Math 335 Practice Exam 2 Key

1.) *[6 points]* Classify each differential equation below as PDE or ODE, linear or nonlinear, and specify the order.

a.) $x^3 y'' - y' \cos 2x = 4y$

2nd order linear ODE

b.) $f_{xx} = x^2 + y^2$

2nd order linear ODE

c.) $f_x f_y = x^3 + 2y + 3$

1st order nonlinear PDE

2.) *[4 points]* Find all singular points of the ODE below and classify the points as regular or irregular.

$$\left(x^2 - 9\right)y'' - (x+3)y' + 4y = 0$$

$$y'' - \frac{\cancel{x+3}}{\cancel{(x+3)}(x-3)}y' + \frac{4}{x^2-9}y = 0$$

$$y'' - \underbrace{\frac{1}{x-3}}_{P} y' + \underbrace{\frac{4}{x^2-9}}_{Q} y = 0$$

singular points $x = \pm 3$

$(x-3)P(x) = 1, \quad (x-3)^2 Q(x) = \frac{4(x-3)}{x+3} \Rightarrow x = 3$ regular

$(x+3)P(x) = \frac{x+3}{x-3}, \quad (x+3)^2 Q(x) = \frac{x+3}{x-3} \Rightarrow x = -3$ regular

3.) *[10 points]* Find the first 5 terms (through x^4) of the series solution about x=0 of the ODE

$$3y'' - 4y' + x^2 y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$3\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4\sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} 3n(n-1) a_n x^{n-2}}_{} - \underbrace{\sum_{n=1}^{\infty} 4n a_n x^{n-1}}_{} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$k+2 = n-2$

$k+4 = n$

$k+3 \quad n-1$

$n=2 \rightarrow k=-2$

$m+2 = n-1$

$m+3 = n$

$n=1 \rightarrow m=-2$

$$\sum_{k=-2}^{\infty} 3(k+4)(k+3) a_{k+4} x^{k+2} - \sum_{m=-2}^{\infty} 4(m+3) a_{m+3} x^{m+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$6a_2 + 18a_3 x - 4a_1 - 8a_2 x$$
$$+ \sum_{n=0}^{\infty} \left[3(n+4)(n+3)a_{n+4} - 4(n+3)a_{n+3} + a_n\right] x^{n+2} = 0$$

$\underline{x^0}$: $6a_2 - 4a_1 = 0 \Rightarrow 6a_2 = 4a_1 \Rightarrow a_2 = \frac{2}{3}a_1$

$\underline{x^1}$: $18a_3 - 8a_2 = 0 \Rightarrow a_3 = \frac{4}{9}a_2 = \frac{4}{9}\left(\frac{2}{3}a_1\right) = \frac{8}{27}a_1$

$\underline{x^2}$: $36a_4 - 12a_3 + a_0 = 0 \Rightarrow a_4 = \frac{1}{3}a_3 - \frac{1}{36}a_0$

$\Rightarrow a_4 = \frac{1}{3}a_3 - \frac{1}{36}a_0 = \frac{1}{3}\left(\frac{8}{27}a_1\right) - \frac{1}{36}a_0 = \frac{8}{81}a_1 - \frac{1}{36}a_0$

$$y = a_0 + a_1 x + \frac{2}{3}a_1 x^2 + \frac{8}{27}a_1 x^3 + \left(\frac{8}{81}a_1 - \frac{1}{36}a_0\right)x^4 + \cdots$$

4.) *[10 points]* Use your answer to #3 to find the solution of the Initial Value Problem

$$3y'' - 4y' + x^2 y = 0, \qquad y(0) = 0, \qquad y'(0) = 4$$

$$y(0) = a_0 = 0$$

$$y'(0) = a_1 = 4$$

$$y = 0 + 4x + \frac{2}{3}(4)x^2 + \frac{8}{27}(4)x^3 + \left(\frac{8}{81}(4) - \frac{1}{36}(0)\right)x^4 + \cdots$$

$$\boxed{y = 4x + \frac{8}{3}x^2 + \frac{32}{27}x^3 + \frac{32}{81}x^4 + \cdots}$$

5.) *[20 points]* Note x=0 is a regular singular point of the ODE

$$2xy'' + 5y' + xy = 0 .$$

Find the indicial roots of the ODE and the general recurrence relation in terms of n and r.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + 5 \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + x \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r) c_n x^{n+r-1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+1}} = 0$$

$m+r-1 = n+r+1$

$m-1 = n+1$

$m-2 = n$

$n=0 \rightarrow m=2$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r) c_n x^{n+r-1} + \sum_{m=2}^{\infty} c_{m-2} x^{m+r-1} = 0$$

$$2r(r-1) c_0 x^{r-1} + 2(1+r) r c_1 x^{r} + 5 r c_0 x^{r-1} + 5(1+r) c_1 x^{r} + \sum_{n=2}^{\infty} \left[2(n+r)(n+r-1) c_n + 5(n+r) c_n + c_{n-2} \right] x^{n+r-1} = 0$$

$\underline{x^{r-1}}$: $2r(r-1)c_0 + 5rc_0 = 0$

$$[2r^2 - 2r + 5r]c_0 = 0$$

$$2r^2 + 3r = 0$$

$$r(2r+3) = 0$$

$$\boxed{r = 0, -\frac{3}{2}}$$

$\underline{x^{r}}$: $2(1+r)rc_1 + 5(1+r)c_1 = 0$

$$[2r + 2r^2 + 5 + 5r]c_1 = 0$$

$$[2r^2 + 7r + 5]c_1 = 0$$

$$\boxed{c_1 = 0}$$

$\underline{x^{n+r-1}}$: $2(n+r)(n+r-1)c_n + 5(n+r)c_n + c_{n-2} = 0$

$$\boxed{c_n = -\frac{c_{n-2}}{2(n+r)(n+r-1) + 5(n+r)}}$$

for $n \geq 2$