times^* . Let us see an example of such graded endomorphism.

Example 3.10. *If m odd, $n > 2$ and $k = 1$, then $P(m,n,1)$ is fibered by the complex projective space $\mathbb{C}P^{n-1}$ over the real projective space $\mathbb{R}P^m$. In this case, $H_\times^* \cong \mathbb{Q}[u, c_1]/\langle u^2, c_1^n \rangle$ and using (10) and Theorem 2.4, the rational cohomology ring*

$$H^*(P(m,n,1); \mathbb{Q}) \cong \mathbb{Q}[u, b]/\langle u^2, b^{\lfloor(n+1)/2\rfloor} \rangle,$$

where u is a generator of $H^m(\mathbb{R}P^m; \mathbb{Q})$ and b restricts to $c_1^2 \in H^2(\mathbb{C}P^{n-1}; \mathbb{Q})$ under the fiber inclusion.

Consider the endomorphism

$$\phi: H^*(P(m,n,1); \mathbb{Q}) \rightarrow H^*(P(m,n,1); \mathbb{Q}), \quad \text{defined by} \quad u \mapsto u, b \mapsto -b.$$

Then ϕ is a well-defined graded endomorphism but it cannot be a restriction of a graded endomorphism of H_\times^* because any such map induces $c_1^2 \mapsto \lambda^2 c_1^2$ for some $\lambda \in \mathbb{Q}$, and $\lambda^2 \neq -1$.

The following corollary helps us to understand the relationship between the automorphisms of $H^*(P(m,n,k))$ with the automorphisms of H_\times^* .

Corollary 3.11. *Let f^* be an automorphism of $H^*(P(m,n,k); \mathbb{Q})$ induced by a continuous function f on $P(m,n,k)$ and assume that $n > 2$. Then \tilde{f}^* is an automorphism of H_\times^* , where \tilde{f} is as in Remark 3.7.*

Moreover there exist $\lambda, \mu \in \mathbb{Q} \setminus \{0\}$ such that $\tilde{f}^*(u) = \mu u$ and $\tilde{f}^*(c_i)$ is of the form given in (2) of Theorem 3.1.

The induced diagram in cohomology implies the following commutative diagram.

$$\begin{array}{ccc} \prod_{i=1}^m u_i & \xrightarrow{\tilde{f}^*} & \prod_{i=1}^m P_i(u_1, \dots, u_m) \\ \uparrow (q \times \text{id})^* & & \uparrow (q \times \text{id})^* \\ u & \xrightarrow{f^*} & f^*(u) \end{array}$$

This implies that $f^*(u)$ does not contain any nonzero element from $H^*(\mathbb{C}G_{n,k}; \mathbb{Z})$. Thus, $f^*(u) = \mu u$ for some $\mu \in \mathbb{Z}$. \square

4. COINCIDENCE THEORY OF $P(m, n, k)$

In this section, we study the *coincidence theory* of generalized Dold spaces $P(m, n, k)$ defined in Subsection 2.4. We establish the necessary conditions for a generalized Dold space $P(S, X)$ defined in (5) to satisfy the coincidence property.

4.1. Let us recall certain definitions that will be required in the rest of this section.

Definition 4.1. Let (X, g) be a pair, where g is a continuous map on a topological space X . The pair (X, g) is said to have the **coincidence property** (in short, CP) if, for every continuous map $f : X \rightarrow X$, there exists a point $x \in X$ such that $f(x) = g(x)$.

If we consider g to be the identity map on X , then the notion of coincidence reduces to that of a fixed point, resulting in the following definition.

Definition 4.2. A topological space X is said to have **fixed-point property** (FPP) if every continuous map $f : X \rightarrow X$ admits a fixed-point; that is, there exists $x \in X$ such that $f(x) = x$.

The following proposition provides a criteria in terms of the fiber X and the base space $Y := S/\sim_\alpha$, allowing one to infer the coincidence properties of the total space $P(S, X)$.

Proposition 4.3. Let $(P(S, X), g)$ be a pair, where g is a continuous map on the generalized Dold space $P(S, X)$. Then $(P(S, X), g)$ does not have the CP if one of the following hold:

- (1) The continuous map g is a fiber bundle map and the pair $(Y, p \circ g \circ s)$ does not have the CP, where $Y = S/\sim_\alpha$ and s denotes a section of the X -bundle projection p defined in (7) and (6).
- (2) There exists a σ -equivariant map f (i.e. $f \circ \sigma = \sigma \circ f$) on X and a $\alpha \times \sigma$ -equivariant map \tilde{g} on $S \times X$ inducing g such that $\text{id}_S \times f$ coincides with neither \tilde{g} nor $(\alpha \times \sigma) \circ \tilde{g}$.

Proof. (1) Suppose that the pair $(Y, p \circ g \circ s)$ does not have the CP. Then there exists a continuous map $f : Y \rightarrow Y$ such that

$$(19) \quad f(x) \neq p \circ g \circ s(x), \forall x \in Y.$$

We are given that g is a fiber bundle map, which implies that there exist $g_1 : Y \rightarrow Y$, satisfying $p \circ g = g_1 \circ p$. Consider $p \circ g \circ s = g_1 \circ p \circ s = g_1$. Thus, $p \circ g = g_1 \circ p$ implies

$$p \circ g(x) = p \circ g \circ s \circ p(x), \forall x \in P(S, X).$$

Define the map $\phi := s \circ f \circ p$ on $P(S, X)$. We claim that $\phi(y) \neq g(y)$, $\forall y \in P(S, X)$. Suppose there exist $y \in P(S, X)$ such that $\phi(y) = g(y)$, then

$$p \circ g \circ s(p(y)) = p \circ g(y) = p \circ s \circ f \circ p(y) = f(p(y)),$$

which contradicts (19).

(2) Let G denote the group of deck transformations of the double covering $\pi : S \times X \rightarrow P(S, X)$, generated by the free involution $\alpha \times \sigma$. The proof then follows from a general observation that if for two G -equivariant maps $\tilde{\phi}, \tilde{\psi}$ on $S \times X$, the maps $\tilde{\phi}$ and $t \cdot \tilde{\psi}$ have no point of coincidence, for any $t \in G$; then the maps they induce on the orbit space $P(S, X)$, namely ϕ, ψ , are also coincidence-free. \square

In particular if we take g to be the identity map in Proposition 4.3, we recover Proposition 7.2.1 of [M], which proves that if the base Y does not have the FPP, or there exist a σ -equivariant map on the fibre X with no fixed point then $P(S, X)$ does not have the FPP. As a consequence, $P(m, n)$ does not have the FPP if either m or n is odd.

4.2. Let us recall a well known result in coincidence theory, the Lefschetz Coincidence Theorem, which will be used to prove results in the rest of this paper.

For a closed oriented manifold M of dimension n , let $[M] \in H^n(M; \mathbb{Q})$ denote a chosen fundamental class. Then we have the Poincaré duality isomorphism $D_M : H^k(M; \mathbb{Q}) \rightarrow H_{n-k}(M; \mathbb{Q})$, defined by

$$(20) \quad D_M(\alpha) = [M] \frown \alpha, \quad \forall \alpha \in H^k(M; \mathbb{Z}).$$

Theorem 4.4 (Lefschetz Coincidence Theorem). *Let f, g be two continuous maps on a compact, connected, oriented manifold M of dimension n . The Lefschetz coincidence number is defined as*

$$L(f, g) := \sum_{i=0}^n (-1)^i \text{tr}(D_M \circ g^* \circ D_M^{-1} \circ f_* : H_i(M; \mathbb{Q}) \longrightarrow H_i(M; \mathbb{Q})).$$

If $L(f, g) \neq 0$, then there exists $x \in M$ such that $f(x) = g(x)$.

When $g = \text{id}_M$, the theorem reduces to the Lefschetz Fixed-Point Theorem for M .

To study the coincidence theory of generalized Dold spaces fibred by complex Grassmannians over real projective spaces, it is helpful to first understand the coincidence theory of complex Grassmannians, a topic of independent interest. We now prove the following lemma (cf. Theorem 2, [GH1]) to prove Proposition 4.6.

Lemma 4.5. *Let d_{2i} be the $2i$ -th Betti number of a complex Grassmannian $\mathbb{C}G_{n,k}$ with $d = k(n - k)$ even. Then the sum $\sum_{i=0}^d d_{2i} \lambda^i \neq 0$, $\forall \lambda \in \mathbb{Q}$.*

Proof. Let us consider the sum $\sum_{i=0}^d d_{2i} \lambda^i$, when λ is an integer. Clearly,

$$\sum_{i=0}^d d_{2i} \lambda^i \equiv 1 \pmod{\lambda}.$$

Hence, $\sum_{i=0}^d d_{2i} \lambda^i \neq 0$, if $\lambda \neq \pm 1$. When $\lambda = 1$, the sum is also positive and therefore nonzero. It remains to consider the case where $\lambda = -1$. Let $\chi(\mathbb{R}G_{n,k})$

denote the Euler-Poincaré characteristic of $\mathbb{R}G_{n,k}$ and be defined by

$$\chi(X) := \sum_{i \geq 0} \dim H^i(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$$

where $\mathbb{R}G_{n,k}$ denotes the Grassmannian of real k -planes in \mathbb{R}^n . Now we observe that $\sum_{i=0}^d d_{2i}(-1)^i = \chi(\mathbb{R}G_{n,k})$ where $d_{2i} = \dim H^{2i}(\mathbb{C}G_{n,k}; \mathbb{Q}) = \dim H^i(\mathbb{R}G_{n,k}; \mathbb{Z}_2)$. It is a well known fact that $\chi(\mathbb{R}G_{n,k}) \neq 0$ if $k(n-k)$ is even.

Let us move to the other case where $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$. Suppose $\sum_{i=0}^d d_{2i}\lambda^i = 0$ for some $\lambda = \frac{p}{q}$ where p and q are coprime integers. Since $d_0 = d_d = 1$, using the rational root theorem $p|1$ and $q|1$. Hence, $\lambda = \pm 1$, which is a contradiction. Therefore, we conclude that $\sum_{i=0}^d d_{2i}\lambda^i \neq 0$ for all $\lambda \in \mathbb{Q}$. \square

Denote the i -th homology groups $H_i(\mathbb{C}G_{n,k}; \mathbb{Q})$, $H_i(\mathbb{S}^m; \mathbb{Q})$ and $H_i(\mathbb{S}^m \times \mathbb{C}G_{n,k}; \mathbb{Q})$, by $H_i^{\mathbb{C}G}$, $H_i^{\mathbb{S}}$ and H_i^{\times} , respectively. Let d denote the complex dimension of $\mathbb{C}G_{n,k}$, given by $d = k(n-k)$. Then we have the following proposition.

Proposition 4.6. *Consider a complex Grassmannian $\mathbb{C}G_{n,k}$ such that the hypothesis (3) is satisfied and $k(n-k)$ is even. Let g be a continuous map on $\mathbb{C}G_{n,k}$ with nonzero Brouwer degree. Then the pair $(\mathbb{C}G_{n,k}, g)$ has the coincidence property.*

Proof. Self-maps with nonzero Brouwer degree induces automorphisms in the rational cohomology algebra. Using Theorem 2.3 part (i), there exist a nonzero rational λ such that $g^*(c_i) = \lambda^i c_i, \forall i \in I$. Let f be a continuous map on $\mathbb{C}G_{n,k}$ and using Theorem 2.3 part (i), there exists $\mu \in \mathbb{Q}$ such that

$$f^*(c_i) = \mu^i c_i, \forall i \in I.$$

Then by the Universal Coefficient Theorem, $\text{Hom}_{\mathbb{Q}}(H_i^{\mathbb{C}G}; \mathbb{Q}) \cong H_{\mathbb{C}G}^i$ non-canonically which implies that

$$\begin{aligned} \varphi \circ f_* &= f^*(\varphi), \forall \varphi \in \text{Hom}_{\mathbb{Q}}(H_{2i}^{\mathbb{C}G}, \mathbb{Q}) \cong H_{\mathbb{C}G}^{2i}. \\ \varphi(f_*(x)) &= (f^*(\varphi))(x) = \mu^i \varphi(x) = \varphi(\mu^i x), \forall x \in H_{2i}^{\mathbb{C}G}. \end{aligned}$$

The last equation implies that $f_*(x) = \mu^i x, \forall x \in H_{2i}^{\mathbb{C}G}$. Now observe that $D \circ g^* \circ D^{-1} \circ f_* : H_{2i}^{\mathbb{C}G} \rightarrow H_{2i}^{\mathbb{C}G}$ is given by

$$D \circ g^* \circ D^{-1} \circ f_*(x) = D \circ g^* \circ D^{-1}(\mu^i x) = \mu^i D \circ g^*(D^{-1}x) = \mu^i D(\lambda^{d-i} D^{-1}x) = \mu^i \lambda^{d-i} x.$$

Thus for $x \in H_{2i}^{\mathbb{C}G}$, the Lefschetz coincidence number is given by

$$\begin{aligned} L(f, g) &= \sum_{i=0}^d (-1)^{2i} \text{tr}(D \circ g^* \circ D^{-1} \circ f_*(x)) \\ &= \sum_{i=0}^d d_{2i} \mu^i \lambda^{d-i} \\ &= \lambda^d \sum_{i=0}^d d_{2i} (\mu/\lambda)^i \neq 0 \quad (\because \lambda \neq 0) \end{aligned}$$

where d_{2i} denotes $\dim_{\mathbb{Q}} H_{\mathbb{C}G}^{2i}$ and the last equation holds by using Lemma 4.5. Therefore, using Theorem 4.4 the pair $(\mathbb{C}G_{n,k}, g)$ has the coincidence property. \square

4.3. Denote by $H_*^{\times} = \bigoplus_{i \geq 0} H_i^{\times}$, $H_*^{\mathbb{C}G} = \bigoplus_{i \geq 0} H_i^{\mathbb{C}G}$, $H_*^{\mathbb{S}} = \bigoplus_{i \geq 0} H_i^{\mathbb{S}}$ and ϑ the fundamental class $[\mathbb{S}^m] \in H_m^{\mathbb{S}}$. Let $\{v_q\}$ be a homogeneous basis of $H_*^{\mathbb{C}G}$, and let $\{\delta_{v_q}\}$ denote the corresponding dual basis of $\text{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q}) \cong H_{\mathbb{C}G}^*$, such that $\delta_{v_q}(v_p) = \delta_{qp}$ where δ_{qp} is the Kronecker delta function. Without loss of generality, assume that $1 = v_0 \in \{v_i\}$ represents the generator of $H_0^{\mathbb{C}G} \cong \mathbb{Q}$.

Over \mathbb{Q} , the Künneth Theorem yields the following decompositions

$$(21) \quad H_i^\times \cong H_i^{\mathbb{C}G} \oplus (\vartheta \otimes H_{i-m}^{\mathbb{C}G}), \quad H_\times^i \cong H_{\mathbb{C}G}^i \oplus uH_{\mathbb{C}G}^{i-m},$$

where $u \in H_\times^m \cong \text{Hom}(H_m^\times, \mathbb{Q})$ corresponds to the element $\delta_{\vartheta \otimes 1}$.

Using (21), we can extend the chosen basis $\{v_q\}$ of $H_*^{\mathbb{C}G}$ to $\{v_q\} \cup \{\vartheta \otimes v_q\}$ of H_\times^* such that the corresponding dual basis can also be extended from $\{\delta_{v_q}\}$ of $\text{Hom}(H_*^{\mathbb{C}G}; \mathbb{Q})$ to $\{\delta_{v_q}\} \cup \{\delta_{\vartheta \otimes v_q}\}$ of $\text{Hom}(H_\times^*; \mathbb{Q})$ satisfying:

$$(22) \quad \delta_{v_q}(v_p) = \delta_{qp}, \quad \delta_{v_q}(\vartheta \otimes v_p) = 0, \quad \delta_{\vartheta \otimes v_q}(v_p) = 0, \quad \delta_{\vartheta \otimes v_q}(\vartheta \otimes v_p) = \delta_{qp}.$$

Let f be a continuous function on $P(m, n, k)$. Using Remark 3.7 and the Universal Coefficient Theorem, there exist a lift \tilde{f} on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ satisfying

$$(23) \quad \varphi \circ \tilde{f}_* = \tilde{f}^*(\varphi), \forall \varphi \in \text{Hom}_{\mathbb{Q}}(H_{2i}^{\mathbb{C}G}, \mathbb{Q}) \cong H_{\mathbb{C}G}^{2i}.$$

Poincaré duality on $\mathbb{S}^m \times \mathbb{C}G_{n,k}$ can be described in terms of the duality on the Grassmannian factor. Let $D_{\mathbb{C}G}: H_{\mathbb{C}G}^i \rightarrow H_{2d-i}^{\mathbb{C}G}$ be the Poincaré duality isomorphism defined in (20) for $\mathbb{C}G_{n,k}$, where $d = k(n-k)$. The Poincaré duality isomorphism on the product is then determined on the basis elements by

$$(24) \quad D: H_\times^j \rightarrow H_{m+2d-j}^\times, \quad \delta_{v_i} \mapsto \vartheta \otimes D_{\mathbb{C}G}(\delta_{v_i}) \quad \text{and} \quad \delta_{\vartheta \otimes v_i} \mapsto D_{\mathbb{C}G}(\delta_{v_i}).$$

We are now ready to establish the following lemmas, which will be useful in the sequel.

Lemma 4.7. *Let f be a continuous function on $P(m, n, k)$ and \tilde{f} be the lift defined in Remark 3.7 such that $\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$ and $k < n - k$. Then there exist $\lambda \in \mathbb{Q} \setminus \{0\}$ and $\mu \in \mathbb{Q}$ such that the induced map \tilde{f}_* on H_\times^* is of the following form.*

- (1) Either $\tilde{f}_*(\vartheta \otimes x) = \mu\lambda^i(\vartheta \otimes x)$, $\forall x \in H_{2i}^{\mathbb{C}G}$ or $\tilde{f}_*(\vartheta \otimes x) \in H_*^{\mathbb{C}G}$, $\forall x \in H_*^{\mathbb{C}G}$.
- (2) $\tilde{f}_*(x) = \lambda^i x + \vartheta \otimes y$, for some $y \in H_{2i-m}^{\mathbb{C}G}$, $\forall x \in H_{2i}^{\mathbb{C}G}$.

Moreover, $y = 0$ in (2) if $\tilde{f}_*(\vartheta \otimes x) = \mu\lambda^i(\vartheta \otimes x)$, $\forall x \in H_{2i}^{\mathbb{C}G}$.

Proof. Using Theorem 3.1, there exist $\lambda \in \mathbb{Q} \setminus \{0\}$ such that $\tilde{f}^*(c_i) = \lambda^i c_i$, $\forall i \in I$ and either $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ or $\tilde{f}^*(u) \in H_{\mathbb{C}G}^*$. It is sufficient to prove the result for the chosen basis $\{v_q\} \cup \{\vartheta \otimes v_q\}$ of H_\times^* .

Let us consider the first case where $\tilde{f}^*(u) = \mu u$. Using $H_{\mathbb{C}G}^* \cong \text{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have

$$(25) \quad \tilde{f}^*(\delta_{v_p}) = \lambda^i \delta_{v_p}, \quad \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta \otimes v_p}) = \mu \lambda^i (\delta_{\vartheta \otimes v_p}), \quad \forall v_p \in H_{2i}^{\mathbb{C}G}.$$

If m is odd, then the coefficient of any basis element $v_p \in H_*^{\mathbb{C}G}$ in $\tilde{f}_*(\vartheta \otimes v_q)$ and $\vartheta \otimes v_p$ in $\tilde{f}_*(v_q)$ is zero because \tilde{f}_* is a graded map. Let us consider the case where $m = 2s$. By (23) and (25), the coefficient of a basis element $v_p \in H_{2i+m}^{\mathbb{C}G}$ in $\tilde{f}_*(\vartheta \otimes v_q)$ written as a \mathbb{Q} -linear combination of the basis elements from $\{v_q\} \cup \{\vartheta \otimes v_q\}$ is the following:

$$\delta_{v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{v_p})(\vartheta \otimes v_q) = \lambda^{i+s} \delta_{v_p}(\vartheta \otimes v_q) = 0, \quad \forall v_q \in H_{2i}^{\mathbb{C}G}$$

and the coefficient of a basis element $\vartheta \otimes v_p \in \vartheta \otimes H_{2i}^{\mathbb{C}G}$ in $\tilde{f}_*(\vartheta \otimes v_q)$ is

$$\delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{\vartheta \otimes v_p})(\vartheta \otimes v_q) = \mu \lambda^i \delta_{\vartheta \otimes v_p}(\vartheta \otimes v_q) = \mu \lambda^i \delta_{pq}, \quad \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

This implies that

$$\tilde{f}_*(\vartheta \otimes v_q) = \mu \lambda^i (\vartheta \otimes v_q), \quad \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

Using similar calculations given above, it is easy to show that

$$\delta_{v_p} \circ \tilde{f}_*(v_q) = \lambda^i \delta_{pq}, \quad \forall v_q \in H_{2i}^{\mathbb{C}G}, \quad \delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(v_q) = 0, \quad \forall v_q \in H_{2i}^{\mathbb{C}G}.$$

Therefore, $\tilde{f}_*(v_q) = \lambda^i v_q, \forall v_q \in H_{2i}^{\mathbb{C}G}$.

If $\tilde{f}^*(u) \in H_{\mathbb{C}G}^*$. Again using $H_{\mathbb{C}G}^* \cong \text{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have

$$(26) \quad \tilde{f}^*(\delta_{v_p}) = \lambda^i \delta_{v_p}, \quad \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta \otimes v_p}) \in H_{\mathbb{C}G}^*, \quad \forall v_p \in H_{2i}^{\mathbb{C}G}.$$

By (23) and (26), we get $\delta_{v_p} \circ \tilde{f}_*(v_q) = \lambda^i \delta_{pq}, \forall v_q \in H_{2i}^{\mathbb{C}G}$, which implies that $\tilde{f}_*(x) = \lambda^i x + \vartheta \otimes y$, for some $y \in H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}$. Note that $\tilde{f}^*(\delta_{\vartheta \otimes v_p}) \in H_{\mathbb{C}G}^*$ and equal to some $\sum a_j \delta_{v_j}$. Then

$$(27) \quad \delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = \tilde{f}^*(\delta_{\vartheta \otimes v_p})(\vartheta \otimes v_q) = \sum a_j \delta_{v_j}(\vartheta \otimes v_q) = 0.$$

Hence, $\tilde{f}_*(\vartheta \otimes v_q) \in H_*^{\mathbb{C}G}$ for all $\vartheta \otimes v_q \in \vartheta \otimes H_*^{\mathbb{C}G}$. \square

Lemma 4.8. *Assume that the hypothesis (3) is satisfied. Let f be a continuous function on $P(m, n, k)$ and \tilde{f} be the lift defined in Remark 3.7 such that $\tilde{f}^*(c_1) = au, a \in \mathbb{Q}$. Then the induced map \tilde{f}_* on $H_*^{\mathbb{C}G}$ is of the following form.*

- (1) $\tilde{f}_*(x) \in \vartheta \otimes H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}, \forall i > 0$.
- (2) $\tilde{f}_*(\vartheta \otimes 1) = \mu(\vartheta \otimes 1) + y, y \in H_m^{\mathbb{C}G},$
 $\tilde{f}_*(\vartheta \otimes x) \in H_{2i+m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}, i > 0 \text{ if } \tilde{f}^*(u) = \mu u, \mu \in \mathbb{Q}$

Proof. Using Proposition 3.3, we have $\tilde{f}^*(c_i) = uP_i$, for some $P_i \in H_{\mathbb{C}G}^*$ and either $\tilde{f}^*(u) = \mu u, \mu \in \mathbb{Q}$ or $\tilde{f}^*(u) \in H_{\mathbb{C}G}^*$.

Let us consider the first case where $\tilde{f}^*(u) = \mu u$. Using $H_{\mathbb{C}G}^* \cong \text{Hom}(H_*^{\mathbb{C}G}, \mathbb{Q})$, we have for $i > 0$

$$(28) \quad \tilde{f}^*(\delta_{v_p}) = \sum a_{jp} \delta_{\vartheta \otimes v_j}, \quad \forall v_p \in H_{2i}^{\mathbb{C}G}, \quad \tilde{f}^*(\delta_{\vartheta \otimes 1}) = \mu \delta_{\vartheta \otimes 1}, \quad \tilde{f}^*(\delta_{\vartheta \otimes v_p}) = 0, \quad \forall v_p \in H_{2i}^{\mathbb{C}G}.$$

Using (23), (28) and similar calculations given in the proof of Lemma 4.7, we have for $v_p \neq 1$

$$\delta_{v_p} \circ \tilde{f}_*(v_q) = 0, \quad \forall v_q \in H_{2i}^{\mathbb{C}G}, \quad \delta_{\vartheta \otimes v_p} \circ \tilde{f}_*(\vartheta \otimes v_q) = 0, \quad \forall v_q \in H_{2i}^{\mathbb{C}G}$$

that concludes the result.

When $\tilde{f}^*(u) \in H_{\mathbb{C}G}^*$ then also we have $\tilde{f}^*(\delta_{v_p}) = \sum a_{jp} \delta_{\vartheta \otimes v_j}, \forall v_p \in H_{2i}^{\mathbb{C}G}, \forall i > 0$ which implies that $\delta_{v_p} \circ \tilde{f}_*(v_q) = 0, \forall v_q \in H_{2i}^{\mathbb{C}G}, \forall i > 0$. \square

4.4. The following theorems provide a criteria for the existence of coincidence points between a pair of continuous functions on $P(m, n, k)$.

Theorem 4.9. *Let $P(m, n, k)$ be a generalized Dold manifold with $k < n - k$ and $k(n - k)$ even. Let f and g be two continuous maps on $P(m, n, k)$ and \tilde{f}, \tilde{g} be their lifts as defined in Remark 3.7 such that*

- (1) g^* is an automorphism of $H^*(P(m, n, k); \mathbb{Q})$.
- (2) $\tilde{f}^*(c_1) \neq au, a \in \mathbb{Q}$.
- (3) $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ if m is odd.

where s denotes a section of the X -bundle projection p defined in (7) and (6). Then, there is a point of coincidence of f and g .

Proof. Using Corollary 3.11, we have \tilde{g}^* is an automorphism on H_{\times}^* given by $\tilde{g}^*(c_i) = \lambda_1^i c_i$, and $\tilde{g}^*(u) = \mu_1 u$ for some $\lambda_1, \mu_1 \in \mathbb{Q} \setminus \{0\}$ if $k < n - k$.

Using Lemma 4.7, there exist $\lambda \in \mathbb{Q} \setminus \{0\}$ and $\mu \in \mathbb{Q}$ such that \tilde{f}_* is of the following form,

$$(29) \quad \begin{aligned} \tilde{f}_*(x) &= \lambda^i x + \vartheta \otimes y, \text{ for some } y \in H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G} \\ \tilde{f}_*(\vartheta \otimes x) &= \mu \lambda^i (\vartheta \otimes x), \text{ or } \tilde{f}_*(\vartheta \otimes x) = z, \text{ for some } z \in H_{2i+m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G} \end{aligned}$$

To prove that f has a point of coincidence with g , it is sufficient to prove that either \tilde{f} or the composition $\theta \circ \tilde{f}$ has a point of coincidence with \tilde{g} where $\theta = \alpha \times \sigma$ defined in Section 2.4. By Theorem 4.4, we need to compute $L(\tilde{f}, \tilde{g})$ and $L(\theta \circ \tilde{f}, \tilde{g})$.

For $x \in H_{2i}^{\mathbb{C}G}$, we have

$$\begin{aligned} D\tilde{g}^* D^{-1} \tilde{f}_*(x) &= \mu_1 \lambda^i \lambda_1^{d-i} x + \vartheta \otimes y' \text{ for some } y' \in H_{2i-m}^{\mathbb{C}G} \\ D\tilde{g}^* D^{-1} \tilde{f}_*(\vartheta \otimes x) &= \mu \lambda^i \lambda_1^{d-i} (\vartheta \otimes x) + z' \text{ for some } z' \in H_{2i+m}^{\mathbb{C}G}. \end{aligned}$$

where $z' = 0$ or $\mu = 0$ depending on the image of $\tilde{f}_*(\vartheta \otimes x)$. Recall that d_{2i} denote the dimension $\dim H_{\mathbb{C}G}^{2i}$. The Lefschetz number $L(\tilde{f}, \tilde{g})$ is

$$L(\tilde{f}, \tilde{g}) = (\mu_1 + \mu) \sum_{i=0}^{k(n-k)} d_{2i} \lambda^i \lambda_1^{d-i}.$$

Using the Lemma 4.5 and the fact that $\lambda_1 \neq 0$, the sum

$$\sum_{i=0}^{k(n-k)} d_{2i} \lambda^i \lambda_1^{d-i} = \lambda_1^d \sum_{i=0}^{k(n-k)} d_{2i} (\lambda/\lambda_1)^i \neq 0,$$

Since $\tilde{f} \circ \theta = \theta \circ \tilde{f}$, it follows that

$$(\theta \circ \tilde{f})^*(c_i) = (-1)^i \tilde{f}^*(c_i), \forall i \in I, \quad (\theta \circ \tilde{f})^*(u) = \begin{cases} -\tilde{f}^*(u), & \text{if } m \text{ is even,} \\ \tilde{f}^*(u), & \text{if } m \text{ is odd.} \end{cases}$$

If m is even, then

$$\begin{aligned} D\tilde{g}^* D^{-1} (\theta \circ \tilde{f})_*(x) &= \mu_1 (-\lambda)^i \lambda_1^{d-i} x + \vartheta \otimes y'' \text{ for some } y'' \in H_{2i-m}^{\mathbb{C}G} \\ D\tilde{g}^* D^{-1} (\theta \circ \tilde{f})_*(\vartheta \otimes x) &= -\mu (-\lambda)^i \lambda_1^{d-i} \vartheta \otimes x + z'' \text{ for some } z'' \in H_{2i+m}^{\mathbb{C}G}. \end{aligned}$$

Thus, the Lefschetz number is

$$L(\theta \circ \tilde{f}, \tilde{g}) = (\mu_1 - \mu) \sum_{i=0}^{k(n-k)} d_{2i} (-\lambda)^i \lambda_1^{d-i}.$$

Also, using $\mu_1 \neq 0$ and Lemma 4.5 it follows that that either $L(\tilde{f}, \tilde{g})$ or $L(\theta \circ \tilde{f}, \tilde{g})$ is nonzero.

If m is odd, $L(\theta \circ \tilde{f}, \tilde{g}) = (\mu_1 + \mu) \sum_{i=0}^{k(n-k)} d_{2i} (-\lambda)^i \lambda_1^{d-i}$. Using Lemma 4.5 and $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ that is $\mu_1 \neq -\mu$, we have both $L(\tilde{f}, \tilde{g})$ and $L(\theta \circ \tilde{f}, \tilde{g})$ are nonzero. This ensures that there exist a point of conincidence between f and g . \square

Theorem 4.10. *Let $P(m, n, k)$ be a generalized Dold manifold with $k(n - k)$ even and assume that the hypothesis (3) is satisfied. Let g and f are two continuous maps on $P(m, n, k)$ and \tilde{g}, \tilde{f} be their lifts as defined in Remark 3.7 such that*

- (1) g^* is an automorphism of $H^*(P(m, n, k); \mathbb{Q})$.

- (2) $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ if $\tilde{f}^*(H_{\mathbb{C}G}^*) \not\subseteq H_{\mathbb{C}G}^*$ and m is even.
(3) $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ if m is odd.

s denotes a section of the X -bundle projection p defined in (7) and (6). Then, there is a point of coincidence of f and g .

Proof. If $\tilde{f}^*(c_1) \neq au$, $a \in \mathbb{Q}$ then we have the result by Theorem 4.9.

Let us consider the other case when $\tilde{f}^*(c_1) = au$, $a \in \mathbb{Q}$, using Theorem 3.3 we have $\tilde{f}^*(c_i) = uP_i$, for some $P_i \in H_{\mathbb{C}G}^{2i-m}$.

If $P_i \neq 0$ for some i in I then $\tilde{f}^*(H_{\mathbb{C}G}^*) \not\subseteq H_{\mathbb{C}G}^*$. Since \tilde{f}^* is graded and by (2) we have $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$. Using Lemma 4.8, \tilde{f}_* is of the following form,

$$(30) \quad \begin{aligned} \tilde{f}_*(x) &= \vartheta \otimes y, \text{ for some } y \in H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}, i > 0 \\ \tilde{f}_*(\vartheta \otimes x) &= \mu(\vartheta \otimes x) + z, \text{ for some } z \in H_{2i+m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G} \end{aligned}$$

where $\mu = 0$ if $i > 0$. By Corollary 3.11, we have \tilde{g}^* is an automorphism on $H_{\mathbb{X}}^*$ given by $\tilde{g}^*(c_i) = \lambda_1^i c_i$, and $\tilde{g}^*(u) = \mu_1 u$ for some $\lambda_1, \mu_1 \in \mathbb{Q} \setminus \{0\}$. Using Theorem 4.4 and the similar calculations as done in the proof of Theorem 4.9, we get

$$L(\tilde{f}, \tilde{g}) = (\mu_1 + \mu)d_0\lambda_1^d, \quad L(\theta \circ \tilde{f}, \tilde{g}) = \begin{cases} (\mu_1 - \mu)d_0\lambda_1^d, & \text{if } m \text{ is even,} \\ (\mu_1 + \mu)d_0\lambda_1^d, & \text{if } m \text{ is odd.} \end{cases}$$

Using $\lambda_1 \neq 0$ and $\mu_1 \neq 0$, either $L(\tilde{f}, \tilde{g})$ or $L(\theta \circ \tilde{f}, \tilde{g})$ is non zero if m is even. Using $\deg(p \circ g \circ s) \neq -\deg(p \circ f \circ s)$ i.e. $\mu_1 \neq -\mu$ we have $L(\tilde{f}, \tilde{g}) = L(\theta \circ \tilde{f}, \tilde{g}) \neq 0$. Hence, we get the result.

Let us consider the case when $P_i = 0$, $\forall i \in I$, if $\tilde{f}^*(u) = \mu u$, $\mu \in \mathbb{Q}$ then the proof remains exactly the same as given above. We need to focus on the case when $\tilde{f}^*(u) \in H_{\mathbb{C}G}^*$. Using Lemma 4.8 and (27), we have

$$\tilde{f}_*(x) = \vartheta \otimes y, \text{ for some } y \in H_{2i-m}^{\mathbb{C}G}, \forall x \in H_{2i}^{\mathbb{C}G}, i > 0, \quad \tilde{f}_*(\vartheta \otimes x) \in H_*^{\mathbb{C}G}, \forall x \in H_*^{\mathbb{C}G}.$$

This is exactly the same if we take $\mu = 0$ in (30). The rest of the calculations also remains the same and we get the result. \square

Remark 4.11. There are many situations when the map f satisfies the required hypothesis (2) considered in Theorem 4.9 or Theorem 4.10. Some of them are as follows:

- (1) The lift \tilde{f} stabilizes a copy of Grassmannian, i.e., $\tilde{f}(\{x_0\} \times \mathbb{C}G_{n,k}) \subseteq \{x_0\} \times \mathbb{C}G_{n,k}$ for some $x_0 \in \mathbb{S}^m$.
- (2) The map $p_1 \circ \tilde{f}^* \circ i_1 : H_{\mathbb{C}G}^* \rightarrow H_{\mathbb{C}G}^*$ is an automorphism, equivalently, $f^*(c_1^2) = \lambda^2 c_1^2$, $\lambda \in \mathbb{Q} \setminus \{0\}$, where p_1 and i_1 are defined in (11).
- (3) The map $p_2 \circ \tilde{f} \circ i_1 : \mathbb{S}^m \rightarrow \mathbb{C}G_{n,k}$ is rationally null homotopic, where p_2 is the projection onto the second summand and i_1 is the inclusion into the first summand.

Under the assumption $m > 2k$, any continuous map f on the generalized Dold space $P(m, n, k)$, the lift \tilde{f} (from Remark 3.7) satisfies $\tilde{f}^*(c_i) = \lambda^i c_i$ for all $i \in I$. Hence condition (2) of Theorem 4.10 may be omitted, and one obtains the following consequence.

Corollary 4.12. *Let $P(m, n, k)$ be a generalized Dold space with m and $k(n - k)$ both even. Assume $m > 2k$, and the hypothesis (3) is satisfied. Then, for any continuous function g on $P(m, n, k)$ that induces an automorphism on $H^*(P(m, n, k); \mathbb{Q})$, the pair $(P(m, n, k), g)$ has the coincidence property.*

In particular, for $g = \text{id}$, the space $P(m, n, k)$ has the fixed-point property.

In Theorem 4.10, the first assumption that g^* is an automorphism of $H^*(P(m, n, k); \mathbb{Q})$ can be relaxed by assuming μ is nonzero, which leads to the following proposition.

Proposition 4.13. *Let $P(m, n, k)$ be a generalized Dold manifold with $k(n - k)$ even and assume that the hypothesis (3) is satisfied. Let g and f are two continuous maps on $P(m, n, k)$ and \tilde{g}, \tilde{f} be their lifts as defined in Remark 3.7 such that*

- (1) $\tilde{g}^*(H_{\mathbb{C}G}^*) = H_{\mathbb{C}G}^*$.
- (2) $\tilde{f}^*(u) = \mu u, \mu \in \mathbb{Q} \setminus \{0\}$

Then, there is a point of coincidence of f and g .

The proof of the above proposition is similar to the proof of Theorem 4.10. Therefore, we omit the details.

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