Math 335
Chapter 12: Fourier Series

The *Fourier series* expansion of f(x) on the interval (-L,L) is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where the coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,.$$

Dirichlet's Theorem If f(x) is bounded and has a finite number of discontinuities and extrema on the interval (-L,L), then the Fourier series will converge at every point in (-L,L) to the value

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right).$$

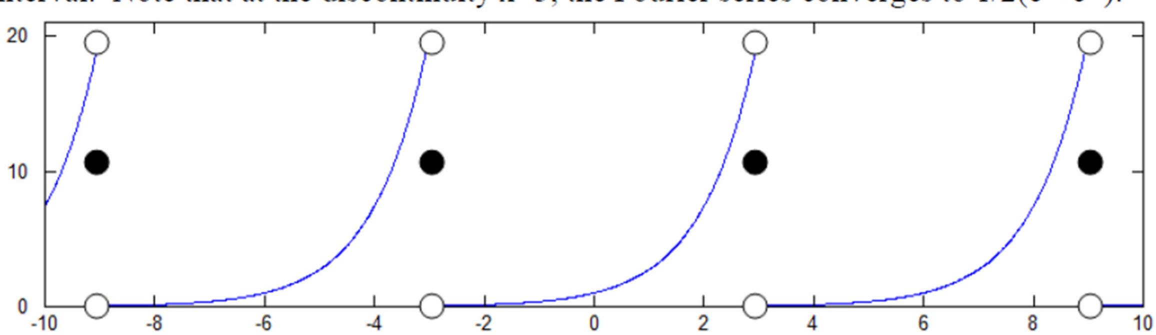
Note this theorem applies even at a discontinuity (jump), where the Fourier series converges to the average of the left and right one-side limits.

Interesting Properties of Fourier Series

- Odd/Even: If the function f(x) is odd, then we should get only sine terms and all $a_n=0$. If f(x) is even, then we get only cosine terms and all $b_n=0$.
- Average Value: The first term $\frac{1}{2}a_0$ is the average value of the function over the interval [-L,L].
- Periodicity: The Fourier series is 2L-periodic.
- Approximation: The truncated finite Fourier series is an approximation of the function f(x) on (-L,L). The more terms of the Fourier series we use, the better the approximation will get.
- Continuity: Unlike power series, Fourier series can represent a function f(x) that contains a discontinuity.

The Graph of a Fourier Series

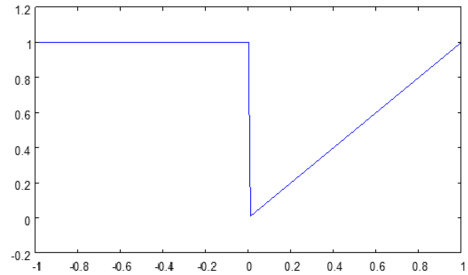
You should have intuition about what the graph of a Fourier series looks like. For example, suppose we find the Fourier series for $f(x) = e^x$ on the interval (-3,3). The graph will look the graph of e^x on (-3,3) and then repeated periodically outside that interval. Note that at the discontinuity x=3, the Fourier series converges to $1/2(e^3+e^{-3})$.



<u>Example</u>: The function

$$f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \le x < 1 \end{cases}$$

is shown at right.



We proved in lecture that this function has the Fourier series representation on (-1,1) as

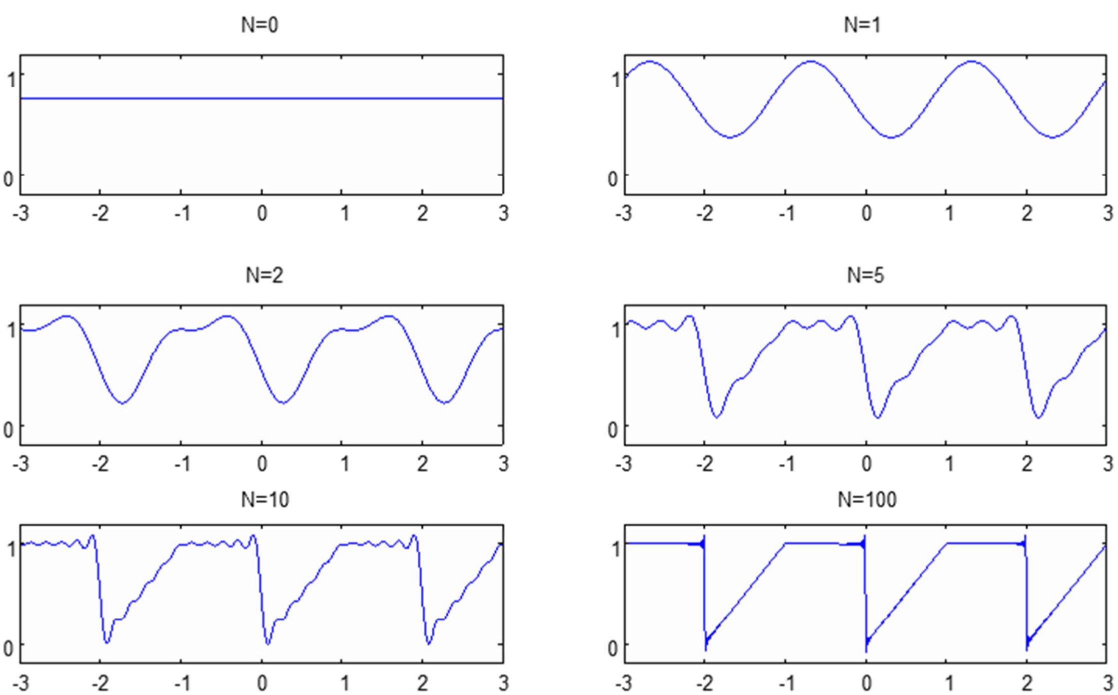
$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2\ \pi^2} \cos n\pi x \ - \ \frac{1}{n\pi} \sin\ n\pi x \right]$$

In practice, we cannot add up infinitely many terms. So we use a partial sum that adds up the terms until some value n=N. This will give us an approximation of $f(x)$.

$$f(x) \approx \frac{3}{4} + \sum_{n=1}^{N} \left[\frac{(-1)^n - 1}{n^2\ \pi^2} \cos n\pi x \ - \ \frac{1}{n\pi} \sin\ n\pi x \right]$$

As we add more terms, the approximation should more closely resemble $f(x)$ on (-1,1).

Outside the interval (-1,1), the Fourier series will simply repeat the graph.



Near a jump discontinuity, the Fourier series starts to oscillate. This behavior is called the <u>Gibbs phenomenon</u>. This phenomenon is responsible for "ringing" artifacts in signals or images that have been compressed.