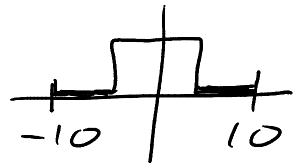
Math 335, Fall 2014 **Exam 3 Key**

1.) Let f(x) be the top-hat function

$$f(x) = \begin{cases} 4 & \text{if } -5 < x < 5 \\ 0 & \text{if } x \le -5 \text{ or } x \ge 5 \end{cases}$$



a.) *[10 points]* Compute the Fourier series of f(x) on the interval $(-10,10)$.

$L = 10$

$f(x)$ even $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{10}\int_{-5}^{5} 4\,dx$$

$$= \frac{1}{10}\left[4x\right]_{-5}^{5} = \frac{1}{10}\left[20+20\right] = 4$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{1}{10}\int_{-5}^{5} 4\cos\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{1}{10}\left[4\,\frac{10}{n\pi}\sin\left(\frac{n\pi x}{10}\right)\right]_{-5}^{5}$$

$$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\sin\left(-\frac{n\pi}{2}\right)$$

$\sin(-x) = -\sin(x)$

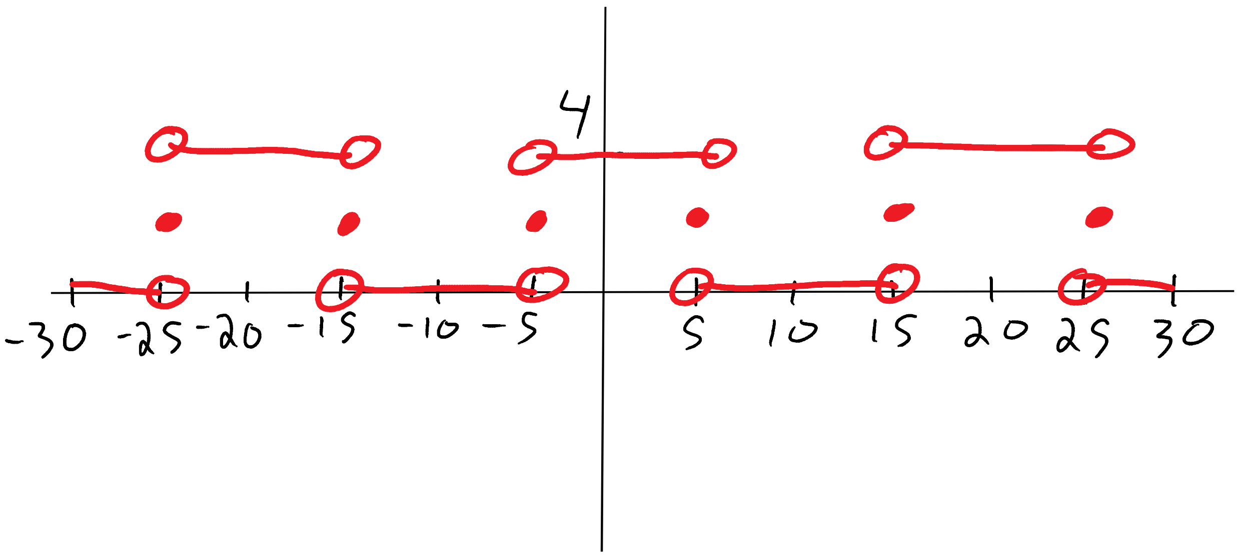
$$= \frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{10}\right)$$

$$= \boxed{2 + \sum_{n=1}^{\infty}\frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{10}\right)}$$

#1 continued...

b.) *[3 points]* Sketch the Fourier series that you computed in part (a) on the axes below for $-30 \le x \le 30$. Label the y-axis and clearly indicate function values at discontinuities with open or dark circles.



c.) *[1 point]* What value does the Fourier series converge to at $x = 5$?

$$\frac{1}{2}\left[f(x^-) + f(x^+)\right] = \frac{1}{2}\left[4 + 0\right] = \boxed{2}$$

d.) *[1 point]* What value does the Fourier series converge to at $x = 10$?

$$\boxed{0}$$

$\swarrow L = \pi$

2.) *[10 points]* Find the Fourier Cosine Series on $(0, \pi)$ for $f(x) = e^x$. You may make use of the following integration formula:

$$\int e^x \cos(nx)\, dx = \frac{e^x}{1+n^2}[\cos(nx) + n \sin(nx)]$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{\pi}\int_0^\pi e^x\,dx$$

$$= \frac{2}{\pi} e^x \Big|_0^\pi = \frac{2}{\pi}\left[e^\pi - e^0\right] = \frac{2(e^\pi - 1)}{\pi}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi e^x \cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos(nx) + n\sin(nx)\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{e^\pi}{1+n^2}\left(\underbrace{\cos(n\pi)}_{(-1)^n} + n\underbrace{\sin(n\pi)}_{0}\right) - \frac{e^0}{1+n^2}\left(\underbrace{\cos(0)}_{1} + n\underbrace{\sin(0)}_{0}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{e^\pi}{1+n^2}(-1)^n - \frac{1}{1+n^2}\right]$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^\infty a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \boxed{\frac{e^\pi - 1}{\pi} + \sum_{n=1}^\infty \frac{2}{\pi(1+n^2)}\left[e^\pi(-1)^n - 1\right]\cos(nx)}$$

3.) *[10 points]* Consider the following 3rd-order PDE in variables t and z:

$$u_{ttt} = 9u_{zz}$$

Assume the solution to this PDE is separable. Find the product solution $u(t, z)$ for the case when the separation constant $\lambda = 0$. Show all work.

Let $u(t,z) = v(t)\,w(z)$.

$$u_{ttt} = 9u_{zz}$$

$$(vw)_{ttt} = 9(vw)_{zz}$$

$$v_{ttt}\,w = 9\,v\,w_{zz}$$

$$\frac{v_{ttt}}{v} = \frac{9w_{zz}}{w} = -\lambda$$

$\underline{\lambda = 0}$

$$\frac{v_{ttt}}{v} = 0$$

$$v_{ttt} = 0$$

Integrate 3 times.

$$v = At^2 + Bt + C$$

$$\frac{9w_{zz}}{w} = 0$$

$$w_{zz} = 0$$

Integrate 2 times.

$$w = Dz + E$$

$$u = vw = (At^2 + Bt + C)(Dz + E)$$

$$= \boxed{C_1 t^2 z + C_2 t z + C_3 z + C_4 t^2 + C_5 t + C_6}$$

4.) *[15 points]* Solve the boundary value problem below on the interval $0 \le x \le 3$.

$u_{tt} = 25u_{xx}$
$u(0,t) = 0, \quad u(3,t) = 0 \qquad$ for all $t \ge 0$
$u(x,0) = 0$ for $0 \le x \le 3$
$\frac{\partial u}{\partial t}(x,0) = \begin{cases} -2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 \le x < 3 \end{cases} \leftarrow g(x)$

Wave Equation
$a = 5$
$L = 3$
$f(x) = 0$

$$f(x) = 0 \Rightarrow A_n = 0$$

$$B_n = \frac{2}{n\pi 5} \int_0^3 f(x) \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{5n\pi} \int_0^1 -2 \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= -\frac{4}{5n\pi} \left[-\frac{3}{n\pi} \cos\left(\frac{n\pi x}{3}\right)\right]_0^1$$

$$= \frac{12}{5n^2\pi^2} \left[\cos\left(\frac{n\pi}{3}\right) - \cancel{\cos(0)}^{\,1}\right]$$

$$= \frac{12}{5n^2\pi^2} \left[\cos\left(\frac{n\pi}{3}\right) - 1\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{12}{5n^2\pi^2} \left[\cos\left(\frac{n\pi}{3}\right) - 1\right] \sin\left(\frac{5n\pi t}{3}\right) \sin\left(\frac{n\pi x}{3}\right)}$$