ELEC 312
Systems I

System Stability
(Derived from Notes by Dr. Robert Barsanti)

(Images from Nise, 7^{th} Edition)

Required Reading: Chapter 5, *Control Systems Engineering*

March 2, 2015

System Stability

Stability is the most important system specification.

If a system is unstable, transient response and steady-state error analyses are useless.

An unstable system cannot be designed for a specific transient response or steady-state error requirement.

There are many definitions for stability, depending upon the kind of system or the point of view.

System Stability

We can control the output of a system if the steady-state response consists of only the zero-state (forced) response.

However, the total response of a system is the sum of the zero-state (forced) and zero-input (natural) responses, or

$$y(t) = y_F(t) + y_N(t).$$

Stability Definitions: Stable vs. Unstable vs. Marginally Stable

- An LTI system is **stable** if the zero-input (natural) response approaches zero as time approaches infinity.
- An LTI system is **unstable** if the zero-input (natural) response grows without bound as time approaches infinity.
- An LTI system is **marginally stable** if the zero-input (natural) response neither decays nor grows but remains constant or oscillates as time approaches infinity.

The definition of stability implies that only the zero-state (forced) response remains as the zero-input (natural) response approaches zero.

Bounded-Input, Bounded-Output (BIBO) Stability Definitions: Stable vs. Unstable vs. Marginally Stable

- An LTI system is **stable** if **every** bounded input yields a bounded output.
- An LTI system is **unstable** if **any** bounded input yields an unbounded output.
- An LTI system is **marginally stable** if **some** bounded inputs yield bounded outputs, but **other** bounded inputs yield unbounded outputs.

Bounded-Input, Bounded-Output (BIBO) Stability Definitions: Stable System Example

Consider the system with transfer function

$$G(s) = \frac{1}{s+1}.$$

Is this system stable?

Note that the system has a single pole at $s = -1$.

Consider an arbitrary bounded input $x(t)$ with Laplace transform $X(s)$. Then the output of this system is given by

$$Y(s) = G(s)X(s) = \frac{1}{s+1}X(s) = \frac{C_1}{s+1} + C_2X(s)$$
$$\implies y(t) = C_1e^{-t} + C_2x(t).$$

Note that the term e^{-t} decays to zero as time approaches infinity.

Therefore, if $x(t)$ is bounded, then $y(t)$ is also bounded.

Therefore, the system represented by $G(s)$ is **stable**.

Bounded-Input, Bounded-Output (BIBO) Stability Definitions: Unstable System Example

Consider the system with transfer function

$$G(s) = \frac{1}{s-1}.$$

Is this system stable?

Note that the system has a single pole at $s = 1$.

Consider an arbitrary bounded input $x(t)$ with Laplace transform $X(s)$. Then the output of this system is given by

$$Y(s) = G(s)X(s) = \frac{1}{s-1}X(s) = \frac{C_1}{s-1} + C_2X(s)$$
$$\implies y(t) = C_1e^{t} + C_2x(t).$$

Note that the term e^{t} grows without bound as time approaches infinity.

Therefore, even though $x(t)$ is bounded, the output $y(t)$ is always unbounded.

Therefore, the system represented by $G(s)$ is **unstable**.

Bounded-Input, Bounded-Output (BIBO) Stability Definitions: Marginally-Stable System Example

Consider the system with transfer function

$$G(s) = \frac{s}{s^2+1}.$$

Is this system stable?

Note that the system has a pair of imaginary conjugate poles at $s = \pm j$.

Consider a step input $x(t) = u(t)$ (which is bounded) with Laplace transform $X(s) = \frac{1}{s}$. Then the output of this system is given by

$$Y(s) = G(s)X(s) = \frac{s}{s^2+1} \cdot \frac{1}{s} = \frac{1}{s^2+1}$$
$$\implies y(t) = \sin(t).$$

Note that $y(t)$ is also bounded.

Bounded-Input, Bounded-Output (BIBO) Stability Definitions: Marginally-Stable System Example (continued)

Now consider a sinusoidal input $x(t) = \sin(t)$ (which is bounded) with Laplace transform $X(s) = \frac{1}{s^2+1}$. Then the output of this system is given by

$$Y(s) = G(s)X(s) = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$
$$\Longrightarrow y(t) = \frac{1}{2} t \sin(t).$$

Note that $y(t)$ is unbounded.

Therefore, we have found that one bounded input produces a bounded output, while another bounded input produces an unbounded output.

Therefore, the system represented by $G(s)$ is **marginally stable**.

Conditions for Stability in Time Domain and Complex Domain

Consider a general causal LTI system given by

$$a_n \frac{d^n y(t)}{dt^n} + \ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + \ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

with a Laplace transform given by

$$Y(s)\left[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right] = X(s)\left[b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right]$$

and transfer function given by

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}.$$

For a causal system, the **region of convergence (ROC)** lies to the right of the pole(s) with the largest real part. If $G(s)$ is reduced, then the **denominator** of $G(s)$ is a polynomial in s and is the **characteristic polynomial** of the system represented by $G(s)$.

Conditions for Stability in Time Domain and Complex Domain

If $G(s)$ is reduced, then we can factor

$$G(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where the z_i's are the system **zeros** ($|G(z_i)| = 0$) for $1 \le i \le m$, and the p_i's are the system **poles** ($|G(p_i)| = \infty$) for $1 \le i \le n$.

Using partial-fraction expansion, we have

$$G(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n},$$

and the system impulse response is given by

$$g(t) = \left[C_1 e^{p_1 t} + C_2 e^{p_2 t} + \ldots + C_n e^{p_n t}\right] u(t),$$

where the p_i's are **complex**, in general.

Conditions for Stability in Time Domain and Complex Domain

Recall that that any output of the system

$$Y(s) = Y_N(s) + Y_F(s) = \underbrace{\frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}}_{Y_N(s)} + Y_F(s)$$
$$= \underbrace{\frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}}_{\text{System Terms}} + \text{Input Terms}$$

It is the nature of the system poles that determines the zero-input (natural) response of the system. The relationship between system response and system pole location is shown below:

Real:	$\frac{C}{s-p_i}$	$p_i < 0$	$\Rightarrow Ce^{p_i t}u(t)$	$\Rightarrow$ Decaying exponential
		$p_i = 0$	$\Rightarrow Cu(t)$	$\Rightarrow$ Constant
		$p_i > 0$	$\Rightarrow Ce^{p_i t}u(t)$	$\Rightarrow$ Growing exponential
Imaginary:	$\frac{C}{s-p_i}$	$p_i = j\omega$	$\Rightarrow Ce^{j\omega t}u(t)$	$\Rightarrow$ Sinusoidal
Complex:	$\frac{C}{s-p_i}$	$p_i = \sigma + j\omega$, $\sigma < 0$	$\Rightarrow Ce^{\sigma t}e^{j\omega t}u(t)$	$\Rightarrow$ Decaying Sinusoidal
		$p_i = \sigma + j\omega$, $\sigma > 0$	$\Rightarrow Ce^{\sigma t}e^{j\omega t}u(t)$	$\Rightarrow$ Growing Sinusoidal

Conditions for Stability in Time Domain and Complex Domain

Therefore, the zero-input (natural) response of an LTI system will have the form

$$y_N(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} + \ldots + C_n e^{p_n t}$$

where the p_i's are the poles of $G(s)$. Note that

- If all poles have real parts that are negative, then all terms in $y_N(t)$ decay. Therefore, the zero-input (natural) response approaches zero as time approaches infinity, and the LTI system is **stable**.
- If any pole p_i has a real part that is positive, then the term $e^{p_i t}$ grows without bound. Therefore, the zero-input (natural) response grows without bound as time approaches infinity, and the LTI system is **unstable**.
- If all poles have real parts that are non-positive, but at least one pole p_i has a real part that is zero, then the term $e^{p_i t}$ either is constant or oscillates. Therefore, the zero-input (natural) response neither decays nor grows (but remains constant or oscillates) as time approaches infinity, and the LTI system is **marginally stable**.

Conditions for Stability in Time Domain and Complex Domain

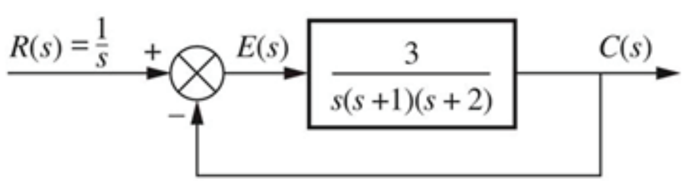
1. The **form** of the system response is determined by the system poles {roots of characteristic equation}.
2. For the above causal system to be **stable**, all poles $\{p_i\}$ must have $\text{Re}\{p_i\} < 0$, i.e. HAVE NEGATIVE REAL PARTS.
   Therefore, **ALL POLES MUST LIE IN THE LEFT-HALF PLANE**.
3. Since the ROC of a causal system lies to the right of the **dominant** pole, a causal stable LTI system must have a ROC that includes the **imaginary** axis.

Note that the **dominant pole** of a causal, stable LTI system

- is the smallest magnitude $-\text{Re}\{p_i\}$, or $\min[-\text{Re}\{p_i\}]$, or the largest $\text{Re}\{p_i\}$.
- is dominant since its associated exponential term will decay the slowest.
- lies closest to the imaginary axis.

Stability: Example 1

Consider the system below:



Determine the poles of the forward-path transfer function $G(s)$ and if $G(s)$ is stable. Are these the system poles?

Stability: Example 1 (continued)

Using MATLAB Code:

```
G = tf(3,[1 3 2 0])
p = pole(G)
if (isstable(G))
    disp('System G(s) is Stable!')
else
    disp('System G(s) is NOT Stable!')
end
```

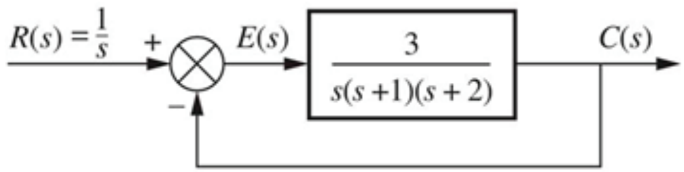
MATLAB Output:

```
G =

        3
-----------------
s^3 + 3 s^2 + 2 s

Continuous-time transfer function.
```

```
p =

     0
    -2
    -1

System G(s) is NOT Stable!
```

Stability: Example 1 (continued)

Consider the system below:



Determine the closed-loop transfer function $G_e(s) = \frac{C(s)}{R(s)}$, the system poles, and if $G_e(s)$ is stable.

Stability: Example 1 (continued)

Using MATLAB Code:

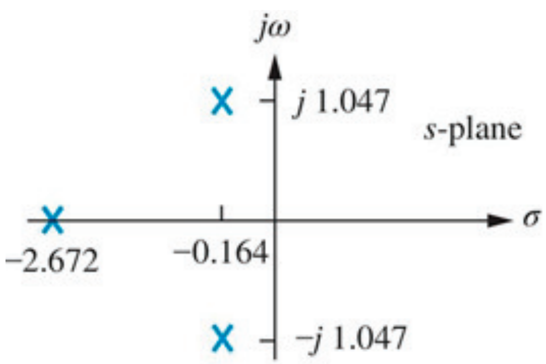
```
G = tf(3,[1 3 2 0]);
Ge = feedback(G,1)
p = pole(Ge)
if (isstable(Ge))
    disp('System G_e(s) is Stable!')
else
    disp('System G_e(s) is NOT Stable!')
end
```

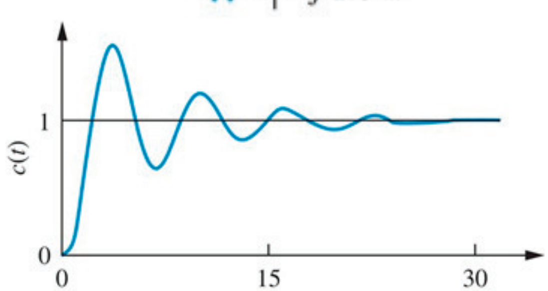
MATLAB Output:

```
Ge =

          3
---------------------
s^3 + 3 s^2 + 2 s + 3

Continuous-time transfer function.
```

```
p =

  -2.6717 + 0.0000i
  -0.1642 + 1.0469i
  -0.1642 - 1.0469i

System G_e(s) is Stable!
```


Stability: Example 1 (continued)

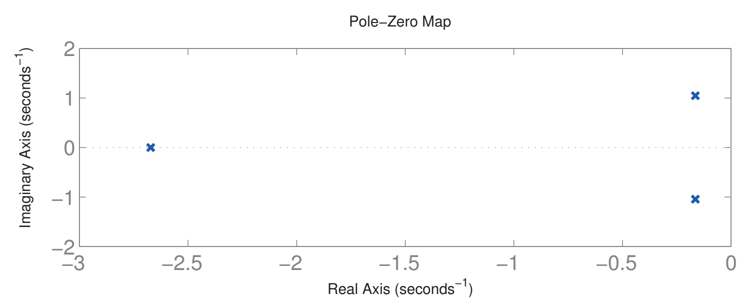


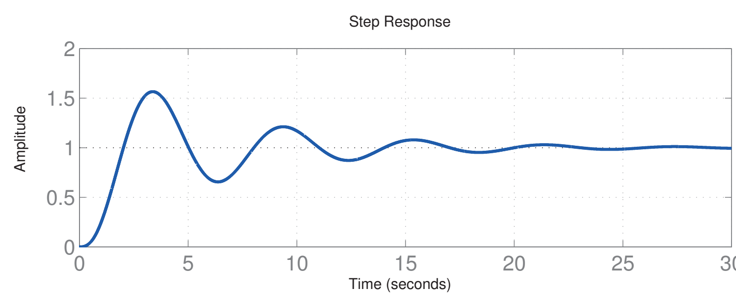


Using MATLAB:

```
G = tf(3,[1 3 2 0]);
Ge = feedback(G,1);
pzmap(Ge)
step(Ge)
```

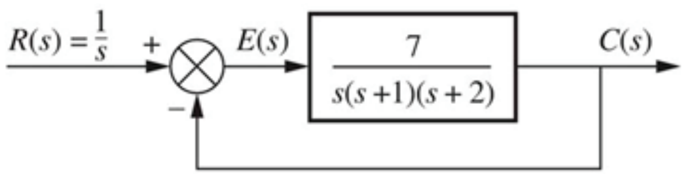

Stability: Example 1 (continued)





Stability: Example 2

Consider the system below:



Determine the poles of the forward-path transfer function $G(s)$ and if $G(s)$ is stable. Are these the system poles?

Stability: Example 2 (continued)

Using MATLAB Code:

```
G = tf(7,[1 3 2 0])
p = pole(G)
if (isstable(G))
    disp('System G(s) is Stable!')
else
    disp('System G(s) is NOT Stable!')
end
```

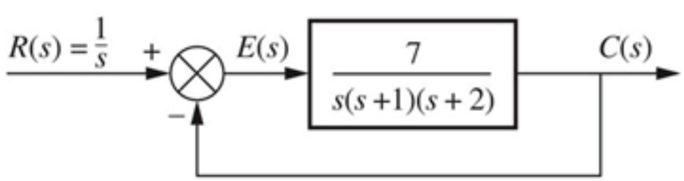
MATLAB Output:

```
G =

         7
-----------------
s^3 + 3 s^2 + 2 s

Continuous-time transfer function.
```

```
p =

     0
    -2
    -1

System G(s) is NOT Stable!
```


Stability: Example 2 (continued)

Consider the system below:



Determine the closed-loop transfer function $G_e(s) = \frac{C(s)}{R(s)}$, the system poles, and if $G_e(s)$ is stable.

Stability: Example 2 (continued)

Using MATLAB Code:

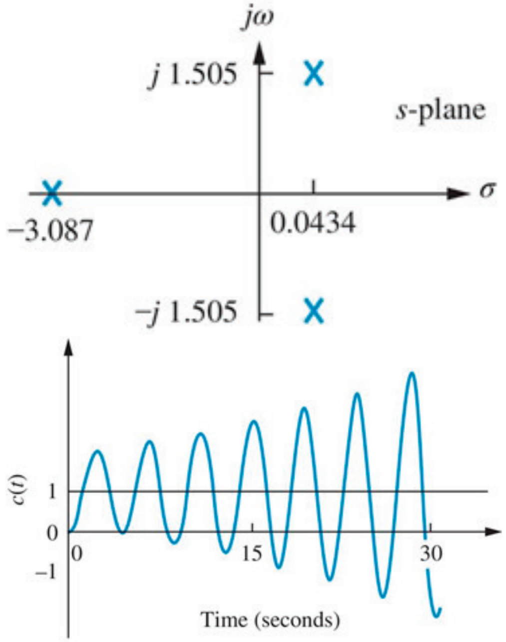
```
G = tf(7,[1 3 2 0]);
Ge = feedback(G,1)
p = pole(Ge)
if (isstable(Ge))
    disp('System G_e(s) is Stable!')
else
    disp('System G_e(s) is NOT Stable!')
end
```

MATLAB Output:

```
Ge =

           7
---------------------
s^3 + 3 s^2 + 2 s + 7

Continuous-time transfer function.
```

```
p =

  -3.0867 + 0.0000i
   0.0434 + 1.5053i
   0.0434 - 1.5053i

System G_e(s) is NOT Stable!
```

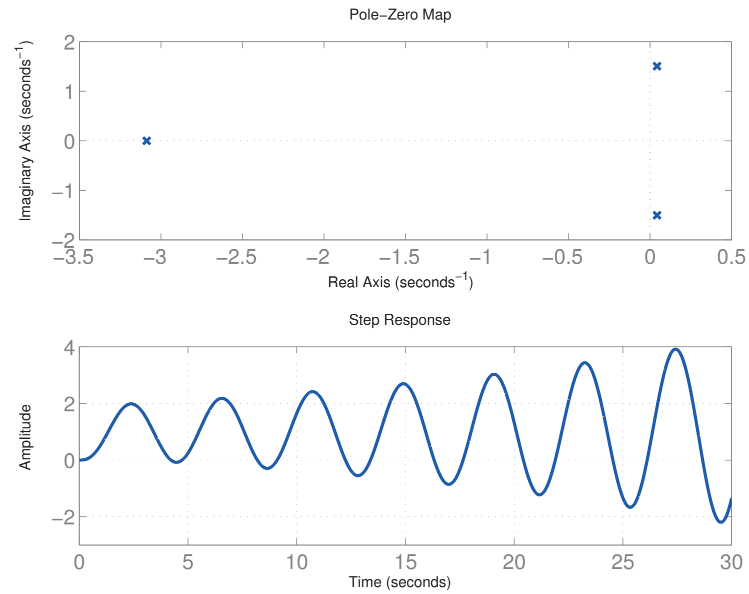
Stability: Example 2 (continued)



Using MATLAB:

```
G = tf(7,[1 3 2 0]);
Ge = feedback(G,1);
pzmap(Ge)
step(Ge)
```


Stability: Example 2 (continued)



Stability: Example 3

Determine if the system specified by the LCCDE

$$y''(t) + 5y'(t) + 4y(t) = x(t)$$

is stable.

Stability: Example 3 (continued)

Using MATLAB Code:

```
G = tf(1,[1 5 4])
p = pole(G)
if (isstable(G))
    disp('System G(s) is Stable!')
else
    disp('System G(s) is NOT Stable!')
end
```

MATLAB Output:

```
G =

        1
  -------------
  s^2 + 5 s + 4

Continuous-time transfer function.
```

```
p =

  -4
  -1

System G(s) is Stable!
```

Stability: Example 3 (continued)

Given $y(0) = 0$ and $y'(0) = 1$ for the LCCDE

$$y''(t) + 5y'(t) + 4y(t) = x(t),$$

determine the natural response.

Stability: Example 3 (continued)

Using MATLAB Code:

```
syms y(t) x(t)
Dy = diff(y);
D2y = diff(y,2);
x(t) = 0;
y(t) = dsolve(D2y+5*Dy+4*y==x(t),y(0)==0,Dy(0)==1);
pretty(y(t))
```

MATLAB Output:

```
exp(-t)   exp(-4 t)
------- - ---------
   3          3
```


Stability: Example 3 (continued)

Given $y(0) = 0$ and $y'(0) = 1$ for the LCCDE

$$y''(t) + 5y'(t) + 4y(t) = x(t),$$

use the final value theorem to determine $\lim_{t\to\infty} y_N(t)$.

Stability: Example 3 (continued)

Using MATLAB Code:

```
syms y(t) x(t)
Dy = diff(y);
D2y = diff(y,2);
x(t) = 0;
y(t) = dsolve(D2y+5*Dy+4*y==x(t),y(0)==0,Dy(0)==1);

fvYtime = double(y(Inf));
fprintf('lim y(t) as t->Inf is %5.3f\n',fvYtime)

syms Y(s)
Y(s) = laplace(y);
fvYcomplex = double(subs(s*Y(s),0));
fprintf('lim sY(s) as s->0 is %5.3f\n',fvYcomplex)
```

MATLAB Output:

```
lim y(t) as t->Inf is 0.000
lim sY(s) as s->0 is 0.000
```

Stability: Example 4

Is the system specified by transfer function $G(s) = \frac{s^2}{s^2+2s+2}$ stable?

Stability: Example 4 (continued)

Using MATLAB Code:

```
G = tf([1 0 0],[1 2 2])
p = pole(G)
if (isstable(G))
    disp('System G(s) is Stable!')
else
    disp('System G(s) is NOT Stable!')
end
```

MATLAB Output:

```
G =

       s^2
  -------------
  s^2 + 2 s + 2

Continuous-time transfer function.
```

```
p =

  -1.0000 + 1.0000i
  -1.0000 - 1.0000i

System G(s) is Stable!
```


Stability: Example 5

Is the system specified by transfer function $G(s) = \frac{s}{s^2-2s+1}$ stable?

Stability: Example 5 (continued)

Using MATLAB Code:

```
G = tf([1 0],[1 -2 1])
p = pole(G)
if (isstable(G))
    disp('System G(s) is Stable!')
else
    disp('System G(s) is NOT Stable!')
end
```

MATLAB Output:

```
G =

        s
  -------------
  s^2 - 2 s + 1

Continuous-time transfer function.
```

```
p =

     1
     1

System G(s) is NOT Stable!
```

Stability: Example 6

Is the system specified by transfer function $G(s) = \frac{s+7}{s^3-s^2-s-1}$ stable?

Stability: Example 6 (continued)

Using MATLAB Code:

```
G = tf([1 0],[1 -2 1])
p = pole(G)
if (isstable(G))
    disp('System G(s) is Stable!')
else
    disp('System G(s) is NOT Stable!')
end
```

MATLAB Output:

```
        s + 7
  -----------------
  s^3 - s^2 - s - 1

Continuous-time transfer function.
```

```
p =

   1.8393 + 0.0000i
  -0.4196 + 0.6063i
  -0.4196 - 0.6063i

System G(s) is NOT Stable!
```


Conditions for Stability in the Complex Domain: Observations

To be stable, the denominator (or characteristic polynomial) $D(s)$ of the proper, irreducible transfer function $G(s)$ must have factored form

$$D(s) = (s-p_1)(s-p_2)\cdots = (s+a)(s+b)(s+c)\cdots,$$

where $a > 0, b > 0, c > 0, \ldots$ since $p_1 < 0, p_2 < 0, \ldots$.

1. The product of such terms is a polynomial with **all positive coefficients.**
2. **No term of the polynomial can be missing** since that would imply cancellation between positive and negative coefficients or imaginary axis roots in the factors.
3. Therefore, a sufficient condition for a system to be **unstable is that all signs** in $D(s)$ are not the same.
4. If **powers of s are missing**, the system is **unstable**.

Determining Stability Using the Routh-Hurwitz Criterion

Recall:

1. The stability of a system can be determined by analyzing the **poles** of a proper and irreducible transfer function $G(s)$.
2. For the system to be **stable**, all **poles** of $G(s)$ must lie strictly in the **left-half complex plane**.
3. A **necessary condition** for a system to be stable is that **all coefficients** of the denominator polynomial $D(s)$ of proper and irreducible transfer function $G(s) = \frac{N(s)}{D(s)}$ be **positive** and not zero.
4. However, fact (3) above is **not sufficient** for stability.

The **Routh-Hurwitz criterion** is a method of determining stability without solving for the roots.

The idea is to form a table and use reduction/computation to assess the location of the roots of any polynomial.

Determining Stability Using the Routh-Hurwitz Criterion: Demonstrative Example

Consider an LTI system with the characteristic polynomial

$$D(s) = 2s^5 + s^4 + 7s^3 + 3s^2 + 4s + 1.5.$$

1. Form first two rows of table using every other coefficient of s. Note that the odd powers of s end up on the row next to the highest odd power of s, and the even powers of s end up on the row next to the highest even power of s,

s^5	2	7	4
s^4	1	3	1.5
s^3			
s^2			
s^1			
s^0			

2. Determine ratio of FIRST elements of row 1 and row 2. In our example, we will place this value to the left of the s^4 row, so we will label it as $k_4 = \frac{2}{1} = 2$.

3. Compute next lower row as

$$s^3 \text{ row} = s^5 \text{ row} - k_4 \cdot s^4 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	2	7	4
$k_4 = 2$	s^4	1	3	1.5
	s^3	7-2(3)=1	4-2(1.5)=1	
	s^2			
	s^1			
	s^0			

4. Determine ratio of FIRST elements of row 2 and row 3. In our example, we will place this value to the left of the s^3 row, so we will label it as $k_3 = \frac{1}{1} = 1$.

5. Compute next lower row as

$$s^2 \text{ row} = s^4 \text{ row} - k_3 \cdot s^3 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	2	7	4
$k_4 = 2$	s^4	1	3	1.5
$k_3 = 1$	s^3	1	1	
	s^2	3-1(1)=2	1.5-1(0)=1.5	
	s^1			
	s^0			

6. Determine ratio of FIRST elements of row 3 and row 4. In our example, we will place this value to the left of the s^2 row, so we will label it as $k_2 = \frac{1}{2} = 0.5$.

7. Compute next lower row as

$$s^1 \text{ row} = s^3 \text{ row} - k_2 \cdot s^2 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	2	7	4
$k_4 = 2$	s^4	1	3	1.5
$k_3 = 1$	s^3	1	1	
$k_2 = 0.5$	s^2	2	1.5	
	s^1	1-0.5(1.5) = 0.25		
	s^0			

8. Determine ratio of FIRST elements of row 4 and row 5. In our example, we will place this value to the left of the s^1 row, so we will label it as $k_1 = \frac{2}{0.25} = 8$.

9. Compute final row as

$$s^0 \text{ row} = s^2 \text{ row} - k_1 \cdot s^1 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	2	7	4
$k_4 = 2$	s^4	1	3	1.5
$k_3 = 1$	s^3	1	1	
$k_2 = 0.5$	s^2	2	1.5	
$k_1 = 8$	s^1	0.25		
	s^0	1.5-8(0)=1.5		

10. Look at the first column of the finished table. The number of sign changes in the first column corresponds to the number of poles (or roots of the denominator polynomial $D(s)$) that are in the right half-plane.

	s^5	2	7	4
$k_4 = 2$	s^4	1	3	1.5
$k_3 = 1$	s^3	1	1	
$k_2 = 0.5$	s^2	2	1.5	
$k_1 = 8$	s^1	0.25		
	s^0	1.5		

Note that there are no sign changes in the first column above. Therefore, no roots of the characteristic polynomial lie in the right-half plane, all roots of the characteristic polynomial lie in the left-half plane, and the system is **stable**.

- If the number of sign changes in the first column is ZERO, then all roots of the characteristic polynomial lie in the left-half plane, and the system is **stable**.
- If the number of sign changes in the first column is positive, then at least one of the roots of the characteristic polynomial lies in the right-half plane, and the system is **unstable**.

Determining Stability Using the Routh-Hurwitz Criterion: Example 1

Determine if the LTI system with characteristic polynomial

$$D(s) = 2s^4 + 2s^3 + 3s + 2$$

is stable.

Determining Stability Using the Routh-Hurwitz Criterion: Example 1 (continued)

Determining Stability Using the Routh-Hurwitz Criterion: Example 2

Determine if the LTI system with transfer function

$$G(s) = \frac{1}{s^4 + s^3 + s^2 + s + 1}$$

is stable.

Determining Stability Using the Routh-Hurwitz Criterion: Example 2 (continued)

Determining Stability Using the Routh-Hurwitz Criterion: Example 3

Determine if the LTI system with transfer function

$$G(s) = \frac{s^2 + s + 1}{2s^4 + 2s^3 + s^2 + 3s + 2}$$

is stable.

Determining Stability Using the Routh-Hurwitz Criterion: Example 3 (continued)

Routh-Hurwitz Criterion Special Cases

There are two special cases that may occur when implementing the Routh-Hurwitz criterion:

1. The Routh table will have a zero **only in the first column** of a row.
2. The Routh table will have an **entire row** of zeros.

Routh-Hurwitz Criterion Special Cases: Zero Only in the First Column

Two techniques for completing the Routh table when there is a zero only in the first column:

1. **Epsilon Method**:
   - If the first element of a row is zero, division by zero would be required to form the next row.
   - To avoid this, we replace the zero with ϵ in the first column.
   - The value ϵ is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the fist column can be determined.
2. **Reverse Coefficients**:
   - If we create a new polynomial from the original polynomial by writing its coefficients in reverse order, we now have a polynomial that has the reciprocal roots of the original polynomial.
   - The roots of the new polynomial are distributed the same (right half-plane, left half-plane, or imaginary axis) as the roots of the original polynomial.
   - Therefore , it is possible that the Routh table for the new polynomial will not have a zero in the first column.

Routh-Hurwitz Criterion Special Cases: Zero Only in the First Column – Epsilon Method Demonstrative Example

Consider an LTI system with the transfer function

$$G(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

1. Form first two rows of table using every other coefficient of s. Note that the odd powers of s end up on the row next to the highest odd power of s, and the even powers of s end up on the row next to the highest even power of s,

s^5	1	3	5
s^4	2	6	3
s^3			
s^2			
s^1			
s^0			

2. Determine ratio of FIRST elements of row 1 and row 2. In our example, we will place this value to the left of the s^4 row, so we will label it as $k_4 = \frac{1}{2} = 0.5$.

3. Compute next lower row as

$$s^3 \text{ row} = s^5 \text{ row} - k_4 \cdot s^4 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	3	5
$k_4 = 0.5$	s^4	2	6	3
	s^3	3-0.5(6)=0	5-0.5(3)=3.5	
	s^2			
	s^1			
	s^0			

4. Note that there is a zero in the first column. We replace this zero with a small positive quantity $\epsilon > 0$.

	s^5	1	3	5
$k_4 = 0.5$	s^4	2	6	3
	s^3	ϵ	3.5	
	s^2			
	s^1			
	s^0			

5. Determine ratio of FIRST elements of row 2 and row 3. In our example, we will place this value to the left of the s^3 row, so we will label it as $k_3 = \frac{2}{\epsilon}$.

6. Compute next lower row as

$$s^2 \text{ row} = s^4 \text{ row} - k_3 \cdot s^3 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	3	5
$k_4 = 0.5$	s^4	2	6	3
$k_3 = 2/\epsilon$	s^3	ϵ	3.5	
	s^2	$6 - \left(\frac{2}{\epsilon}\right)(3.5) = \frac{6\epsilon - 7}{\epsilon}$	$3 - \left(\frac{2}{\epsilon}\right)(0) = 3$	
	s^1			
	s^0			

Note that

$$\lim_{\epsilon \to 0^+} \frac{6\epsilon - 7}{\epsilon} = -\infty,$$

which is obviously **negative**.

7. Determine ratio of FIRST elements of row 3 and row 4. In our example, we will place this value to the left of the s^2 row, so we will label it as $k_2 = \frac{\epsilon}{\frac{6\epsilon-7}{\epsilon}} = \frac{\epsilon^2}{6\epsilon-7}$.

8. Compute next lower row as

$$s^1 \text{ row} = s^3 \text{ row} - k_2 \cdot s^2 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	3	5
$k_4 = 0.5$	s^4	2	6	3
$k_3 = 2/\epsilon$	s^3	ϵ	3.5	
$k_2 = \frac{\epsilon^2}{6\epsilon-7}$	s^2	$\frac{6\epsilon-7}{\epsilon}$	3	
	s^1	$3.5 - \frac{\epsilon^2}{6\epsilon-7}(3) = \frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$		
	s^0			

Note that

$$\lim_{\epsilon \to 0^+} \frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14} = \frac{-49}{-14} = 3.5,$$

which is obviously **positive**.

9. Determine ratio of FIRST elements of row 4 and row 5. In our example, we will place this value to the left of the s^1 row, so we will label it as $k_1 = \frac{\frac{\epsilon^2}{6\epsilon-7}}{\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}} = \frac{2\epsilon^2}{42\epsilon-49-6\epsilon^2}$.

10. Compute final row as

$$s^0 \text{ row} = s^2 \text{ row} - k_1 \cdot s^1 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	3	5
$k_4 = 0.5$	s^4	2	6	3
$k_3 = 2/\epsilon$	s^3	ϵ	3.5	
$k_2 = \frac{\epsilon^2}{6\epsilon-7}$	s^2	$\frac{6\epsilon-7}{\epsilon}$	3	
$k_1 = \frac{2\epsilon^2}{42\epsilon-49-6\epsilon^2}$	s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$		
	s^0	$3 - \frac{2\epsilon^2}{42\epsilon-49-6\epsilon^2}(0)$		

11. Look at the first column of the finished table. The number of sign changes in the first column corresponds to the number of poles (or roots of the denominator polynomial $D(s)$) that are in the right half-plane.

			$\epsilon > 0$	$\epsilon < 0$
	s^5	1	+	+
$k_4 = 0.5$	s^4	2	+	+
$k_3 = 2/\epsilon$	s^3	ϵ	+	-
$k_2 = \frac{\epsilon^2}{6\epsilon-7}$	s^2	$\frac{6\epsilon-7}{\epsilon}$	-	+
$k_1 = \frac{2\epsilon^2}{42\epsilon-49-6\epsilon^2}$	s^1	$\frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14}$	+	+
	s^0	3	+	+

Note that there are two sign changes in the first column above, no matter if we consider ϵ to be positive or negative. Therefore, two roots of the characteristic polynomial lie in the right-half plane, and the system is **unstable**.

Routh-Hurwitz Criterion Special Cases: Zero Only in the First Column – Reverse Coefficients Method Demonstrative Example

Consider an LTI system with the transfer function

$$G(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}.$$

1. Form new denominator polynomial by reversing the coefficients of the original polynomial, or $D(s) = 3^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$.

2. Form first two rows of table using every other coefficient of s. Note that the odd powers of s end up on the row next to the highest odd power of s, and the even powers of s end up on the row next to the highest even power of s,

s^5	3	6	2
s^4	5	3	1
s^3			
s^2			
s^1			
s^0			

3. Determine ratio of FIRST elements of row 1 and row 2. In our example, we will place this value to the left of the s^4 row, so we will label it as $k_4 = \frac{3}{5} = 0.6$.

4. Compute next lower row as

$$s^3 \text{ row} = s^5 \text{ row} - k_4 \cdot s^4 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	3	6	2
$k_4 = 0.6$	s^4	5	3	1
	s^3	6-0.6(3)=4.2	2-0.6(1)=1.4	
	s^2			
	s^1			
	s^0			

5. Determine ratio of FIRST elements of row 2 and row 3. In our example, we will place this value to the left of the s^3 row, so we will label it as $k_3 = \frac{5}{4.2} = 1.1905$.

6. Compute next lower row as

$$s^2 \text{ row} = s^4 \text{ row} - k_3 \cdot s^3 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	3	6	2
$k_4 = 0.6$	s^4	5	3	1
$k_3 = 1.905$	s^3	4.2	1.4	
	s^2	$3 - 1.1905(1.4) = 1.3333$	$1 - 1.905(0) = 1$	
	s^1			
	s^0			

7. Determine ratio of FIRST elements of row 3 and row 4. In our example, we will place this value to the left of the s^2 row, so we will label it as $k_2 = \frac{4.2}{1.3333} = 3.15$.

8. Compute next lower row as

$$s^1 \text{ row} = s^3 \text{ row} - k_2 \cdot s^2 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	3	6	2
$k_4 = 0.6$	s^4	5	3	1
$k_3 = 1.905$	s^3	4.2	1.4	
$k_2 = 3.15$	s^2	1.3333	1	
	s^1	$1.4 - 3.15(1) = -1.75$		
	s^0			

9. Determine ratio of FIRST elements of row 4 and row 5. In our example, we will place this value to the left of the s^1 row, so we will label it as $k_1 = \frac{1.3333}{-1.75} = -0.7619$.

10. Compute final row as

$$s^0 \text{ row} = s^2 \text{ row} - k_1 \cdot s^1 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	3	6	2
$k_4 = 0.6$	s^4	5	3	1
$k_3 = 1.905$	s^3	4.2	1.4	
$k_2 = 3.15$	s^2	1.3333	1	
$k_1 = -0.7619$	s^1	-1.75		
	s^0	$1 + 0.7619(0) = 1$		

11. Look at the first column of the finished table. The number of sign changes in the first column corresponds to the number of poles (or roots of the denominator polynomial $D(s)$) that are in the right half-plane.

	s^5	3	6	2
$k_4 = 0.6$	s^4	5	3	1
$k_3 = 1.905$	s^3	4.2	1.4	
$k_2 = 3.15$	s^2	1.3333	1	
$k_1 = -0.7619$	s^1	-1.75		
	s^0	1		

Note that there are two sign changes in the first column above. Therefore, two roots of the characteristic polynomial lie in the right-half plane, and the system is **unstable**.

Also note that this method is usually computationally easier than the epsilon method previously described.

Routh-Hurwitz Criterion Special Cases: Entire Row is Zero

The technique for completing the Routh table when there is a entire row of zeros is as follows:

- Once a row of zeros is encountered, write a polynomial $P(s)$ using the row directly above the row of zeros. For example, if the row directly above the row of zeros is $\boxed{s^4 \;\; 2 \;\; 3 \;\; 4}$, then $P(s) = 2s^4 + 3s^2 + 4$.
- Differentiate this polynomial with respect to s, or

$$\frac{dP(s)}{ds} = 8s^3 + 6s + 0.$$

- Use the coefficients of $dP(s)/ds$ to replace the row of zeros. The row of zeros using the example above would be $\boxed{s^3 \;\; 8 \;\; 3 \;\; 0}$.

Routh-Hurwitz Criterion Special Cases: Zero Only in the First Column – Epsilon Method Demonstrative Example

Consider an LTI system with the transfer function

$$G(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

1. Form first two rows of table using every other coefficient of s. Note that the odd powers of s end up on the row next to the highest odd power of s, and the even powers of s end up on the row next to the highest even power of s,

s^5	1	6	8
s^4	7	42	56
s^3			
s^2			
s^1			
s^0			

2. Determine ratio of FIRST elements of row 1 and row 2. In our example, we will place this value to the left of the s^4 row, so we will label it as $k_4 = \frac{1}{7} = 0.1429$.

3. Compute next lower row as

$$s^3 \text{ row} = s^5 \text{ row} - k_4 \cdot s^4 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	6	8
$k_4 = 0.1429$	s^4	7	42	56
	s^3	6-0.1429(42)=0	8-0.1429(56)=0	
	s^2			
	s^1			
	s^0			

4. Note that there is an entire row of zeros in the s^3 row.

5. We replace this entire row of zeros with $\boxed{s^3 \;\; 28 \;\; 84 \;\; 0}$ because

$$\frac{dP(s)}{ds} = \frac{d}{ds}\left[7s^4 + 42s^2 + 56\right] = 28s^3 + 84s + 0.$$

6. Now we have

	s^5	1	6	8
$k_4 = 0.1429$	s^4	7	42	56
	s^3	28	84	
	s^2			
	s^1			
	s^0			

7. Determine ratio of FIRST elements of row 2 and row 3. In our example, we will place this value to the left of the s^3 row, so we will label it as $k_3 = \frac{7}{28} = 0.25$.

8. Compute next lower row as

$$s^2 \text{ row} = s^4 \text{ row} - k_3 \cdot s^3 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	6	8
$k_4 = 0.1429$	s^4	7	42	56
$k_3 = 0.25$	s^3	28	84	
	s^2	$42 - 0.25(84) = 21$	$56 - 0.25(0) = 56$	
	s^1			
	s^0			

9. Determine ratio of FIRST elements of row 3 and row 4. In our example, we will place this value to the left of the s^2 row, so we will label it as $k_2 = \frac{28}{21} = 1.3333$.

10. Compute next lower row as

$$s^1 \text{ row} = s^3 \text{ row} - k_2 \cdot s^2 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	6	8
$k_4 = 0.1429$	s^4	7	42	56
$k_3 = 0.25$	s^3	28	84	
$k_2 = 1.3333$	s^2	21	56	
	s^1	$84 - 1.3333(56) = 9.3333$		
	s^0			

11. Determine ratio of FIRST elements of row 4 and row 5. In our example, we will place this value to the left of the s^1 row, so we will label it as $k_1 = \frac{21}{9.3333} = 2.25$.

12. Compute final row as

$$s^0 \text{ row} = s^2 \text{ row} - k_1 \cdot s^1 \text{ row}.$$

Note that first element will always be zero, so discard it.

	s^5	1	6	8
$k_4 = 0.1429$	s^4	7	42	56
$k_3 = 0.25$	s^3	28	84	
$k_2 = 1.3333$	s^2	21	56	
$k_1 = 2.25$	s^1	9.3333		
	s^0	$56 - 2.25(0) = 56$		

13. Look at the first column of the finished table. The number of sign changes in the first column corresponds to the number of poles (or roots of the denominator polynomial $D(s)$) that are in the right half-plane.

	s^5	1	6	8
$k_4 = 0.1429$	s^4	7	42	56
$k_3 = 0.25$	s^3	28	84	
$k_2 = 1.3333$	s^2	21	56	
$k_1 = 2.25$	s^1	9.3333		
	s^0	56		

Note that there are no sign changes in the first column above. Therefore, no roots of the characteristic polynomial lie in the right-half plane, and the system is **stable**.

Routh-Hurwitz Criterion Special Cases: Example 1

Determine if the LTI system with transfer function

$$G(s) = \frac{s^3 + 7s^2 - 21s + 10}{s^6 + s^5 - 6s^4 - s^2 - s + 6}$$

is stable.

Determining Stability Using the Routh-Hurwitz Criterion: Example 1 (continued)

Determining Stability Using the Routh-Hurwitz Criterion: Example 1 (continued)

Routh-Hurwitz Criterion Special Cases: Example 2

Determine how many poles of the following transfer function are in the left-half plane, the right-half plane, or on the imaginary axis:

$$G(s) = \frac{s^3 + 7s^2 - 21s + 10}{s^6 + s^5 - 6s^4 - s^2 - s + 6}$$

Determining Stability Using the Routh-Hurwitz Criterion: Example 2 (continued)

Determining Stability Using the Routh-Hurwitz Criterion: Example 2 (continued)