Math 335, Fall 2014 Exam 2 Key

1.) *[10 points]* Combine the series below into one series. Clearly indicate and simplify any extra terms you pulled out of the series.

$$\sum_{k=0}^{\infty} a_k x^k - 4\sum_{k=0}^{\infty} a_{k+1}x^{k+1}$$

$n=k+1,\ k=0 \Rightarrow n=1$

$$\sum_{k=0}^{\infty} a_k x^k - \sum_{n=1}^{\infty} 4a_n x^n$$

$$\boxed{a_0 - \sum_{n=1}^{\infty} 3a_n x^n}$$

$$(x+3)\sum_{k=2}^{\infty} a_{k-1}x^k + 2\sum_{k=0}^{\infty} a_k x^k$$

$$\sum_{k=2}^{\infty} a_{k-1} x^{k+1} + \sum_{k=2}^{\infty} 3a_{k-1}x^k + \sum_{k=0}^{\infty} 2a_k x^k$$

$n=k+1,\ n-2=k-1,\ k=2 \Rightarrow n=3$

$$\sum_{n=3}^{\infty} a_{n-2}x^n + \sum_{k=2}^{\infty} 3a_{k-1}x^k + \sum_{k=0}^{\infty} 2a_k x^k$$

Pull out $k=2$ term. Pull out $k=0,1,2$ terms.

$$\boxed{3a_1x^2 + 2a_0 + 2a_1x + 2a_2x^2 + \sum_{n=3}^{\infty}\left[a_{n-2} + 3a_{n-1} + 2a_n\right]x^n}$$

2.) *[10 points]* On the homework, you showed that the power series solution about $x = -2$ to they Airy equation $y'' - xy = 0$ is given by

$$y = a_0 + a_1(x+2) - a_0(x+2)^2 + \left(\frac{1}{6}a_0 - \frac{1}{3}a_1\right)(x+2)^3 + \cdots$$

where a_0 and a_1 are unknown constants. Use this information to find the first 4 terms of the specific power series solution about $x = -2$ to the initial value problem

$$y'' - xy = 0, \qquad y(-2) = 3, \qquad y'(-2) = 9$$

Write your answer in the blanks below.

$\underline{y(-2)=3} \Rightarrow 3 = a_0 + a_1\underbrace{(-2+2)}_{0} - a_0\underbrace{(-2+2)^2}_{0} + \cdots$

$3 = a_0$

$\underline{y'(-2)=9} \Rightarrow 9 = a_1 - 2a_0\underbrace{(-2+2)}_{0} + 3\left(\frac{1}{6}a_0 - \frac{1}{3}a_1\right)\underbrace{(-2+2)^2}_{0} +$

$9 = a_1$

$(x+2)^2$ Term: $-a_0 = -3$

$(x+2)^3$ Term: $\frac{1}{6}a_0 - \frac{1}{3}a_1 = \frac{1}{6}(3) - \frac{1}{3}(9)$

$= \frac{1}{2} - 3 = -\frac{5}{2}$

$y =$ ___3___ $+$ ___9___ $(x+2) +$ ___-3___ $(x+2)^2 +$ ___$-\frac{5}{2}$___ $(x+2)^3 + \cdots$

3.) *[10 points]* Find the first 5 terms (through x^4) of the power series solution about x=0 of the ODE

$$y'' + xy' - 3y = 0$$

Write your coefficients in the blanks below in terms of a_0 and a_1.

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$\hookrightarrow n-2=k, \ n=k+2, \ n-1=k+1, \ n=2 \Rightarrow k=0$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3 a_n x^n = 0$$

Pull out $k=0$. Pull out $n=0$.

$$\underbrace{(2)(1)a_2 x^0 - 3a_0 x^0}_{=0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+2)(n+1)a_{n+2} + n a_n - 3a_n\right]}_{=0} x^n = 0$$

$2a_2 - 3a_0 = 0$

$\Rightarrow a_2 = \frac{3}{2} a_0$

$(n+2)(n+1)a_{n+2} = -n a_n + 3a_n$

$$\boxed{a_{n+2} = \frac{3-n}{(n+2)(n+1)} a_n}$$

$\underline{n=1}$ $\quad a_3 = \frac{3-1}{(1+2)(1+1)} a_1 = \frac{1}{3} a_1$

$\underline{n=2}$ $\quad a_4 = \frac{3-2}{(2+2)(2+1)} a_2 = \frac{1}{12}\left(\frac{3}{2} a_0\right) = \frac{1}{8} a_0$

$y = a_0 + a_1 x +$ $\underline{\frac{3}{2} a_0}$ $x^2 +$ $\underline{\frac{1}{3} a_1}$ $x^3 +$ $\underline{\frac{1}{8} a_0}$ $x^4 + \cdots$

4.) *[20 points]* Note x=0 is a regular singular point of the ODE

$$2xy'' - y' + 2y = 0$$

Using the Method of Frobenius about x=0, find the indicial roots of the ODE and the general recurrence relation in terms of n and r. *(You do not need to find the Frobenius series solutions. The back of this page is blank if you need more room.)*

$$y=\sum_{n=0}^{\infty} c_n x^{n+r}, \quad y'=\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, \quad y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$n+r=k+r-1,\ n=k-1,\ n=0 \Rightarrow k=1$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{k=1}^{\infty} 2c_{k-1} x^{k+r-1} = 0$$

Pull out $n=0$ terms.

$$\underbrace{2r(r-1)c_0 x^{r-1} - r c_0 x^{r-1}}_{=0} + \sum_{n=1}^{\infty}\Big[\underbrace{2(n+r)(n+r-1)c_n - (n+r)c_n + 2c_{n-1}}_{=0}\Big]x^{n} = 0$$

$$2r(r-1) - r = 0$$
$$2r^2 - 2r - r = 0$$
$$2r^2 - 3r = 0$$
$$r(2r-3) = 0$$
$$\boxed{r = 0, \tfrac{3}{2}}$$

$$\big[2(n+r)(n+r-1) - (n+r)\big]c_n = -2c_{n-1}$$

$$\boxed{c_n = \frac{-2}{2(n+r)(n+r-1)-(n+r)}\,c_{n-1}}$$