ELEC 312
Systems I

Transient Response Analysis
(Derived from Notes by Dr. Robert Barsanti)

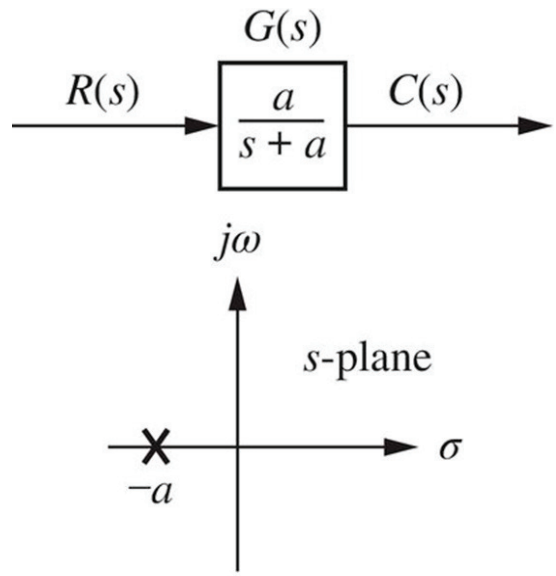
(Images from Nise, 7^{th} Edition)

Required Reading: Chapter 4, *Control Systems Engineering*

February 5, 2015

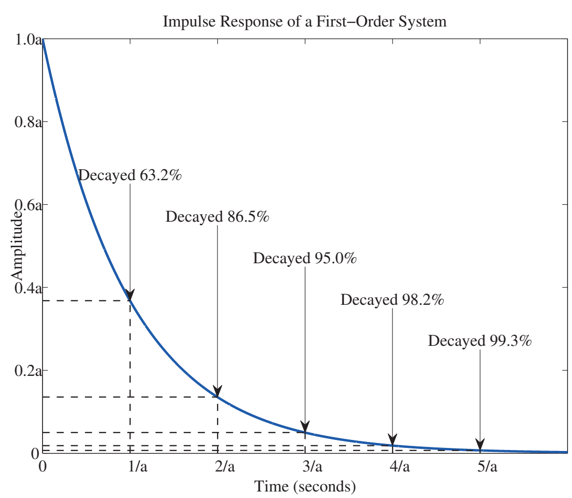
First-Order Systems

Consider the following first-order system:



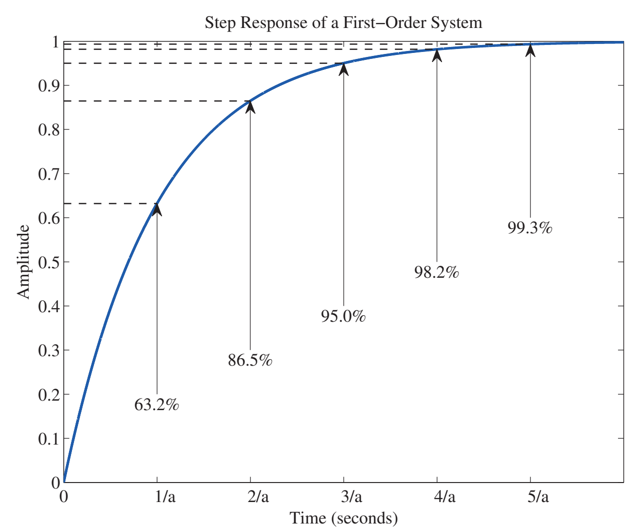
IMPULSE Response of First-Order System

$$Y(s) = G(s)X(s) = G(s)\cdot 1 = \frac{a}{s+a} \overset{\mathcal{L}}{\longleftrightarrow} y(t) = g(t) = \mathcal{L}^{-1}\left\{\frac{a}{s+a}\right\} = ae^{-at}u(t)$$



STEP Response of First-Order System

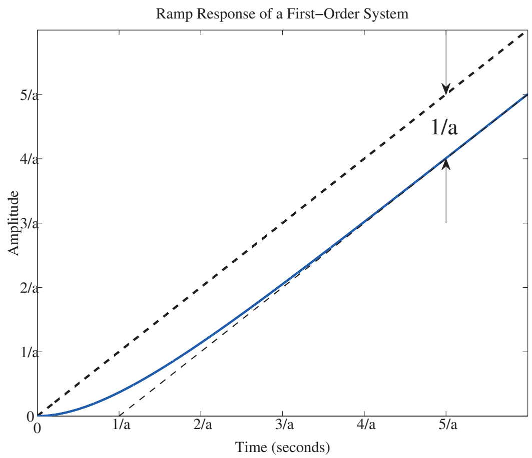
$$Y(s) = G(s)X(s) = \frac{a}{s+a}\cdot\frac{1}{s} = \frac{1}{s} - \frac{1}{s+a} \overset{\mathcal{L}}{\longleftrightarrow} y(t) = \left[1 - e^{-at}\right]u(t)$$



RAMP Response of First-Order System

$$Y(s) = G(s)X(s) = \frac{a}{s+a} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{1/a}{s} + \frac{1/a}{s+a}$$

$$\stackrel{\mathcal{L}}{\longleftrightarrow} y(t) = \left[t - \frac{1}{a} + \frac{1}{a}e^{-at}\right]u(t) = \left[t - \frac{1}{a}\left(1 - e^{-at}\right)\right]u(t)$$



First-Order System: Transient Response Specifications

1. **Time Constant (τ or T_c)** - Time required for the step response to rise to 63.212% of its final value.

$$\boxed{T_c = \frac{1}{a} = \tau}$$

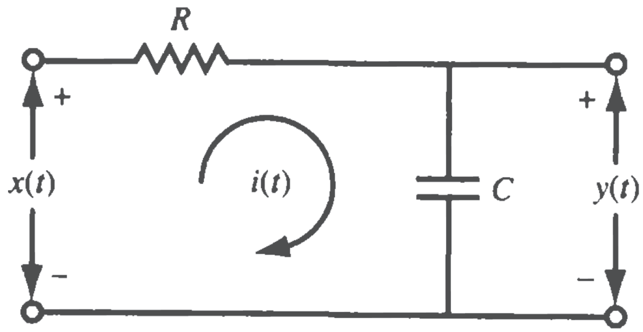
2. **Rise Time (T_r)** - Time required for the step response to rise from 10% to 90% of its final value.

$$\boxed{T_r = \frac{2.2}{a} = 2.2\tau}$$

3. **Settling Time (T_s)** - Time required for the step response to obtain and stay within 2% of its final value.

$$\boxed{T_s = \frac{4}{a} = 4\tau}$$

First-Order System: Example 1

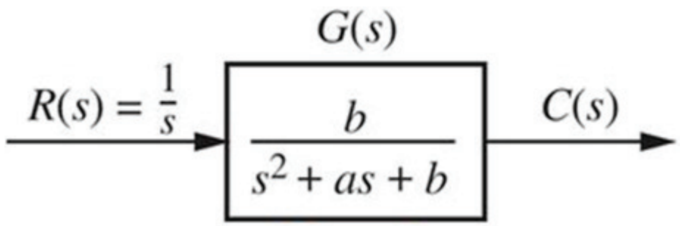


Determine the time constant, rise time, and settling time for the above first-order RC circuit if $R = 20$ kΩ and $C = 1$ μF.

First-Order System: Example 1 (continued)

Second-Order Systems

Consider the following general second-order system:

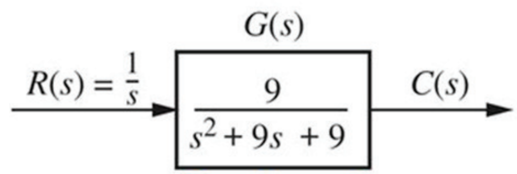


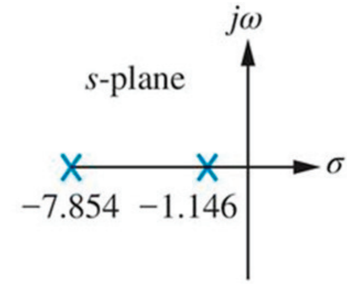
The system represented by $G(s)$ above has no zeros and has characteristic equation $s^2 + as + b = 0$, which yields poles at

$$s = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}.$$

Second-Order System: OVERDAMPED Responses

Consider the following second-order system:



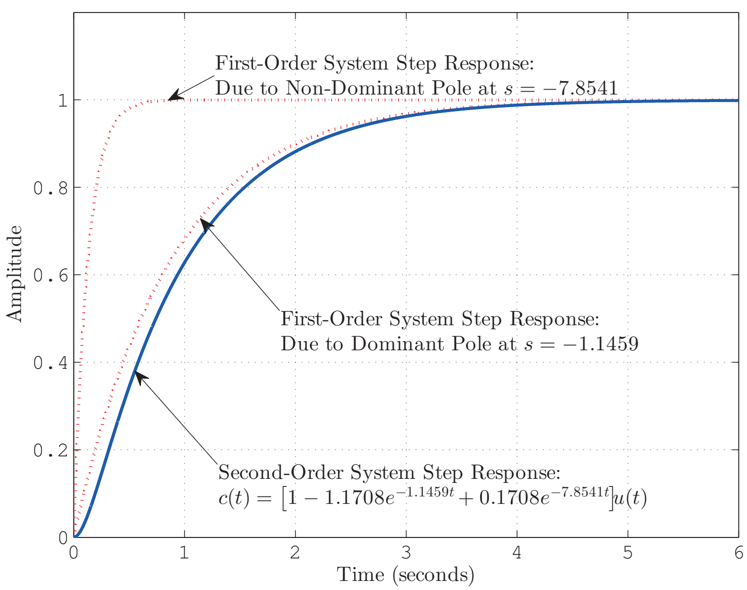


The system represented by $G(s)$ above has no zeros and has characteristic equation $s^2 + as + b = 0$, which yields poles at:

$$s = \frac{-9 \pm \sqrt{9^2 - 4(9)}}{2} = -\frac{9}{2} \pm \sqrt{\left(\frac{9}{2}\right)^2 - 9} = \frac{-9 \pm 3\sqrt{5}}{2} = -7.8541, -1.1459.$$

A second-order system with real, distinct poles has an **overdamped** response.

Second-Order System: OVERDAMPED Step Response



Second-Order System: OVERDAMPED Responses

Overdamped responses have the following characteristics:

- **Poles**: Two distinct real at $s = -\sigma_1$ and $s = -\sigma_2$

- **Natural response**: Two exponentials with time constants equal to the reciprocal of the magnitude of the pole locations, or

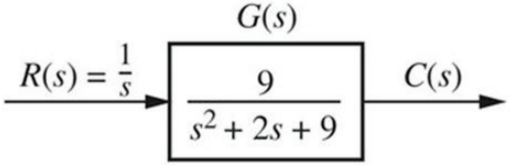
$$c(t) = C_1 e^{-\sigma_1 t} + C_2 e^{-\sigma_2 t}.$$

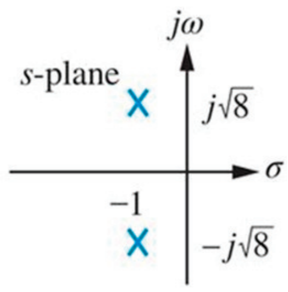
- **Step response**: Step function plus two exponentials with time constants equal to the reciprocal of the magnitude of the pole locations, or

$$c(t) = C_1 + C_2 e^{-\sigma_1 t} + C_3 e^{-\sigma_2 t}.$$

Second-Order System: UNDERDAMPED Responses

Consider the following second-order system:



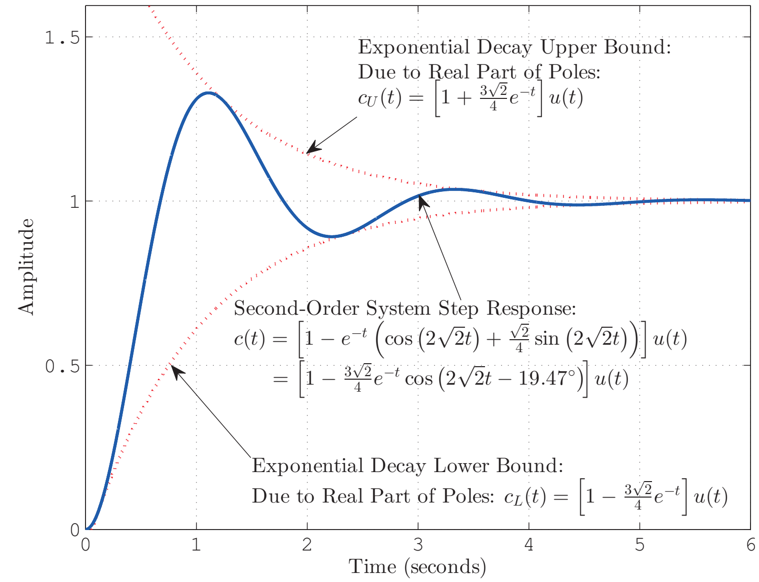


The system represented by $G(s)$ above has no zeros and has characteristic equation $s^2 + as + b = 0$, which yields poles at:

$$s = \frac{-2 \pm \sqrt{2^2 - 4(9)}}{2} = -1 \pm j2\sqrt{2} = -1 \pm j2.8284.$$

A second-order system with complex conjugate poles has an **underdamped** response.

Second-Order System: UNDERDAMPED Step Response



Second-Order System: UNDERDAMPED Responses

Underdamped responses have the following characteristics:

- **Poles**: Two complex conjugates at $s = -\sigma_d \pm j\omega_d$

- **Natural response**: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of σ_d, which is called the **exponential damping frequency** and is the magnitude of the real part of the poles. The radian frequency of the sinusoid, given by the **damped frequency of oscillation** ω_d, is equal to the imaginary part of the poles. The natural response is of the form

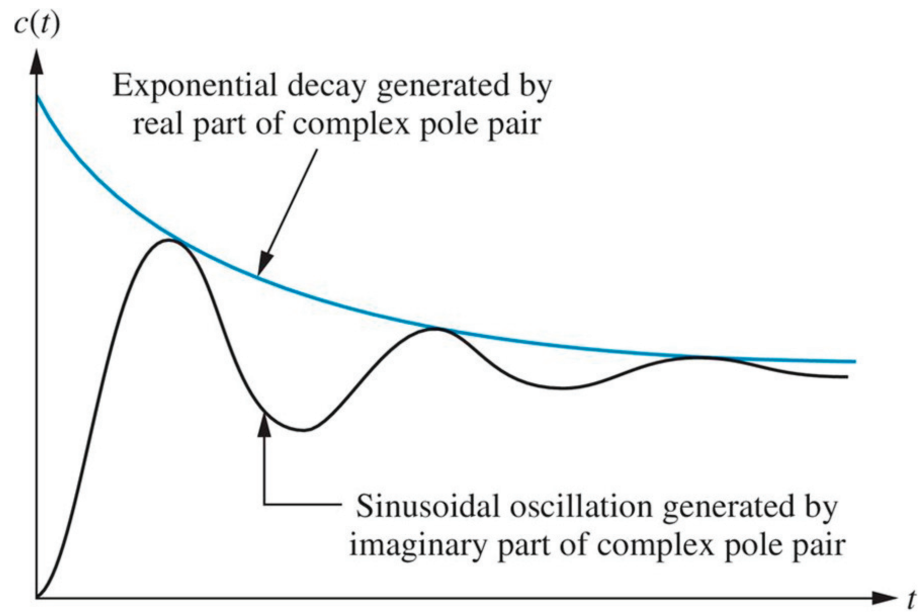
$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi).$$

- **Step response**: Step function plus damped sinusoid (described above), or

$$\begin{aligned} c(t) &= C_1 + e^{-\sigma_d t}\left[C_2\cos(\omega_d t) + C_3\sin(\omega_d t)\right] \\ &= C_1 + A \cdot e^{-\sigma_d t}\cos(\omega_d t - \phi) \end{aligned}$$

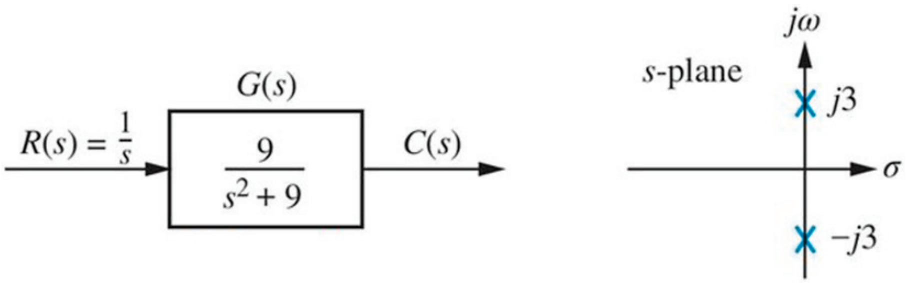
where $A = \sqrt{C_2^2 + C_3^2}$ and $\phi = \tan^{-1}\left[\frac{C_3}{C_2}\right]$.

Second-Order System: UNDERDAMPED Step Response



Second-Order System: UNDAMPED Responses

Consider the following second-order system:

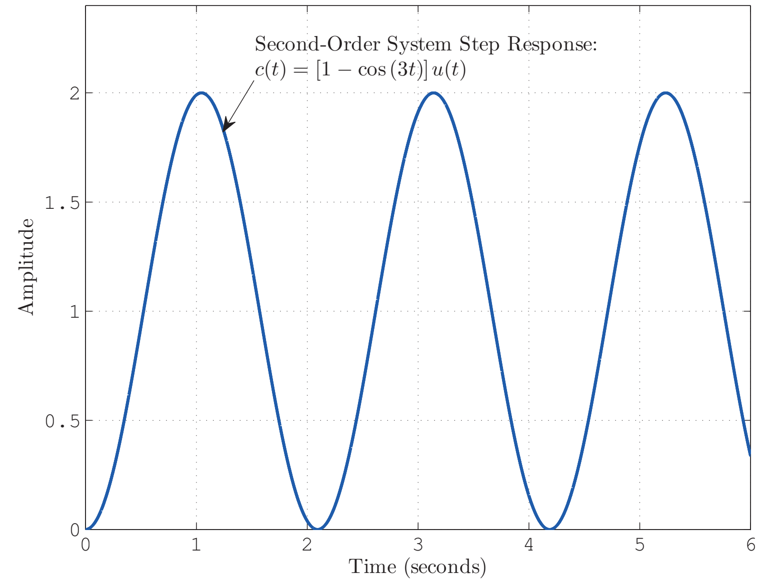


The system represented by $G(s)$ above has no zeros and has characteristic equation $s^2 + as + b = 0$, which yields poles at:

$$s = \frac{0 \pm \sqrt{0^2 - 4(9)}}{2} = \pm j3.$$

A second-order system with imaginary conjugate poles has an **undamped** response.

Second-Order System: UNDAMPED Step Response



Second-Order System: UNDAMPED Responses

Undamped responses have the following characteristics:

- **Poles**: Two imaginary conjugates at $s = \pm j\omega_n$
- **Natural response**: Undamped sinusoid with radian frequency given by the **undamped frequency**, or **natural frequency of oscillation**, ω_n, which is equal to the imaginary part of the poles. The natural response is of the form

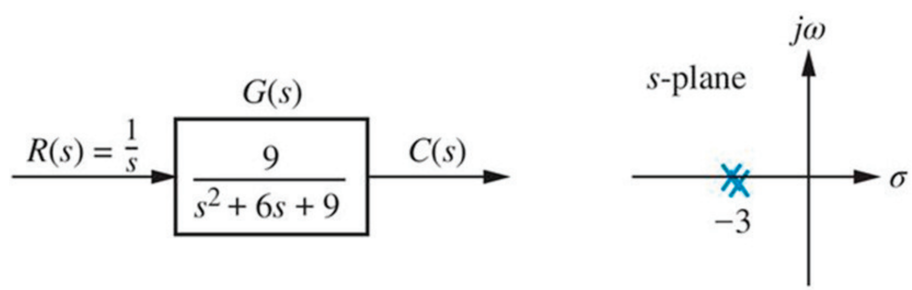
$$c(t) = A\cos(\omega_n t - \phi).$$

- **Step response**: Step function plus undamped sinusoid (described above), or

$$c(t) = C_1 + A\cos(\omega_n t - \phi).$$

Second-Order System: CRITICALLY-DAMPED Responses

Consider the following second-order system:

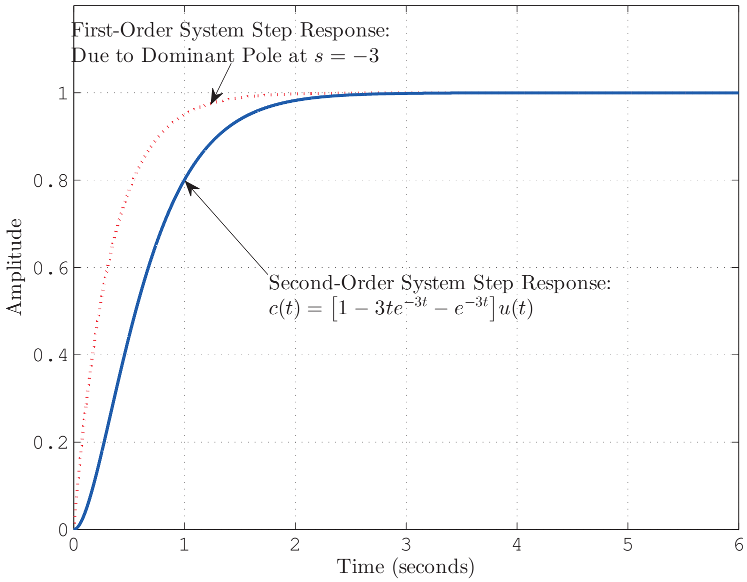


The system represented by $G(s)$ above has no zeros and has characteristic equation $s^2 + as + b = 0$, which yields poles at:

$$s = \frac{-6 \pm \sqrt{6^2 - 4(9)}}{2} = -3, -3$$

A second-order system with real, repeated poles has an **critically-damped** response.

Second-Order System: CRITICALLY-DAMPED Step Response



Second-Order System: CRITICALLY-DAMPED Responses

Critically-damped responses have the following characteristics:

- **Poles**: Two repeated real at $s = -\sigma_d$
- **Natural response**: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location. The natural response is of the form

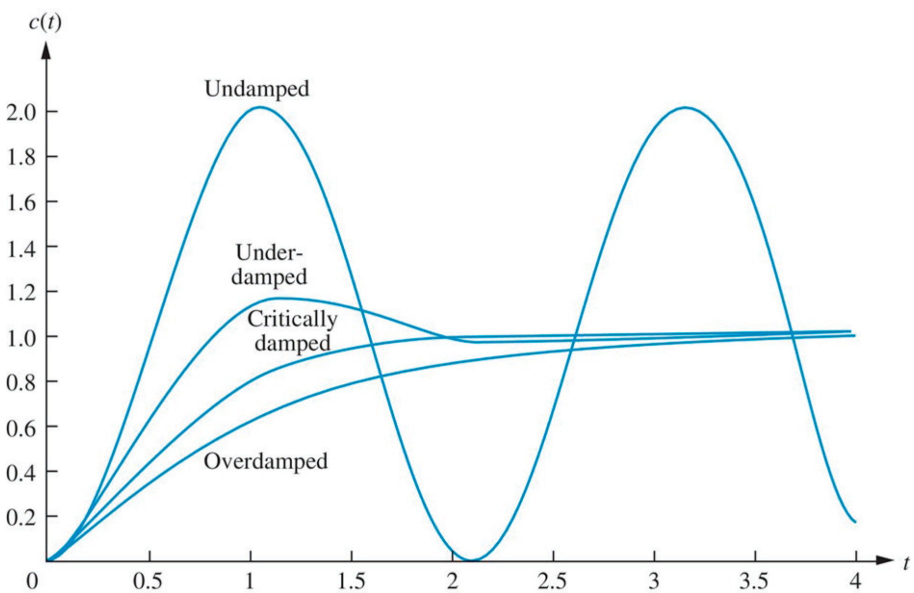
$$c(t) = C_1 e^{-\sigma_d t} + C_2 t e^{-\sigma_d t}.$$

- **Step response**: Step function plus an exponential whose time constant is equal to the reciprocal of the pole location and another term that is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location. The step response is of the form

$$c(t) = C_1 + C_2 e^{-\sigma_d t} + C_3 e^{-\sigma_d t}.$$

- **Fastest response without overshoot!**

Second-Order System Step Responses



General Second-Order Systems: Natural Frequency

The **natural frequency** of a second-order system is the frequency of oscillation of the system <u>without</u> damping. We denote natural frequency as ω_n.

Consider the general second-order system

$$G(s) = \frac{b}{s^2 + as + b}.$$

If $G(s)$ produces an undamped response, then the system poles are purely imaginary, $a = 0$, and

$$G(s) = \frac{b}{s^2 + b}.$$

The poles of the system $G(s)$ are given by $s = \pm j\sqrt{b} = \omega_n$. Therefore,

$$b = \omega_n^2.$$

General Second-Order Systems: Damping Ratio

The **damping ratio** is the ratio of the exponential decay frequency of the envelope to the natural frequency, or

$$\zeta = \frac{\text{Exponential damping frequency}}{\text{Natural frequency (rad/second)}} = \frac{\sigma_d}{\omega_n}$$

Consider the general second-order system

$$G(s) = \frac{b}{s^2 + as + b}.$$

If $G(s)$ produces an underdamped response, then the system poles are complex conjugates and are given by

$$s = -\sigma_d \pm j\omega_d = -\frac{a}{2} \pm j\sqrt{b - \left(\frac{a}{2}\right)^2}.$$

. Therefore,

$$\zeta = \frac{\sigma_d}{\omega_n} = \frac{a/2}{\omega_n} \Rightarrow a = 2\zeta\omega_n.$$

General Second-Order Systems

Therefore, the transfer function for a general second-order system is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The system poles are given by

$$\begin{aligned} s_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \end{aligned}$$

Now, let us examine the different responses due to different amounts of damping, organized according to the damping ratio, ζ.

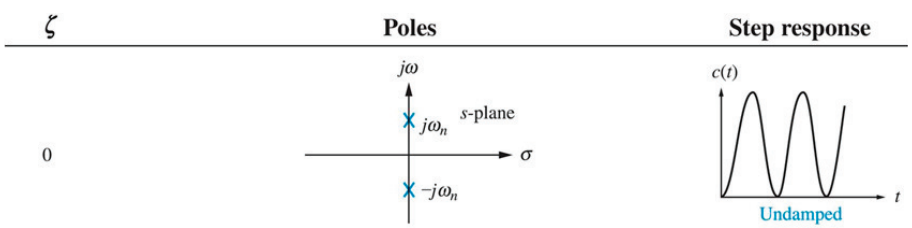
General Second-Order Systems: UNDAMPED Responses ($\zeta = 0$)

If $\zeta = 0$, then we have undamped responses, and the transfer function for the general second-order system reduces to

$$G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2},$$

which has system poles given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \boxed{\pm j\omega_n.}$$



General Second-Order Systems: UNDERDAMPED Responses ($0 < \zeta < 1$)

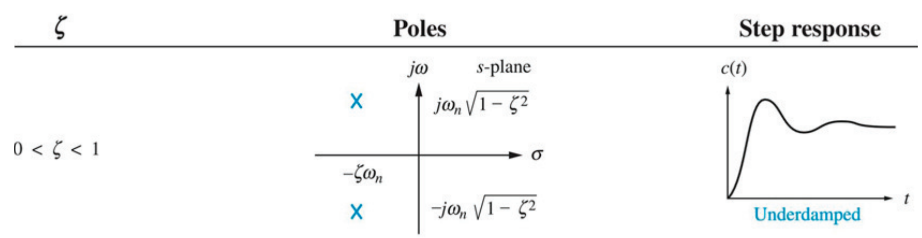
If $0 < \zeta < 1$, then we have underdamped responses, and the system poles are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \boxed{-\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}} = -\sigma_d \pm j\omega_d,$$

where

$\sigma_d = \zeta\omega_n =$ **exponential damping frequency** and

$\omega_d = \omega_n\sqrt{1 - \zeta^2} =$ **damped frequency of oscillation**.



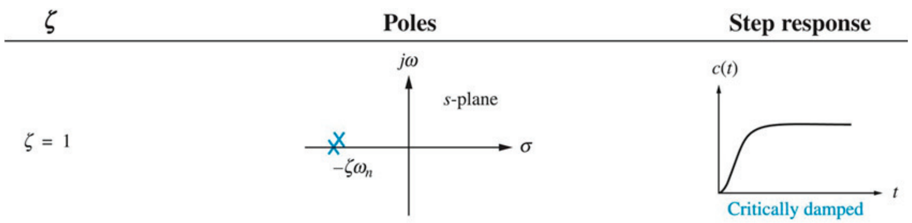
General Second-Order Systems: CRITICALLY-DAMPED Responses $(\zeta = 1)$

If $\zeta = 1$, then we have critically-damped responses, and the system poles are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \boxed{-\zeta\omega_n.} = -\sigma_d,$$

where

$$\sigma_d = \zeta\omega_n.$$



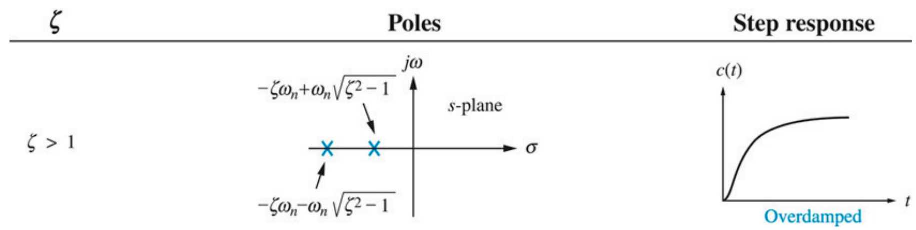
General Second-Order Systems: OVERDAMPED Responses $(\zeta > 1)$

If $\zeta > 1$, then we have overdamped responses, and the system poles are given by

$$s_{1,2} = \sigma_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_1 = \sigma_1 = \boxed{-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \text{ (dominant pole)}}$$

$$s_2 = \sigma_2 = \boxed{-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \text{ (non-dominant pole).}}$$



General Second-Order Systems: UNDERDAMPED Step Response $(0 < \zeta < 1)$

If $0 < \zeta < 1$, then

$$s_{1,2} = -\sigma_d \pm j\omega_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \text{ and}$$

$$Y_F(s) = G(s)X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{1}{s}$$

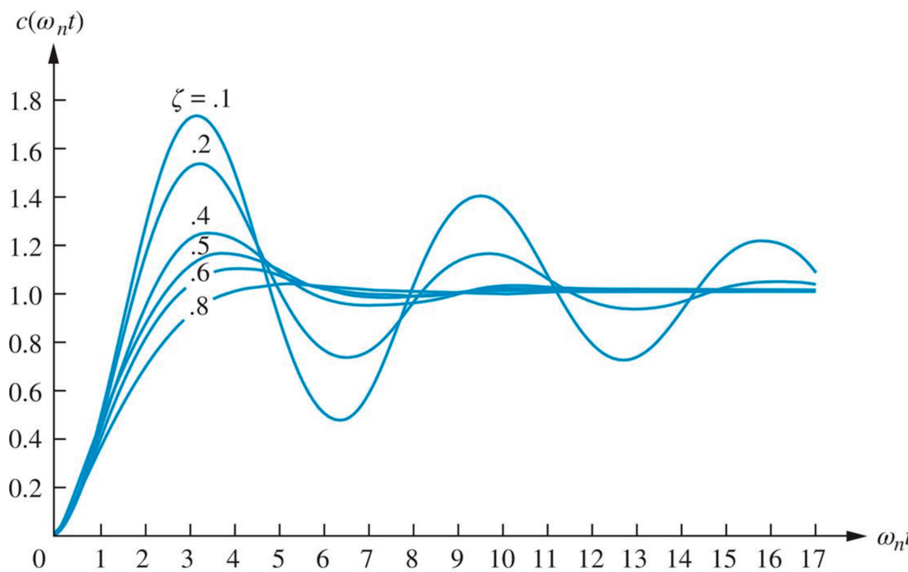
$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} - \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}.,$$

Therefore, the **underdamped** step response of a second-order system is given by

$$y_F(t) = \boxed{1 - e^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]}$$

$$= \boxed{1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right) \text{ where } \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}.$$

General Second-Order Systems: UNDERDAMPED Step Responses $(0 < \zeta < 1)$

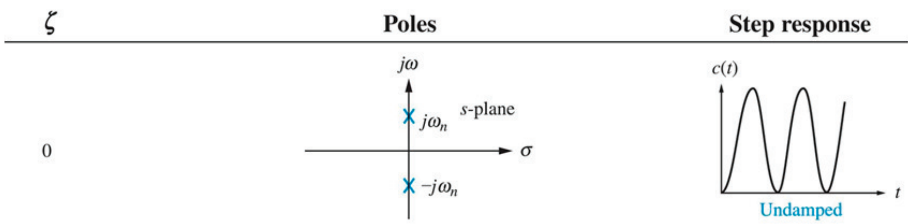


General Second-Order Systems: UNDAMPED Step Response ($\zeta = 0$)

If $\zeta = 0$, then we have a boundary case of the underdamped step response.

Therefore, the **undamped** step response of a second-order system is given by

$$y_F(t) = \left[1 - e^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]\right]_{\zeta=0}$$
$$= \boxed{1 - \cos\left(\omega_n t\right).}$$



General Second-Order Systems: CRITICALLY-DAMPED Step Response ($\zeta = 1$)

If $\zeta = 1$, then we have a boundary case of the underdamped step response, where

$$Y_F(s) = \left[\frac{1}{s} - \frac{s+\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}\right]_{\zeta=1}$$
$$= \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}.$$

Therefore, the **critically-damped** step response of a second-order system is given by

$$y_F(t) = \boxed{1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}.}$$

General Second-Order Systems: OVERDAMPED Step Response ($\zeta > 1$)

If $\zeta > 1$, then

$$s_1 = -\sigma_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1},$$
$$s_2 = -\sigma_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}, \text{ and}$$
$$Y_F(s) = G(s)X(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\cdot\frac{1}{s} = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{\frac{1}{\sigma_1}}{s+\sigma_1} - \frac{\frac{1}{\sigma_2}}{s+\sigma_2}\right].$$

Therefore, the **overdamped** step response of a second-order system is given by

$$y_F(t) = \boxed{1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{e^{-\sigma_1 t}}{\sigma_1} - \frac{e^{-\sigma_2 t}}{\sigma_2}\right].}$$

General Second-Order Systems: Example 1(a)

Classify the step response and find the damping ratio, the natural frequency, the damped frequency of oscillation, and/or the exponential damping frequency for the system given by

$$G(s) = \frac{100}{s^2 + 100}.$$

General Second-Order Systems: Example 1(b)

Classify the step response and find the damping ratio, the natural frequency, the damped frequency of oscillation, and/or the exponential damping frequency for the system given by

$$G(s) = \frac{100}{s^2 + 10s + 100}.$$

General Second-Order Systems: Example 1(c)

Classify the step response and find the damping ratio, the natural frequency, the damped frequency of oscillation, and/or the exponential damping frequency for the system given by

$$G(s) = \frac{100}{s^2 + 20s + 100}.$$

General Second-Order Systems: Example 1(d)

Classify the step response and find the damping ratio, the natural frequency, the damped frequency of oscillation, and/or the exponential damping frequency for the system given by

$$G(s) = \frac{100}{s^2 + 40s + 100}.$$

Second-Order Systems: Transient Response Specifications

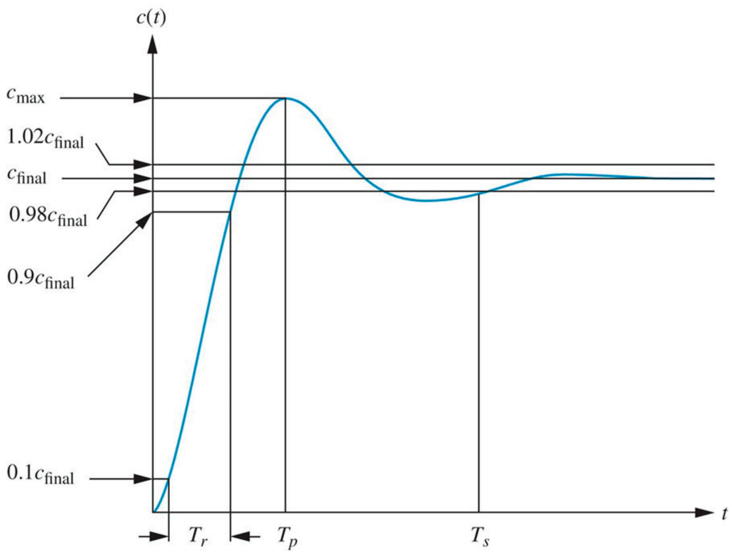
1. **Peak Time (T_p)**: Time required to reach the first, or maximum, peak.

2. **Percent Overshoot ($\%OS$)**: Amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

3. **Settling Time (T_s)**: Time required for the transient response's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state, or final, value.

4. **Rise Time (T_r)**: Time required for the waveform to go from 10% of the final value to 90% of the final value.

General Second-Order Systems: Transient Response Specifications



General Second-Order Systems: Transient Response Specifications: Peak Time

For the underdamped $(0 < \zeta < 1)$ step response, the peak time, T_p, is found by differentiating

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$$

and finding the first zero crossing after $t = 0$. Therefore,

$$\begin{aligned} c'(t) &= \zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \\ &\quad - e^{-\zeta\omega_n t}\left[-\omega_n\sqrt{1-\zeta^2}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) + \zeta\omega_n\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \\ &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right). \end{aligned}$$

General Second-Order Systems: Transient Response Specifications: Peak Time

Note that $c'(t) = 0$ implies that $\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) = 0$. The first zero crossing after $t = 0$ is when $t = T_p$ and

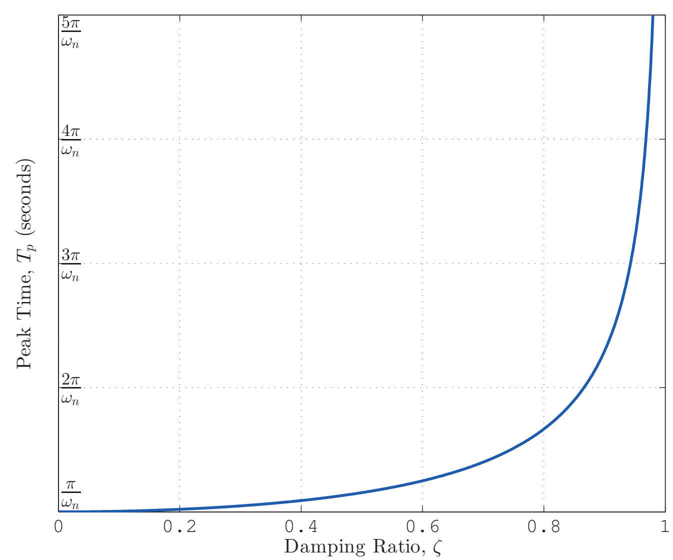
$$\omega_n\sqrt{1-\zeta^2}T_p = \pi, \text{ or } \boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}.}$$

For the undamped $(\zeta = 0)$ step response, the peak time, T_p, is given by

$$T_p = \left.\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right|_{\zeta=0} = \boxed{\frac{\pi}{\omega_n}.}$$

For the critically-damped $(\zeta = 1)$ and overdamped $(\zeta > 1)$ step responses, the peak time, T_p, makes no sense as the response never achieves a peak. It can therefore (for limiting purposes) be considered to be $T_p = \infty$.

General Second-Order Systems: Transient Response Specifications: Peak Time



General Second-Order Systems: Transient Response Specifications: Percent Overshoot

For the underdamped $(0 < \zeta < 1)$ step response, the percent overshoot, $\%OS$, is found given by

$$\boxed{\%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} \times 100\%.}$$

The term c_{max} is found by evaluating $c(t)$ at the peak time $c(T_p)$, or

$$\begin{aligned} c_{\text{max}} = c(T_p) &= 1 - e^{-\left[\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right]}\left[\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\pi)\right] \\ &= 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}. \end{aligned}$$

For a unit-step input, $c_{\text{final}} = 1$ and

$$\boxed{\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%.}$$

Notice that the percent overshoot is a function only of the damping ratio, ζ.

General Second-Order Systems: Transient Response Specifications: Percent Overshoot

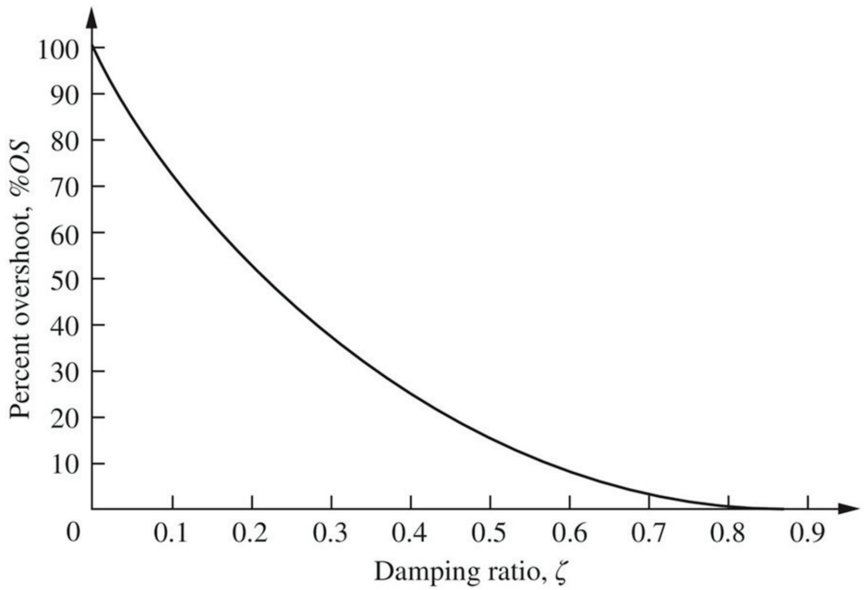
For the underdamped $(0 < \zeta < 1)$ step response, the damping ratio, ζ, can be determined from the percent overshoot, $\%OS$, by

$$\boxed{\zeta = \frac{-\ln\left(\frac{\%OS}{100\%}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100\%}\right)}}.}$$

For the undamped $(\zeta = 0)$ step response, the percent overshoot, $\%OS$, makes no sense as the response never achieves steady state. The percent overshoot can therefore (for limiting purposes) be considered to be $\%OS = 100\%$.

For the critically-damped $(\zeta = 1)$ and overdamped $(\zeta > 1)$ step responses, the response never overshoots the steady-state value. Therefore, the percent overshoot (for limiting purposes) is $\%OS = 0$.

General Second-Order Systems: Transient Response Specifications: Percent Overshoot



General Second-Order Systems: Transient Response Specifications: Settling Time

For the underdamped $(0 < \zeta < 1)$ step response, the settling time, T_s, is the time it takes for the amplitude of the decaying sinusoid

$$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right)$$

to reach 0.02, or

$$\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n T_s} = 0.02.$$

This equation is a conservative estimate, since we are assuming that $\cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right) = 1$ at the settling time.

Solving for T_s, we have

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \boxed{\frac{4}{\zeta\omega_n}.}$$

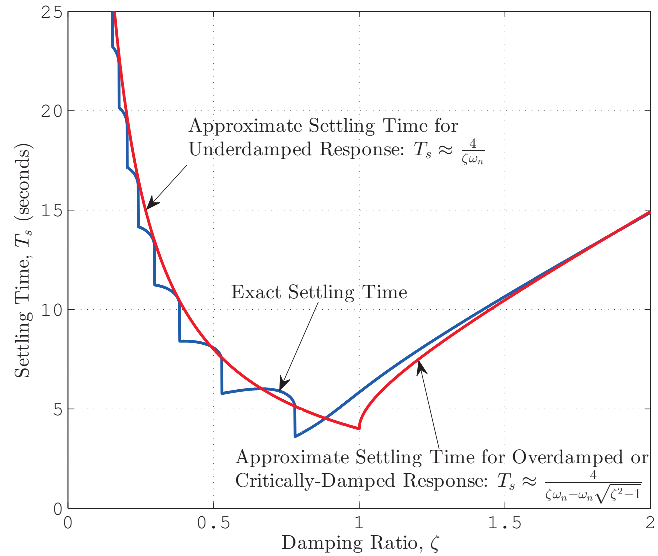
General Second-Order Systems: Transient Response Specifications: Settling Time

For the undamped $(\zeta = 0)$ step response, the settling time, T_s, makes no sense as the response never actually settles. The settling time can therefore (for limiting purposes) be considered to be $T_s = \infty$.

For the critically-damped $(\zeta = 1)$ and overdamped $(\zeta > 1)$ step responses, the previous approximation for the setting time does not work. However, since there is no oscillation, we can approximate the second-order system response using a first-order system with a pole equal to the dominant pole of the second-order system. Therefore, for critically-damped $(\zeta = 1)$ and overdamped $(\zeta > 1)$ step responses, the settling time, T_s, is approximately

$$\boxed{T_s \approx \frac{4}{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}.}$$

General Second-Order Systems: Transient Response Specifications: Settling Time



General Second-Order Systems: Transient Response Specifications: Rise Time

A precise analytical relationship between rise time, T_r, and damping ratio, ζ, cannot be found.

For the underdamped $(0 < \zeta < 1)$ step response, the rise time, T_r, can be approximated by

$$\boxed{T_r \approx \frac{1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1}{\omega_n}.}$$

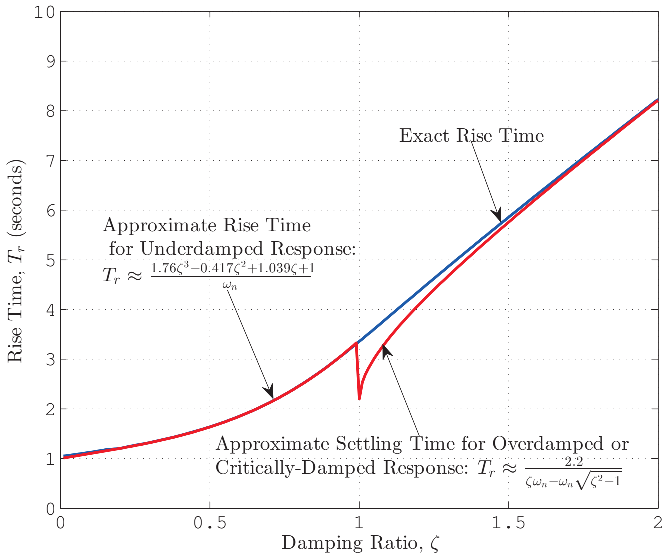
For the undamped $(\zeta = 0)$ step response, the rise time, T_r, makes no sense as the response never actually achieves a final value. The settling time can therefore (for limiting purposes) be considered to be $T_r = \infty$.

General Second-Order Systems: Transient Response Specifications: Rise Time

For the critically-damped $(\zeta = 1)$ and overdamped $(\zeta > 1)$ step responses, the previous approximation for the rise time does not work. However, since there is no oscillation, we can approximate the second-order system response using a first-order system with a pole equal to the dominant pole of the second-order system. Therefore, for critically-damped $(\zeta = 1)$ and overdamped $(\zeta > 1)$ step responses, the rise time, T_r, is approximately

$$\boxed{T_r \approx \frac{2.2}{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}.}$$

General Second-Order Systems: Transient Response Specifications: Rise Time



General Second-Order Systems: Transient Response Specifications: Example 2(a)

Determine the peak time, percent overshoot, settling time, and/or rise time for the system given by

$$G(s) = \frac{100}{s^2 + 100}.$$

General Second-Order Systems: Transient Response Specifications: Example 2(b)

Determine the peak time, percent overshoot, settling time, and/or rise time for the system given by

$$G(s) = \frac{100}{s^2 + 10s + 100}.$$

General Second-Order Systems: Transient Response Specifications: Example 2(c)

Determine the peak time, percent overshoot, settling time, and/or rise time for the system given by

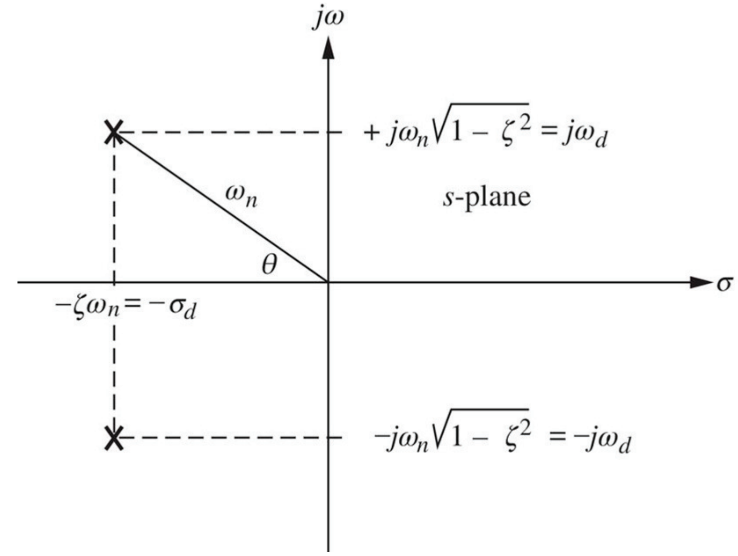
$$G(s) = \frac{100}{s^2 + 20s + 100}.$$

General Second-Order Systems: Transient Response Specifications: Example 2(d)

Determine the peak time, percent overshoot, settling time, and/or rise time for the system given by

$$G(s) = \frac{100}{s^2 + 40s + 100}.$$

General Second-Order Systems: Transient Response Specifications: Pole Locations



General Second-Order Systems: Transient Response Specifications: Pole Locations

Recall that the peak time, T_p, of the underdamped step response for a second-order system is given by

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d},$$

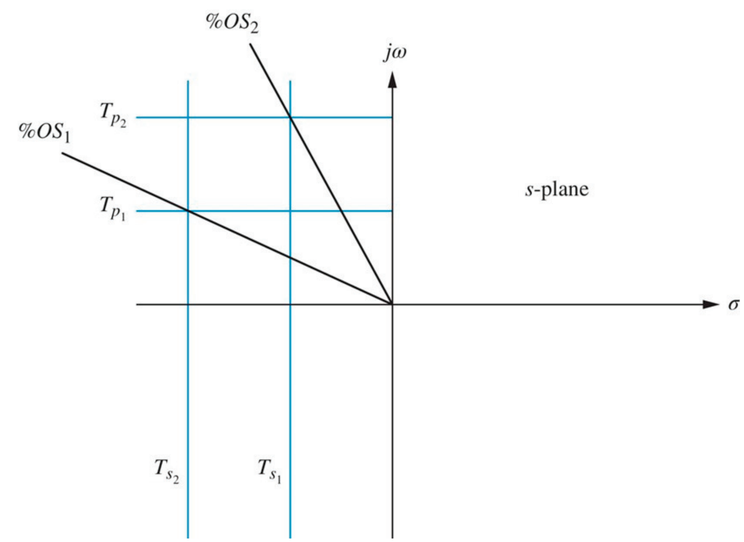
where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the imaginary part of the pole and is called the **damped frequency of oscillation**.

Also, the settling time, T_s, of the underdamped step response for a second-order system is given by

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d},$$

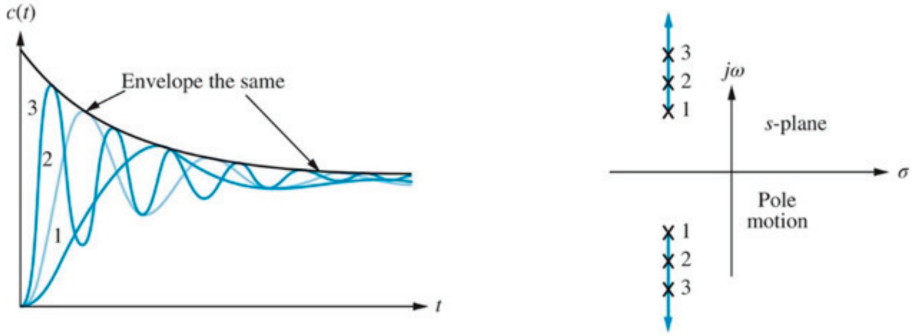
where $\sigma_d = \zeta\omega_n$ is the magnitude of the real part of the pole and is called the **exponential damping frequency**.

General Second-Order Systems: Transient Response Specifications: Pole Locations



Note that $T_{s_2} < T_{s_1}$, $T_{p_2} < T_{p_1}$, and $\%OS_1 < \%OS_2$.

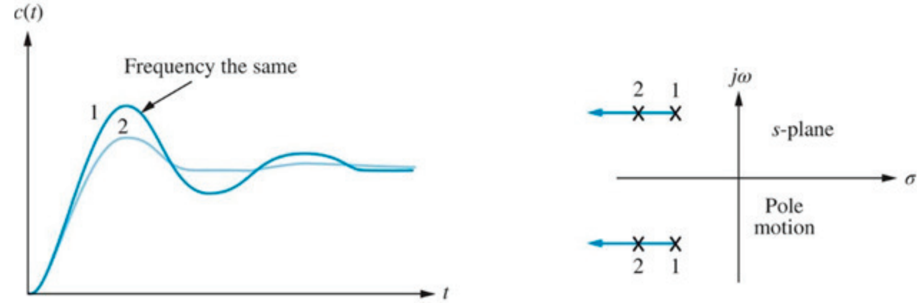
General Second-Order Systems: Transient Response Specifications: Pole Locations



Moving from 1 to 2 to 3 increases θ (and therefore decreases ζ) and increases ω_n and ω_d while keeping σ_d constant.

Therefore, moving from 1 to 2 to 3 decreases the peak time (T_p), decreases the rise time (T_r), but increases the percent overshoot $(\%OS)$ while keeping the settling time (T_s) constant.

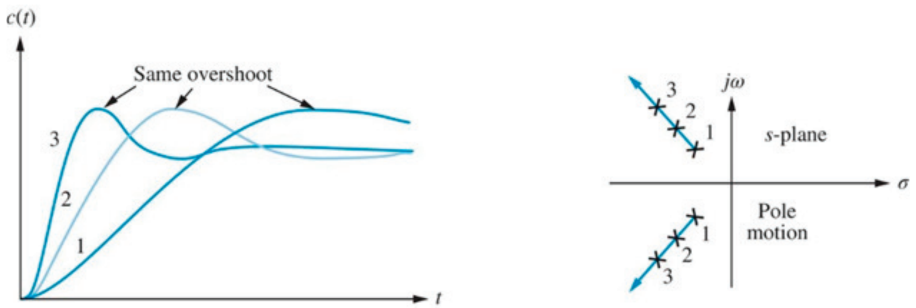
General Second-Order Systems: Transient Response Specifications: Pole Locations



Moving from 1 to 2 decreases θ (and therefore increases ζ) and increases ω_n and σ_d while keeping ω_d constant.

Therefore, moving from 1 to 2 decreases the settling time (T_s), increases the rise time (T_r), but decreases the percent overshoot $(\%OS)$ while keeping the peak time (T_p) constant.

General Second-Order Systems: Transient Response Specifications: Pole Locations



Moving from 1 to 2 to 3 increases σ_d, ω_d, and ω_n, while keeping θ (and therefore ζ) constant.

Therefore, moving from 1 to 2 to 3 decreases the settling time (T_s), decreases the rise time (T_r), and decreases the peak time (T_p) while keeping the percent overshoot $(\%OS)$ constant.

General Second-Order Systems: Transient Response Specifications: Example 3

Given poles $s_{1,2} = -3 \pm j4$, classify the step response and find the damping ratio, the natural frequency, the damped frequency of oscillation, the exponential damping frequency, peak time, percent overshoot, settling time, and/or rise time.