Lecture 18: Fourier Series

Eevee's Goals for the Day

- Define the Fourier Series representation
- Practice calculating Fourier Series
- Introduce some basic properties of Fourier Series

12.1 Orthogonal functions

Recall Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if

$$\vec{u} \cdot \vec{v} = 0.$$

Def Two functions $f(x)$ and $g(x)$ are orthogonal on $[a,b]$ if

$$\int_a^b f(x)\, g(x)\, dx = 0.$$

The functions sine and cosine are orthogonal.

Look on $(-\pi, \pi)$.

$$\int_{-\pi}^{\pi} \cos(nx) \sin(mx)\, dx$$

Trig Identity: Product-to-Sum

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin(nx+mx) - \sin(nx-mx)\right] dx$$

$$= \frac{1}{2}\left[-\frac{\cos(nx+mx)}{n+m} + \frac{\cos(nx-mx)}{n-m} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[-\frac{\cancel{\cos(n+m)\pi}}{n+m} + \frac{\cancel{\cos(n-m)\pi}}{n-m} \right.$$

$\cos(-x)=\cos x$

$$\left. + \frac{\cancel{\cos(-(n+m)\pi)}}{n+m} - \frac{\cancel{\cos(-(n-m)\pi)}}{n-m} \right]$$

$$= 0$$

$\Rightarrow \cos(nx)$ and $\sin(mx)$ are orthogonal if $m \neq n$

Is cosine orthogonal to itself?

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(nx-mx) + \cos(nx+mx)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(nx-mx)}{n-m} + \frac{\sin(nx+mx)}{n+m} \right]_{-\pi}^{\pi}$$

$= 0$ Because $\sin(k\pi) = 0$ for any integer k.

What happens if $m = n$?

$$\int_{-\pi}^{\pi} \cos(nx)\cos(nx)\,dx$$

$$= \int_{-\pi}^{\pi} \cos^2(nx)\,dx$$

Table of Integrals

$$= \frac{1}{2}x + \frac{1}{4n}\sin(2nx) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\pi + \frac{1}{4n}\cancel{\sin(2n\pi)}^{\,0} - \frac{1}{2}(-\pi) - \frac{1}{4n}\cancel{\sin(-2n\pi)}^{\,0}$$

$$= \pi$$

Cosine is not orthogonal to itself if the frequencies are equal.

Orthogonality on $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

Orthogonality on $(-L, L)$ ($L > 0$ constant)

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

12.2 Fourier Series

Power Series $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Power series have difficulties when $f(x)$ is

1. periodic
2. discontinuous

Represent a function using sines and cosines.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

↳ Fourier Series

Derivation of Fourier series on $(-\pi, \pi)$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Multiply by $\cos(mx)$ and integrate.

$$\int_{-\pi}^{\pi} \cos(mx) f(x) = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n \underbrace{\cos(nx)\cos(mx)}_{0 \text{ unless } m=n} + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \underbrace{\sin(nx)\cos(mx)}_{0}$$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m \int_{-\pi}^{\pi} \cos(mx)\cos(mx)\,dx$$

$$= a_m \pi$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$$

Similarly

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx$$

Def The **Fourier Series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

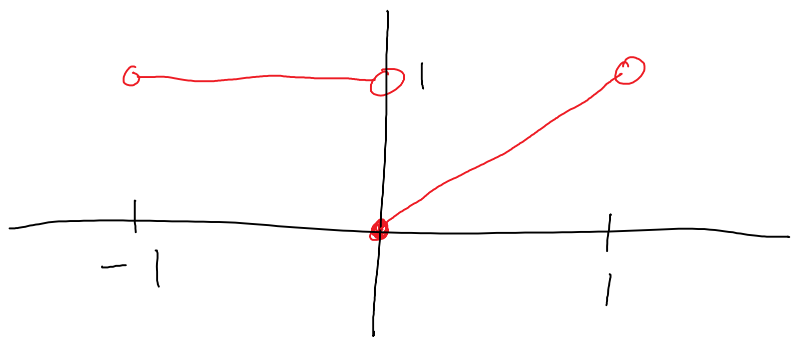
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note Unlike power series, we can represent a discontinuous function as a Fourier series.

Ex Find Fourier series on $(-1,1)$ of

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ x & \text{if } 0 \le x < 1 \end{cases}$$



$(-1,1)$

$L = 1$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx$$

$$= x\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{1}$$

$$= [0 - -1] + \left[\frac{1}{2} - 0\right]$$

$$= \frac{3}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad L=1$$

$$= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{0} 1\cos(n\pi x)\,dx + \int_0^1 x\cos(n\pi x)\,dx$$

$$\left\{\begin{array}{ll} u = x & v = \frac{1}{n\pi}\sin(n\pi x) \\ du = dx & dv = \cos(n\pi x)\,dx \\ uv & -\int v\,du \end{array}\right.$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} + x\frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 - \int_0^1 \frac{1}{n\pi}\sin(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} + \frac{x}{n\pi}\sin(n\pi x)\Big|_0^1 + \frac{1}{(n\pi)^2}\cos(n\pi x)\Big|_0^1$$

$$= \frac{1}{n\pi}\underbrace{\sin 0}_{0} - \frac{1}{n\pi}\underbrace{\sin(-n\pi)}_{0} + \frac{1}{n\pi}\underbrace{\sin(n\pi)}_{0} - \frac{0}{n\pi}\underbrace{\sin(0)}_{0} + \frac{1}{(n\pi)^2}\underbrace{\cos(n\pi)}_{(-1)^n} - \frac{1}{(n\pi)^2}\underbrace{\cos 0}_{1}$$

$$= \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx + \underbrace{\int_0^1 x\sin(n\pi x)\,dx}_{\text{Integration by Parts}}$$

$$= -\frac{1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \overbrace{\qquad\qquad}$$

$$= -\frac{1}{n\pi}\cancel{\cos 0}^{\,1} + \frac{1}{n\pi}\cancel{\cos(-n\pi)}^{\,(-1)^n} - \frac{1}{n\pi}(-1)^n$$

$$= -\frac{1}{n\pi} + \cancel{\frac{1}{n\pi}(-1)^n} - \cancel{\frac{1}{n\pi}(-1)^n}$$

$$= -\frac{1}{n\pi}$$

Corrected mistake in lecture.

Plug these values into Fourier Series.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + \sum_{n=1}^{\infty} b_n\sin(n\pi x)$$

$$\boxed{= \frac{3}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) + \sum_{n=1}^{\infty} -\frac{1}{n\pi}\sin(n\pi x)}$$