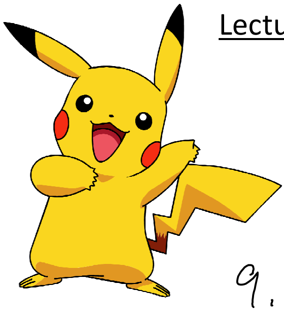


Lecture 2: Vectors

Pikachu's Goals for the Day
- Review how to draw and understand vectors
- Review vector arithmetic: addition, scalar multiplication, dot & cross product
- Define a vector function and its application to motion on a curve

9.1 Vector Calculus

Def A vector is a list of numbers.

$$\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$$

We concentrate on 3D vectors.

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

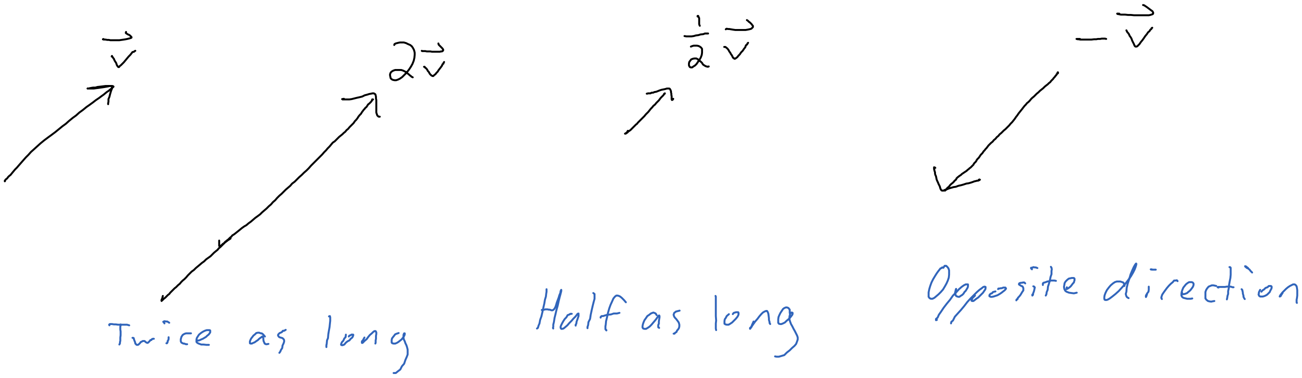
$$\vec{v} = v_1 i + v_2 j + v_3 k$$

$$i = \langle 1, 0, 0 \rangle \qquad j = \langle 0, 1, 0 \rangle \qquad k = \langle 0, 0, 1 \rangle$$

Vector Arithmetic

1. Scalar multiplication

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$$

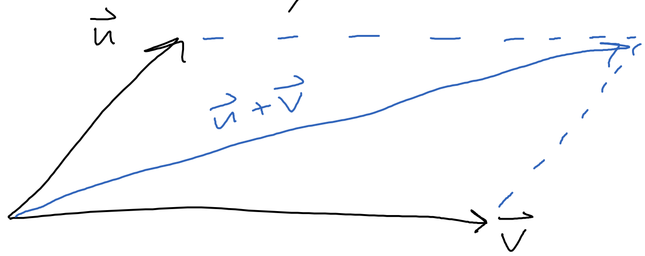


Multiplying by a constant scales the length of the vector.

(2) Vector addition

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

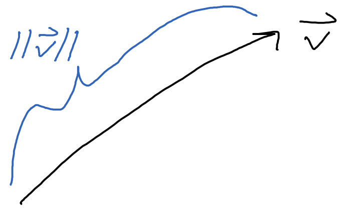
Note: We can only add vectors of same #components.



Parallelogram Law

(3) Magnitude (Length)

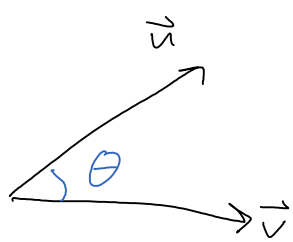
$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$



(4) Dot Product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Note: The dot product gives a scalar.



The angle between vectors $\vec{u}$ and $\vec{v}$ is given by

$$\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \; \|\vec{v}\|}}$$

Ex Find the angle between $\vec{a} = \langle 2, 1, 0 \rangle$ and $\vec{b} = \langle -1, 4, 6 \rangle$.

$$\vec{a} \cdot \vec{b} = (2)(-1) + (1)(4) + (0)(6) = -2 + 4 + 0 = 2$$

$$\|\vec{a}\| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$$

$$\|\vec{b}\| = \sqrt{(-1)^2 + 4^2 + 6^2} = \sqrt{53}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} = \frac{2}{\sqrt{5}\sqrt{53}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}\sqrt{53}}\right)}$$

$$\boxed{\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0}$$

5 Cross Product

The cross product is only for 3D vectors.

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

← Determinant of a 3×3 matrix

$$= +i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

2×2 Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex Compute $\vec{a}\times\vec{b}$ for $\vec{a} = \langle 2,1,0\rangle$ and $\vec{b} = \langle -1,4,6\rangle$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -1 & 4 & 6 \end{vmatrix}$$

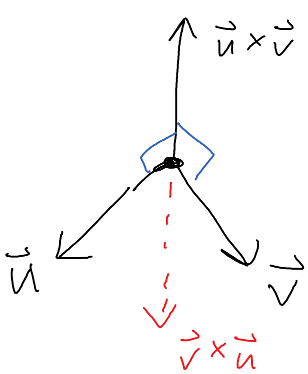
$$= i\begin{vmatrix} 1 & 0 \\ 4 & 6 \end{vmatrix} - j\begin{vmatrix} 2 & 0 \\ -1 & 6 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix}$$

$$= i(6-0) - j(12-0) + k(8+1)$$

$$= 6i - 12j + 9k$$

$$= \boxed{\langle 6, -12, 9\rangle}$$

The cross product $\vec{u}\times\vec{v}$ is perpendicular to both vectors $\vec{u}$ and $\vec{v}$.



Right-hand rule

$$(\vec{u}\times\vec{v})\cdot\vec{u} = 0$$

$$(\vec{u}\times\vec{v})\cdot\vec{v} = 0$$

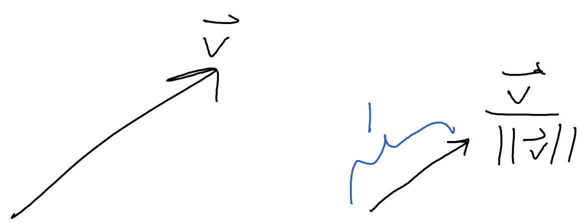
$$\boxed{\vec{u} \times \vec{v} = \vec{0} \quad \Longleftrightarrow \quad \vec{u} \parallel \vec{v}}$$

6 Unit Vector

A <u>unit vector</u> has length one: $\|\vec{u}\| = 1$.

To make a vector into a unit vector by dividing by its magnitude.

The unit vector version of a vector is called the "direction" of the vector.



<u>Ex</u> Find the direction of $\vec{w} = \langle 3, 2, -1 \rangle$.

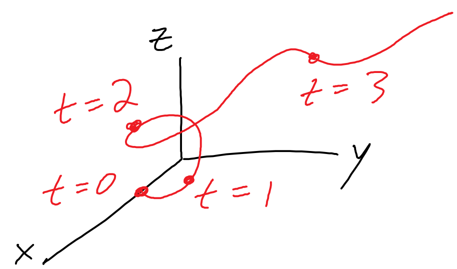
$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{\langle 3, 2, -1 \rangle}{\sqrt{3^2 + 2^2 + (-1)^2}} = \frac{\langle 3, 2, -1 \rangle}{\sqrt{14}} = \boxed{\left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}} \right\rangle}$$

Def A <u>vector function</u> is a vector with each component being a function of a common variable.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Applications

① Position $\vec{r}(t)$



② Velocity $\vec{v}(t)$

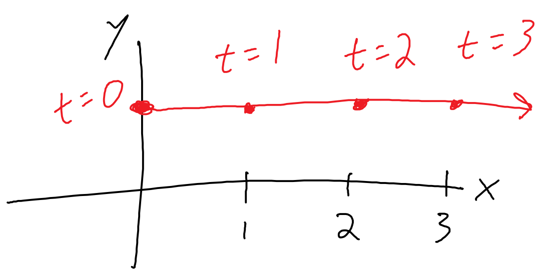
$\|\vec{v}\|$ = speed

③ Force over time $\vec{F}(t)$

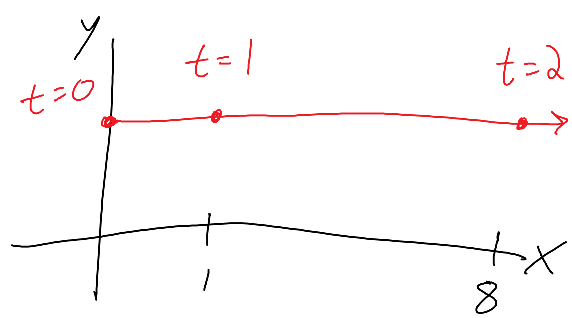
$\|\vec{F}\|$ = total force

2D Vector Functions

$\vec{r}(t) = \langle t, 2 \rangle$

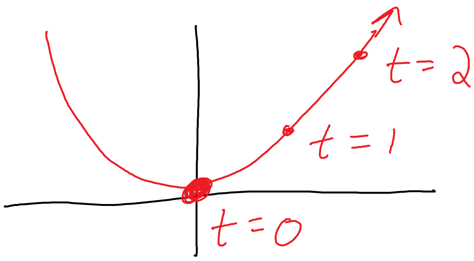


$\vec{r}(t) = \langle t^3, 2 \rangle$



Accelerating

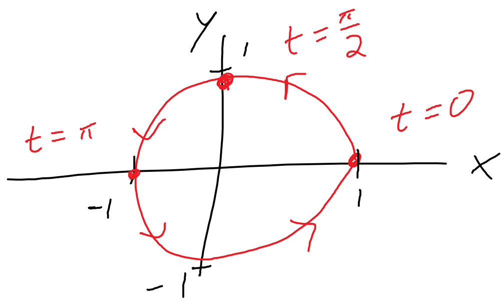
$\vec{r}(t) = \langle t, t^2 \rangle$



To write $y = f(x)$ as a vector function

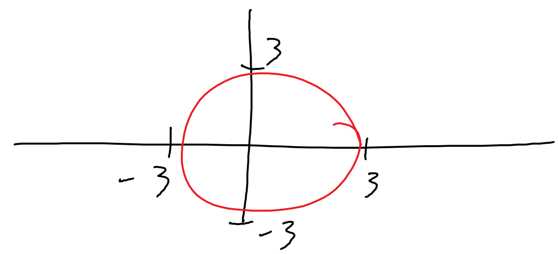
$$\vec{r}(t) = \langle t, f(t) \rangle$$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

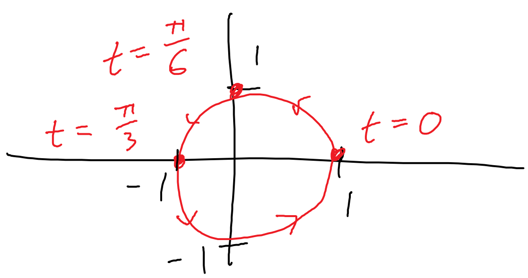


Circle of radius 3

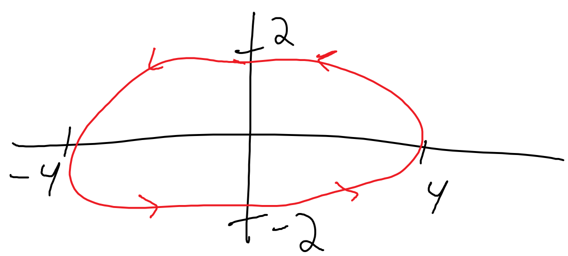
$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$



$\vec{r}(t) = \langle \cos 3t, \sin 3t \rangle$

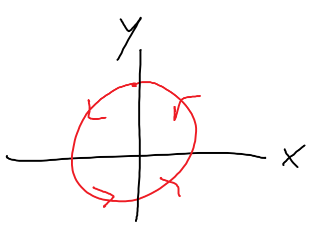


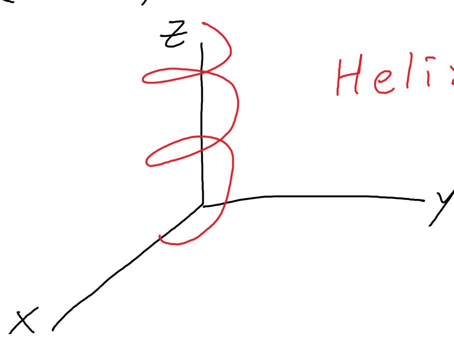
$\vec{r}(t) = \langle 4\cos t, 2\sin t \rangle$



3D Vectors

Ex Draw $\quad \vec{r}(t) = \langle \cos t, \sin t, t \rangle$





Helix

Ex How do you make a tornado?

$$\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t \rangle$$

A line in 3D that starts at a point P and is parallel to vector $\vec{v}$ is given by

$$\vec{r}(t) = P + t\,\vec{v}$$

Application Motion

Position $\vec{r}(t)$

Velocity $\vec{v}(t) = \vec{r}\,'(t)$

Acceleration $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

Ex Pikachu starts at $(3,2,0)$ at time $t=0$ and runs with velocity

$$\vec{v}(t) = \langle 3t,\ e^t,\ t^2+1 \rangle.$$

Find Pikachu's position.

$$\vec{r}(t) = \int \vec{v}(t)\, dt$$

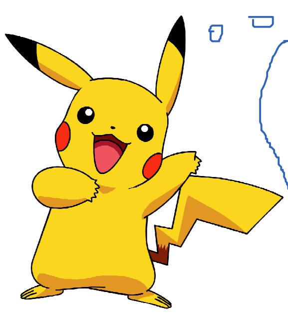
$$= \left\langle \int 3t\, dt,\ \int e^t\, dt,\ \int t^2+1\ dt \right\rangle$$

$$= \left\langle \frac{3}{2}t^2 + C_1,\ e^t + C_2,\ \frac{1}{3}t^3 + t + C_3 \right\rangle$$

$$\vec{r}(0) = (3, 2, 0) = \langle C_1,\ 1 + C_2,\ C_3 \rangle$$

$$\Rightarrow C_1 = 3,\ C_2 = 1,\ C_3 = 0$$

$$\boxed{\vec{r}(t) = \left\langle \frac{3}{2}t^2 + 3,\ e^t + 1,\ \frac{1}{3}t^3 + t \right\rangle}$$



A common mistake is to assume that the starting point tells us the values of the constants directly.

We actually have to plug t=0 into the vector function and then solve 3 equations separately to find the values of C1, C2, C3.