THE CITADEL
THE MILITARY COLLEGE OF SOUTH CAROLINA

Dr. Gregory J. Mazzaro

Spring 2015

ELEC 318 – *Electromagnetic Fields*

Lecture 4(x)

Exam #1
Discussion

ELEC 318 Exam #1: **Problem #1**

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1. A sphere of radius 6 cm contains a volume charge density equal to $\dfrac{1}{\pi}\cos^2\theta \quad (\mathrm{C/m^3})$.

Determine the total charge contained in the sphere.

Example: Volume Charge Density

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An electron beam shaped like a circular cylinder of radius r_0 carries a charge density

$$\rho_v = -\frac{\rho_0}{1+r^2} \quad \left(\frac{\mathrm{C}}{\mathrm{m}^3}\right)$$

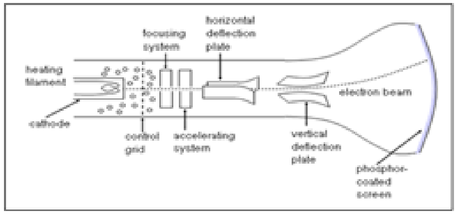
where ρ_0 is a positive constant and the beam is along the z axis.
Determine the total charge contained in length L of the beam.

$$Q = \int_v \rho_v \, dv$$

$$= \int_{z=0}^{z=L} \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=r_0} \left(-\frac{\rho_0}{1+r^2}\right) r \, dr \, d\phi \, dz$$

$$= -2\pi \cdot L \cdot \rho_0 \cdot \int_{r=0}^{r=r_0} r\left(1+r^2\right)^{-1} dr$$

$$= -2\pi \cdot L \cdot \rho_0 \cdot \left[\frac{1}{2}\ln\left(1+r^2\right)\right]_0^{r_0} = \boxed{-\pi L \rho_0 \ln\left(1+r_0^2\right)}$$



Cathode-Ray-Tube (CRT) television

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Lecture 4(d)
Slide #13, 14

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1. A sphere of radius 6 cm contains a volume charge density equal to $\frac{1}{\pi}\cos^2\theta$ $(\mathrm{C/m^3})$.

Determine the total charge contained in the sphere.

textbook, page 147

Example 3-6: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density ρ_v given by

$$\rho_v = 4\cos^2\theta \qquad (\mathrm{C/m^3}).$$

Find the total charge Q contained in the sphere.

Solution:

$$Q = \int_{v} \rho_v \, d\upsilon$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{2\times10^{-2}} (4\cos^2\theta) R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= 4 \int_{0}^{2\pi} \int_{0}^{\pi} \left(\frac{R^3}{3}\right)\Bigg|_{0}^{2\times10^{-2}} \sin\theta \cos^2\theta \, d\theta \, d\phi$$

$$= \frac{32}{3} \times 10^{-6} \int_{0}^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\Bigg|_{0}^{\pi} d\phi$$

$$= \frac{64}{9} \times 10^{-6} \int_{0}^{2\pi} d\phi$$

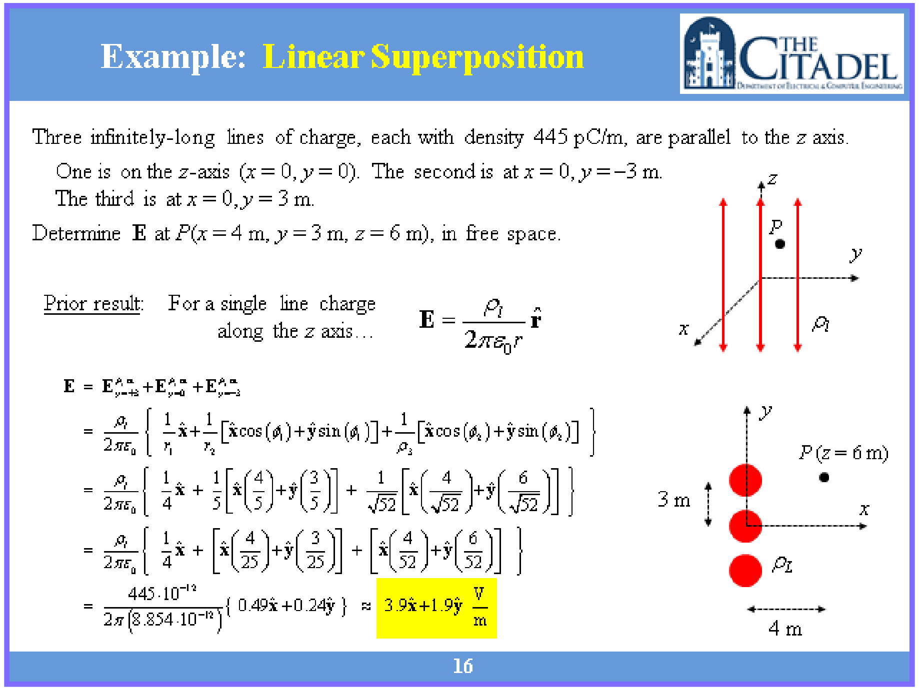
$$= \frac{128\pi}{9} \times 10^{-6} = 44.68 \qquad (\mu\mathrm{C}).$$

ELEC 318 Exam #1: Problem #2

2. A positive 490-nC charge is located at (12 m, 5 m, 0).
A positive 334-nC charge is located at (8 m, –6 m, 0).

Determine the force experienced by a negative 2-μC charge located at the origin, in free space.

Write your answer with appropriate units, in the appropriate direction.



Example: Linear Superposition

Three infinitely-long lines of charge, each with density 445 pC/m, are parallel to the z axis.
One is on the z-axis ($x=0, y=0$). The second is at $x=0, y=-3$ m.
The third is at $x=0, y=3$ m.
Determine $\mathbf{E}$ at $P(x=4 \text{ m}, y=3 \text{ m}, z=6 \text{ m})$, in free space.

Prior result: For a single line charge along the z axis… $\mathbf{E} = \dfrac{\rho_l}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$

$$\mathbf{E} = \mathbf{E}_{y=+3} + \mathbf{E}_{y=0} + \mathbf{E}_{y=-3}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\left\{ \frac{1}{r_1}\hat{\mathbf{x}} + \frac{1}{r_2}\left[\hat{\mathbf{x}}\cos(\phi_1) + \hat{\mathbf{y}}\sin(\phi_1)\right] + \frac{1}{\rho_3}\left[\hat{\mathbf{x}}\cos(\phi_2) + \hat{\mathbf{y}}\sin(\phi_2)\right] \right\}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\left\{ \frac{1}{4}\hat{\mathbf{x}} + \frac{1}{5}\left[\hat{\mathbf{x}}\left(\frac{4}{5}\right) + \hat{\mathbf{y}}\left(\frac{3}{5}\right)\right] + \frac{1}{\sqrt{52}}\left[\hat{\mathbf{x}}\left(\frac{4}{\sqrt{52}}\right) + \hat{\mathbf{y}}\left(\frac{6}{\sqrt{52}}\right)\right] \right\}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\left\{ \frac{1}{4}\hat{\mathbf{x}} + \left[\hat{\mathbf{x}}\left(\frac{4}{25}\right) + \hat{\mathbf{y}}\left(\frac{3}{25}\right)\right] + \left[\hat{\mathbf{x}}\left(\frac{4}{52}\right) + \hat{\mathbf{y}}\left(\frac{6}{52}\right)\right] \right\}$$

$$= \frac{445\cdot 10^{-12}}{2\pi\left(8.854\cdot 10^{-12}\right)}\left\{ 0.49\hat{\mathbf{x}} + 0.24\hat{\mathbf{y}} \right\} \approx 3.9\hat{\mathbf{x}} + 1.9\hat{\mathbf{y}} \ \frac{\text{V}}{\text{m}}$$

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Lecture 4(d)
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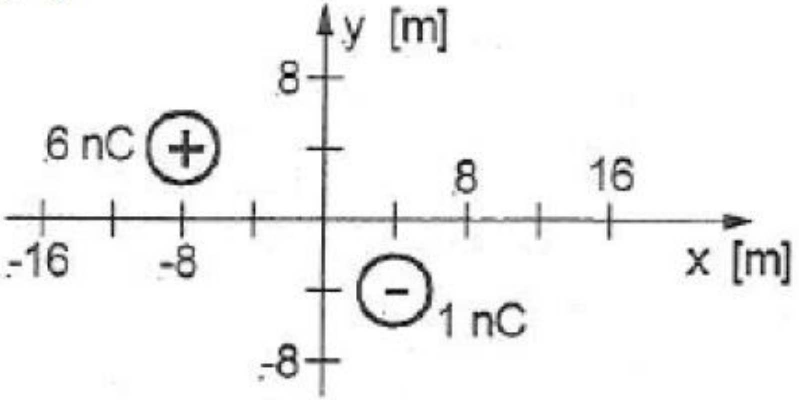
2. A positive 490-nC charge is located at (12 m, 5 m, 0).
A positive 334-nC charge is located at (8 m, –6 m, 0).

Determine the force experienced by a negative 2-μC charge located at the origin, in free space.

Write your answer with appropriate units, in the appropriate direction.

Exam #1 review packet, #17

Given the point charge distribution below (in the *x*-*y* plane), compute the electric field vector that would be measured at the observation point (8, 0, 1).



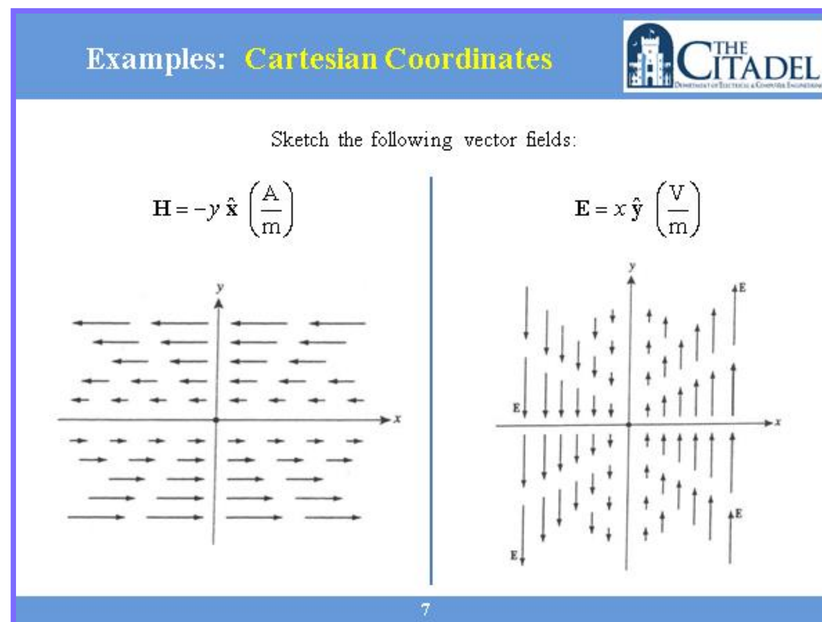
ELEC 318 Exam #1: Problem #3

3. An electric field intensity is equal to $\begin{cases} 5\, e^{-y}\, \hat{\mathbf{x}} \ \ \mathrm{V/m} & y \geq 0 \\ 5\, \hat{\mathbf{x}} \ \ \mathrm{V/m} & y < 0 \end{cases}$.

(a) Sketch this field in the *x*-*y* plane. Clearly indicate where the field is strongest and where the field is weakest. Account for all four quadrants and the axes.

(b) Determine the amount of work required to move a positive 7-mC charge from $P(r = 4\text{ cm},\ \phi = -60°,\ z = 0)$ to $Q(r = 8\text{ cm},\ \phi = -120°,\ z = 0)$ in this field.

Lecture 3(b)
Slide #6, 7



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3. An electric field intensity is equal to $\begin{cases} 5\, e^{-y}\, \hat{\mathbf{x}} \ \ \mathrm{V/m} & y \geq 0 \\ 5\, \hat{\mathbf{x}} \ \ \mathrm{V/m} & y < 0 \end{cases}$.

(a) Sketch this field in the x-y plane. Clearly indicate where the field is strongest and where the field is weakest. Account for all four quadrants and the axes.

(b) Determine the amount of work required to move a positive 7-mC charge from $P(r = 4\text{ cm}, \phi = -60°, z = 0)$ to $Q(r = 8\text{ cm}, \phi = -120°, z = 0)$ in this field.

HW #3

6. The electric field as a function of location is $20R\sin\theta\,\hat{\mathbf{R}} + 10R\cos\theta\,\hat{\boldsymbol{\theta}}$ V/m. Calculate the work required to move a charge of 10 nC…

(a) from $A(5, 30°, 0°)$ to $B(5, 90°, 0°)$, and

(b) from $C(10, 30°, 0°)$ to $A(5, 30°, 0°)$.

Exam #1 review packet #38

Calculate the work required to bring a 2-μC charge from $(r = 5\text{ m}, \phi = \pi/4, z = 3\text{ m})$ to $(r = 1\text{ m}, \phi = \pi/2, z = 6\text{ m})$ in the presence of this field: $\mathbf{E} = \dfrac{30}{r^2}\hat{\mathbf{r}} \left(\dfrac{\mathrm{V}}{\mathrm{m}}\right)$.

Express your answer in Joules.

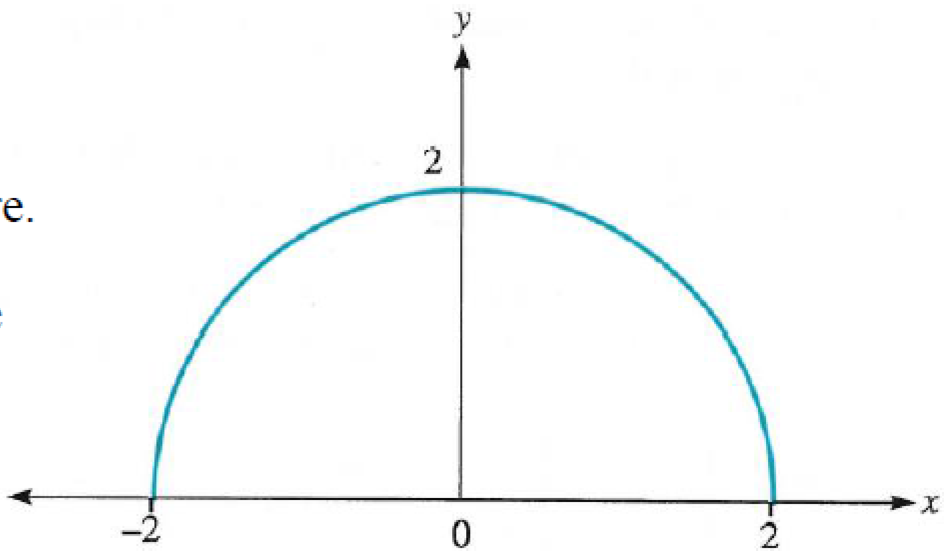
4. A circular ring of radius $b = 4$ m , in the x-y plane and centered on the origin, carries a uniform line charge density of 2.77 nC/m .

Calculate the electric field intensity directly above the center of the ring, at a height $h = 3$ m .

Homework #3

1. A point charge Q is located at point $P(0, -4, 0)$, while a 10 nC charge is uniformly distributed along a semicircular ring as shown in the figure.

   Determine the value of Q such that the electric field at the origin is zero.



ELEC 318 Exam #1: Problem #4

textbook, pages 185-186

Example 4-4: Electric Field of a Ring of Charge

A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the x–y plane as shown in Fig. 4-6. Determine the electric field intensity $\mathbf{E}$ at a point $P = (0, 0, h)$ along the axis of the ring at a distance h from its center.

Solution: We start by considering the electric field generated by a differential ring segment with cylindrical coordinates $(b, \phi, 0)$ in Fig. 4-6(a). The segment has length $dl = b\,d\phi$ and contains charge $dq = \rho_\ell\,dl = \rho_\ell b\,d\phi$. The distance vector $\mathbf{R}'_1$ from segment 1 to point $P = (0, 0, h)$ is

$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h,$$

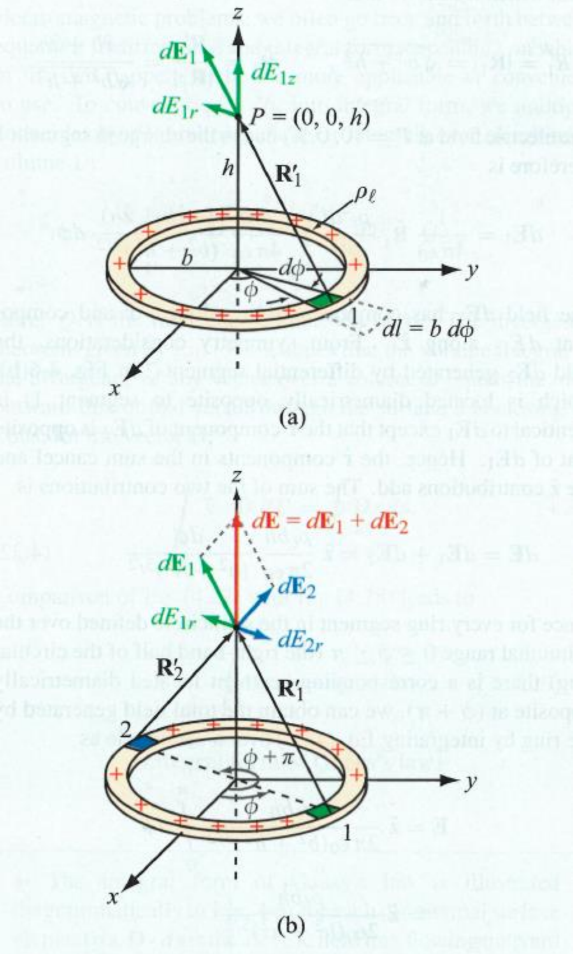


Figure 4-6 Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

from which it follows that

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}\,, \qquad \hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}\,.$$

The electric field at $P = (0, 0, h)$ due to the charge in segment 1 therefore is

$$d\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\,\hat{\mathbf{R}}'_1\,\frac{\rho_\ell\,dl}{R_1'^{\,2}} = \frac{\rho_\ell b}{4\pi\epsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}}\,d\phi.$$

The field $d\mathbf{E}_1$ has component dE_{1r} along $-\hat{\mathbf{r}}$ and component dE_{1z} along $\hat{\mathbf{z}}$. From symmetry considerations, the field $d\mathbf{E}_2$ generated by differential segment 2 in Fig. 4-6(b), which is located diametrically opposite to segment 1, is identical to $d\mathbf{E}_1$ except that the $\hat{\mathbf{r}}$ component of $d\mathbf{E}_2$ is opposite that of $d\mathbf{E}_1$. Hence, the $\hat{\mathbf{r}}$ components in the sum cancel and the $\hat{\mathbf{z}}$ contributions add. The sum of the two contributions is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\epsilon_0}\,\frac{d\phi}{(b^2 + h^2)^{3/2}}\,. \qquad (4.22)$$

Since for every ring segment in the semicircle defined over the azimuthal range $0 \le \phi \le \pi$ (the right-hand half of the circular ring) there is a corresponding segment located diametrically opposite at $(\phi + \pi)$, we can obtain the total field generated by the ring by integrating Eq. (4.22) over a semicircle as

$$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\epsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi d\phi$$

$$= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\epsilon_0 (b^2 + h^2)^{3/2}}$$

ELEC 318 Exam #1: Problem #5

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5. A spherical shell extending from inner radius $a = 12$ m to outer radius $b = 30$ m surrounds a charge-free cavity. The shell contains a constant volume charge density of 44.27 pC/m^3.

Determine the electric field intensity at $P(24\text{ m}, 70°, 40°)$. Assume $\varepsilon = \varepsilon_0$.

Homework #3

4. A volume charge density as a function of location is $\dfrac{50e^{-R}}{R}\ \dfrac{\text{nC}}{\text{m}^3}$.

Solve for the electric field everywhere, **in free space.**

Exam #1 review packet, #36

A spherical shell extending from inner radius a to outer radius b surrounds a charge-free cavity.

If this geometry contains a volume charge density given by $\rho_v = \begin{cases} -\dfrac{\rho_0}{R^2} & a \le R \le b \\ 0 & R < a, R > b \end{cases}$,

determine **E** everywhere. Assume $\varepsilon = \varepsilon_0$.

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ELEC 318 – *Electromagnetic Fields*

Lecture 4(e)

The Laplacian & Laplace's Equation

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171 Moultrie Street, Charleston, SC 29409

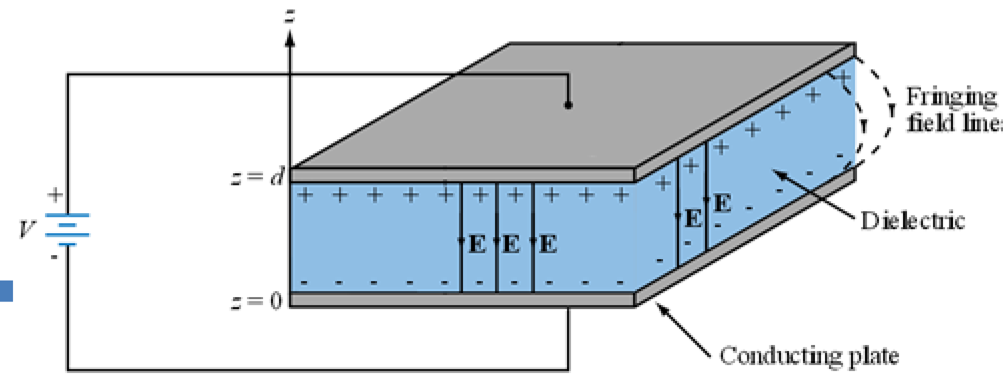
Laplacian

Laplacian of a *scalar* field (V) at a point (P)

-- a *scalar field*; a measure of the *curvature* of V summed along all three dimensions

$$\nabla^2 V$$

-- equals the divergence of the gradient of V



$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

-- most useful for determining electric potential in space

Poisson's Equation, Laplace's Equation

starting with Gauss' Law,

substituting **E** for **D**,

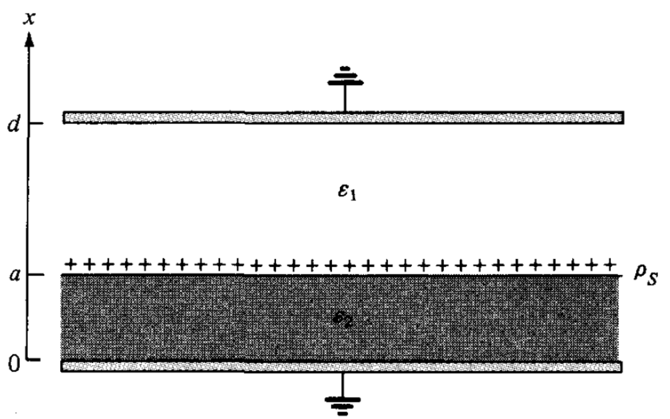
substituting V for **E**, and

taking the divergence of the gradient…

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \mathbf{E} = \rho_v / \varepsilon$$

$$\nabla \cdot \left(-\nabla V \right) = \rho_v / \varepsilon$$



$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

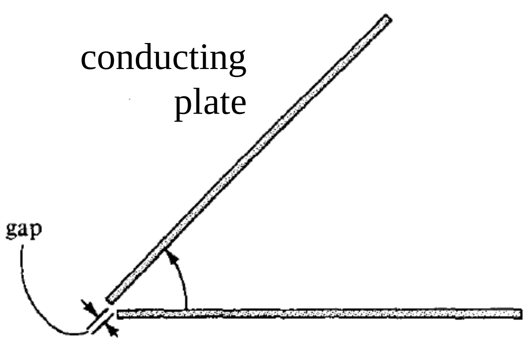
Poisson's Equation
(charge present)

$$\boxed{\nabla^2 V = 0}$$

Laplace's Equation
(charge-free)

→ another way to solve for potential, electric field, work

→ useful for more complicated EM geometries
(e.g. capacitors containing multiple dielectrics)
(e.g. capacitors that are *not* parallel plates)

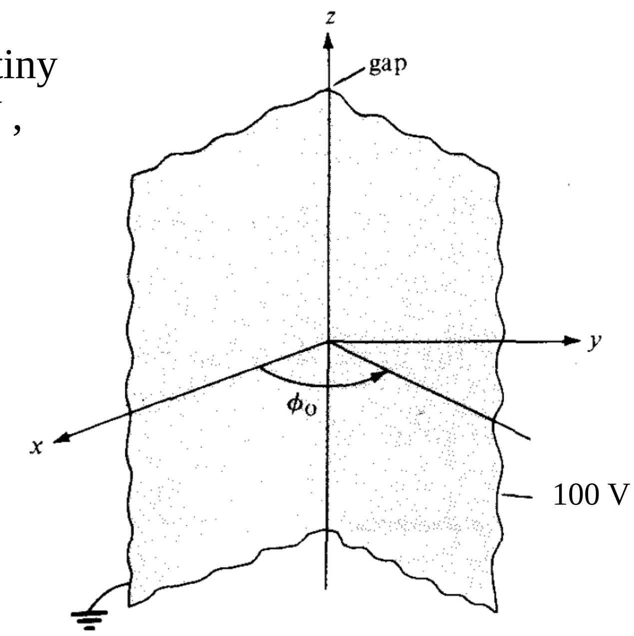


Example: Laplace's Equation

Semi-infinite conducting plates at $\phi = 0$ and $\phi = \pi/6$ are separated by a tiny insulating gap along $z = 0$. If $V(\phi = 0) = 0$ and $V(\phi = \pi/6) = 100$ V , calculate V and **E** everywhere between the plates.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$$



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ELEC 318 – *Electromagnetic Fields*

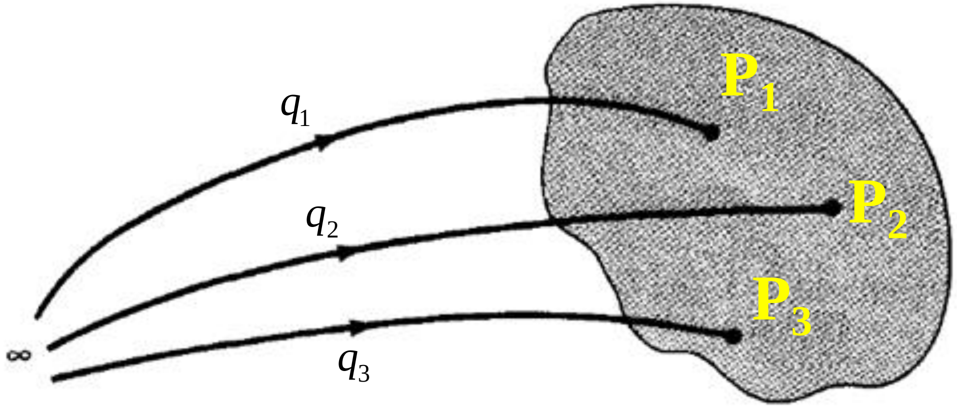
<u>Lecture 4(f)</u>

Electrostatic Energy & Capacitance

Electrostatic Energy Storage

Derivation of the total energy stored in an electrostatic field:

(1) Assume no charge exists in a region of space, then bring charges in, one at a time. Calculate the work required to bring each new charge into that space:



$$W_E = W_1 + W_2 + W_3$$
$$= 0 + q_2 V_{21} + q_3\left(V_{31} + V_{32}\right)$$

V_{MN} = potential of q_M in the presence of q_N

In reverse order, $\quad W_E = 0 + q_2 V_{23} + q_1\left(V_{12} + V_{13}\right)$

(2) Add these two equations for W_E, divide by 2…

$$W_E = \frac{1}{2}\left[q_1 V_1 + q_2 V_2 + q_3 V_3\right]$$

For more than 3 charges,

$$W_E = \frac{1}{2}\sum_{k=1}^{N} q_k V_k$$

(3) Rewriting this result for W_E in integral form (derived in your book)…

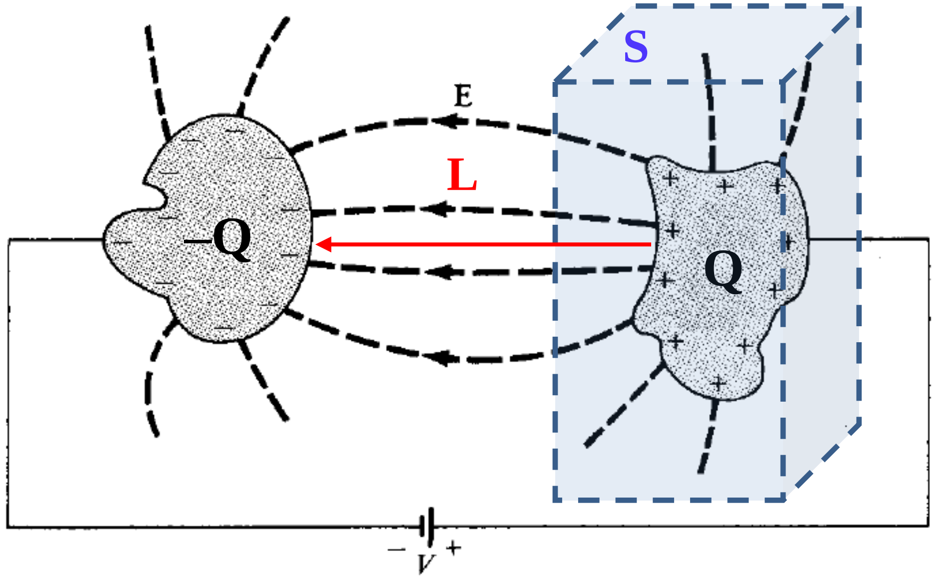
$$dW_E = \frac{1}{2} dq \cdot V \quad \Rightarrow \quad W_E = \frac{1}{2}\int_v \rho_v \cdot V\, dv = \frac{1}{2}\int_v (\nabla\cdot\mathbf{D}) V\, dv$$
$$= \frac{1}{2}\int_v (\mathbf{D}\cdot\mathbf{E})\, dv = \frac{1}{2}\int_v \varepsilon_0 \left|\mathbf{E}\right|^2 dv$$

Capacitance

capacitance

-- a measure of the <u>ability</u> of a structure to store electrostatic energy

-- computed as the ratio of *charge induced* (Q) on the *positive* conductor to the *voltage applied* (V) across the two conductors



$$C = \frac{Q}{V} = \frac{\varepsilon \oint_S \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}}$$

Gauss' Law, applied to the conductor holding Q

line integral of **E**, from Q to $-Q$

<u>Note</u>: For the same applied voltage, if Q induced is larger, $|\mathbf{E}|$ is larger, and W_E is larger.

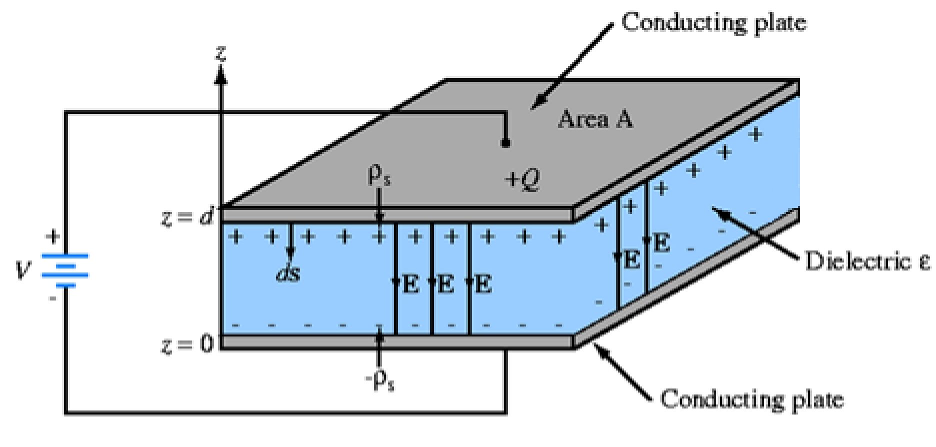
i.e. For a higher capacitance, when V is applied, W_E is larger.

$$W_E = \frac{1}{2}\int dq \cdot V = \frac{1}{2} V \int_v \rho_v \, dv = \frac{1}{2} V \cdot Q = \frac{1}{2} V(CV) = \frac{1}{2} CV^2$$

Example: Capacitance, Planar

Determine the capacitance of this planar structure (in terms of d, A, and ε) .

Assume that the plates are large enough to neglect fringing.



$$C = \frac{Q}{V} = \frac{\varepsilon \oint_S \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}}$$

Method 1:

Assume Q on each plate, solve for V, take the ratio.

$$C = \frac{Q}{\int_L \mathbf{E} \cdot d\mathbf{l}}$$

Method 2:

Assume V applied, solve for Q on each plate, take the ratio.

$$C = \frac{\varepsilon \oint_S \mathbf{E} \cdot d\mathbf{S}}{V}$$