Math 335 HW 4 NAME: KEY

Due Wednesday 9/24 5:15pm

Practice Problems *(Do not turn in.)*
Sec 9.8 #3, 7, 17, 21, 31
Sec 9.9 #3, 13, 15, 19

Print out this assignment and write all work directly on this worksheet. Do not attach extra pages. Show all work. Your answers must be clear and legible. All pages must be stapled. Homework may be submitted within 24 hours of the due date with an automatic 2 point deduction. After Thursday 5:00pm, no late homework will be accepted for any reason.

1.) *[5 points]* Evaluate $\int_C x + y + z\, ds$ where C is the line segment from (1,2,3) to (1,0,-1).

First parametrize the path C.

$$C: \vec{r}(t) = \langle 1, 2-2t, 3-4t \rangle \qquad 0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 0, -2, -4 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{0^2 + (-2)^2 + (-4)^2} = \sqrt{20}$$

$$\int_C x+y+z\, ds = \int_0^1 (1+2-2t+3-4t)\sqrt{20}\, dt$$

$$= \sqrt{20} \int_0^1 -6t + 6\, dt$$

$$= \sqrt{20}\left[-3t^2 + 6t\right]_0^1$$

$$= \sqrt{20}\left[-3 + 6 + 0 - 0\right]$$

$$= \boxed{3\sqrt{20}}$$

2.) *[6 points]* The coiled tail of a Tepig can be parametrized in cm by

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, t\,\rangle, \quad 0 \le t \le 3\pi$$

a.) Find the length of Tepig's tail.

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$$

$$|r'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (1)^2} = \sqrt{5}$$

$$\int_C ds = \int_0^{3\pi} \sqrt{5}\,dt = \sqrt{5}\,t\Big|_0^{3\pi} = \boxed{3\pi\sqrt{5}\ \text{cm}}$$

b.) If the linear density of the tail in g/cm is $\rho(x, y, z) = x^2y + z^2$, find the total mass of Tepig's tail.

$$\int_C x^2y + z^2\,ds = \int_0^{3\pi} \left[(2\cos t)^2(2\sin t) + (t)^2\right]\sqrt{5}\,dt$$

$$= \sqrt{5}\left[-\frac{8}{3}\cos^3 t + \frac{1}{3}t^3\right]_0^{3\pi}$$

$$= \sqrt{5}\left[\frac{8}{3} + \frac{1}{3}(3\pi)^3 + \frac{8}{3} - 0\right]$$

$$= \boxed{\sqrt{5}\left[\frac{16}{3} + 9\pi^3\right]}$$

3.) *[9 points]* Tepig is swimming in the ocean near Kanto Route 20. The 3-dimensional velocity field of the ocean currents is given by

$$\vec{F} = \langle z^2, 2y, 2xz + 6z \rangle$$

a.) Prove $\vec{F}$ is a conservative vector field.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2y & 2xz + 6z \end{vmatrix}$$

$$= \langle 0 - 0, -(2z - 2z), 0 - 0 \rangle$$

$$= \langle 0, 0, 0 \rangle \quad \Rightarrow \vec{F} \text{ conservative}$$

b.) Find a potential function f(x,y,z) for $\vec{F}$. *(Answers may vary up to a constant.)*

$$\vec{F} = \langle \underbrace{z^2}_{① \frac{\partial f}{\partial x}}, \underbrace{2y}_{② \frac{\partial f}{\partial y}}, \underbrace{2xz + 6z}_{③ \frac{\partial f}{\partial z}} \rangle$$

① $f = \int z^2 \, dx = xz^2 + g_1(y,z)$

② $f = \int 2y \, dy = y^2 + g_2(x,z)$

③ $f = \int 2xz + 6z \, dz = xz^2 + 3z^2 + g_3(x,y)$

$$\boxed{f(x,y,z) = xz^2 + y^2 + 3z^2}$$

#3 continued...

c.) Tepig swims from the point (0,0,1) to (2,3,4). Calculate the flow along Tepig's path.

Since $\vec{F}$ is conservative, we can use the Fundamental Theorem of Line Integrals.

Use $f(x,y,z)$ from part (b)

$$\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds$$

$$= f(2,3,4) - f(0,0,1)$$

$$= \left[2(4)+3^2+3(4)^2\right] - \left[0(1)^2+0^2+3(1)^2\right]$$

$$= 32+9+48-0-0-3$$

$$= \boxed{86}$$

Alternatively, since conservative vector fields are path independent, you could assume the path is a straight line and find a parametrization between the two points. Plugging this in and integrating as in #1 will also give the answer 86. But it's a A LOT more work!

d.) Tepig decides to make a full lap. He swims from (0,0,1) to (2,3,4) and then back to (0,0,1). Calculate the circulation.

Since the curve is closed and $\vec{F}$ is conservative, the Loop Property tells us the circulation is zero.

$$\int_C \vec{F}\cdot\vec{T}\,ds = \boxed{0}$$