Lecture 21: Separable PDEs

Machop's Goals for the Day

- Review solving 2nd order ODEs
- Discuss how to classify PDEs
- Outline a method for solving separable PDEs

Ch. 13 Boundary Value Problems

13.1 Separable PDEs

An ordinary differential equation (ODE) has all derivatives with respect to the same variable.

$f(x)$: $$\frac{d^2 f}{dx^2} - 3x \frac{df}{dx} = 2$$

$$f'' - 3x f' = 2$$

A partial differential equation (PDE) has derivatives in more than one variables.

$f(x,y)$: $$\frac{\partial^2 f}{\partial x^2} - 3x \frac{\partial f}{\partial y} = 2$$

$$f_{xx} - 3x f_y = 2$$

Review Solving ODEs

I. 1st Order ODE

① Separation of Variables

② Integrating Factor

II. 2nd order ODE

Linear 2nd order ODE w/ constant coefficients

$$ay'' + by' + cy = 0$$ Homogeneous

Characteristic Equation

$$ar^2 + br + c = 0$$

Find roots.

3 Cases for Roots

① 2 real roots r_1, r_2 (Distinct roots)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

② 1 real root r (Repeated root)

$$y = C_1 e^{rx} + C_2 x e^{rx}$$

③ 2 complex roots $r = a \pm bi$

$$y = C_1 e^{ax} \sin(bx) + C_2 e^{ax} \cos(bx)$$

<u>Note</u> If the roots are $\pm R$, then the solution is

$$y = C_1 e^{Rx} + C_2 e^{-Rx}$$

we can rewrite this as

$$y = B_1 \sinh(Rx) + B_2 \cosh(Rx)$$

<u>Ex</u> Solve $y'' - 2y' + 10y = 0$.

$$r^2 - 2r + 10 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} = \frac{2 \pm \sqrt{-36}}{2}$$

$$= \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$\boxed{y = C_1 e^{x} \sin(3x) + C_2 e^{x} \cos(3x)}$$

A 2nd order linear PDE with constant coefficients has the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

<u>Def</u> We say a PDE is ...

Hyperbolic if $B^2 - 4AC > 0$

Elliptic if $B^2 - 4AC < 0$

Parabolic if $B^2 - 4AC = 0$

<u>3 Fundamental PDEs</u>

1. Heat Equation

$$u_t = k u_{xx}$$

Rearrange $\quad \underset{A = k}{k u_{xx}} \; \underset{E = -1}{- u_t} = 0 \qquad B, C, D, F, G = 0$

$B^2 - 4AC = 0 \quad \Rightarrow$ Parabolic

(2) Wave Equation

$$u_{tt} = a^2 u_{xx}$$

Rewrite $\quad \underset{A=a^2}{a^2 u_{xx}} - \underset{C=-1}{u_{tt}} = 0$

$$B^2 - 4AC = 0^2 - 4(a^2)(-1) = 4a^2 > 0 \Rightarrow \text{Hyperbolic}$$

(3) Laplace's Equation

$$\underset{A=1}{u_{xx}} + \underset{C=1}{u_{yy}} = 0$$

$$B^2 - 4AC = 0^2 - 4(1)(1) = -4 < 0 \Rightarrow \text{Elliptic}$$

Def We say a solution $u(x,y)$ is <u>separable</u> if it can be written as the product of single variable functions.

$$u(x,y) = v(x)\,w(y) \qquad \text{Product Solution}$$

Ex $u(x,y) = \underbrace{x^2}_{u(x)} \underbrace{\cos y}_{w(y)}$ separable

$u(x,y) = x^2 + \cos y$ not separable

$u(x,y) = \cos(xy)$ not separable

Idea: Solve PDE by assuming the solution is separable.

Fact If a function of x equals a function of y, then both functions must equal a constant.

$$f(x) = g(y) = \underbrace{-\lambda}_{\text{Separation Constant}}$$

Ex Solve $u_{xx} = u_y$.

Assume solution is separable.

$$u(x,y) = v(x)\, w(y)$$

Plug this into the PDE.

$$u_{xx} = u_y$$

$$(vw)_{xx} = (vw)_y$$

$$v_{xx}\, w = v\, w_y$$

$$\underbrace{\frac{v_{xx}}{v}}_{\text{Function of } x} = \underbrace{\frac{w_y}{w}}_{\text{Function of } y} = \underbrace{-\lambda}_{\text{Separation Constant}}$$

3 Cases: λ could be zero, positive, or negative.

① $\underline{\lambda = 0}$

$$\frac{v_{xx}}{v} = 0$$

$$v_{xx} = 0$$

$$\int v_{xx}\,dx = \int 0\,dx$$

$$v_x = C_1$$

$$\int v_x\,dx = \int C_1\,dx$$

$$v = C_1 x + C_2$$

$$\frac{w_y}{w} = 0$$

$$w_y = 0$$

$$\int w_y\,dy = \int 0\,dy$$

$$w = C_3$$

$$\begin{aligned} u_1(x,y) &= v(x)\,w(y) \\ &= (C_1 x + C_2)(C_3) \\ &= C_1 C_3 x + C_2 C_3 \\ &= \boxed{A_1 x + A_2} \end{aligned}$$

② $\underline{\lambda = \alpha^2}$

$$\frac{v_{xx}}{v} = -\alpha^2$$

$$v_{xx} = -\alpha^2 v$$

$$v_{xx} + \alpha^2 v = 0$$

$$\Downarrow$$

$$r^2 + \alpha^2 = 0$$

$$r^2 = -\alpha^2$$

$$r = \pm \alpha i$$

$$v = C_1 \sin(\alpha x) + C_2 \cos(\alpha x)$$

$$\frac{w_y}{w} = -\alpha^2$$

$$w_y = -\alpha^2 w$$

$$w_y + \alpha^2 w = 0$$

$$\Downarrow$$

$$\text{I.F.} \quad e^{\int \alpha^2 dy} = e^{\alpha^2 y}$$

$$e^{\alpha^2 y}\left[w_y + \alpha^2 w\right] = 0$$

$$\int \frac{d}{dy}\left[e^{\alpha^2 y} w\right] = \int 0$$

$$e^{\alpha^2 y} w = C_3$$

$$w = C_3 e^{-\alpha^2 y}$$

$$u_2(x,y) = v(x)\, w(y)$$

$$= \left[C_1 \sin \alpha x + C_2 \cos \alpha x\right]\left[C_3 e^{-\alpha^2 y}\right]$$

$$= \boxed{B_1 e^{-\alpha^2 y} \sin \alpha x + B_2 e^{-\alpha^2 y} \cos \alpha x}$$

③ $\lambda = -\alpha^2$

$$\frac{v_{xx}}{v} = \alpha^2$$

$$v_{xx} = \alpha^2 v$$

$$v_{xx} - \alpha^2 v = 0$$

$$\Downarrow$$

$$r^2 - \alpha^2 = 0$$

$$r^2 = \alpha^2$$

$$r = \pm\alpha$$

$$v = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$$

or

$$v = C_1 \sinh \alpha x + C_2 \cosh \alpha x$$

$$\frac{w_y}{w} = \alpha^2$$

$$w_y = \alpha^2 w$$

$$\frac{dw}{dy} = \alpha^2 w$$

$$\int \frac{1}{w} dw = \int \alpha^2 dy$$

$$\ln w = \alpha^2 y + C_3$$

$$w = e^{\alpha^2 y + C_3}$$

$$w = C_4 e^{\alpha^2 y}$$

$$u_3(x,y) = v(x)\, w(y)$$

$$= \left[C_1 e^{\alpha x} + C_2 e^{-\alpha x}\right]\left[C_4 e^{\alpha^2 y}\right]$$

$$= \boxed{D_1 e^{\alpha x} e^{\alpha^2 y} + D_2 e^{-\alpha x} e^{\alpha^2 y}}$$

or

$$u_3(x,y) = D_1 \sinh(\alpha x) e^{\alpha^2 y} + D_2 \cosh(\alpha x) e^{\alpha^2 y}$$

Your textbook likes to use the hyperbolic trig functions sinh and cosh.

We'll talk more about these functions later when we get to Laplace's Equation.

The general solution of the PDE $u_{xx} = u_y$ is

$$u = u_1 + u_2 + u_3$$

If we were given initial conditions or boundary values, we could decide which parts of the solution to keep and which to throw away.