Lecture 25: Laplace's Equation

Mewtwo's Goals for the Day

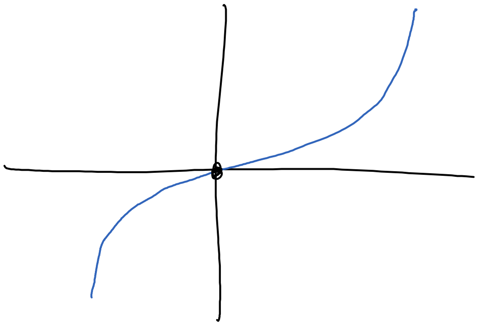
- Discuss the hyperbolic trig functions
- Define Laplace's Equation and the Dirichlet Problem
- Derive Dirchlet's Solution to Laplace's Equation

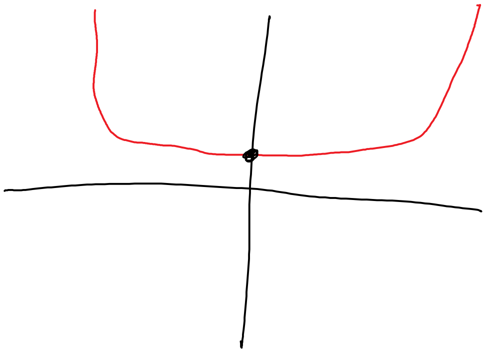
13.5 Laplace's Equation

Hyperbolic Trig Functions

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$





$$\sinh 0 = 0 \qquad \cosh 0 = 1$$

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$e^{x} = \cosh x + \sinh x = \frac{e^{x} + \cancel{e^{-x}}}{2} + \frac{e^{x} - \cancel{e^{-x}}}{2}$$

$$e^{-x} = \cosh x - \sinh x = \frac{\cancel{e^{x}} + e^{-x}}{2} - \frac{\cancel{e^{x}} - e^{-x}}{2}$$

Hyperbolic trig functions are useful in solving 2nd-order ODEs.

Ex Solve $y'' - k^2 y = 0$ (k constant)

$$r^2 - k^2 = 0$$

$$r^2 = k^2$$

$$r = \pm k$$

$$\boxed{y = C_1 e^{kt} + C_2 e^{-kt}}$$

$$y = C_1 \left[\cosh(kt) + \sinh(kt)\right] + C_2 \left[\cosh(kt) - \sinh(kt)\right]$$

$$= \underbrace{(C_1 + C_2)}_{B_1} \cosh(kt) + \underbrace{(C_1 - C_2)}_{B_2} \sinh(kt)$$

$$= \boxed{B_1 \cosh(kt) + B_2 \sinh(kt)}$$

Def A function $u(x,y)$ is harmonic if it satisfies Laplace's Equation

$$u_{xx} + u_{yy} = 0.$$

Notation

$$u_{xx}+u_{yy} = \nabla \cdot \nabla u = \nabla^2 u = \underset{\substack{\uparrow \\ \text{The Laplacian}}}{\Delta} u$$

Laplace's Equation: $\nabla^2 u = 0$

Applications

① Steady-state of 2D Heat Equation

1D: $u_t = k u_{xx}$

2D: $u_t = k(u_{xx} + u_{yy})$

Steady-state is reached when $u_t = 0$

$$\Rightarrow \nabla^2 u = 0$$

② Electromagnetic potentials are harmonic.

Gauss' Law

$$\underset{\substack{\uparrow \\ \text{electric field}}}{\nabla \cdot E} = \frac{\rho}{\varepsilon_0} \quad \begin{array}{l} \leftarrow \text{charge density} \\ \nwarrow \text{vacuum permitivitty constant} \end{array}$$

A potential V satisfies $E = -\nabla V$.

$$\nabla \cdot E = \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

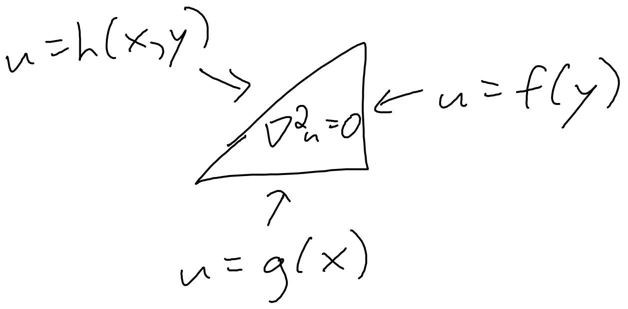
In free space, the charge $\rho = 0$.

$$\nabla^2 V = 0$$

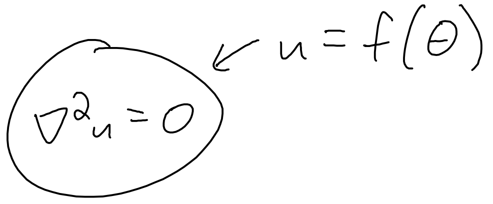
The Dirichlet Problem

Find a solution to Laplace's Equation on a domain D with prescribed boundary values.

Triangle

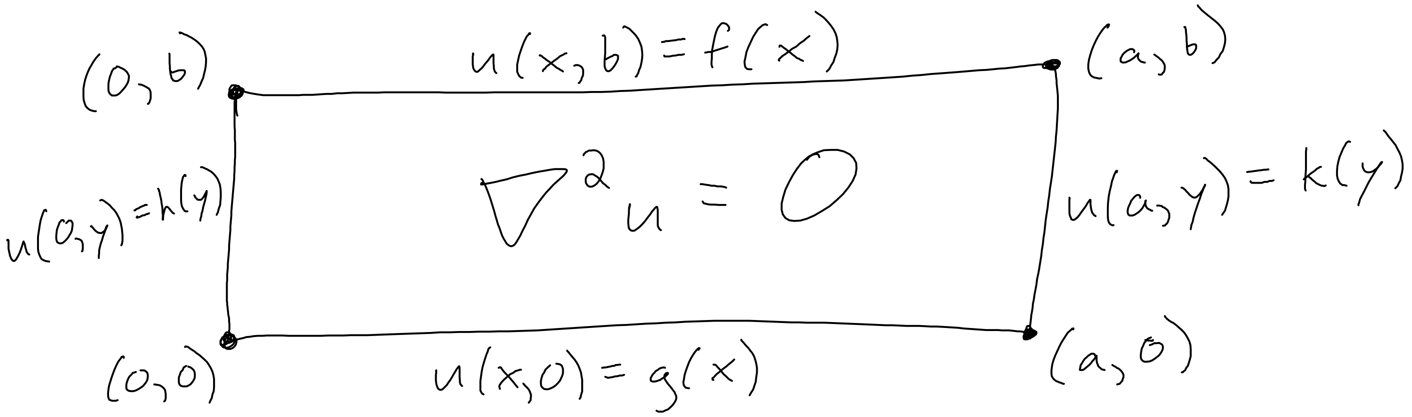


Circle

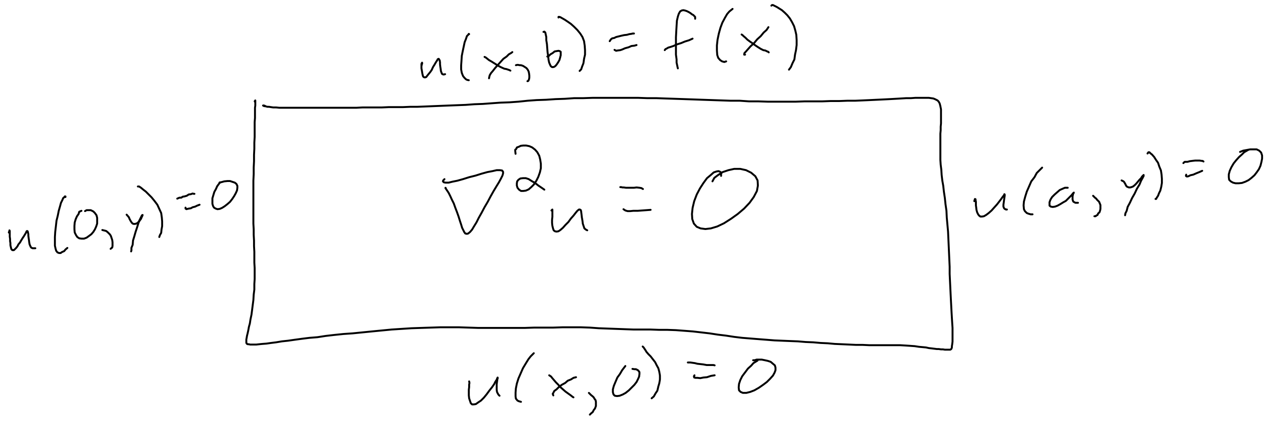


Today: Look at rectangular domain.

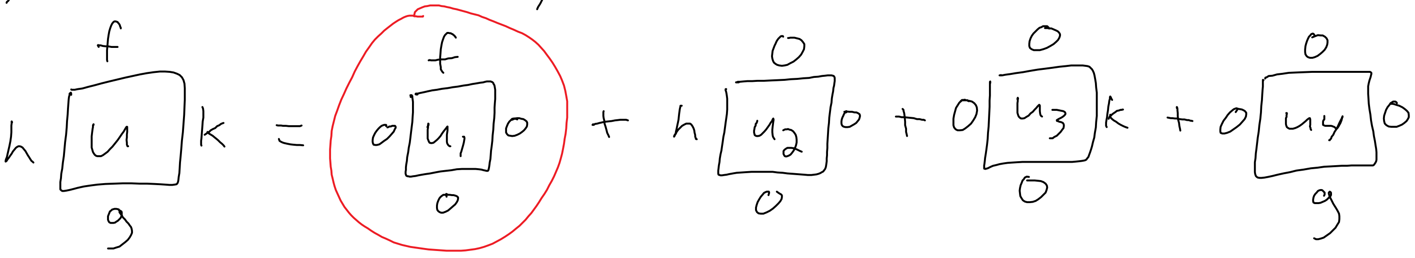
$$0 \le x \le a, \qquad 0 \le y \le b$$



We will solve the simpler problem where 3 sides are zero.



Note To solve the general problem, add the solutions to simpler problems.



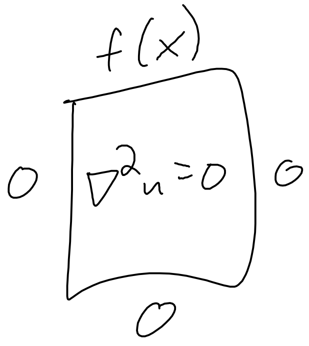
$$u = u_1 + u_2 + u_3 + u_4$$

Ex Solve

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ \text{Left} \quad u(0,y) = 0 \\ \text{Right} \quad u(a,y) = 0 \\ \text{Bottom} \quad u(x,0) = 0 \\ \text{Top} \quad u(x,b) = f(x) \end{cases}$$

Laplace's Equation

Dirichlet BCs



Assume the solution is separable,

$$u(x,y) = v(x)\, w(y)$$

Plug this into the PDE.

$$u_{xx}+u_{yy}=0$$

$$(vw)_{xx}+(vw)_{yy}=0$$

$$v_{xx}w+vw_{yy}=0$$

$$v_{xx}w=-vw_{yy}$$

$$-\frac{v_{xx}}{v}=\frac{w_{yy}}{w}=-\lambda$$

↗ separation constant

BCs

Left: $v(0)=0$

Right: $v(a)=0$

Bottom: $w(0)=0$

Three Cases: λ is zero, positive, and negative.

① $\underline{\lambda=0}$ ⇒ $u=0$ Trivial

② $\underline{\lambda>0}$ ⇒ $u=0$ Trivial

③ $\underline{\lambda<0}$ Assume $\lambda=-\alpha^2 \Rightarrow -\lambda=\alpha^2$

$$-\frac{v_{xx}}{v}=\alpha^2$$

$$-v_{xx}=\alpha^2 v$$

$$0=v_{xx}+\alpha^2 v$$

$$\frac{w_{yy}}{w}=\alpha^2$$

$$w_{yy}=\alpha^2 w$$

$$w_{yy}-\alpha^2 w=0$$

$$r^2-\alpha^2=0$$

$$0 = v_{xx} + \alpha^2 v$$

$$0 = r^2 + \alpha^2$$

$$r^2 = -\alpha^2$$

$$r = \pm \alpha i$$

$$v(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

$\underline{v(0)=0:}$ $\quad 0 = C_1 \cancelto{1}{\cos(0)} + C_2 \cancelto{0}{\sin(0)}$

$$0 = C_1$$

$\underline{v(a)=0:}$ $\quad 0 = C_2 \sin(\alpha a)$

$$\alpha a = n\pi$$

$$\alpha = \frac{n\pi}{a}$$

$$v(x) = C_n \sin\left(\frac{\pi n x}{a}\right)$$

w_{yy} α

$$r^2 - \alpha^2 = 0$$

$$r^2 = \alpha^2$$

$$r = \pm \alpha$$

$$w(y) = C_3 e^{\alpha y} + C_4 e^{-\alpha y}$$

↙ Hyperbolic Trig

$$w(y) = B_1 \cosh(\alpha y) + B_2 \sinh(\alpha y)$$

$\underline{w(0)=0:}$ $\quad 0 = B_1 \cancelto{1}{\cosh(0)} + B_2 \cancelto{0}{\sinh(0)}$

$$B_1 = 0$$

$$w(y) = B_2 \sinh(\alpha y)$$

$$\alpha = \frac{n\pi}{a}$$

Product solution

$$u(x,y) = v(x)\, w(y)$$

$$= \sum_{n=1}^{\infty} D_n \sin\left(\frac{\pi n x}{a}\right) \sinh\left(\frac{\pi n y}{a}\right)$$

Top: $u(x,b) = f(x)$

$$u(x,b) = \sum_{n=1}^{\infty} \underline{D_n} \sin\left(\frac{\pi n x}{a}\right) \underline{\sinh\left(\frac{\pi n b}{a}\right)} = f(x)$$

Fourier Sine Series

$$D_n \sinh\left(\frac{\pi n b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{\pi n x}{a}\right) dx$$

$$D_n = \frac{2}{a \sinh\left(\frac{\pi n b}{a}\right)} \underbrace{\int_0^a f(x) \sin\left(\frac{\pi n x}{a}\right) dx}_{A_n}$$

Dirichlet's Solution

$$u(x,y) = \sum_{n=1}^{\infty} \frac{A_n \sinh\left(\frac{\pi n y}{a}\right) \sin\left(\frac{\pi n x}{a}\right)}{\sinh\left(\frac{\pi n b}{a}\right)}$$

$$A_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n \pi x}{a}\right) dx$$

Phew! That's a lot of math!

Way to end the semester on a high note!

I could ask you on the final exam to derive the formulas for a rectangle with one of the other sides being non-zero (left, right, or bottom).