ELEC 309
Signals and Systems

Complex-Domain Analysis of Continuous-Time Signals using the Laplace Transform

(Chapter 3, *Schaum's Outline of Signals and Systems*)

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The Laplace Transform: Introduction

Consider a continuous-time LTI system with an input $x(t) = e^{st}$, where s is a complex variable. Therefore,

$$x(t) = e^{st} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t),$$

where the output is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st}.$$

where the function $H(s)$ is referred to as the Laplace transform of $h(t)$ and is given by

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt. \tag{1}$$

The Laplace Transform: Bilateral Definition

For a general continuous-time signal $x(t)$, the **bilateral** (or **two-sided**) Laplace transform $X(s)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt. \tag{2}$$

The variable s is complex, in general, and is expressed as

$$s = \sigma + j\omega.$$

The Laplace Transform: Region of Convergence

The range of values of the complex variables s for which the bilateral Laplace transform converges is called the **region of convergence (ROC)**.

The **ROC** is illustrated in the following examples.

The Laplace Transform: ROC Example 1

Determine the bilateral Laplace transform and region of convergence for the signal

$$x(t) = e^{-at}u(t) \text{ for } a \text{ real.}$$

The Laplace Transform: ROC Example 1

The Laplace Transform: ROC Example 2

Determine the bilateral Laplace transform and region of convergence for the signal

$$x(t) = -e^{-at}u(-t) \text{ for } a \text{ real.}$$

The Laplace Transform: ROC Example 2

The Laplace Transform: Poles and Zeros of $X(s)$

The function $X(s)$ will typically be a rational function in s, or

$$X(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n},$$

where m and n are positive integers and the coefficients a_k and b_k are real constants.

$X(s)$ is a called a **proper** rational function if $n > m$ and an **improper** rational function if $n \leq m$.

The Laplace Transform: Poles and Zeros of $X(s)$

A rational function $X(s)$ can also be written as

$$X(s) = \frac{b_0}{a_0}\left[\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}\right].$$

The roots, z_k, of the numerator polynomial are called the **zeros** of $X(s)$ because $X(s) = 0$ for those values of s.

The roots, p_k, of the denominator polynomial are called the **poles** of $X(s)$ because $X(s) \to \infty$ for those values of s.

The Laplace Transform: Poles and Zeros of $X(s)$

The poles of $X(s)$ lie outside the ROC since $X(s)$ does not converge at the poles (by definition). Traditionally, an "$\times$" is used to indicate each pole location in the s-plane.

The zeros may lie inside or outside the ROC. Traditionally, an "$\circ$" is used to indicate each zero location in the s-plane.

Except for the scale factor $\left(\frac{b_0}{a_0}\right)$, $X(s)$ is completely specified by its poles and zeros.

The Laplace Transform: Poles and Zeros of $X(s)$

A compact representation of $X(s)$ in the s-plane is to show the location of poles and zeros in addition to the ROC.

This is illustrated in the following example.

The Laplace Transform: Poles and Zeros - Example

Determine the bilateral Laplace transform $X(s)$ and region of convergence for the signal

$$x(t) = \left(e^{-t} + e^{-3t}\right) u(t).$$

Plot the poles, zeros, and ROC for $X(s)$.

The Laplace Transform: Poles and Zeros - Example

The Laplace Transform: Properties of the ROC

The ROC of $X(s)$ depends on the nature of $x(t)$. If we assume that $X(s)$ is a rational function of s, then there are five interesting properties of the ROC.

Property 1: The ROC does not contain any poles.

Property 2: If $x(t)$ is a **finite-duration** signal ($x(t) = 0$ except for a finite interval $t_1 \leq t \leq t_2$), then the ROC is the entire s-plane (except possibly $s = 0$ or infinite values of s).

Property 3: If $x(t)$ is a **right-sided** signal ($x(t) = 0$ for $t < t_1 < \infty$), then the ROC is of the form

$$\text{Re}(s) > \sigma_{\text{max}},$$

where σ_{max} equals the maximum real part of any of the poles of $X(s)$.

Therefore, the ROC is a half-plane to the right of the vertical line

$$\text{Re}(s) = \sigma_{\text{max}}$$

in the s-plane and thus to the right of all of the poles of $X(s)$.

Property 4: If $x(t)$ is a **left-sided** signal ($x(t) = 0$ for $t > t_2 > -\infty$), then the ROC is of the form

$$\text{Re}(s) < \sigma_{\text{min}},$$

where σ_{min} equals the minimum real part of any of the poles of $X(s)$.

Therefore, the ROC is a half-plane to the left of the vertical line

$$\text{Re}(s) = \sigma_{\text{min}}$$

in the s-plane and thus to the left of all of the poles of $X(s)$.

Property 5: If $x(t)$ is a **two-sided** signal ($x(t)$ is an infinite-duration signal that is neither right-side nor left-sided), then the ROC is of the form

$$\sigma_1 < \text{Re}(s) < \sigma_2$$

where σ_1 and σ_2 are the real parts of the two poles of $X(s)$.

Therefore, the ROC is a vertical strip in the s-plane between the vertical lines

$$\text{Re}(s) = \sigma_1 \text{ and } \text{Re}(s) = \sigma_2.$$

The Laplace Transform: Unilateral Definition

The **unilateral** (or **one-sided**) Laplace transform is formally defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt, \tag{3}$$

where the lower limit of 0^- is used to include signals existing strictly at time 0 (such as $\delta(t)$).

The Laplace Transform: Unilateral Definition

Informally, we simply write Equation 3 as

$$\boxed{X(s) = \int_{0}^{\infty} x(t)e^{-st}dt. \qquad (4)}$$

Note that the bilateral and unilateral Laplace transforms are equivalent if $x(t) = 0$ for $t < 0$, and the unilateral Laplace transform ignores $x(t)$ for $t < 0$.

In other words, the unilateral Laplace transform is sufficient for **right-sided** signals (typical real-world signals).

The Laplace Transform: Unilateral Definition

The unilateral Laplace transform is useful for calculating the complete response of an LTI system to a causal input (typical real-world signal). It can take into account both the zero-state response of the **causal** system due to a **causal** input, as well as the zero-input response due to **nonzero** initial conditions.

> From this point forward, we will ONLY use the **unilateral** Laplace transform. All references henceforth to the Laplace transform refer only to the **unilateral** Laplace transform.

The Laplace Transform: Representation

Equation 4 is sometimes considered an operator that transforms a signal $x(t)$ into a function $X(s)$ symbolically represented by

$$X(s) = \mathcal{L}\left\{x(t)\right\},$$

and the signal $x(t)$ and its Laplace transform $X(s)$ are said to form a **Laplace transform pair** denoted as

$$x(t) \longleftrightarrow X(s).$$

See Table 3-1 (page 105 of *Schaum's Outline of Signals and Systems*) for a listing of Laplace transform pairs.

Laplace Transform Pairs: Unit Impulse Function $\delta(t)$

The Laplace transform of a unit impulse function is given by

$$\mathcal{L}\left\{\delta(t)\right\} = \int_0^\infty \delta(t)e^{-st}dt = 1 \text{ for all } s.$$

Therefore, the Laplace transform pair for a unit impulse function is

$$\delta(t) \longleftrightarrow 1 \text{ with ROC} = \text{all } s.$$

Laplace Transform Pairs: Unit Step Function $u(t)$

The Laplace transform of a unit step function is given by

$$\mathcal{L}\left\{u(t)\right\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt =$$
$$= -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \text{ for Re}\left\{s\right\} > 0.$$

Therefore, the Laplace transform pair for a unit step function is

$$u(t) \longleftrightarrow \frac{1}{s} \text{ with ROC} = \text{Re}\left\{s\right\} > 0.$$

Laplace Transform Pairs: Other Common Signals

Table 1: Some Laplace Transform Pairs

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\text{Re}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > -\text{Re}\{a\}$

Laplace Transform Pairs: Other Common Signals

Table 2: Some Laplace Transform Pairs

$x(t)$	$X(s)$	ROC
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > -\text{Re}\{a\}$
$\cos(\omega_0 t)\, u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\, u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > -\text{Re}\{a\}$
$e^{-at}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > -\text{Re}\{a\}$

Laplace Transform Pairs: Example

Determine the Laplace transform of

$$x(t) = e^{-t}\cos(2t)u(t).$$

Properties of the Laplace Transform: Linearity

If

$$x_1(t) \longleftrightarrow X_1(s) \text{ with ROC} = R_1 \text{ and}$$
$$x_2(t) \longleftrightarrow X_2(s) \text{ with ROC} = R_2,$$

then

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longleftrightarrow \alpha_1 X_1(s) + \alpha_2 X_2(s)$$

with ROC$= R' \supset R_1 \cap R_2$.

(The ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually, the ROC will simply be $R' = R_1 \cap R_2$.)

Properties of the Laplace Transform: Time Shifting

If

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R,$$

then

$$x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$$

with ROC$= R' = R$.

(The ROC of the resultant Laplace transform is unaffected by the time-shifting operation.)

Properties of the Laplace Transform: Shifting in the s-Domain

If

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R,$$

then

$$e^{s_0 t}x(t) \longleftrightarrow X(s-s_0)$$

with ROC$= R' = R + \mathrm{Re}\,(s_0)$.

(The ROC of the resultant Laplace transform is shifted by an amount equal to $\mathrm{Re}\,(s_0)$.)

Properties of the Laplace Transform: Time Scaling

If

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R,$$

then

$$x(at) \longleftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

with ROC$= R' = aR$.

(The ROC of the resultant Laplace transform is the original ROC scaled by the constant a.)

Properties of the Laplace Transform: Time Reversal

If

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R,$$

then

$$x(-t) \longleftrightarrow X\left(-s\right)$$

with ROC$= R' = -R$.

(The ROC of the resultant Laplace transform is the original ROC with the reversal of both the real (σ) and imaginary ($j\omega$) axes in the s-plane.)

Properties of the Laplace Transform: Differentiation in the Time Domain

If

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R,$$

then

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s) - x(0)$$

with ROC$= R' \supset R$.

(The ROC of the resultant Laplace transform is the original ROC unless there is a pole-zero cancellation at $s = 0$.)

Properties of the Laplace Transform: Differentiation in the Time Domain

This property can be extended by repeated application. For example,

$$\frac{d^2x(t)}{dt^2} \longleftrightarrow s^2X(s) - sx(0) - x'(0),$$

$$\frac{d^3x(t)}{dt^3} \longleftrightarrow s^3X(s) - s^2x(0) - sx'(0) - x''(0), \text{ or}$$

$$\frac{d^4x(t)}{dt^4} \longleftrightarrow s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0).$$

Properties of the Laplace Transform: Differentiation in the Time Domain

A general formula for determining the Laplace transform of the n^{th} derivative of $x(t)$ is given by

$$\frac{d^nx(t)}{dt^n} \longleftrightarrow s^nX(s) - s^{n-1}x(0) - \cdots - x^{(n-1)}(0)$$

where

$$x^{(r)}(0) = \left.\frac{d^rx(t)}{dt^r}\right|_{t=0}.$$

Properties of the Laplace Transform: Differentiation in the s-Domain

If

$$x(t) \longleftrightarrow X(s) \text{ with ROC} = R,$$

then

$$-tx(t) \longleftrightarrow \frac{dX(s)}{ds}$$

with ROC$= R' = R$.

(The ROC of the resultant Laplace transform is the original ROC.)

Properties of the Laplace Transform: Integration in the Time Domain

If $x(t) \longleftrightarrow X(s)$ with ROC $= R$, then

$$\int_0^t x(\tau)d\tau \longleftrightarrow \frac{1}{s}X(s) \text{ and}$$
$$\int_{-\infty}^t x(\tau)d\tau \longleftrightarrow \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^0 x(\tau)d\tau$$

with ROC$= R' = R \cap \{\text{Re}(s) > 0\}$.

(The ROC of the resultant Laplace transform follows from the possible introduction of an additional pole at $s = 0$ by the multiplication of $1/s$.)

Properties of the Laplace Transform: Convolution

If

$$x_1(t) \longleftrightarrow X_1(s) \text{ with ROC} = R_1 \text{ and}$$
$$x_2(t) \longleftrightarrow X_2(s) \text{ with ROC} = R_2,$$

then

$$x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s)$$

with ROC$= R' \supset R_1 \cap R_2$.

(The ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually, the ROC will simply be $R' = R_1 \cap R_2$.)

Properties of the Laplace Transform: Example 1

Find the Laplace transform of

$$x(t) = u(t-5).$$

Properties of the Laplace Transform: Example 1

Properties of the Laplace Transform: Example 2

Find the Laplace transform of

$$x(t) = e^{-2t}\left[u(t) - u(t-5)\right].$$

Properties of the Laplace Transform: Example 2

Properties of the Laplace Transform: Example 3

Find the Laplace transform of

$$x(t) = \delta(-2t+3).$$

Properties of the Laplace Transform: Example 3

The Inverse Laplace Transform: Definition

The **inverse Laplace transform** is the inversion of the Laplace transform to determine the signal $x(t)$ from its Laplace transform $X(s)$.

It is symbolically denoted as

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}.$$

The Inverse Laplace Transform: Line-Integral Formula

The **inverse Laplace transform** can be written as as the evaluation of a line integral in the complex s-plane of the form

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds.$$

The value of the real constant c is selected such that if the ROC of $X(s)$ is $\sigma_1 < \text{Re}(s) < \sigma_2$, then $\sigma_1 < c < \sigma_2$.

The evaluation of the inverse Laplace transform integral requires an understanding of complex variable theory.

The Inverse Laplace Transform: Use of Tables

A simpler method to determine the **inverse Laplace transform** is to express $X(s)$ as a sum

$$X(s) = X_1(s) + \cdots + X_n(s),$$

where $X_1(s), \ldots, X_n(s)$ are functions with known inverse Laplace transforms $x_1(t), \ldots, x_n(t)$ (given in tables of Laplace transforms).

From the linearity property, it follows that

$$x(t) = x_1(t) + \cdots + x_n(t).$$

The Inverse Laplace Transform: Use of Tables Example 1

Find the inverse Laplace transform of

$$X(s) = \frac{1}{s+1} \text{ with ROC} = \text{Re}(s) > -1.$$

The Inverse Laplace Transform: Use of Tables
Example 1

The Inverse Laplace Transform: Use of Tables
Example 2

Find the inverse Laplace transform of

$$X(s) = \frac{s}{s^2+4} \text{ with ROC} = \mathrm{Re}(s) > 0.$$

The Inverse Laplace Transform: Use of Tables
Example 2

The Inverse Laplace Transform: Use of Tables
Example 3

Find the inverse Laplace transform of

$$X(s) = \frac{s+1}{s^2+2s+5} \text{ with ROC} = \mathrm{Re}(s) > -1.$$

The Inverse Laplace Transform: Use of Tables
Example 3

The Inverse Laplace Transform: Partial-Fraction Expansion

If $X(s)$ is of the form

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)},$$

then $X(s)$ is a rational function and a simple technique involving partial-fraction expansion can be used for the inversion of $X(s)$.

If $X(s)$ is a proper rational function $(m < n)$, then the techniques on the following slides can be used to perform partial-fraction expansion.

The Inverse Laplace Transform: Partial-Fraction Expansion
Simple Pole Case

If all the poles $(p_1, \ldots, p_n)$ of $X(s)$ are distinct (all roots of $D(s)$ are different), then $X(s)$ can be written as

$$X(s) = \frac{c_1}{s-p_1} + \cdots + \frac{c_n}{s-p_n}$$

where the coefficients c_k are given by

$$c_k = (s-p_k)\,X(s)\Big|_{s=p_k}$$

The Inverse Laplace Transform: Partial-Fraction Expansion
Simple Pole Case Example

Find the inverse Laplace transform of

$$X(s) = \frac{5s+13}{s(s^2+4s+13)} \text{ with ROC} = \text{Re}(s) > 0.$$

The Inverse Laplace Transform: Partial-Fraction Expansion
Simple Pole Case Example

The Inverse Laplace Transform: Partial-Fraction Expansion
Multiple Pole Case

If $D(s)$ has multiple factors of the form $(s-p_i)^r$, we say that p_i is a **multiple pole** of $X(s)$ with **multiplicity** r. Then, the partial-fraction expansion of $X(s)$ will consist of terms of the form

$$\frac{\lambda_1}{s-p_i}+\frac{\lambda_2}{(s-p_i)^2}+\cdots+\frac{\lambda_r}{(s-p_i)^r}$$

where the coefficients λ_k are determined from the formula

$$\lambda_{r-k}=\frac{1}{k!}\cdot\frac{d^k}{ds^k}\left[(s-p_i)^r X(s)\right]\Big|_{s=p_i}$$

The Inverse Laplace Transform: Partial-Fraction Expansion
Multiple Pole Case Example

Find the inverse Laplace transform of

$$X(s)=\frac{s^2+2s+5}{(s+3)(s+5)^2} \text{ with ROC} = \mathrm{Re}(s) > -3.$$

The Inverse Laplace Transform: Partial-Fraction Expansion
Multiple Pole Case Example

The Inverse Laplace Transform: Partial-Fraction Expansion
Improper Rational Functions

When $X(s)$ is an improper rational function ($m \geq n$), we can rewrite $X(s)$ using polynomial long division as

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

where the quotient $Q(s)$ is a polynomial in s with degree $m - n$ and the remainder $R(s)$ is a polynomial with degree strictly less than n.

Since $R(s)/D(s)$ is proper, we can determine its inverse Laplace transform using tables and partial-fraction expansion.

The Inverse Laplace Transform: Partial-Fraction Expansion
Improper Rational Functions

Since the quotient $Q(s)$ is a polynomial in s, its inverse Laplace transform can be determined from the Laplace transform pair

$$\frac{d^K \delta(t)}{dt^k} \longleftrightarrow s^k \text{ for } k = 1, 2, 3, \ldots$$

The Inverse Laplace Transform: Partial-Fraction Expansion
Improper Rational Function Example 1

Find the inverse Laplace transform of

$$X(s) = \frac{2s+1}{s+2} \text{ with ROC} = \text{Re}(s) > -2.$$

The Inverse Laplace Transform: Partial-Fraction Expansion
Improper Rational Function Example 1

The Inverse Laplace Transform: Partial-Fraction Expansion Improper Rational Function Example 2

Find the inverse Laplace transform of

$$X(s) = \frac{s^3 + 2s^2 + 6}{s^2 + 3s} \text{ with ROC} = \text{Re}(s) > 0.$$

The Inverse Laplace Transform: Partial-Fraction Expansion Improper Rational Function Example 2