4 Convolution

Solutions to Recommended Problems

S4.1

The given input in Figure S4.1-1 can be expressed as linear combinations of $x_1[n]$, $x_2[n]$, $x_3[n]$.

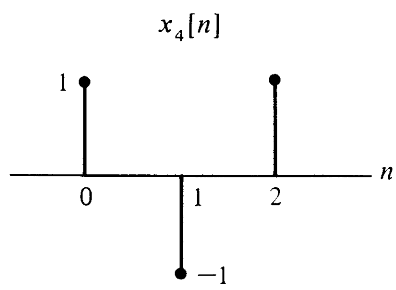

Figure S4.1-1

(a) $x_4[n] = 2x_1[n] - 2x_2[n] + x_3[n]$

(b) Using superposition, $y_4[n] = 2y_1[n] - 2y_2[n] + y_3[n]$, shown in Figure S4.1-2.

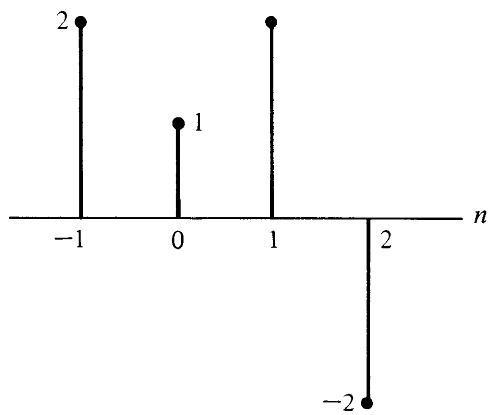

Figure S4.1-2

(c) The system is not time-invariant because an input $x_1[n] + x_1[n - 1]$ does not produce an output $y_1[n] + y_1[n - 1]$. The input $x_1[n] + x_1[n - 1]$ is $x_1[n] + x_1[n - 1] = x_2[n]$ (shown in Figure S4.1-3), which we are told produces $y_2[n]$. Since $y_2[n] \neq y_1[n] + y_1[n - 1]$, this system is not time-invariant.

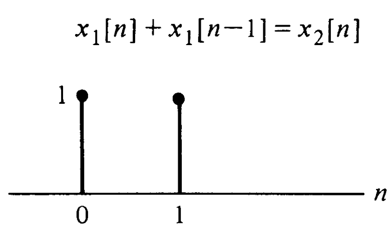

Figure S4.1-3

S4.2

The required convolutions are most easily done graphically by reflecting $x[n]$ about the origin and shifting the reflected signal.

(a) By reflecting $x[n]$ about the origin, shifting, multiplying, and adding, we see that $y[n] = x[n] * h[n]$ is as shown in Figure S4.2-1.

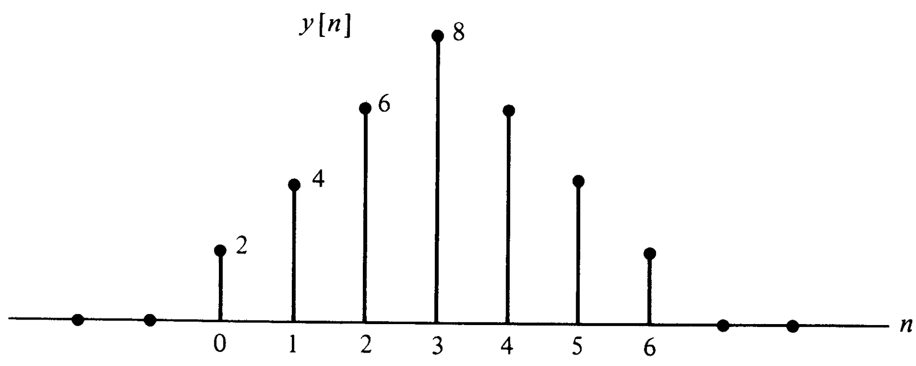

Figure S4.2-1

(b) By reflecting $x[n]$ about the origin, shifting, multiplying, and adding, we see that $y[n] = x[n] * h[n]$ is as shown in Figure S4.2-2.

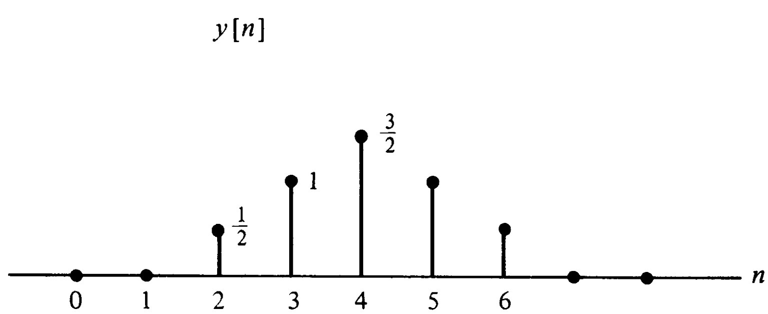

Figure S4.2-2

Notice that $y[n]$ is a shifted and scaled version of $h[n]$.

S4.3

(a) It is easiest to perform this convolution graphically. The result is shown in Figure S4.3-1.

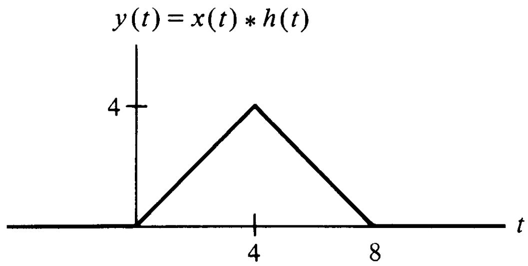

Figure S4.3-1

(b) The convolution can be evaluated by using the convolution formula. The limits can be verified by graphically visualizing the convolution.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h\,(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-(\tau-1)}u(\tau-1)u(t-\tau+1)d\tau \\ &= \begin{cases} \int_{1}^{t+1} e^{-(\tau-1)}d\tau, & t > 0, \\ 0, & t < 0, \end{cases} \end{aligned}$$

Let $\tau' = \tau - 1$. Then

$$y(t) = \begin{cases} \int_{0}^{t} e^{-\tau'}d\tau' \\ 0 \end{cases} = \begin{cases} 1 - e^{-t}, & t > 0, \\ 0, & t < 0 \end{cases}$$

(c) The convolution can be evaluated graphically or by using the convolution formula.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-2)\,d\tau = x(t-2)$$

So $y(t)$ is a shifted version of $x(t)$.

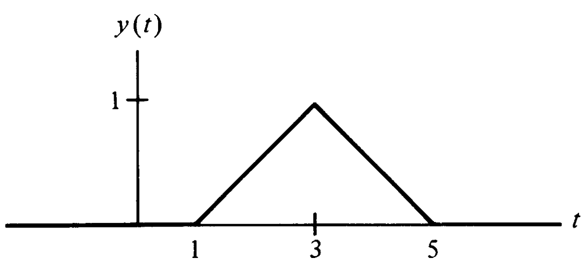

Figure S4.3-2

S4.4

(a) Since $y[n] = \Sigma_{m=-\infty}^{\infty} x[m]h[n - m]$,

$$y[n] = \sum_{m=-\infty}^{\infty} \delta[m - n_0]h[n - m] = h[n - n_0]$$

We note that this is merely a shifted version of $h[n]$.

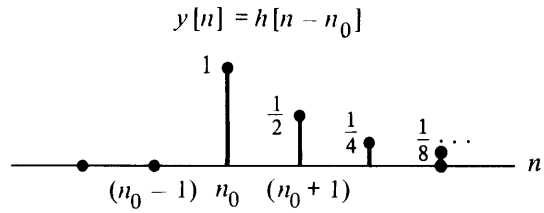

Figure S4.4-1

(b) $y[n] = \Sigma_{m=-\infty}^{\infty} (\frac{1}{2})^m u[m]u[n - m]$

For $n > 0$: $\quad y[n] = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m = \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right),$

$\quad y[n] = 2 - (\frac{1}{2})^n$

For $n < 0$: $\quad y[n] = 0$

Here the identity

$$\sum_{m=0}^{N-1} a^m = \frac{1 - a^N}{1 - a}$$

has been used.

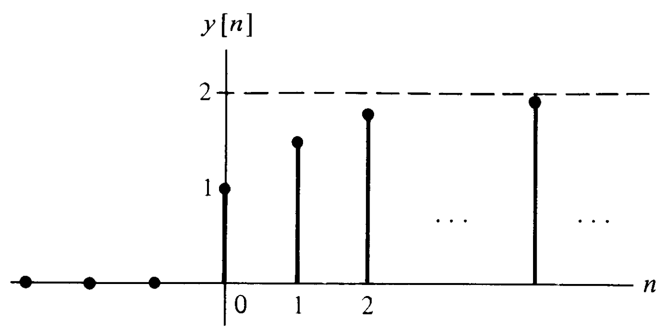

Figure S4.4-2

(c) Reversing the role of the system and the input has no effect on the output because

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] = \sum_{m=-\infty}^{\infty} h[m]x[n - m]$$

The output and sketch are identical to those in part (b).

S4.5

(a) (i) Using the formula for convolution, we have

$$y_1(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)/2}u(t-\tau)\,d\tau$$
$$= \int_{0}^{t} e^{-(t-\tau)/2}\,d\tau, \qquad t > 0,$$
$$= 2e^{-(t-\tau)/2}\Big|_0^t = 2(1-e^{-t/2}), \qquad t > 0,$$
$$y(t) = 0, \qquad t < 0$$

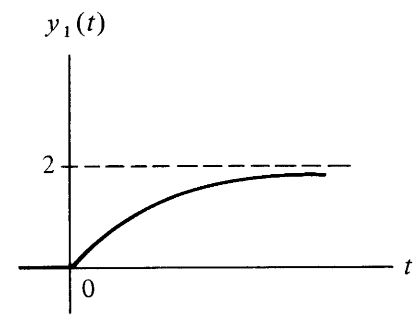

Figure S4.5-1

(ii) Using the formula for convolution, we have

$$y_2(t) = \int_0^t 2e^{-(t-\tau)/2}\,d\tau, \qquad 3 \geq t \geq 0,$$
$$= 4(1-e^{-t/2}), \qquad 3 \geq t \geq 0,$$
$$y_2(t) = \int_0^3 2e^{-(t-\tau)/2}\,d\tau, \qquad t \geq 3,$$
$$= 4e^{-(t-\tau)/2}\Big|_0^3 = 4(e^{-(t-3)/2} - e^{-t/2})$$
$$= 4e^{-t/2}(e^{3/2}-1), \qquad t \geq 3,$$
$$y_2(t) = 0, \qquad t \leq 0$$

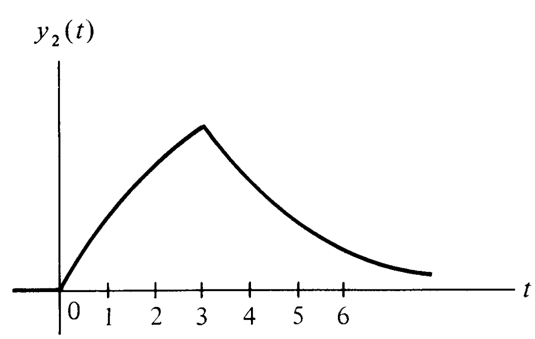

Figure S4.5-2

(b) Since $x_2(t) = 2[x_1(t) - x_1(t-3)]$ and the system is linear and time-invariant, $y_2(t) = 2[y_1(t) - y_1(t-3)]$.

$$
\begin{aligned}
\text{For } 0 \le t \le 3: \quad & y_2(t) = 2y_1(t) = 4(1 - e^{-t/2}) \\
\text{For } 3 \le t: \quad & y_2(t) = 2y_1(t) - 2y_1(t-3) \\
& = 4(1 - e^{-t/2}) - 4(1 - e^{-(t-3)/2}) \\
& = 4e^{-t/2}[e^{3/2} - 1] \\
\text{For } t < 0: \quad & y_2(t) = 0
\end{aligned}
$$

We see that this result is identical to the result obtained in part (a)(ii).

Solutions to Optional Problems

S4.6

(a)

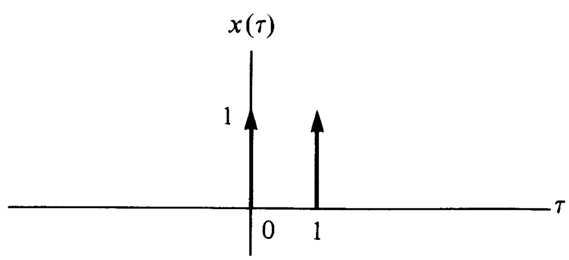

Figure S4.6-1

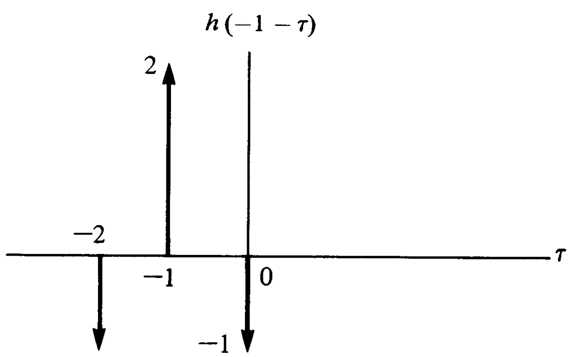

Figure S4.6-2

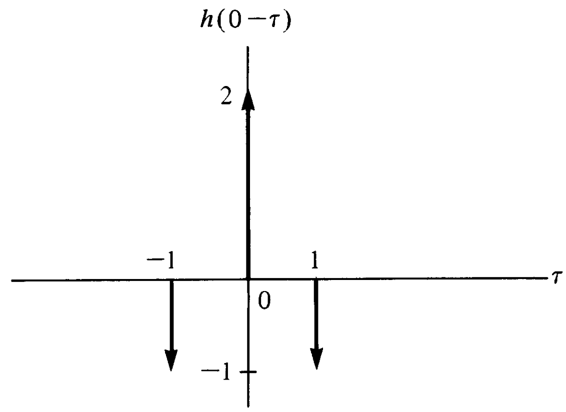

Figure S4.6-3

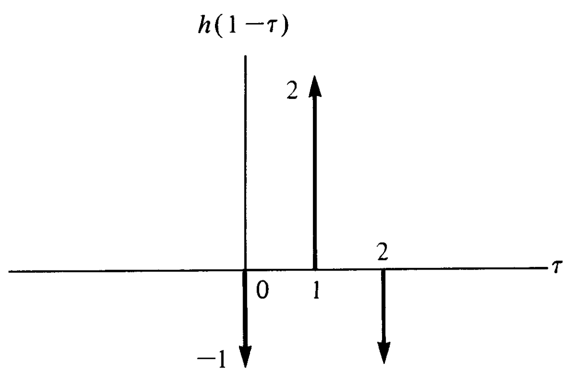

Figure S4.6-4

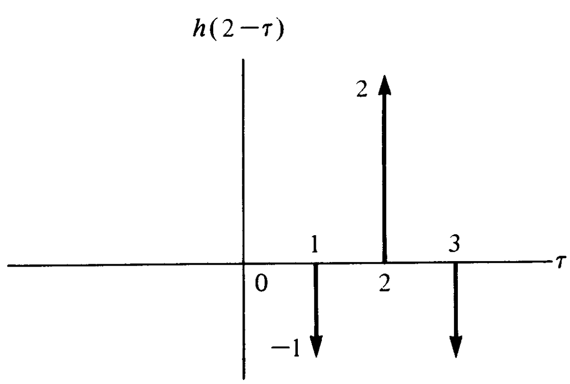

Figure S4.6-5

Using these curves, we see that since $y(t) = x(t) * h(t)$, $y(t)$ is as shown in Figure S4.6-6.

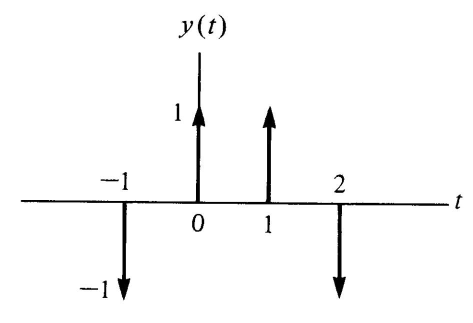

Figure S4.6-6

(b) Consider $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)\,d\tau$.

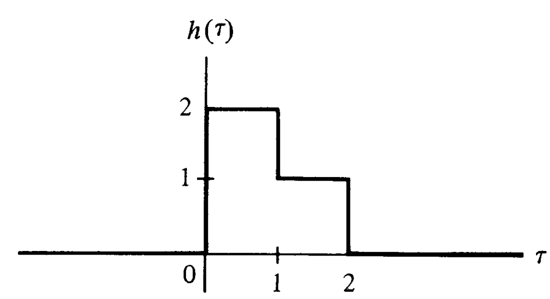

Figure S4.6-7

For $0 < t < 1$, only one impulse contributes.

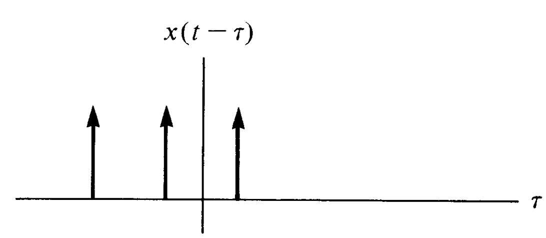

Figure S4.6-8

For $1 < t < 2$, two impulses contribute.

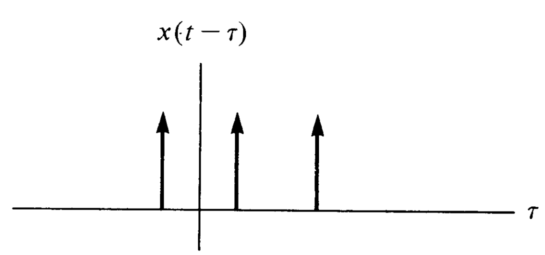

Figure S4.6-9

For $2 < t < 3$, two impulses contribute.

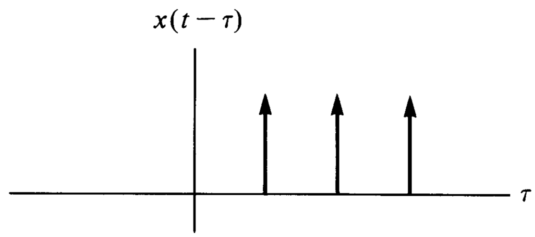

Figure S4.6-10

For $3 < t < 4$, one impulse contributes.

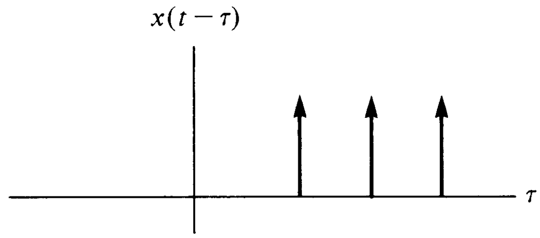

Figure S4.6-11

For $t < 0$ or $t > 4$, there is no contribution, so $y(t)$ is as shown in Figure S4.6-12.

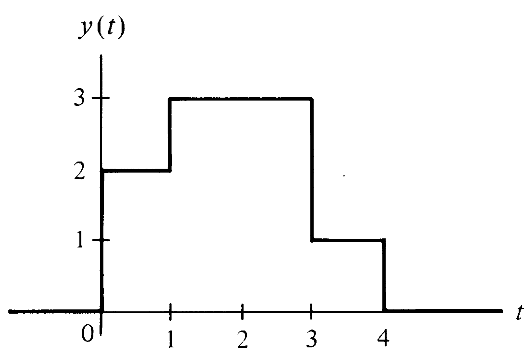

Figure S4.6-12

S4.7

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] \\ &= \sum_{m=-\infty}^{\infty} \alpha^{n-m}\, u[n-m]\beta^m u[m] \\ &= \sum_{m=0}^{n} \alpha^{n-m}\beta^m, \qquad n > 0, \end{aligned}$$

$$y[n] = \alpha^n \sum_{m=0}^{n} \left(\frac{\beta}{\alpha}\right)^m = \alpha^n \left[\frac{1-(\beta/\alpha)^{n+1}}{1-(\beta/\alpha)}\right]$$
$$= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}, \qquad n \geq 0,$$
$$y[n] = 0, \qquad n < 0$$

S4.8

(a) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ is a series of impulses spaced T apart.

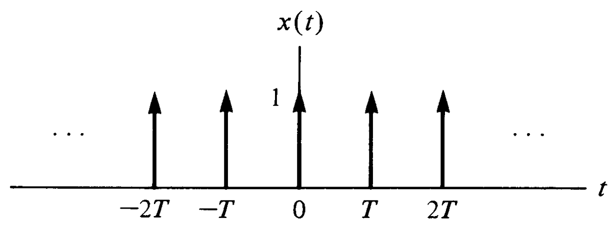

Figure S4.8-1

(b) Using the result $x(t) * \delta(t - t_0) = x(t_0)$, we have

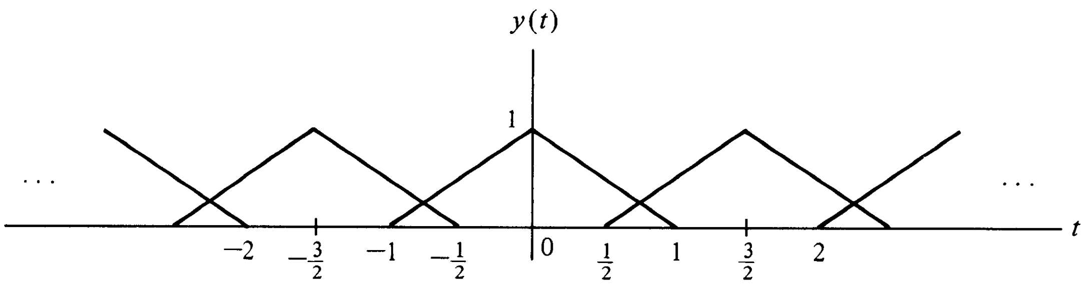

Figure S4.8-2

So $y(t) = x(t) * h(t)$ is as shown in Figure S4.8-3.

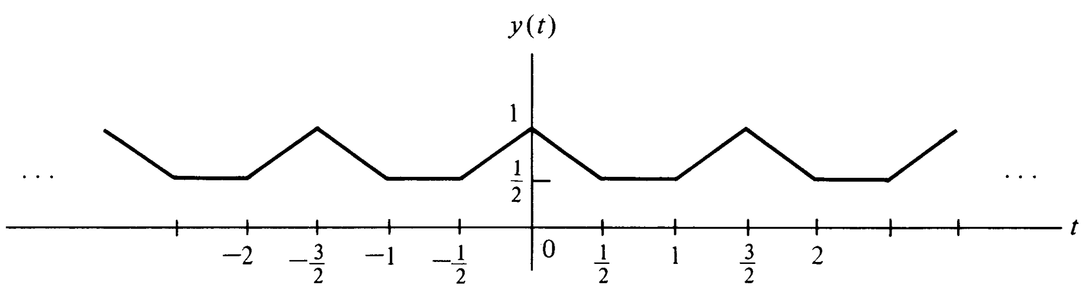

Figure S4.8-3

S4.9

(a) False. Counterexample: Let $g[n] = \delta[n]$. Then

$$x[n] * \{h[n]g[n]\} = x[n] \cdot h[0],$$
$$\{x[n] * h[n]\}g[n] = \delta[n] \cdot [x[n] * h[n]] \Big|_{n=0}$$

and $x[n]$ may in general differ from $\delta[n]$.

(b) True.

$$y(2t) = \int_{-\infty}^{\infty} x(2t - \tau)h(\tau)d\tau$$

Let $\tau' = \tau/2$. Then

$$\begin{aligned} y(2t) &= \int_{-\infty}^{\infty} x(2t - 2\tau')h(2\tau')2\, d\tau' \\ &= 2x(2t) * h(2t) \end{aligned}$$

(c) True.

$$\begin{aligned} y(t) &= x(t) * h(t) \\ y(-t) &= x(-t) * h(-t) \\ &= \int_{-\infty}^{\infty} x(-t + \tau)h(-\tau)\, d\tau = \int_{-\infty}^{\infty} [-x(t - \tau)][-h(\tau)]\, d\tau \\ &= \int_{-\infty}^{\infty} x(t - \tau)h(\tau)\, d\tau \qquad \text{since } x(\cdot) \text{ and } h(\cdot) \text{ are odd functions} \\ &= y(t) \end{aligned}$$

Hence $y(t) = y(-t)$, and $y(t)$ is even.

(d) False. Let

$$\begin{aligned} x(t) &= \delta(t - 1), \\ h(t) &= \delta(t + 1), \\ y(t) &= \delta(t), \qquad Ev\{\, y\,(t)\} = \delta(t) \end{aligned}$$

Then

$$\begin{aligned} x(t) * Ev\{h(t)\} &= \delta(t - 1) * \tfrac{1}{2}[\delta(t + 1) + \delta(t - 1)] \\ &= \tfrac{1}{2}[\delta(t) + \delta(t - 2)], \\ Ev\{x(t)\} * h(t) &= \tfrac{1}{2}[\delta(t - 1) + \delta(t + 1)] * \delta(t + 1) \\ &= \tfrac{1}{2}[\delta(t) + \delta(t + 2)] \end{aligned}$$

But since $\frac{1}{2}[\delta(t - 2) + \delta(t + 2)] \neq 0$,

$$Ev\{y(t)\} \neq x(t) * Ev\{h(t)\} + Ev\{x(t)\} * h(t)$$

S4.10

(a)

$$\begin{aligned} \tilde{y}(t) &= \int_0^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t - \tau)\, d\tau, \\ \tilde{y}(t + T_0) &= \int_0^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t + T_0 - \tau)\, d\tau \\ &= \int_0^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t - \tau)\, d\tau = \tilde{y}(t) \end{aligned}$$

(b)

$$\tilde{y}_a(t) = \int_a^{a+T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau,$$

$$a = kT_0 + b, \qquad \text{where } 0 \le b \le T_0,$$

$$\tilde{y}_a(t) = \int_{kT_0+b}^{(k+1)T_0+b} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau,$$

$$\tilde{y}_a(t) = \int_b^{T_0+b} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau, \qquad \tau' = \tau - b$$

$$= \int_b^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau + \int_{T_0}^{T_0+b} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau$$

$$= \int_b^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau + \int_0^{b} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau$$

$$= \int_0^{T_0} \tilde{x}_1(\tau)\tilde{x}_2(t-\tau)\,d\tau = \tilde{y}(t)$$

(c) For $0 \le t \le \frac{1}{2}$:

$$\tilde{y}(t) = \int_0^t e^{-\tau} d\tau + \int_{1/2+t}^{1} e^{-\tau} d\tau$$

$$= \left(-e^{-\tau}\Big|_0^t\right) + \left(-e^{-\tau}\Big|_{1/2+t}^{1}\right),$$

$$\tilde{y}(t) = 1 - e^{-t} + e^{-(t+1/2)} - e^{-1} = 1 - e^{-1} + (e^{-1/2} - 1)e^{-t}$$

For $\frac{1}{2} \le t \le 1$:

$$\tilde{y}(t) = \int_{t-1/2}^{t} e^{-\tau}\,d\tau = e^{-(t-1/2)} - e^{-t}$$

$$= (e^{1/2} - 1)e^{-t}$$

(d) Performing the periodic convolution graphically, we obtain the solution as shown in Figure S4.10-1.

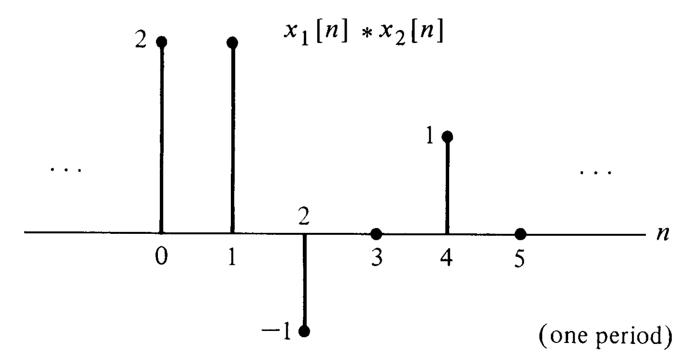

Figure S4.10-1

(e)

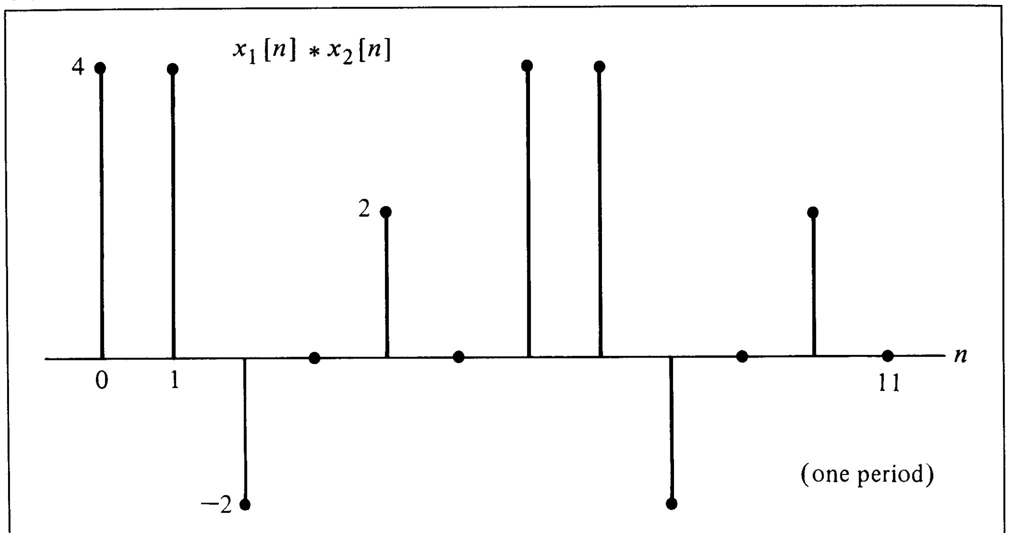

Figure S4.10-2

S4.11

(a) Since $y(t) = x(t) * h(t)$ and $x(t) = g(t) * y(t)$, then $g(t) * h(t) = \delta(t)$. But

$$g(t) * h(t) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} g_k \delta(t - \tau - kT) \sum_{l=0}^{\infty} h_l \delta(\tau - lT)\, d\tau$$
$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} g_k h_l \delta(t - (l + k)T)$$

Let $n = l + k$. Then $l = n - k$ and

$$g(t) * h(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} g_k h_{n-k} \right) \delta(t - nT)$$

So

$$\sum_{k=0}^{n} g_k h_{n-k} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0 \end{cases}$$

Therefore,

$$g_0 = 1/h_0,$$
$$g_1 = -h_1/h_0^2,$$
$$g_2 = \frac{-1}{h_0}\left(\frac{-h_1^2}{h_0^2} + \frac{h_2}{h_0} \right) \cdots$$

(b) We are given that $h_0 = 1$, $h_1 = \frac{1}{2}$, $h_i = 0$. So

$$g_0 = 1,$$
$$g_1 = -\tfrac{1}{2},$$
$$g_2 = +(\tfrac{1}{2})^2,$$
$$g_3 = -(\tfrac{1}{2})^3 \ldots$$

Therefore,

$$g(t) = \sum_{k=0}^{\infty} (-\tfrac{1}{2})^k \delta(t-kT)$$

(c) (i) Each impulse is delayed by T and scaled by α, so

$$h(t) = \sum_{k=0}^{\infty} \alpha^k (t-kT)$$

(ii) If $0 < \alpha < 1$, a bounded input produces a bounded output because

$$y(t) = x(t) * h(t),$$

$$|y(t)| < \sum_{k=0}^{\infty} \alpha^k \left| \int_{-\infty}^{\infty} \delta(\tau - kT) x(t-\tau)\, d\tau \right|$$

$$< \sum_{k=0}^{\infty} \alpha^k \int_{-\infty}^{\infty} \delta(\tau - kT) |x(t-\tau)|\, d\tau$$

Let $M = \max|x(t)|$. Then

$$|y(t)| < M \sum_{k=0}^{\infty} \alpha^k = M \frac{1}{1-\alpha}, \qquad |\alpha| < 1$$

If $\alpha > 1$, a bounded input will no longer produce a bounded output. For example, consider $x(t) = u(t)$. Then

$$y(t) = \sum_{k=0}^{\infty} \alpha^k \int_{-\infty}^{t} \delta(\tau - kT)\, d\tau$$

Since $\int_{-\infty}^{t} \delta(\tau - kT)\, d\tau = u(t-kT)$,

$$y(t) = \sum_{k=0}^{\infty} \alpha^k u(t-kT)$$

Consider, for example, t equal to (or slightly greater than) NT:

$$y(NT) = \sum_{k=0}^{N} \alpha^k$$

If $\alpha > 1$, this grows without bound as N (or t) increases.

(iii) Now we want the inverse system. Recognize that we have actually solved this in part (b) of this problem.

$$g_1 = 1,$$
$$g_2 = -\alpha$$
$$g_i = 0, \qquad i \neq 0, 1$$

So the system appears as in Figure S4.11.

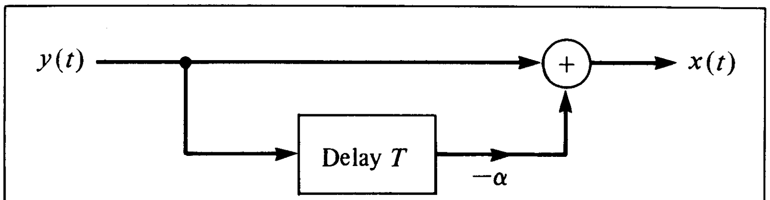

Figure S4.11

(d) If $x[n] = \delta[n]$, then $y[n] = h[n]$. If

$$x[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-N],$$

then

$$y[n] = \tfrac{1}{2}h[n] + \tfrac{1}{2}h[n],$$
$$y[n] = h[n]$$

S4.12

(a) $\delta[n] = \phi[n] - \tfrac{1}{2}\phi[n-1]$,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = \sum_{k=-\infty}^{\infty} x[k](\phi[n-k] - \tfrac{1}{2}\phi[n-k-1]),$$

$$x[n] = \sum_{k=-\infty}^{\infty} (x[k] - \tfrac{1}{2}x[k-1])\phi[n-k]$$

So $a_k = x[k] - \tfrac{1}{2}x[k-1]$.

(b) If $r[n]$ is the response to $\phi[n]$, we can use superposition to note that if

$$x[n] = \sum_{k=-\infty}^{\infty} a_k\phi[n-k],$$

then

$$y[n] = \sum_{k=-\infty}^{\infty} a_k r[n-k]$$

and, from part (a),

$$y[n] = \sum_{k=-\infty}^{\infty} (x[k] - \tfrac{1}{2}x[k-1])r[n-k]$$

(c) $y[n] = \psi[n] * x[n] * r[n]$ when

$$\psi[n] = \delta[n] - \tfrac{1}{2}\delta[n-1]$$

and, from above,

$$\delta[n] = \phi[n] - \tfrac{1}{2}\phi[n-1]$$

So

$$\psi[n] = \phi[n] - \tfrac{1}{2}\phi[n-1] - \tfrac{1}{2}(\phi[n-1] - \tfrac{1}{2}\phi[n-2]),$$
$$\psi[n] = \phi[n] - \phi[n-1] + \tfrac{1}{4}\phi[n-2]$$

(d)

$$\phi[n] \rightarrow r[n],$$
$$\phi[n-1] \rightarrow r[n-1],$$
$$\delta[n] = \phi[n] - \tfrac{1}{2}\phi[n-1] \rightarrow r[n] - \tfrac{1}{2}r[n-1]$$

So

$$h[n] = r[n] - \tfrac{1}{2}r[n-1],$$

where $h[n]$ is the impulse response. Also, from part (c) we know that

$$y[n] = \psi[n] * x[n] * r[n]$$

and if $x[n] = \phi[n]$ produces $r[n]$, it is apparent that $\phi[n] * \psi[n] = \delta[n]$.