Math 335 HW 9 NAME: KEY
Due Wednesday 10/29 5:15pm

Practice Problems *(Do not turn in.)*
Sec 5.2 #3, 15, 19
Sec 5.3 #3, 5, 7

Print out this assignment and write all work directly on this worksheet. Do not attach extra pages. Show all work. Your answers must be clear and legible. All pages must be stapled. Homework may be submitted within 24 hours of the due date with an automatic 2 point deduction. After Thursday 5:00pm, no late homework will be accepted for any reason.

1.) *(Sec 5.2 #23)* We want to find a solution about x=0 for the ODE:

$$9x^2y'' + 9x^2y' + 2y = 0.$$

a.) *[3 points]* Explain why x=0 is a *regular* singular point.

$$y'' + y' + \frac{2}{9x^2}y = 0$$

$$\left.\begin{array}{l} xP(x) = x \\ x^2Q(x) = \frac{2}{9} \end{array}\right] \text{Analytic at } x=0 \Rightarrow \text{regular}$$

#1 continued...

b.) *[5 points]* Since x=0 is a singular point, there will be no power series solution. But since x=0 is a regular singular point, we may be able to find a *Frobenius* series solution. Plug

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

into the ODE

$$9x^2 y'' + 9x^2 y' + 2y = 0.$$

By examining the coefficients of x^r, find the *indicial equation* of the ODE and use this two find the two *indicial roots* r_1 and r_2. Looking at the coefficients of x^{n+r}, you should find a general recurrence relation that depends on n and r, something like $c_n = f(n,r)c_{n-1}$.

$$9x^2 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + 9x^2 \underbrace{\sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}}_{\text{shift to } x^{n+r}} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 9c_n (n+r)(n+r-1) x^{n+r} + \sum_{k=1}^{\infty} 9c_{k-1}(k+r-1)x^{k+r} + \sum_{n=0}^{\infty} 2c_n x^{n+r} = 0$$

$$9c_0 r(r-1)x^r + 2c_0 x^r + \sum_{n=1}^{\infty}\left[9c_n(n+r)(n+r-1) + 9c_{n-1}(n+r-1) + 2c_n\right]x^{n+r} = 0$$

$\underline{x^r}$: $[9r(r-1)+2]c_0 = 0$

$$9r^2 - 9r + 2 = 0$$

$$(3r-1)(3r-2) = 0$$

$\underline{x^{n+r}}$: $9c_n(n+r)(n+r-1) + 9c_{n-1}(n+r-1) + 2c_n = 0$

$r_1 = \underline{\frac{1}{3}}$ $r_2 = \underline{\frac{2}{3}}$ $c_n = \dfrac{-9(n+r-1)}{9(n+r)(n+r-1)+2} c_{n-1}$

#1 continued....

c.) *[3 points]* Plug your first root r_1 into your recurrence relation. This should give you a specific recurrence relation for the first solution

$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$. Your answer should be something like $a_n = f(n)a_{n-1}$.

Plug $r_1 = \frac{1}{3}$ into recurrence.

$$a_n = \frac{-9(n+\frac{1}{3}-1)}{9(n+\frac{1}{3})(n+\frac{1}{3}-1)+2}\, a_{n-1}$$

$$= \frac{-9(n-\frac{2}{3})}{9(n^2-\frac{1}{3}n-\frac{2}{9})+2}\, a_{n-1} = \frac{-9n+6}{9n^2-3n}\, a_{n-1}$$

$$n \geq 1$$

d.) *[3 points]* Use your answer to (c) to write the first 3 terms of the series solution y_1 in terms of the unknown constant a_0.

$\underline{n=1}$ $\quad a_1 = \frac{-9+6}{9-3} a_0 = \frac{-3}{6} a_0 = -\frac{1}{2} a_0$

$\underline{n=2}$ $\quad a_2 = \frac{-9(2)+6}{9(2)^2-3(2)} a_1 = \frac{-12}{30}\left(-\frac{1}{2}a_0\right) = \frac{1}{5} a_0$

$\underline{n=3}$ $\quad a_3 = \frac{-9(3)+6}{9(3)^2-3(3)} a_2 = \frac{-21}{72}\left(\frac{1}{5}a_0\right) = -\frac{7}{120} a_0$

$$y_1 = x^{\frac{1}{3}}\left[a_0 + \left(-\frac{1}{2}a_0\right)x + \frac{1}{5}a_0\, x^2 + \left(-\frac{7}{120}a_0\right)x^3 + \cdots\right]$$

#1 continued....

e.) *[3 points]* Find the recurrence relation for the series solution

$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$ for the your second root r_2. Your answer should be something like $b_n = f(n) b_{n-1}$.

Plug $r_2 = \frac{2}{3}$ into recurrence.

$$b_n = \frac{-9(n+\frac{2}{3}-1)}{9(n+\frac{2}{3})(n+\frac{2}{3}-1)+2} b_{n-1}$$

$$= \frac{-9(n-\frac{1}{3})}{9(n^2+\frac{1}{3}n-\frac{2}{9})+2} b_{n-1} = \frac{-9n+3}{9n^2+3n} b_{n-1}$$

f.) *[3 points]* Use your answer to (e) to write the first 3 terms of the series solution y_2 in terms of the unknown constant b_0.

$\underline{n=1}$ $b_1 = \frac{-9+3}{9+3} b_0 = \frac{-6}{12} b_0 = -\frac{1}{2} b_0$

$\underline{n=2}$ $b_2 = \frac{-9(2)+3}{9(2)^2+3(2)} b_1 = \frac{-15}{42}\left(-\frac{1}{2} b_0\right) = \frac{5}{28} b_0$

$\underline{n=3}$ $b_3 = \frac{-9(3)+3}{9(3)^2+3(3)} b_2 = \frac{-24}{90}\left(\frac{5}{28} b_0\right) = -\frac{1}{21} b_0$

$$y_2 = x^{\frac{2}{3}}\left[b_0 + \underline{-\tfrac{1}{2} b_0}\, x + \underline{\tfrac{5}{28} b_0}\, x^2 + \underline{-\tfrac{1}{21} b_0}\, x^3 + \cdots\right]$$

The general solution of the ODE is then $y = y_1 + y_2$.