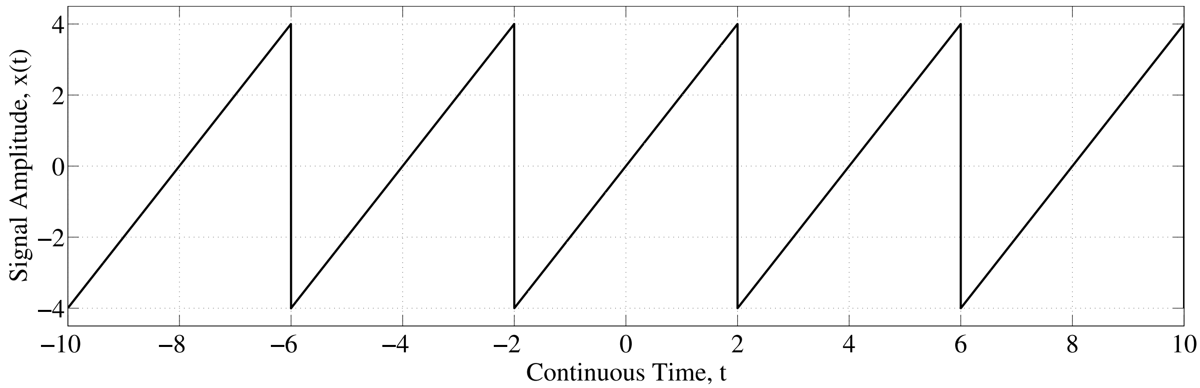
ELEC 309
Signals and Systems
Homework 5 Solutions

Frequency-Domain Analysis of Signals



1. For the periodic continuous-time signal $x(t)$ shown above:

We can write the periodic continuous-time signal above as

$$x(t) = 2t - 4k \text{ for } k - 2 \leq t < k + 2$$

for all integers k. The fundamental period is $T_0 = 4$ seconds, and the fundamental angular frequency is $\omega_0 = 2\pi/T_0 = \pi/2$ rad/sec.

(a) Find the exponential Fourier series representation of $x(t)$.

The average value or dc component is given by

$$D_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = \frac{1}{4}\int_{-2}^{2} 2tdt = \frac{1}{2}\left[\frac{t^2}{2}\right]_{-2}^{2} = \boxed{0,}$$

To find the rest of the exponential Fourier series coefficients, we will using the following identities:

$$\int_a^b xe^{cx}dx = \frac{1}{c^2}e^{cx}\left(cx-1\right)\Big|_a^b \quad \text{with } c = -jn\pi/2$$

$$\cos\left(\theta\right) = \frac{e^{+\theta}+e^{-\theta}}{2} \qquad \text{(Euler's)}$$

$$\sin\left(\theta\right) = \frac{e^{+\theta}-e^{-\theta}}{j2} \qquad \text{(Euler's)}$$

The rest of the exponential Fourier series coefficients are given by

$$
\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{4}\int_{-2}^{2} 2te^{-jn\pi t/2}dt = \frac{1}{2}\int_{-2}^{2} te^{-jn\pi t/2}dt \\
&= \frac{1}{2}\left(\frac{4}{-n^2\pi^2}\right)\left[e^{-jn\pi t/2}\left(-jn\pi t/2-1\right)\right]_{-2}^{2} \\
&= \left(\frac{2}{-n^2\pi^2}\right)\left[e^{-jn\pi}\left(-jn\pi-1\right)-e^{jn\pi}\left(jn\pi-1\right)\right] \\
&= \left(\frac{j2}{n\pi}\right)\left[e^{jn\pi}+e^{-jn\pi}\right]-\left(\frac{2}{n^2\pi^2}\right)\left[e^{jn\pi}-e^{-jn\pi}\right] \\
&= \left(\frac{j4}{n\pi}\right)\left[\frac{e^{jn\pi}+e^{-jn\pi}}{2}\right]-\left(\frac{j4}{n^2\pi^2}\right)\left[\frac{e^{jn\pi}-e^{-jn\pi}}{j2}\right] \\
&= \left(\frac{j4}{n\pi}\right)\cancelto{(-1)^n}{\cos(n\pi)}-\left(\frac{j4}{n^2\pi^2}\right)\cancelto{0}{\sin(n\pi)} = \boxed{\frac{j4\cos(n\pi)}{n\pi}=\frac{j4(-1)^n}{n\pi}}.
\end{aligned}
$$

Therefore, we can represent $x(t)$ using the exponential Fourier series representation given by

$$
x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \boxed{\sum_{n=-\infty, n\neq 0}^{\infty} \frac{j4\cos(n\pi)}{n\pi}e^{jn\pi t/2} = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{j4(-1)^n}{n\pi}e^{jn\pi t/2}}.
$$

(b) Find the trigonometric Fourier series representation of $x(t)$.

Using our identities from the class notes, we have

$$
\begin{aligned}
a_0 &= D_0 = \boxed{0,} \\
a_n &= 2\mathrm{Re}\{D_n\} = \boxed{0,}\ \text{and} \\
b_n &= -2\mathrm{Im}\{D_n\} = \boxed{\frac{-8\cos(n\pi)}{n\pi} = \frac{-8(-1)^n}{n\pi}}.
\end{aligned}
$$

Therefore, we can represent $x(t)$ using the trigonometric Fourier series representation given by

$$
\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) \\
&= \boxed{\sum_{n=1}^{\infty}\frac{-8\cos(n\pi)}{n\pi}\sin(n\pi t/2) = \sum_{n=1}^{\infty}\frac{-8(-1)^n}{n\pi}\sin(n\pi t/2).}
\end{aligned}
$$

(c) Find the harmonic/compact Fourier series representation of $x(t)$.

Using our identities from the class notes, we have

$$C_0 = D_0 = \boxed{0,}$$
$$C_n = 2\left|D_n\right| = \boxed{\frac{8}{n\pi},} \text{ and}$$
$$\theta_n = \angle D_n = \begin{cases} -\pi/2 & \text{for } n \text{ even} \\ \pi/2 & \text{for } n \text{ odd} \end{cases} = \boxed{-\frac{\pi}{2}\left(-1\right)^n = \frac{\pi}{2}\left(-1\right)^{n+1}.}$$

Therefore, we can represent $x(t)$ using the harmonic/compact Fourier series representation given by

$$\begin{aligned} x(t) &= C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) \\ &= \boxed{\sum_{n=1}^{\infty} \frac{8}{n\pi} \cos\left(n\pi t/2 - \frac{\pi}{2}\left(-1\right)^n\right) = \sum_{n=1}^{\infty} \frac{-8\left(-1\right)^n}{n\pi} \cos\left(n\pi t/2 - \pi/2\right).} \end{aligned}$$

(d) Verify Parseval's theorem for $x(t)$, using the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The power in the time domain is given by

$$P_x = \frac{1}{T_0}\int_{T_0} x^2(t)dt = \frac{1}{4}\int_{-2}^{2} 4t^2 dt = \left[\frac{t^3}{3}\right]_{-2}^{2} = \boxed{\frac{16}{3} = 5.3333.}$$

The power in the frequency domain is given by

$$\begin{aligned} P_x &= \sum_{n=-\infty}^{\infty} \left|D_n\right|^2 = 2\sum_{n=1}^{\infty} \frac{16}{n^2\pi^2} = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{32}{\pi^2}\left(\frac{\pi^2}{6}\right) = \boxed{\frac{16}{3} = 5.3333.} \\ &= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{64}{n^2\pi^2} = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{32}{\pi^2}\left(\frac{\pi^2}{6}\right) = \boxed{\frac{16}{3} = 5.3333.} \\ &= C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{64}{n^2\pi^2} = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{32}{\pi^2}\left(\frac{\pi^2}{6}\right) = \boxed{\frac{16}{3} = 5.3333.} \end{aligned}$$

Thus, Parseval's theorem has been verified.

(e) Using MATLAB, write a script m-file to plot the Fourier spectra for the signal. (Plot $|D_n|$ vs. ω and $\angle D_n$ vs. ω (where $\omega = n\omega_0$) on a single figure by using the *subplot* command.) Upload a copy of your MATLAB script m-file to the course website.

MATLAB code to plot the Fourier spectra:

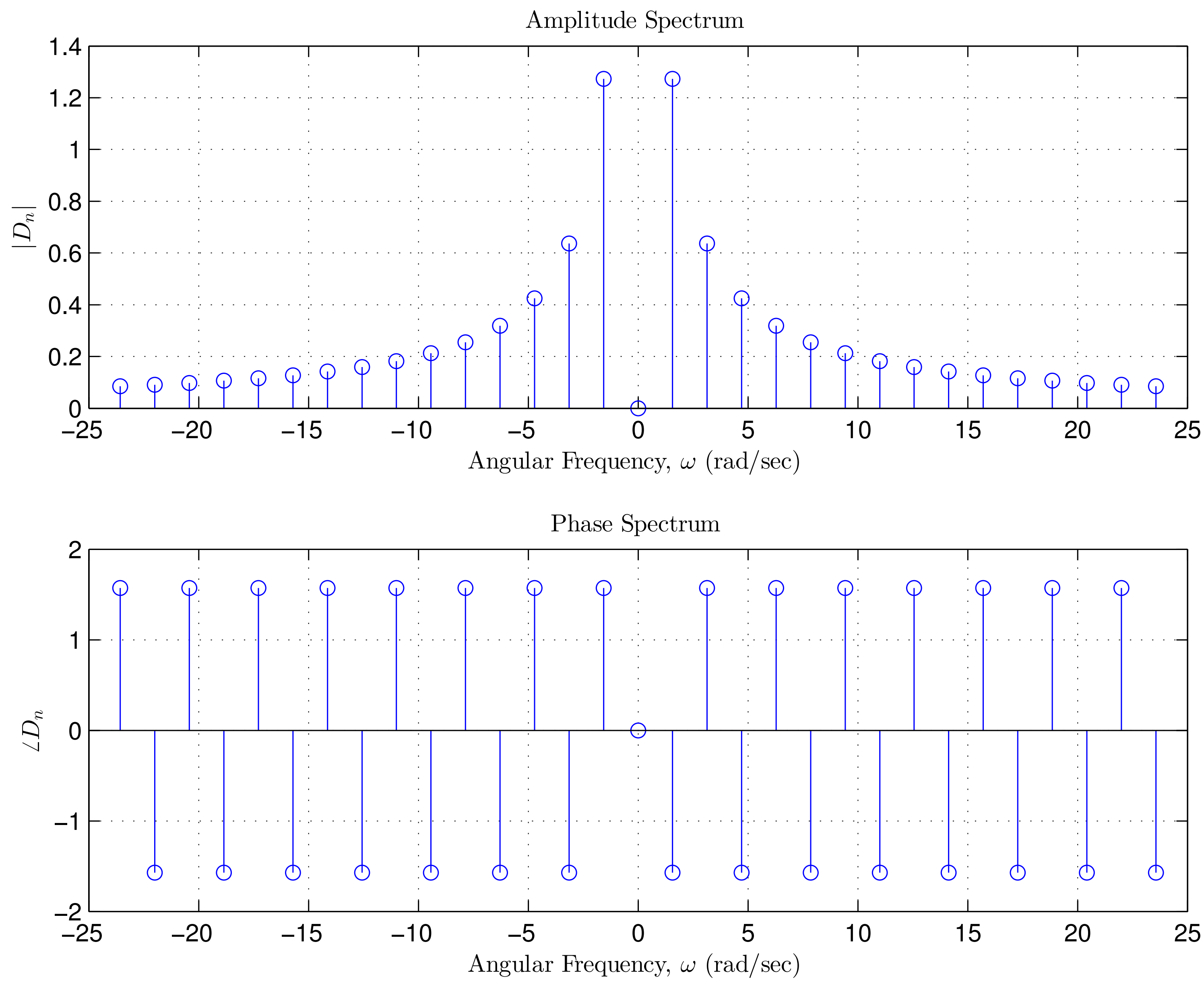
```
% Fundamental period and angular frequency
T0 = 4;
w0 = 2*pi/T0;

% Range of n and omega values
n = -15:15;
w = n*w0;

% Exponential Fourier coefficients determined by hand
Dn = 1i*4*(-1).^n./(n*pi);
% Need to explicitly specify D_0
Dn(n==0) = 0;

% Generate magnitude and phase of complex Fourier coefficients
magDn = abs(Dn);
phaseDn = angle(Dn);

% Plot Fourier spectra
figure(1)
% Plot magnitude spectrum
subplot(2,1,1), stem(w,magDn), grid on
xlabel('Angular Frequency, $$\omega$$ (rad/sec)','Interpreter','LaTeX');
ylabel('$$|D_n|$$','Interpreter','LaTeX');
title('Amplitude Spectrum','Interpreter','LaTeX');
% Plot phase spectrum
subplot(2,1,2), stem(w,phaseDn), grid on
xlabel('Angular Frequency, $$\omega$$ (rad/sec)','Interpreter','LaTeX');
ylabel('$$\angle D_n$$','Interpreter','LaTeX');
title('Phase Spectrum','Interpreter','LaTeX');
```


Amplitude Spectrum
$|D_n|$
Angular Frequency, ω (rad/sec)
Phase Spectrum
$\angle D_n$
Angular Frequency, ω (rad/sec)