Math 335, Fall 2013 Final Exam Key

1.) *[10 points]* The Gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Prove $\Gamma(x+1) = x\,\Gamma(x)$.

Integration by Parts

$u = t^x$ $\qquad v = -e^{-t}$

$du = x t^{x-1} dt$ $\qquad dv = e^{-t} dt$

$$\int u\, dv = uv - \int v\, du$$

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -t^x e^{-t}\Big|_0^\infty - \int_0^\infty (-e^{-t})\, x t^{x-1} dt$$

$$= -\underbrace{\infty^x e^{-\infty}}_{0} + \underbrace{0^x e^{-0}}_{0} + x \int_0^\infty t^{x-1} e^{-t}\, dt$$

$$= x \int_0^\infty t^{x-1} e^{-t}\, dt$$

$$= x\,\Gamma(x)$$

$\square$

2.) *[15 points]* Find the first 5 terms (through x^4) of the series solution about x=0 of the ODE

$$y'' + xy' - 2y = 0$$

Write your coefficients in the blanks below in terms of a_0 and a_1.

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$y''+xy'-2y=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

shift $n-2=k$, $n=k+2$, $n-1=k+1$, $n=2 \to k=0$

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$(2)(1)a_2 x^0 - 2a_0 x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + na_n - 2a_n\right]x^n = 0$$

$$2a_2 - 2a_0 = 0 \Rightarrow a_2 = a_0$$

$\underline{n=1}$ $(3)(2)a_3 + (1)a_1 - 2a_1 = 0$

$$6a_3 - a_1 = 0$$

$$a_3 = \frac{1}{6}a_1$$

$\underline{n=2}$ $(4)(3)a_4 + 2a_2 - 2a_2 = 0$

$$12a_4 = 0$$

$$a_4 = 0$$

$y = a_0 + a_1 x +$ $\underline{\quad a_0 \quad}$ $x^2 +$ $\underline{\quad \frac{1}{6}a_1 \quad}$ $x^3 +$ $\underline{\quad 0 \quad}$ $x^4 + \cdots$

3.) *[15 points]* Note x=0 is a regular singular point of the ODE

$$3xy'' + (2-x)y' - y = 0$$

Using the Method of Frobenius about x=0, find the indicial roots of the ODE and the general recurrence relation in terms of n and r. *(You do not need to find the Frobenius series solutions. Use the back of this page if you need more room for your work.)*

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$3xy'' + (2-x)y' - y = 0$$

$$3x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (2-x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}3(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$
$$- \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Shift $n+r-1 = k+r$, $n+r = k+r+1$, $n = k+1$, $n=0 \rightarrow k=-1$

$$\sum_{k=-1}^{\infty} 3(k+r+1)(k+r)c_{k+1}x^{k+r} + \sum_{k=-1}^{\infty} 2(k+r+1)c_{k+1}x^{k+r}$$
$$-\sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$3r(r-1)c_0 x^{r-1} + 2rc_0 x^{r-1}$$
$$+\sum_{n=0}^{\infty}\left[3(n+r+1)(n+r)c_{n+1} + 2(n+r+1)c_{n+1} - (n+r)c_n - c_n\right]x^{n+r} = 0$$

$$\underline{x^{r-1}}\,\big[3r(r-1) + 2r\big]c_0 = 0 \Rightarrow 3r^2 - 3r + 2r = 0$$

$$3r^2 - r = 0$$

$$r(3r-1) = 0 \Rightarrow \boxed{r = 0, \tfrac{1}{3}}$$

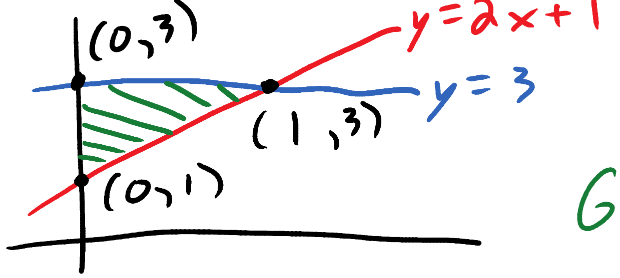
$$\underline{x^{n+r}}\,\big[3(n+r+1)(n+r) + 2(n+r+1)\big]c_{n+1} - \big[(n+r)+1\big]c_n = 0$$

$$\boxed{c_{n+1} = \frac{n+r+1}{3(n+r+1)(n+r) + 2(n+r+1)}\,c_n}$$

4.) *[10 points]* Find the counterclockwise circulation of

$$\vec{F}(x,y) = \langle xy^2 + 5x, 3x - y \rangle$$

around the triangle with sides $x = 0$, $y = 3$, and $y = 2x + 1$.



Closed 2D curve

$\Downarrow$

Green's Theorem!

$$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

$$= \int_0^1 \int_{2x+1}^{3} 3 - 2xy\,dy\,dx$$

$$= \int_0^1 3y - xy^2 \Big|_{y=2x+1}^{y=3}\,dx$$

$$= \int_0^1 9 - 9x - 3(2x+1) + x(2x+1)^2\,dx$$

$$= \int_0^1 9 - 9x - 6x - 3 + 4x^3 + 4x^2 + x\,dx$$

$$= \int_0^1 4x^3 + 4x^2 - 14x + 6\,dx$$

$$= x^4 + \frac{4}{3}x^3 - 7x^2 + 6x \Big|_{x=0}^{x=1}$$

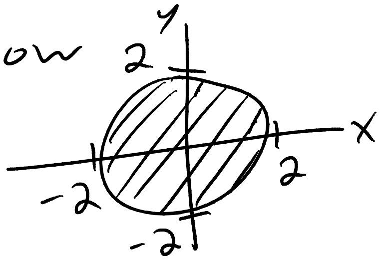
$$= 1 + \frac{4}{3} - 7 + 6$$

$$= \boxed{\frac{4}{3}}$$

5.) *[10 points]* The surface Q is the portion of the paraboloid $z = x^2 + y^2 + 3$ that is over the disk $x^2 + y^2 \leq 4$ in the xy-plane. Compute the surface area of Q.

$f(x,y)$

Shadow



$f_x = 2x, \quad f_y = 2y$

$$S.A. = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \iint_R \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

Polar

$$= \frac{1}{8}\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; 8r\, dr\, d\theta$$

$u = 1 + 4r^2, \quad du = 8r\,dr$

$$= \frac{1}{8}\left[\int_0^{2\pi} d\theta\right]\left[\int_1^{17} \sqrt{u}\, du\right]$$

$$= \frac{1}{8}\left[\theta \Big|_0^{2\pi}\right]\left[\frac{2}{3} u^{3/2} \Big|_1^{17}\right]$$

$$= \frac{1}{8}\left[2\pi\right]\left[\frac{2}{3}(17)^{3/2} - \frac{2}{3}(1)^{3/2}\right]$$

$$= \boxed{\frac{\pi}{6}\left[(17)^{3/2} - 1\right]}$$

6.) *[10 points]* Meowth is trapped inside a spherical Pokeball given by

$$x^2 + y^2 + z^2 = 4.$$

Unhappy with his unfamiliar surroundings, Meowth unleashes an attack with velocity field

$$\vec{F}(x,y,z) = \langle x - 2y, 1 + z^2, 3z \rangle.$$

Compute the outward flux of Meowth's attack through the surface of the Pokeball.

(There is a hard way and an easy way to do this problem. I suggest doing it the easy way, but carefully show work and explain how you arrived at your answer.)

Closed Surface $\Rightarrow$ Divergence Theorem

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x-2) + \frac{\partial}{\partial y}(1+z^2) + \frac{\partial}{\partial z}(3z)$$

$$= 1 + 0 + 3 = 4$$

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_D \nabla\cdot\vec{F}\, dV$$

$$= \iiint_{x^2+y^2+z^2 \le 4} 4 \, dV$$

= 4 (Volume of sphere with radius 2

$$= 4\left(\frac{4}{3}\pi 2^3\right)$$

$$= \boxed{\frac{128\pi}{3}}$$

7.) *[15 points]* Find the Fourier series on $(-10,10)$ of
$$f(x)=\begin{cases}2 & if \quad x<1\\ 3 & if \quad x\geq 1\end{cases}$$

$$a_0=\frac{1}{10}\int_{-10}^{10}f(x)\,dx=\frac{1}{10}\int_{-10}^{1}2\,dx+\frac{1}{10}\int_{1}^{10}3\,dx$$

$$=\frac{1}{10}\left[2x\Big|_{-10}^{1}+3x\Big|_{1}^{10}\right]=\frac{1}{10}\left[(2+20)+(30-3)\right]=\frac{49}{10}$$

$$a_n=\frac{1}{10}\int_{-10}^{10}f(x)\cos\frac{n\pi x}{10}\,dx=\frac{1}{10}\left[\int_{-10}^{1}2\cos\frac{n\pi x}{10}dx+\int_{1}^{10}3\cos\frac{n\pi x}{10}dx\right]$$

$$=\frac{1}{10}\left[\frac{20}{n\pi}\sin\frac{n\pi x}{10}\Big|_{-10}^{1}+\frac{30}{n\pi}\sin\frac{n\pi x}{10}\Big|_{1}^{10}\right]$$

$$=\frac{1}{10}\left[\frac{20}{n\pi}\sin\frac{n\pi}{10}-\frac{20}{n\pi}\underbrace{\sin(-n\pi)}_{0}+\frac{30}{n\pi}\underbrace{\sin(n\pi)}_{0}-\frac{30}{n\pi}\sin\frac{n\pi}{10}\right]$$

$$=\frac{1}{10}\left[-\frac{10}{n\pi}\sin\frac{n\pi}{10}\right]=-\frac{1}{n\pi}\sin\frac{n\pi}{10}$$

$$b_n=\frac{1}{10}\int_{-10}^{10}f(x)\sin\frac{n\pi x}{10}\,dx=\frac{1}{10}\left[\int_{-10}^{1}2\sin\frac{n\pi x}{10}dx+\int_{1}^{10}3\sin\frac{n\pi x}{10}dx\right]$$

$$=\frac{1}{10}\left[-\frac{20}{n\pi}\cos\frac{n\pi x}{10}\Big|_{-10}^{1}-\frac{30}{n\pi}\cos\frac{n\pi x}{10}\Big|_{1}^{10}\right]$$

$$=\frac{1}{10}\left[-\frac{20}{n\pi}\cos\frac{n\pi}{10}+\frac{20}{n\pi}\underbrace{\cos(-n\pi)}_{(-1)^n}-\frac{30}{n\pi}\underbrace{\cos(n\pi)}_{(-1)^n}+\frac{30}{n\pi}\cos\frac{n\pi}{10}\right]$$

$$=\frac{1}{10}\left[\frac{10}{n\pi}\cos\frac{n\pi}{10}-\frac{10}{n\pi}(-1)^n\right]=\frac{1}{n\pi}\cos\frac{n\pi}{10}-\frac{1}{n\pi}(-1)^n$$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{10}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{10}$$

$$\boxed{=\frac{49}{20}+\sum_{n=1}^{\infty}-\frac{1}{n\pi}\sin\frac{n\pi}{10}\cos\frac{n\pi x}{10}+\sum_{n=1}^{\infty}\frac{1}{n\pi}\left[\cos\frac{n\pi}{10}-(-1)^n\right]\sin\frac{n\pi x}{10}}$$

8.) *[15 points]* Solve the following boundary value problem for u(x,t).

$u_t = 4u_{xx}$

$u(0,t) = u(5,t) = 0 \quad \text{for } t \geq 0$

$u(x,0) = 3 \quad \text{for } 0 < x < 5$

Heat Equation

$k=4, \quad L=5, \quad f(x)=3$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{5}\int_0^5 3\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{6}{5}\left[-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^5\right]$$

$$= -\frac{6}{n\pi}\cancel{\cos}(n\pi)^{\nearrow (-1)^n} + \frac{6}{n\pi}\cancel{\cos 0}^{\nearrow 1}$$

$$= -\frac{6}{n\pi}\left[(-1)^n - 1\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$= \boxed{\sum_{n=1}^{\infty} -\frac{6}{n\pi}\left[(-1)^n - 1\right]\sin\left(\frac{n\pi x}{5}\right)e^{-4\left(\frac{n\pi}{5}\right)^2 t}}$$