1) Let w be the displacement, he Equation of Motion is

$$m\ddot{w} + kw = u$$

$$\ddot{w} + 100w = u$$

The equation above can be converted into the State-Space form of

$$x' = \begin{bmatrix} {x_1}' \\ {x_2}' \end{bmatrix} = \begin{bmatrix} \dot{w} \\ \ddot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} w \\ \dot{w} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \cdot u$$

because $\ddot{w} = \frac{u}{m} - \frac{k}{m} \cdot w$

$$y = \ddot{w} = \begin{bmatrix} -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} w \\ \dot{w} \end{bmatrix} + \frac{1}{m} \cdot u$$

so $A_c = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} -100 & 0 \end{bmatrix}, D = 1$

2) With the sampling interval $\Delta t = 0.01\ sec$, applying the analytical power series expressions we obtain the discrete-time matrices A and B:

$$A = e^{A_c \Delta t} = I + A_c \Delta t + \frac{1}{2!}(A_c \Delta t)^2 + \frac{1}{3!}(A_c \Delta t)^3 + \frac{1}{4!}(A_c \Delta t)^4 + \cdots$$

$$B = (I\Delta t + \frac{1}{2!} A_c \Delta t^2 + \frac{1}{3!} A_c^2 \Delta t^3 + \frac{1}{4!} A_c^3 \Delta t^4 + \frac{1}{5!} A_c^4 \Delta t^5 + \cdots) \cdot B_c$$

preserving the first 5 terms, by MATLAB, the solutions are:

$A = \begin{bmatrix} 0.9950 & 0.01 \\ -0.9983 & 0.9950 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, C = \begin{bmatrix} -100 & 0 \end{bmatrix}, D = 1$

The $c2d$ function generates the same matrices.

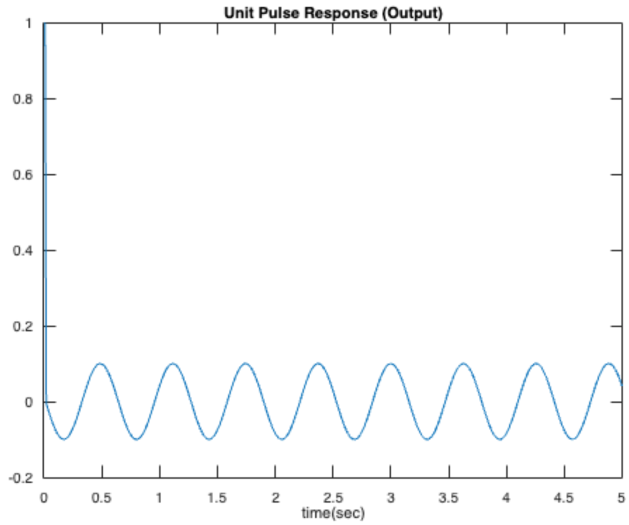
```
a = 2x2
      0.9950    0.0100
     -0.9983    0.9950

b = 2x1
      0.0000
      0.0100

c = 1x2
   -100      0

d = 1
```

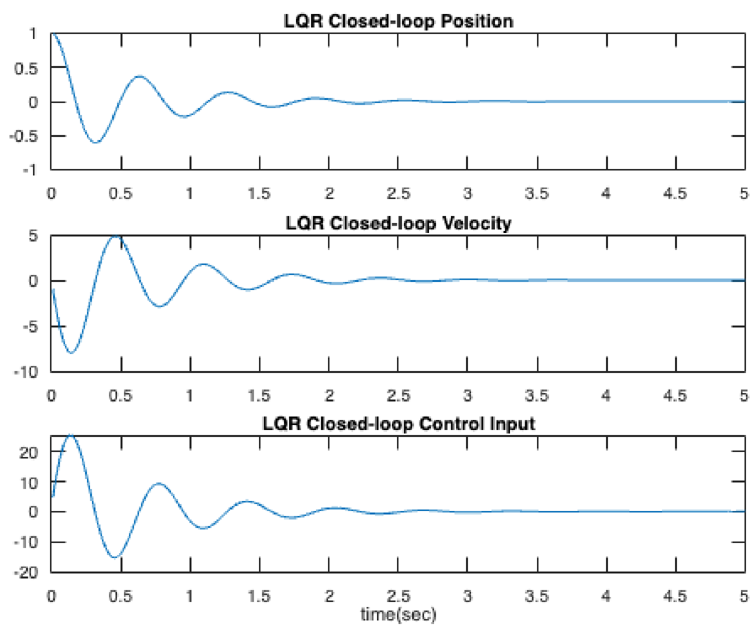
Outputs of the first 5 seconds under the unit pulse are plotted as below



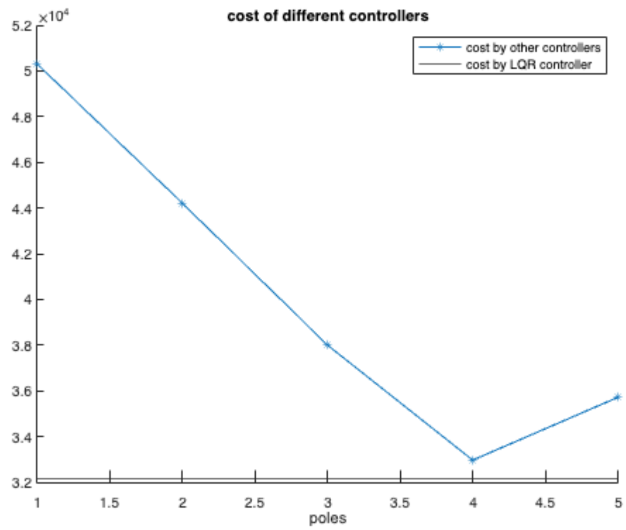
3) The closed-loop state-space model is

$$x(k+1) = (A + BF)x(k)$$
$$y(k) = (C + DF)x(k)$$

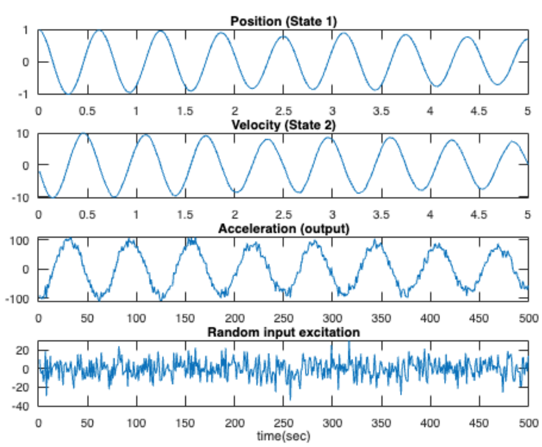
where $u(k) = Fx(k)$ and F is computed such that the cost J is minimized. The states and controlled inputs are plotted as below:

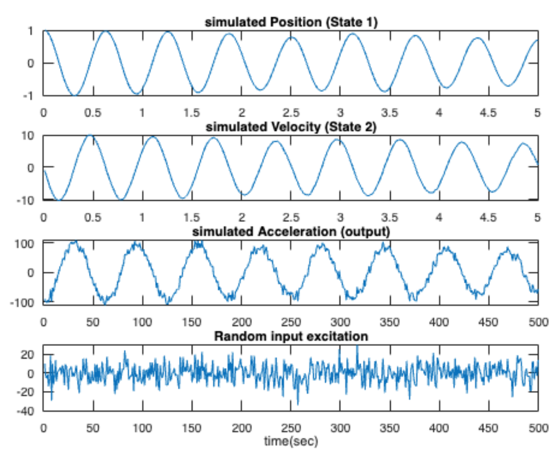


By comparing the cost associated with the LQR optimal controller and other controllers by placing poles inside the unit complex circle, we found the LQR indeed returns the lowest cost.



4) With the same state-space model and random generated inputs, manual coding and simulated methods generate the sample plots.

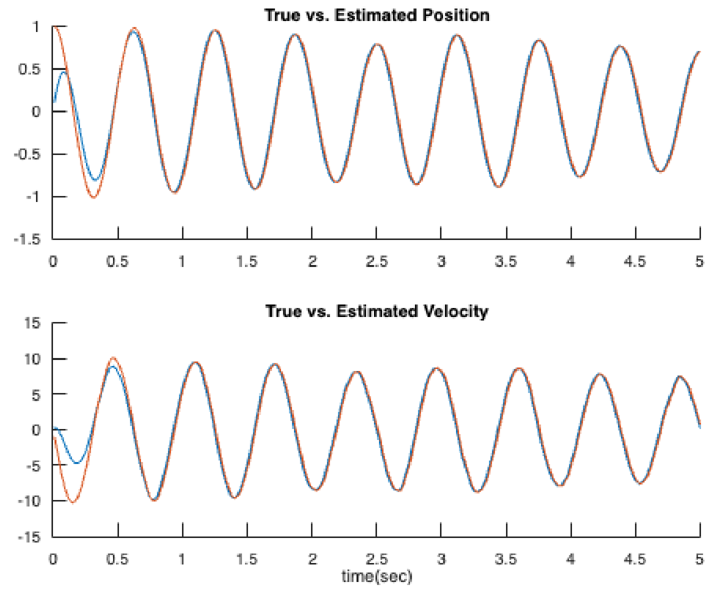




5) The one-equation version of the Luenberger observer with gain M is defined as:

$$\hat{x}(k+1) = (A + MC)\hat{x}(k) + (B + MD)u(k) - My(k)$$

the true versus estimated states comparison is plotted as below and the observe correctly estimates the system states.



6) The model for observed-based state-feedback control system is

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} A & BF \\ -MC & A + MC + BF \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

The results for the first 5 seconds are