Math 335 HW 13 **NAME:** KEY
Due Wednesday 12/3 5:15pm

Practice Problems *(Do not turn in.)*
Sec 13.3 #1, 3, 7
Sec 13.4 #3, 7, 19
Sec 13.5 #5, 7

Print out this page and write all answers directly on this worksheet. Show all work. Your answers must be clear and legible. All pages must be stapled. Homework may be submitted within 24 hours of the due date with an automatic 2 point deduction. After Thursday 5:00pm, no late homework will be accepted for any reason.

1.) *[10 points]* A warm rod of length 10 meters is made of uniform material with thermal diffusivity constant $k = 3$. Initially the rod has a temperature in $°C$ at position x given by

$$f(x) = 2x, \quad 0 < x < 10.$$

The two ends of the rod are dipped in an ice bath kept at $0°C$.

a.) Assuming no heat is gained or lost along the rod, write down the PDE and full set of conditions that describe the temperature u(x,t) at position x and time t. Do not solve.

$$u_t = 3u_{xx}$$

$$u(0,t) = u(10,t) = 0 \quad \text{for } t \geq 0$$

$$u(x,0) = 2x \quad \text{for } 0 < x < 10$$

#1 continued...

b.) Solve for the temperature u(x,t) in the problem you stated in part (a). You do not need to start from the separation of variables, you may use the general formula we produced in lecture.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{10}\int_0^{10} 2x\sin\left(\frac{n\pi x}{10}\right)dx$$

Integration by Parts

$u = 2x$ $\quad v = \frac{-10}{n\pi}\cos\frac{n\pi x}{10}$

$du = 2dx$ $\quad dv = \sin\frac{n\pi x}{10}dx$

$$= \frac{1}{5}\left[-\frac{20x}{n\pi}\cos\frac{n\pi x}{10} + \int\frac{20}{n\pi}\cos\frac{n\pi x}{10}dx\right]$$

$$= \frac{1}{5}\left[-\frac{20x}{n\pi}\cos\frac{n\pi x}{10} + \frac{200}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5}\left[-\frac{200}{n\pi}\cos(n\pi) + \frac{200}{n^2\pi^2}\sin(n\pi)\right.$$

$$\left. + \frac{20(0)}{n\pi}\cos 0 - \frac{200}{n^2\pi^2}\sin 0\right]$$

($\cos(n\pi) \to (-1)^n$; $\sin(n\pi) \to 0$; $20(0) \to 0$; $\sin 0 \to 0$)

$$= -\frac{40}{n\pi}(-1)^n$$

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{40}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{10}\right)e^{-3\left(\frac{n\pi}{10}\right)^2 t}$$

2.) *[10 points]* The displacement u(x,t) of a vibrating guitar string of length 10 inches is modeled by

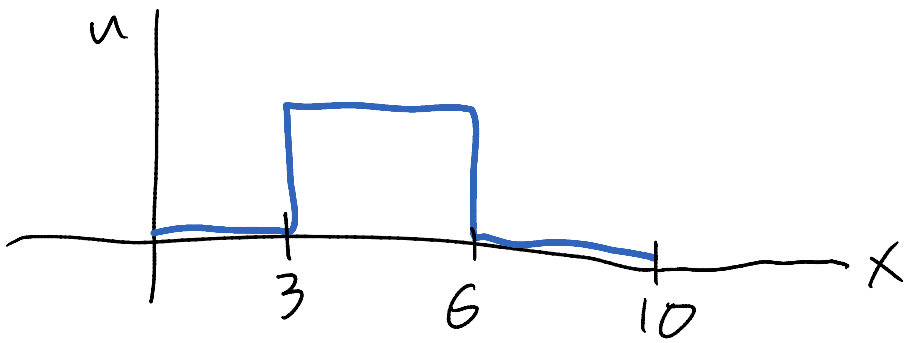
$$u_{tt} = 4u_{xx}$$
$$u(0,t) = u(10,t) = 0, \quad t > 0$$
$$u(x,0) = \begin{cases} 0 & if\ x < 3\ or\ x > 6 \\ 2 & if\ 3 \le x \le 6 \end{cases}$$

a.) Explain in words what the condition $u(0,t) = u(10,t) = 0$ means.

The ends of the string are clamped at the x-axis.

b.) Explain in words what the condition on $u(x,0)$ means.

The initial shape of the string looks like a top hat.



#2 continued...

c.) Solve for u(x,t).

Assuming it is released from rest $\Rightarrow \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \Rightarrow B_n = 0$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$\leftarrow L = 10$

$$= \frac{2}{10}\int_0^{10} u(x,0)\sin\frac{n\pi x}{10}\,dx$$

$\leftarrow u(x,0) = 0$ for $0 < x < 3$ and $6 < x < 10$

$$= \frac{1}{5}\int_3^6 2\sin\frac{n\pi x}{10}\,dx$$

$$= -\frac{2}{5}\,\frac{10}{n\pi}\cos\frac{n\pi x}{10}\Big|_3^6$$

$$= -\frac{4}{n\pi}\cos\frac{3\pi n}{5} + \frac{4}{n\pi}\cos\frac{3\pi n}{10}$$

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}\left[\cos\frac{3\pi n}{5} - \cos\frac{3\pi n}{10}\right]\cos\left(\frac{n\pi t}{5}\right)\sin\left(\frac{n\pi x}{10}\right)$$