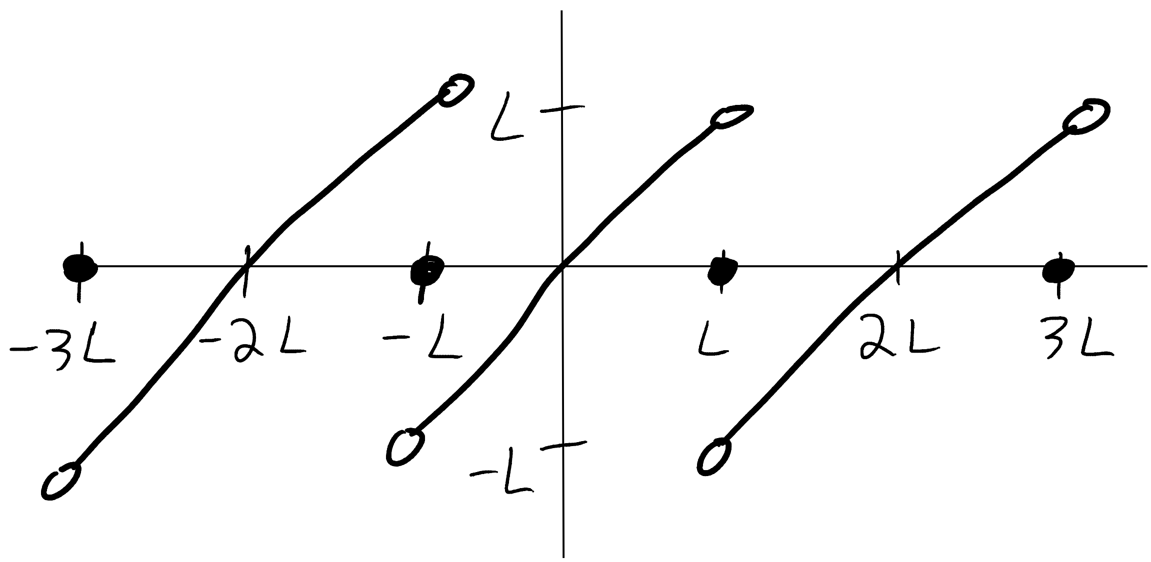
Math 335 HW 11 **NAME:** KEY
Due Wednesday 11/12 5:15pm

Practice Problems *(Do not turn in.)*
Sec 12.3 #11, 15, 19

Print out this page and write all answers directly on this worksheet. Show all work. Your answers must be clear and legible. All pages must be stapled.

1.) *[10 points]* Let $L > 0$ be a fixed constant. Consider the Fourier series for $f(x) = x$ on the general interval $(-L, L)$ centered at the origin.

a.) Sketch the Fourier series for $f(x) = x$ on the interval $(-3L, 3L)$. Label the y-axis. Clearly indicate any discontinuities with open or dark circles.



b.) You should see a jump discontinuity at $x = L$. What value does the Fourier series converge to at $x = L$?

$$\frac{1}{2}\left[f(L^-) + f(L^+)\right] = \frac{1}{2}\left[-L + L\right]$$

$$= \boxed{0}$$

#1 continued...

c.) Find the Fourier series for $f(x) = x$ on the interval $(-L, L)$. Your answer should be in terms of L. *(<u>Hint</u>: Read Paul's Notes.)*

$$f(x) = x \quad \text{odd} \Rightarrow a_n = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\,dx$$

Integration by parts

$$u = x \qquad v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$$

$$du = dx \qquad dv = \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[\frac{-L}{n\pi}x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \int_{-L}^{L}\frac{L}{n\pi}\cos\frac{\pi x}{L}\,dx\right]$$

$$= \frac{1}{L}\left[\frac{-L}{n\pi}x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\Big|_{-L}^{L}\right]$$

$$= \frac{1}{L}\left[\frac{-L^2}{n\pi}\underbrace{\cos n\pi}_{(-1)^n} - \frac{L^2}{n\pi}\underbrace{\cos(-n\pi)}_{(-1)^n} - \frac{L^2}{n^2\pi^2}\underbrace{\sin n\pi}_{0} - \frac{L^2}{n^2\pi^2}\underbrace{\sin(-n\pi)}_{0}\right]$$

$$= -\frac{2L}{n\pi}(-1)^n$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} -\frac{2L}{n\pi}(-1)^n \sin\frac{n\pi x}{L}}$$

$L = \pi$

2.) *[5 points]* Find the Fourier series on $(-\pi, \pi)$ for the top-hat function

$$f(x) = \begin{cases} 0 & x < -2 \\ 1 & -2 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

$f(x)$ even $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-2}^{2} 1\,dx = \frac{4}{\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_{-2}^{2}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin nx \Big|_{-2}^{2}\right]$$

$-\sin(2n)$

$$= \frac{1}{n\pi}\sin 2n - \frac{1}{n\pi}\sin(-2n)$$

$$= \frac{2}{n\pi}\sin 2n$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$= \boxed{\frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(2n)\cos(nx)}$$

3.) *[5 points]* In #1, you should have noticed that since $f(x) = x$ is an odd function, we get the cosine coefficients $a_n = 0$ for the Fourier series on $(-\pi, \pi)$. When the interval is not symmetric about the origin, we may not see the coefficients disappear. The <u>Fourier Cosine Series</u> for $f(x)$ on the "half-range" interval $(0, L)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_0 = \frac{2}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Note that compared to the standard Fourier series formulas for a_n, we simply cut the interval of integration in half and double the coefficients. Use these formulas to find the Fourier Cosine Series for $f(x) = x$ on the half-range interval $(0, \pi)$. $\leftarrow L = \pi$

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^\pi = \frac{2}{\pi}\left[\frac{1}{2}\pi^2\right] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx$$

$u = x \qquad v = \frac{1}{n}\sin(nx)$

$du = dx \qquad dv = \cos(nx)\,dx$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^\pi - \int_0^\pi \frac{1}{n}\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\underbrace{\sin(n\pi)}_{0} + \frac{1}{n^2}\underbrace{\cos(n\pi)}_{(-1)^n} - 0 - \frac{1}{n^2}\underbrace{\cos 0}_{1}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos(nx)}$$