12 Filtering

Solutions to Recommended Problems

S12.1

(a) The impulse response is real because

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega,$$

$$h^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(\omega)e^{-j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(-\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega = h(t)$$

where we used the fact that $H(\omega) = H^*(\omega) = H(-\omega)$.

The impulse response is even because

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega,$$

$$h(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(-\omega)e^{j\omega t}\,d\omega$$

Since $H(-\omega) = H(\omega)$,

$$h(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega$$
$$= h(t)$$

The impulse response is noncausal because $h(-t) = h(t) \neq 0$.

(b) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - 9n)$,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j[(2\pi kt)/T]},$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-j[(2\pi kt)/T]}\,dt$$

Here $T = 9$, so

$$a_k = \frac{1}{9} \quad \text{and} \quad \mathcal{F}\{e^{j[(2\pi kt)/T]}\} = 2\pi\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Consequently, the Fourier transform of the filter input is as shown in Figure S12.1-1.

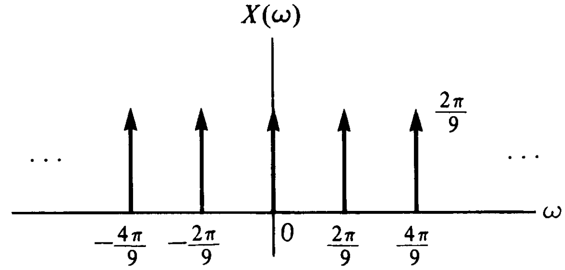

Figure S12.1-1

Since $Y(\omega) = H(\omega)X(\omega)$, the Fourier transform of the filter output is as shown in Figure S12.1-2.

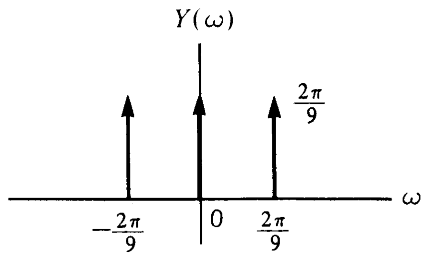

Figure S12.1-2

(c) We determine $y(t)$ by performing an inverse Fourier transform on $Y(\omega)$ as found in part (b). Using superposition, we have

$$y(t) = \frac{1}{9} + \frac{2}{9}\cos\left(\frac{2\pi t}{9}\right)$$

S12.2

From the filter frequency response plots we can determine that

$$H(\omega) = 0.25e^{-j(\pi/8)} \quad \text{at } \omega = \omega_1 = \pi,$$
$$H(\omega) = 0.5e^{-j(\pi/4)} \quad \text{at } \omega = \omega_2 = 2\pi$$

Using superposition, we easily determine $y(t)$ to be

$$y(t) = 0.25 \sin(\pi t + \pi/8) + \cos\left(2\pi t - \frac{7\pi}{12}\right)$$

S12.3

(a) $RC\dfrac{dv_c}{dt} + v_c = v_s$

Taking the Fourier transform of this equation, we have

$$(RCj\omega + 1)V_c(\omega) = V_s(\omega)$$

We now define

$$H_1(\omega) = \frac{V_c(\omega)}{V_s(\omega)} = \frac{1}{1 + j\omega RC}$$

We can see from this expression that $v_c(t)$ is a lowpass version of $v_s(t)$.

The magnitude and phase of $H_1(\omega)$ are given in Figure S12.3-1.

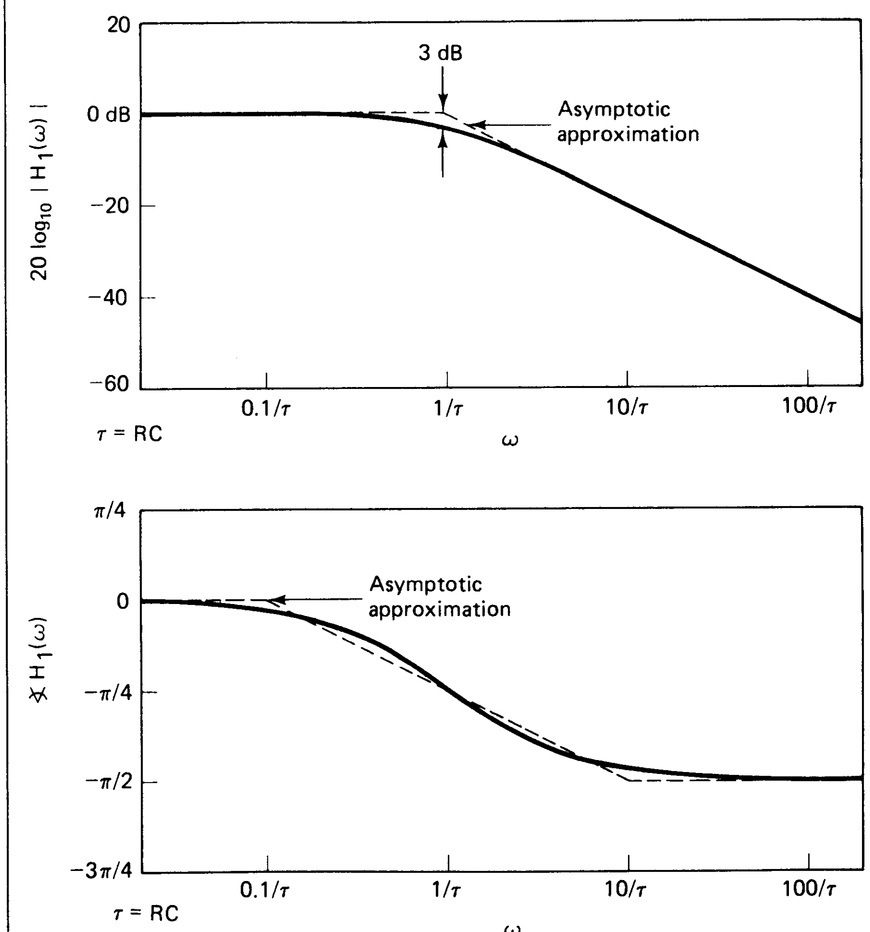

Figure S12.3-1

(b)

$$RC\,\frac{d(v_s - v_r)}{dt} + v_s - v_r = v_s,$$

$$RCj\omega V_s(\omega) - RCj\omega V_r(\omega) - V_r(\omega) = 0,$$

$$(j\omega RC)V_s(\omega) = (1 + j\omega RC)V_r(\omega),$$

$$H_2(\omega) = \frac{V_r(\omega)}{V_s(\omega)} = \frac{j\omega RC}{1 + j\omega RC}$$

The magnitude and phase of $H_2(\omega)$ are given in Figure S12.3-2.

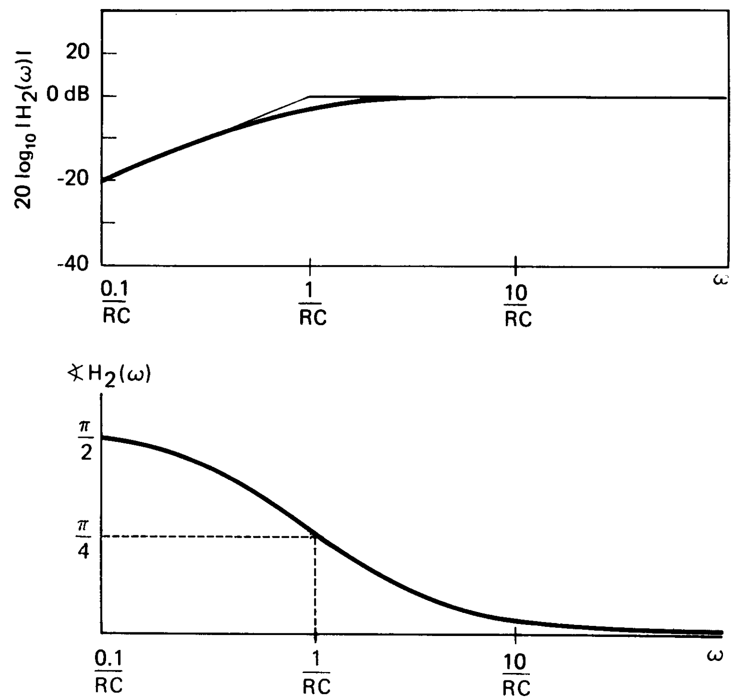

Figure S12.3-2

(c) The cutoff frequencies are $\omega_c = 1/RC$ in both cases.

(d) $\dfrac{V(\omega)}{V_s(\omega)} = 1 - H_1(\omega) = \dfrac{j\omega RC}{1 + j\omega RC} = H_2(\omega)$

This is the same frequency response as sketched in part (b). We have transformed a lowpass into a highpass filter by a feed-forward system. The cutoff frequency, as in part (c), is $\omega_c = 1/RC$.

S12.4

Consider $0 \leq \Omega_0 \leq \pi$. In this range, the gain of the filter $|H(\Omega)|$ is Ω_0. The phase shift for the positive frequency component is $+\pi/2$ and the shift for the negative frequency component is $-\pi/2$. Since

$$x[n] = \cos(\Omega_0 n + \theta) = \frac{1}{2}[e^{j(\Omega_0 n + \theta)} + e^{-j(\Omega_0 n + \theta)}],$$

$$y[n] = \frac{\Omega_0}{2}[e^{j[\Omega_0 n + \theta + (\pi/2)]} + e^{-j[\Omega_0 n + \theta + (\pi/2)]}]$$

$$= j\frac{\Omega_0}{2}[e^{j(\Omega_0 n + \theta)} - e^{-j(\Omega_0 n + \theta)}],$$

$$y[n] = -\Omega_0 \sin(\Omega_0 n + \theta)$$

It is apparent from this expression that $H(\Omega)$ is a discrete-time differentiator. A similar result holds for $-\pi \leq \Omega_0 \leq 0$.

If Ω_0 is outside the range $-\pi \le \Omega_0 \le \pi$, we can express $x[n]$ identically using a Ω_0 within this range. For example,

$$x[n] = \cos\left(\frac{3\pi}{2}n + \theta\right)$$
$$= \cos\left(-\frac{\pi}{2}n + \theta\right),$$
$$y[n] = \frac{\pi}{2}\sin\left(-\frac{\pi}{2}n + \theta\right)$$

S12.5

(a) We see by examining $y_1[n]$ and $y_2[n]$ that $y_1[n]$ averages $x[n]$ and thus tends to suppress changes while $y_2[n]$ tends to suppress components that have not varied from $x[n-1]$ to $x[n]$. Therefore, the $y_1[n]$ system is lowpass and $y_2[n]$ is highpass.

(b) Taking the Fourier transforms yields

$$Y_1(\Omega) = X(\Omega)\left(\frac{1 + e^{-j\Omega}}{2}\right),$$
$$H_1(\Omega) = \frac{1}{2}(1 + e^{-j\Omega})$$

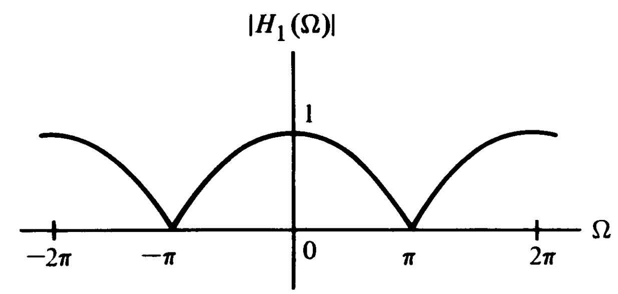

Figure S12.5-1

$$Y_2(\Omega) = X(\Omega)\left(\frac{1 - e^{-j\Omega}}{2}\right),$$
$$H_2(\Omega) = \frac{1}{2}(1 - e^{-j\Omega})$$

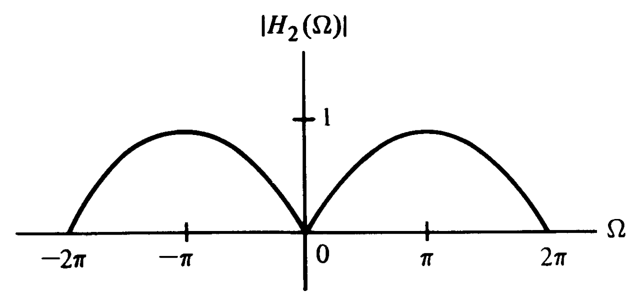

Figure S12.5-2

S12.6

(a) By inspection we see that the impulse response is given by

$$h_1[n] = \frac{1}{2N+1}\sum_{k=-N}^{N} \delta[n-k]$$

(b) $H_2(\Omega) = 1 - \dfrac{1}{2N+1}\left[\dfrac{\sin\left(\Omega\dfrac{2N+1}{2}\right)}{\sin(\Omega/2)}\right]$

(c)

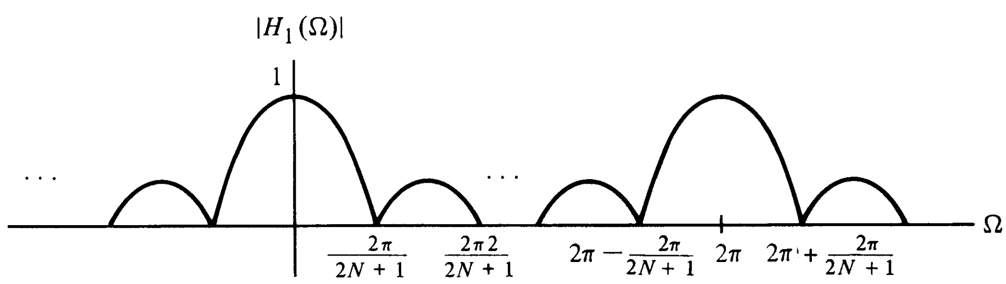

Figure S12.6-1

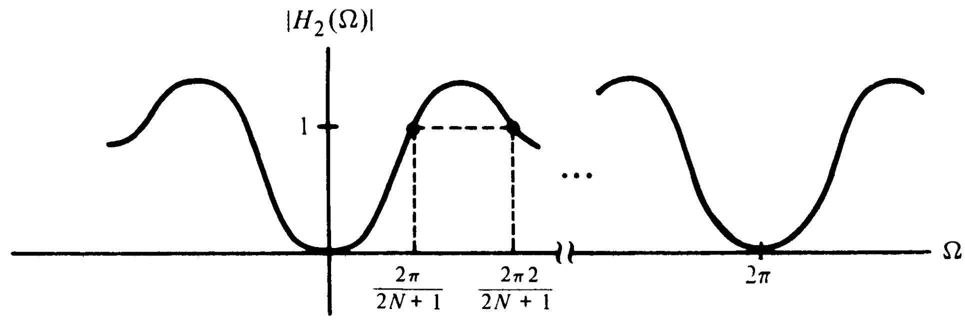

Figure S12.6-2

Zero and one crossings are at

$$\left(\frac{2\pi}{2N+1}\right)k.$$

(d) $H_2(\Omega)$ is an approximation to a highpass filter.

S12.7

(a) From the specification that $H(0) = 1$, we know that

$$H(\omega) = \frac{\alpha}{\alpha + j\omega}$$

(b)
$$|H(\omega)| = \frac{\alpha}{(\alpha^2+\omega^2)^{1/2}},$$

$$\frac{\alpha}{(\alpha^2+4)^{1/2}} = |H(\omega)|\,\bigg|_{\omega=2}$$

The low end specification is satisfied for $\alpha \geq 4$, as shown in Figure S12.7-1.

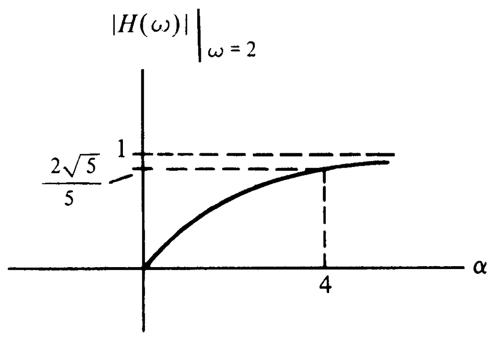

Figure S12.7-1

The high end specification is met for $\alpha \leq 6$, as shown in Figure S12.7-2.

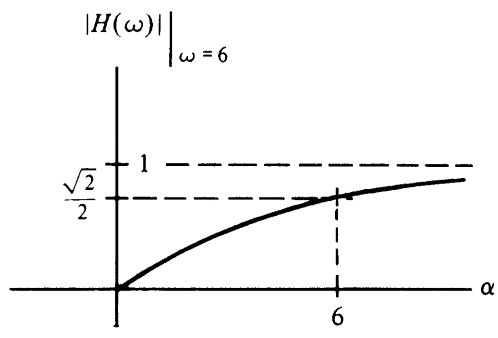

Figure S12.7-2

The range of α such that the total specification is met is $4 \leq \alpha \leq 6$.

Solutions to Optional Problems

S12.8

The easiest method for solving this problem is to recognize that passing $x(t)$ through $H(\omega)$ is equivalent to performing

$$-2\frac{dx(t)}{dt}$$

This is easily seen since

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega,$$

$$-2\frac{dx(t)}{dt} = \frac{1}{2\pi}\int \underbrace{-2j\omega}_{H(\Omega)} X(\omega)e^{j\omega t}\,d\omega$$

so

$$-2\frac{dx(t)}{dt} \longleftrightarrow -2j\omega X(\omega)$$

(a) $-2\dfrac{dx(t)}{dt} = -2\dfrac{de^{jt}}{dt} = -2je^{jt} = y(t)$

(b) $-2\dfrac{dx(t)}{dt} = -2\dfrac{d[(\sin\omega_0 t)u(t)]}{dt} = -2\omega_0(\cos\omega_0 t)u(t)$

(c)
$$\begin{aligned}
X(\omega) &= \frac{1}{j\omega(6+j\omega)} = \frac{\frac{1}{6}}{j\omega} + \frac{-\frac{1}{6}}{6+j\omega},\\
x(t) &= \frac{1}{6}\left[u(t) - \frac{1}{2}\right] - \frac{1}{6}e^{-6t}u(t)\\
-2\frac{dx(t)}{dt} &= -2\left[\frac{1}{6}\delta(t) + e^{-6t}u(t) - \frac{1}{6}e^{-6t}\delta(t)\right]\\
&= -2e^{-6t}u(t)
\end{aligned}$$

Alternatively, for this part it is perhaps simpler to use the fact that

$$\begin{aligned}
Y(\omega) = H(\omega)X(\omega) &= \frac{-2j\omega}{j\omega(6+j\omega)}\\
&= -\frac{2}{6+j\omega}
\end{aligned}$$

so that $y(t) = -2e^{-6t}u(t)$

(d)
$$\begin{aligned}
X(\omega) &= \frac{1}{2+j\omega}\\
x(t) &= e^{-2t}u(t)\\
-2\frac{dx(t)}{dt} &= -2[-2e^{-2t}u(t) + e^{-2t}\delta(t)] = 4e^{-2t}u(t) - 2\delta(t)
\end{aligned}$$

S12.9

(a) $H(\Omega) = H_r(\Omega)e^{-jM\Omega}$

(i) $H_r(\Omega)$ is real and even:

$$h_r[n] \longleftrightarrow H_r(\Omega)$$

From Table 5.1 of the text (page 335), we see that the even part of $h_r[n]$ has a Fourier transform that is the real part of $H_r(\Omega)$. This result is easily verified:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} h_r[-n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} h_r[n]e^{j\Omega n} &= \left(\sum_{n=-\infty}^{\infty} h_r[n]e^{-j\Omega n}\right)^*\\
&= H_r^*(\Omega),
\end{aligned}$$

so

$$\begin{aligned}
\tfrac{1}{2}(h_r[n] + h_r[-n]) &\longleftrightarrow \tfrac{1}{2}[H_r(\Omega) + H_r^*(\Omega)],\\
Ev\{h_r[n]\} &\longleftrightarrow Re\{H_r(\Omega)\}
\end{aligned}$$

Now since

$$Re\{H_r(\Omega)\} = H_r(\Omega),$$

we have that $Ev\{h_r[n]\} = h_r[n]$, i.e., $h_r[n]$ is even, and therefore

$$h_r[n] = h_r[-n]$$

(ii) From Table 5.1,

$$x[n - n_0] \longleftrightarrow e^{-j\Omega n_0},$$

so

$$H_r(\Omega)e^{-j\Omega M} \longleftrightarrow h_r[n - M],$$
$$h[n] = h_r[n - M]$$

(b) $h_r[n] = h_r[-n]$
Since $h[n] = h_r[n - M]$,

$$h[n + M] = h_r[n],$$
$$h[M - n] = h_r[(M - n) - M] = h_r[-n],$$

but

$$h_r[n] = h_r[-n] \Rightarrow h[M - n] = h[M + n]$$

(c) $h[M + n] = h[M - n]$ from part (b). Since $h[n]$ is causal, $h[M - n] = 0$ for $n > M$. But if $h[M + n] = h[M - n]$, then

$$h[M + n] = 0 \qquad \text{for } n > M,$$

so

$$h[n] = 0 \qquad \text{for } n > 2M$$

Summarizing, we have

$$h[n] = 0 \qquad \text{for } n < 0,\ n > 2M$$

S12.10

(a)

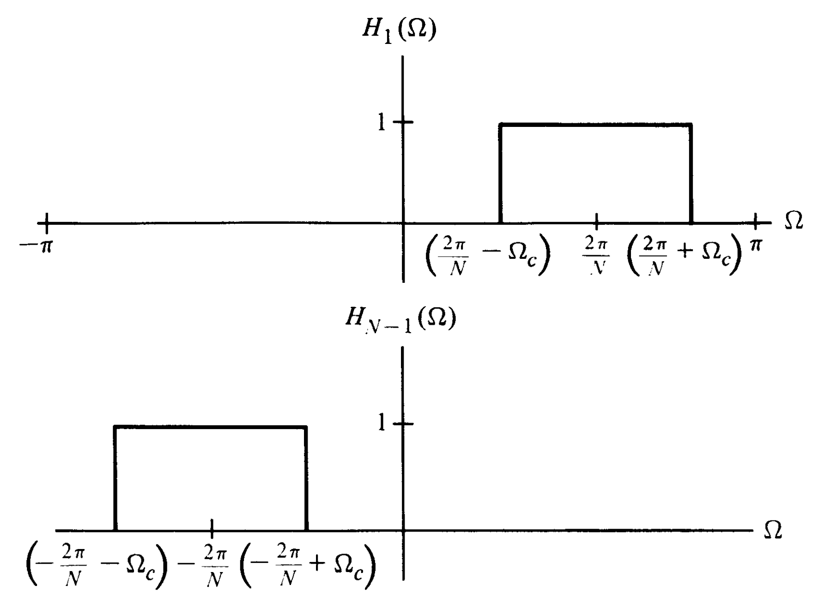

Figure S12.10-1

(b) If the cutoff frequency $\Omega_c = \pi/N$, the total system is an identity system.

(c) $h[n] = \sum_{k=0}^{N-1} h_k[n] = \sum_{k=0}^{N-1} e^{j(2\pi nk/N)} h_0[n]$

$$= \left[\frac{1 - e^{j2\pi n}}{1 - e^{j(2\pi n/N)}}\right] h_0[n],$$

$$h[n] = \begin{cases} Nh_0[n], & n = \text{an integer multiple of } N, \\ 0, & n \neq \text{an integer multiple of } N, \end{cases}$$

so $r[n]$ is as shown in Figure S12.10-2.

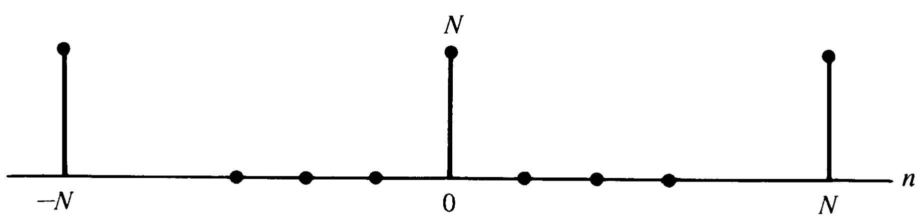

Figure S12.10-2

(d) $h_0[n] = \dfrac{1}{N}, \qquad n = 0,$

$h_0[n] = 0, \qquad n = \text{an integer multiple of } N,$

are the necessary and sufficient conditions.

S12.11

From the system diagram,

$$Y(\omega) = X(\omega)[\alpha - G(\omega)],$$
$$H(\omega) = \alpha - G(\omega)$$

(a) $\sphericalangle H(\omega)$ is 0 for all ω.

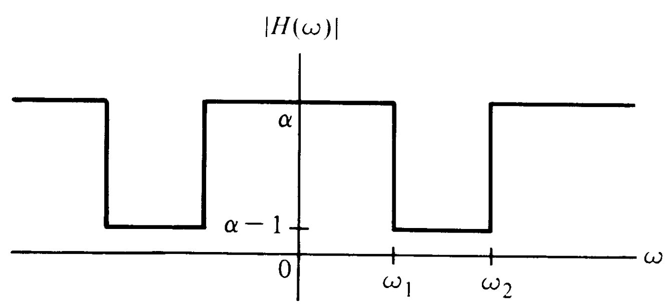

Figure S12.11-1

(b)

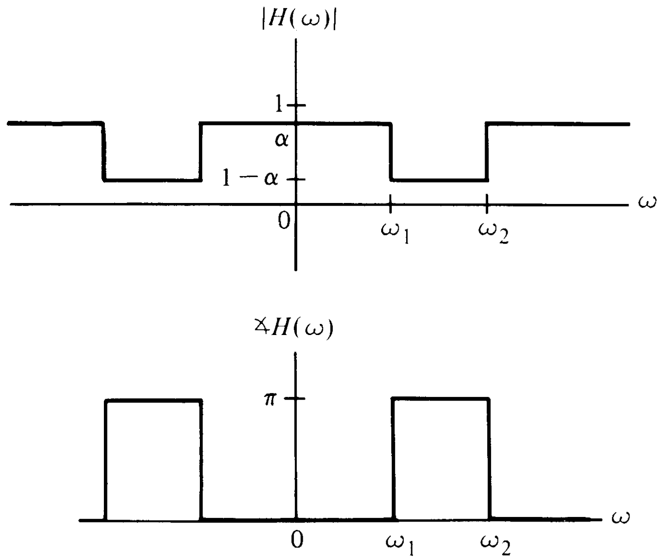

Figure S12.11-2

(c) $\sphericalangle H(\omega)$ is π for all ω.

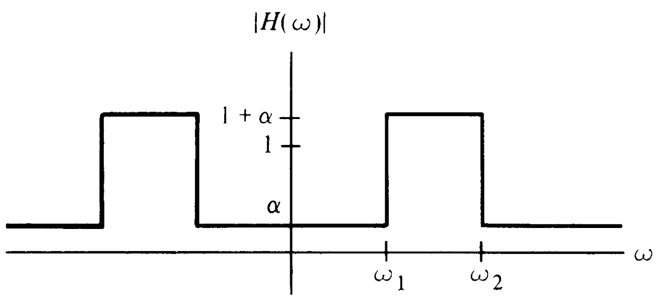

Figure S12.11-3