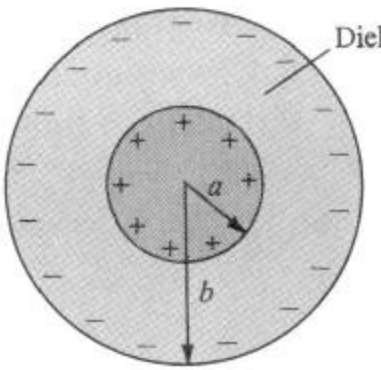
Example: Capacitance

The spherical capacitor (depicted) consists of two concentric, spherical conducting shells. The inner radius is a and the outer radius is b .

The two radii are separated by a dielectric with permittivity ε .

Compute the capacitance of this geometry using Gauss's Law.



Assume charges $+Q$ and $-Q$ on the two shells and solve for V from one shell to the other shell…

$$\iint_S \varepsilon \mathbf{E} \cdot d\mathbf{S} = Q \qquad \& \qquad \mathbf{E} = E_R \hat{\mathbf{R}}$$

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \left(\varepsilon E_R \hat{\mathbf{R}}\right) \cdot \left(R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{R}}\right) = Q$$

$$4\pi R^2 \cdot \varepsilon E_R = Q$$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon R^2} \hat{\mathbf{R}}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \oiint_S \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}}$$

$$V = \int_{r=a}^{r=b} \mathbf{E} \cdot d\mathbf{l} = \int_{R=a}^{R=b} \frac{Q}{4\pi\varepsilon R^2} \hat{\mathbf{R}} \cdot \hat{\mathbf{R}} \, dR$$

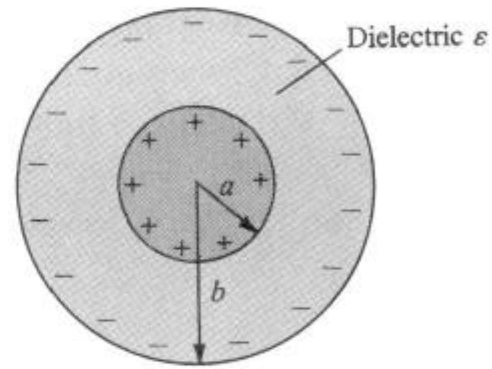
$$= \frac{Q}{4\pi\varepsilon} \int_{R=a}^{R=b} \frac{1}{R^2} dr = \frac{Q}{4\pi\varepsilon} \left[\frac{1}{a} - \frac{1}{b}\right]$$

$$C = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]} = \boxed{\frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}}$$

Example: Potential; Function of Space

The spherical capacitor consists of two concentric, spherical conducting shells. The outer shell is grounded, while the inner shell is charged to V_0 .

Determine the potential everywhere between $R = a$ and $R = b$, and from this function determine the capacitance of the system.



$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$\nabla V = \frac{\partial V}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \oiint_S \mathbf{E}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}}$$

$$\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) = 0 \quad \Rightarrow \quad V = \frac{V_1}{R} + V_2$$

$$V(a) = \frac{V_1}{a} + V_2 = V_0$$

$$V(b) = \frac{V_1}{b} + V_2 = 0$$

$$V_1 = \frac{V_0}{1/a - 1/b} \quad \Rightarrow \quad V(R) = \frac{V_0}{1/a - 1/b}\frac{1}{R} + V_2$$

$$\Rightarrow \quad \mathbf{E} = \frac{V_0}{1/a - 1/b}\cdot\frac{1}{R^2}\hat{\mathbf{R}}$$

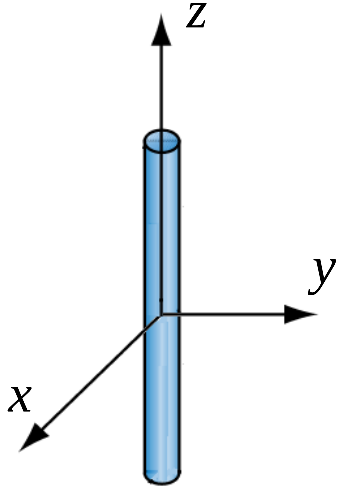
$$\oiint_S \mathbf{E}\cdot d\mathbf{S} = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} \frac{V_0}{1/a - 1/b}\cdot\frac{1}{R^2}\hat{\mathbf{R}}\cdot\hat{\mathbf{R}}\, R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{V_0}{1/a - 1/b}\cdot\left[-\cos\theta\right]_0^{\pi}\cdot 2\pi = \frac{4\pi V_0}{1/a - 1/b}$$

$$C = \frac{\varepsilon \oiint_S \mathbf{E}\cdot d\mathbf{S}}{V_0} = \boxed{\frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}}$$

Example: Charge Density vs. Charge

Calculate the total charge Q contained in a line charge extending from z = –5 m to z = +5 m, and whose charge density is $\rho_l = 2|z|$ (C/m).



$$Q = \int_L \rho_l dl = \int_{-5}^{+5} 2|z| \cdot dz$$

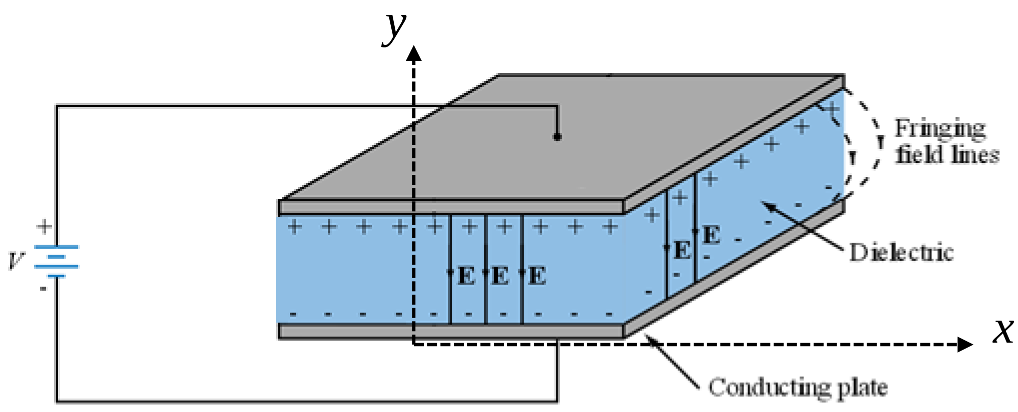
$$= \int_0^{+5} 2(z) \cdot dz + \int_{-5}^{0} 2(-z) \cdot dz$$

$$= 2\left[\frac{1}{2}z^2\right]_0^{+5} + 2\left[\frac{1}{2}z^2\right]_0^{-5} = \boxed{50\ \text{C}}$$

Example: Potential & Electric Field

An electric field in space is defined by

$$\mathbf{E} = -2.5\,\hat{\mathbf{y}}\,\frac{\text{V}}{\text{cm}}$$



Evaluate the potential difference from $P(x = 2\text{ cm}, y = 0)$ to $Q(x = 0, y = 2\text{ cm})$.

$$\begin{aligned}
V_{PQ} &= -\int_P^Q \mathbf{E}\cdot d\mathbf{l} \\
&= -\int_{x=2}^{x=0} \mathbf{E}\cdot\hat{\mathbf{x}}\,dx\Big|_{y=0} - \int_{y=0}^{y=2} \mathbf{E}\cdot\hat{\mathbf{y}}\,dy\Big|_{x=0} \\
&= -\int_{x=2}^{x=0} (-2.5\,\hat{\mathbf{y}})\cdot\hat{\mathbf{x}}\,dx\Big|_{y=0} - \int_{y=0}^{y=2} (-2.5\,\hat{\mathbf{y}})\cdot\hat{\mathbf{y}}\,dy\Big|_{x=0} \\
&= 0 + \left(2.5\,\frac{\text{V}}{\text{cm}}\right)(2\text{ cm}) = 5\text{ V}
\end{aligned}$$

$$\begin{aligned}
V_{PQ} &= -\int_P^Q \mathbf{E}\cdot d\mathbf{l} \\
&= -\int_{\phi=0}^{\phi=\pi/2} (2.5)\left(\sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}}\right)\cdot\hat{\boldsymbol{\phi}}\,r\,d\phi\Big|_{r=2} \\
&= -5\int_{\phi=0}^{\phi=\pi/2} \cos\phi\cdot d\phi = -5\left[-\sin\phi\right]_0^{\pi/2} = 5\ \text{V}
\end{aligned}$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$

Example: Electrostatic Energy

Along the surface of a conducting sphere is a uniform charge density of 10 nC/m^2. The sphere has a radius of 10 cm .

Calculate the electrostatic energy that is stored in this system. Assume $\varepsilon = \varepsilon_0$.

$$Q = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \rho_s \, R^2 \sin\theta \, d\theta \, d\phi$$

$$= 4\pi (0.10)^2 \left(10 \cdot 10^{-9}\right) = 1.26 \text{ nC}$$

$$\oint_S \varepsilon \mathbf{E} \cdot d\mathbf{S} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \varepsilon_0 \, E_R \cdot R^2 \sin\theta \, d\theta \, d\phi$$

$$= 4\pi R^2 \varepsilon_0 E_R$$

$$\mathbf{E} = \frac{1.26 \cdot 10^{-9}}{4\pi \left(8.854 \cdot 10^{-12}\right) R^2} \hat{\mathbf{R}}$$

$$W_E = \frac{1}{2} \int_v \varepsilon \left|\mathbf{E}\right|^2 dv$$

$$\mathbf{E} = \frac{1}{2} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{R=0.1}^{R=\infty} \left(8.854 \cdot 10^{-12}\right) \left[\frac{1.26 \cdot 10^{-9}}{4\pi \left(8.854 \cdot 10^{-12}\right) R^2} \right]^2 R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= \frac{\left(1.26 \cdot 10^{-9}\right)^2}{32\pi^2 \left(8.854 \cdot 10^{-12}\right)} \int_{\phi=0}^{\phi=2\pi} d\phi \int_{\theta=0}^{\theta=\pi} \sin\theta d\theta \int_{R=0.1}^{R=\infty} \frac{1}{R^2} \, dR = \boxed{71 \text{ nJ}}$$