Math 335 Practice Exam 3 Key

1.) *[10 points]* A metal bar of length π initially has temperature $u(x,0) = -x$ at position x, where x=0 is the left end of the bar. At time t=0, the two ends of the bar are wrapped in ice cubes with constant temperature 0 degrees Celsius. Find the temperature u(x,t) of the bar assuming a thermal diffusivity constant k=3.

$$\begin{cases} u_t = 3u_{xx} \\ u(x,0) = -x \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} -x\sin(nx)\,dx$$

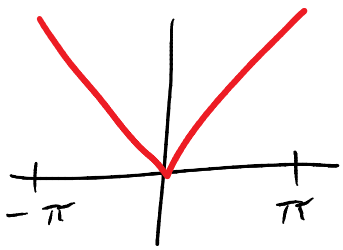
$$= \frac{2}{\pi}\left[\frac{1}{n}x\cos(nx) - \frac{1}{n}\sin(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi n}\Big[\pi \underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\sin(n\pi)}_{0} + 0\underbrace{\cos(0)}_{1} - \underbrace{\sin(0)}_{0}\Big]$$

$$= \frac{2}{n}(-1)^n$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin(nx)\, e^{-3n^2 t}}$$

2.) *[10 points]* Find the Fourier series of $f(x) = |x|$ on the interval $(-\pi, \pi)$.



$f(x) = |x|$ even $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}x^2\Big|_{-\pi}^{0} + \frac{1}{2}x^2\Big|_{0}^{\pi}\right] = \frac{1}{\pi}\left[\frac{1}{2}\pi^2 + \frac{1}{2}\pi^2\right] = \pi$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x}{n}\sin nx - \frac{1}{n^2}\cos nx\Big|_{-\pi}^{0}\right.$$

$$\left. + \frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\Big|_{0}^{\pi}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$f(x) = \boxed{\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}\cos nx}$$

What value does the Fourier series converge to at $x = \pi$?

$$\frac{1}{2}\left[f(\pi^-) + f(\pi^-)\right] = \frac{1}{2}\left[\pi + \pi\right] = \boxed{\pi}$$

3.) *[10 points]* State in words the physical situation that the equations below describe:

$$u_{tt} = 4u_{xx}, \quad u(0,t) = 0, \quad u(10,t) = 0, \quad u(x,0) = 20, \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

Then solve the boundary value problem.

A vibrating string with tension constant a=2 ($a^2 = 4$) has length 10 and the ends are clamped at the x-axis. The string is held at a position 20 above equilibrium and released from rest (zero velocity).

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{10}\int_0^{10} 20\sin\frac{n\pi x}{10}\,dx$$

$$= -4\left(\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10}\Big|_0^{10}$$

$$= -\frac{40}{n\pi}\overset{(-1)^n}{\cos n\pi} - \frac{40}{n\pi}\overset{1}{\cos 0}$$

$$= -\frac{40}{n\pi}\left((-1)^n - 1\right)$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \Rightarrow B_n = 0$$

Only have to worry about B_n if there is initial velocity.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} -\frac{40}{n\pi}\left((-1)^n - 1\right)\cos\left(\frac{n\pi 2t}{10}\right)\sin\left(\frac{n\pi x}{10}\right)}$$

4.) *[10 points]* The ends of a metal bar of length L with thermal diffusivity constant k are insulated so that no heat flows in or out through the ends. Assuming the initial temperature profile of the bar is $u(x,0) = f(x)$ and no heat is lost or gained along the length of the bar, use separation of variables to derive a solution to the one-dimensional heat equation. You may assume that when you set the equations equal to a separation constant $-\lambda$, the cases $\lambda = 0$ and $\lambda = -\alpha^2$ give trivial solutions.

This is known as the Heat-Neumann model.

$$u_t = ku_{xx}, \quad u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Assume $u(x,t) = v(t)w(x)$ and plug this into PDE to get

$$\frac{v_t}{kv} = \frac{w_{xx}}{w} = -\lambda$$

<u>Case 1</u>: $\lambda = 0 \quad \rightarrow u = 0$ Trivial solution

<u>Case 2</u>: $\lambda = -\alpha^2 \quad \rightarrow u = 0$ Trivial solution

<u>Case 3</u>: : $\lambda = \alpha^2$

$$\frac{v_t}{kv} = -\alpha^2 \Rightarrow \frac{dv}{dt} = -k\alpha^2 v$$

$$\Rightarrow \int \frac{1}{v}\, dv = \int -k\alpha^2\, dt$$

$$\Rightarrow \ln v = -k\alpha^2 t + C$$

$$\Rightarrow v = C_1 e^{-k\alpha^2 t}$$

$$\frac{w_{xx}}{w} = -\alpha^2 \Rightarrow w_{xx} = -\alpha^2 w$$

$$\Rightarrow w_{xx} + \alpha^2 w = 0$$

$$\Rightarrow r^2 + \alpha^2 = 0 \Rightarrow r = \pm \alpha i$$

$$\Rightarrow w = C_2 \cos(\alpha x) + C_3 \sin(\alpha x)$$

$$u = vw = e^{-k\alpha^2 t}\left[C_2 \cos(\alpha x) + C_3 \sin(\alpha x)\right]$$

$$\frac{\partial u}{\partial x} = e^{-k\alpha^2 t}\left[-C_2 \alpha \sin(\alpha x) + C_3 \alpha \cos(\alpha x)\right]$$

Plug in boundary conditions at ends.

$\underline{x=0}$ $\quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = e^{-k\alpha^2 t}[C_3 \alpha] = 0$

$$\Rightarrow C_3 = 0$$

$\underline{x=L}$ $\quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = e^{-k\alpha^2 t}[-C_2 \alpha \sin \alpha L] = 0$

Choose $\alpha L = n\pi$

$$\Rightarrow \alpha = \frac{n\pi}{L}$$

So the solution is

$$\boxed{u = \sum_{n=1}^{\infty} a_n \, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)}$$

Using a Fourier series cosine expansion for $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ gives

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

5.) *[10 points]* A string of length π has tension constant *a* and has its two ends clamped at the x-axis. Assuming the string is released from rest, the vertical displacement u(x,t) of the string is modeled by

$$u_{tt} = a^2 u_{xx}, \qquad u(0,t) = u(\pi,t) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

Suppose the string has initial shape

$$u(x,0) = \sin(kx)$$

for some integer $k \geq 1$. Prove that the string has $k-1$ stationary points in the open interval $(0,\pi)$.

Proof

The general solution with initial shape f(x) and initial velocity 0 is

$$u = \sum_{n=1}^{\infty} A_n \sin(nx)\cos(nat), \qquad A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\, dx$$

In our case, plugging in $f(x) = \sin(kx)$ and using the orthogonality of the sine functions that you proved in HW 11 gives

$$A_n = \frac{2}{\pi}\int_0^{\pi} \sin(kx)\sin(nx)\,dx = \begin{cases} 1 & if \quad k = n \\ 0 & otherwise \end{cases}$$

So the solution is

$$u = \sin(kx)\cos(kat)$$

A stationary point is a point along the string that never moves, so $\frac{\partial u}{\partial t} = 0$ for all t.

Taking the derivative with respect to t and setting it equal to zero gives

$$\frac{\partial u}{\partial t} = -ka\sin(kx)\sin(kat) = 0 \quad \text{for all t}$$

So the term involving x must be zero:

$$\sin(kx) = 0 \quad \rightarrow \quad kx = \pi n \quad \rightarrow \quad x = \frac{\pi n}{k} = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \ldots, \frac{(k-1)\pi}{k}, \ \pi$$

This gives us k-1 values in the range $(0,\pi)$.

Q.E.D.