ELEC 309
Signals and Systems
Homework 8 Solutions

Complex-Domain Analysis of Signals

1. Determine the unilateral Laplace transform of the following continuous-time signals using **only** basic Laplace transforms and Laplace transform properties (**do not use the unilateral Laplace transform integral!**):

(a)

$$x(t) = u(t-1) * e^{-2t}u(t-1)$$

$$\begin{aligned}
X(s) &= \mathcal{L}\left\{x(t)\right\} \\
&= \mathcal{L}\left\{u(t-1)\right\} \cdot \mathcal{L}\left\{e^{-2t}u(t-1)\right\} \quad \text{(Time-Convolution Property)} \\
&= \mathcal{L}\left\{u(t-1)\right\} \cdot \mathcal{L}\left\{e^{-2}e^{-2(t-1)}u(t-1)\right\} \\
&= \mathcal{L}\left\{u(t-1)\right\} \cdot e^{-2}\mathcal{L}\left\{e^{-2(t-1)}u(t-1)\right\} \quad \text{(Linearity Property)} \\
&= e^{-s}\mathcal{L}\left\{u(t)\right\} \cdot e^{-2}e^{-s}\mathcal{L}\left\{e^{-2t}u(t)\right\} \quad \text{(Time-Shifting Property)} \\
&= e^{-2s-2} \cdot \frac{1}{s} \cdot \frac{1}{s+2} \\
&= \boxed{\frac{e^{-2s-2}}{s(s+2)} \text{ with ROC: } \operatorname{Re}\left\{s\right\} > 0}
\end{aligned}$$

MATLAB code to check our answer:

```
syms t x1(t) x2(t) s X(s)
x1(t) = 1;
X1(s) = exp(-s)*laplace(x1(t));
x2(t) = exp(-2*t);
X2(s) = exp(-s-2)*laplace(x2(t));
X(s) = X1(s)*X2(s);
X(s) = simplify(X(s));
pretty(X(s))
```

Command Windows output of MATLAB code:

```
exp(- 2 s - 2)
--------------
   s (s + 2)
```

(b)

$$x(t) = \int_0^t e^{-3\tau} \cos(2\tau)\, d\tau$$

$$\begin{aligned}
X(s) &= \mathcal{L}\left\{x(t)\right\} \\
&= \mathcal{L}\left\{\int_0^t e^{-3\tau} \cos(2\tau)\, d\tau\right\} \\
&= \frac{1}{s}\mathcal{L}\left\{e^{-3t}\cos(2t)\right\} \quad \text{(Time-Integration Property)} \\
&= \frac{1}{s} \cdot \frac{s+3}{(s+3)^2 + 2^2} \\
&= \boxed{\frac{s+3}{s(s^2+6s+13)} \text{ with ROC: } \operatorname{Re}\{s\} > 0}
\end{aligned}$$

MATLAB code to check our answer:

```
syms t x(t) tau s X(s)
x(t) = int(exp(-3*tau)*cos(2*tau),tau,0,t);
X(s) = laplace(x(t));
X(s) = simplify(X(s));
pretty(X(s))
```

Command Windows output of MATLAB code:

```
      s + 3
-----------------
    2
s (s  + 6 s + 13)
```

(c)

$$x(t) = t\frac{d}{dt}\left[e^{-t}\cos(t)u(t)\right]$$

$$\begin{aligned}
X(s) &= \mathcal{L}\left\{x(t)\right\} \\
&= \mathcal{L}\left\{t\frac{d}{dt}\left[e^{-t}\cos(t)u(t)\right]\right\} \\
&= -\frac{d}{ds}\mathcal{L}\left\{\frac{d}{dt}\left[e^{-t}\cos(t)u(t)\right]\right\} \quad (s\text{-Domain-Differentiation Property}) \\
&= -\frac{d}{ds}\left[s\mathcal{L}\left\{e^{-t}\cos(t)u(t)\right\} - \left[e^{-t}\cos(t)u(t)\right]_{t=0}\right] \quad (\text{Time-Differentiation Property}) \\
&= -\frac{d}{ds}\left[\frac{s(s+1)}{(s+1)^2+1^2} - 1\right] \\
&= -\frac{d}{ds}\left[\frac{s^2+s}{s^2+2s+2}\right] \\
&= \boxed{-\frac{s^2+4s+2}{\left(s^2+2s+2\right)^2} \text{ with ROC: } \operatorname{Re}\left\{s\right\} > -1}
\end{aligned}$$

MATLAB code to check our answer:

```
syms t x(t) tau s X(s)
x(t) = t*diff(exp(-t)*cos(t));
X(s) = laplace(x(t));
X(s) = simplify(X(s));
pretty(X(s))
```

Command Windows output of MATLAB code:

```
      2
     s  + 4 s + 2
- ---------------
     2           2
  (s  + 2 s + 2)
```

2. Determine the continuous-time signals for the following unilateral Laplace transforms:

(a)

$$X(s) = \frac{s+3}{s^2+3s+2}$$

$$X(s) = \frac{s+3}{(s+1)(s+2)} = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$
$$\Rightarrow s+3 = C_1(s+2) + C_2(s+1)$$
$$\Rightarrow \text{Let } s=-1\text{: } 2 = C_1 \Rightarrow \boxed{C_1 = 2}$$
$$\Rightarrow \text{Let } s=-2\text{: } 1 = -C_2 \Rightarrow \boxed{C_2 = -1}$$
$$\Rightarrow X(s) = \frac{2}{s+1} - \frac{1}{s+2}$$
$$\Rightarrow x(t) = \boxed{2e^{-t}u(t) - e^{-2t}u(t)}$$

MATLAB code to check our answer:

```
syms s X(s) t x(t)
X(s) = (s+3)/(s^2+3*s+2);
x(t) = ilaplace(X(s));
pretty(x(t))
```

Command Windows output of MATLAB code:

```
2 exp(-t) - exp(-2 t)
```

(b)

$$X(s) = \frac{3s^2 + 10s + 10}{(s+2)(s^2 + 6s + 10)}$$

$$\begin{aligned}
X(s) &= \frac{C_1}{s+2} + \frac{C_2 s + C_3}{(s+3)^2 + 1^2} \\
\Rightarrow 3s^2 + 10s + 10 &= C_1(s^2 + 6s + 10) + (C_2 s + C_3)(s+2) \\
\Rightarrow \text{Let } s = -2:\ \ 2 &= 2C_1 \Rightarrow \boxed{C_1 = 1} \\
\Rightarrow \text{Let } s = 0:\ \ 10 &= 10 + 2C_3 \Rightarrow \boxed{C_3 = 0} \\
\Rightarrow \text{Let } s = 1:\ \ 23 &= 17 + 3C_2 \Rightarrow \boxed{C_2 = 2} \\
\Rightarrow X(s) &= \frac{1}{s+2} + \frac{2s}{(s+3)^2 + 1^2} \\
&= \frac{1}{s+2} + 2\left[\frac{s+3}{(s+3)^2 + 1^2}\right] - 6\left[\frac{1}{(s+3)^2 + 1^2}\right] \\
\Rightarrow x(t) &= \boxed{e^{-2t}u(t) + 2e^{-3t}\cos(t)u(t) - 6e^{-3t}\sin(t)u(t)}
\end{aligned}$$

MATLAB code to check our answer:

```
syms s X(s) t x(t)
X(s) = (3*s^2+10*s+10)/((s+2)*(s^2+6*s+10));
x(t) = ilaplace(X(s));
pretty(x(t))
```

Command Windows output of MATLAB code:

```
exp(-2 t) + exp(-3 t) (cos(t) - 3 sin(t)) 2
```

(c)

$$X(s) = \frac{s^2-3}{(s+2)(s^2+2s+1)}$$

$$X(s) = \frac{s^2-3}{(s+2)(s+1)^2} = \frac{C_1}{s+2} + \frac{C_2}{s+1} + \frac{C_3}{(s+1)^2}$$

$$\Rightarrow s^2 - 3 = C_1(s+1)^2 + C_2(s+2)(s+1) + C_3(s+2)$$

$$\Rightarrow \text{Let } s=-2\text{:}\ \ 1 = C_1 \Rightarrow \boxed{C_1 = 1}$$

$$\Rightarrow \text{Let } s=-1\text{:}\ \ -2 = C_3 \Rightarrow \boxed{C_3 = -2}$$

$$\Rightarrow \text{Let } s=0\text{:}\ \ -3 = 1 + 2C_2 - 4 \Rightarrow \boxed{C_2 = 0}$$

$$\Rightarrow X(s) = \frac{1}{s+2} - \frac{2}{(s+1)^2}$$

$$\Rightarrow x(t) = \boxed{e^{-2t}u(t) - 2te^{-t}u(t)}$$

MATLAB code to check our answer:

```
syms s X(s) t x(t)
X(s) = (s^2-s)/((s+2)*(s^2+2*s+1));
x(t) = ilaplace(X(s));
pretty(x(t))
```

Command Windows output of MATLAB code:

```
exp(-2 t) - 2 t exp(-t)
```

3. Determine the unilateral z transform of the following discrete-time signals using **only** basic z transforms and z transform properties (**do not use the unilateral z transform summation!**):

(a)

$$x[n] = u[-n]$$

$$\begin{aligned} X(z) &= \mathcal{Z}\left\{x[n]\right\} \\ &= \mathcal{Z}\left\{u[n]\right\}\Big|_{\text{Replace } z \text{ with } z^{-1}} \qquad \text{(Time-Reversal Property)} \\ &= \left[\frac{z}{z-1}\right]_{\text{Replace } z \text{ with } z^{-1}} \\ &= \frac{z^{-1}}{z^{-1}-1} \\ &= \boxed{\frac{1}{1-z} \text{ with ROC: } |z| < 1} \end{aligned}$$

MATLAB code to check our answer:

```
syms n x(n) z X(z)
x(n) = 1;
X(z) = ztrans(x(n));
X(z) = simplify(X(1/z));
pretty(X(z))
```

Command Windows output of MATLAB code:

```
    1
- -----
  z - 1
```

(b)

$$x[n] = \left(\frac{1}{2}\right)^n u[n] * 2^n u[-n-1]$$

$$\begin{aligned}
X(z) &= \mathcal{Z}\left\{x[n]\right\} \\
&= \mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\} \cdot \mathcal{Z}\left\{2^n u[-n-1]\right\} \quad \text{(Time-Convolution Property)} \\
&= \left(\frac{z}{z-\frac{1}{2}}\right)\left(-\frac{z}{z-2}\right) \\
&= \boxed{\frac{-z^2}{\left(z-\frac{1}{2}\right)(z-2)} \text{ with ROC: } \frac{1}{2} < |z| < 2}
\end{aligned}$$

MATLAB code to check our answer:

```
syms n x1(n) x2(n) z X1(z) X2(z) X(z)
x1(n) = (1/2)^n;
X1(z) = ztrans(x1(n));
x2(n) = (1/2)^(n+1);
X2(z) = ztrans(x2(n));
X2(z) = (z^-1)*X2(z);           % Time-Shifting
X2(z) = X2(1/z);                % Time-Reversal
X(z) = X1(z)*X2(z);             % Time-Convolution
X(z) = simplify(X(z));
pretty(X(z))
```

Command Windows output of MATLAB code:

```
          2
       2 z
- --------------
     2
  2 z  - 5 z + 2
```

(c)

$$x[n] = n\left(\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n-2]\right)$$

$$\begin{aligned}
X(z) &= \mathcal{Z}\left\{x[n]\right\} \\
&= -z\frac{d}{dz}\left[\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^n u[n-2]\right\}\right] \quad \text{(z-Domain-Differentiation Property)} \\
&= -z\frac{d}{dz}\left[\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\} \cdot \mathcal{Z}\left\{\frac{1}{16}\left(\frac{1}{4}\right)^{n-2} u[n-2]\right\}\right] \quad \text{(Time-Convolution Property)} \\
&= -z\frac{d}{dz}\left[\left(\frac{z}{z-\frac{1}{2}}\right) \cdot \frac{1}{16}z^{-2}\left(\frac{z}{z-\frac{1}{4}}\right)\right] \\
&= -\frac{z}{16}\frac{d}{dz}\left[\left(z^2-\frac{3}{4}z+\frac{1}{8}\right)^{-1}\right] \\
&= -\frac{z}{2}\frac{d}{dz}\left[\left(8z^2-6z+1\right)^{-1}\right] = \frac{z}{2}\left[\frac{16z-6}{\left(8z^2-6z+1\right)^2}\right] \\
&= \boxed{\frac{z\left(8z-3\right)}{\left(8z^2-6z+1\right)^2} \text{ with ROC: } |z| > \frac{1}{2}}
\end{aligned}$$

MATLAB code to check our answer:

```
syms n x1(n) x2(n) z X1(z) X2(z) X(z)
x1(n) = (1/2)^n;
X1(z) = ztrans(x1(n));
x2(n) = (1/16)*(1/4)^n;
X2(z) = (z^-2)*ztrans(x2(n));   % Time-Shifting
X3(z) = X1(z)*X2(z);            % Time-Convolution
x(n) = n*iztrans(X3(z));
X(z) = ztrans(x(n));
X(z) = simplify(X(z),100);
pretty(X(z))
```

Command Windows output of MATLAB code:

```
     z (8 z - 3)
---------------------
         2          2
(2 z - 1)  (4 z - 1)
```