

Math 335
Chapter 12: Fourier Series



The Fourier series expansion of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where the coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx.$$

Dirichlet's Theorem

If $f(x)$ is bounded and has a finite number of discontinuities and extrema on the interval $(-L, L)$, then the Fourier series will converge at every point in $(-L, L)$ to the value

$$\frac{1}{2}(f(x^+) + f(x^-)).$$

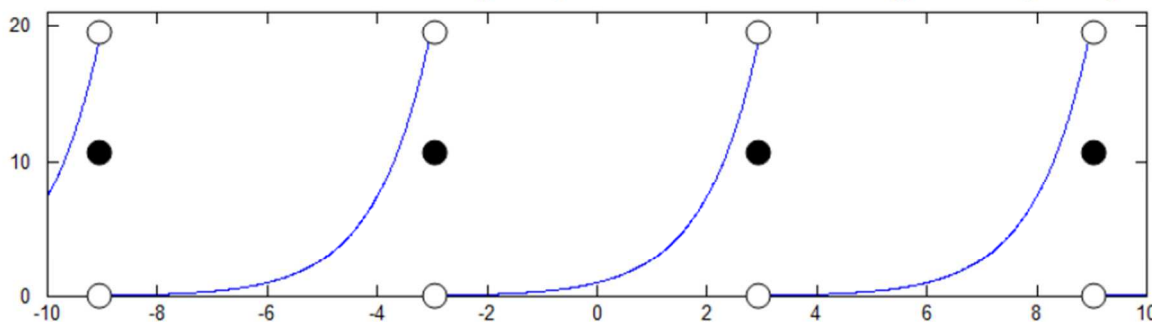
Note this theorem applies even at a discontinuity (jump), where the Fourier series converges to the average of the left and right one-side limits.

Interesting Properties of Fourier Series

- Odd/Even: If the function $f(x)$ is odd, then we should get only sine terms and all $a_n=0$. If $f(x)$ is even, then we get only cosine terms and all $b_n=0$.
- Average Value: The first term $\frac{1}{2}a_0$ is the average value of the function over the interval $[-L, L]$.
- Periodicity: The Fourier series is $2L$ -periodic.
- Approximation: The truncated finite Fourier series is an approximation of the function $f(x)$ on $(-L, L)$. The more terms of the Fourier series we use, the better the approximation will get.
- Continuity: Unlike power series, Fourier series can represent a function $f(x)$ that contains a discontinuity.

The Graph of a Fourier Series

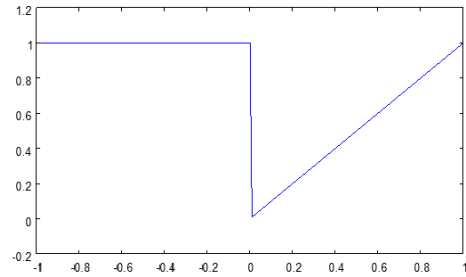
You should have intuition about what the graph of a Fourier series looks like. For example, suppose we find the Fourier series for $f(x) = e^x$ on the interval $(-3, 3)$. The graph will look the graph of e^x on $(-3, 3)$ and then repeated periodically outside that interval. Note that at the discontinuity $x=3$, the Fourier series converges to $1/2(e^3 + e^{-3})$.



Example: The function

$$f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \leq x < 1 \end{cases}$$

is shown at right.



We proved in lecture that this function has the Fourier series representation on $(-1,1)$ as

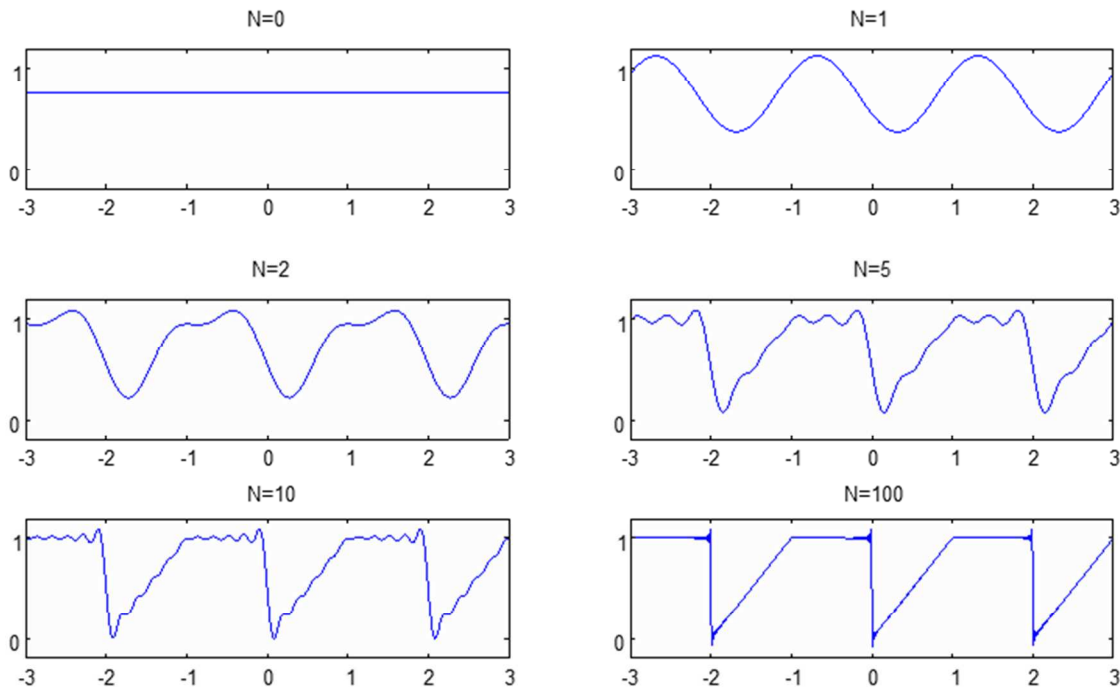
$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2 \pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]$$

In practice, we cannot add up infinitely many terms. So we use a partial sum that adds up the terms until some value $n=N$. This will give us an approximation of $f(x)$.

$$f(x) \approx \frac{3}{4} + \sum_{n=1}^N \left[\frac{(-1)^n - 1}{n^2 \pi^2} \cos n\pi x - \frac{1}{n\pi} \sin n\pi x \right]$$

As we add more terms, the approximation should more closely resemble $f(x)$ on $(-1,1)$.

Outside the interval $(-1,1)$, the Fourier series will simply repeat the graph.



Near a jump discontinuity, the Fourier series starts to oscillate. This behavior is called the Gibbs phenomenon. This phenomenon is responsible for "ringing" artifacts in signals or images that have been compressed.