1) Let $\cos\big(x(t)\big) - x^3(t) + u(t) = -10x(t)$, we get $u(t) = -10x(t) - \cos\big(x(t)\big) + x^3(t)$. In discrete settings, $u(k) = -10x(k) - \cos\big(x(k)\big) + x^3(k)$. This is the controller for the non-linear system.

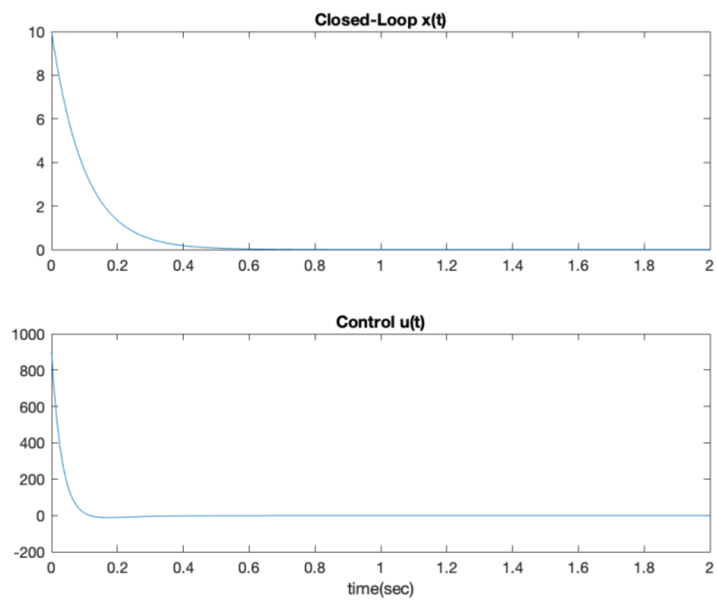
With the finite difference approximation, the first order derivative is approximated by

$$\frac{x(k+1) - x(k)}{\Delta t} = cosx(k) - x^3(k) + u(k)$$

$$x(k+1) = \Delta t\big(cosx(k) - x^3(k) + u(k)\big) + x(k)$$

$$x(k+1) = \Delta t\big(-10x(k)\big) + x(k) = (1 - 10\Delta t)x(k)$$

The closed-loop $x(t)$ and the controller $u(k)$ are plotted as below:



2) $\begin{bmatrix}\dot{x_1}(t)\\ \dot{x_2}(t)\end{bmatrix} = \begin{bmatrix}sinx_2(t)\\ -x_1^2(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u(t)$, where $f(x) = \begin{bmatrix}sinx_2(t)\\ -x_1^2(t)\end{bmatrix}$, $g(x) = \begin{bmatrix}0\\1\end{bmatrix}$. To design the controller of $u(x(t))$, the four equations must hold:

$$\frac{\partial z_1}{\partial x_1}sinx_2 - \frac{\partial z_1}{\partial x_2}x_1^2 = z_2 \;\; (1)$$

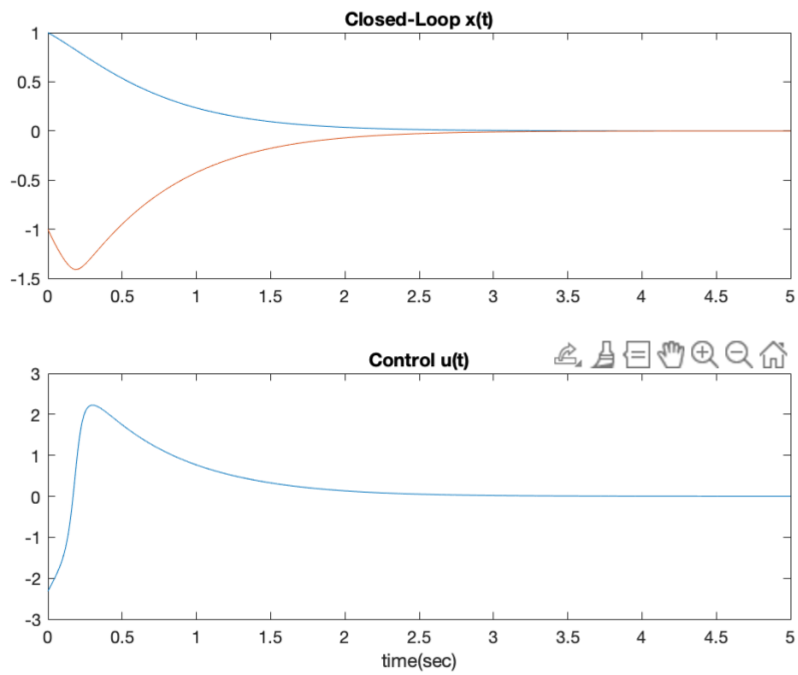
$$\frac{\partial z_2}{\partial x_1}sinx_2 - \frac{\partial z_2}{\partial x_2}x_1^2 = -\frac{\alpha}{\beta} \;\; (2)$$

$$\frac{\partial z_1}{\partial x_1}0 - \frac{\partial z_1}{\partial x_2}1 = 0 \;\; (3)$$

$$\frac{\partial z_2}{\partial x_1}0 - \frac{\partial z_2}{\partial x_2}1 = \frac{1}{\beta} \;\; (4)$$

Where $u(t) = \alpha(x) + \beta(x)v(t)$, and $v(t)$is determined by the linear system $\dot{z}(t) = Az(t) + Bv(t)$. Given $x_1(t)$, it's easy to find $z_1(t)$ and $z_2(t)$. Given $z(t)$, with (3) and (1), and the fact that $z_1(t) = x_1(t)$, we can find the unique $x(t)$. Solving $(1) - (4)$, we get $z_1 = x_1, z_2 = sinx_2, \alpha = x_1^2, \beta = \frac{1}{cosx_2}$. So $u(t) = x_1^2 + \frac{1}{cosx_2}(f_1 z_1 + f_2 z_2) = x_1^2 + \frac{1}{cosx_2}(f_1 x_1 + f_2 sinx_2)$, where F = $[f_1\ f_2]$ is obtained by placing the eigenvalues of $A + BF$ to the negative half of the complex plane. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The finite difference approximation yields $x(k+1) = \Delta t(f(x) + g(x)u(k)) + x(k)$.

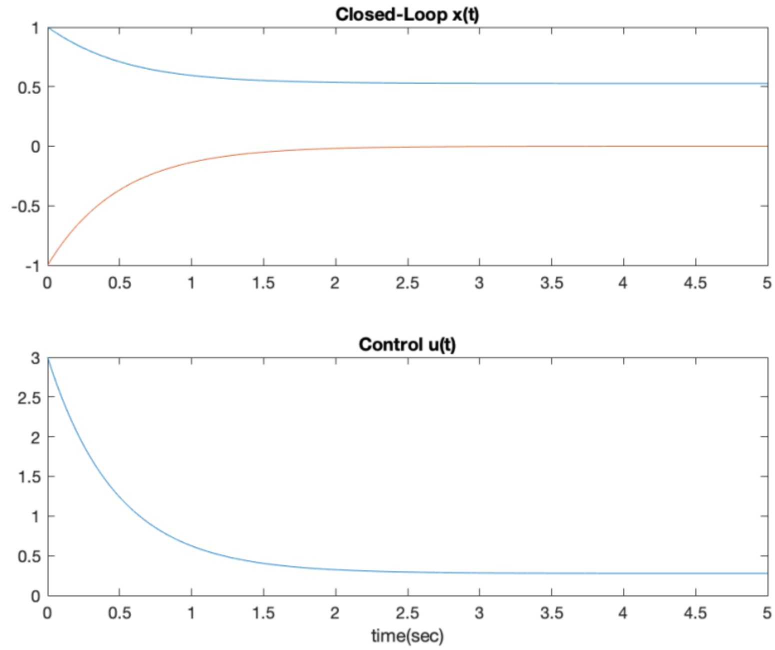
The closed-loop $x(t)$ and the controller $u(k)$ are plotted as below:



3) $\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} sinx_2(t) \\ -x_1^2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$, where $f(x) = \begin{bmatrix} sinx_2(t) \\ -x_1^2(t) \end{bmatrix}$, $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Given $y(t) = x_2(t)$, $\dot{y(t)} = \dot{x_2}(t) = -x_1^2(t) + u(t)$. Let $-x_1^2(t) + u(t) = v(t)$, so $u(t) = v(t) + x_1^2(t)$. Suppose that $y(t)$ is driven by $\dot{y(t)} + 2y(t) = 0$, let $v(t) = -2y(t) = -2x_2(t)$. So $u(t) = x_1^2(t) - 2x_2(t)$.

The closed-loop $x(t)$ and the controller $u(k)$ are plotted as below:



4) $\begin{bmatrix}\dot{x_1}(t)\\ \dot{x_2}(t)\end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) + 2(1 - x_1^2(t))x_2(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} u(t)$, where $f(x) = \begin{bmatrix} x_2(t) \\ -x_1(t) + 2(1 - x_1^2(t))x_2(t)\end{bmatrix}$, $g(x) = \begin{bmatrix}0\\1\end{bmatrix}$. Since $y(t) = x_1(t)$, $y(\dot{t}) = \dot{x_1}(t) = x_2(t)$, $y(\ddot{t}) = \dot{x_2}(t) = -x_1(t) + 2\left(1 - x_1^2(t)\right)x_2(t) + u(t)$. Let $y(\ddot{t}) = v(t)$, $u(t) = x_1(t) - 2\left(1 - x_1^2(t)\right)x_2(t) + v(t)$.

For the linear system $\begin{bmatrix} y(\dot{t}) \\ \\ y(\ddot{t})\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix} y(t) \\ \\ y(\dot{t})\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} v(t)$. F = [$f_1$ f_2] is obtained by placing the eigenvalues of $A + BF$ to the negative half of the complex plane. $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $B = \begin{bmatrix}0\\1\end{bmatrix}$. $u(t) = x_1(t) - 2\left(1 - x_1^2(t)\right)x_2(t) + \left(f_1 y(t) + f_2\, y(\dot{t})\right) = x_1(t) - 2\left(1 - x_1^2(t)\right)x_2(t) + (f_1 x_1(t) + f_2\, x_2(t))$.

The closed-loop $x(t)$ and the controller $u(k)$ are plotted as below: