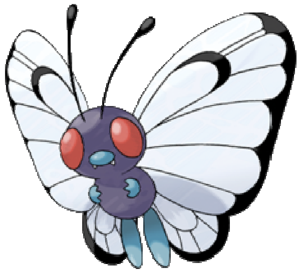


Lecture 3: Arc Length & Curvature

Butterfree's Goals for the Day

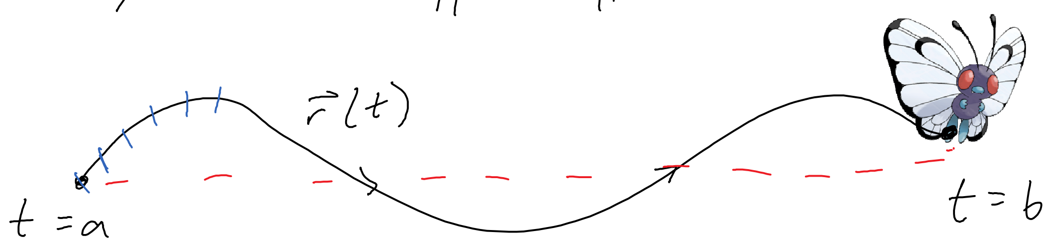
- Learn how to calculate the arc length and curvature of a moving object
- Describe the Frenet frame (T, N, B)
- Introduce multivariable functions and partial derivatives

9.3 Curvature + Components of Acceleration

Position $\vec{r}(t)$

Velocity $\vec{v}(t) = \vec{r}\,'(t)$

Speed $\|\vec{v}(t)\|$



How far did Butterfree fly?

The distance is not equal to the straight line distance between start and end points.

Distance = Rate × Time

$= \text{Sum}\left(\begin{matrix}\text{speed at} \\ \text{time } t\end{matrix} \times \begin{matrix}\text{small time} \\ \text{window}\end{matrix} \right)$

$$= \int_a^b \|\vec{v}(t)\|\, dt$$

Def The <u>arc length</u> of a curve $\vec{r}(t)$, $a \le t \le b$ is given by

$$L = \int_a^b \|\vec{v}(t)\| \, dt$$

Ex Compute the length of the curve

$$\vec{r}(t) = \langle t^2, \tfrac{1}{3}t^3, \tfrac{3}{2}t^2 \rangle \quad 0 \le t \le 2$$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, t^2, 3t \rangle$$

$$L = \int_a^b |\vec{v}(t)| dt$$

$$= \int_0^2 \sqrt{(2t)^2 + (t^2)^2 + (3t)^2} \, dt$$

$$= \int_0^2 \sqrt{4t^2 + t^4 + 9t^2} \, dt$$

$$= \int_0^2 \sqrt{13t^2 + t^4} \, dt$$

$$= \int_0^2 \sqrt{t^2(13 + t^2)} \, dt$$

$u = 13 + t^2$

$du = 2t\,dt$

$$= \frac{1}{2}\int_0^2 2t \sqrt{13 + t^2} \, dt$$

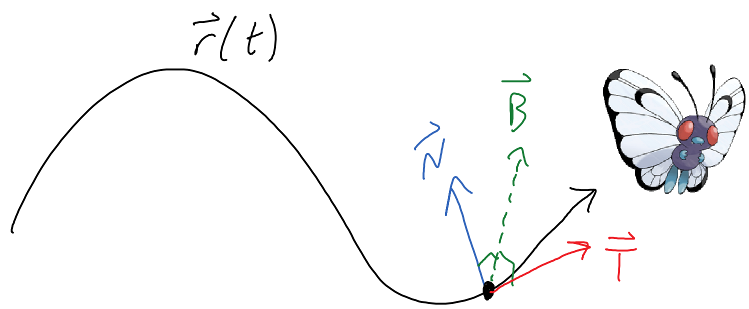
$$= \frac{1}{2}\int_{13}^{17} \sqrt{u} \, du$$

$t = 0 \to u = 13$

$t = 2 \to u = 17$

$$= \cancel{\frac{1}{2}}\,\frac{\cancel{2}}{3}\,u^{3/2}\Big|_{13}^{17}$$

$$= \boxed{\frac{1}{3}(17)^{3/2} - \frac{1}{3}(13)^{3/2}}$$



$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}}{|\vec{v}|}$$ (Centrifugal Force)

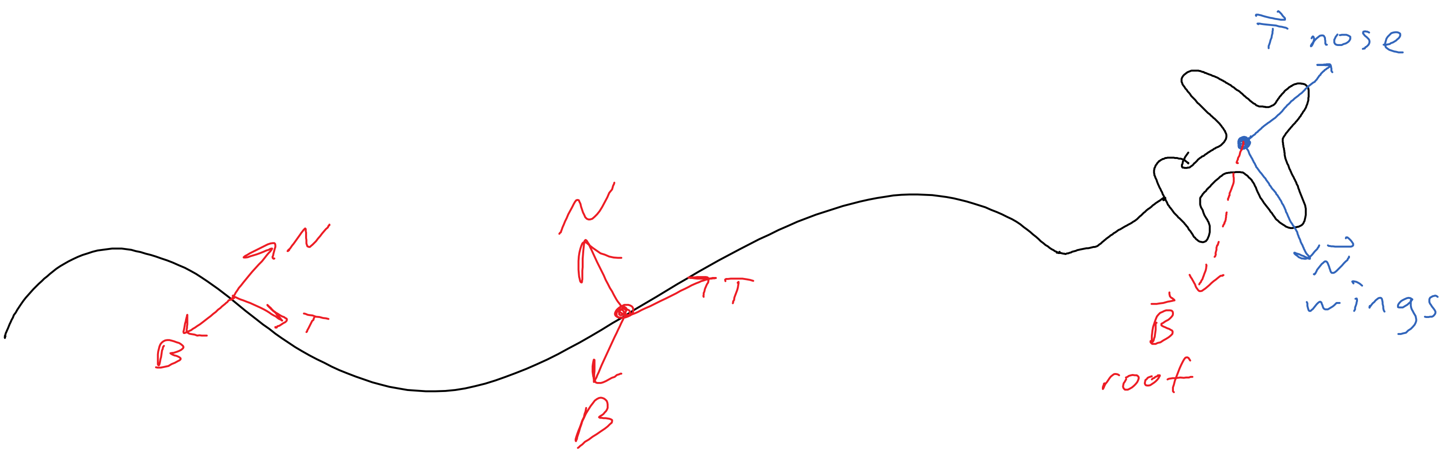
Unit Tangent Vector $\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}}{|\vec{v}|}$ (Centrifugal Force)

Unit Normal Vector $\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|}$ (Centripetal Force)

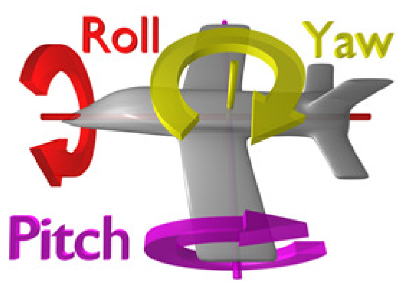
Unit Binormal Vector $\vec{B} = \vec{T} \times \vec{N}$

Def The **Frenet Frame** is the set of vectors that describe the relative coordinate axes along a path $\vec{r}(t)$.

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}, \quad \vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|}, \quad \vec{B} = \vec{T} \times \vec{N}$$

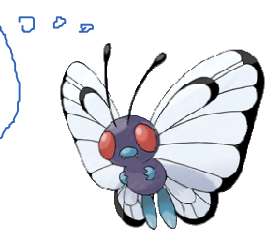


Rotations around Frenet Frame vectors

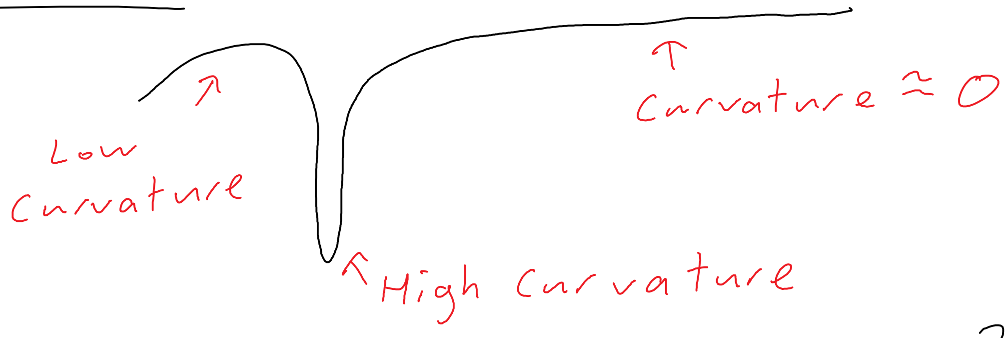


Calculating the Frenet Frame is very important for aerospace engineers who want to simulate aircraft flight. This tells us how the aircraft would twist and turn along a turbulent path.

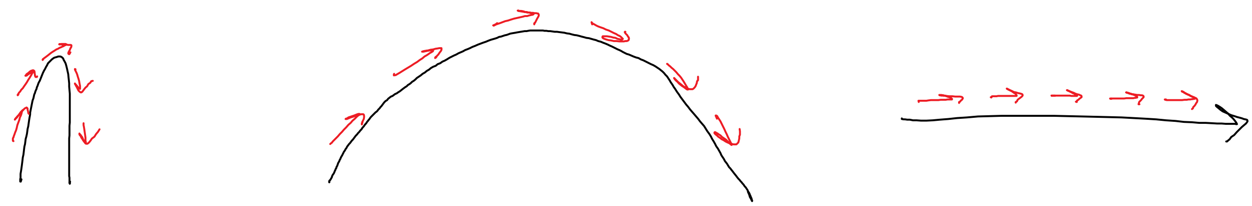
It's lso important for video game designers.



Curvature



How do we measure curvature?



Look at how fast the tangent vector is changing: $\|\vec{T}'\|$

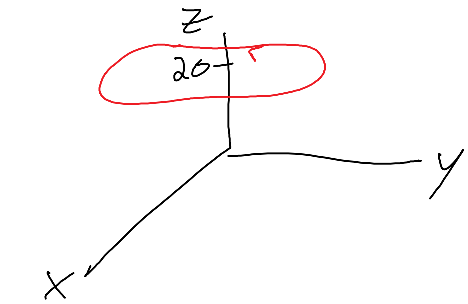
We have to normalize for speed along curve.

$$\frac{\|\vec{T}'\|}{\|\vec{v}\|}$$ ← Normalize by speed

Def The <u>curvature</u> is defined as

$$K = \frac{\|\vec{T}'\|}{\|\vec{v}\|}$$

Ex Butterfree flies at altitude of 20m in a circle of radius $R>0$. Find curvature of Butterfree's path.



$$\vec{r}(t) = \langle R\cos t,\ R\sin t,\ 20 \rangle$$

$$K = \frac{\|\vec{T}'\|}{\|\vec{v}\|}$$

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\vec{v} = \vec{r}\,'$$

$$\vec{v} = \vec{r}\,'(t) = \langle -R\sin t,\ R\cos t,\ 0 \rangle$$

$$\|\vec{v}\| = \sqrt{(-R\sin t)^2 + (R\cos t)^2 + (0)^2}$$

$$= \sqrt{R^2\sin^2 t + R^2\cos^2 t}$$

$$= \sqrt{R^2(\underbrace{\cancel{\sin^2 t + \cos^2 t}}_{1})}$$

$$= R$$

$$\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -R\sin t, R\cos t, 0\rangle}{R} = \langle -\sin t, \cos t, 0\rangle$$

$$\vec{T}' = \langle -\cos t, -\sin t, 0\rangle$$

$$\|\vec{T}'\| = \sqrt{(-\cos t)^2 + (-\sin t)^2 + (0)^2} = 1$$

$$K = \frac{\|\vec{T}'\|}{\|\vec{v}\|} = \boxed{\frac{1}{R}}$$

<u>Fact</u> The curvature of a circle of radius R is $\frac{1}{R}$.

Bigger circle $\Rightarrow$ Smaller curvature

9.4 Partial Derivatives

1 Variable $f(x)$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2 Variables $f(x, y)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

<u>Ex</u> Compute first partial derivatives of

$$f(x,y) = x^2 \sin y + 3x - 4y$$

$$\frac{\partial f}{\partial x} = 2x \sin y + 3$$

$$\frac{\partial f}{\partial y} = x^2 \cos y - 4$$

Notation

Leibnitz $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial y^2},\ \frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial^2 f}{\partial y \partial x}$$

Subscript f_x, f_y

$$f_{xx},\ f_{yy},\ f_{yx},\ f_{xy}$$

$\underline{\text{Ex}}$ $f(x,y) = e^{x^2 y} + \cos(x+3y)$

Compute first and second partial derivatives.

$f_x = 2xye^{x^2y} - \sin(x+3y)$

$f_y = x^2 e^{x^2y} - 3\sin(x+3y)$

$f_{xx} = 2ye^{x^2y} + 4x^2y^2e^{x^2y} - \cos(x+3y)$

$f_{yy} = x^4 e^{x^2y} - 9\cos(x+3y)$

$f_{xy} = 2xe^{x^2y} + 2x^3ye^{x^2y} - 3\cos(x+3y)$

same!

$f_{yx} = 2xe^{x^2y} + 2x^3ye^{x^2y} - 3\cos(x+3y)$

Theorem Mixed Derivatives Theorem

If $f(x,y)$ and its derivatives are continuous on an open domain, then

$$f_{xy} = f_{yx}$$

This theorem extends to higher order derivatives.
For example, you get the same answer if you differentiate with respect to x twice and y once, regardless of the order in which you did it.

$$f_{xxy} = f_{yxx} = f_{xyx}$$