ELEC 309
Signals and Systems
Homework 7 Solutions

Frequency-Domain Analysis of Signals and Systems

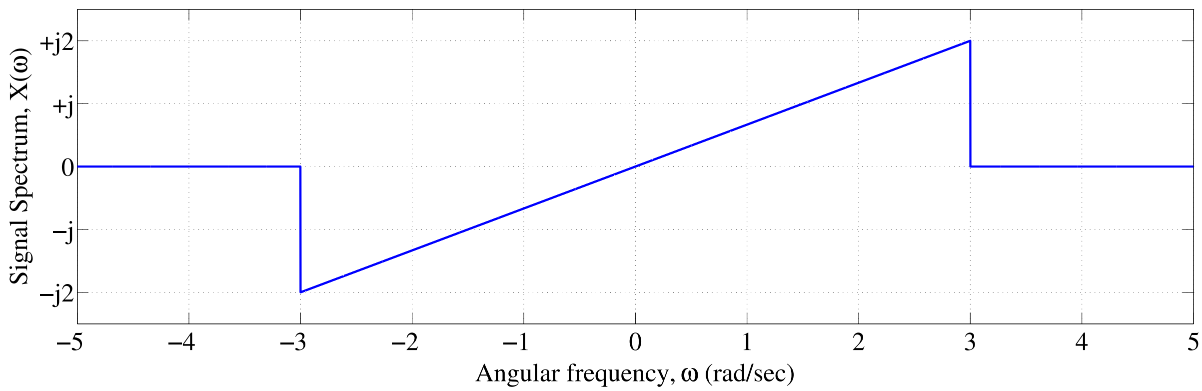


Figure 1: Signal spectrum $X(\omega)$ for Problem 1

1. For the Fourier spectrum $X(\omega)$ shown in Figure 1:

   (a) Determine a single mathematical expression for the Fourier spectrum $X(\omega)$.
   Hint: You should use the ***rect***() *function discussed in class!]*

$$X(\omega) = \frac{2}{3} j\omega \mathrm{rect}\left(\frac{\omega}{6}\right)$$

   (b) Find signal $x(t)$ by using the inverse Fourier transform integral (Equation 33 in the class notes).

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{+j\omega t} d\omega = \frac{1}{2\pi}\int_{-3}^{3} \frac{2}{3} j\omega e^{+j\omega t} d\omega = \frac{j}{3\pi}\int_{-3}^{3} \omega e^{+j\omega t} d\omega \\
&= \frac{j}{3\pi}\left(\frac{1}{(jt)^2}\right)\left[e^{+j\omega t}(j\omega t - 1)\right]_{-3}^{3} = \frac{\left[e^{+j\omega t}(j+\omega t)\right]_{-3}^{3}}{3\pi t^2} = \frac{e^{+j3t}(j+3t) - e^{-j3t}(j-3t)}{3\pi t^2} \\
&= \frac{-2\left[\frac{e^{+j3t}-e^{-j3t}}{j2}\right] + 6t\left[\frac{e^{+j3t}+e^{-j3t}}{2}\right]}{3\pi t^2} = \boxed{\frac{6t\cos(3t) - 2\sin(3t)}{3\pi t^2}}
\end{aligned}$$

(c) Find signal $x(t)$ by using the table of Fourier transform pairs and the Differentiation in the Time Domain property (Equation 40 in the class notes).

Combining the Fourier transform pair $\frac{W}{\pi}\text{sinc}(Wt) \longleftrightarrow \text{rect}\big(\frac{\omega}{2W}\big)$ with the Time-Differentiation property $\left(\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)\right)$ and the linearity property $(\alpha x(t) \longleftrightarrow \alpha X(\omega))$, we have

$$x(t) = \frac{2}{3}\frac{d}{dt}\left[\frac{3}{\pi}\text{sinc}(3t)\right] \longleftrightarrow \frac{2}{3} \cdot j\omega \cdot \text{rect}\Big(\frac{\omega}{6}\Big) = X(w).$$

Therefore,

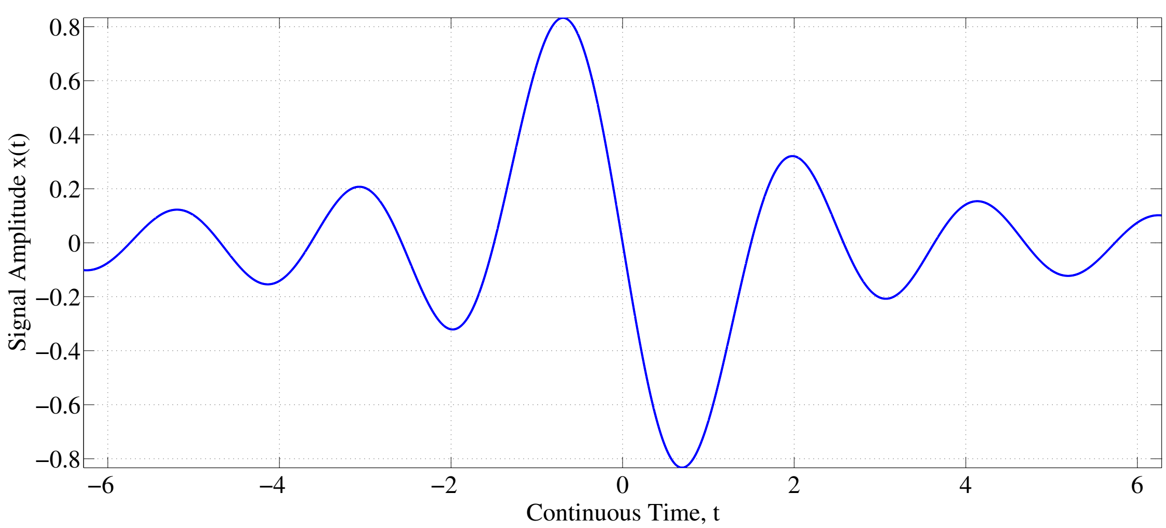
$$\begin{aligned} x(t) &= \frac{2}{3}\frac{d}{dt}\left[\frac{3}{\pi}\text{sinc}(3t)\right] = \frac{2}{3\pi}\frac{d}{dt}\left[\frac{\sin(3t)}{t}\right] = \frac{2}{3\pi}\left[\frac{t \cdot 3\cos(3t) - \sin(3t) \cdot 1}{t^2}\right] \\ &= \boxed{\frac{6t\cos(3t) - 2\sin(3t)}{3\pi t^2}}. \end{aligned}$$

MATLAB code to check our answers from parts (b) and (c):

```
syms w t x(t) X(w)
X(w) = 2*1i/3*w*(heaviside(w+3)-heaviside(w-3));
x(t) = simplify(ifourier(X(w),t));
pretty(x(t))
```

Command Windows output of MATLAB code:

```
  sin(3 t) 2 - t cos(3 t) 6
- -------------------------
                 2
           3 pi t
```



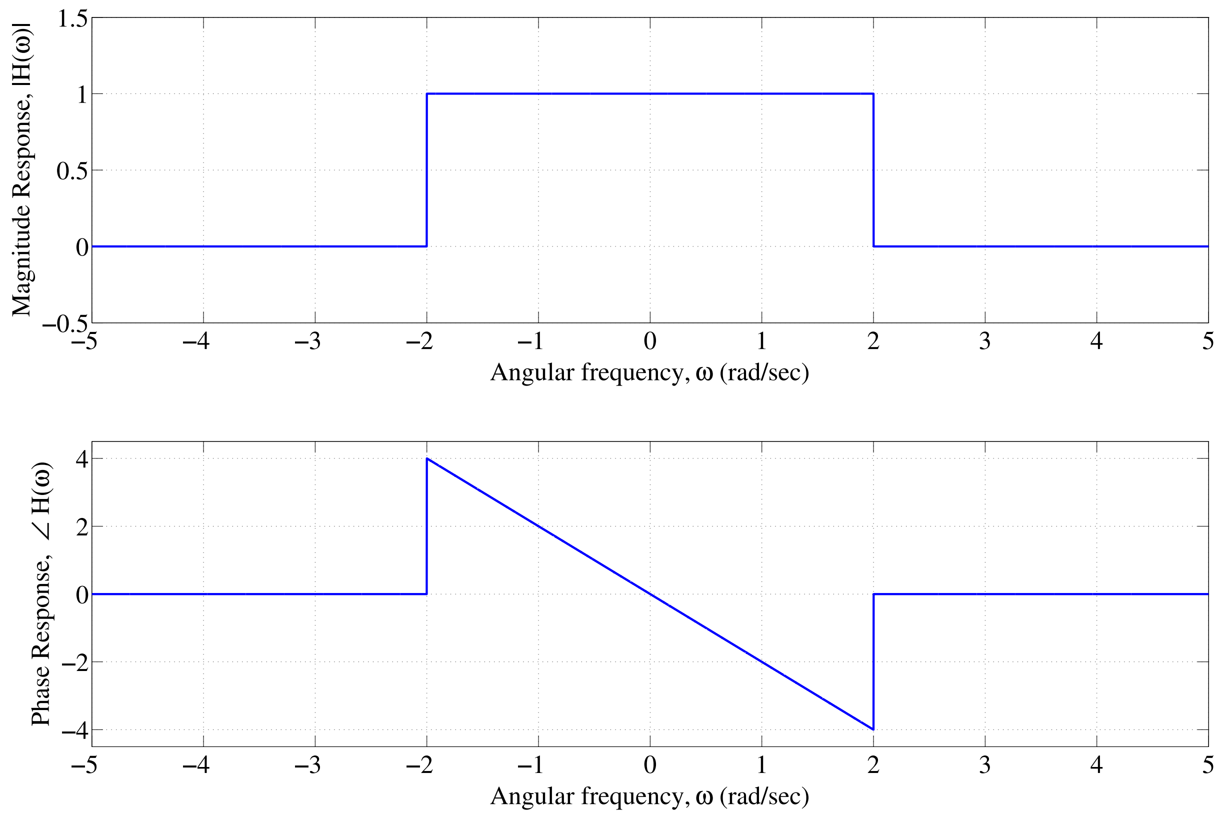


Figure 2: Frequency response $H(\omega)$ for Problem 1

(d) Determine a single mathematical expression for the frequency response $H(\omega)$ shown in Figure 2. *Hint: You should use the **rect**() function discussed in class!]*

$$H(\omega) = \text{rect}\left(\frac{\omega}{4}\right) e^{-j2\omega}$$

(e) For the LTI system with frequency response $H(\omega)$ shown in Figure 2, determine the impulse response $h(t)$ by using the table of Fourier transform pairs and the Time Shifting property (Equation 35 in the class notes).

Combining the Fourier transform pair $\frac{W}{\pi}\text{sinc}(Wt) \longleftrightarrow \text{rect}\left(\frac{\omega}{2W}\right)$ with the time-shifting property $(x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega))$, we have

$$h(t) = \frac{2}{\pi}\text{sinc}(2[t-2]) \longleftrightarrow \text{rect}\left(\frac{\omega}{4}\right) \cdot e^{-j\omega \cdot 2} = H(w).$$

Therefore,

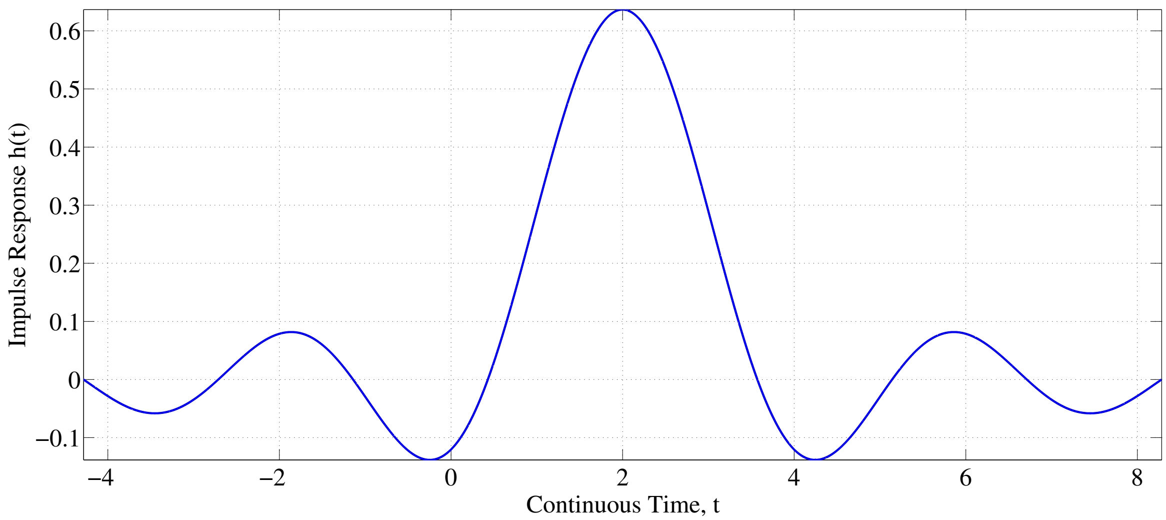
$$h(t) = \boxed{\frac{2}{\pi}\text{sinc}(2t-4) = \frac{\sin(2t-4)}{\pi(t-2)}}.$$

MATLAB code to check our answers from part (e):

```
syms w t h(t) H(w)
H(w) = exp(-1i*2*w)*(heaviside(w+2)-heaviside(w-2));
h(t) = simplify(ifourier(H(w),t));
pretty(h(t))
```

Command Windows output of MATLAB code:

```
sin(2 t - 4)
------------
 pi (t - 2)
```



(f) For the LTI system with frequency response $H(\omega)$ shown in Figure 2, determine if the system is causal.

Since $h(t)$ is everlasting and therefore $h(t) \neq 0$ for $t < 0$, then the LTI system represented by $H(\omega)$ is $\boxed{\text{noncausal.}}$

(g) For the LTI system with frequency response $H(\omega)$ shown in Figure 2, determine the output Fourier spectrum $Y(\omega)$ if the input signal to the LTI system is $x(t)$.

$$Y(\omega) = X(\omega)\, H(\omega) = \frac{2}{3} j\omega \text{rect}\left(\frac{\omega}{6}\right) \cdot \text{rect}\left(\frac{\omega}{4}\right) e^{-j2\omega} = \boxed{\frac{2}{3} j\omega \text{rect}\left(\frac{\omega}{4}\right) e^{-j2\omega}}$$

(h) For the LTI system with frequency response $H(\omega)$ shown in Figure 2, determine the output signal $y(t)$ if the input signal to the LTI system is $x(t)$.

Combining the Fourier transform pair $\frac{W}{\pi}\text{sinc}(Wt) \longleftrightarrow \text{rect}\left(\frac{\omega}{2W}\right)$ with the time-differentiation property $\left(\frac{dy(t)}{dt} \longleftrightarrow j\omega Y(\omega)\right)$, the linearity property $(\alpha y(t) \longleftrightarrow \alpha Y(\omega))$, and the time-shifting property $(y(t-t_0) \longleftrightarrow e^{-j\omega t_0}Y(\omega))$, we have

$$y(t) = \frac{2}{3}\frac{d}{dt}\left[\frac{2}{\pi}\text{sinc}(2(t-2))\right] \longleftrightarrow \frac{2}{3}\cdot j\omega\cdot\text{rect}\left(\frac{\omega}{4}\right)\cdot e^{-j2\omega} = Y(w).$$

Therefore,

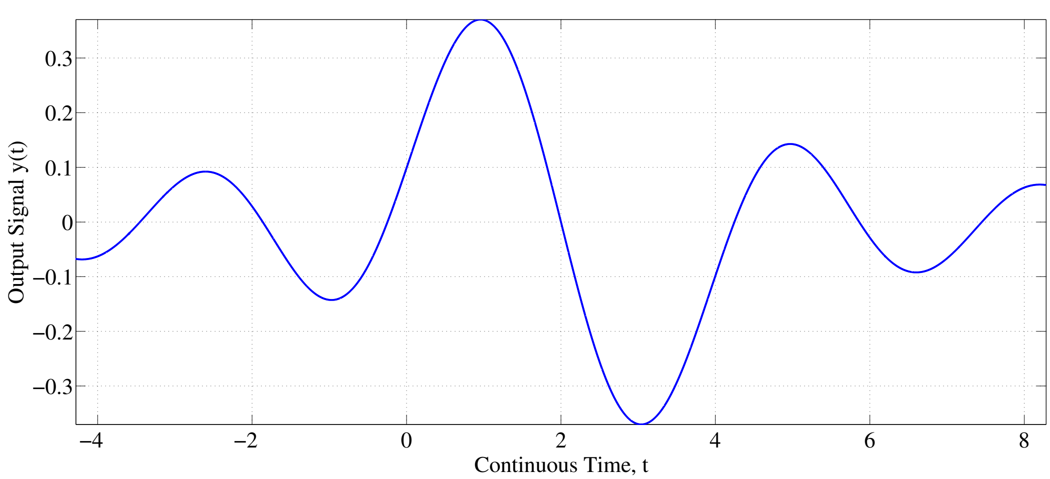
$$\begin{aligned}
y(t) &= \frac{2}{3}\frac{d}{dt}\left[\frac{2}{\pi}\text{sinc}(2t-4)\right] = \frac{2}{3\pi}\frac{d}{dt}\left[\frac{\sin(2t-4)}{t-2}\right] \\
&= \frac{2}{3\pi}\left[\frac{(t-2)\cdot 2\cos(2t-4) - \sin(2t-4)\cdot 1}{(t-2)^2}\right] \\
&= \boxed{\frac{4(t-2)\cos(2(t-2)) - 2\sin(2(t-2))}{3\pi(t-2)^2}} = \boxed{\frac{(4t-8)\cos(2t-4) - 2\sin(2t-4)}{3\pi(t-2)^2}}.
\end{aligned}$$

MATLAB code to check our answers from part (h):

```
syms w t y(t) Y(w)
Y(w) = (2/3)*1i*w*exp(-1i*2*w)*(heaviside(w+2)-heaviside(w-2));
y(t) = simplify(ifourier(Y(w),t));
pretty(y(t))
```

Command Windows output of MATLAB code:

```
  cos(2 t - 4) 8 + sin(2 t - 4) 2 - t cos(2 t - 4) 4
- --------------------------------------------------
                                     2
                        3 pi (t - 2)
```



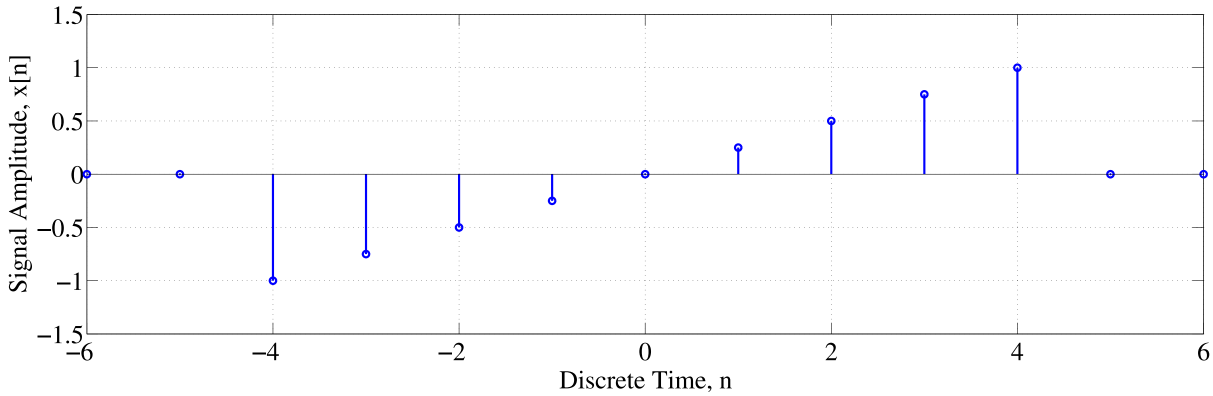


Figure 3: Discrete-time signal $x[n]$ for Problem 2

2. For the discrete-time signal $x[n]$ shown in Figure 3:

   (a) Determine the Fourier spectrum $X(\Omega)$. Simplify your answer as much as possible.

Note that

$$x[n] = \sum_{k=-4}^{4} \frac{k}{4}\delta[n-k] = \frac{1}{4}\sum_{k=-4}^{4} k\delta[n-k]$$

Combining the Fourier transform pair $\delta[n-k] \longleftrightarrow e^{-jk\Omega}$ with the linearity property, we have

$$\begin{aligned}
X(\Omega) &= \frac{1}{4}\sum_{k=-4}^{4} k e^{-jk\Omega} \\
&= -e^{+j4\Omega} - \frac{3}{4}e^{+j3\Omega} - \frac{1}{2}e^{+j2\Omega} - \frac{1}{4}e^{+j\Omega} + \frac{1}{4}e^{-j\Omega} + \frac{1}{2}e^{=j2\Omega} + \frac{3}{4}e^{-j3\Omega} + e^{-j4\Omega} \\
&= -j2\left[\frac{e^{+j4\Omega} - e^{-j4\Omega}}{j2}\right] - j\frac{3}{2}\left[\frac{e^{+j3\Omega} - e^{-j3\Omega}}{j2}\right] \\
&\quad - j\left[\frac{e^{+j2\Omega} - e^{-j2\Omega}}{j2}\right] - j\frac{1}{2}\left[\frac{e^{+j\Omega} - e^{-j\Omega}}{j2}\right] \\
&= -j2\sin(4\Omega) - j\frac{3}{2}\sin(3\Omega) - j\sin(2\Omega) - j\frac{1}{2}\sin(\Omega)
\end{aligned}$$

(b) Determine the fundamental period for the spectrum $X(\Omega)$.

Note that

$$\begin{aligned}
-j2\sin(4\Omega) &\text{ has fundamental period } \frac{\pi}{2},\\
-j\frac{3}{2}\sin(3\Omega) &\text{ has fundamental period } \frac{2\pi}{3},\\
-j\sin(2\Omega) &\text{ has fundamental period } \pi, \text{ and}\\
-j\frac{1}{2}\sin(\Omega) &\text{ has fundamental period } 2\pi.
\end{aligned}$$

Therefore, $X(\Omega)$ has fundamental period $\boxed{2\pi.}$