Lecture 6: Line Integrals

Tepig's Goals for the Day

- Define the line integral of a function
- Discuss the geometric interpretation of line integrals
- Discuss applications to distance and mass calculations

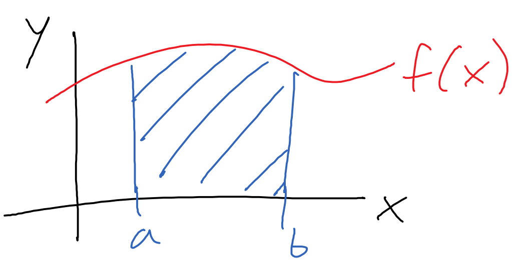
9.8 Line Integrals

The line integral (path integral) of a function $f(x,y)$ over a curve C with arc length parameter s is written

$$\int_C f(x,y)\,ds.$$

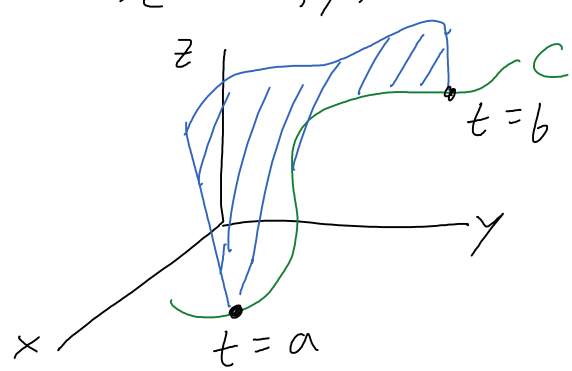
Integral

$$\int_a^b f(x)\,dx$$

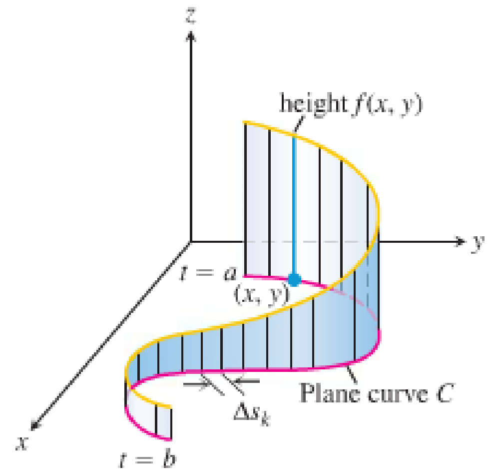


Line Integral

$$\int_C f(x,y)\,ds$$



Think of $\int_C f(x,y)\,ds$ as being the area of a curved wall.



Applications

① $f(x,y)$ = linear density of a wire (g/cm, lb/ft)

$$\int_C f(x,y)\,ds = \text{mass of wire}$$

You can the calculate center of mass and moments of inertia.

$$\bar{x} = \frac{\int_C x f(x,y)\,ds}{\int_C f(x,y)\,ds}$$

x-coord of centroid

② $f(x,y,z)$ = force exerted on an object moving along a path C

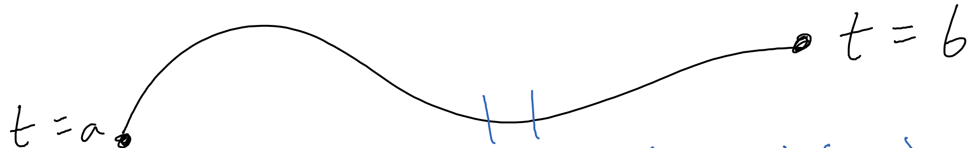
$$\int_C f(x,y,z)\,ds = \text{Total work done on object}$$

③ $f(x,y,z) = 1$

$$\int_C ds = \text{Arc length of curve } C$$

To calculate a line integral, we need a parametrization of the curve C.

$$C: \quad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b$$

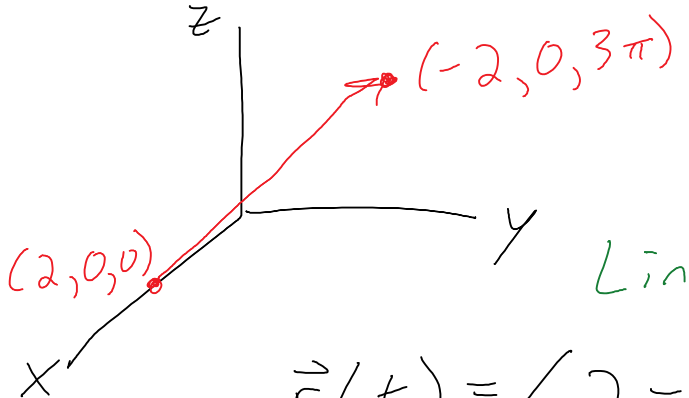


$$ds = (\text{speed})(\Delta t) = \|\vec{r}\,'(t)\|\, dt$$

$$\boxed{\int_C f(x,y,z)\, ds = \int_a^b f(t)\, \|\vec{r}\,'(t)\|\, dt}$$

Recall: Arc length $L = \int_a^b \|\vec{r}\,'(t)\|\, dt$

Ex Compute $\int_{C_1} x+2y+z\,ds$ where C_1 is the line segment joining the point $(2,0,0)$ to $(-2,0,3\pi)$.



$f(x,y,z) = x+2y+z$

Parametrize C_1

Line $\vec{r} = \vec{x}_1 + t(\vec{x}_2 - \vec{x}_1)$, $0 \le t \le 1$

$$\vec{r}(t) = \langle \underbrace{2-4t}_{x}, \underbrace{0}_{y}, \underbrace{3\pi t}_{z} \rangle, \quad 0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle -4, 0, 3\pi \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-4)^2 + 0^2 + (3\pi)^2} = \sqrt{16+9\pi^2}$$

$$\int_C (x+2y+z)\,ds = \int_0^1 (2-4t+2(0)+3\pi t)\sqrt{16+9\pi^2}\,dt$$

$$= \sqrt{16+9\pi^2}\int_0^1 (2-4t+3\pi t)\,dt$$

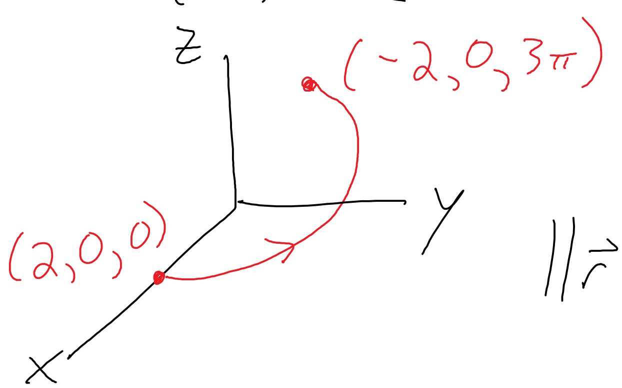
$$= \sqrt{16+9\pi^2}\left[2t - 2t^2 + \frac{3\pi}{2}t^2\right]_0^1$$

$$= \sqrt{16+9\pi^2}\left[\cancel{2} - \cancel{2} + \frac{3\pi}{2} - 0 + 0 - 0\right]$$

$$= \boxed{\frac{3\pi}{2}\sqrt{16+9\pi^2}}$$

<u>Ex</u> Find $\int_{C_2} x+2y+z\,ds$ where C_2 is the portion of the helix

$$\vec{r}(t) = \langle \overset{x}{2\cos t}, \overset{y}{2\sin t}, \overset{z}{3t} \rangle, \quad 0 \le t \le \pi$$



$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 3 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2}$$

$$= \sqrt{4+9} = \sqrt{13}$$

$$\int_{C_2} (x+2y+z)\,ds = \int_0^{\pi} \left(2\cos t + 2(2\sin t) + 3t\right)\sqrt{13}\,dt$$

$$= \sqrt{13}\left[2\sin t - 4\cos t + \frac{3}{2}t^2\right]_0^{\pi}$$

$$= \sqrt{13}\left[2\sin\underset{0}{\pi} - 4\cos\underset{-1}{\pi} + \frac{3}{2}\pi^2 - 2\sin\underset{0}{0} + 4\cos\underset{1}{0} - \underset{0}{\frac{3}{2}(0)^2}\right]$$

$$= \sqrt{13}\left[4 + \frac{3}{2}\pi^2 + 4\right]$$

$$= \boxed{\sqrt{13}\left[8 + \frac{3}{2}\pi^2\right]}$$

Note The curves C_1 and C_2 in the last two examples join the same points.

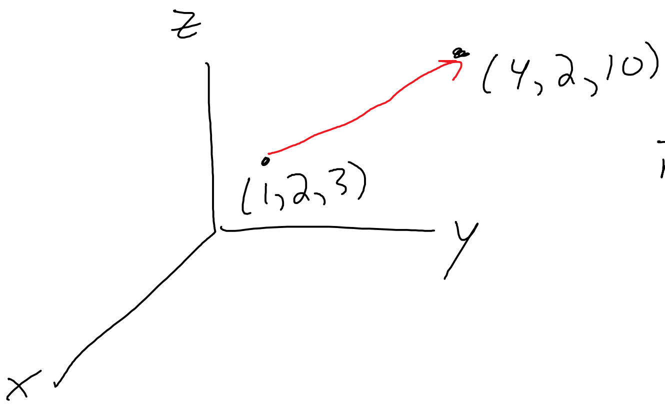
$$\int_{C_1}(x+2y+z)\,ds \neq \int_{C_2}(x+2y+z)\,ds$$

Ex The linear density of a wire in g/cm is

$$f(x,y,z) = x^2 z + 3y.$$

The wire is a straight line joining the point $(1,2,3)$ to $(4,2,10)$.

What is the mass of the wire?



$$\vec{r}(t) = \langle \overbrace{1+3t}^{x}, \overbrace{2}^{y}, \overbrace{3+7t}^{z} \rangle$$

$$0 \le t \le 1$$

$$\vec{r}\,'(t) = \langle 3, 0, 7 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{3^2+0^2+7^2} = \sqrt{58}$$

$$\text{Mass} = \int_C f(x,y,z)\,ds$$

$$= \int_C x^2 z + 3y \; ds$$

$$= \int_0^1 \left[(1+3t)^2(3+7t) + 3(2) \right] \sqrt{58}\, dt$$

$$= \sqrt{58} \int_0^1 \left[63t^3 + 69t^2 + 25t + 9 \right] dt$$

$$= \sqrt{58} \left[\frac{63}{4}t^4 + 23t^3 + \frac{25}{2}t^2 + 9t \right]_0^1$$

$$= \sqrt{58} \left[\frac{63}{4} + 23 + \frac{25}{2} + 9 \right]$$

$$= \sqrt{58}\left[\frac{63}{4} + \frac{92}{4} + \frac{50}{4} + \frac{36}{4}\right]$$

$$= \boxed{\frac{241\sqrt{58}}{4} \text{ grams}}$$

In the examples we did, the speed $||r'(t)||$ ended up being a constant.

The speed may end up being a function of time t. In which case, the integral will be a little nastier to compute.