Lecture 22: Boundary Value Problems (BVPs)

Meowth's Goals for the Day
- Learn how to set up BVPs
- Describe the applications of 3 classical PDEs
- Visualize solutions using Matlab

13.2 Classical PDEs and BVPs

When we specify data values for a DE, there are 2 types of conditions.

$u(x, t)$

$\uparrow$ space $\quad \uparrow$ time

① Initial condition (IC)

$$u(x, 0) = f(x)$$

Specify value at time $t = 0$

② Boundary condition (BC)

$$u(0, t) = f(t)$$

Specify value at position $x = 0$

i.) Dirichlet BC

$$u(x_0, t) = f(t)$$

ii.) Neumann BC

$$\frac{\partial u}{\partial x}(x_0, t) = f(t)$$

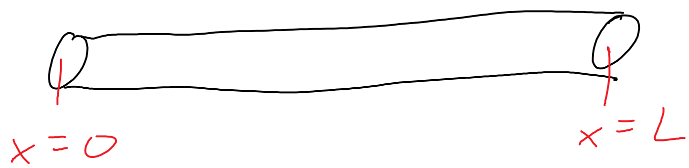
iii) Robin BC

$$\frac{\partial u}{\partial x}(x_0, t) + c\, u(x_0, t) = f(t)$$

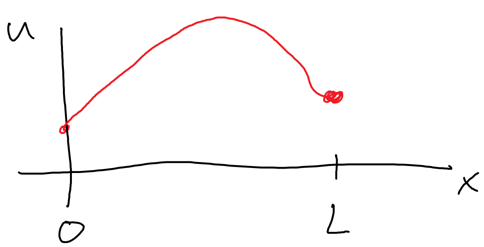
I. The Heat Equation

A metal bar of uniform material has length L.

$0 \leq x \leq L$

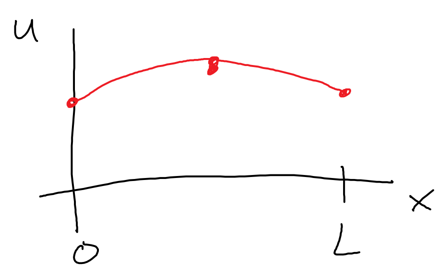


Let $u(x,t)$ be temperature of bar at position x and time t.



$u(x,0)$

Initial temperature profile



$u(x,1)$

Temperature at time $t=1$

1D Heat Equation: $u_t = k\, u_{xx}$

$\hookrightarrow$ Thermal diffusivity constant

$$k = \frac{c}{\gamma \rho} = \frac{\text{conductivity}}{(\text{specific heat})(\text{density})}$$

IC: Specify the initial temperature profile

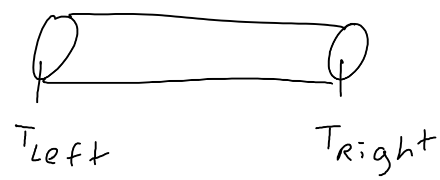
$$u(x,0) = f(x)$$

BC: i.) Dirichlet BC

Fix the temperature at ends of bar.

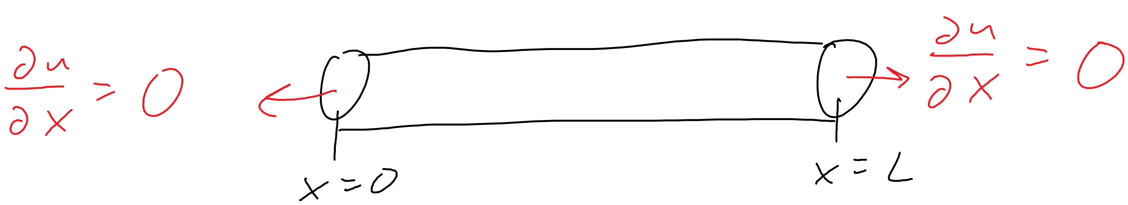
$$u(0,t) = T_{Left}$$

$$u(L,t) = T_{Right}$$



ii.) Neumann BC

The ends of the bar are insulated.



$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

iii.) Robin BC

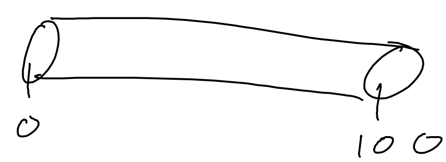
Newton's Law of Cooling: $\frac{\partial u}{\partial t} = c\left(u - u_{Ambient}\right)$

$$\frac{\partial u}{\partial t} - cu = -c\, u_{Ambient}$$

This conditions holds for all $0 \le x \le L$.

Ex A bar of length 100m has diffusivity constant 41. The left end of the bar starts $-10°C$ and is heating up $2°C$ per second. The right end of the bar is insulated.

$$u_t = k u_{xx}$$



$$\boxed{u_t = 41\, u_{xx}}$$

Left: $\boxed{u(0,t) = -10 + 2t}$

Right: $\boxed{\frac{\partial u}{\partial x}(100,t) = 0}$

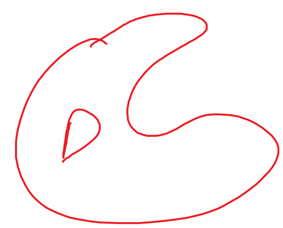
2D Heat Equation: $u_t = k(u_{xx} + u_{yy})$

If we let the 2D Heat Equation run until it reaches steady-state, then we reach a temperature distribution $u(x,y)$ where $\frac{\partial u}{\partial t} = 0$.

$$\Rightarrow u_{xx} + u_{yy} = 0$$

II. Laplace's Equation

$$u(\underbrace{x,y}_{\text{2D space}})$$

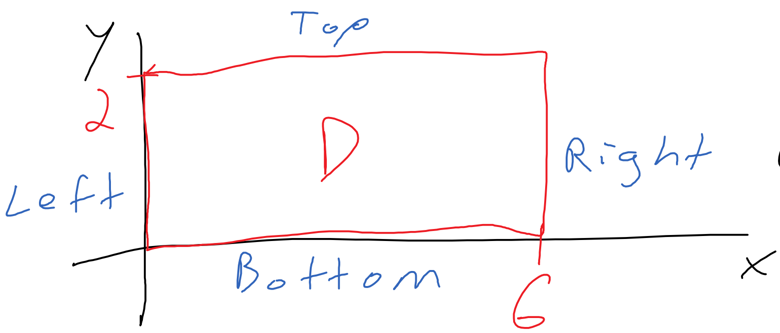


$$u_{xx} + u_{yy} = 0 \quad \text{on some domain } D$$

Application: Laplace's Equation is the steady-state of the 2D Heat Equation.

An electrostatic field on a charged plate satisfies Laplace's Equation.

Generally, we solve Laplace's Equation on a rectangular domain D.



$0 \le x \le 6$

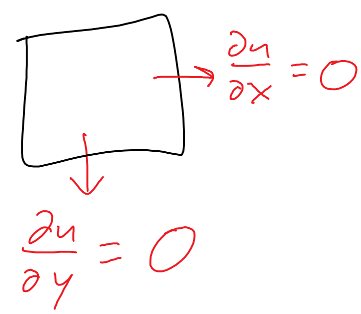
$0 \le y \le 2$

Dirichlet conditions specify values along a side.

Left: $u(0, y) = f(y) \qquad 0 \le y \le 2$

Top: $u(x, 2) = g(x) \qquad 0 \le x \le 6$

Neumann conditions specify how u flows through the boundary.

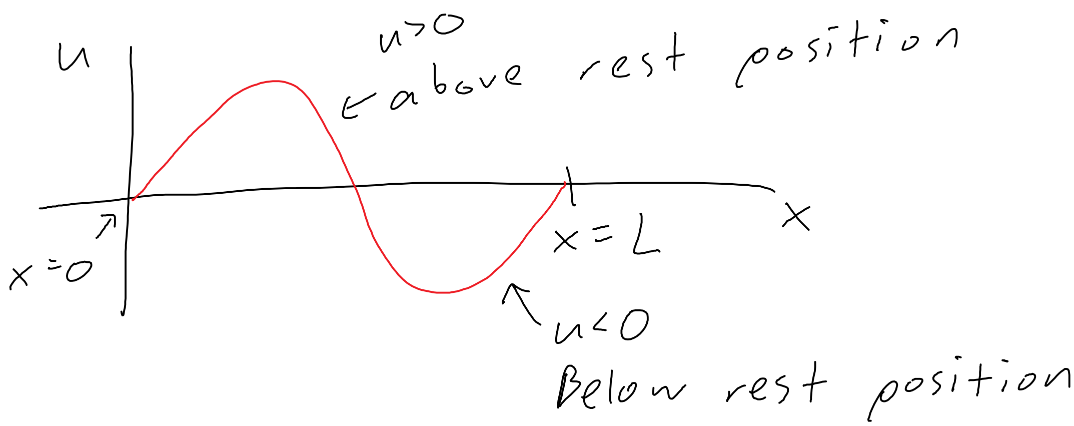


Right: $\frac{\partial u}{\partial x}(6, y) = 0$

Bottom: $\frac{\partial u}{\partial y}(x, 0) = 0$

III. The Wave Equation

$u(x,t)$ = vertical displacement of a vibrating string



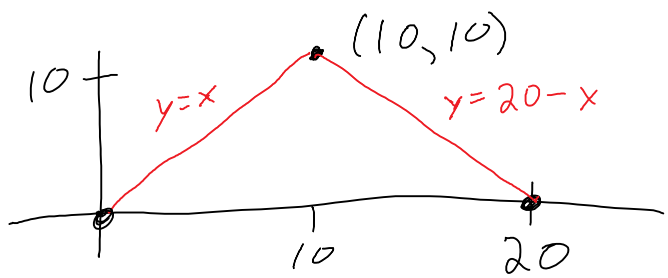
Assuming no external forces...

$$u_{tt} = \underbrace{a^2}_{\text{Tension constant}} u_{xx}$$

$$a^2 = \frac{T}{\rho} = \frac{\text{Tension}}{\text{density}}$$

<u>Ex</u> Plucked String

A guitar string of length 20 has tension constant 4 and the ends are clamped to the x-axis. The string is held at the center at a height 10 above rest. Then it is released.



$u(x,0)$

$$u_{tt} = 4u_{xx}$$

$a^2 = 4$

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(20,t) &= 0 \end{aligned}\right] \text{ clamp ends}$$

$$u(x,0) = \begin{cases} x & 0 < x < 10 \\ 20 - x & 10 < x < 20 \end{cases}$$