6 Systems Represented by Differential and Difference Equations

Solutions to Recommended Problems

S6.1

We substitute $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ into the homogeneous differential equation

$$\frac{dy_3(t)}{dt} + ay_3(t) = \frac{d}{dt}[\alpha y_1(t) + \beta y_2(t)] + a[\alpha y_1(t) + \beta y_2(t)]$$

Since differentiation is distributive, we can express the preceding equation as

$$\alpha \frac{dy_1(t)}{dt} + \beta \frac{dy_2(t)}{dt} + a\alpha y_1(t) + a\beta y_2(t)$$
$$= \alpha \left[\frac{dy_1(t)}{dt} + ay_1(t)\right] + \beta \left[\frac{dy_2(t)}{dt} + ay_2(t)\right]$$

However, since both $y_1(t)$ and $y_2(t)$ satisfy the homogeneous differential equation, the right side of the equation is zero. Therefore,

$$\frac{dy_3(t)}{dt} + ay_3(t) = 0$$

S6.2

(a) We are assuming that $y(t) = e^{st}$. Substituting in the differential equation yields

$$\frac{d^2}{dt^2}(e^{st}) + 3\frac{d}{dt}(e^{st}) + 2e^{st} = 0$$

so that

$$s^2e^{st} + 3se^{st} + 2e^{st} = e^{st}(s^2 + 3s + 2) = 0$$

For any finite s, e^{st} is not zero. Therefore, s must satisfy

$$0 = s^2 + 3s + 2 = (s + 1)(s + 2), \qquad s = -1, -2$$

(b) From the answer to part (a), we know that both $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$ satisfy the homogeneous LCCDE. Therefore,

$$y_3(t) = K_1e^{-t} + K_2e^{-2t},$$

for any constants K_1, K_2, will also satisfy the equation.

S6.3

(a) Assuming $y(t)$ of the form

$$y(t) = Ke^{st},$$

we substitute into the LCCDE, setting $x[n] = 0$:

$$0 = \frac{dy(t)}{dt} + \frac{1}{2}y(t) = Kse^{st} + K\frac{1}{2}e^{st} = Ke^{st}\left(s + \frac{1}{2}\right)$$

Since $K \neq 0$ and $e^{st} \neq 0$, s must equal $-\frac{1}{2}$. K then becomes arbitrary, so the family of $y(t)$ that satisfies the homogeneous equation is

$$y(t) = Ke^{-t/2}$$

(b) Substituting into eq. (P6.3-1) $y_1(t) = Ae^{-t}$ for $t > 0$, we find

$$\frac{dy_1(t)}{dy} + \frac{1}{2}y_1(t) = -Ae^{-t} + \frac{1}{2}Ae^{-t} = e^{-t}, \qquad t > 0$$

Since e^{-t} never equals zero, we can divide it out. This gives us an equation for A,

$$-A + \frac{A}{2} = 1 \qquad \text{as } A = -2$$

(c) For $y_1(t) = (2e^{-t/2} - 2e^{-t})u(t)$,

$$\frac{dy_1(t)}{dt} = \begin{cases} [2\left(-\frac{1}{2}\right)e^{-t/2} - 2(-1)e^{-t}], & t > 0 \\ 0, \quad t \leq 0, \end{cases}$$

$$\frac{dy_1(t)}{dt} + \frac{1}{2}y_1(t) = \begin{cases} (-e^{-t/2} + 2e^{-t}) + \frac{1}{2}(2e^{-t/2} - 2e^{-t}) = e^{-t}, & t > 0 \\ 0, \quad t < 0 \end{cases}$$
$$= x(t)$$

S6.4

(a) Note that since $y[n]$ is delayed by one sample by the delay element, we can label the block diagram as shown in Figure S6.4.

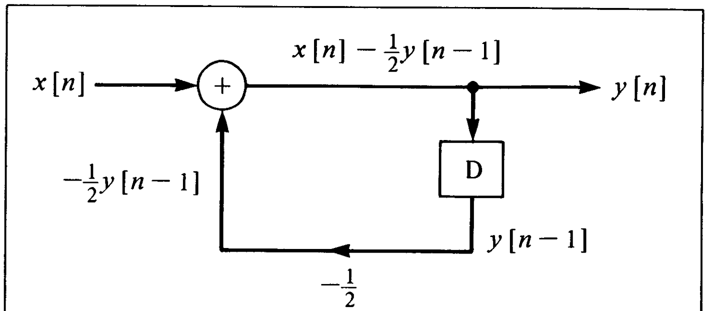

Figure S6.4

Thus $y[n] = x[n] - \frac{1}{2}y[n - 1]$, or $y[n] + \frac{1}{2}y[n - 1] = x[n]$.

(b) Since the system is assumed to be causal, $y[n]$ must be zero before a nonzero input is applied. Therefore, $x[n] = 0$ for $n < 0$, and consequently $y[n]$ must be zero for $n < 0$. Thus, $y[-5] = 0$.

(c) Since $x[n] = \delta[n] = 0$ for $n < 0$, $y[n]$ must also equal zero for $n < 0$. For $n = 0$, we have $y[0] + \frac{1}{2}y[-1] = 1$ or, substituting for $y[n]$,

$$K\alpha^0u[0] + \tfrac{1}{2}K\alpha^{-1}u[-1] = 1,$$
$$K + \tfrac{1}{2} \cdot 0 = 1, \quad \text{or} \quad K = 1$$

For $n > 0$, we have

$$y[n] + \tfrac{1}{2}y[n - 1] = 0 \quad \text{or} \quad \alpha^n + \tfrac{1}{2}\alpha^{n-1} = 0$$

since $K = 1$. Thus, α must equal $-\frac{1}{2}$ for $\alpha^n + \frac{1}{2}\alpha^{n-1}$ to equal 0 for all $n > 0$. Therefore, $y[n] = (-\frac{1}{2})^n u[n]$. Substituting into the left side of the difference equation, we have

$$(-\tfrac{1}{2})^n\, u[n] + \tfrac{1}{2}(-\tfrac{1}{2})^{n-1}\, u[n-1] = (-\tfrac{1}{2})^n\, u[n] - (-\tfrac{1}{2})^n\, u[n-1]$$

$$= \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

(d) We can successively calculate $y[n]$ by noting that $y[-1] = 0$ and that

$$y[n] = -\tfrac{1}{2}y[n-1] + \delta[n]$$

So

$$\begin{aligned} n &= 0, & y[0] &= -\tfrac{1}{2}\cdot 0 + 1 = 1 \\ n &= 1, & y[1] &= -\tfrac{1}{2}\cdot 1 + 0 = -\tfrac{1}{2} \\ n &= 2, & y[2] &= -\tfrac{1}{2}\cdot(-\tfrac{1}{2}) + 0 = \tfrac{1}{4} \end{aligned}$$

We see that these correspond to the answer to part (c).

S6.5

(a) Performing the manipulations in inverse order to that done in the lecture (see Figure S6.5-1) yields the system shown in Figure S6.5-2.

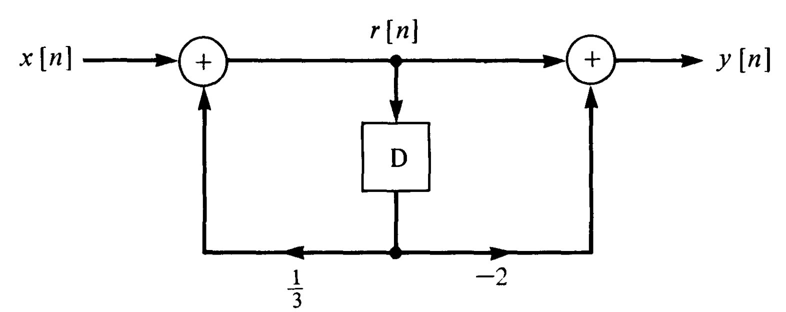

Figure S6.5-1

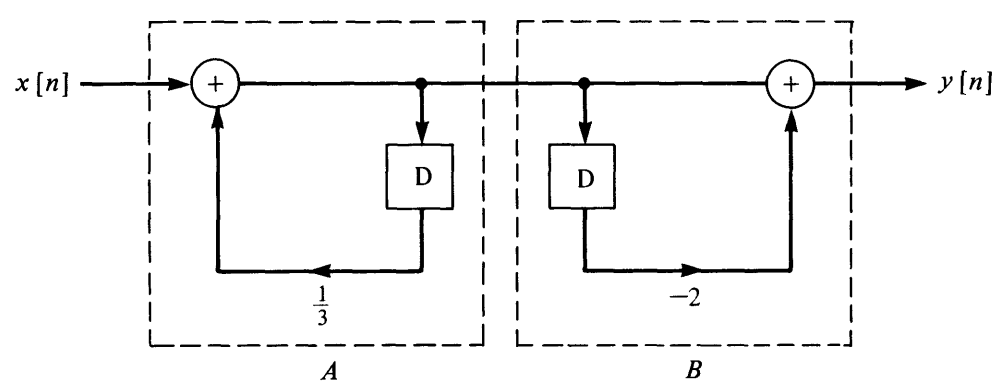

Figure S6.5-2

Since the system is linear and time-invariant, we can exchange the order of the two boxes A and B, yielding the direct form I shown in Figure S6.5-3.

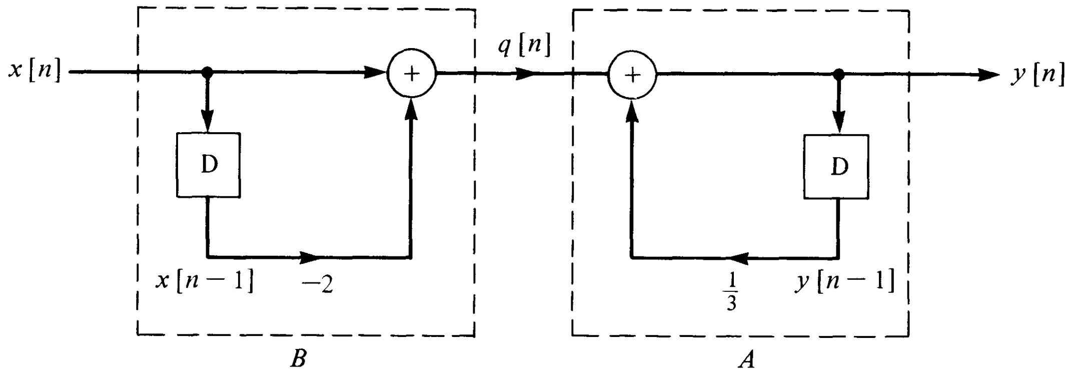

Figure S6.5-3

(b) From the direct form I, we see that the intermediate variable $q[n]$ is related to $x[n]$ by

$$q[n] = x[n] - 2x[n-1]$$

The signal $y[n]$ can be described in terms of $q[n]$ and $y[n-1]$ as

$$y[n] = q[n] + \tfrac{1}{3}y[n-1]$$

Combining the two equations yields

$$y[n] = \tfrac{1}{3}y[n-1] + x[n] - 2x[n-1], \quad \text{or}$$
$$y[n] - \tfrac{1}{3}y[n-1] = x[n] - 2x[n-1]$$

(c) (i) Figure S6.5-4 shows that if we concentrate on the right half of the diagram of direct form II given in Figure P6.5, we see the relation

$$y[n] = r[n] - 2r[n-1]$$

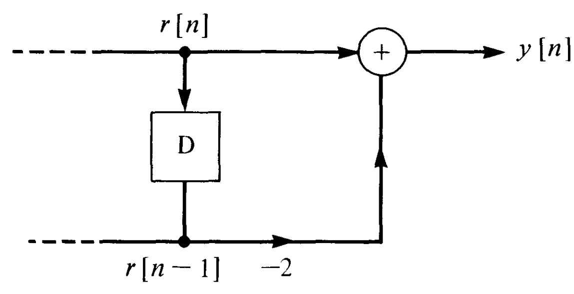

Figure S6.5-4

(ii) Similarly, Figure S6.5-5 shows that if we concentrate on the first half of the diagram, we obtain the relation

$$r[n] = x[n] + \tfrac{1}{3}r[n-1], \quad \text{or} \quad x[n] = r[n] - \tfrac{1}{3}r[n-1]$$

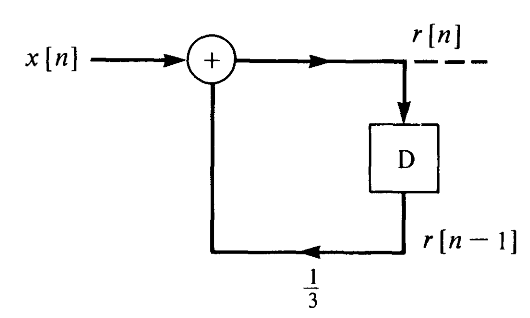

Figure S6.5-5

(iii) From the two equations obtained in parts (i) and (ii),

$$x[n] = r[n] - \tfrac{1}{3}r[n-1] \qquad \text{(S6.5-1)}$$

and

$$y[n] = r[n] - 2r[n-1], \qquad \text{(S6.5-2)}$$

we solve for $r[n]$, obtaining

$$r[n] = \tfrac{6}{5}x[n] - \tfrac{1}{5}y[n]$$

Substituting $r[n]$ into eq. (S6.5-1), we have

$$x[n] = \tfrac{6}{5}x[n] - \tfrac{1}{5}y[n] - \tfrac{1}{3}\{\tfrac{6}{5}x[n-1] - \tfrac{1}{5}y[n-1]\},$$

which simplifies to

$$y[n] - \tfrac{1}{3}y[n-1] = x[n] - 2x[n-1]$$

S6.6

(a) Integrating both sides of eq. (P6.6-1) yields

$$y(t) + a\int y(t)\,dt = bx(t) + c\int x(t)\,dt, \quad \text{or}$$
$$y(t) = -a\int y(t)\,dt + bx(t) + c\int x(t)\,dt$$

Thus, we set up the direct form I in Figure S6.6-1.

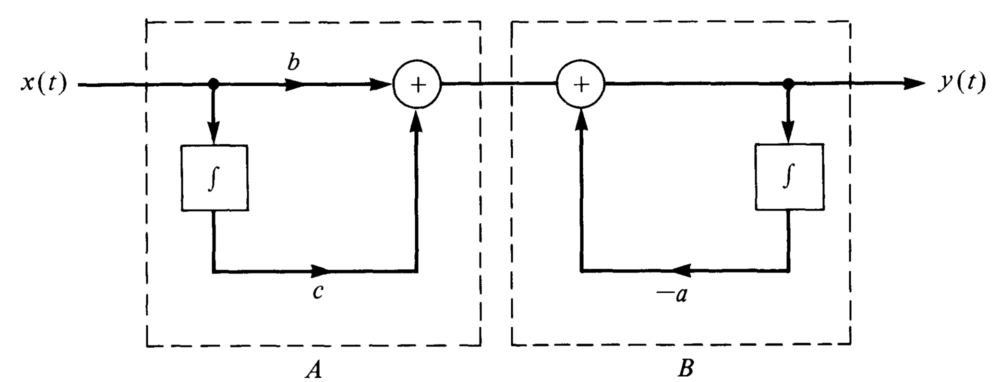

Figure S6.6-1

(b) Since we are told that the system is linear and time-invariant, we can interchange boxes A and B, as shown in Figure S6.6-2.

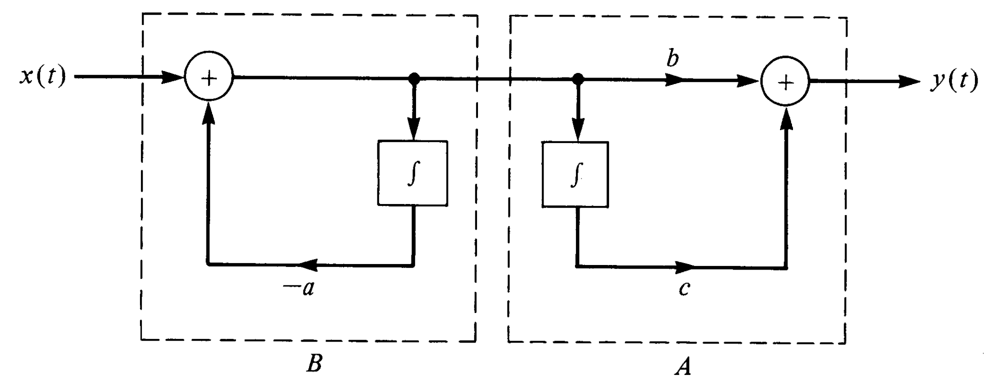

Figure S6.6-2

Combining the two integrators yields the final answer, shown in Figure S6.6-3.

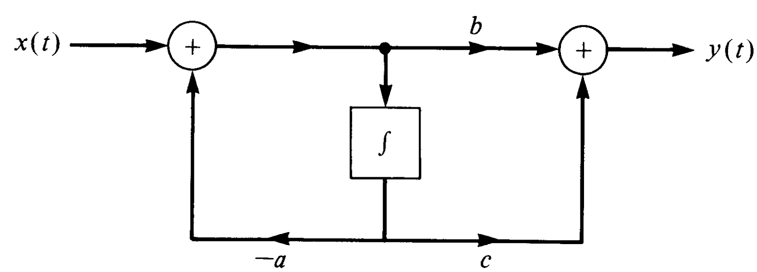

Figure S6.6-3

Solutions to Optional Problems

S6.7

(a) In Figure S6.7 we convert the block diagram from Figure P6.7 to direct form I.

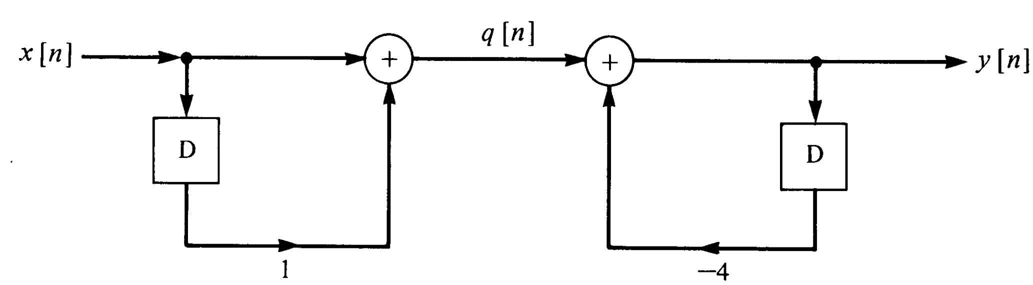

Figure S6.7

$q[n]$ is given by

$$q[n] = x[n] + x[n - 1]$$

while

$$y[n] = q[n] - 4y[n - 1]$$

Substituting for $q[n]$ yields

$$y[n] + 4y[n - 1] = x[n] + x[n - 1]$$

(b) The relation between $x[n]$ and $r[n]$ is $r[n] = -4r[n - 1] + x[n]$. For such a simple equation, we solve it recursively when $\delta[n] = x[n]$.

n	$\delta[n]$	$r[n-1]$	$r[n]$
<0	0	0	0
0	1	0	1
1	0	1	-4
2	0	-4	16
3	0	16	-64

We see that $r[n] = (-4)^n u[n]$.

(c) $y[n]$ is related to $r[n]$ by

$$y[n] = r[n] + r[n - 1]$$

Now $y[n] = h[n]$, the impulse response, when $x[n] = \delta[n]$, and

$$h[n] = (-4)^n u[n] + (-4)^{n-1}u[n - 1]$$

This expression for $h[n]$ can be further simplified:

$$h[n] = (-4)^n u[n] + (-4)^{n-1}u[n - 1]$$

or

$$h[n] = \begin{cases} 0, & n < 0, \\ 1, & n = 0 \end{cases}$$

For $n > 0$,

$$\begin{aligned} h[n] &= (-4)^n + (-4)^{n-1} \\ &= -3(-4)^{n-1} \end{aligned}$$

Thus,

$$h[n] = \delta[n] - 3(-4)^{n-1}u[n - 1]$$

S6.8

Note that the system in Figure P6.8 is not in any standard form. Relating $r(t)$ to $x(t)$ first, we have

$$\int a[x(t) + r(t)]\,dt = r(t), \qquad \text{or} \tag{S6.8-1}$$

$$\frac{dr(t)}{dt} - ar(t) = ax(t),$$

represented in the system shown in Figure S6.8.

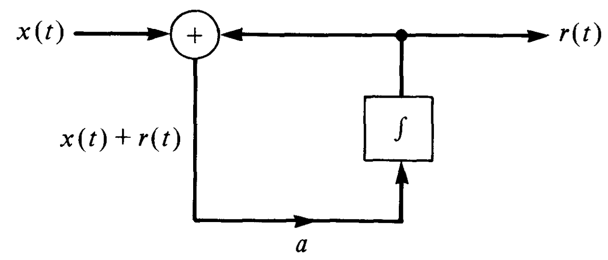

Figure S6.8

The signal $y(t)$ is related to $r(t)$ as follows:

$$r(t) + b\int r(t)\,dt = y(t), \qquad \text{or} \tag{S6.8-2}$$
$$\frac{dr(t)}{dt} + br(t) = \frac{dy(t)}{dt}$$

Solving for $dr(t)/dt$ in eqs. (S6.8-1) and (S6.8-2) and equating, we obtain

$$ar(t) + ax(t) = -br(t) + \frac{dy(t)}{dt}$$

Therefore,

$$r(t) = \frac{-a}{a+b}x(t) + \frac{1}{a+b}\frac{dy(t)}{dt} \tag{S6.8-3}$$

We now substitute eq. (S6.8-3) into eq. (S6.8-1) (or eq. S6.8-2), which, after simplification, yields

$$\frac{dy^2(t)}{dt^2} - a\frac{dy(t)}{dt} = a\frac{dx(t)}{dt} + abx(t)$$

S6.9

(a) Substituting $y[n] = Az_0^n$ into the homogeneous LCCDE, we have

$$Az_0^n - \tfrac{1}{2}Az_0^{n-1} = 0$$

Dividing by Az_0^{n-1} yields

$$z_0 - \tfrac{1}{2} = 0, \qquad \text{or} \qquad z_0 = \tfrac{1}{2}$$

(b) For the moment, assume that the input is $\hat{x}[n] = Ke^{j\Omega_0 n}u[n]$ and the resulting output is $\hat{y}[n] = Ye^{j\Omega_0 n}u[n]$. Thus,

$$\hat{y}[n] - \tfrac{1}{2}\hat{y}[n-1] = \hat{x}[n]$$

Substituting for $\hat{y}[n]$ and $\hat{x}[n]$ yields

$$Ye^{j\Omega_0 n} - \tfrac{1}{2}Ye^{j\Omega_0(n-1)} = Ke^{j\Omega_0 n} \qquad \text{for } n \geq 1$$

Dividing by $e^{j\Omega_0 n}$, we get

$$Y - \tfrac{1}{2}e^{-j\Omega_0} \cdot Y = K$$

Thus

$$Y = \frac{K}{1 - \frac{1}{2}e^{-j\Omega_0}} = \frac{K}{\sqrt{\frac{5}{4} - \cos\Omega_0}\; e^{+j\tan^{-1}[(\sin\Omega_0)/(2-\cos\Omega_0)]}}, \qquad \text{or}$$

$$Y = \frac{K}{\sqrt{\frac{5}{4} - \cos\Omega_0}}\, e^{-j\tan^{-1}[(\sin\Omega_0)/(2-\cos\Omega_0)]}$$

Therefore,

$$y[n] = Re[Ye^{j\Omega_0 n}u[n]] = \frac{K}{\sqrt{\frac{5}{4} - \cos\Omega_0}} Re[e^{j(\Omega_0 n - \tan^{-1}[(\sin\Omega_0)/(2-\cos\Omega_0)])}u[n]]$$

$$= B\cos(\Omega_0 n + \theta), \qquad \text{where } B = \frac{K}{\sqrt{\frac{5}{4} - \cos\Omega_0}},$$

$$\theta = -\tan^{-1}\left(\frac{\sin\Omega_0}{2 - \cos\Omega_0}\right)$$

S6.10

The important observation to make is that if $[d^i r(t)]/dt^i$ is the input to the system H, then $[d^i s(t)]/dt^i$ will be the output. Suppose that we construct a signal

$$q(t) = \sum_{i=1}^{M} a_i \frac{d^i r(t)}{dt^i}$$

The response of H to the excitation $q(t)$ is

$$p(t) = \sum_{i=1}^{M} a_i \frac{d^i s(t)}{dt^i}$$

However, $q(t) = 0$ for all t. Therefore, $p(t) = 0$ for all t. Thus,

$$\sum_{i=1}^{M} a_i \frac{d^i s(t)}{dt^i} = 0$$

S6.11

(a) Substituting $y(t) = Ae^{s_0 t}$ into the homogeneous LCCDE, we have

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} a_k \frac{d^k}{dt^k}(Ae^{s_0 t}) = 0$$

$$= \left(\sum_{k=0}^{N} a_k s_0^k\right) Ae^{s_0 t} = 0$$

Since $A \neq 0$ and $e^{s_0 t} \neq 0$, we get

$$p(s_0) = \sum_{k=0}^{N} a_k s_0^k = 0$$

(b) Here we need to use a rather subtle trick. Note that

$$Ate^{st} = \frac{d}{ds}(Ae^{st})$$

Using this alternative form for Ate^{st}, we obtain

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k}\left(\frac{d}{ds} Ae^{st}\right) = \frac{d}{ds}\left[\sum_{k=0}^{N} a_k \frac{d^k}{dt^k}(Ae^{st})\right]$$

$$= \frac{d}{ds}[p(s)Ae^{st}] = Atp(s) + A\frac{dp(s)}{ds}e^{st}$$

For $s = s_0$, $p(s_0) = 0$. Also, since $p(s)$ is of the form

$$p(s) = (s - s_0)^2 q(s),$$

we have

$$\left.\frac{dp(s)}{ds}\right|_{s=s_0} = 0$$

Therefore, $Ate^{s_0 t}$ satisfies the homogeneous LCCDE.

(c) Substituting $y(t) = e^{st}$, we get the characteristic equation

$$s^2 + 2s + 1 = 0, \qquad \text{or} \qquad s_0 = -1$$

Thus, $y(t) = K_1 e^{-t} + K_2 te^{-t}$. For $y(0) = 1$ and $y'(0) = 1$, we need $K_1 = 1$ and $K_2 - K_1 = 1$, or $K_2 = 2$. Thus,

$$y(t) = e^{-t} + 2te^{-t}$$