

22 The z-Transform

Recommended Problems

P22.1

An LTI system has an impulse response $h[n]$ for which the z -transform is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

- (a) Plot the pole-zero pattern for $H(z)$.
- (b) Using the fact that signals of the form z^n are eigenfunctions of LTI systems, determine the system output for all n if the input $x[n]$ is

$$x[n] = \left(\frac{3}{4}\right)^n + 3(2)^n$$

P22.2

Consider the sequence $x[n] = 2^n u[n]$.

- (a) Is $x[n]$ absolutely summable?
- (b) Does the Fourier transform of $x[n]$ converge?
- (c) For what range of values of r does the Fourier transform of the sequence $r^{-n}x[n]$ converge?
- (d) Determine the z -transform $X(z)$ of $x[n]$, including a specification of the ROC.
- (e) $X(z)$ for $z = 3e^{j\Omega}$ can be thought of as the Fourier transform of a sequence $x_1[n]$, i.e.,

$$\begin{aligned} 2^n u[n] &\xleftrightarrow{\mathcal{Z}} X(z), \\ x_1[n] &\xleftrightarrow{\mathcal{F}} X(3e^{j\Omega}) = X_1(e^{j\Omega}) \end{aligned}$$

Determine $x_1[n]$.

P22.3

Shown in Figure P22.3 is the pole-zero plot for the z -transform $X(z)$ of a sequence $x[n]$.

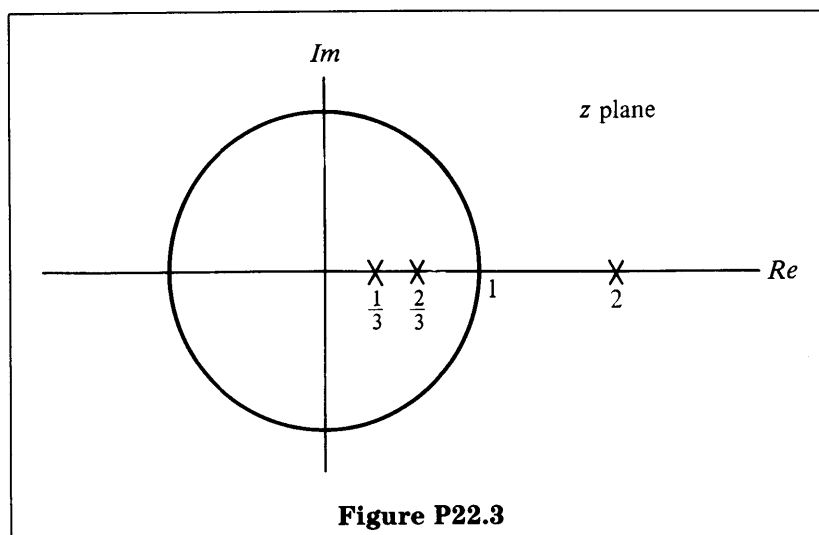


Figure P22.3

Determine what can be inferred about the associated region of convergence from each of the following statements.

- (a) $x[n]$ is right-sided.
- (b) The Fourier transform of $x[n]$ converges.
- (c) The Fourier transform of $x[n]$ does not converge.
- (d) $x[n]$ is left-sided.

P22.4

- (a) Determine the z -transforms of the following two signals. Note that the z -transforms for both have the same algebraic expression and differ only in the ROC.

(i) $x_1[n] = (\frac{1}{2})^n u[n]$

(ii) $x_2[n] = -(\frac{1}{2})^n u[-n - 1]$

- (b) Sketch the pole-zero plot and ROC for each signal in part (a).

- (c) Repeat parts (a) and (b) for the following two signals:

(i) $x_3[n] = 2u[n]$

(ii) $x_4[n] = -(2)^n u[-n - 1]$

- (d) For which of the four signals $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ in parts (a) and (c) does the Fourier transform converge?

P22.5

Consider the pole-zero plot of $H(z)$ given in Figure P22.5, where $H(a/2) = 1$.

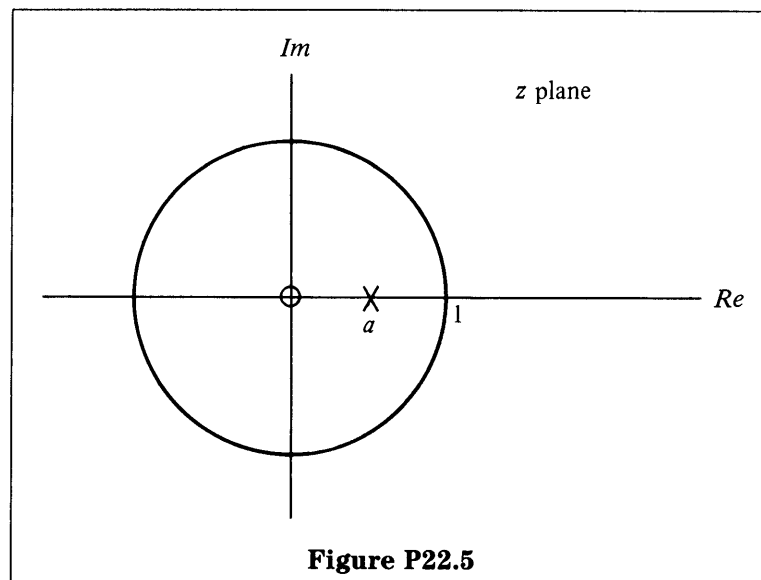


Figure P22.5

- (a) Sketch $|H(e^{j\omega})|$ as the number of zeros at $z = 0$ increases from 1 to 5.
- (b) Does the number of zeros affect $\angle H(e^{j\omega})$? If so, specifically in what way?
- (c) Find the region of the z plane where $|H(z)| = 1$.

P22.6

Determine the z -transform (including the ROC) of the following sequences. Also sketch the pole-zero plots and indicate the ROC on your sketch.

- (a) $(\frac{1}{3})^n u[n]$
- (b) $\delta[n + 1]$

P22.7

For each of the following z -transforms determine the inverse z -transform.

- (a) $X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$
- (b) $X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \quad |z| > \frac{1}{2}$
- (c) $X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \quad |z| > \left| \frac{1}{a} \right|$

Optional Problems

P22.8

In this problem we study the relation between the z -transform, the Fourier transform, and the ROC.

- (a) Consider the signal $x[n] = u[n]$. For which values of r does $r^{-n}x[n]$ have a converging Fourier transform?
- (b) In the lecture, we discussed the relation between $X(z)$ and $\mathcal{F}\{r^{-n}x[n]\}$. For each of the following values of r , sketch where in the z plane $X(z)$ equals the Fourier transform of $r^{-n}x[n]$.
 - (i) $r = 1$
 - (ii) $r = \frac{1}{2}$
 - (iii) $r = 3$
- (c) From your observations in parts (a) and (b), sketch the ROC of the z -transform of $u[n]$.

P22.9

- (a) Suppose $X(z)$ on the circle $z = 2e^{j\Omega}$ is given by

$$X(2e^{j\Omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}}$$

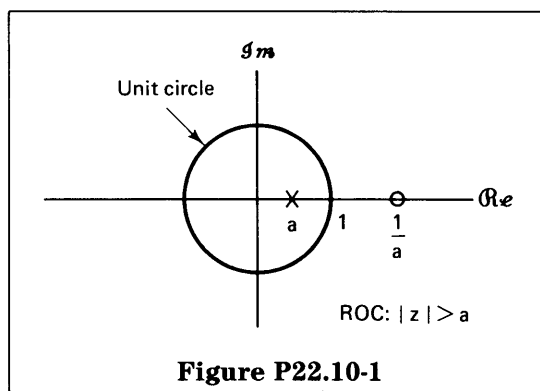
Using the relation $X(re^{j\Omega}) = \mathcal{F}\{r^{-n}x[n]\}$, find $2^{-n}x[n]$ and then $x[n]$, the inverse z -transform of $X(z)$.

- (b) Find $x[n]$ from $X(z)$ below using partial fraction expansion, where $x[n]$ is known to be causal, i.e., $x[n] = 0$ for $n < 0$.

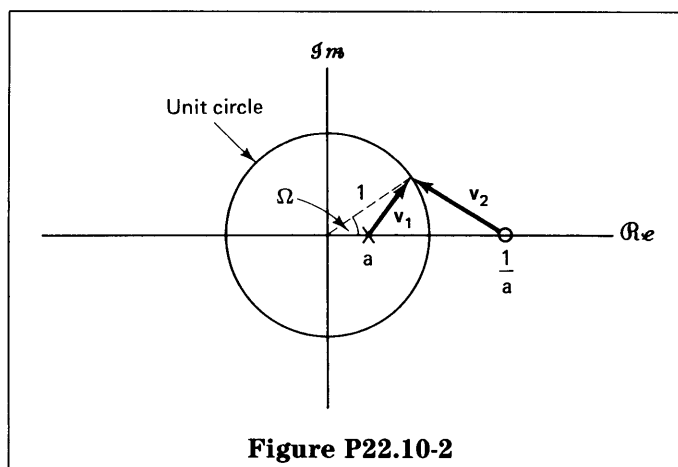
$$X(z) = \frac{3 + 2z^{-1}}{2 + 3z^{-1} + z^{-2}}$$

P22.10

A discrete-time system with the pole-zero pattern shown in Figure P22.10-1 is referred to as a first-order all-pass system because the magnitude of the frequency response is a constant, independent of frequency.



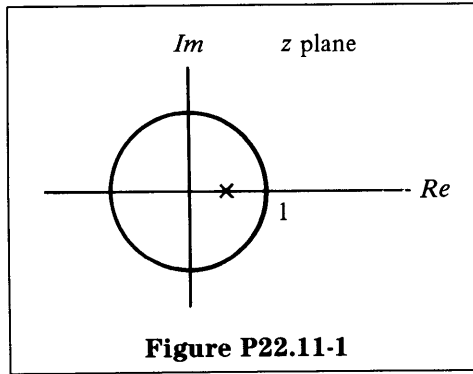
- (a) Demonstrate algebraically that $|H(e^{j\Omega})|$ is constant.
- (b) To demonstrate the same property geometrically, consider the vector diagram in Figure P22.10-2. Show that the length of v_2 is proportional to the length of v_1 independent of Ω by following these two steps:
- (i) Express the length of v_1 using the law of cosines and the fact that it is one leg of a triangle for which the other two legs are the unit vector and a vector of length a .
 - (ii) In a manner similar to that in step (i), determine the length of v_2 and show that it is proportional in length to v_1 independent of Ω .



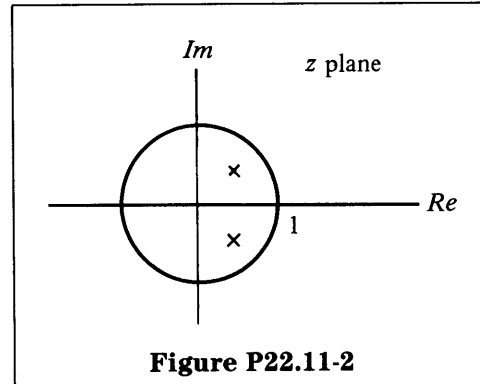
P22.11

Parts (a)–(e) (Figures P22.11-1 to P22.11-5) give pole-zero plots, and parts (i)–(iv) (Figures P22.11-6 to P22.11-9) give sketches of possible Fourier transform magnitudes. Assume that for all the pole-zero plots, the ROC includes the unit circle. For each pole-zero plot (a)–(e), specify which one *if any* of the sketches (i)–(iv) could represent the associated Fourier transform magnitude. More than one pole-zero plot may be associated with the same sketch.

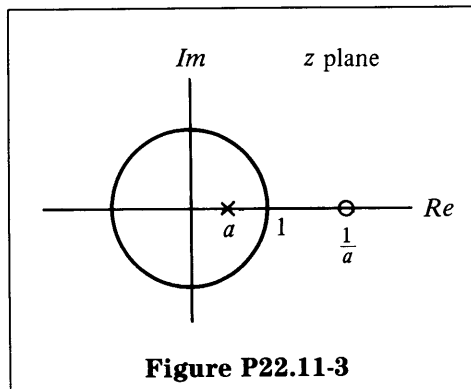
(a)



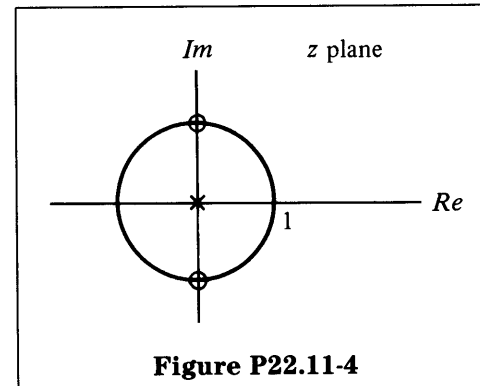
(b)



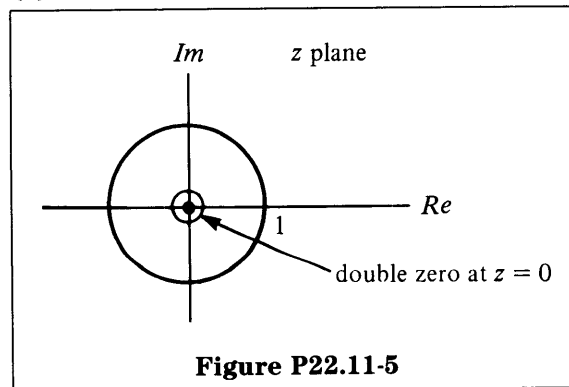
(c)



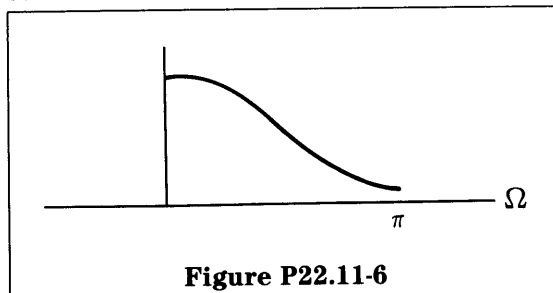
(d)



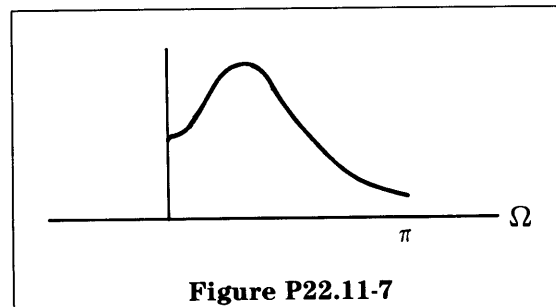
(e)



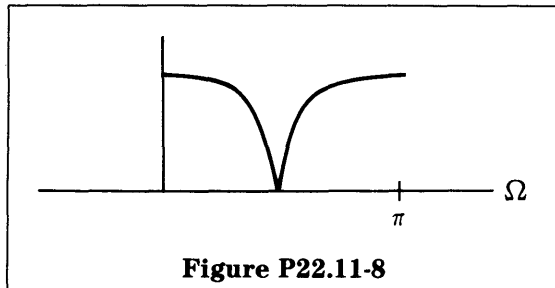
(i)



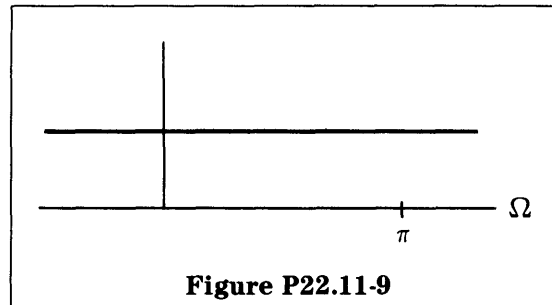
(ii)



(iii)



(iv)



P22.12

Determine the z -transform for the following sequences. Express all sums in closed form. Sketch the pole-zero plot and indicate the ROC. Indicate whether the Fourier transform of the sequence exists.

(a) $(\frac{1}{2})^n \{u[n] - u[n - 10]\}$

(b) $(\frac{1}{2})^{|n|}$

(c) $7 \left(\frac{1}{3}\right)^n \cos \left[\frac{2\pi n}{6} + \frac{\pi}{4} \right] u[n]$

(d) $x[n] = \begin{cases} 0, & n < 0 \\ 1, & 0 \leq n \leq 9 \\ 0, & 9 < n \end{cases}$

P22.13

Using the power-series expansion

$$\log(1 - w) = - \sum_{i=1}^{\infty} \frac{w^i}{i}, \quad |w| < 1,$$

determine the inverse of the following z -transforms.

(a) $X(z) = \log(1 - 2z), \quad |z| < \frac{1}{2}$

(b) $X(z) = \log(1 - \frac{1}{2}z^{-1}), \quad |z| > \frac{1}{2}$

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