ELEC 309
Signals and Systems

Complex-Domain Analysis of Discrete-Time Signals using the z Transform

(Chapter 4, *Schaum's Outline of Signals and Systems*)

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The z Transform: Introduction

Consider a discrete-time LTI system with an input $x[n] = z^n$, where z is a complex variable. Therefore,

$$x[n] = z^n \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n],$$

where the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z) z^n.$$

where the function $H(z)$ is referred to as the z **transform** of $h[n]$ and is given by

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}. \tag{1}$$

The z Transform: Bilateral Definition

For a general discrete-time signal $x[n]$, the **bilateral** (or **two-sided**) z transform $X(z)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \tag{2}$$

The variable z is complex, in general, and is typically expressed in polar/exponential form as

$$z = re^{j\Omega}$$

where r is the magnitude of z and Ω is the angle of z.

The z Transform: Region of Convergence

The range of values of the complex variables z for which the z transform converges is called the **region of convergence (ROC)**.

The **ROC** is illustrated in the following examples.

The z Transform: ROC Example 1

Determine the z transform and region of convergence for the signal

$$x[n] = a^n u[n] \text{ for } a \text{ real.}$$

The z Transform: ROC Example 1

The z Transform: ROC Example 2

Determine the z transform and region of convergence for the signal

$$x[n] = -a^n u[-n-1] \text{ for } a \text{ real.}$$

The z Transform: ROC Example 2

The z Transform: Poles and Zeros of $X(z)$

The function $X(z)$ will typically be a rational function in z. That is,

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n},$$

where m and n are positive integers and the coefficients a_k and b_k are real constants.

$X(z)$ is a called a **proper** rational function if $n > m$ and an **improper** rational function if $n \leq m$.

The z Transform: Poles and Zeros of $X(z)$

A rational function $X(z)$ can also be written as

$$X(z) = \frac{b_0}{a_0}\left[\frac{(z - z_1)\cdots(z - z_m)}{(z - p_1)\cdots(z - p_n)}\right].$$

The roots, z_k, of the numerator polynomial are called the **zeros** of $X(z)$ because $X(z) = 0$ for those values of z.

The roots, p_k, of the denominator polynomial are called the **poles** of $X(z)$ because $X(z) \to \infty$ for those values of z.

The z Transform: Poles and Zeros of $X(z)$

The poles of $X(z)$ lie outside the ROC since $X(z)$ does not converge at the poles (by definition). Traditionally, an "$\times$" is used to indicate each pole location in the complex z-plane.

The zeros may lie inside or outside the ROC. Traditionally, an "$\circ$" is used to indicate each zero location in the complex z-plane.

Except for the scale factor $\left(\frac{b_0}{a_0}\right)$, $X(z)$ is completely specified by its poles and zeros.

The z Transform: Poles and Zeros of $X(z)$

A compact representation of $X(z)$ in the complex z-plane is to show the location of poles and zeros in addition to the ROC.

This is illustrated in the following example.

The z Transform: Poles and Zeros - Example

Determine the z transform $X(z)$ and region of convergence for the signal

$$x[n] = \left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] u[n].$$

Plot the poles, zeros, and ROC for $X(z)$.

The z Transform: Poles and Zeros - Example

The z Transform: Properties of the ROC

The ROC of $X(z)$ depends on the nature of $x[n]$. If we assume that $X(z)$ is a rational function of z, then there are five interesting properties of the ROC.

Property 1: The ROC does not contain any poles.

Property 2: If $x[n]$ is a **finite-duration** sequence ($x[n] = 0$ except for a finite interval $N_1 \leq n \leq N_2$) and $X(z)$ converges for some value of z, then the ROC is the entire z-plane (except possibly $z = 0$).

Property 3: If $x[n]$ is a **right-sided** sequence ($x[n] = 0$ for $n < N_1 < \infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| > r_{\text{max}}$$

where r_{max} equals the largest magnitude of any of the poles of $X(z)$.

Therefore, the ROC is the exterior of the circle

$$|z| = r_{\text{max}}$$

in the z-plane.

Property 4: If $x[n]$ is a **left-sided** sequence ($x[n] = 0$ for $n > N_2 > -\infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| < r_{\text{min}} \text{ or } 0 < |z| < r_{\text{min}}$$

where r_{min} equals the smallest magnitude of any of the poles of $X(z)$.

Therefore, the ROC is the interior of the circle

$$|z| = r_{\text{min}}$$

in the z-plane with the possible exception of $z = 0$.

Property 5: If $x[n]$ is a **two-sided** sequence ($x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of z, then the ROC is of the form

$$r_1 < |z| < r_2$$

where r_1 and r_2 are the magnitudes of the two poles of $X(z)$.

Therefore, the ROC is an annular ring in the z-plane between the circles

$$|z| = r_1 \text{ and } |z| = r_2$$

not containing any poles.

The z Transform: Unilateral Definition

The **unilateral** (or **one-sided**) z transform is defined as

$$\boxed{X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}. \qquad (3)}$$

Note that the bilateral and unilateral z transforms are equivalent if $x[n] = 0$ for $n < 0$, and the unilateral z transform ignores $x[n]$ for $n < 0$.

In other words, the unilateral z transform is sufficient for **right-sided** signals (typical real-world signals).

The z Transform: Unilateral Definition

The unilateral z transform is useful for calculating the complete response of an LTI system to a causal input (typical real-world signal). It can take into account both the zero-state response of the **causal** system due to a **causal** input, as well as the zero-input response due to **nonzero** initial conditions.

From this point forward, we will ONLY use the **unilateral** z transform. All references henceforth to the z transform refer only to the **unilateral** z transform.

The z Transform: Representation

Equation 3 is sometimes considered an operator that transforms a signal $x[n]$ into a function $X(z)$ symbolically represented by

$$X(z) = \mathcal{Z}\{x[n]\},$$

and the signal $x[n]$ and its z transform $X(z)$ are said to form a z-transform pair denoted as

$$x[n] \longleftrightarrow X(z).$$

See Table 4-1 (page 153 of *Schaum's Outline of Signals and Systems*) for a listing of z transform pairs.

z-Transform Pairs: Unit Impulse Function $\delta[n]$

The z transform of a unit impulse function is given by

$$\mathcal{Z}\{\delta[n]\} = \sum_{n=0}^{\infty} \delta[n] z^{-n} = z^{-0} = 1 \text{ for all } z.$$

Therefore, the z-transform pair for a unit impulse function is

$$\delta[n] \longleftrightarrow 1 \text{ with ROC = all } z.$$

z-Transform Pairs: Unit Step Function $u[n]$

The z transform of a unit step function is given by

$$\mathcal{Z}\{u[n]\} = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$
$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \text{ for } |z| > 1.$$

Therefore, the z-transform pair for a unit step function is

$$u[n] \longleftrightarrow \frac{z}{z-1} \text{ with ROC} = |z| > 1.$$

z-Transform Pairs: Other Common Signals

Table 1: Some z-Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	All z except 0 if $m > 0$ or ∞ if $m < 0$
$u[n]$	$\frac{1}{1-z^{-1}}$ or $\frac{z}{z-1}$	$\|z\| > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$ or $\frac{z}{z-a}$	$\|z\| > \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{\left(1-az^{-1}\right)^2}$ or $\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$

z-Transform Pairs: Other Common Signals

Table 2: Some z-Transform Pairs

$x[n]$	$X(z)$	ROC
$\cos(\Omega_0 n)\, u[n]$	$\frac{z^2-\cos(\Omega_0)z}{z^2-2\cos(\Omega_0)z+1}$	$\lvert z\rvert > 1$
$\sin(\Omega_0 n)\, u[n]$	$\frac{\sin(\Omega_0)z}{z^2-2\cos(\Omega_0)z+1}$	$\lvert z\rvert > 1$
$r^n\cos(\Omega_0 n)\, u[n]$	$\frac{z^2-r\cos(\Omega_0)z}{z^2-2r\cos(\Omega_0)z+r^2}$	$\lvert z\rvert > r$
$r^n\sin(\Omega_0 n)\, u[n]$	$\frac{r\sin(\Omega_0)z}{z^2-2r\cos(\Omega_0)z+r^2}$	$\lvert z\rvert > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

z Transform Pairs: Example

Determine the z transform of

$$x[n] = \left(\frac{1}{2}\right)^n \cos(2n)\, u[n].$$

z Transform Pairs: Example

Properties of the z Transform: Linearity

If

$$x_1[n] \longleftrightarrow X_1(z) \text{ with ROC} = R_1 \text{ and}$$
$$x_2[n] \longleftrightarrow X_2(z) \text{ with ROC} = R_2,$$

then

$$a_1x_1[n] + a_2x_2[n] \longleftrightarrow a_1X_1(z) + a_2X_2(z)$$

with ROC$= R' \supset R_1 \cap R_2$.

(The ROC of the resultant z transform is at least as large as the region in common between R_1 and R_2. Usually, the ROC will simply be $R' = R_1 \cap R_2$.)

Properties of the z Transform: Time Shifting

If $x[n] \longleftrightarrow X(z)$ with ROC$= R$, then for $m \geq 0$,

$$x[n-m] \longleftrightarrow z^{-m}X(z) + z^{-m+1}x[-1] + z^{-m+2}x[-2] + \cdots + x[-m]$$

$$\longleftrightarrow z^{-m}X(z) + z^{-m}\sum_{i=1}^{m} z^{i}x[-i]$$

and

$$x[n+m] \longleftrightarrow z^{m}X(z) - z^{m}x[0] - z^{m-1}x[1] - \cdots - zx[m-1]$$

$$\longleftrightarrow z^{m}X(z) - z^{m}\sum_{i=0}^{m-1} z^{-i}x[i]$$

with ROC$= R' = R \cap \{0 < |z| < \infty\}$.
(The ROC of the resultant z transform is the original ROC minus $z = 0$ and/or infinite values of z.)

Properties of the z Transform: Time Shifting Special Cases

$x[n-1]u[n] \longleftrightarrow z^{-1}X(z) + x[-1]$ with ROC $= R' = R \cap \{0 < |z|\}$

$x[n+1]u[n] \longleftrightarrow zX(z) - zx[0]$ with ROC $= R' = R \cap \{|z| < \infty\}$

For z transform:

- z^{-1} often called **unit-delay operator**
- z often called **unit-advance operator**

For Laplace transform:

- s^{-1} corresponds to time-domain integration
- s corresponds to time-domain differentiation

Properties of the z Transform: Multiplication by z_0^n

If

$$x[n] \longleftrightarrow X(z) \text{ with ROC} = R,$$

then

$$z_0^n x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right)$$

with ROC$= R' = |z_0|R$.

(The ROC of the resultant z transform expands or contracts by the factor $|z_0|$. A pole at $z = p_k$ or a zero at $z = z_k$ moves to $z = z_0 p_k$ or $z = z_0 z_k$, respectively.)

Properties of the z Transform: Multiplication by z_0^n Special Case

$$e^{j\Omega_0 n}x[n] \longleftrightarrow X\left(e^{-j\Omega_0}z\right)$$

with ROC$= R' = R$.

(All poles and zeros are simply rotated by the angle Ω, and the ROC of the resultant z transform is unchanged.)

Properties of the z Transform: Time Reversal

If

$$x[n] \longleftrightarrow X(z) \text{ with ROC} = R,$$

then

$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$$

with ROC$= R' = \frac{1}{R}$.
(The ROC of the resultant z transform is the inversion of the original ROC, indicating the fact that a right-sided sequence becomes left-sided if time-reversed, and vice versa. A pole at $z = p_k$ or a zero at $z = z_k$ moves to $z = 1/p_k$ or $z = 1/z_k$, respectively, after time reversal.)

Properties of the z Transform: Multiplication by n or Differentiation in the z-Domain

If

$$x[n] \longleftrightarrow X(z) \text{ with ROC} = R,$$

then

$$nx[n] \longleftrightarrow -z\frac{dX(z)}{dz}$$

with ROC$= R' = R$.

(The ROC of the resultant z transform is the original ROC.)

Properties of the z Transform: Accumulation

If $x[n] \longleftrightarrow X(z)$ with ROC $= R$, then

$$\sum_{k=0}^{n} x[k] \longleftrightarrow \frac{1}{1-z^{-1}}X(z) = \frac{z}{z-1}X(z) \text{ and}$$

$$\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1-z^{-1}}X(z) = \frac{z}{z-1}X(z) + \frac{z}{z-1}\sum_{k=-\infty}^{0} x[k]$$

with ROC$= R' \supset R \cap \{|z| > 1\}$.

(Note that $\sum_{k=-\infty}^{n} x[k]$ is the discrete-time counterpart to integration in the discrete-time domain and is called **accumulation**.)

Properties of the z Transform: Convolution

If

$$x_1[n] \longleftrightarrow X_1(z) \text{ with ROC} = R_1 \text{ and}$$
$$x_2[n] \longleftrightarrow X_2(z) \text{ with ROC} = R_2,$$

then

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$$

with ROC$= R' \supset R_1 \cap R_2$.

(The ROC of the resultant z transform is at least as large as the region in common between R_1 and R_2. Usually, the ROC will simply be $R' = R_1 \cap R_2$.)

Properties of the z Transform: Example 1

Find the z transform of

$$x[n] = u\left[n - 5\right].$$

Properties of the z Transform: Example 1

Properties of the z Transform: Example 2

Find the z transform of

$$x[n] = \left(\frac{1}{2}\right)^n \left(u[n] - u[n-5]\right).$$

Properties of the z Transform: Example 2

The Inverse z Transform: Definition

The **inverse z transform** is the inversion of the z transform to determine the signal $x[n]$ from its z transform $X(z)$.

It is symbolically denoted as

$$x[n] = \mathcal{Z}^{-1}\left\{X(z)\right\}.$$

The Inverse z Transform: Contour-Integral Formula

The **inverse z transform** can be written formally as as the evaluation of an integral in the complex z-plane of the form

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz.$$

where C is a counterclockwise contour of integration enclosing the origin.

The evaluation of the inverse z transform integral requires an understanding of complex variable theory.

The Inverse z Transform: Use of Tables

A simpler method to determine the **inverse z transform** is to express $X(z)$ as a sum

$$X(z) = X_1(z) + \cdots + X_m(z),$$

where $X_1(z), \ldots, X_m(z)$ are functions with known inverse z transforms $x_1[n], \ldots, x_m[n]$ (given in tables of z transforms).

From the linearity property, it follows that

$$x[n] = x_1[n] + \cdots + x_m[n].$$

The Inverse z Transform: Use of Tables Example 1

Find the inverse z transform of

$$X(z) = \frac{z}{z-\frac{1}{2}} \text{ with ROC} = |z| > \frac{1}{2}.$$

The Inverse z Transform: Use of Tables
Example 1

The Inverse z Transform: Use of Tables
Example 2

Find the inverse z transform of

$$X(z) = z^{-2} \text{ with ROC} = |z| > 0.$$

The Inverse z Transform: Use of Tables
Example 2

The Inverse z Transform: Use of Tables
Example 3

Find the inverse z transform of

$$X(z) = \frac{1 - \frac{1}{(5z)^5}}{1 - 0.2z^{-1}} \text{ with ROC} = |z| > 0.$$

The Inverse z Transform: Use of Tables
Example 3

The Inverse z Transform: Partial-Fraction Expansion

If $X(z)$ is of the form

$$X(z) = \frac{N(z)}{D(z)} = k\frac{(z - z_1)\cdots(z - z_m)}{(z - p_1)\cdots(z - p_n)},$$

then $X(z)$ is a rational function and a simple technique involving partial-fraction expansion can be used for the inversion of $X(z)$.

The Inverse z Transform: Partial-Fraction Expansion
Simple Pole Case for $m \leq n$

If $n \geq m$ and all the poles $(p_1, \ldots, p_n)$ of $X(z)$ are distinct (all roots of $D(z)$ are different), then

$$\begin{aligned}\frac{X(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \cdots + \frac{c_n}{z - p_n} \\ &= \frac{c_0}{z} + \sum_{k=1}^{n} \frac{c_k}{z - p_k}\end{aligned}$$

where the coefficients are given by

$$c_0 = X(z)\Big|_{z=0} \text{ and } c_k = (z - p_k)\,\frac{X(z)}{z}\Big|_{z=p_k}.$$

The Inverse z Transform: Partial-Fraction Expansion
Simple Pole Case for $m \leq n$

Therefore, we have

$$X(z) = c_0 + c_1\frac{z}{z - p_1} + c_2\frac{z}{z - p_2} + \cdots + c_n\frac{z}{z - p_n} \tag{4}$$

$$= c_0 + \sum_{k=1}^{n} c_k\frac{z}{z - p_k} \tag{5}$$

We can then use our tables of z transforms and infer the ROC for each term from the overall ROC of $X(z)$ to determine the overall inverse z-transform.

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case Example 1

Find the inverse z transform of

$$X(z) = \frac{z}{2z^2 - 3z + 1} \text{ with ROC} = |z| > 1$$

using the partial-fraction expansion technique.

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case Example 1

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case Example 2

Find the inverse z transform of

$$X(z) = \frac{3}{z - 2} \text{ with ROC} = |z| > 2$$

using the partial-fraction expansion technique.

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case Example 2

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case for $m > n$

If $m > n$ and all the poles $(p_1, \ldots, p_n)$ of $X(z)$ are distinct (all roots of $D(z)$ are different), then a polynomial of z of order $(m - n)$ must be added to the right-hand side of Equation 4.

Therefore, for $m > n$, the complete partial-fraction expansion would have the form

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} c_k \frac{z}{z - p_k}$$

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case for $m > n$ Example 1

Find the inverse z transform of

$$X(z) = \frac{2z^3 - 5z^2 + z + 3}{(z-1)(z-2)} \text{ with ROC} = |z| > 2$$

using the partial-fraction expansion technique.

The Inverse z Transform: Partial-Fraction Expansion Simple Pole Case for $m > n$ Example 1

The Inverse z Transform: Partial-Fraction Expansion Multiple Pole Case

If $D(z)$ has multiple factors of the form $(z - p_i)^r$, we say that p_i is a **multiple pole** of $X(z)$ with **multiplicity** r. Then, the partial-fraction expansion of $X(z)$ will consist of terms of the form

$$\frac{\lambda_1}{z - p_i} + \frac{\lambda_2}{(z - p_i)^2} + \cdots + \frac{\lambda_r}{(z - p_i)^r}$$

where the coefficients λ_k are determined from the formula

$$\lambda_{r-k} = \frac{1}{k!} \cdot \frac{d^k}{dz^k} \left[(z - p_i)^r \frac{X(z)}{z} \right] \Bigg|_{z=p_i}.$$

The Inverse z Transform: Partial-Fraction Expansion Multiple Pole Case Example

Find the inverse z transform of

$$X(z) = \frac{z}{(z-1)(z-2)^2} \text{ with ROC} = |z| > 2$$

using the partial-fraction expansion technique.

The Inverse z Transform: Partial-Fraction Expansion Multiple Pole Case Example

The Inverse z Transform: Power Series Expansion

The defining expression for the z transform is a power series where the sequence values $x[n]$ are the coefficients of z^{-n}. Therefore, if $X(z)$ is given as a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + \cdots ,$$

we can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}.

The Inverse z Transform: Power Series Expansion

This approach **may not** provide a closed-form solution.

It is very useful for a finite-length sequence where $X(z)$ may have not simpler form than a polynomial in z^{-1}.

For rational z-transforms, a power series expansion can be obtained by long division.

The Inverse z Transform: Partial-Fraction Expansion Power Series Expansion Example

Find the inverse z transform of

$$X(z) = \frac{z}{2z^2 - 3z + 1} \text{ with ROC} = |z| > 1$$

using the power series expansion technique.

The Inverse z Transform: Partial-Fraction Expansion Power Series Expansion Example