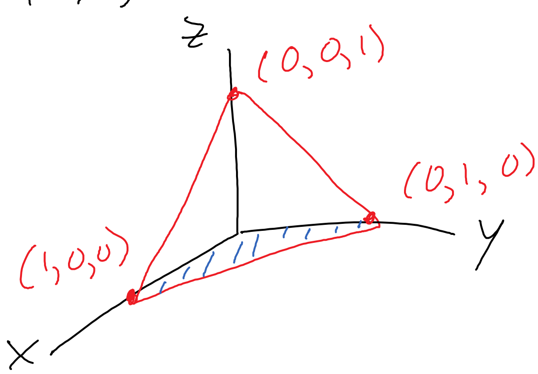
Lecture 10: The Divergence Theorem

Wobbuffett's Goals for the Day

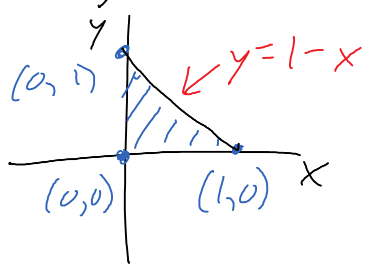
- Practice setting up triple integrals
- Introduce the concept of surface integrals
- State the Divergence Theorem and use it to calculate surface flux

9.15 Triple Integrals

Ex Calculate the volume under the plane $x+y+z=1$ that lies in the first octant.



shadow



$0 \le z \le 1-x-y$

$0 \le y \le 1-x$

$0 \le x \le 1$

$$V = \iiint_D dV = \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} dz \right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[z \Big|_{z=0}^{z=1-x-y} \right] dy\, dx$$

$$= \int_0^1 \left[\int_0^{1-x} 1-x-y \quad dy \right] dx$$

$$= \int_0^1 y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=1-x} \, dx$$

$$= \int_0^1 (1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 1 - x - x + x^2 - \frac{1}{2}(1 - 2x + x^2) \, dx$$

$$= \int_0^1 \frac{1}{2}x^2 - x + \frac{1}{2} \, dx$$

$$= \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x \Big|_{x=0}^{x=1}$$

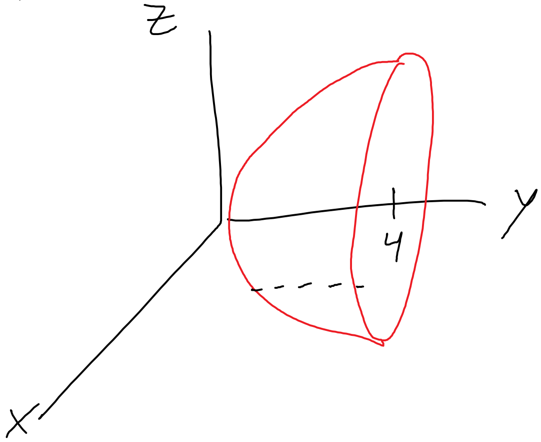
$$= \frac{1}{6} \cancel{- \frac{1}{2}} \cancel{+ \frac{1}{2}}$$

$$= \boxed{\frac{1}{6}}$$

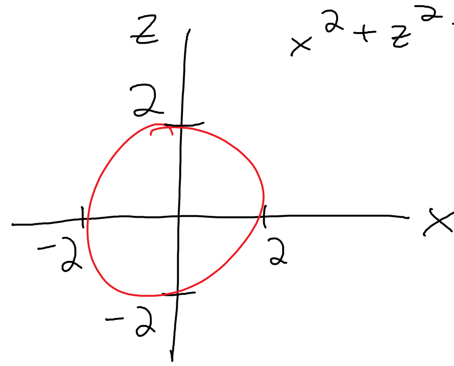
Note We only integrate when we are calculating volumes.

Note We can use polar coordinates on any 2 variables.

Ex Calculate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is the region enclosed by the parabaloid $y = x^2 + z^2$ and the plane $y = 4$.



$x^2 + z^2 = 4$



$x, z \to$ Polar

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

Parabaloid $y = x^2 + z^2$ Plane $y = 4$

$$r^2 \le y \le 4$$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 \underbrace{r}_{\sqrt{x^2+z^2}}\ \underbrace{r}_{\text{Jacobian}}\, dy\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\, dy\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2 y \Big|_{y=r^2}^{y=4}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 4r^2 - r^4\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{4}{3}r^3 - \frac{1}{5}r^5 \Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \frac{32}{3} - \frac{32}{5} \, d\theta$$

$$= \int_0^{2\pi} \frac{64}{15} \, d\theta$$

$$= \frac{64}{15}\theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{64}{15}(2\pi)$$

$$= \boxed{\frac{128\pi}{15}}$$

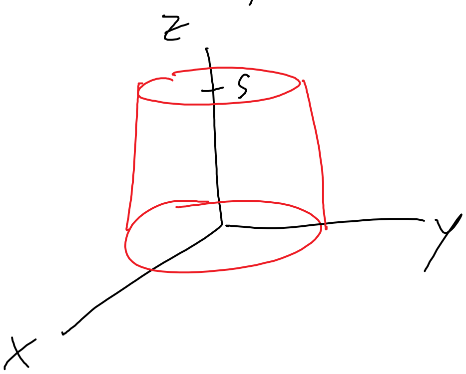
Rectangular: $\displaystyle\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2} \, dy\,dx\,dz$

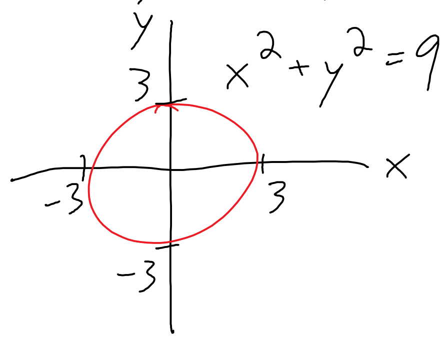
Note: If the limits are constants and the integrand can be written as a product, then you can break up the integrals as a product.

$$\int_a^b \int_c^d f(x)\,g(y)\,dy\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]$$

e.g. $$\int_1^3 \int_2^6 x^2 y\,dy\,dx = \left[\int_1^3 x^2\,dx\right]\left[\int_2^6 y\,dy\right]$$

<u>Ex</u> Calculate $\iiint_F 4xz^2\,dV$ where F is the cylinder $x^2+y^2 \le 9$, $0 \le z \le 5$.





$0 \le z \le 5$

Polar: $x, y \to r, \theta$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\iiint_E 4xz^2\,dV = \int_0^5 \int_0^{2\pi} \int_0^3 \underbrace{4r\cos\theta\, z^2}_{4xz^2}\, \underbrace{r}_{\text{Jacobian}}\,dr\,d\theta\,dz$$

$$= 4\left[\int_0^5 z^2\,dz\right]\left[\int_0^{2\pi} \cos\theta\,d\theta\right]\left[\int_0^3 r^2\,dr\right]$$

$$= 4\left[\tfrac{1}{3}z^3\Big|_0^5\right]\left[\sin\theta\Big|_0^{2\pi}\right]\left[\tfrac{1}{3}r^3\Big|_0^3\right]$$

$$= 4\left[\frac{125}{3}\right]\left[\quad 0 \quad\right]\left[\quad 9 \quad\right]$$

$$= \boxed{0}$$

We integrated $4x^2z$ over a cylinder.

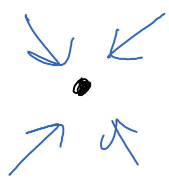
For the front half of the cylinder, $4x^2z > 0$.
For the back half of the cylinder, $4x^2z < 0$.

Since the integrand and volume is symmetric about the x-axis, the positive values end up canceling the negative values and we get a total of zero.

9.16 The Divergence Theorem

Recall Divergence

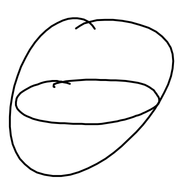
$\nabla \cdot \vec{F}$ measures the rate expansion of $\vec{F}$ at each point.



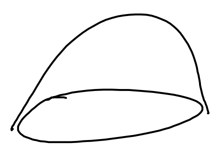
$\nabla \cdot \vec{F} < 0$ compression

$\nabla \cdot \vec{F} > 0$ expansion

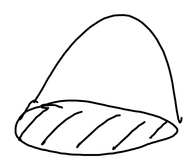
Def A surface is closed if it completely encloses a 3D volume.



Sphere
closed



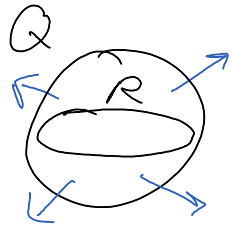
Hemisphere
Not closed



Hemisphere + Base
Closed

Idea $\vec{F}$ = velocity of gas particles

Closed surface Q enclosing a region R



Rate gas flows out through surface Q = Rate gas in region R is expanding

$$\iint_Q \vec{F}\cdot\vec{n}\, dS = \iiint_R \nabla\cdot\vec{F}\, dV$$

Theorem The Divergence Theorem

Let $\vec{F}$ be a 3D differentiable vector field.

Let Q be a closed surface enclosing a 3D region R.

If $\vec{n}$ is the outward pointing normal to Q, then

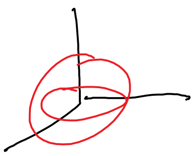
$$\iint_Q \vec{F}\cdot\vec{n}\, dS = \iiint_R \nabla\cdot\vec{F}\, dV$$

<u>Ex</u> Find outward flux of

$$\vec{F} = \langle x, y, 3z \rangle$$

through the sphere

$$x^2 + y^2 + z^2 = 4.$$



Sphere of radius 2

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(3z) = 1 + 1 + 3 = 5$$

$$\text{Flux} = \iint_Q \vec{F} \cdot \vec{n}\, dS = \iiint_R \nabla \cdot \vec{F}\, dV$$

$$= \iiint_{x^2+y^2+z^2 \le 4} 5\, dV$$

$$= 5 \left[\text{Volume of sphere} \right]$$

$$= 5 \left[\frac{4}{3}\pi (2)^3 \right]$$

$$= \boxed{\frac{160\pi}{3}}$$

Fluid flows outward since flux is positive