



**THE
CITADEL**
THE MILITARY COLLEGE OF SOUTH CAROLINA

Dr. Gregory J. Mazzaro
Spring 2015

ELEC 318 – *Electromagnetic Fields*

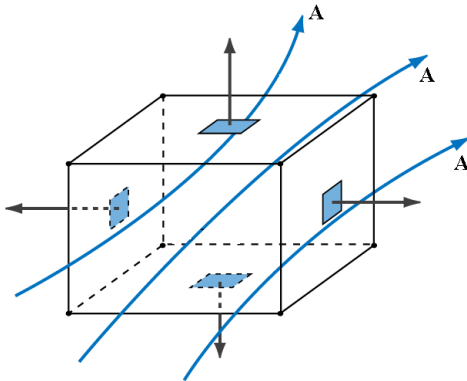
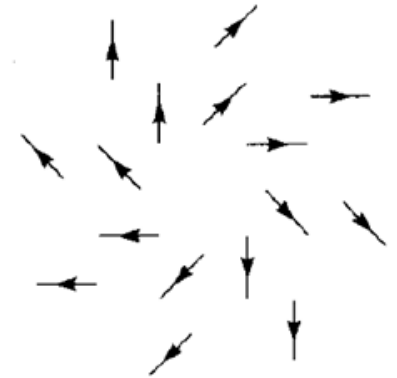
Lecture 3(d)

**Review of Vector Calculus:
Del Operator, Grad, Div, Curl**

∇ – the Del Operator

del operator

- vector differential operator
- allows us to generate 3 different *directional derivatives*: grad, div, curl



integral form,
closed surface

differential form,
at a point

$$\oint_S \mathbf{A} \cdot d\mathbf{S} \Rightarrow \nabla \cdot \mathbf{A}$$

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}} \\ &= \frac{\partial}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial}{\partial z} \hat{\mathbf{z}}\end{aligned}$$

$$= \frac{\partial}{\partial R} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi} \hat{\boldsymbol{\phi}}$$

→ derived in textbook

Gradient

gradient of a *scalar* field (V)

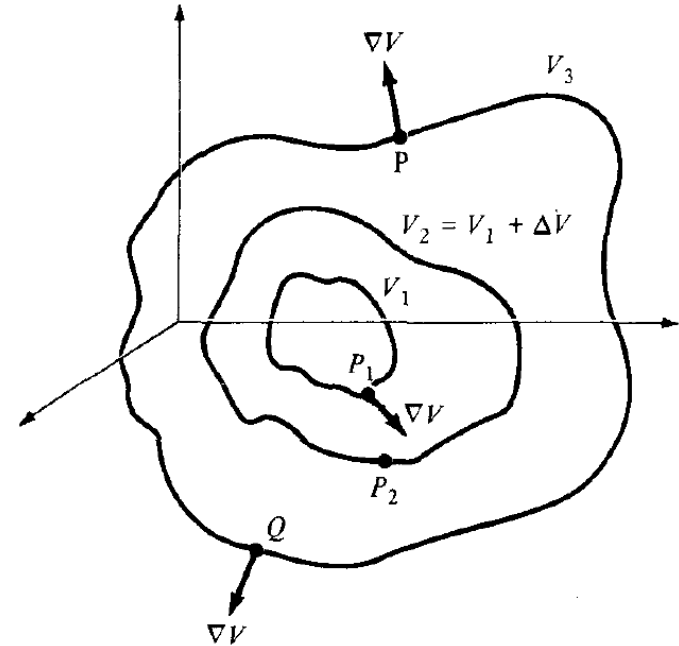
-- a *vector field* that represents the maximum spatial rate of increase of V

-- allows us to answer the questions

“In which direction does V change?”

“How quickly does V change from point-to-point”

$$\nabla V$$



Example: $\mathbf{E} = -\nabla V$

$$\begin{aligned}\nabla V &= \frac{\partial V}{\partial x} \hat{\mathbf{x}} + \frac{\partial V}{\partial y} \hat{\mathbf{y}} + \frac{\partial V}{\partial z} \hat{\mathbf{z}} &= \frac{\partial V}{\partial r} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z} \hat{\mathbf{z}}\end{aligned}$$

Example: Gradient (1 of 2)

Find ∇V if

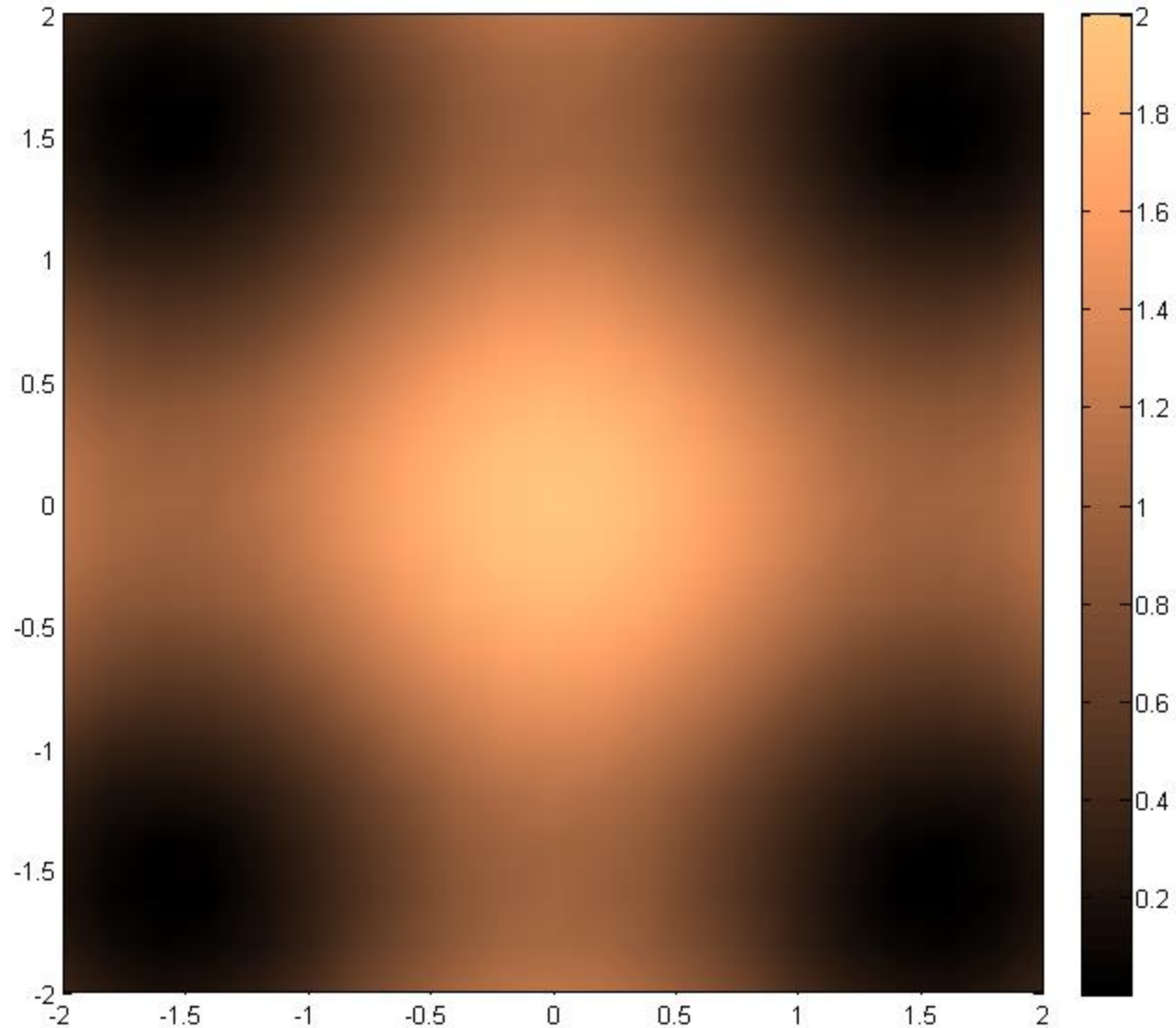
$$V = \cos^2(x) + \cos^2(y)$$

```
[x,y] = meshgrid(-2:.2:2,-2:.2:2);  
V = (cos(x)).^2 + (cos(y)).^2;  
[u,v] = gradient(V,1,1);
```

```
figure(1)  
pcolor(x,y,V);  
colormap copper; shading interp;  
axis image; colorbar
```

```
figure(2)  
quiver(x,y,u,v,'k','Linewidth',2);  
axis image
```

```
figure(3)  
quiver(x,y,u,v,'w','Linewidth',2)  
hold on; axis image;  
pcolor(x,y,V);  
colormap copper; shading interp;  
colorbar; hold off;
```



Example: Gradient (1 of 2)

Find ∇V if

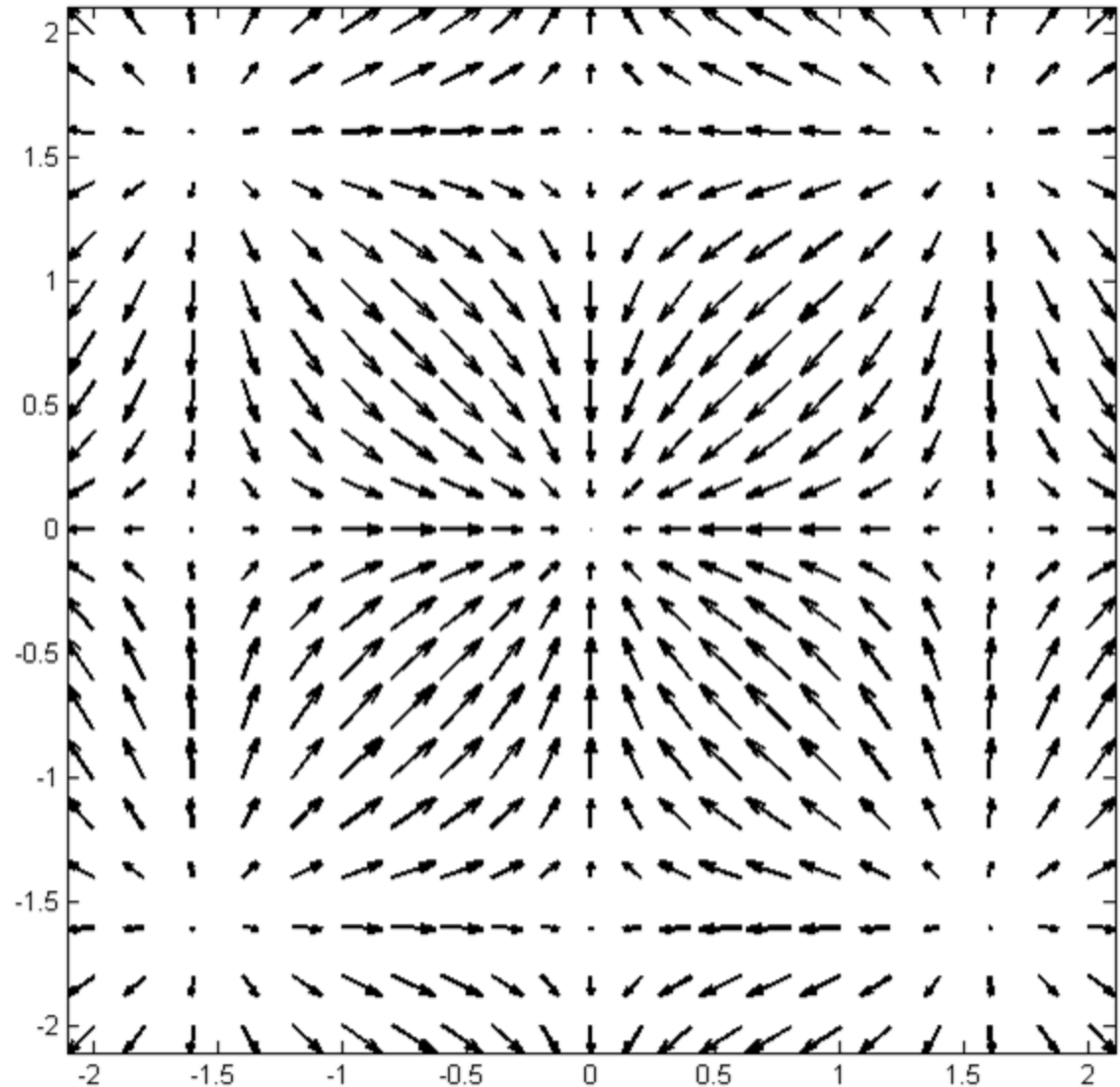
$$V = \cos^2(x) + \cos^2(y)$$

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V = (cos(x)).^2 + (cos(y)).^2;  
[u,v] = gradient(V,1,1);
```

```
figure(1)  
pcolor(x,y,V);  
colormap copper; shading interp;  
axis image; colorbar
```

```
figure(2)  
quiver(x,y,u,v,'k','Linewidth',2);  
axis image
```

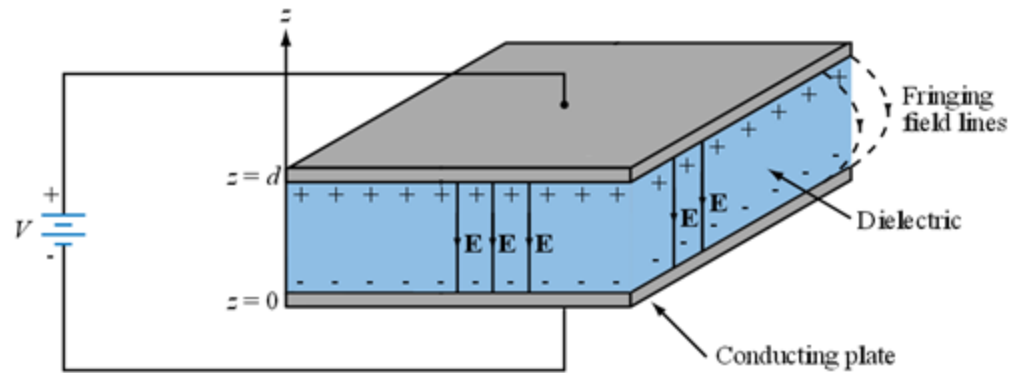
```
figure(3)  
quiver(x,y,u,v,'w','Linewidth',2)  
hold on; axis image;  
pcolor(x,y,V);  
colormap copper; shading interp;  
colorbar; hold off;
```



Example: Gradient (2 of 2)

$$\begin{aligned}\nabla V &= \frac{\partial V}{\partial x} \hat{\mathbf{x}} + \frac{\partial V}{\partial y} \hat{\mathbf{y}} + \frac{\partial V}{\partial z} \hat{\mathbf{z}} &= \frac{\partial V}{\partial r} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \\ &= \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z} \hat{\mathbf{z}}\end{aligned}$$

Estimate the quantity $-\nabla V$
within the parallel plates
for $V = 5 \text{ V}$, $d = 0.025 \text{ mm}$.



Divergence

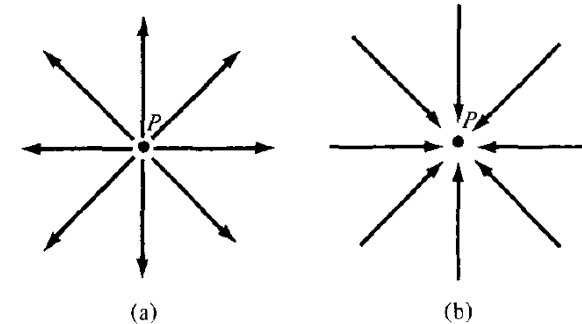
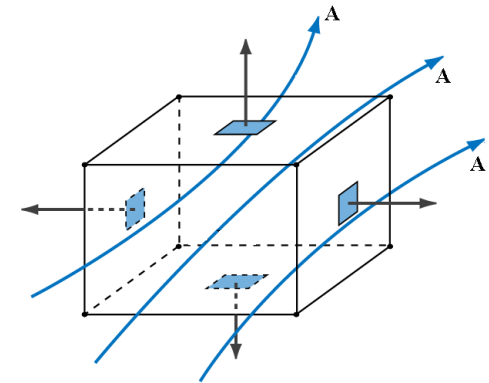
divergence of a *vector* field (\mathbf{A}) at a point (P)

-- a *scalar field*; a measure of the *flux* of \mathbf{A}
per unit volume outward from each point P

$$\nabla \cdot \mathbf{A}$$

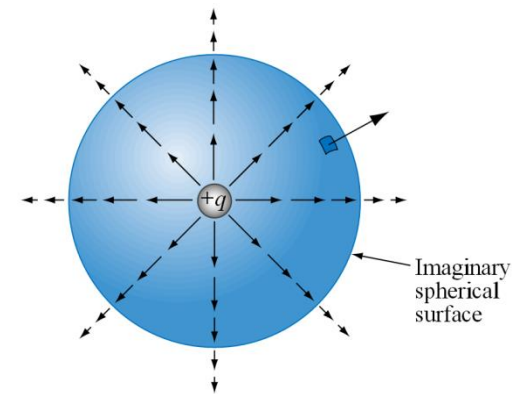
-- allows us to answer the question

“What is the net amount of \mathbf{A} that exits a point (P)?”



$$\nabla \cdot \mathbf{A} = 0 \Rightarrow \text{"solenoidal"}$$

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z \\ &= \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi} A_\phi\end{aligned}$$



Example: Divergence (1 of 2)

Find $\nabla \cdot \mathbf{E}$ if

$$\mathbf{E} = e^{-x^2/10} \cos(x) \hat{\mathbf{x}} + e^{-y^2/10} \cos(y) \hat{\mathbf{y}}$$

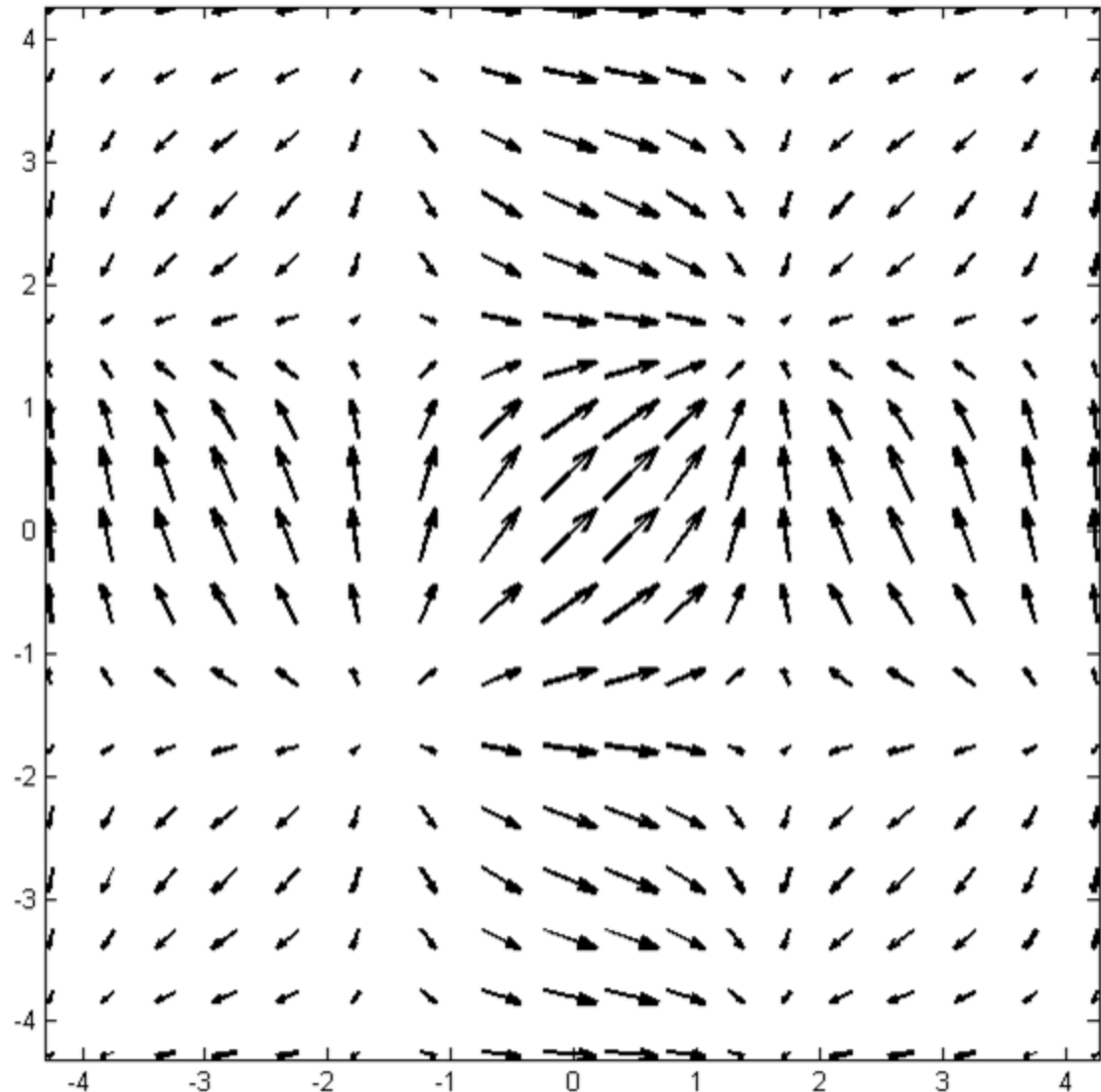
```
[x,y] = meshgrid(-4.25:.5:4.25,...  
                -4.25:.5:4.25);
```

```
u = exp(-x.^2/10).*cos(x);  
v = exp(-y.^2/10).*cos(y);
```

```
div = divergence(x,y,u,v);
```

```
figure(1)  
quiver(x,y,u,v,'k','Linewidth',2);  
axis image
```

```
figure(2)  
pcolor(x,y,div);  
colormap copper; shading interp;  
axis image; colorbar
```

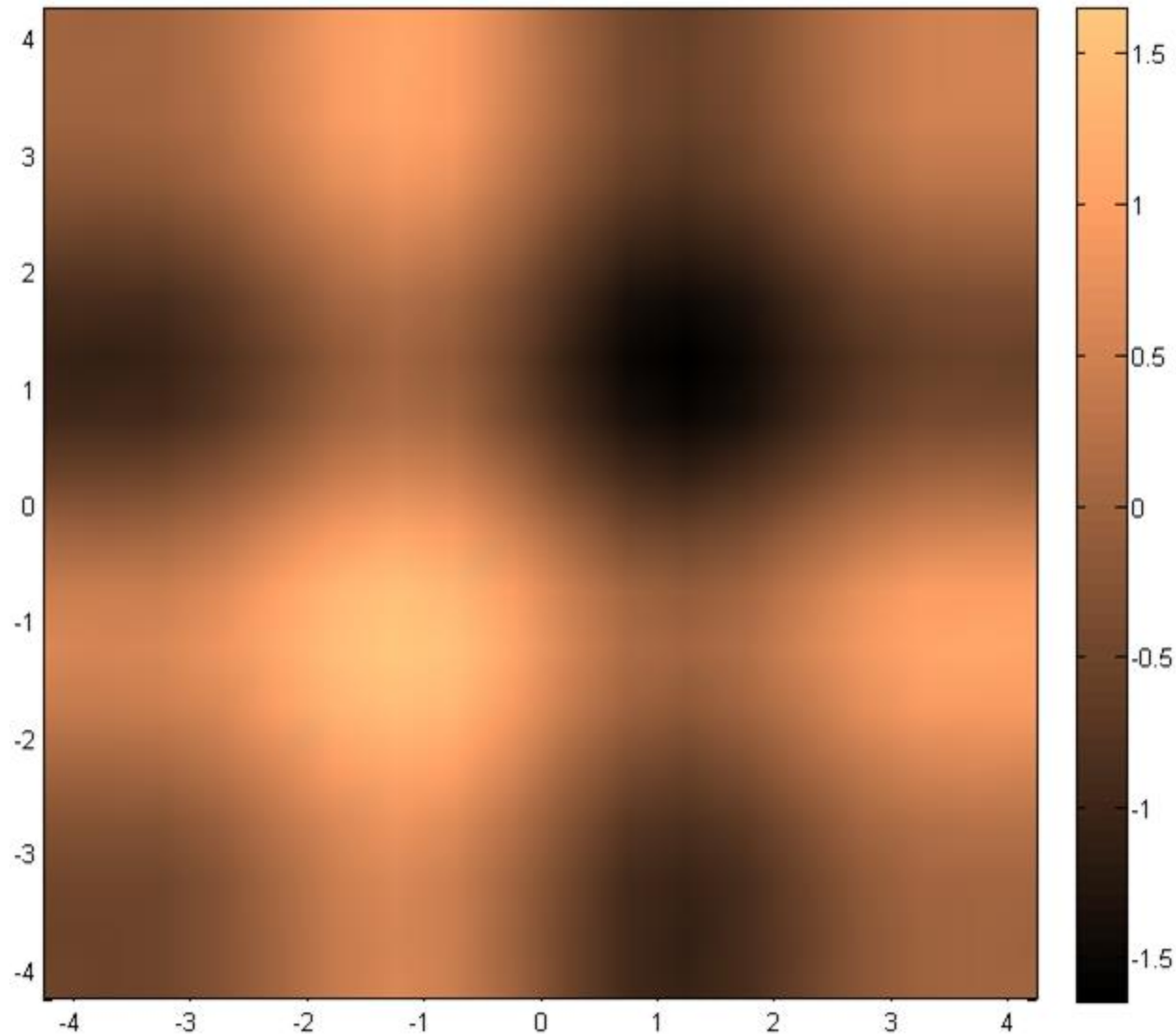


Example: Divergence (1 of 2)

Find $\nabla \cdot \mathbf{E}$ if

$$\mathbf{E} = e^{-x^2/10} \cos(x) \hat{\mathbf{x}} + e^{-y^2/10} \cos(y) \hat{\mathbf{y}}$$

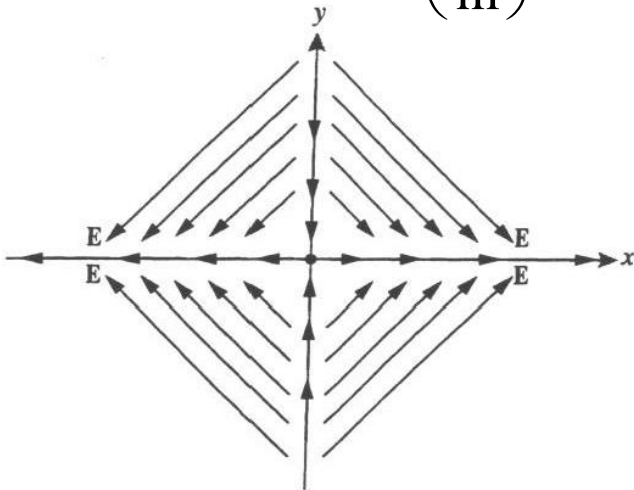
```
[x,y] = meshgrid(-4.25:.5:4.25,...  
                -4.25:.5:4.25);  
  
u = exp(-x.^2/10).*cos(x);  
v = exp(-y.^2/10).*cos(y);  
  
div = divergence(x,y,u,v);  
  
figure(1)  
quiver(x,y,u,v,'k','Linewidth',2);  
axis image  
  
figure(2)  
pcolor(x,y,div);  
colormap copper; shading interp;  
axis image; colorbar
```



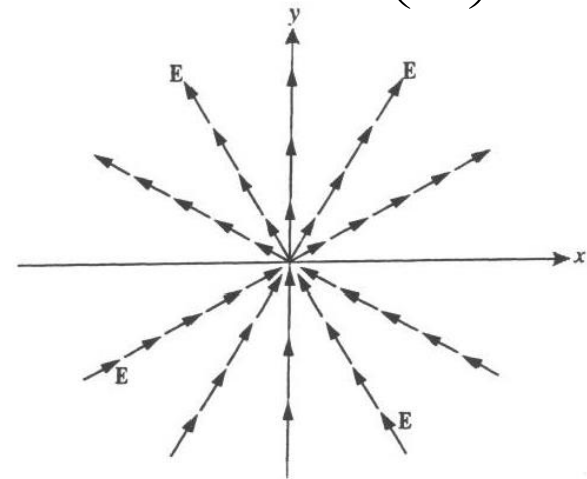
Example: Divergence (2 of 2)

Write the divergence of the following vector fields:

$$\mathbf{E} = x \hat{\mathbf{x}} - y \hat{\mathbf{y}} \left(\frac{\text{V}}{\text{m}} \right)$$



$$\mathbf{E} = \sin \phi \hat{\mathbf{r}} \left(\frac{\text{V}}{\text{m}} \right)$$

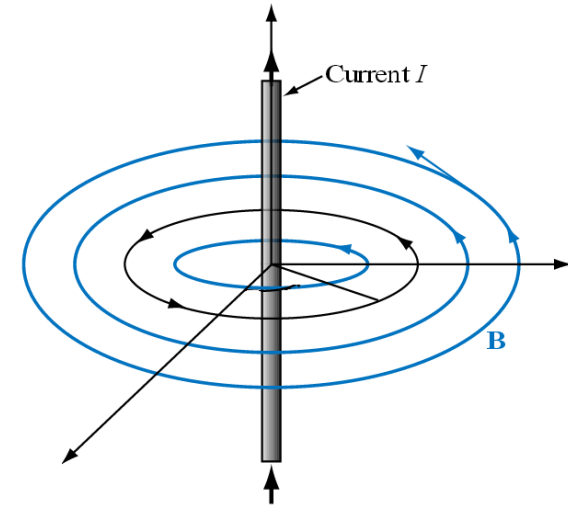


Curl

curl of a *vector field* (\mathbf{A}) at a point (P)

-- a *vector field*; a measure of the *circulation* of \mathbf{A} per unit area at each point P

$$\nabla \times \mathbf{A}$$



-- allows us to answer the question

“What is the net amount of \mathbf{A} that rotates in space about a point (P)?”

$$\nabla \times \mathbf{A} = 0 \Rightarrow \text{"conservative"}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \hat{\mathbf{r}} & r \hat{\phi} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & r A_\phi & A_z \end{vmatrix} = \frac{1}{R^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{R}} & R \hat{\theta} & R \sin \theta \hat{\phi} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & R A_\theta & R \sin \theta A_\phi \end{vmatrix}$$

Example: Curl (1 of 2)

Find $\nabla \times \mathbf{H}$ if

$$\mathbf{H} = \sin\left(y + \frac{\pi}{2}\right) \hat{\mathbf{x}} + \sin\left(x + \frac{\pi}{2}\right) \hat{\mathbf{y}}$$

```
[x,y] = meshgrid(-3:.4:3,-3:.4:3);
```

```
u = sin(y+pi/2);
```

```
v = sin(x+pi/2);
```

```
[curlz,cav]= curl(x,y,u,v);
```

```
figure(1)
```

```
quiver(x,y,u,v,'k','Linewidth',2)
```

```
axis image
```

```
figure(2)
```

```
pcolor(x,y,abs(cav));
```

```
colormap copper; shading interp;
```

```
axis image; colorbar
```

```
figure(3)
```

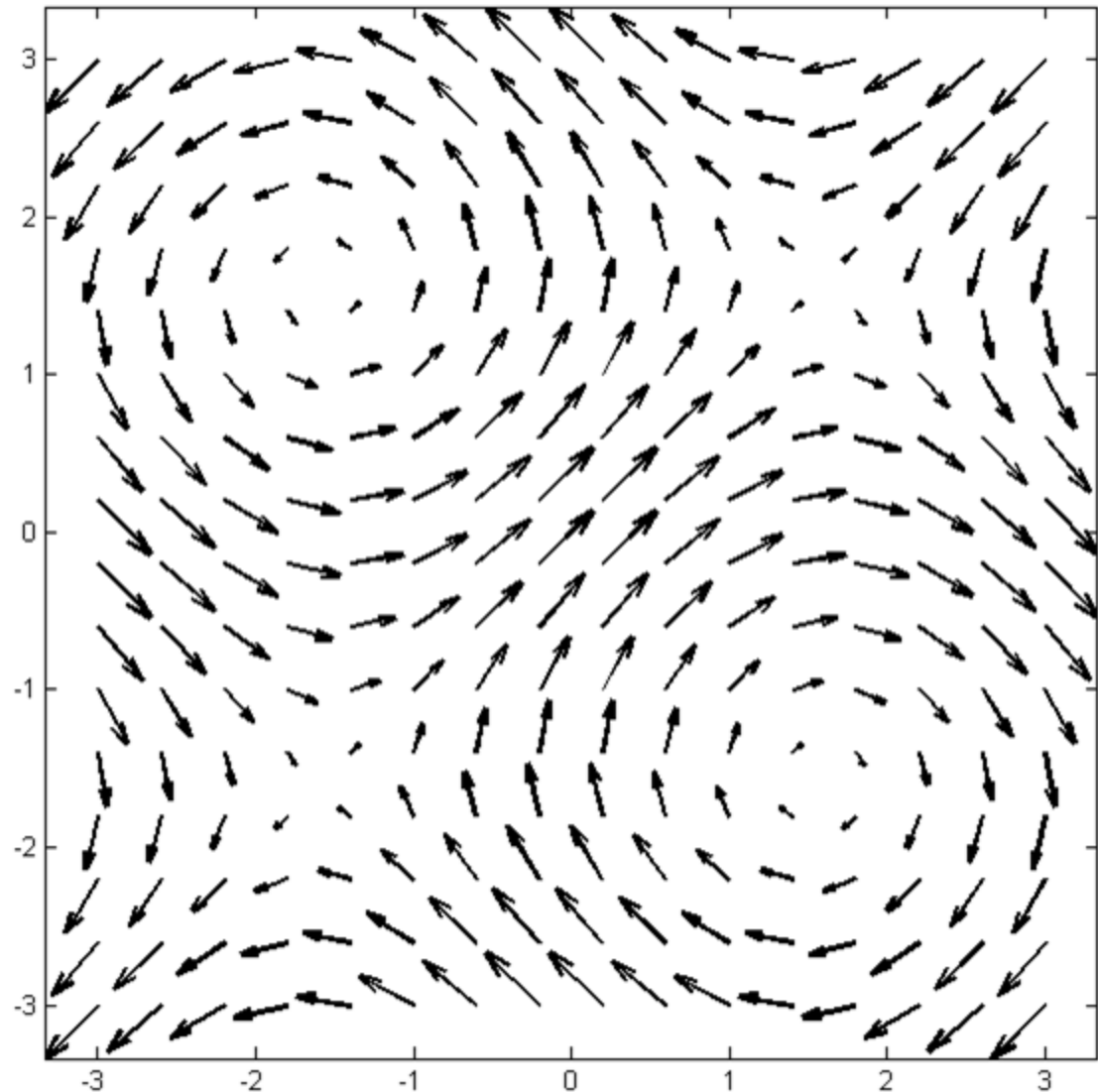
```
quiver(x,y,u,v,'w','Linewidth',2)
```

```
hold on; axis image;
```

```
pcolor(x,y,abs(cav));
```

```
colormap copper; shading interp;
```

```
Colorbar; hold off;
```

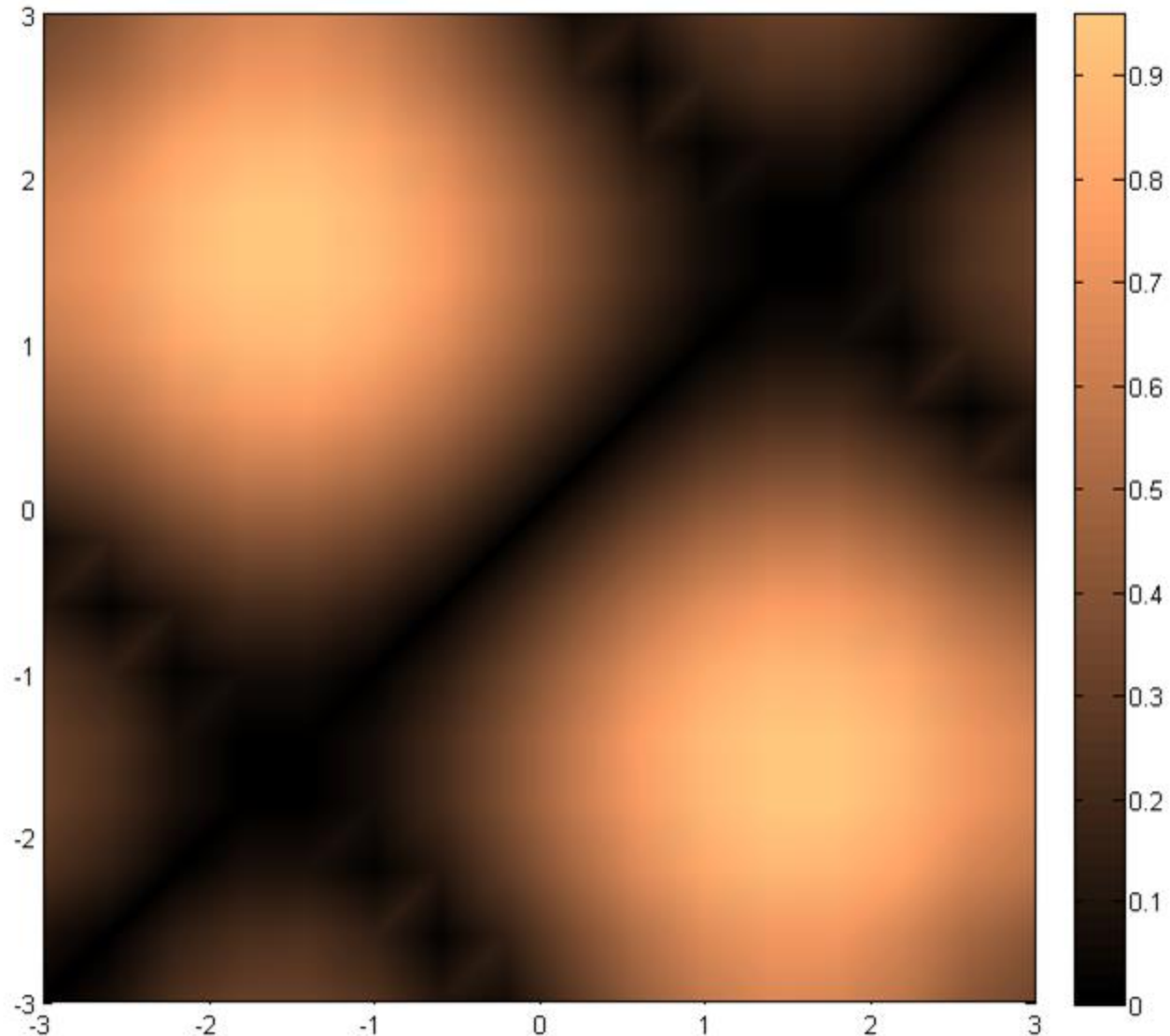


Example: Curl (1 of 2)

Find $\nabla \times \mathbf{H}$ if

$$\mathbf{H} = \sin\left(y + \pi/2\right) \hat{\mathbf{x}} \\ + \sin\left(x + \pi/2\right) \hat{\mathbf{y}}$$

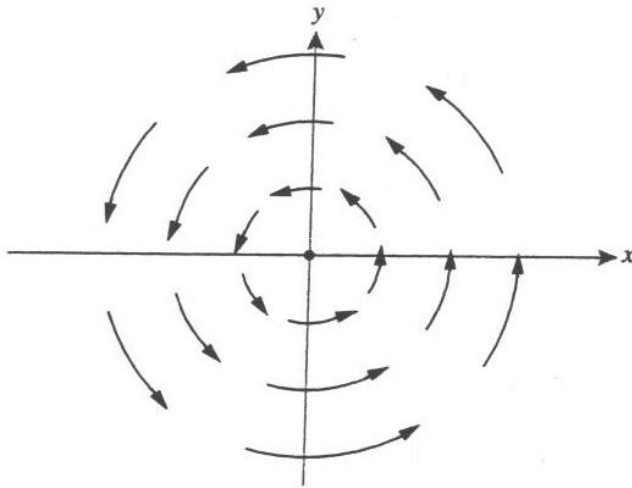
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[x,y] = meshgrid(-3:.4:3,-3:.4:3);  
  
u = sin(y+pi/2);  
v = sin(x+pi/2);  
  
[curlz,cav]= curl(x,y,u,v);  
  
figure(1)  
quiver(x,y,u,v,'k','Linewidth',2)  
axis image  
  
figure(2)  
pcolor(x,y,abs(cav));  
colormap copper; shading interp;  
axis image; colorbar  
  
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quiver(x,y,u,v,'w','Linewidth',2)  
hold on; axis image;  
pcolor(x,y,abs(cav));  
colormap copper; shading interp;  
Colorbar; hold off;
```



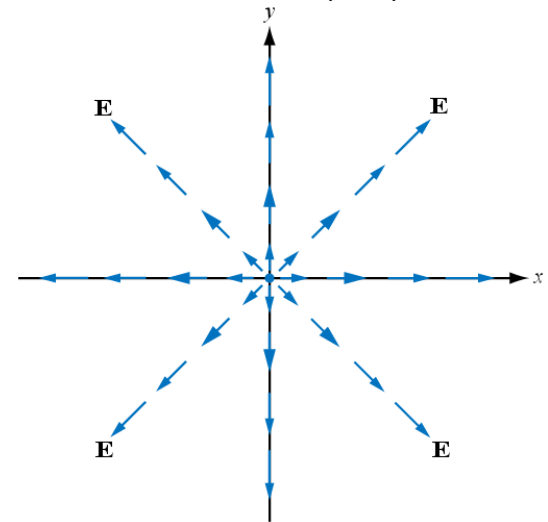
Example: Curl (2 of 2)

Write the curl of the following vector fields:

$$\mathbf{H} = r \hat{\phi} \left(\frac{\text{A}}{\text{m}} \right)$$



$$\mathbf{E} = r \hat{\mathbf{r}} \left(\frac{\text{V}}{\text{m}} \right)$$



To be studied **outside of class**



- Divergence Theorem
- Stokes' Theorem
- Laplacian