19 Discrete-Time Sampling

Solutions to Recommended Problems

S19.1

$x[n]$ is given by

$$x[n] = (-1)^n = e^{j\pi n}$$

Hence, the Fourier transform of $x[n]$ is

$$X(\Omega) = \sum_{k=-\infty}^{\infty} \delta(\Omega - \pi - 2\pi k)$$

Now $p[n]$ can be written as

$$p[n] = \frac{1 + (-1)^n}{2}$$

Hence, its Fourier transform is given by

$$P(\Omega) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) + \frac{1}{2} \sum_{k=-\infty}^{\infty} \delta(\Omega - \pi - 2\pi k)$$

It is clear that $x_p[n] = p[n]$. Hence

$$X_p(\Omega) = P(\Omega)$$

S19.2

(a) $x_s[n]$ is $x[n]$ "stretched" by interspersing with zeros, as indicated in Figure S19.2-1.

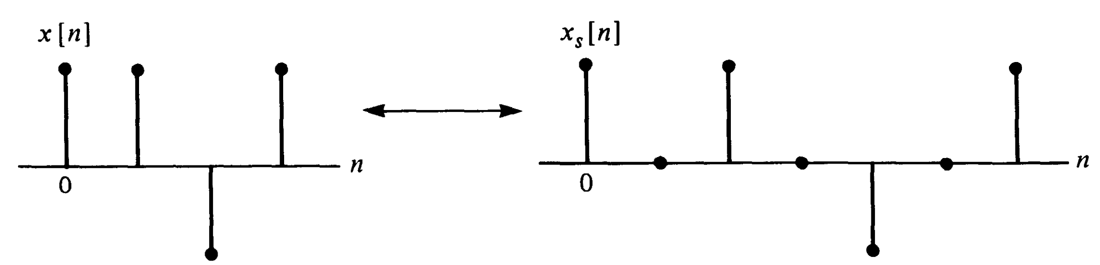

Figure S19.2-1

$$\begin{aligned} X_s(\Omega) &= \sum_{n=-\infty}^{\infty} x_s[n] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x_s[2n] e^{-j\Omega 2n} + \sum_{n=-\infty}^{\infty} x_s[2n+1] e^{-j\Omega(2n+1)} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(2\Omega)n} + 0 \\ &= X(2\Omega) \end{aligned}$$

(b) $x_d[n] = x[2n]$,

$$\begin{aligned} X_d(\Omega) &= \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x[2n]e^{-j\Omega n} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} (x[n] + (-1)^n x[n])e^{-j\Omega n/2} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega/2)n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} x[n]e^{-j[(\Omega/2)-\pi]n} \\ &= \frac{1}{2}X\left(\frac{\Omega}{2}\right) + \frac{1}{2}X\left(\frac{\Omega}{2} - \pi\right) \end{aligned}$$

(c) $X_s(\Omega) = X(2\Omega)$

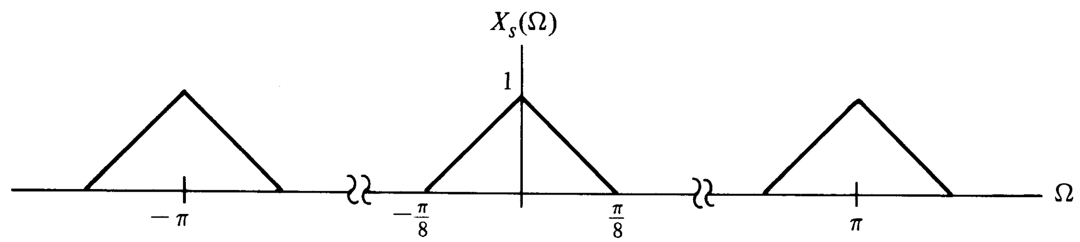

Figure S19.2-2

$$X_d(\Omega) = \frac{1}{2}X\left(\frac{\Omega}{2}\right) + \frac{1}{2}X\left(\frac{\Omega}{2} - \pi\right)$$

$\frac{1}{2}X(\Omega/2)$ is indicated in Figure S19.2-3. Therefore, $X_d(\Omega)$ is as shown in Figure S19.2-4.

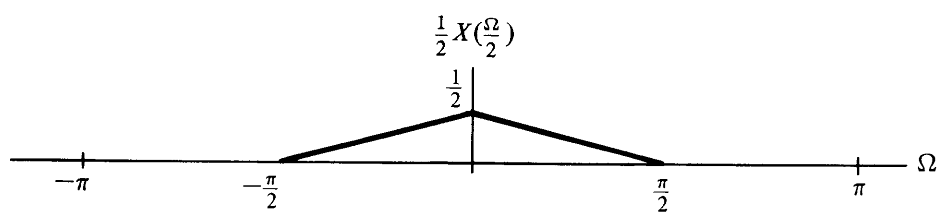

Figure S19.2-3

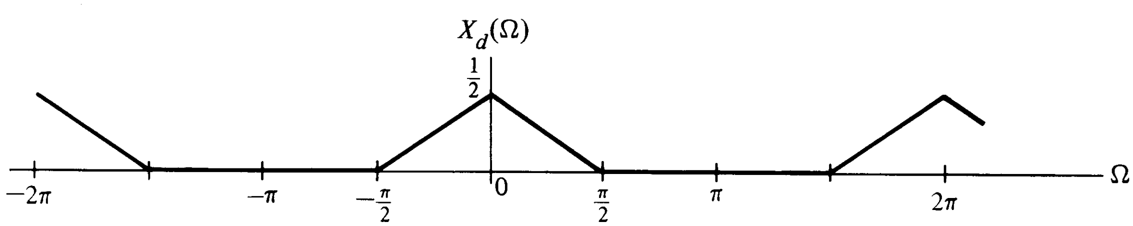

Figure S19.2-4

S19.3

(a) For $N = 1$, $p[n] = 1$. Hence

$$P(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

as shown in Figure S19.3-1.

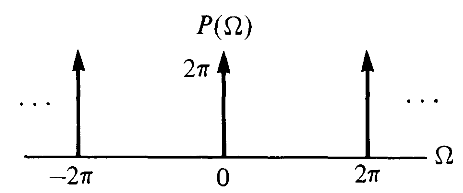

Figure S19.3-1

For $N = 2$,

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2k]$$

Hence

$$P(\Omega) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \pi k),$$

shown in Figure S19.3-2.

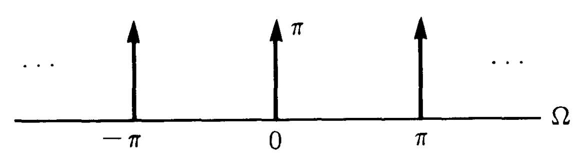

Figure S19.3-2

For $N = L$,

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - Lk]$$

Hence

$$P(\Omega) = \frac{2\pi}{L} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{L}\right),$$

shown in Figure S19.3-3.

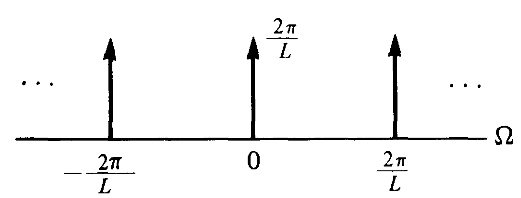

Figure S19.3-3

(b) $X_p(\Omega)$, the spectrum of $x_p[n]$, is proportional to the periodic convolution of $P(\Omega)$ and $X(\Omega)$. Consequently, with $P(\Omega)$ as indicated in Figure S19.3-3 and $X(\Omega)$ as indicated in Figure S19.3-4, $X_p(\Omega)$ is shown in Figure S19.3-5. In order that $x[n]$ be reconstructible from $x_p[n]$ using an ideal lowpass filter, aliasing must be avoided, which requires that

$$\Omega_M < \frac{2\pi}{N} - \Omega_M, \quad \text{or} \quad \Omega_M < \frac{\pi}{N}$$

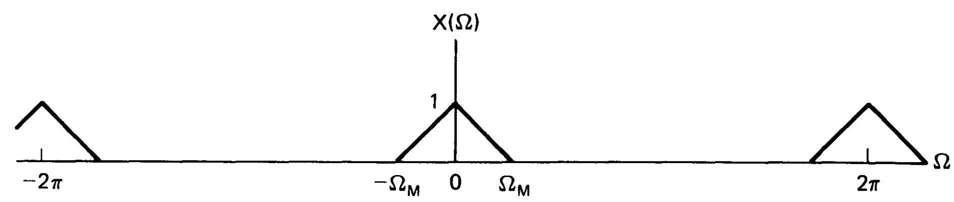

Figure S19.3-4

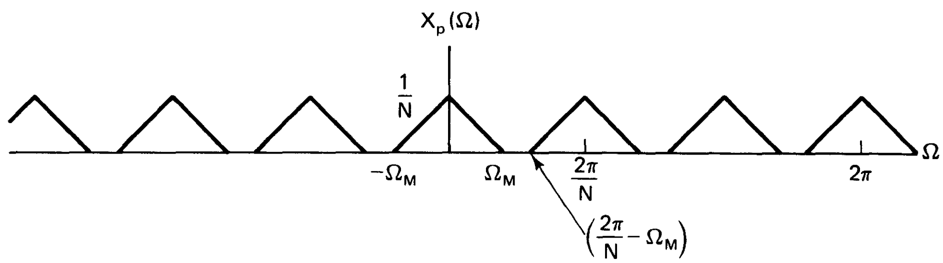

Figure S19.3-5

(i) $\Omega_M = 3\pi/10$. Therefore, to avoid aliasing,

$$\frac{\pi}{N} > \frac{3\pi}{10} \quad \text{or} \quad N < \frac{10}{3}$$

Since N must be an integer, we require that $N \leq 3$. For $N = 3$, the cutoff frequency of the lowpass filter must be greater than $3\pi/10$ and less than

$$\frac{2\pi}{3} - \frac{3\pi}{10} = \left(\frac{11}{3}\right)\frac{\pi}{10}$$

(ii) $\Omega_M = 3\pi/5$. To avoid aliasing,

$$\frac{\pi}{N} > \frac{3\pi}{5} \quad \text{or} \quad N < \frac{5}{3}$$

Since N must be a positive integer, this requires that $N = 1$, i.e., $x[n]$ cannot be sampled.

S19.4

(a) The sampling period T_1 is 3 ms for the system in Figure P19.4-2 to be equivalent to the one in Figure P19.4-1.

(b) $X(\Omega)$ is sketched in Figure S19.4-1.

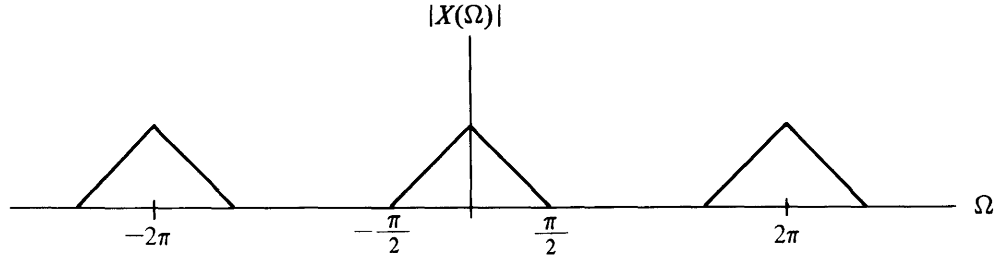

Figure S19.4-1

From the result of part (a), $Y(\Omega)$ is as shown in Figure S19.4-2.

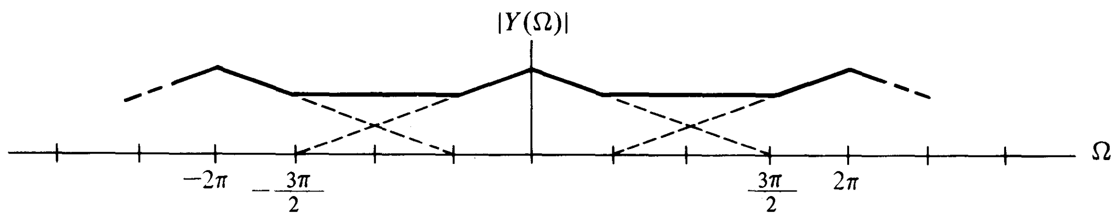

Figure S19.4-2

S19.5

(a) Consider $x_{d_1}[n]$ and $x_{d_2}[n]$, and let

$$x_{d_3}[n] = x_{d_1}[n] + \alpha x_{d_2}[n]$$

Then

$$x_{p_3}[n] = \begin{cases} x_{d_1}[n/N] + \alpha x_{d_2}[n/N], & n = 0, \pm N, \dots, \\ 0, & \text{otherwise} \end{cases}$$

But

$$x_{p_1}[n] = \begin{cases} x_{d_1}[n/N], & n = 0, \pm N, \dots, \\ 0, & \text{otherwise} \end{cases}$$

And

$$\alpha x_{p_2}[n] = \begin{cases} \alpha x_{d_2}[n/N], & n = 0, \pm N, \dots, \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$x_{p1}[n] + \alpha x_{p2}[n] = \begin{cases} x_{d1}[n/N] + \alpha x_{d2}[n/N], & n = 0, \pm N, \dots, \\ 0, & \text{otherwise} \end{cases}$$

and

$$x_{p3}[n] = x_{p1}[n] + \alpha x_{p2}[n]$$

So system A is linear.

(b) Take $x_{d1}[n]$ as shown in Figure S19.5-1, with $N = 4$.

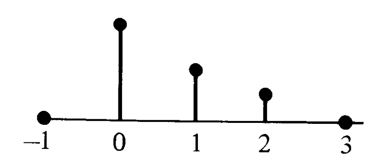

Figure S19.5-1

Then $x_{p1}[n]$ is as shown in Figure S19.5-2.

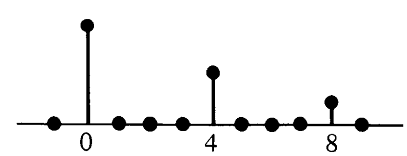

Figure S19.5-2

Take $x_{d2}[n] = x_{d1}[n + 1]$. Then $x_{p1}[n]$ is as shown in Figure S19.5-3.

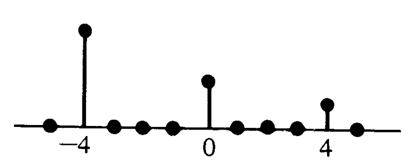

Figure S19.5-3

Hence, system A is not time-invariant.

(c) $x_p[n] = \begin{cases} x_d[n/N], & n = 0, \pm N, \dots, \\ 0, & \text{otherwise} \end{cases}$

Hence

$$X_p(\Omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega(Nn)} = X_d(N\Omega),$$

as shown in Figure S19.5-4.

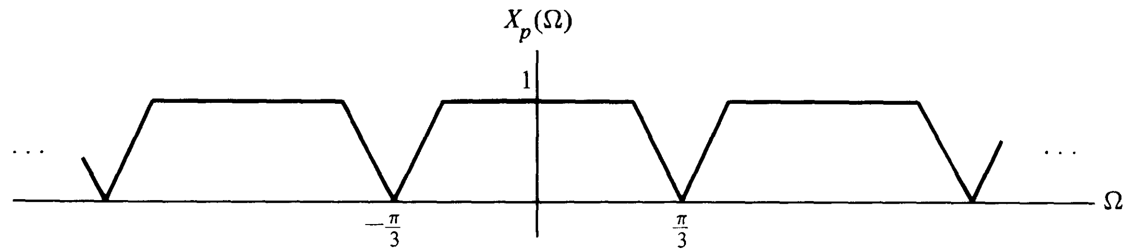

Figure S19.5-4

(d) $X(\Omega)$ is as shown in Figure S19.5-5 for exact bandlimited interpolation.

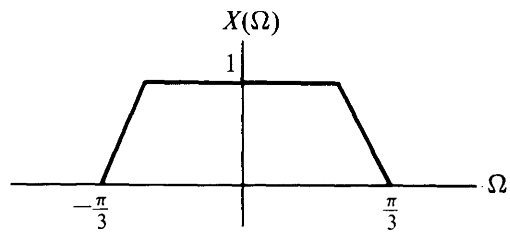

Figure S19.5-5

S19.6

(a) $x_p(t)$ is sketched in Figure S19.6-1, and $Y_p(\omega)$ is sketched in Figure S19.6-2.

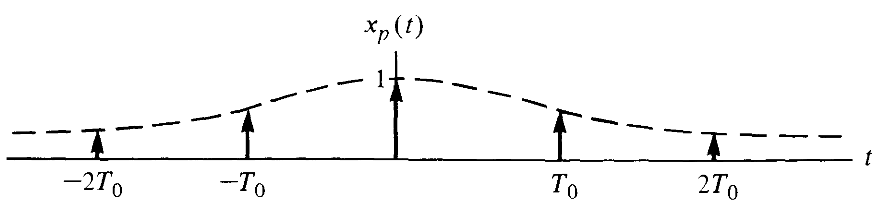

Figure S19.6-1

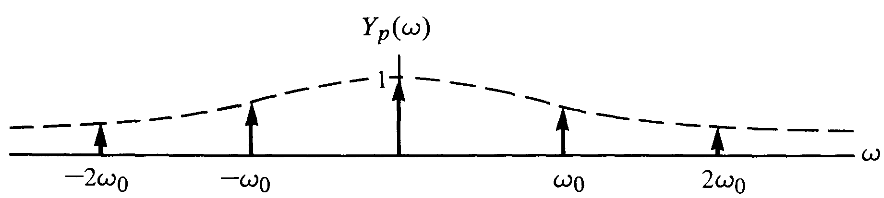

Figure S19.6-2

(b) $X_p(\omega)$ is sketched in Figure S19.6-3, and $y_p(t)$ is sketched in Figure S19.6-4.

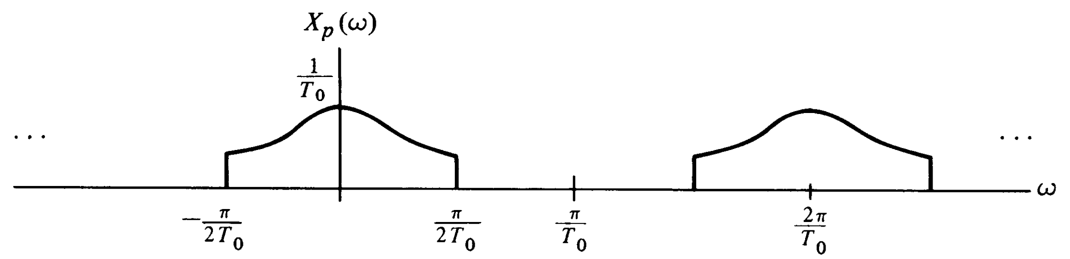

Figure S19.6-3

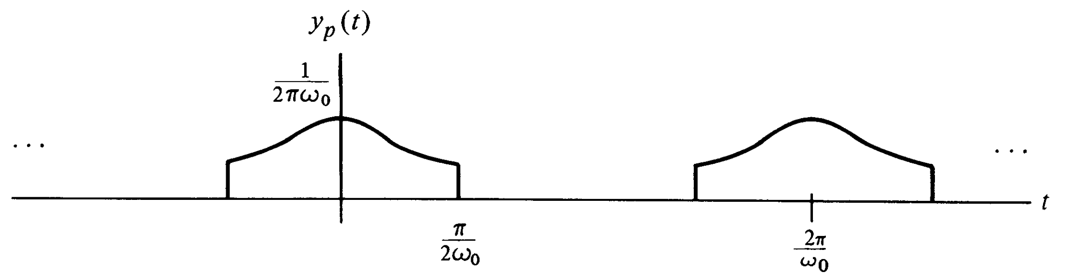

Figure S19.6-4

(c) Yes, $y_p(t)$ is periodic and this is reflected in $Y_p(\omega)$, which contains impulses.

Solutions to Optional Problems

S19.7

(a) $$x_p[n] = \begin{cases} x[n], & n = 0, \pm 2, \pm 4, \dots, \\ 0, & n = \pm 1, \pm 3, \pm 5, \dots \end{cases}$$

This is sketched in Figure S19.7-1.

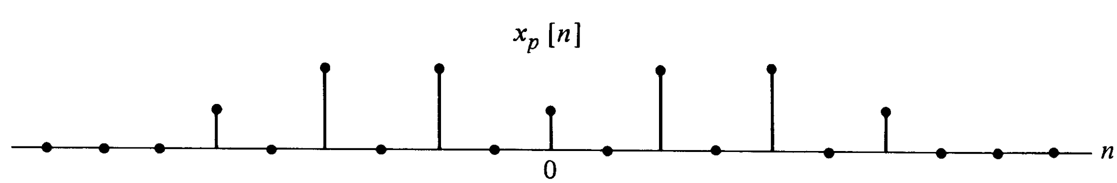

Figure S19.7-1

Similarly, $x_d[n] = x[2n]$, as shown in Figure S19.7-2.

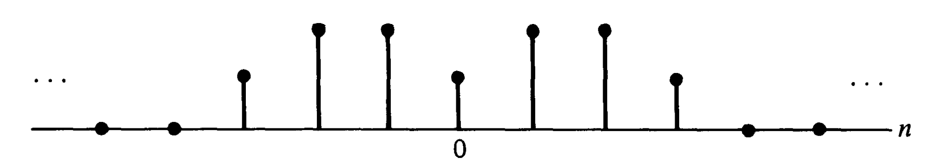

Figure S19.7-2

(b) $X_p(\Omega)$ is obtained as follows:

$$x_p[n] = \tfrac{1}{2}x[n] + \tfrac{1}{2}(-1)^n x[n]$$
$$X_p(\Omega) = \tfrac{1}{2}X(\Omega) + \tfrac{1}{2}X(\Omega - \pi)$$

and

$$X_d(\Omega) = \frac{1}{2}X\left(\frac{\Omega}{2}\right) + \frac{1}{2}X\left(\frac{\Omega}{2} - \pi\right),$$

which are shown in Figures S19.7-3 and S19.7-4. See Problem P19.2(b).

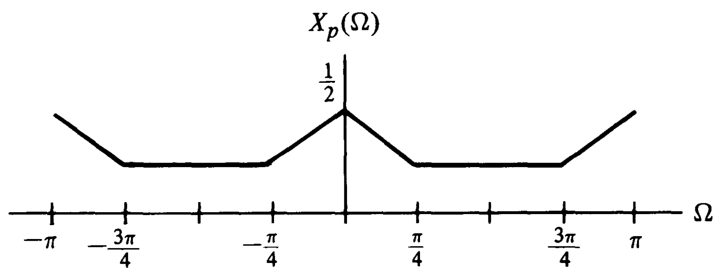

Figure S19.7-3

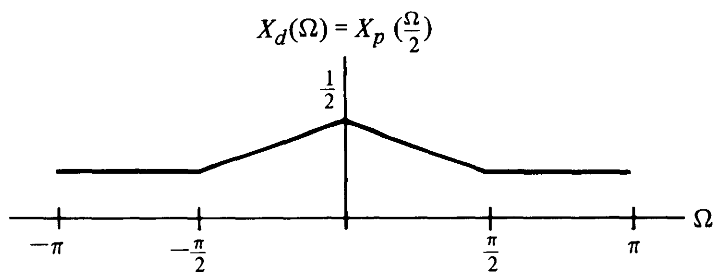

Figure S19.7-4

S19.8

(a) We know that the Fourier transform of $p[n]$ is given by

$$\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\Omega - \frac{2\pi k}{N}\right)$$

Aliasing will be just avoided when the sampled spectra will look as shown in Figure S19.8-1.

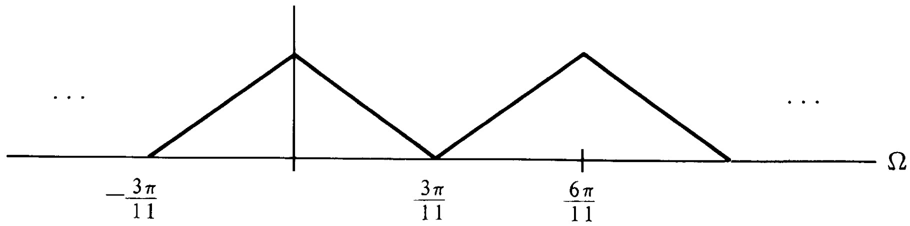

Figure S19.8-1

Hence, we require that

$$\frac{2\pi}{N} > \frac{6\pi}{11}, \qquad \text{or} \qquad N < \frac{22}{6} \Rightarrow N \le 3$$

Consequently, aliasing is avoided if $1 \le N \le 3$. $X_p(\Omega)$ for $N = 1$, 2, and 3 are shown in Figure S19.8-2.

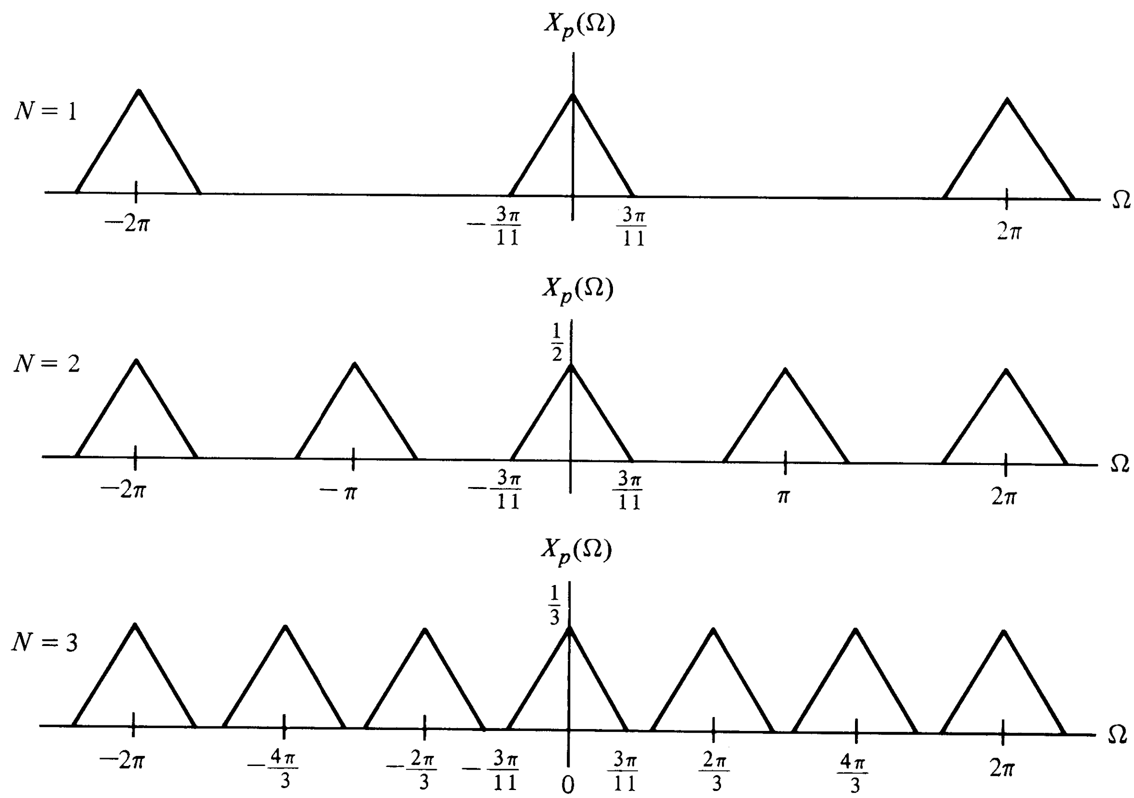

Figure S19.8-2

(b) An appropriate $H(\Omega)$ is shown in Figure S19.8-3.

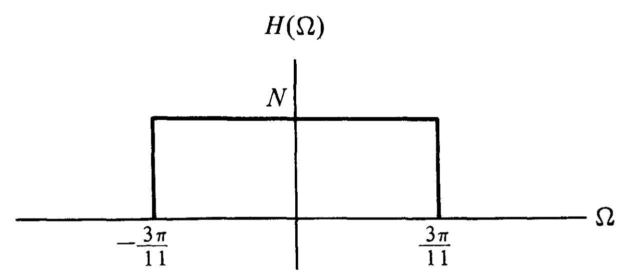

Figure S19.8-3

S19.9

(a) $$y[n] = \frac{x[3n] + x[3n+1] + x[3n+2]}{3}$$

$$y[-4] = 0$$
$$y[-3] = 0$$
$$y[-2] = \frac{1}{3}x[-4] = \frac{1}{3}$$
$$y[-1] = \frac{x[-3] + x[-2] + x[-1]}{3} = \frac{2}{3}$$
$$y[0] = \frac{x[0] + x[1] + x[2]}{3} = \frac{4}{3}$$
$$y[1] = \frac{x[3] + x[4] + x[5]}{3} = \frac{1}{3}$$
$$y[2] = 0$$
$$y[3] = 0$$

Hence, $y[n]$ can be sketched as in Figure S19.9.

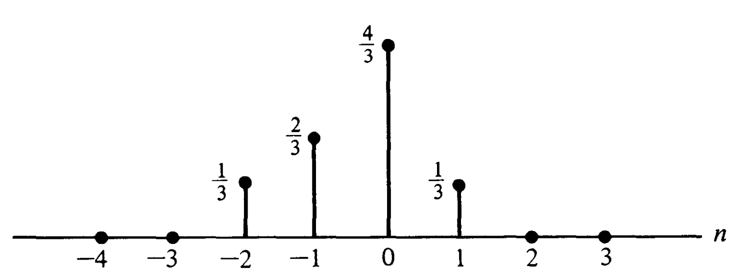

Figure S19.9

(b) If

$$z[n] = \tfrac{1}{3}[x[n] + x[n+1] + x[n+2]], \qquad \text{for all } n$$

and

$$y[n] = z[3n],$$

we have expressed the processing as a combination of filtering and decimati

S19.10

If $h[0] = 1$ and $h[n] = 0$ for $n = kN$, $k \neq 0$, it is easy to see that the samples $x_0[n]$ that came from $x[n]$ will be unaffected. Hence,

$$y[kN] = x[k], \qquad \text{for all } k$$