

# 4 Convolution

## Recommended Problems

### P4.1

This problem is a simple example of the use of superposition. Suppose that a discrete-time linear system has outputs  $y[n]$  for the given inputs  $x[n]$  as shown in Figure P4.1-1.

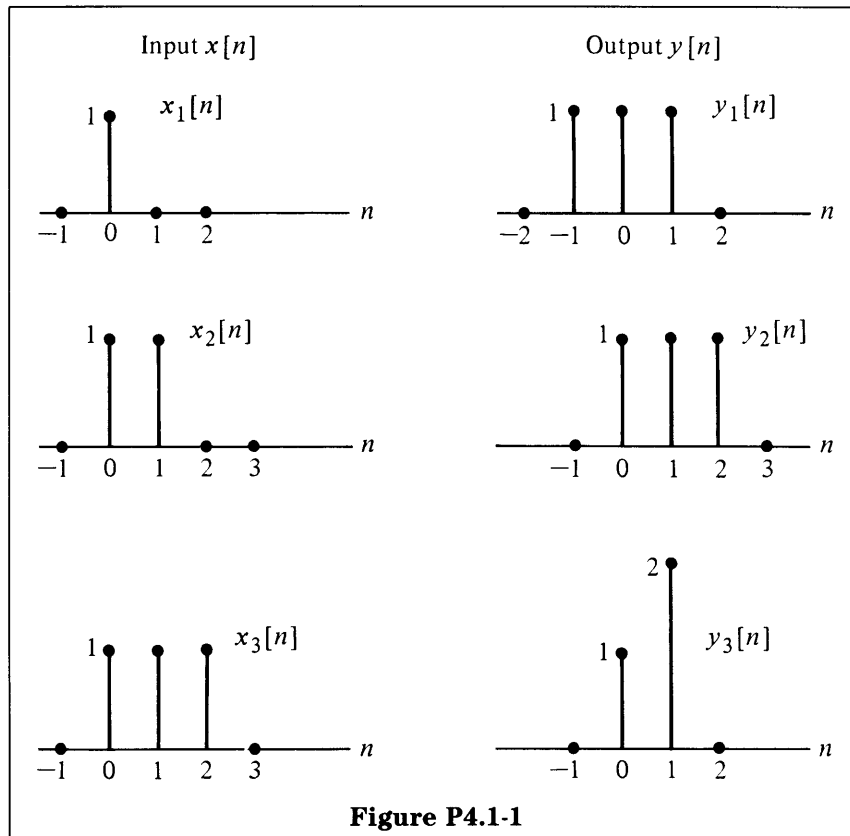


Figure P4.1-1

Determine the response  $y_4[n]$  when the input is as shown in Figure P4.1-2.

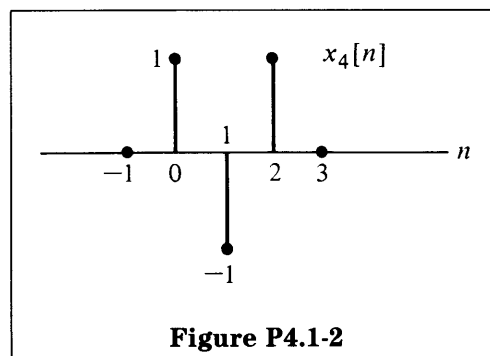


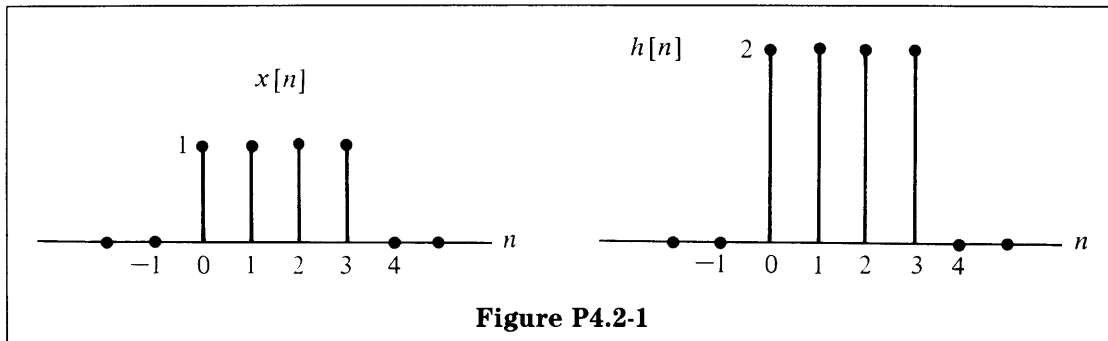
Figure P4.1-2

- Express  $x_4[n]$  as a linear combination of  $x_1[n]$ ,  $x_2[n]$ , and  $x_3[n]$ .
- Using the fact that the system is linear, determine  $y_4[n]$ , the response to  $x_4[n]$ .
- From the input-output pairs in Figure P4.1-1, determine whether the system is time-invariant.

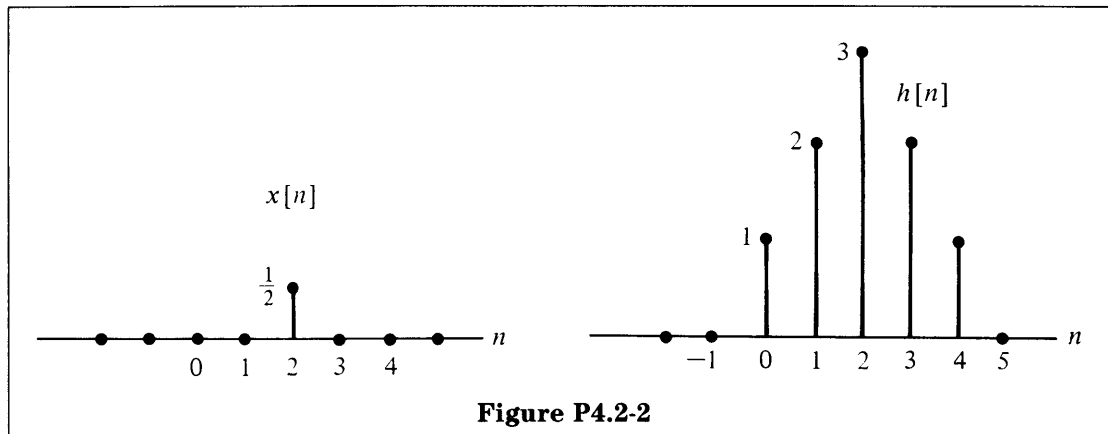
**P4.2**

Determine the discrete-time convolution of  $x[n]$  and  $h[n]$  for the following two cases.

(a)



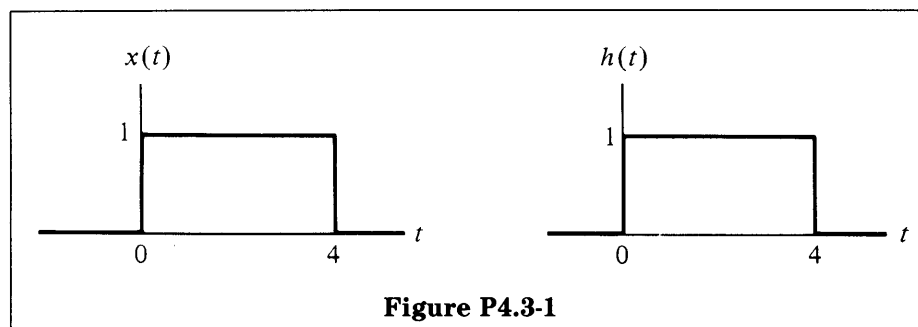
(b)

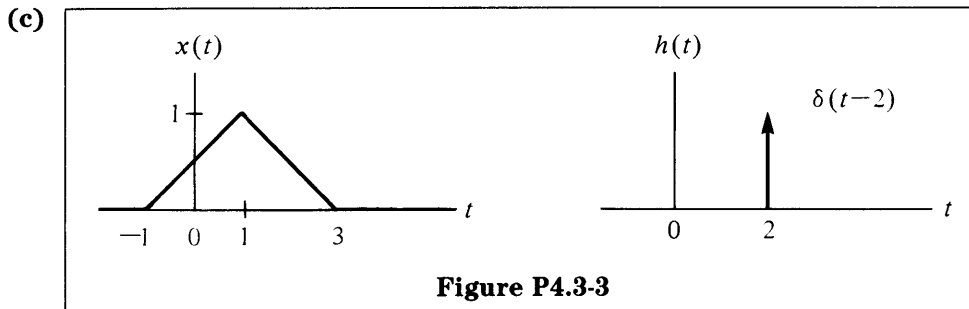
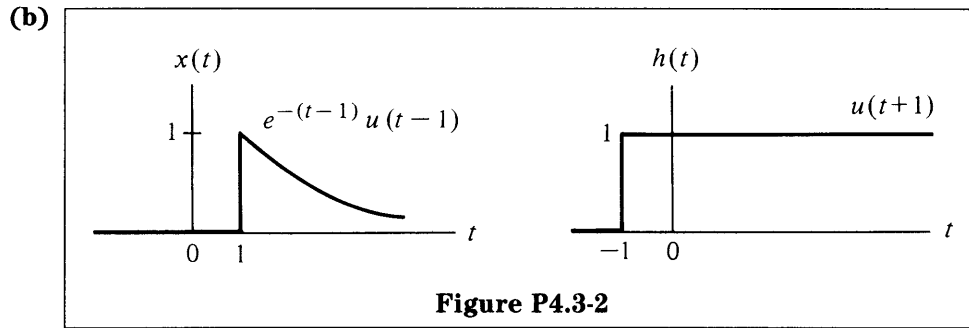


**P4.3**

Determine the continuous-time convolution of  $x(t)$  and  $h(t)$  for the following three cases:

(a)



**P4.4**

Consider a discrete-time, linear, shift-invariant system that has unit sample response  $h[n]$  and input  $x[n]$ .

- (a) Sketch the response of this system if  $x[n] = \delta[n - n_0]$ , for some  $n_0 > 0$ , and  $h[n] = (\frac{1}{2})^n u[n]$ .
- (b) Evaluate and sketch the output of the system if  $h[n] = (\frac{1}{2})^n u[n]$  and  $x[n] = u[n]$ .
- (c) Consider reversing the role of the input and system response in part (b). That is,

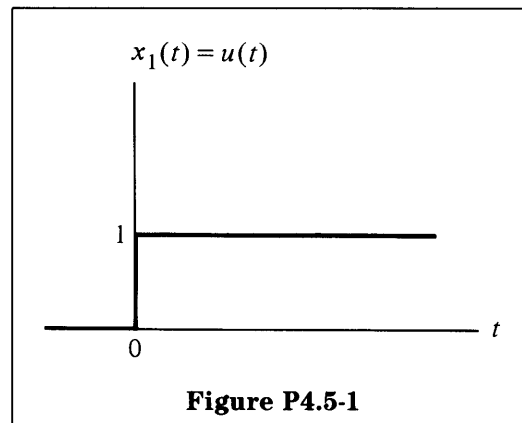
$$\begin{aligned} h[n] &= u[n], \\ x[n] &= (\tfrac{1}{2})^n u[n] \end{aligned}$$

Evaluate the system output  $y[n]$  and sketch.

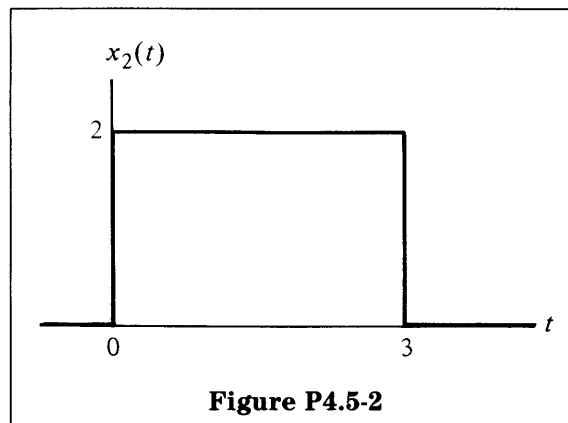
**P4.5**

- (a) Using convolution, determine and sketch the responses of a linear, time-invariant system with impulse response  $h(t) = e^{-t/2} u(t)$  to each of the two inputs  $x_1(t)$ ,  $x_2(t)$  shown in Figures P4.5-1 and P4.5-2. Use  $y_1(t)$  to denote the response to  $x_1(t)$  and use  $y_2(t)$  to denote the response to  $x_2(t)$ .

(i)



(ii)



(b)  $x_2(t)$  can be expressed in terms of  $x_1(t)$  as

$$x_2(t) = 2[x_1(t) - x_1(t - 3)]$$

By taking advantage of the linearity and time-invariance properties, determine how  $y_2(t)$  can be expressed in terms of  $y_1(t)$ . Verify your expression by evaluating it with  $y_1(t)$  obtained in part (a) and comparing it with  $y_2(t)$  obtained in part (a).

## Optional Problems

### P4.6

Graphically determine the continuous-time convolution of  $h(t)$  and  $x(t)$  for the cases shown in Figures P4.6-1 and P4.6-2.

(a)

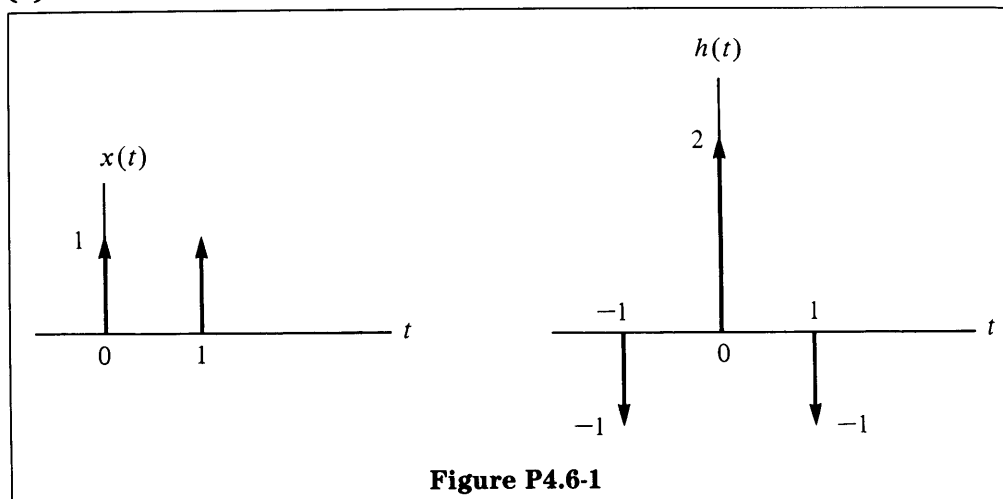


Figure P4.6-1

(b)

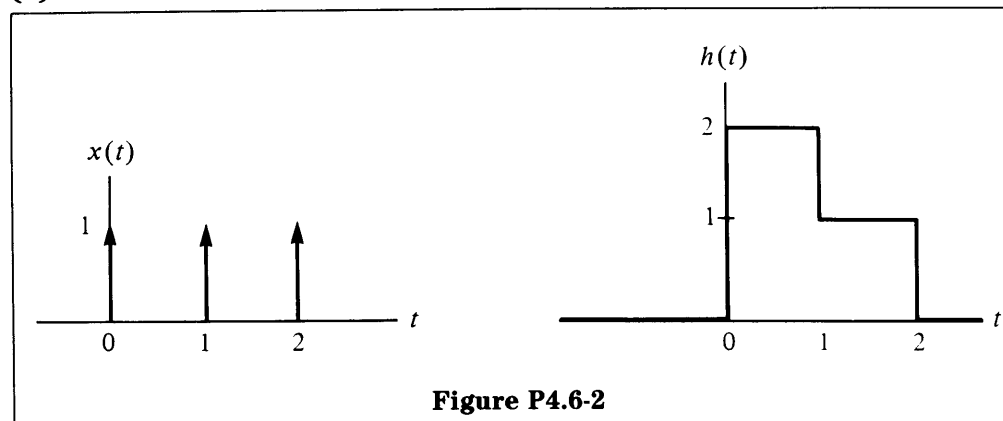


Figure P4.6-2

**P4.7**

Compute the convolution  $y[n] = x[n] * h[n]$  when

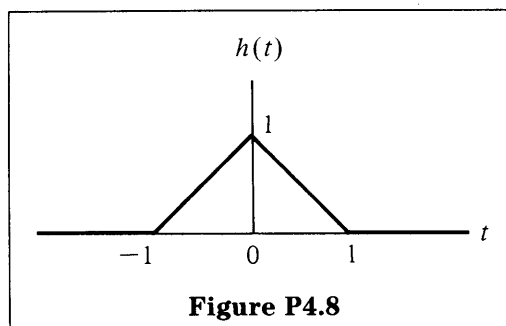
$$\begin{aligned} x[n] &= \alpha^n u[n], & 0 < \alpha < 1, \\ h[n] &= \beta^n u[n], & 0 < \beta < 1 \end{aligned}$$

Assume that  $\alpha$  and  $\beta$  are not equal.

**P4.8**

Suppose that  $h(t)$  is as shown in Figure P4.8 and  $x(t)$  is an impulse train, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



- (a) Sketch  $x(t)$ .
- (b) Assuming  $T = \frac{3}{2}$ , determine and sketch  $y(t) = x(t) * h(t)$ .

#### P4.9

Determine if each of the following statements is true in general. Provide proofs for those that you think are true and counterexamples for those that you think are false.

- (a)  $x[n] * \{h[n]g[n]\} = \{x[n] * h[n]\}g[n]$
- (b) If  $y(t) = x(t) * h(t)$ , then  $y(2t) = 2x(2t) * h(2t)$ .
- (c) If  $x(t)$  and  $h(t)$  are odd signals, then  $y(t) = x(t) * h(t)$  is an even signal.
- (d) If  $y(t) = x(t) * h(t)$ , then  $Ev\{y(t)\} = x(t) * Ev\{h(t)\} + Ev\{x(t)\} * h(t)$ .

#### P4.10

Let  $\tilde{x}_1(t)$  and  $\tilde{x}_2(t)$  be two periodic signals with a common period  $T_0$ . It is not too difficult to check that the convolution of  $\tilde{x}_1(t)$  and  $\tilde{x}_2(t)$  does not converge. However, it is sometimes useful to consider a form of convolution for such signals that is referred to as *periodic convolution*. Specifically, we define the periodic convolution of  $\tilde{x}_1(t)$  and  $\tilde{x}_2(t)$  as

$$\tilde{y}(t) = \int_0^{T_0} \tilde{x}_1(\tau) \tilde{x}_2(t - \tau) d\tau = \tilde{x}_1(t) \otimes \tilde{x}_2(t) \quad (\text{P4.10-1})$$

Note that we are integrating over exactly one period.

- (a) Show that  $\tilde{y}(t)$  is periodic with period  $T_0$ .
- (b) Consider the signal

$$\tilde{y}_a(t) = \int_a^{a+T_0} \tilde{x}_1(\tau) \tilde{x}_2(t - \tau) d\tau,$$

where  $a$  is an arbitrary real number. Show that

$$\tilde{y}(t) = \tilde{y}_a(t)$$

*Hint:* Write  $a = kT_0 - b$ , where  $0 \leq b < T_0$ .

- (c) Compute the periodic convolution of the signals depicted in Figure P4.10-1, where  $T_0 = 1$ .

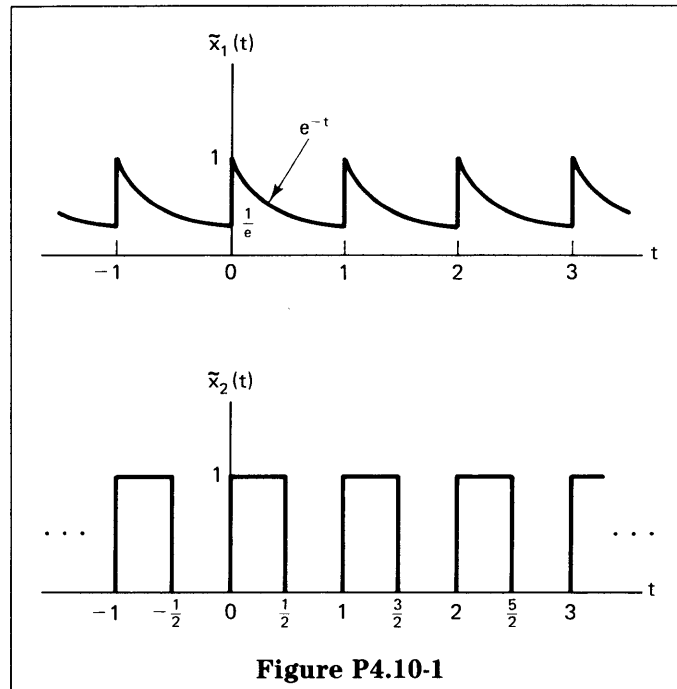


Figure P4.10-1

- (d) Consider the signals  $x_1[n]$  and  $x_2[n]$  depicted in Figure P4.10-2. These signals are periodic with period 6. Compute and sketch their periodic convolution using  $N_0 = 6$ .

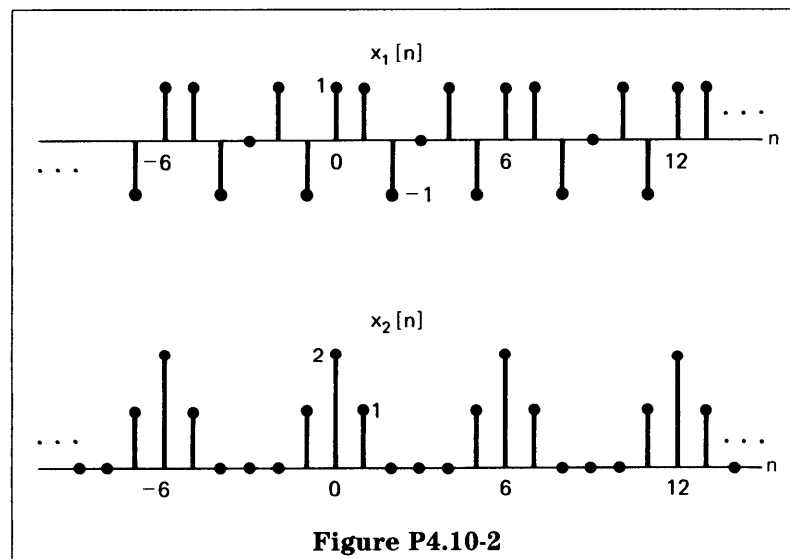


Figure P4.10-2

- (e) Since these signals are periodic with period 6, they are also periodic with period 12. Compute the periodic convolution of  $x_1[n]$  and  $x_2[n]$  using  $N_0 = 12$ .

#### P4.11

One important use of the concept of inverse systems is to remove distortions of some type. A good example is the problem of removing echoes from acoustic signals. For example, if an auditorium has a perceptible echo, then an initial acoustic impulse is

followed by attenuated versions of the sound at regularly spaced intervals. Consequently, a common model for this phenomenon is a linear, time-invariant system with an impulse response consisting of a train of impulses:

$$h(t) = \sum_{k=0}^{\infty} h_k \delta(t - kT) \quad (\text{P4.11-1})$$

Here the echoes occur  $T$  s apart, and  $h_k$  represents the gain factor on the  $k$ th echo resulting from an initial acoustic impulse.

- (a) Suppose that  $x(t)$  represents the original acoustic signal (the music produced by an orchestra, for example) and that  $y(t) = x(t) * h(t)$  is the actual signal that is heard if no processing is done to remove the echoes. To remove the distortion introduced by the echoes, assume that a microphone is used to sense  $y(t)$  and that the resulting signal is transduced into an electrical signal. We will also use  $y(t)$  to denote this signal, as it represents the electrical equivalent of the acoustic signal, and we can go from one to the other via acoustic-electrical conversion systems.

The important point to note is that the system with impulse response given in eq. (P4.11-1) is invertible. Therefore, we can find an LTI system with impulse response  $g(t)$  such that

$$y(t) * g(t) = x(t)$$

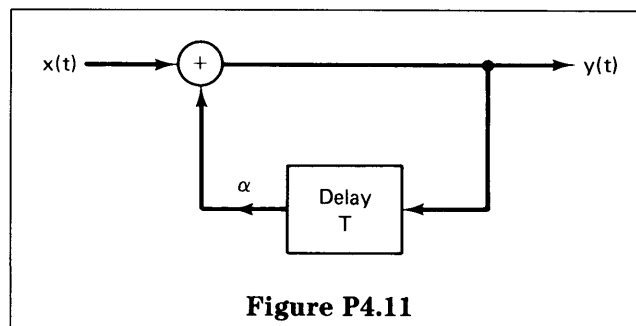
and thus, by processing the electrical signal  $y(t)$  in this fashion and then converting back to an acoustic signal, we can remove the troublesome echoes.

The required impulse response  $g(t)$  is also an impulse train:

$$g(t) = \sum_{k=0}^{\infty} g_k \delta(t - kT)$$

Determine the algebraic equations that the successive  $g_k$  must satisfy and solve for  $g_1$ ,  $g_2$ , and  $g_3$  in terms of the  $h_k$ . [Hint: You may find part (a) of Problem 3.16 of the text (page 136) useful.]

- (b) Suppose that  $h_0 = 1$ ,  $h_1 = \frac{1}{2}$ , and  $h_i = 0$  for all  $i \geq 2$ . What is  $g(t)$  in this case?
- (c) A good model for the generation of echoes is illustrated in Figure P4.11. Each successive echo represents a feedback version of  $y(t)$ , delayed by  $T$  s and scaled by  $\alpha$ . Typically  $0 < \alpha < 1$  because successive echoes are attenuated.



- (i) What is the impulse response of this system? (Assume initial rest, i.e.,  $y(t) = 0$  for  $t < 0$  if  $x(t) = 0$  for  $t < 0$ .)
- (ii) Show that the system is stable if  $0 < \alpha < 1$  and unstable if  $\alpha > 1$ .
- (iii) What is  $g(t)$  in this case? Construct a realization of this inverse system using adders, coefficient multipliers, and  $T$ -s delay elements.



Although we have phrased this discussion in terms of continuous-time systems because of the application we are considering, the same general ideas hold in discrete time. That is, the LTI system with impulse response

$$h[n] = \sum_{k=0}^{\infty} h_k \delta[n - kN]$$

is invertible and has as its inverse an LTI system with impulse response

$$g[n] = \sum_{k=0}^{\infty} g_k \delta[n - kN]$$

It is not difficult to check that the  $g_i$  satisfy the same algebraic equations as in part (a).

- (d) Consider the discrete-time LTI system with impulse response

$$h[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

This system is *not* invertible. Find two inputs that produce the same output.

#### P4.12

Our development of the convolution sum representation for discrete-time LTI systems was based on using the unit sample function as a building block for the representation of arbitrary input signals. This representation, together with knowledge of the response to  $\delta[n]$  and the property of superposition, allowed us to represent the system response to an arbitrary input in terms of a convolution. In this problem we consider the use of other signals as building blocks for the construction of arbitrary input signals.

Consider the following set of signals:

$$\begin{aligned} \phi[n] &= \left(\frac{1}{2}\right)^n u[n], \\ \phi_k[n] &= \phi[n - k], \quad k = 0, \pm 1, \pm 2, \pm 3, \dots \end{aligned}$$

- (a) Show that an arbitrary signal can be represented in the form

$$x[n] = \sum_{k=-\infty}^{+\infty} a_k \phi[n - k]$$

by determining an explicit expression for the coefficient  $a_k$  in terms of the values of the signal  $x[n]$ . [*Hint*: What is the representation for  $\delta[n]$ ?]

- (b) Let  $r[n]$  be the response of an LTI system to the input  $x[n] = \phi[n]$ . Find an expression for the response  $y[n]$  to an arbitrary input  $x[n]$  in terms of  $r[n]$  and  $x[n]$ .
- (c) Show that  $y[n]$  can be written as

$$y[n] = \psi[n] * x[n] * r[n]$$

by finding the signal  $\psi[n]$ .

- (d) Use the result of part (c) to express the impulse response of the system in terms of  $r[n]$ . Also, show that

$$\psi[n] * \phi[n] = \delta[n]$$

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