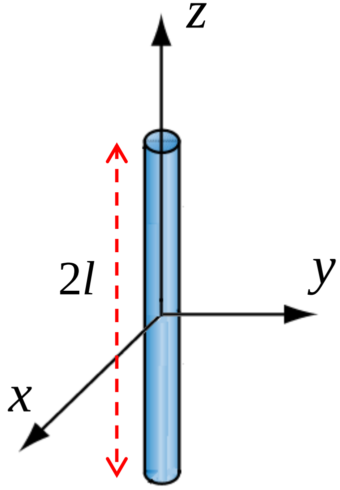
Example: Line Charge, Finite

Calculate the electric field **E** at any point *P* in the *r*-ϕ plane due to a line charge of length $2l$, centered on the origin and extending along the *z* axis, with a constant charge density ρ_l .



$$\mathbf{R}-\mathbf{R}' = r\,\hat{\mathbf{r}} - z'\,\hat{\mathbf{z}} \;\;, \quad dl = dz'$$

$$\mathbf{E} = \int_L \frac{\rho_l dl}{4\pi\varepsilon_0}\frac{r\,\hat{\mathbf{r}} - z'\,\hat{\mathbf{z}}}{\left(r^2+z'^2\right)^{3/2}}$$

$$= \frac{\rho_l dz'}{4\pi\varepsilon_0}\left\{\int_{-l}^{+l}\frac{r\,\hat{\mathbf{r}}}{\left(r^2+z'^2\right)^{3/2}} - \int_{-l}^{+l}\frac{z'\,\hat{\mathbf{z}}}{\left(r^2+z'^2\right)^{3/2}}\right\}$$

$$= \hat{\mathbf{r}}\;\frac{\rho_l\cdot r}{4\pi\varepsilon_0}\int_{-l}^{+l}\frac{dz'}{\left(r^2+z'^2\right)^{3/2}} = \boxed{\frac{\rho_l}{2\pi\varepsilon_0}\cdot\frac{l}{r\sqrt{r^2+l^2}}\,\hat{\mathbf{r}}}$$

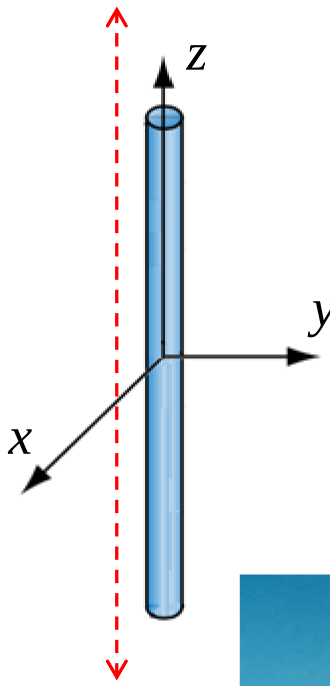
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_L \rho_l dl\,\frac{\mathbf{R}-\mathbf{R}'}{\left|\mathbf{R}-\mathbf{R}'\right|^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_S \rho_s dS\,\frac{\mathbf{R}-\mathbf{R}'}{\left|\mathbf{R}-\mathbf{R}'\right|^3}$$

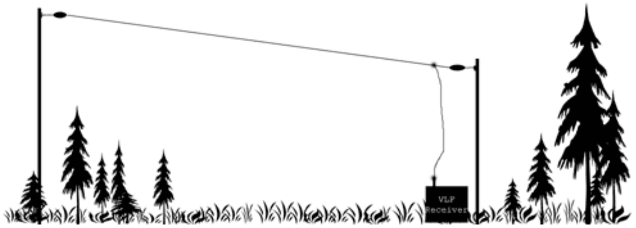
$$= \frac{1}{4\pi\varepsilon_0}\int_V \rho_v dV\,\frac{\mathbf{R}-\mathbf{R}'}{\left|\mathbf{R}-\mathbf{R}'\right|^3}$$

Example: Line Charge, Infinite

Calculate the electric field **E** at any point $P\ (r, \phi, z)$ due to a line charge of <u>infinite</u> length along the z axis, with a constant charge density ρ_l .



Very-Low-Frequency (VLF) antennas



Dr. Gregory J. Mazzaro

Spring 2015

ELEC 318 – *Electromagnetic Fields*

Lecture 4(b)

Electrostatic Fields: Gauss' Law & Electric Potential

THE CITADEL, THE MILITARY COLLEGE OF SOUTH CAROLINA
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Electric Flux Density

electric flux density, D (C/m^2)

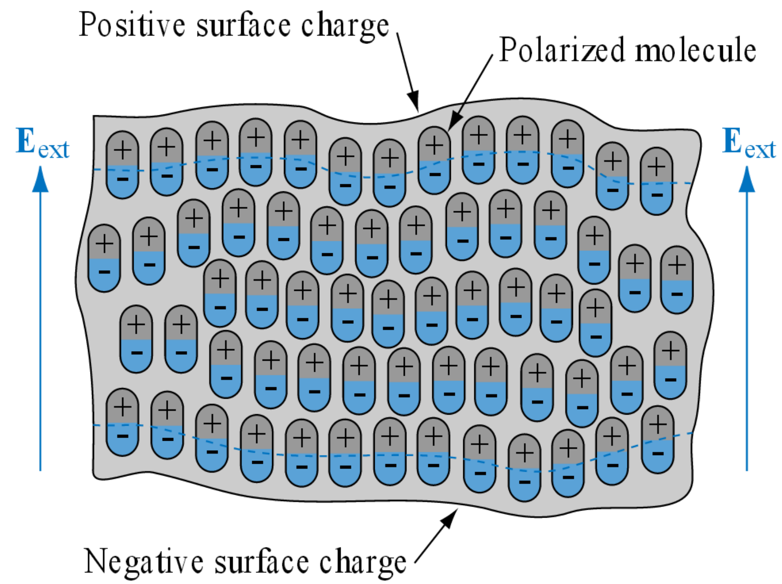
$$\mathbf{D} = \varepsilon \mathbf{E}$$

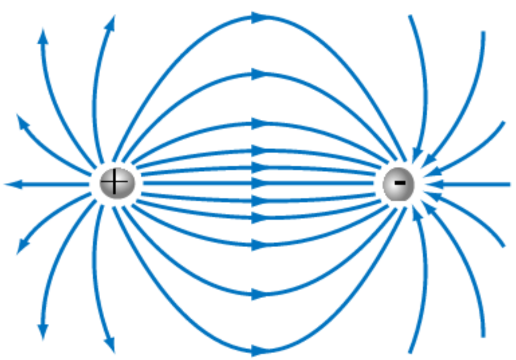
-- similar to electric *field intensity*, but it encompasses permittivity, ε

-- for a given **E** field, a measure of how *polarized* a particular material is

-- useful for particular electrostatic calculations (e.g. for analyzing fields at boundaries between material media)

-- 1 line of flux begins on a 1-C positive charge & terminates on a 1-C negative charge

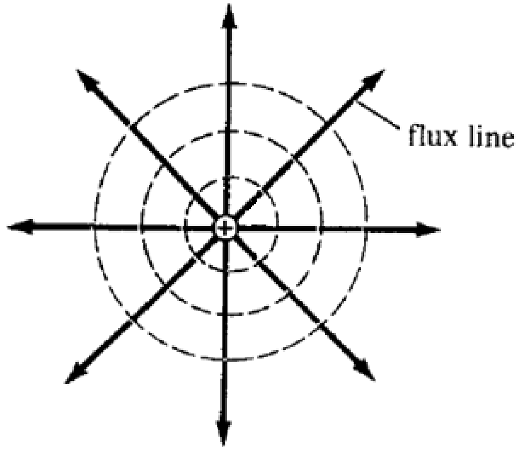




near a single point charge q in free space ε_0 …

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{R} - \mathbf{R}'}{\left|\mathbf{R} - \mathbf{R}'\right|^3}$$

$$\mathbf{D} = \frac{q}{4\pi} \frac{\mathbf{R} - \mathbf{R}'}{\left|\mathbf{R} - \mathbf{R}'\right|^3}$$



Gauss' Law

Gauss's Law: -- 1 of 4 of Maxwell's Equations
-- 1 of 2 that govern the behavior of *electric fields*
-- can be written in 2 ways:

Q = charge contained in a "Gaussian" surface
$\mathbf{D}$ = electric flux density
$d\mathbf{S}$ is *normal* to the surface and directed *outward*
ρ_V = volume charge density (C/m^3)

$$\nabla \cdot \mathbf{D} = \rho_v$$

differential form

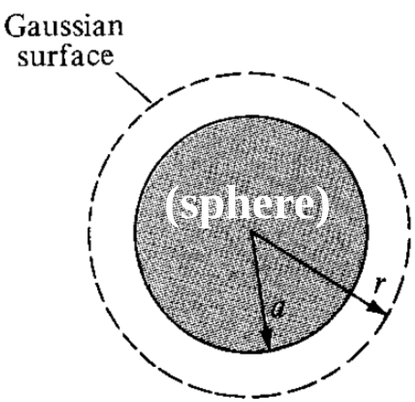
$$\int_V \rho_V \, dV = Q$$

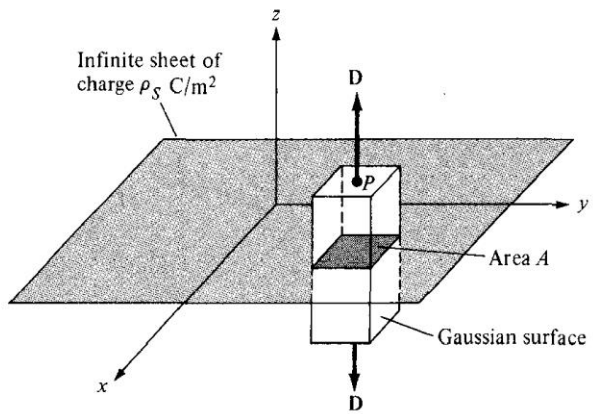
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{D} \, dV$$

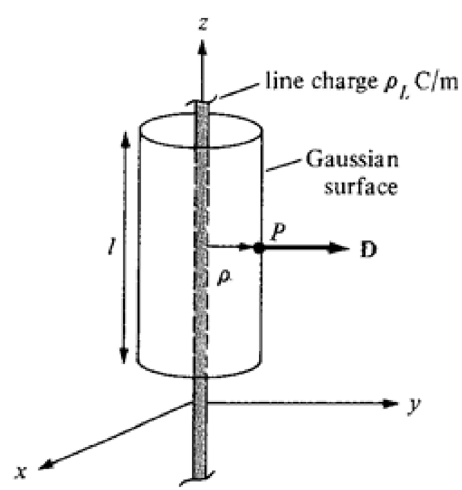
(Divergence Theorem)

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

integral form







→ **an alternative to Coulomb's Law for determining *electric field*, under <u>symmetry</u>**

Gauss' Law, Integral Form: "How-To"

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

(1) Identify the symmetry of the problem (e.g. planar, cylindrical, spherical) .

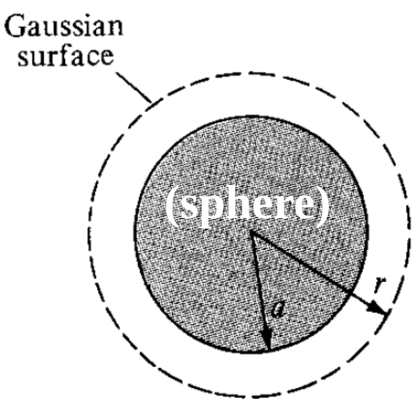
(2) Express **D** as a vector according to the symmetry (e.g. $\mathbf{D} = \mathrm{D}_z\mathbf{z}$, $\mathbf{D} = \mathrm{D}_R\mathbf{R}$) .

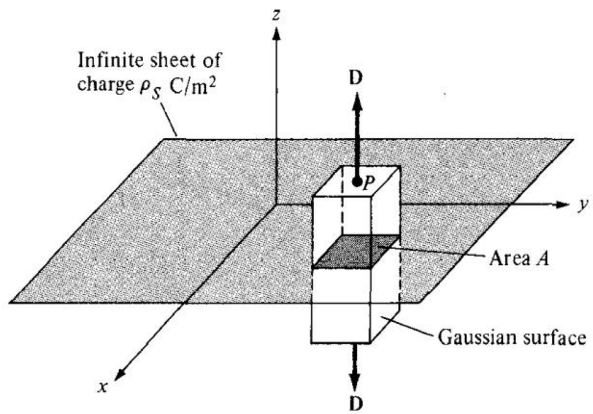
(3) Draw a Gaussian surface in accordance with the symmetry, which (a) passes through the observation point and (b) is perfectly perpendicular or parallel to **D** .

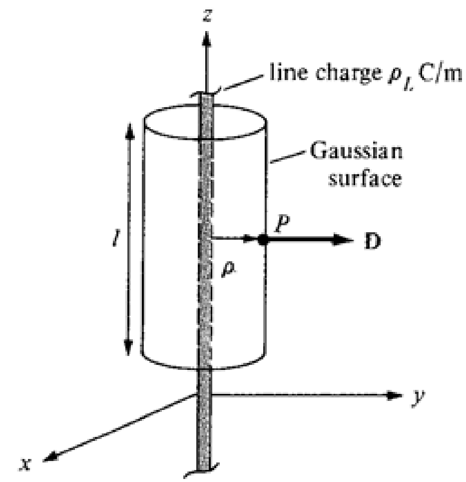
(4) Solve for charge enclosed by the Gaussian surface as the (line/surface/volume) integral of charge.

(5) Evaluate the flux integral, set the result equal to the charge enclosed, and solve for the scalar component(s) of **D** .

(6) Write **D** as a vector using (2) . Solve for **E** by dividing the result by ε .





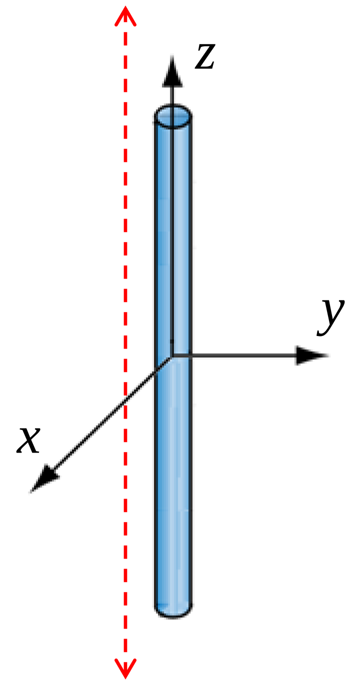


→ an alternative to Coulomb's Law for determining *electric field*, under <u>symmetry</u>

Example: Line Charge, Infinite

Calculate the electric field **E** at any point P (r, ϕ, z) due to a line charge of infinite length along the z axis, with a constant charge density ρ_l , in free space.

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$



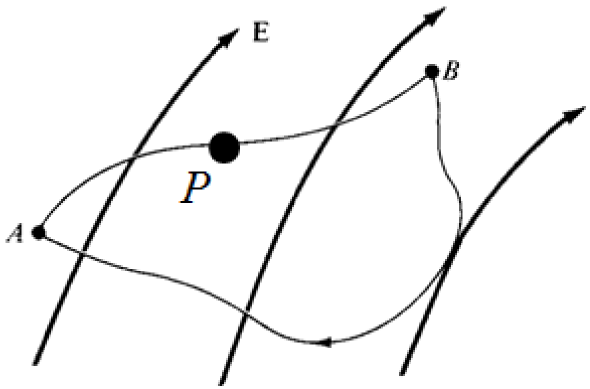
Electric (Scalar) Potential

electric potential (V)

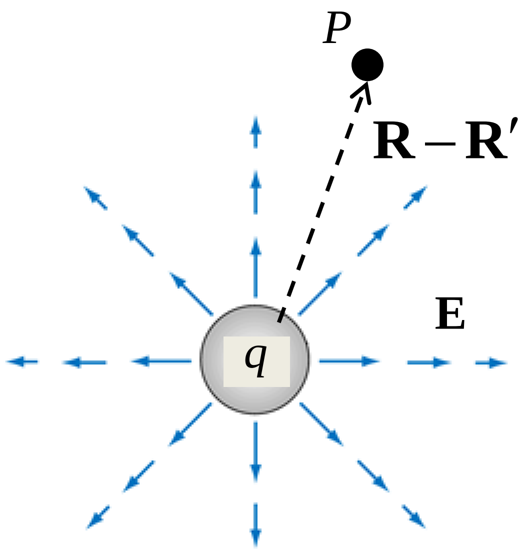
-- a measure of *work* required to move a unit charge through an **E** field:

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{l} = -q\int_A^B \mathbf{E}\cdot d\mathbf{l}$$

$$V = \frac{W}{q} \quad \Rightarrow \quad V_{AB} = -\int_A^B \mathbf{E}\cdot d\mathbf{l}$$



for a point charge q…



$$V_{\text{point}} = -\int_{\infty}^{|\mathbf{R}-\mathbf{R}'|} \mathbf{E}_{\text{point}}\cdot d\mathbf{l}$$

$$= -\int_{\infty}^{|\mathbf{R}-\mathbf{R}'|} \left(\frac{q}{4\pi\varepsilon_0}\frac{\mathbf{R}-\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|^3}\right)\cdot\left(\frac{\mathbf{R}-\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|}\right)dr$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\left\{\frac{1}{|\mathbf{R}-\mathbf{R}'|}-\frac{1}{\infty}\right\}$$

$$V_{\text{point}} = \frac{q}{4\pi\varepsilon_0|\mathbf{R}-\mathbf{R}'|}$$

V for a charge distribution:

$$dV = \frac{dq}{4\pi\varepsilon_0|\mathbf{R}-\mathbf{R}'|}$$

Electric (Scalar) Potential

electric potential (V)

-- a measure of *work* required to move a unit charge through an **E** field:

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{l} = -q\int_A^B \mathbf{E}\cdot d\mathbf{l}$$

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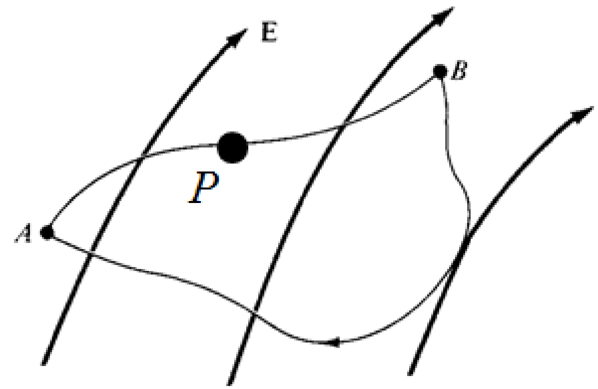


for a **static** electric field, V_{AB} is <u>independent of the path chosen from A to B</u>

$$V_{AP} - V_{PB} = V_{AB} \quad \forall \; P(x,y,z)$$

(Faraday's Law)

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \longrightarrow \boxed{\nabla\times\mathbf{E} = 0} \longrightarrow \boxed{\oint_L \mathbf{E}\cdot d\mathbf{l} = 0}$$

no time variation (i.e. *static* fields)

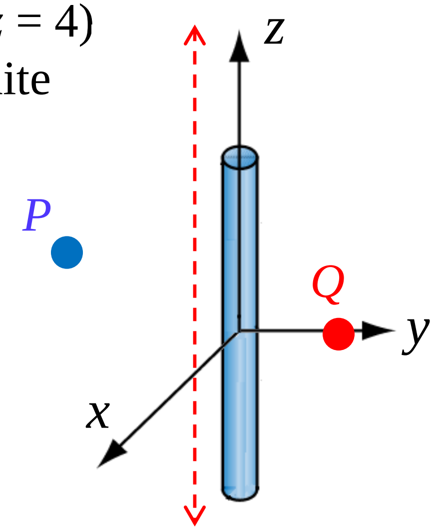
(Stokes' Theorem)

$$\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = \oint_L \mathbf{E}\cdot d\mathbf{l}$$

Example: V from E & Work

Calculate the difference in electric potential from point $P(x = 4, y = 0, z = 4)$ to point $Q(x = 0, y = 2, z = 0)$ in the presence of a line charge of infinite length along the z axis with a constant charge density of 16 pC/m .

Determine the work required to move 3 mC from P to Q .



$$V_{PQ} = -\int_P^Q \mathbf{E} \cdot d\mathbf{l}$$

$$V = \frac{W}{q}$$

$$\mathbf{E}_{\text{line charge}}^{\text{infinite}} = \frac{\rho_l}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$$

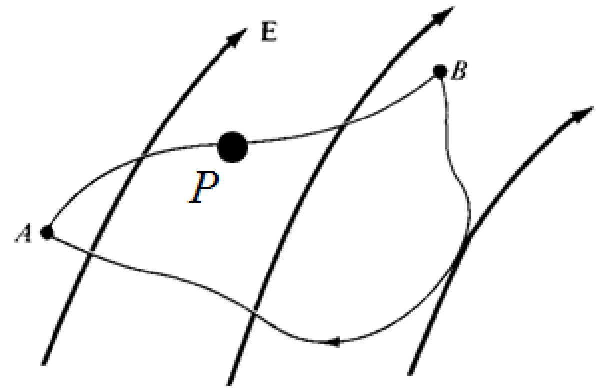
Electric (Scalar) Potential

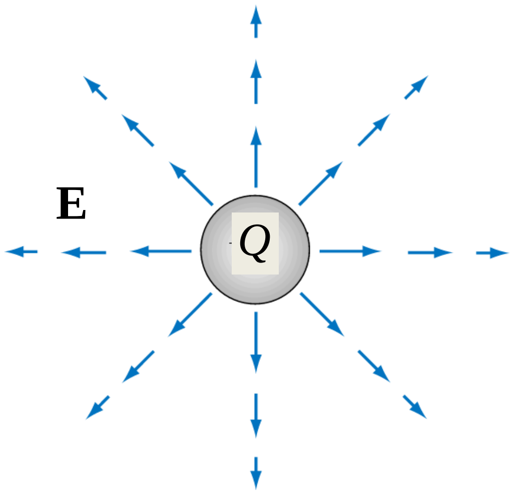
electric potential (V)

-- a measure of *work* required to move a unit charge through an **E** field:

$$W = -\int_A^B \mathbf{F}\cdot d\mathbf{l} = -q\int_A^B \mathbf{E}\cdot d\mathbf{l}$$

$$V = \frac{W}{q} \quad \Rightarrow \quad \boxed{V_{AB} = -\int_A^B \mathbf{E}\cdot d\mathbf{l}}$$





$$dV = -\mathbf{E}\cdot d\mathbf{l} = -E_x dx - E_y dy - E_z dz$$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

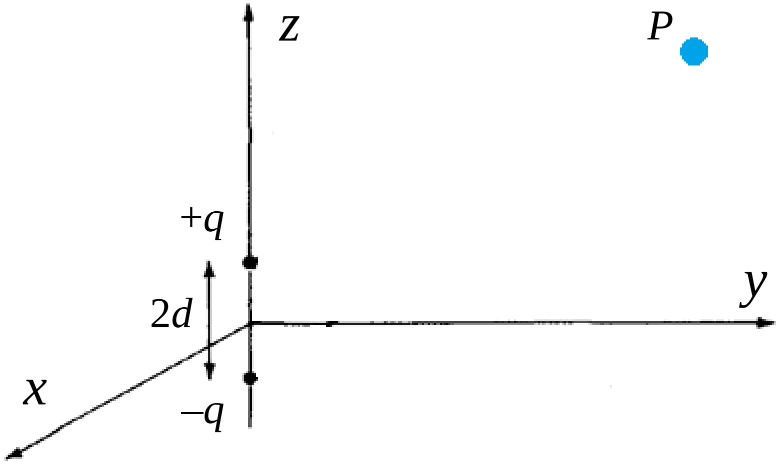
$$\mathbf{E} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial y} - \frac{\partial E_z}{\partial z}$$

$$\boxed{\mathbf{E} = -\nabla V}$$ "**E** from *V*"

→ **It's usually easier to calculate *V*, then take its gradient rather than to compute *E* directly from Coulomb's Law.**

Example: *E* from *V* , Dipole

For the dipole depicted, determine **E** at all points $P(R, \theta, \phi)$. You may assume that $R >> 2d$.



$$\mathbf{E} = -\nabla V$$

$$\begin{aligned}\nabla V &= \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \\ &= \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \\ &= \frac{\partial V}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}\end{aligned}$$

To be studied outside of class

- dipole moment
- Poisson’s equation & Laplace’s equation