Name ______________________________

1. A current density is equal to $8R\cos\phi\,\hat{\mathbf{R}} - 10R^2\sin\theta\,\hat{\boldsymbol{\phi}} + 12R^3\sin\phi\,\hat{\boldsymbol{\theta}}\ \left(\mathrm{mA/m^2}\right)$. Determine the current crossing through the surface given by $R = 5\ \mathrm{m}$, $0 \le \theta \le 60°$, $0 \le \phi \le 30°$.

$$I = \iint \vec{J} \cdot d\vec{s}$$

$$= \int_{\phi=0°}^{\phi=30°} \int_{\theta=0°}^{\theta=60°} \vec{J} \cdot \hat{R}\ R^2 \sin\theta\ d\theta\ d\phi$$

$$= \int_0^{30°} \int_0^{60°} 8R^3 \cos\phi\ \sin\theta\ d\theta\ d\phi \Big|_{R=5} \times 10^{-3}$$

$$= (8)(5)^3 \int_0^{30°} \cos\phi\ d\phi \int_0^{60°} \sin\theta\ d\theta \times 10^{-3}$$

$$= 1000 \left[+\sin\phi\right]_0^{30°} \left[-\cos\theta\right]_0^{60°} \times 10^{-3}$$

$$= (1)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \tfrac{1}{4} = 0.25\ \mathrm{A}$$

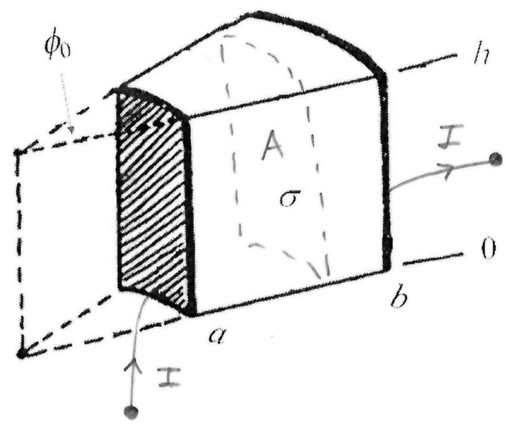
$$= \boxed{250\ \mathrm{mA}}$$

Name ____

2. A cylindrical-wedge resistor is drawn in the figure.

Its ends are capped by thin metal plates at radii $a = 2$ cm and $b = 4$ cm .
The conductivity of the material between the plates is $\sigma = 1.1 \times 10^6$ S/m .
The height of the resistor is $h = 6$ cm and the angle of the wedge is $\phi_0 = \pi/6$.

Determine the resistance of this structure from radius a to radius b
(from the front to the back, in the figure).



front cross section
$\neq$ back cross section

$$\therefore \quad R = \frac{\int \vec{E} \cdot d\vec{\ell}}{\iint \sigma \vec{E} \cdot d\vec{s}}$$

must be used

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assuming current I into front and out back...

$$\vec{J} = \frac{I}{A}\hat{r} \; ; \quad A = r \phi_0 h$$

$$\vec{J} = I/r\phi_0 h \; \hat{r}$$

$$\vec{E} = \vec{J}/\sigma = \frac{I}{\sigma \phi_0 h r}\hat{r}$$

$$\int \vec{E} \cdot d\vec{\ell} = \int_a^b \frac{I}{\sigma \phi_0 h r}\hat{r} \cdot \hat{r}\, dr$$

$$= \frac{I}{\sigma \phi_0 h}\int_a^b \frac{1}{r}\, dr = \frac{I \ln(b/a)}{\sigma \phi_0 h} = V$$

$$R = \frac{V}{I} = \frac{\ln(b/a)}{\sigma \phi_0 h} = \frac{\ln(4/2)}{(1.1 \times 10^6)(\pi/6)(.06)} \approx \boxed{20\ \mu\Omega}$$

Name ______________________________

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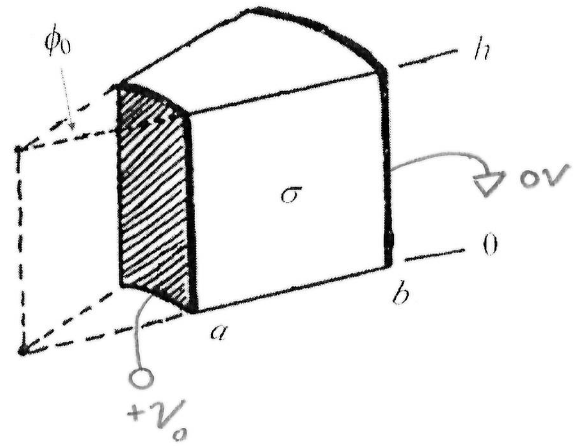


alternative : Laplace

$$\nabla^2 V = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V}{\partial r}$$

$$\Rightarrow \quad V = V_1 \ln(r) + V_2$$

$$\left.\begin{aligned} V(r=a) &= V_0 \\ V(r=b) &= 0 \end{aligned}\right\} \qquad V_1 = \frac{-V_0}{\ln(b/a)}$$

$$V = \frac{-V_0}{\ln(b/a)} \ln(r) + V_2$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} = \frac{V_0}{\ln(b/a)} \cdot \frac{1}{r}\hat{r}$$

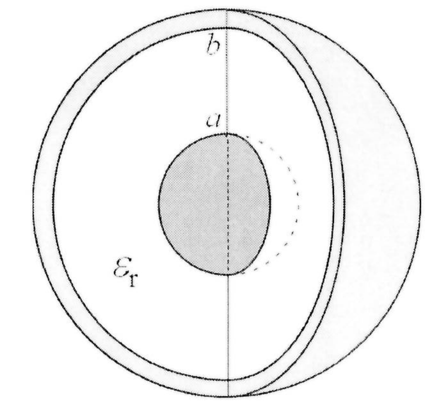
$$\iint \sigma \vec{E} \cdot d\vec{s} = \int_{\phi=0}^{\phi_0} \int_{z=0}^{h} \sigma \frac{V_0}{\ln(b/a)} \cdot \frac{1}{r} \cdot r \, dr \, d\phi$$

$$= \frac{\sigma V_0 \phi_0 h}{\ln(b/a)} \quad ; \quad R = \frac{V_0}{\dfrac{\sigma V_0 \phi_0 h}{\ln(b/a)}}$$

$$R = \frac{\ln(b/a)}{\sigma \phi_0 h} = \frac{\ln(4/2)}{(1.1\times10^6)(\pi/6)(.06)} \approx \boxed{20\ \mu\Omega}$$

Name ______________________________

3. A spherical capacitor has two concentric spherical conductors: an inner conductor at radius $a = 4$ m and an outer conductor at radius $b = 6$ m .



An open-cut view of the capacitor is shown in the figure. (The actual capacitor is closed all the way around.)

The inner conductor is held at a potential $V_a = 1$ V and the outer conductor is held at a potential $V_b = 3$ V .

The dielectric constant of the material inside the capacitor is $\varepsilon_r = 5$. There is no charge in the dielectric.

Write a complete expression for the potential everywhere between the two conductors.

$$\nabla^2 V = 0 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right)$$

$$\Rightarrow \quad V = \frac{V_1}{R} + V_2$$

$$V(R=4) = 1 = \frac{V_1}{4} + V_2$$

$$V(R=6) = 3 = \frac{V_1}{6} + V_2 \qquad \text{subtract}$$

$$V_1\left(\tfrac{1}{4} - \tfrac{1}{6}\right) = -2 \quad \Rightarrow \quad V_1 = -24$$

$$1 = \frac{-24}{4} + V_2 \quad \Rightarrow \quad V_2 = 7$$

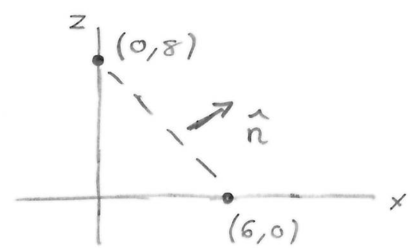
$$\boxed{V = \frac{-24}{R} + 7} \quad \text{(Volts)}$$

Name ______________________________

4. The boundary between two regions of space is defined by $8x - 6z = 48$ m.

The region including the origin is air, where the electric field intensity is $125\,\hat{\mathbf{x}} - 75\,\hat{\mathbf{y}} + 50\,\hat{\mathbf{z}}$ V/m.

Determine the electric field intensity in the second region, where the permittivity is $2\varepsilon_0$. The boundary is charge-free.



$$\hat{n} = \frac{8\hat{x} - 6\hat{z}}{\sqrt{8^2+6^2}} = \frac{4}{5}\hat{x} - \frac{3}{5}\hat{z}$$

$$\vec{E}_{1n} = (\vec{E}\cdot\hat{n})\,\hat{n}$$

$$= \left[125\left(\tfrac{4}{5}\right) + 0 + 50\left(\tfrac{-3}{5}\right)\right]\left[\tfrac{4}{5}\hat{x} - \tfrac{3}{5}\hat{z}\right]$$

$$= 70\left[\tfrac{4}{5}\hat{x} - \tfrac{3}{5}\hat{z}\right] = 56\hat{x} - 42\hat{z} \quad \text{V/m}$$

$$\vec{E}_{1t} = \vec{E}_1 - \vec{E}_{1n} = 69\hat{x} - 75\hat{y} + 92\hat{z} \quad \text{V/m}$$

$$= \vec{E}_{2t}$$

$$\varepsilon_1\vec{E}_{1n} = \varepsilon_2\vec{E}_{2n} \Rightarrow \vec{E}_{2n} = \frac{\varepsilon_1}{\varepsilon_2}\vec{E}_{1n}$$

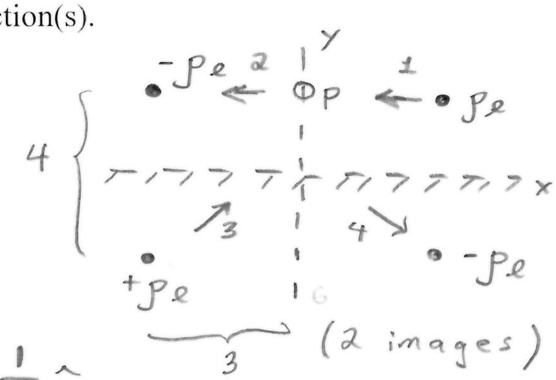
$$\vec{E}_{2n} = \frac{1}{2}\left[56\hat{x} - 42\hat{z}\right] = 28\hat{x} - 21\hat{z}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} = \boxed{97\hat{x} - 75\hat{y} + 71\hat{z} \quad \text{V/m}}$$

Name ______________________________

5. An infinite line carrying a charge density of +913 pC/m is located at $x = 3,\ y = 2$ m .
Another infinite line carrying a charge density of –913 pC/m is located at $x = -3,\ y = 2$ m .
A grounded (perfect) conductor occupies $y \le 0$. Assume $\varepsilon = \varepsilon_0$.

(a) Determine the electric field intensity at the point ($x = 0$ m, $y = 2$ m, $z = 4$ m).
Express your answer in V/m, in the appropriate direction(s).



$$\vec{E} = \frac{\rho_\ell}{2\pi\varepsilon_0 r}\,\hat{r} \qquad \text{with 4 lines}$$

$$\vec{E}_1 = \frac{+\rho_\ell}{2\pi\varepsilon_0(3)} \cdot \frac{-3\hat{x}}{3} = \frac{-\rho_\ell}{2\pi\varepsilon_0} \cdot \frac{1}{3}\hat{x}$$

$$\vec{E}_2 = \frac{-\rho_\ell}{2\pi\varepsilon_0(3)} \cdot \frac{+3\hat{x}}{3} = \frac{-\rho_\ell}{2\pi\varepsilon_0} \cdot \frac{1}{3}\hat{x}$$

$$\vec{E}_3 = \frac{+\rho_\ell}{2\pi\varepsilon_0(5)} \cdot \frac{3\hat{x}+4\hat{y}}{5} = \frac{+\rho_\ell}{2\pi\varepsilon_0}\left[\frac{3}{25}\hat{x} + \frac{4}{25}\hat{y}\right]$$

$$\vec{E}_4 = \frac{-\rho_\ell}{2\pi\varepsilon_0(5)} \cdot \frac{-3\hat{x}+4\hat{y}}{5} = \frac{+\rho_\ell}{2\pi\varepsilon_0}\left[\frac{3}{25}\hat{x} - \frac{4}{25}\hat{y}\right]$$

$$\vec{E}_{total} = \frac{\rho_\ell}{2\pi\varepsilon_0}\,\hat{x}\left[\frac{6}{25} - \frac{2}{3}\right] \approx \boxed{-7\,\hat{x}\ \text{V/m}}$$

(b) Determine the electric field intensity at the point ($x = 1$ m, $y = -2$ m, $z = 0$ m).

point (1, -2, 0) is inside the grounded conductor

$$\therefore\ \boxed{\vec{E} = 0}$$