Lecture 12: Stokes' Theorem

**Haunter's Goals for the Day**

- Practice calculating the flux across a surface using surface integrals
- State Stokes' Theorem and its meaning
- Discuss how Stokes' Theorem simplifies certain line integral calculations

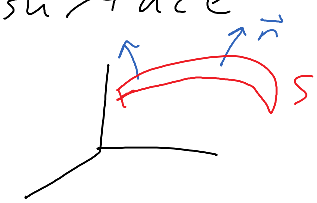
9.13 Surface Integrals

Recall surface integral of $z=f(x,y)$

$$\iint_S g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\,dA$$

Application Flux through surface

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$$




If $\vec{F}$ is the velocity field of a fluid, then the flux is the volume of fluid flowing through $S$ in the direction of $\vec{n}$ per unit time.

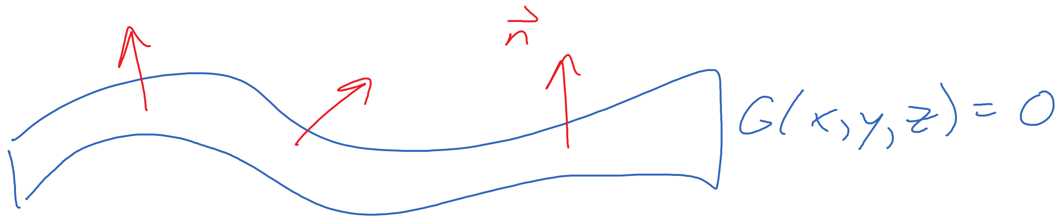
Flux > 0 $\Rightarrow$ Fluid flows in direction of $\vec{n}$

Flux < 0 $\Rightarrow$ Fluid flows against $\vec{n}$

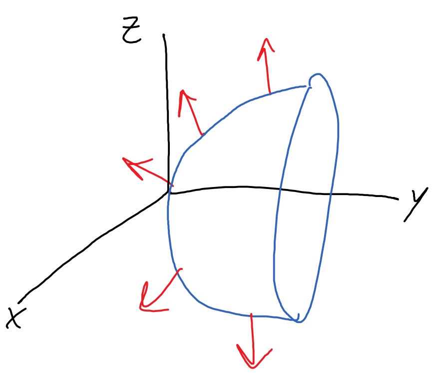
Air flow, electric flux, magnetic flux, heat flux, ...

Unit normal vector $\vec{n}$ to a surface $G(x,y,z)=0$ is given by

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|}$$




Ex Find outward normal to the surface $y = x^2 + 9z^2$.




Write $G(x,y,z)=0$

$$y - x^2 - 9z^2 = 0$$

$$\nabla G = \langle -2x,\ 1,\ -18z \rangle$$

This points inward.

So choose $-\nabla G = \langle 2x, -1, 18z \rangle$

Make it a unit vector.

$$\frac{-\nabla G}{\|-\nabla G\|} = \frac{\langle 2x,-1,18z\rangle}{\sqrt{(2x)^2+(-1)^2+(18z)^2}} = \boxed{\frac{\langle 2x,-1,18z\rangle}{\sqrt{4x^2+1+324z^2}}}$$

# Steps for Computing Flux Through a Surface

① Find normal vector $\vec{n}$

Surface $G(x,y,z)=0$

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|} \quad \text{or} \quad -\frac{\nabla G}{\|\nabla G\|}$$

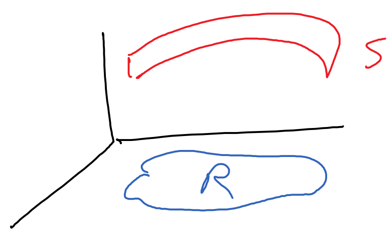
② Compute dot product $\vec{F}\cdot\vec{n}$

③ Compute Jacobian

surface $z = f(x,y)$

$$\sqrt{1 + f_x^2 + f_y^2}$$

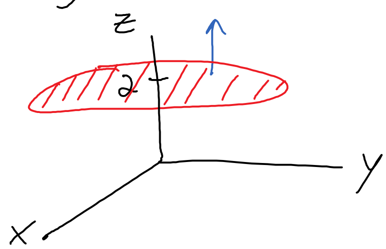
④ Determine shadow region $R$ in $xy$-plane




⑤ Integrate

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R (\vec{F}\cdot\vec{n})\sqrt{1+f_x^2+f_y^2}\, dA$$

Ex Calculate upward flux of $\vec{F} = \langle x, 0, z^2 \rangle$ through the disk of radius 1 centered at the origin at height $z = 2$.




① Normal vector $\vec{n}$

$$z = 2 \Rightarrow \underbrace{z - 2}_{G(x,y,z)} = 0$$

$$\nabla G = \langle 0, 0, 1 \rangle$$

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{\langle 0, 0, 1 \rangle}{1} = \langle 0, 0, 1 \rangle$$

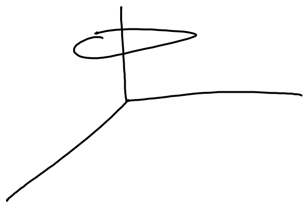
② Dot product $\vec{F} \cdot \vec{n}$

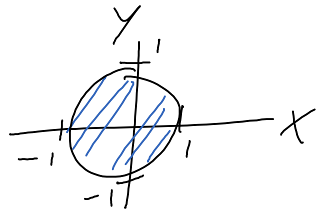
$$\vec{F} \cdot \vec{n} = \langle x, 0, z^2 \rangle \cdot \langle 0, 0, 1 \rangle = z^2$$

③ Jacobian of $z = 2$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 0^2 + 0^2} = 1$$

④ Shadow Region






$x^2+y^2\le 1$

Polar

$0\le\theta\le 2\pi$

$0\le r\le 1$

⑤ Integrate

$$\text{Flux} = \iint_R (\vec{F}\cdot\vec{n})\sqrt{1+f_x^2+f_y^2}\,dA$$

$z=2 \Rightarrow z^2=4$

$$= \iint_R (4)(1)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 4r\,dr\,d\theta$$

$$= 4\left[\int_0^{2\pi} d\theta\right]\left[\int_0^1 r\,dr\right]$$

$$= 4\left[\theta\Big|_0^{2\pi}\right]\left[\tfrac{1}{2}r^2\Big|_0^1\right]$$

$$= 4(2\pi)\left[\tfrac{1}{2}\right]$$

$$= \boxed{4\pi}$$

$\vec{F}$ flows upward

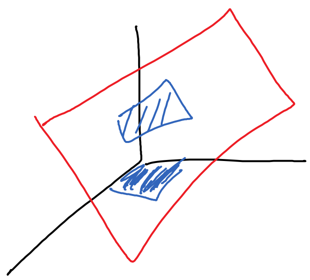
$\underline{\text{Ex}}$ Set up the double integral to find upward flux of

$$\vec{F} = \langle 2xy, 2yz, 2xz \rangle$$

through the portion of the plane

$$x + y + 3z = 2$$

that is above the square $0 \le x \le 1,\ 0 \le y \le 1$.



① Normal

$$\underbrace{x + y + 3z - 2}_{G(x,y,z)} = 0$$

$$\nabla G = \langle 1, 1, 3 \rangle \quad \text{Upwards}$$

$$\vec{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{\langle 1, 1, 3 \rangle}{\sqrt{1^2 + 1^2 + 3^2}} = \left\langle \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle$$

② $\vec{F} \cdot \vec{n} = \langle 2xy, 2yz, 2xz \rangle \cdot \left\langle \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle$

$$= \frac{1}{\sqrt{11}} \left[ 2xy + 2yz + 6xz \right]$$

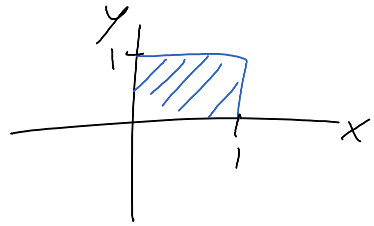
③ Jacobian $\quad x + y + 3z = 2$

$$z = -\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{11}{9}} = \frac{\sqrt{11}}{3}$$

④ Shadow Region

$0 \le x \le 1, \quad 0 \le y \le 1$




⑤ Integrate

$$\text{Flux} = \iint_R (\vec{F} \cdot \vec{n}) \sqrt{1 + f_x^2 + f_y^2} \, dx\,dy$$

$$\boxed{= \int_0^1 \int_0^1 \frac{1}{\sqrt{11}} \left[ 2xy + 2y\left(-\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}\right) + 6x\left(-\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}\right) \right] \frac{\sqrt{11}}{3} \, dx\,dy}$$

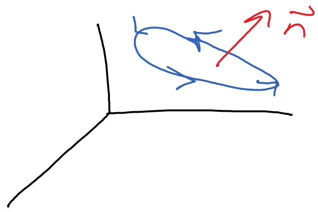
This would actually not be totally horrible to integrate.

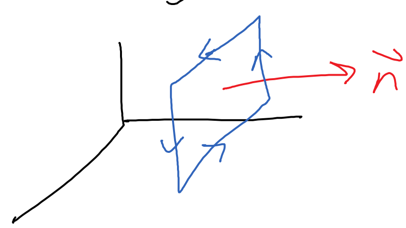
You just have to first do the algebra to multiply out and collect terms.

This would be do-able as a homework problem.

# 9.14 Stokes' Theorem

**Def** A normal vector $\vec{n}$ to a surface is *positively oriented* if the circulation around the closed curve on the perimeter follows the right-hand rule.

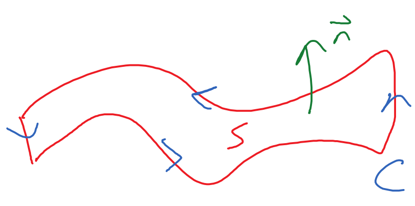


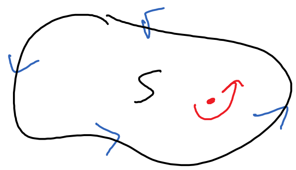

**Theorem** Stokes' Theorem

Let $S$ be a piecewise smooth orientable surface. Let $C$ be the piecewise smooth closed boundary curve of $S$. Let $\vec{n}$ be the unit normal vector to $S$ with positive orientation compatible with $C$. Let $\vec{F}$ be a smooth vector field. Then

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$



Idea  Flat surface $S$



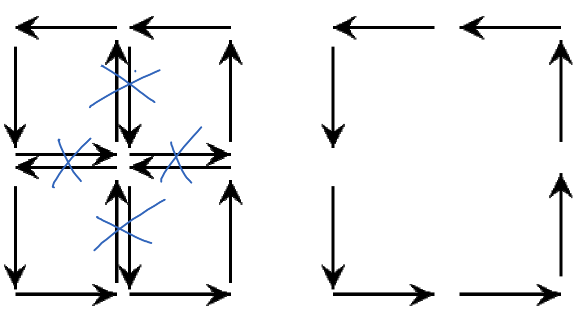

Recall $\nabla\times\vec{F}$ measures rotation.

$(\nabla\times\vec{F})\cdot\vec{n}$ is the normal component of rotation

"micro circulation"

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS = \text{Total circulation on } S$$

$= \text{Circulation around perimeter of } S$

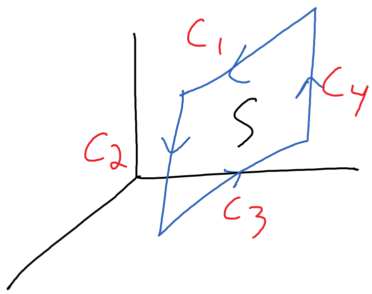


$$= \oint_C \vec{F}\cdot\vec{T}\, ds$$

The micro-circulations in the interior cancel each other out.

This leaves us with just the circulation around the boundary C.

We use Stokes' Theorem to calculate line integral around a complicated path.




Calculate circulation.

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} \vec{F}\cdot\vec{T}\,ds + \int_{C_2} \vec{F}\cdot\vec{T}\,ds + \int_{C_3} \vec{F}\cdot\vec{T}\,ds + \int_{C_4} \vec{F}\cdot\vec{T}\,ds$$

Use Stokes' Theorem

$$\text{Circulation} = \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$$

So we could calculate the circulation over a rectangular path in two ways.

1.) Parametrize each line segment and compute 4 line integrals.

OR

2.) Use Stokes' Theorem and calculate one surface integral.

Surface integrals are generally harder to compute than line integrals. So we have to choose between computing 4 easy integrals or 1 hard integral.

We would only use Stokes' Theorem if the path is sufficiently complicated.