Dr. Gregory J. Mazzaro

Spring 2015

ELEC 318 – *Electromagnetic Fields*

Lecture 3(d)

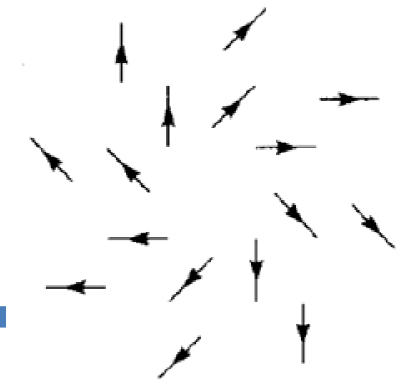
Review of Vector Calculus: Del Operator, Grad, Div, Curl

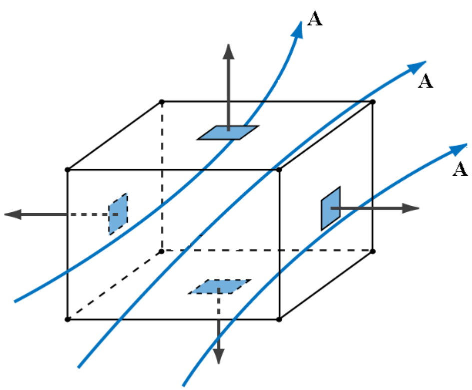
∇ – the Del Operator

del operator

-- vector differential operator
-- allows us to generate 3 different *directional derivatives*: grad, div, curl

$$\nabla$$





integral form, closed surface

differential form, at a point

$$\oint_S \mathbf{A} \cdot d\mathbf{S} \quad \Rightarrow \quad \nabla \cdot \mathbf{A}$$

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} = \frac{\partial}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$= \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$$

→ derived in textbook

Gradient

gradient of a *scalar* field (V)

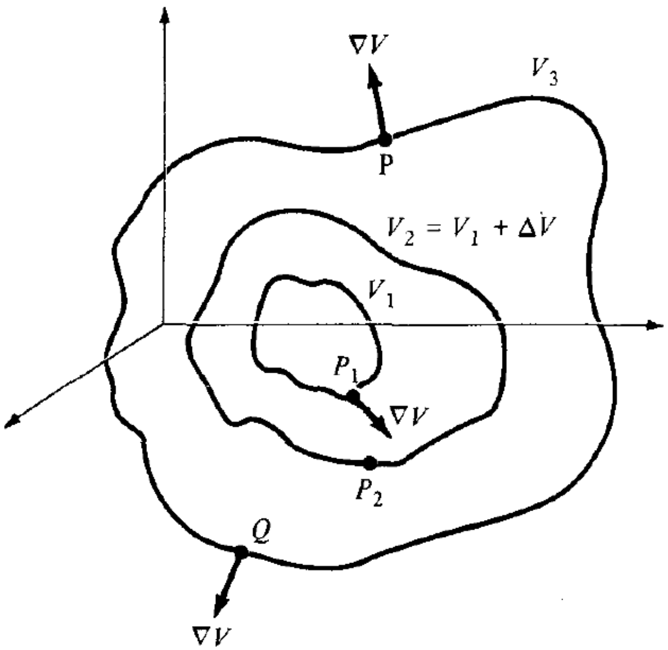
-- a *vector field* that represents the maximum spatial rate of increase of V

-- allows us to answer the questions

"In which direction does V change?"

"How quickly does V change from point-to-point"

$$\nabla V$$



Example: $\mathbf{E} = -\nabla V$

$$\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} = \frac{\partial V}{\partial r}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$$

Example: Gradient (1 of 2)

Find ∇V if

$$V = \cos^2(x) + \cos^2(y)$$

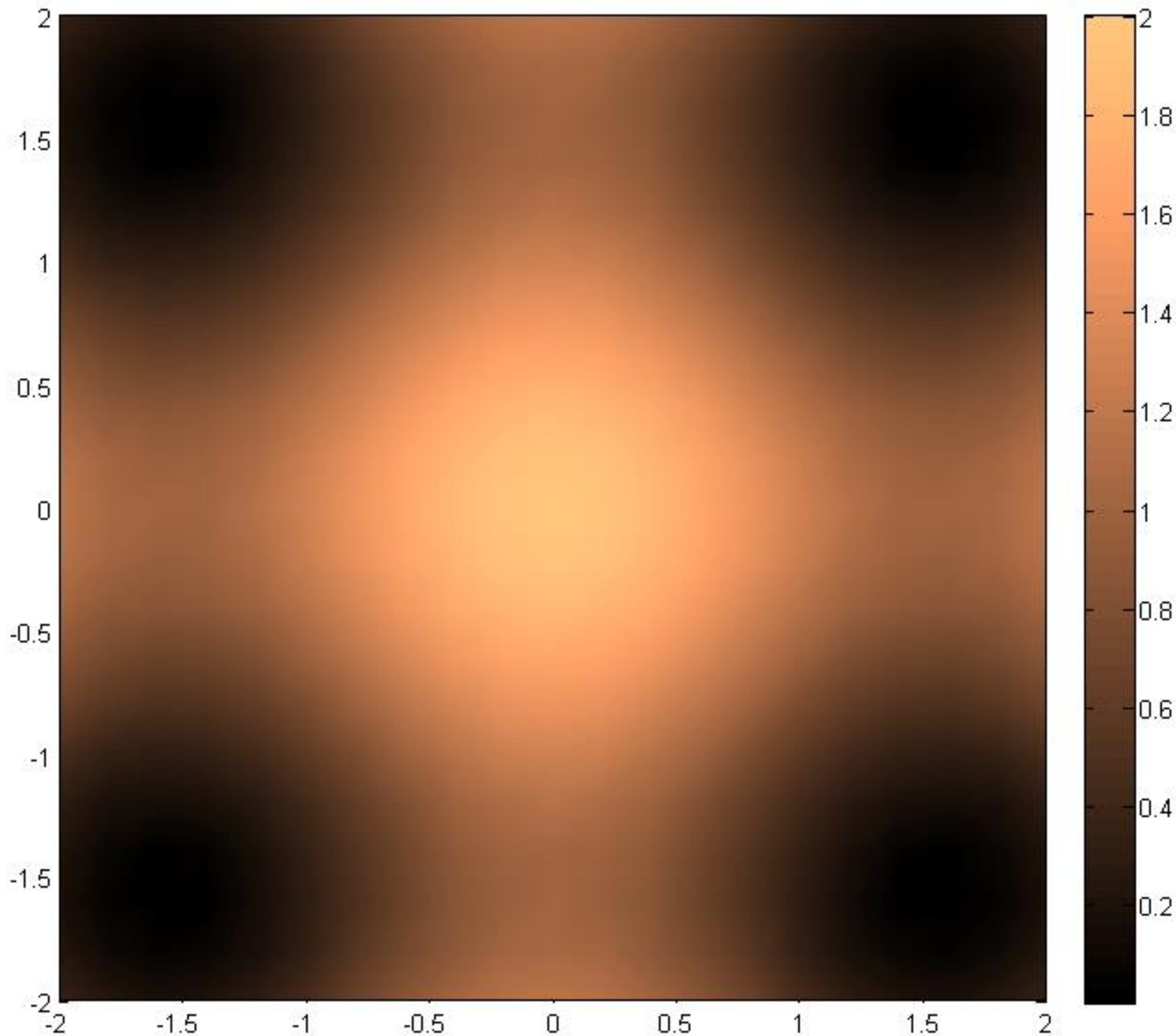
```
[x,y] = meshgrid(-2:.2:2,-2:.2:2);

V = (cos(x)).^2 + (cos(y)).^2;
[u,v] = gradient(V,1,1);

figure(1)
pcolor(x,y,V);
colormap copper; shading interp;
axis image; colorbar

figure(2)
quiver(x,y,u,v,'k','Linewidth',2);
axis image

figure(3)
quiver(x,y,u,v,'w','Linewidth',2)
hold on; axis image;
pcolor(x,y,V);
colormap copper; shading interp;
colorbar; hold off;
```



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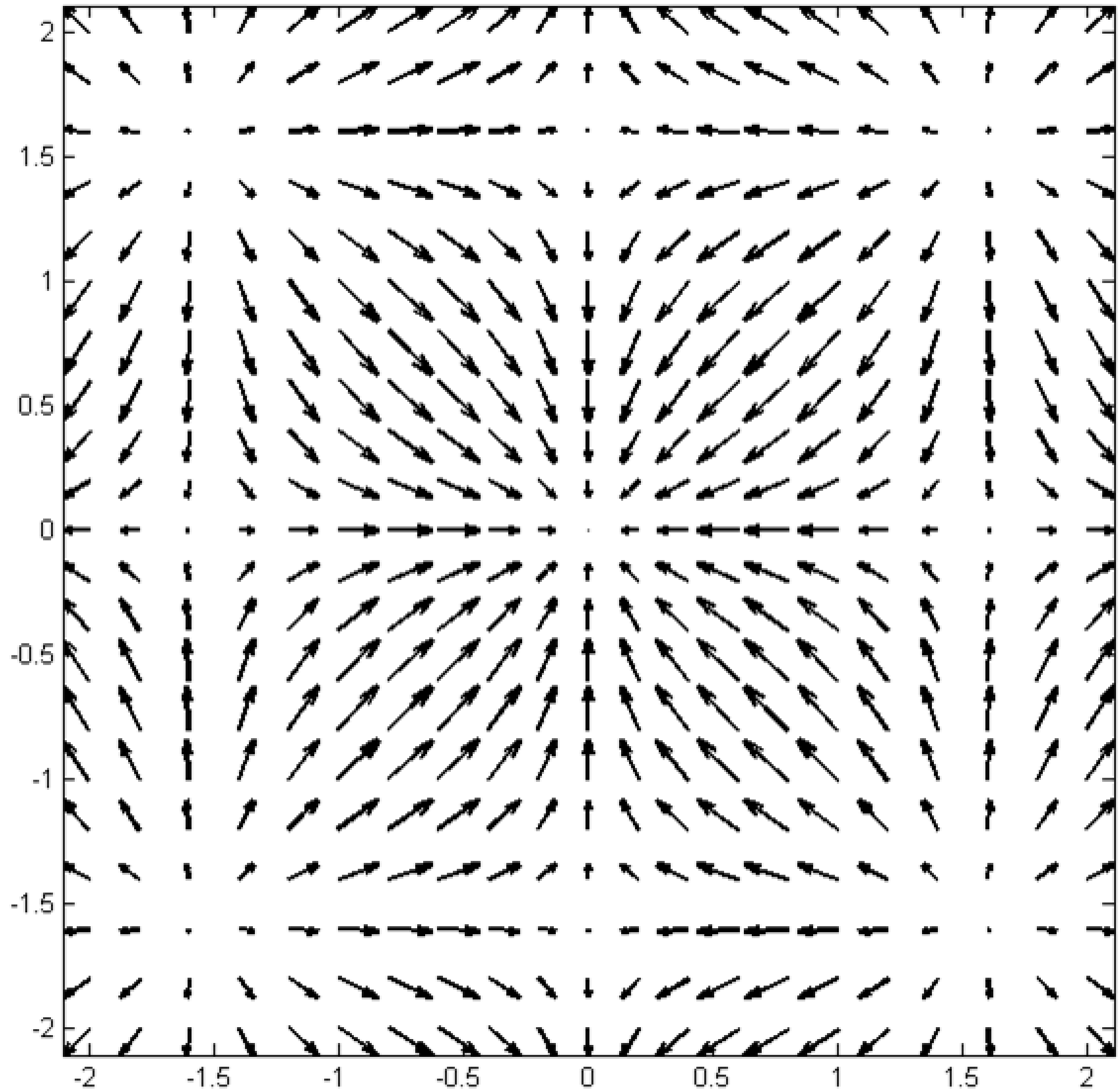
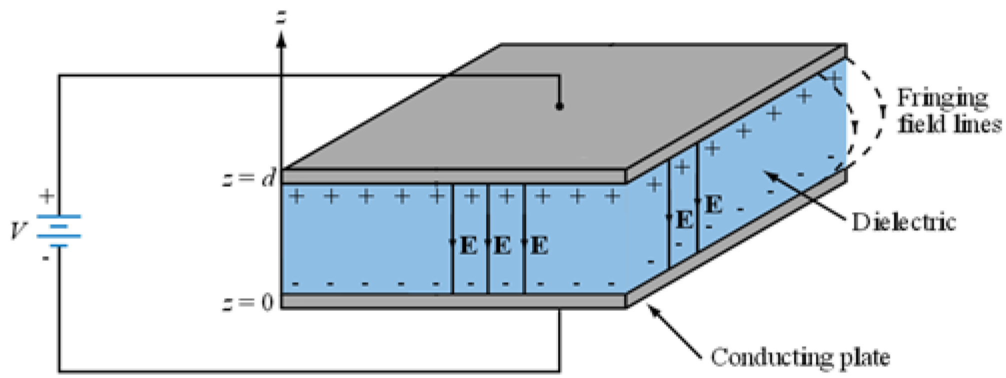


Example: Gradient (2 of 2)

$$\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \qquad = \frac{\partial V}{\partial r}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$$

Estimate the quantity $-\nabla V$ within the parallel plates for $V = 5$ V, $d = 0.025$ mm.



Divergence

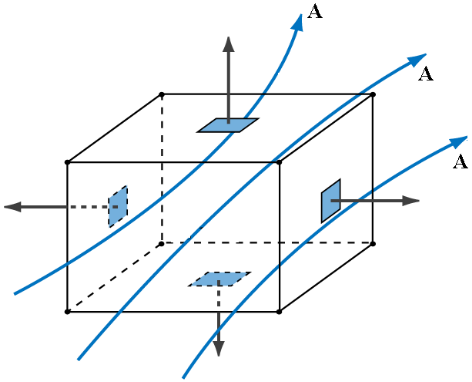
divergence of a *vector* field (**A**) at a point (*P*)

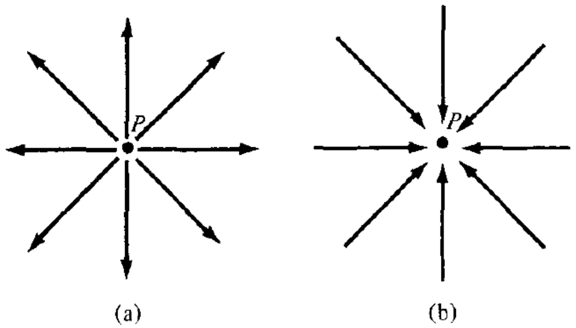
-- a *scalar field*; a measure of the *flux* of **A** per unit volume outward from each point *P*

$$\nabla \cdot \mathbf{A}$$

-- allows us to answer the question

"What is the net amount of **A** that exits a point (*P*)?"



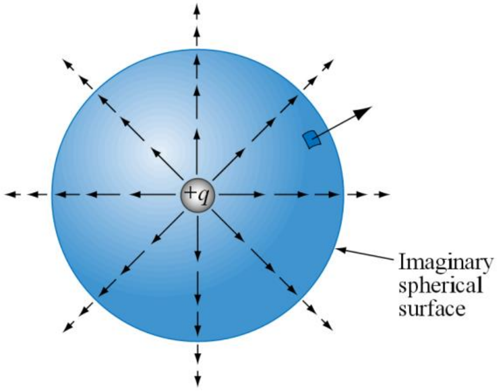


$$\nabla \cdot \mathbf{A} = 0 \quad \Rightarrow \quad \text{"solenoidal"}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}A_\phi$$



Example: Divergence (1 of 2)

Find $\nabla \bullet \mathbf{E}$ if

$$\mathbf{E} = e^{-x^2/10}\cos(x)\,\hat{\mathbf{x}} + e^{-y^2/10}\cos(y)\,\hat{\mathbf{y}}$$

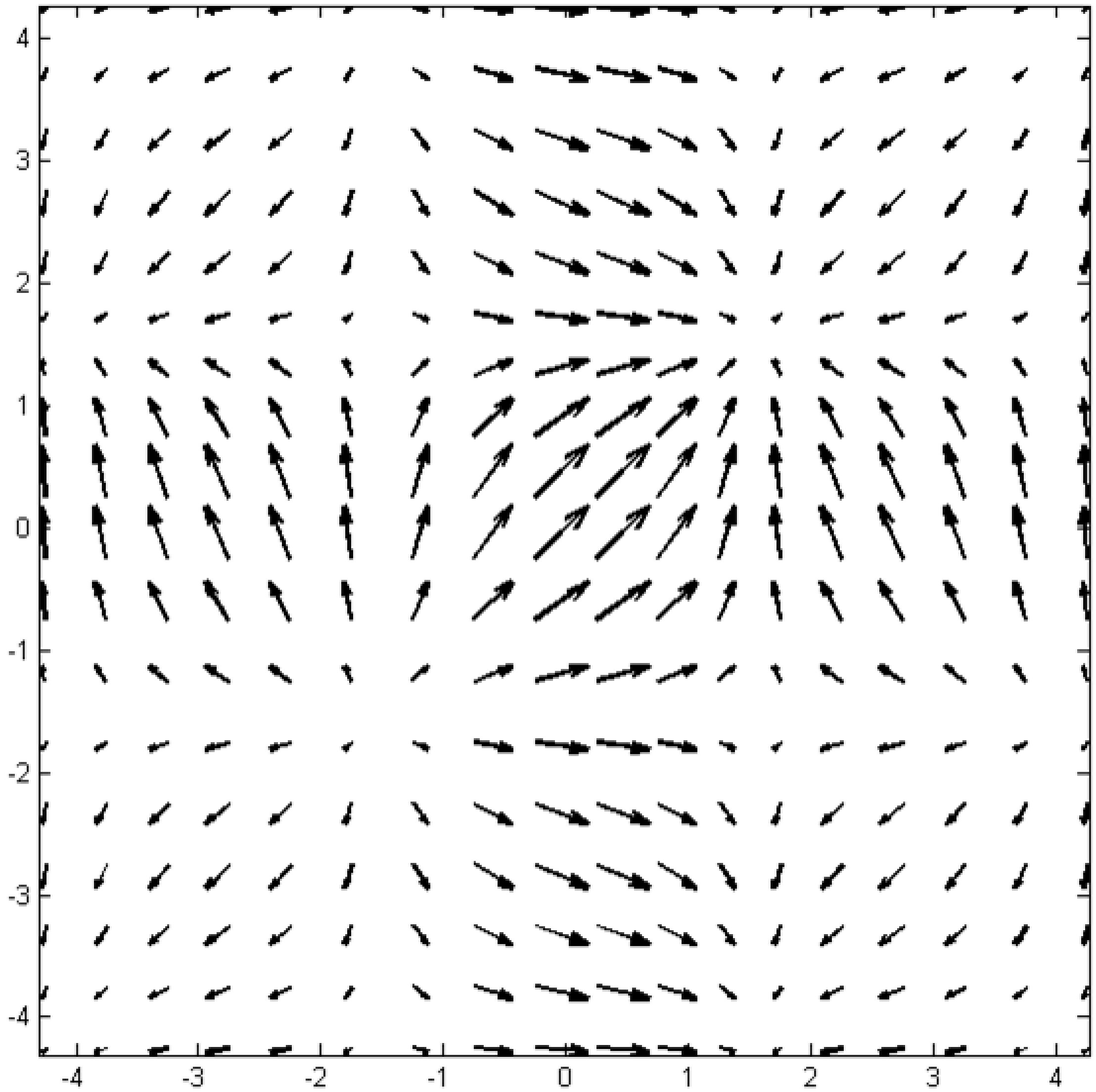
```
[x,y] = meshgrid(-4.25:.5:4.25,...
                 -4.25:.5:4.25);

u = exp(-x.^2/10).*cos(x);
v = exp(-y.^2/10).*cos(y);

div = divergence(x,y,u,v);

figure(1)
quiver(x,y,u,v,'k','Linewidth',2);
axis image

figure(2)
pcolor(x,y,div);
colormap copper; shading interp;
axis image; colorbar
```



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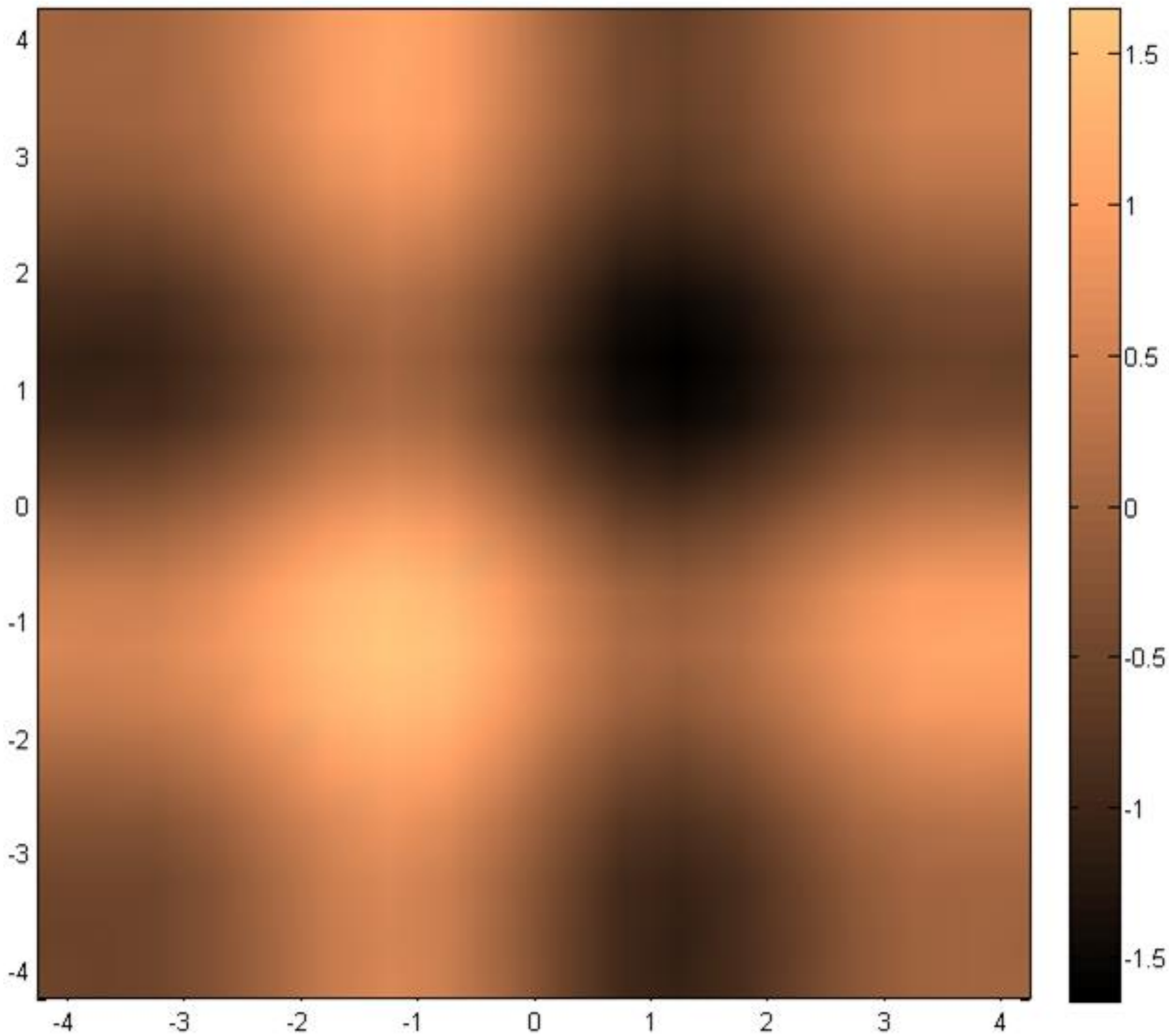
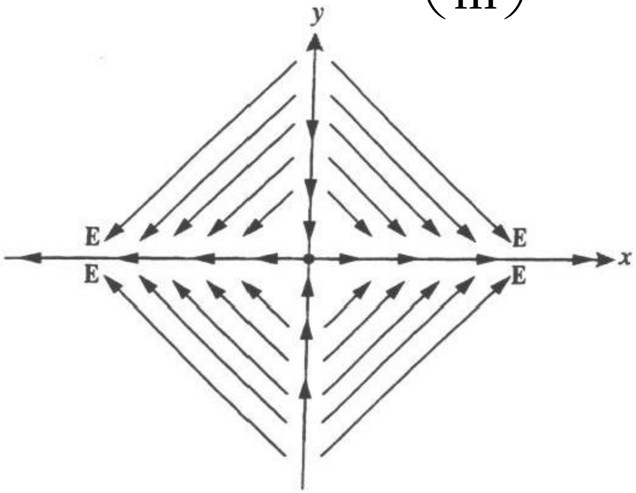
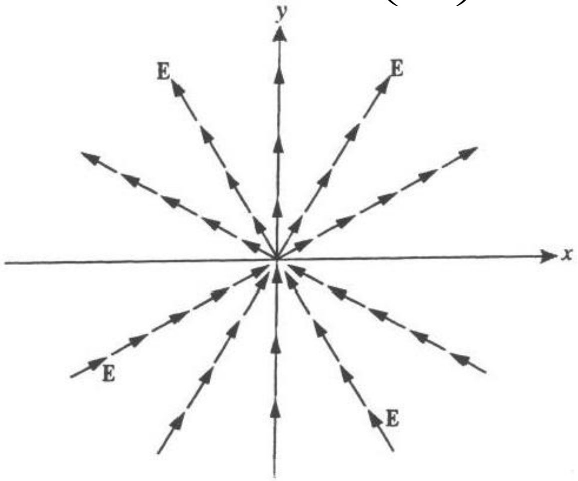


Write the divergence of the following vector fields:

$$\mathbf{E} = x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}} \ \left(\frac{\text{V}}{\text{m}}\right)$$



$$\mathbf{E} = \sin\phi\,\hat{\mathbf{r}} \ \left(\frac{\text{V}}{\text{m}}\right)$$



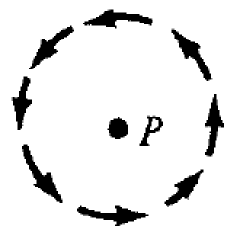
Curl

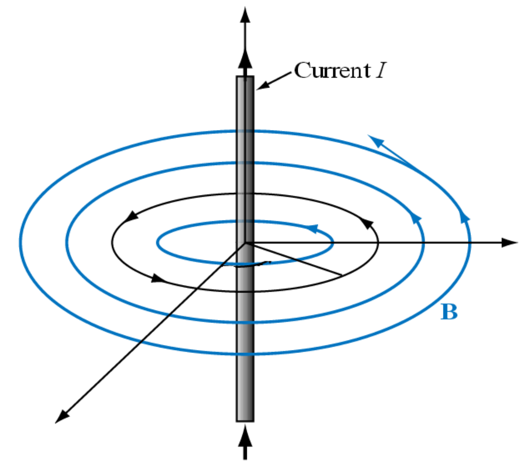
curl of a *vector* field (**A**) at a point (*P*)

-- a *vector field*; a measure of the *circulation* of **A** per unit area at each point *P*

-- its vector directions are the *axes of rotation* of the field

$$\nabla \times \mathbf{A}$$





-- allows us to answer the question

"What is the net amount of **A** that rotates in space about a point (*P*)?"

$$\nabla \times \mathbf{A} = 0 \quad \Rightarrow \quad \text{"conservative"}$$

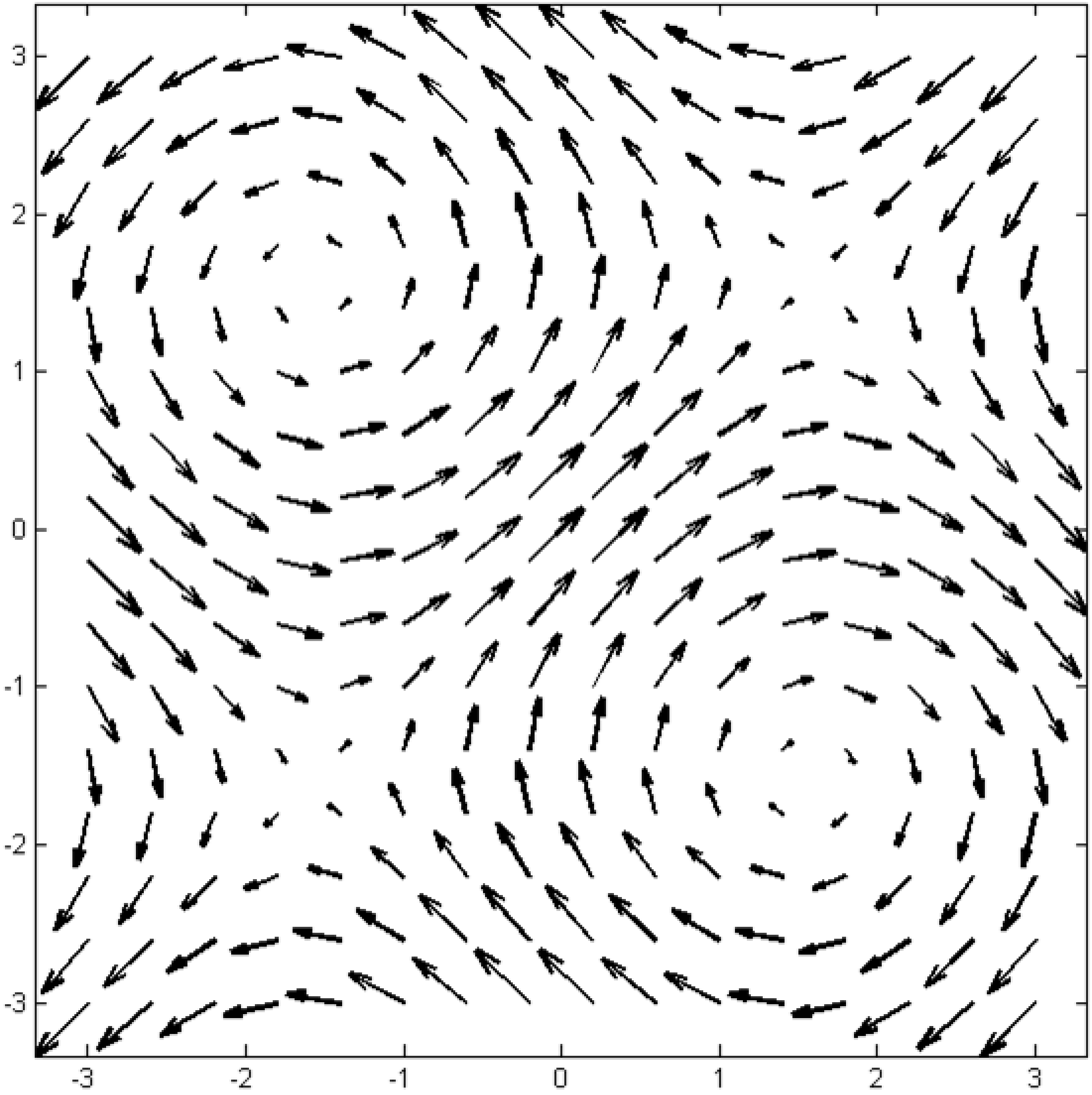
$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\,\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & R\,\hat{\boldsymbol{\theta}} & R\sin\theta\,\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & RA_\theta & R\sin\theta A_\phi \end{vmatrix}$$

Example: Curl (1 of 2)

Find $\nabla \times \mathbf{H}$ if

$$\mathbf{H} = \sin\left(y + \pi/2\right)\hat{\mathbf{x}} + \sin\left(x + \pi/2\right)\hat{\mathbf{y}}$$

```
[x,y] = meshgrid(-3:.4:3,-3:.4:3);

u = sin(y+pi/2);
v = sin(x+pi/2);

[curlz,cav]= curl(x,y,u,v);

figure(1)
quiver(x,y,u,v,'k','Linewidth',2)
axis image

figure(2)
pcolor(x,y,abs(cav));
colormap copper; shading interp;
axis image; colorbar

figure(3)
quiver(x,y,u,v,'w','Linewidth',2)
hold on; axis image;
pcolor(x,y,abs(cav));
colormap copper; shading interp;
Colorbar; hold off;
```



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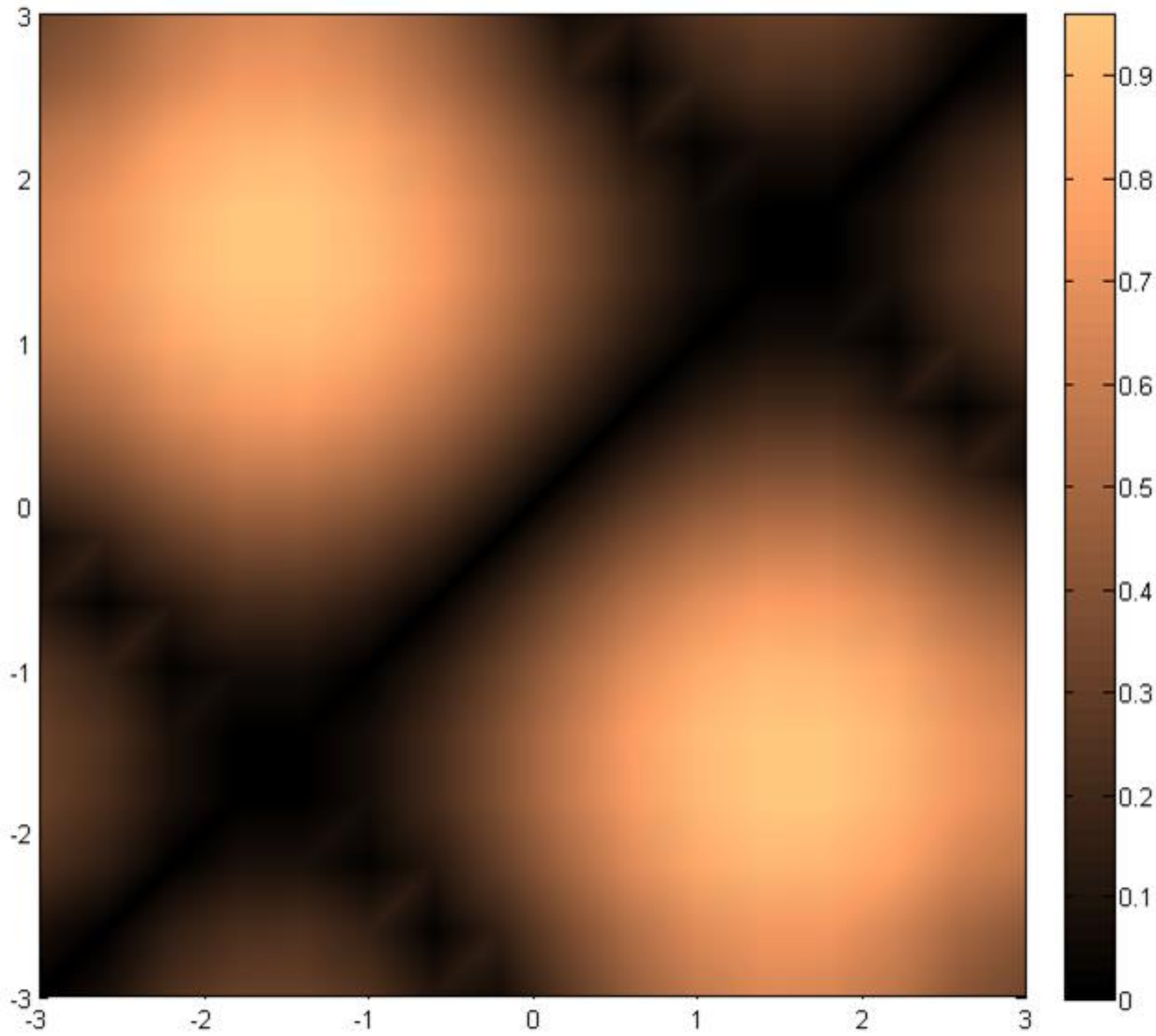
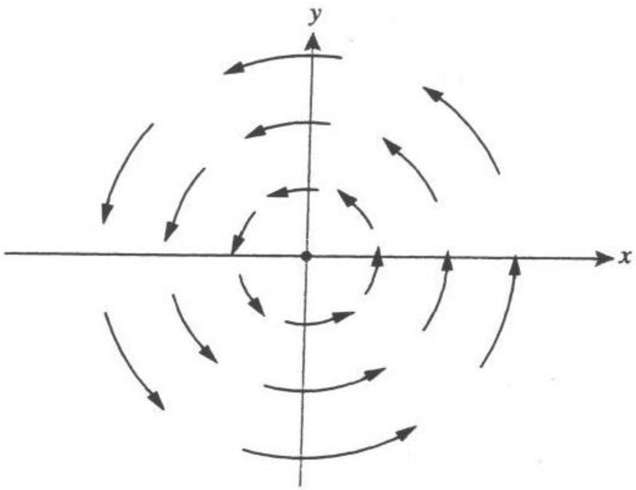
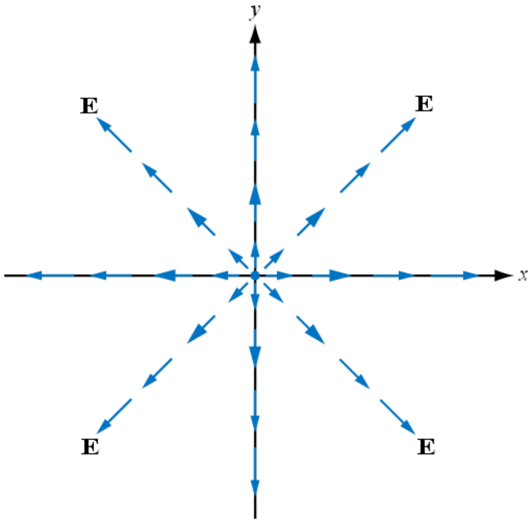


Write the curl of the following vector fields:

$$\mathbf{H} = r\,\hat{\boldsymbol{\phi}}\left(\frac{\mathrm{A}}{\mathrm{m}}\right)$$



$$\mathbf{E} = r\,\hat{\mathbf{r}}\left(\frac{\mathrm{V}}{\mathrm{m}}\right)$$



To be studied outside of class

- Divergence Theorem
- Stokes' Theorem
- Laplacian