THE CITADEL
THE MILITARY COLLEGE OF SOUTH CAROLINA

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ELEC 318 – *Electromagnetic Fields*

Lecture 7(a)

Time-Harmonic
E and **H** Fields

Wave Equation Solution

Assuming no conductivity ($\sigma = 0$)… $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mu_0 \mathbf{H}$ $\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\varepsilon_0 \mathbf{E}$

Substitution of Ampere's Law into Faraday's Law (or vice versa) gives…

$$\nabla^2 \mathbf{E} + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

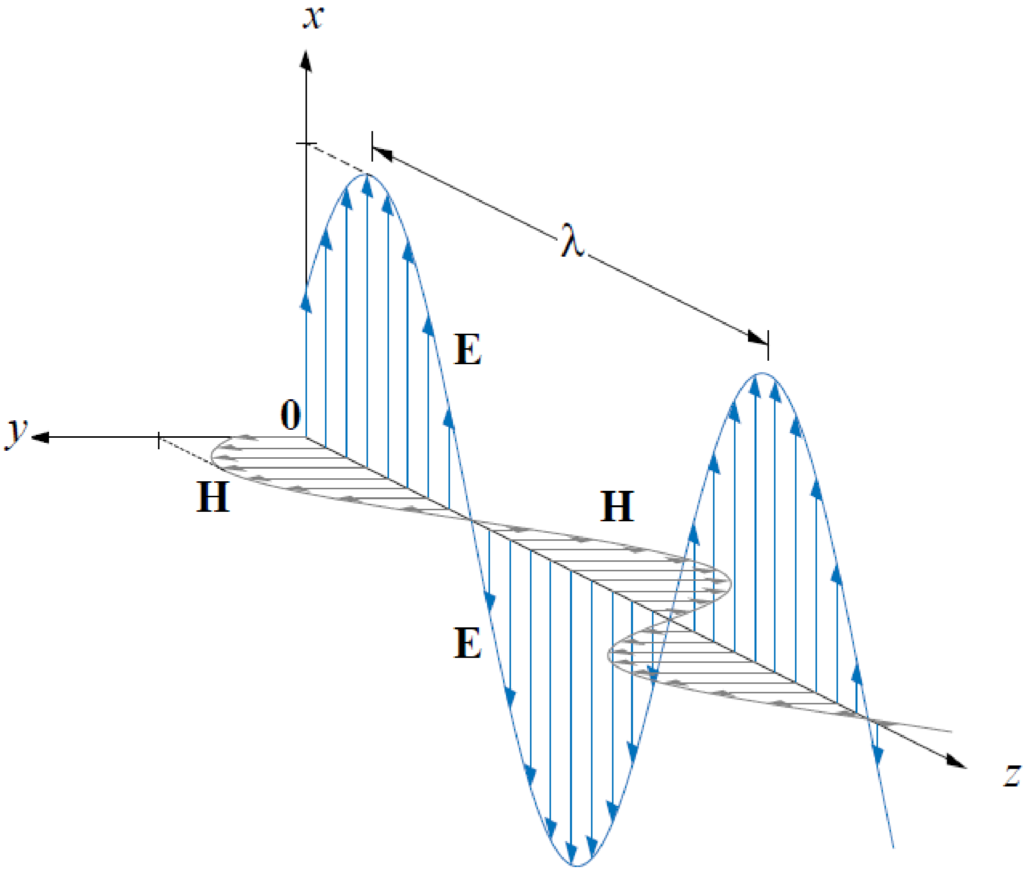
The form of the solution that satisfies this differential equation is

$$\mathbf{E}(z,t) = E_0 \cos(\omega t - kz + \phi_0)\,\hat{\mathbf{x}}$$

$$\frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

…which is the electric field vector associated with a wave that is propagating in the *z* direction, with velocity ω/β

Wave Equation Solution



$$\mathbf{E}(z,t) = E_0 \cos(\omega t - kz + \phi_0)\,\hat{\mathbf{x}}$$

$$u = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \lambda = \frac{2\pi}{k}$$

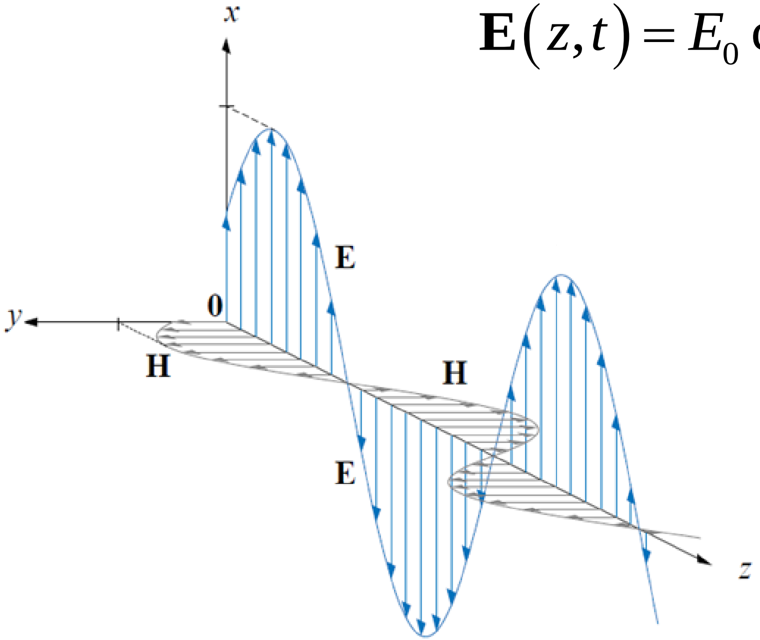
…which is the electric field vector associated with a wave that is propagating in the *z* direction, with velocity ω/k

The solutions to Maxwell's Equations vary sinusoidally with time…

…therefore the field equations and solutions may be written as **phasors**.

$$f(t) = F_0 \cos(\omega t + \theta)\,\hat{\mathbf{n}} \quad \Leftrightarrow \quad \tilde{\mathbf{F}} = F_0 e^{j\theta}\,\hat{\mathbf{n}}$$

$$\mathbf{E}(z,t) = E_0 \cos(\omega t - kz + \phi_0)\,\hat{\mathbf{x}} \quad \Leftrightarrow \quad \tilde{\mathbf{E}}(z) = E_0 e^{j(-kz+\phi_0)}\,\hat{\mathbf{x}}$$



$$\nabla \cdot \mathbf{D} = \rho_v \quad \Leftrightarrow \quad \nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_v$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Leftrightarrow \quad \nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Leftrightarrow \quad \nabla \cdot \tilde{\mathbf{B}} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \Leftrightarrow \quad \nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}$$

Example: Plane Wave

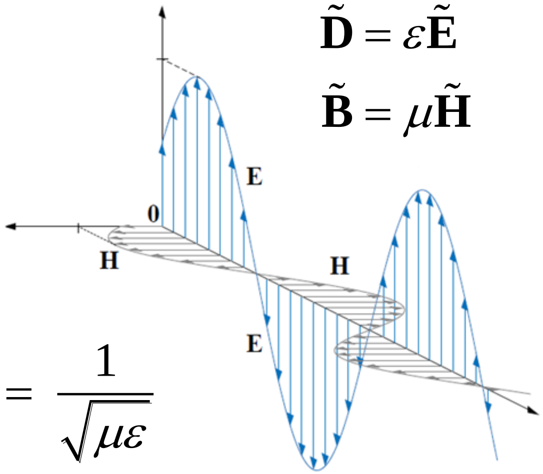
In a medium characterized by $\sigma = 0$, $\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$, and

$$\mathbf{E}(z,t) = 20\sin\left(10^8 t - kz\right)\hat{\mathbf{y}}\ \ \mathrm{V/m}$$

Calculate k and $\mathbf{H}(z, t)$.

$$\tilde{\mathbf{D}} = \varepsilon\tilde{\mathbf{E}}$$

$$\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}}$$



$$u = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\nabla\times\tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}$$

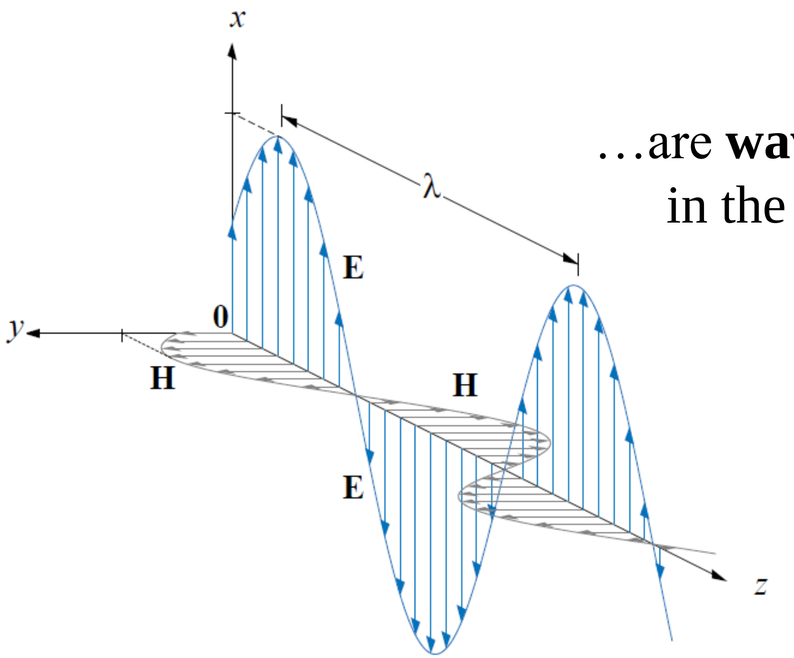
$$\nabla\times\tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}}$$

$$\nabla\times\mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

Waves -- In General

The time-varying fields that satisfy Maxwell's Equations…

$$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}_v$$
$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}$$
$$\nabla \cdot \tilde{\mathbf{B}} = 0$$
$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega \tilde{\mathbf{D}}$$



…are **waves**, which may be written in the time domain or as phasors:

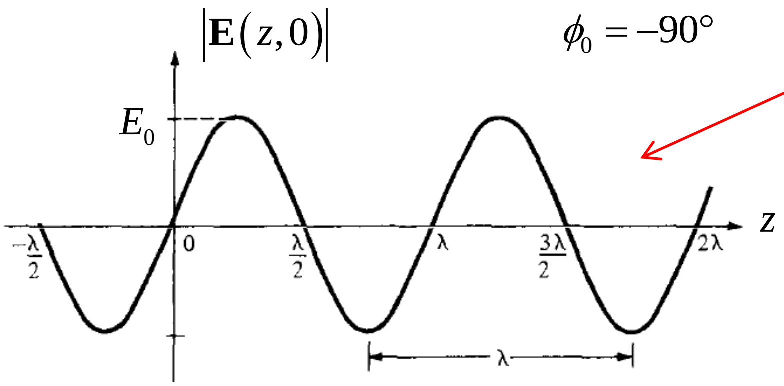
$$\mathbf{E}(z,t) = E_0 \cos(\omega t - kz + \phi_0)\,\hat{\mathbf{x}}$$
$$\tilde{\mathbf{E}} = E_0 e^{-jkz + j\phi_0}\,\hat{\mathbf{x}}$$

$$u = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

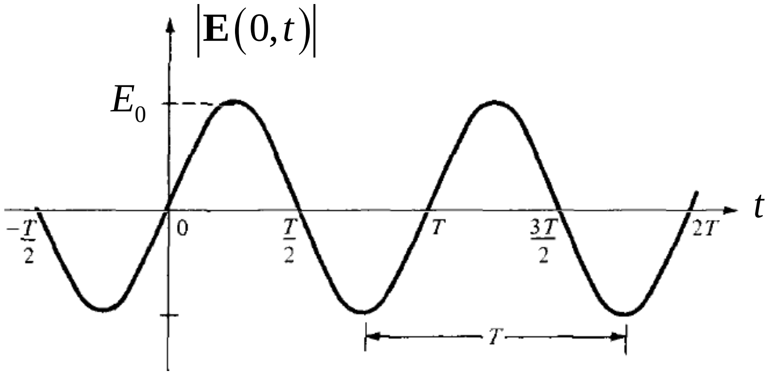
$$\lambda = 2\pi/k$$
$$f = \omega/2\pi$$
$$T = 1/f$$

E (above) is for a wave propagating in the *z* direction, with **velocity** u, **frequency** f, **radian frequency** ω, **period** T, **wavelength** λ, and **wave number** k.

Waves -- In General



"snapshot in time"



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E (above) is for a wave propagating in the *z* direction, with **velocity** u , **frequency** f , **radian frequency** ω , **period** T , **wavelength** λ , and **wave number** k .

Waves -- In General

velocity u (m/s) : "speed" of a wave

frequency f (Hz) : "repetition rate", in time, at a given position (cycles/second)

period T (s) : *time* between consecutive maxima, at a given position

wavelength λ (m) : *distance* between consecutive maxima, at a given time

radian frequency ω (rad/s) :
"repetition rate" in *time*,
alternate representation

wave number k (rad/m) :
"repetition rate" in *space*,
for a given (fixed) *time*

$$\mathbf{E}(z,t) = E_0 \cos(\omega t - kz + \phi_0)\,\hat{\mathbf{x}}$$

$$\tilde{\mathbf{E}} = E_0 e^{-jkz + j\phi_0}\,\hat{\mathbf{x}}$$

$$u = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda = 2\pi/k$$
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$$T = 1/f$$

ε = electric permittivity

μ = magnetic permeability → for the material in which the wave propagates