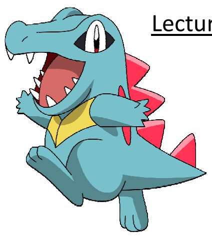


Lecture 8: Double & Triple Integrals

Totodile's Goals for the Day
- Practice finding potential functions for conservative vector fields
- Discuss path independence
- Review how to set up double and triple integrals
- Review polar coordinates

9.9 Independence of the Path

Recall A vector field $\vec{F}$ is conservative if there exists a potential function f such that $\nabla f = \vec{F}$.

Test 3D: $\nabla \times \vec{F} = \vec{0} \Rightarrow \vec{F}$ conservative

Most natural vector fields are conservative (electromagnetic, gravitational).

Conservative vector fields are sometimes called irrotational.

Ex Determine if the vector field is conservative and if so, find the potential function.

$$\vec{F} = \langle e^x \cos y,\ -e^x \sin y - z,\ -y + 3z^2 \rangle$$

i.) Test for conservative.

curl

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y & -e^x \sin y - z & -y + 3z^2 \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -e^x\sin y - z & -y+3z^2 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ e^x\cos y & -y+3z^2 \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^x\cos y & -e^x\sin y - z \end{vmatrix}$$

$$= i(-1 - -1) - j(0 - 0) + k(-e^x\sin y + e^x\sin y)$$

$$= \langle 0, 0, 0 \rangle$$

$\Rightarrow \vec{F}$ is conservative

ii.) Find potential function.

$$\vec{F} = \langle \underbrace{e^x\cos y}_{\frac{\partial f}{\partial x}}, \underbrace{-e^x\sin y - z}_{\frac{\partial f}{\partial y}}, \underbrace{-y+3z^2}_{\frac{\partial f}{\partial z}} \rangle$$

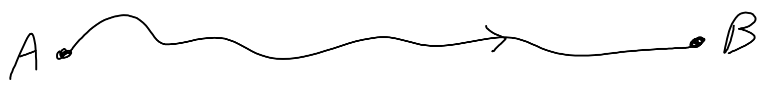
1. $f = \int e^x\cos y \, dx = e^x\cos y + g_1(y,z)$
2. $f = \int -e^x\sin y - z \, dy = e^x\cos y - yz + g_2(x,z)$
3. $f = \int -y + 3z^2 \, dz = -yz + z^3 + g_3(x,y)$

$$\boxed{f(x,y,z) = e^x\cos y - yz + z^3}$$

Potential functions may differ up to a constant.

Theorem (Fundamental Theorem of Line Integrals)

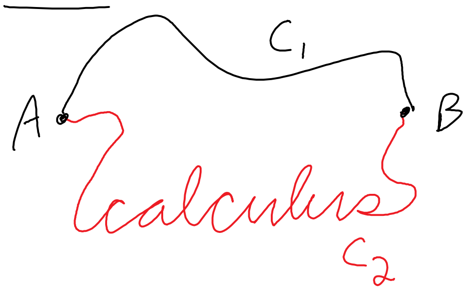
Let C be a smooth curve joining the point A to the point B.



If $\vec{F}$ is a conserative and continuous vector field with potential function f, then

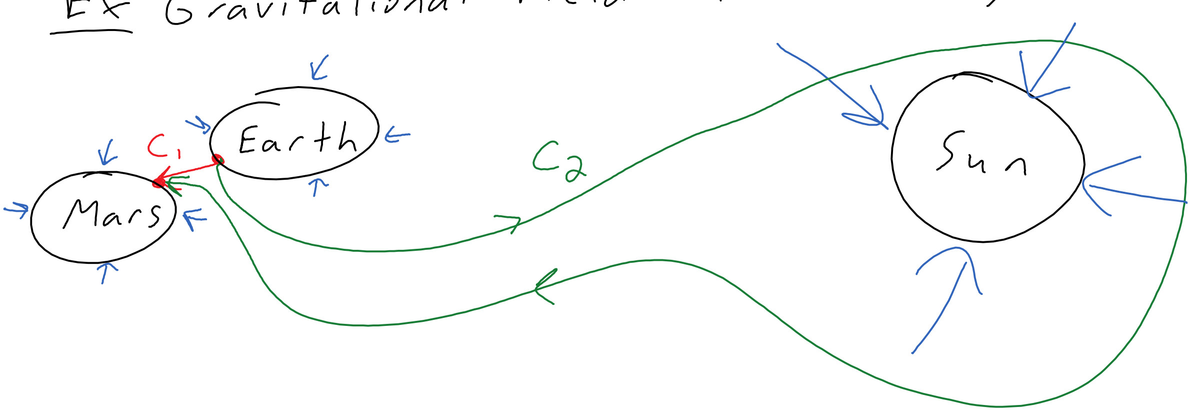
$$\int_C \vec{F}\cdot\vec{T}\,ds = \underset{\text{End}}{f(B)} - \underset{\text{Start}}{f(A)}$$

<u>Note</u> Conservative vector field $\Rightarrow$ Path Independence



$$\int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$$

<u>Ex</u> Gravitational Field of Solar System



Work done by field on rocket $= \int_{C_1} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} \vec{F}\cdot\vec{T}\,ds$

<u>Ex</u> Find the work done on an object moving from (1,0,2) to (3,0,5) by the vector field

$$\vec{F} = \langle e^x \cos y, -e^x \sin y - z, -y + 3z^2 \rangle.$$

We found the potential function

$$f(x,y,z) = e^x \cos y - yz + z^3.$$

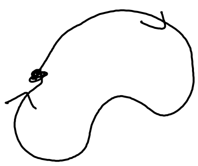
$$\text{Work} = \int_C \vec{F} \cdot \vec{T}\, ds = \overset{\text{End}}{f(3,0,5)} - \overset{\text{Start}}{f(1,0,2)}$$

$$= \left[e^3 \cos 0 - (0)(5) + 5^3\right] - \left[e^1 \cos 0 - (0)(2) + 2^3\right]$$

$$= \left[e^3 + 125\right] - \left[e + 8\right]$$

$$= \boxed{e^3 - e + 117}$$

What if the curve is closed?



$$\int_C \vec{F}\cdot\vec{T}\,ds = f(\text{End}) - f(\text{start}) = 0$$

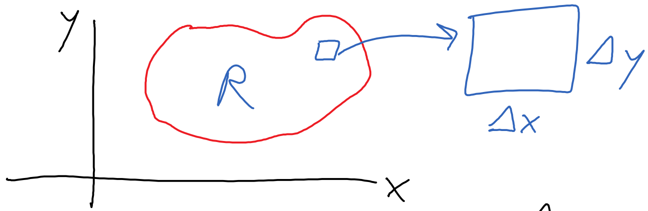
<u>Corollary</u> (Loop Property)

If $\vec{F}$ is conservative and C is a closed curve, then

$$\oint_C \vec{F}\cdot\vec{T}\,ds = 0$$

The circulation in a conservative vector field is always zero.

9.10 Double Integrals

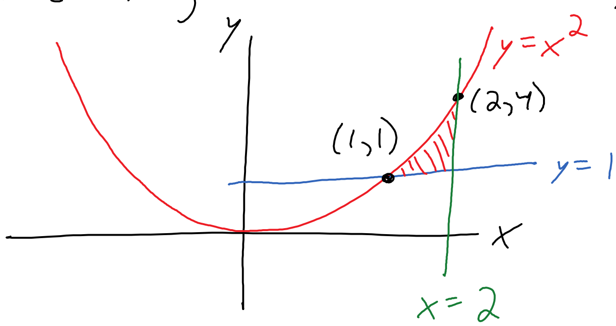


Area of $R = \sum\sum \Delta x \Delta y$

$\downarrow \Delta x \to 0,\ \Delta y \to 0$

Area of $R = \underbrace{\iint_R dx\,dy}_{dA \text{ area element}}$

Ex Calculate $\iint_R x^2 y \, dA$ where R is the region bounded by $y = x^2$, $x = 2$, and $y = 1$.



$$\int_1^4 \int_{\sqrt{y}}^2 x^2 y \, dx\, dy$$

$$\iint_R x^2 y \, dA = \int_1^2 \left[\int_1^{x^2} x^2 y \, dy \right] dx$$

$$= \int_1^2 \left[\tfrac{1}{2} x^2 y^2 \Big|_{y=1}^{y=x^2} \right] dx$$

$$= \int_1^2 \left[\tfrac{1}{2} x^2 (x^2)^2 - \tfrac{1}{2} x^2 (1)^2 \right] dx$$

$$= \int_1^2 \tfrac{1}{2} x^6 - \tfrac{1}{2} x^2 \, dx$$

$$= \tfrac{1}{14} x^7 - \tfrac{1}{6} x^3 \Big|_{x=1}^{x=2}$$

$$= \tfrac{1}{14}(2)^7 - \tfrac{1}{6}(2)^3 - \tfrac{1}{14}(1)^7 + \tfrac{1}{6}(1)^3$$

$$= \frac{64}{7} - \frac{4}{3} - \frac{1}{14} + \frac{1}{6}$$

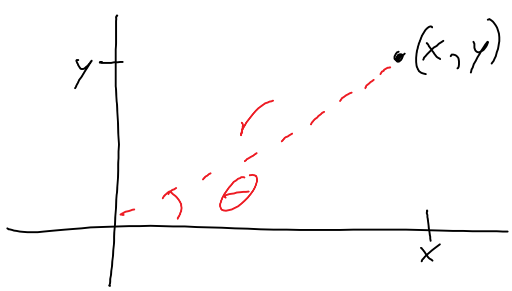
$$= \frac{384}{42} - \frac{56}{42} - \frac{3}{42} + \frac{7}{42}$$

$$= \frac{332}{42}$$

$$= \boxed{\frac{166}{21}}$$

9.11 Double Integrals in Polar Coordinates

Polar Coordinates



$$x = r\cos\theta$$

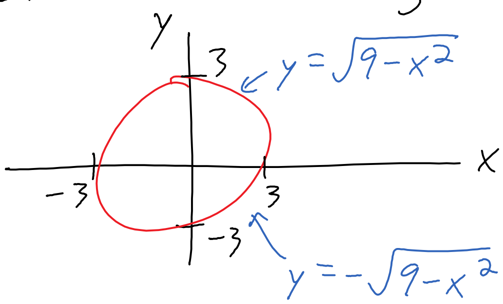
$$y = r\sin\theta$$

$$\tan\theta = \frac{y}{x}$$

$$r^2 = x^2 + y^2$$

$$\iint_R f(x,y)\,dA = \iint_{R'} f(r,\theta)\,\underbrace{r}_{\text{Jacobian}}\,dr\,d\theta$$

Ex Find area of circle of radius 3 centered at origin.



Rectangular

$$A = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$$

Polar

$$A = \int_0^{2\pi}\int_0^3 r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 \Big|_{r=0}^{r=3} d\theta$$

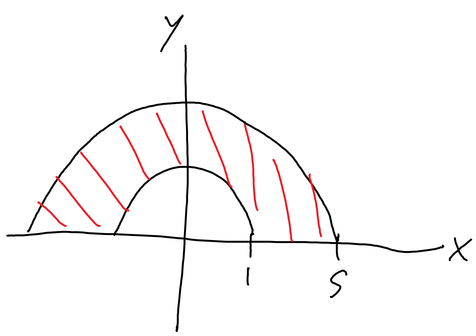
$$= \int_0^{2\pi} \frac{9}{2}\,d\theta$$

$$= \frac{9}{2}\theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \boxed{9\pi}$$

Note Polar coordinates are ideal for integrating circular regions.

Ex Integrate $e^{x^2+y^2}$ over the region between the circles of radius 1 and 5 centered at the origin for $y \geq 0$.



$$\iint_R e^{x^2+y^2}\, dA$$

Polar

$$\int_0^{\pi}\int_1^5 e^{r^2}\, r\, dr\, d\theta$$

$u = r^2$
$du = 2r\,dr$
$\int e^{r^2} r\,dr = \frac{1}{2}e^{r^2}$

$$= \int_0^{\pi} \frac{1}{2}e^{r^2}\Big|_{r=1}^{r=5}\, d\theta$$

$$= \int_0^{\pi} \frac{1}{2}e^{25} - \frac{1}{2}e\; d\theta$$

$$= \left(\frac{1}{2}e^{25} - \frac{1}{2}e\right)\int_0^{\pi} d\theta$$

$$= \left(\frac{1}{2}e^{25} - \frac{1}{2}e\right)\theta\Big|_0^{\pi}$$

$$= \boxed{\left(\frac{1}{2}e^{25} - \frac{1}{2}e\right)\pi}$$