ELEC 312
Systems I

Root Locus Design
(Part 2)

(Adapted from Notes by Dr. Robert Barsanti)

(Images from Nise, 7^{th} Edition)

Required Reading: Chapter 9,
Control Systems Engineering

April 20, 2015

Improving Transient Response via Cascade Compensation: Lead Compensation

While the ideal derivative compensator (PD) can improve the transient response of the system, it has **two disadvantages**:

1. It requires an active circuit to perform differentiation.
2. Differentiation of high-frequency noise can lead to large unwanted signals.

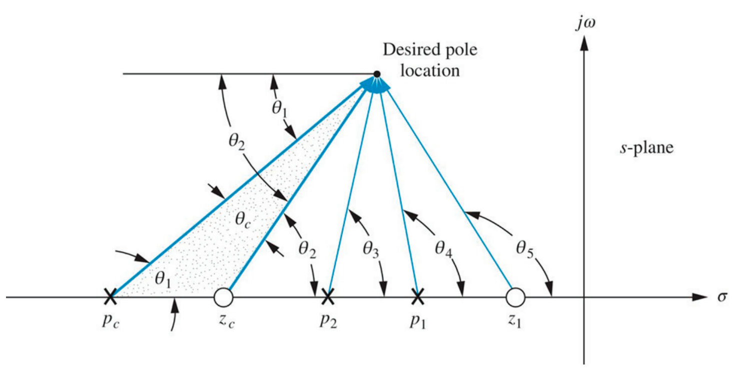
The **lead compensator** is a **passive** network used to overcome the disadvantages of ideal differentiation and retain the ability to **improve** the **transient response**.

When passive networks are used, a single zero can not be produced–rather, a compensator zero and a **compensator pole** result.

A disadvantage of lead compensation is that the additional pole prevents reduction of branches of the root locus that cross the imaginary axis into the right-half plane, while the addition of a single zero of the PD controller tends to reduce the number of branches of the root locus crossing in the right-half plane.

Improving Transient Response via Cascade Compensation: Lead Compensation

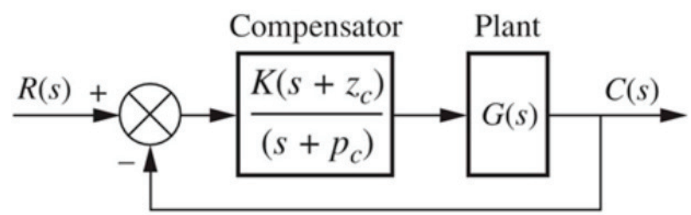
If we place the pole to the left of the zero, then the angular contribution of the compensator is still positive and thus approximates the single zero of the PD compensator.



$$\underbrace{\theta_2 - \theta_1}_{\theta_c} - \theta_3 - \theta_4 + \theta_5 = (2k+1)180^\circ$$

(The angular contribution of the pole (θ_1) subtracts from that of the zero (θ_2), but the net angular contribution θ_c remains positive (just like the ideal PD single zero).)

Improving Transient Response via Cascade Compensation: Lead Compensator Transfer Function



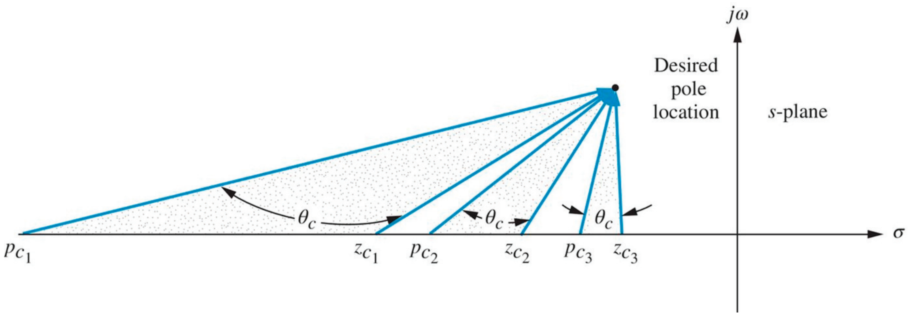
Therefore, a **lead compensator** has transfer function of form

$$G_c(s) = K\left(\frac{s+z_c}{s+p_c}\right),$$

where $-p_c < -z_c$ (i.e. pole is to the **left** of the zero).

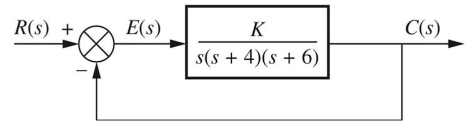
Improving Transient Response via Cascade Compensation: Lead Compensator Design

1. Select desired closed-loop pole location to meet transient response specifications.
2. Sum contributions from uncompensated poles and zeros.
3. Difference between $180°$ and sum of angles of uncompensated open-loop poles and zero is the angular contribution θ_c required of the compensator.



For design, we arbitrarily select either the lead compensator pole or zero and find the angular contribution at the design point. We can then find the required angle contribution of the remaining compensator pole or zero.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1



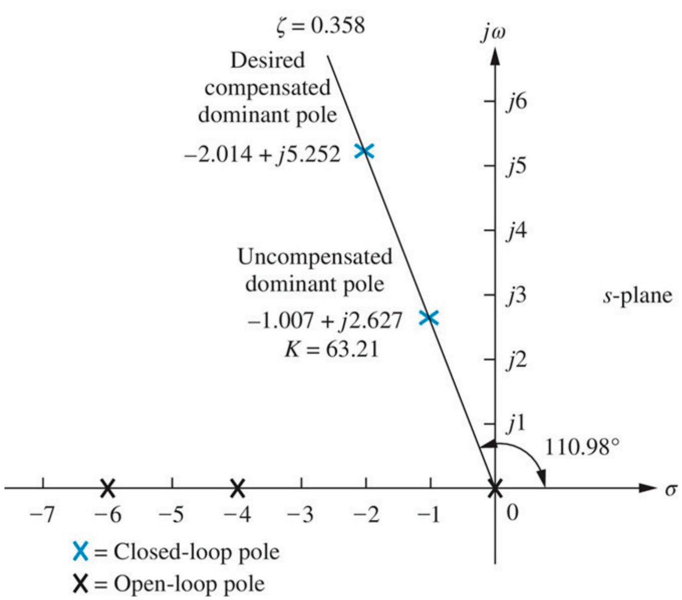
Design three lead compensators for the system above that will reduce the settling time by a factor of 2 while maintaining 30% overshoot. Compare the system characteristics between the three designs.

The value of the damping ratio ζ to yield 30% overshoot is given by

$$\zeta = \frac{-\ln\left(\frac{30\%}{100\%}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{30\%}{100\%}\right)}} = 0.358.$$

The root locus for the uncompensated system above with a 0.358-damping-ratio line is shown on the following slide.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1



Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1

The dominant, second-order pair of poles that yields $\zeta = 0.358$ is $s = -1.007 \pm j2.627$.

The settling time of the uncompensated system (assuming that the third-order system is approximately second-order) is given by

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} = \frac{4}{1.007} = 3.972 \text{ seconds.}$$

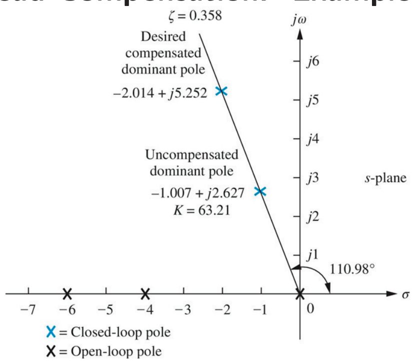
To reduce the settling time by a factor of 2 requires that the new settling time be

$$T_s = \frac{3.972 \text{ seconds}}{2} = 1.986 \text{ seconds,}$$

and the new dominant, second-order pair of poles that maintains $\zeta = 0.358$ is $s = -2.014 \pm j5.252$.

The desired compensated dominant pole is superimposed on the root locus for the uncompensated system above with a 0.358-damping-ratio line on the previous slide.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1

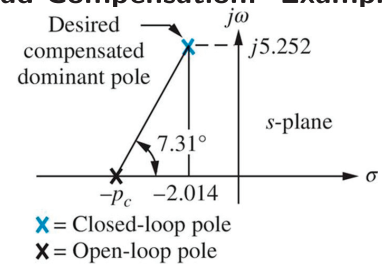


The angular contribution required of the compensator is given by

$$\theta_c = -180^\circ + \underbrace{180 - \tan^{-1}\left(\frac{5.252}{2.014}\right)}_{\text{from pole at } s=0} + \underbrace{\tan^{-1}\left(\frac{5.252}{4-2.014}\right)}_{\text{from pole at } s=-4} + \underbrace{\tan^{-1}\left(\frac{5.252}{6-2.014}\right)}_{\text{from pole at } s=-6}$$

$$= -180^\circ + 110.98^\circ + 69.29^\circ + 52.80^\circ = 53.07^\circ.$$

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1



Part (a): Assume a compensator zero at $s = -5$ $(z_c = 5)$. Note that $\theta_c = \theta_z - \theta_p$ so the location for the compensator pole is determined from

$$\theta_p = \theta_z - \theta_c = \tan^{-1}\left(\frac{5.252}{5-2.014}\right) - 53.07^\circ = 60.38^\circ - 53.07^\circ = 7.31^\circ.$$

Therefore, the location of the compensator pole when the compensator zero is at $s = -5$ is given by

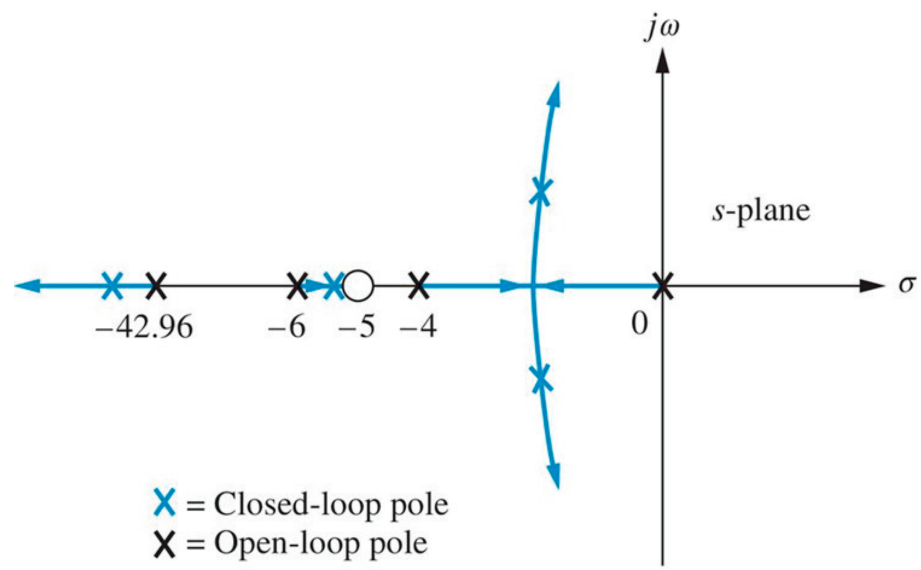
$$\tan(7.31^\circ) = \frac{5.252}{p_c - 2.014} \Rightarrow p_c = 2.014 + \frac{5.252}{\tan(7.31^\circ)} = 42.96.$$

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1

From the root locus plot (on the following slide), we note that:

- Even though the system is a fourth-order system, we can model it as approximately a second-order system due to:
  - One closed-loop pole being so far (at least 5 times the largest time constant) from the two dominant closed-loop poles.
  - One closed-loop pole being effectively canceled out by the closed-loop zero.
- It is determined that for $\zeta = 0.358$, the open-loop gain must be $K = 1423$.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1



Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1

Part (b): Assume a compensator zero at $s = -4$ ($z_c = 4$). Note that $\theta_c = \theta_z - \theta_p$ so the location for the compensator pole is determined from

$$\theta_p = \theta_z - \theta_c = \tan^{-1}\left(\frac{5.252}{4 - 2.014}\right) - 53.07^\circ = 60.38^\circ - 53.07^\circ = 16.22^\circ.$$

Therefore, the location of the compensator pole when the compensator zero is at $s = -4$ is given by

$$\tan(16.22^\circ) = \frac{5.252}{p_c - 2.014} \Rightarrow p_c = 2.014 + \frac{5.252}{\tan(16.22^\circ)} = 20.09.$$

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1

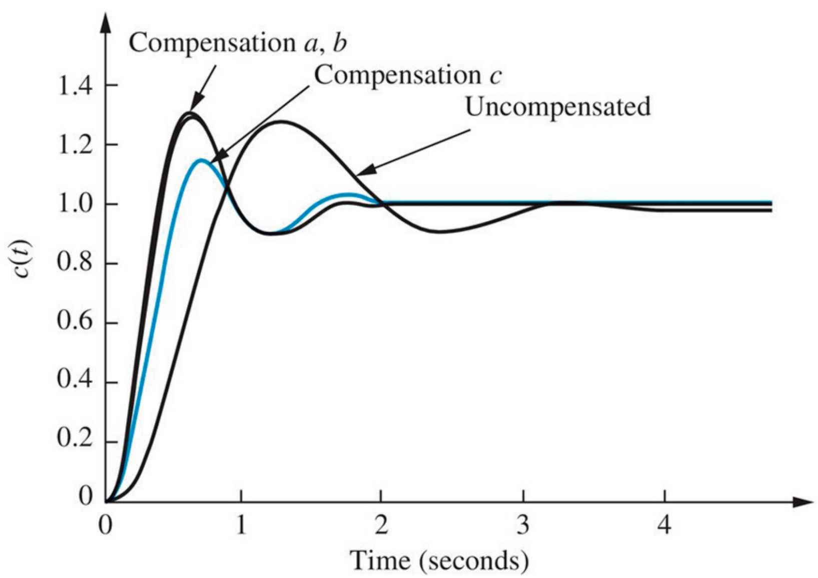
Part (c): Assume a compensator zero at $s = -2$ ($z_c = 2$). Note that $\theta_c = \theta_z - \theta_p$ so the location for the compensator pole is determined from

$$\theta_p = \theta_z - \theta_c = 180 - \tan^{-1}\left(\frac{5.252}{2.014 - 2}\right) - 53.07^\circ = 60.38^\circ - 53.07^\circ = 37.08^\circ.$$

Therefore, the location of the compensator pole when the compensator zero is at $s = -2$ is given by

$$\tan(37.08^\circ) = \frac{5.252}{p_c - 2.014} \Rightarrow p_c = 2.014 + \frac{5.252}{\tan(37.08^\circ)} = 8.97.$$

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1



Improving Transient Response via Cascade Compensation: Lead Compensation: Example 1

	Uncompensated	Compensation a	Compensation b	Compensation c
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$	$\frac{K(s+5)}{s(s+4)(s+6)(s+42.96)}$	$\frac{K(s+4)}{s(s+4)(s+6)(s+20.09)}$	$\frac{K(s+2)}{s(s+4)(s+6)(s+8.971)}$
Dominant poles	$-1.007 \pm j2.627$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$
K	63.21	1423	698.1	345.6
ζ	0.358	0.358	0.358	0.358
ω_n	2.813	5.625	5.625	5.625
$\%OS^*$	30 (28)	30 (30.7)	30 (28.2)	30 (14.5)
T_s^*	3.972 (4)	1.986 (2)	1.986 (2)	1.986 (1.7)
T_p^*	1.196 (1.3)	0.598 (0.6)	0.598 (0.6)	0.598 (0.7)
K_v	2.634	6.9	5.791	3.21
$e(\infty)$	0.380	0.145	0.173	0.312
Other poles	-7.986	$-43.8, -5.134$	-22.06	$-13.3, -1.642$
Zero	None	-5	None	-2
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK	No pole-zero cancellation

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 2

A unity-feedback system with the forward transfer function

$$G(s) = \frac{K}{s(s+7)}$$

is operating with a closed-loop step response that has 15% overshoot. Evaluate the settling time.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 2

A unity-feedback system with the forward transfer function

$$G(s) = \frac{K}{s(s+7)}$$

is operating with a closed-loop step response that has 15% overshoot. Design a lead compensator to decrease the settling time by factor of three. Choose the compensator's zero to be at -10.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 2

A unity-feedback system with the forward transfer function

$$G(s) = \frac{K}{s(s+7)}$$

is operating with a closed-loop step response that has 15% overshoot. Evaluate the settling time for your compensated system.

Improving Transient Response via Cascade Compensation: Lead Compensation: Example 2

A unity-feedback system with the forward transfer function

$$G(s) = \frac{K}{s(s+7)}$$

is operating with a closed-loop step response that has 15% overshoot. Evaluate how much improvement in settling time was realized.

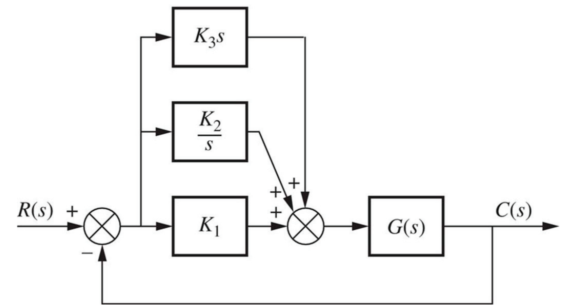
Improving Steady-State Error and Transient Response: PID Controller Design

We can design an active circuit to improve both the steady-state error and transient response for a given system by following the general steps below:

1. Improve the transient response of the uncompensated system by designing an active PD controller.
2. Improve the steady-state error of the PD-compensated system by designing an active PI controller.

The resulting compensator is called a **proportional-plus-integral-plus-derivative (PID)** controller.

Improving Steady-State Error and Transient Response: PID Controller Design



Therefore, a **PID controller** has transfer function of form

$$G_c(s) = K_1 + \frac{K_2}{s} + K_3 s = \frac{K_1 s + K_2 + K_3 s^2}{s} = \frac{K_3\left(s^2 + \frac{K_1}{K_3}s + \frac{K_2}{K_3}\right)}{s},$$

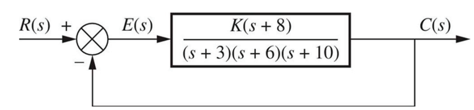
which has two zeros plus a pole at the origin. One zero and the pole at the origin can be designed as the PI compensator; the other zero can be designed as the PD compensator.

Improving Steady-State Error and Transient Response: PID Controller Design

We can design an active circuit to improve both the steady-state error and transient response for a given system by following the specific steps below:

1. Evaluate the performance of the uncompensated system to determine how much improvement in transient response is required.
2. Design the PD controller to meet the transient response specifications. The design includes the zero location and the loop gain.
3. Simulate the system to be sure all requirements have been met.
4. Redesign if the simulation shows that requirements have not been met.
5. Design the PI controller to yield the required steady-state error.
6. Determine the gains K_1, K_2, and K_3.
7. Simulate the system to be sure all requirements have been met.
8. Redesign if simulation shows that requirements have not been met.

Improving Steady-State Error and Transient Response: PID Controller Design: Example 1



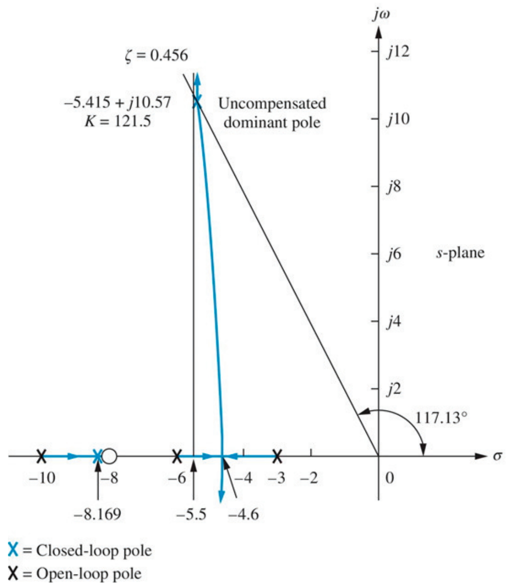
Given the system above, design a PID controller so that the system can operate with a peak time that is two-thirds that of the uncompensated system at 20% overshoot and with zero steady-state error for a step input.

Step 1: The value of the damping ratio ζ to yield 20% overshoot is given by

$$\zeta = \frac{-\ln\left(\frac{20\%}{100\%}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{20\%}{100\%}\right)}} = 0.456.$$

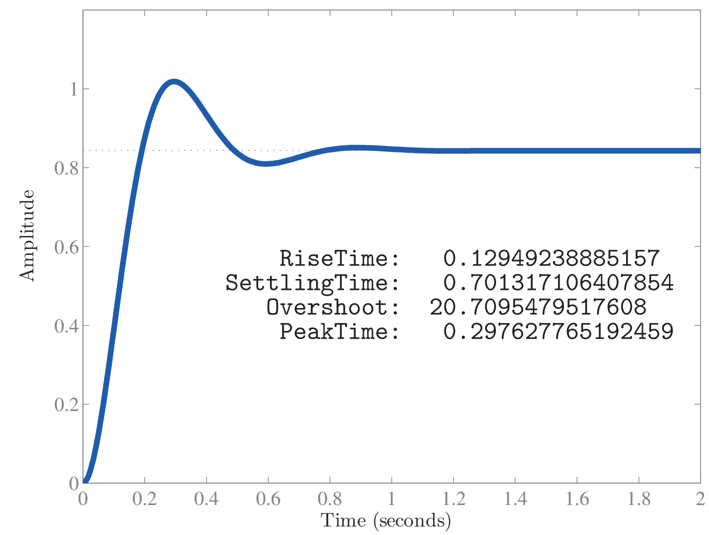
The root locus for the uncompensated system above with a 0.456-damping-ratio line is shown on the following slide.

Improving Steady-State Error and Transient Response: PID Controller Design: Example 1



Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

Step Response for $G(s) = \dfrac{121.5(s+8)}{(s+3)(s+6)(s+10)}$



Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

The dominant, second-order pair of poles that yields $\zeta = 0.456$ is $s = -5.415 \pm j10.57$.

The peak time of the uncompensated system (assuming that the third-order system is approximately second-order) is given by

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{10.57} = 0.297 \text{ seconds.}$$

Step 2: To have a new peak time that is two-thirds of the old peak time, the new peak time should be

$$T_p = \frac{2}{3}(0.297 \text{ seconds}) = 0.198 \text{ seconds,}$$

and the new dominant, second-order pair of poles that maintains $\zeta = 0.456$ is $s = -8.13 \pm j15.87$.

Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

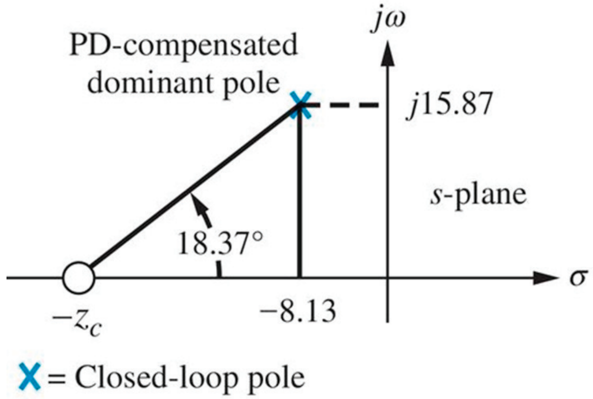
The angular contribution required of the compensator is given by

$$\theta_c = -180^\circ - \left[\underbrace{180 - \tan^{-1}\left(\frac{15.87}{8.13-8}\right)}_{\text{from zero at } s=-8}\right] + \underbrace{180 - \tan^{-1}\left(\frac{15.87}{8.13-3}\right)}_{\text{from pole at } s=-3}$$

$$+ \underbrace{180 - \tan^{-1}\left(\frac{15.87}{8.13-6}\right)}_{\text{from pole at } s=-6} + \underbrace{\tan^{-1}\left(\frac{15.87}{10-8.13}\right)}_{\text{from pole at } s=-10}$$

$$= -180^\circ - 90.47^\circ + 107.91^\circ + 97.64^\circ + 83.28^\circ = 18.37^\circ.$$

Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

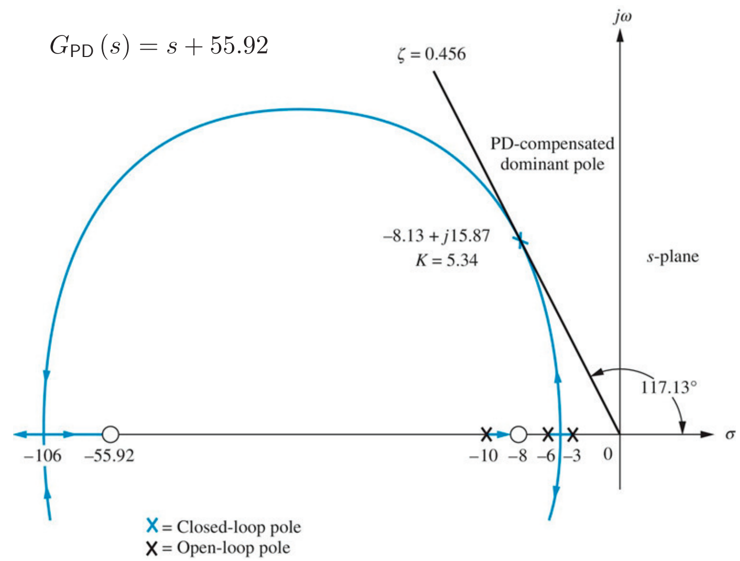


The location for the compensator zero is is given by

$$\tan(18.37°) = \frac{15.87}{z_c - 8.13} \Rightarrow z_c = 8.13 + \frac{15.87}{\tan(18.37°)} = 55.92.$$

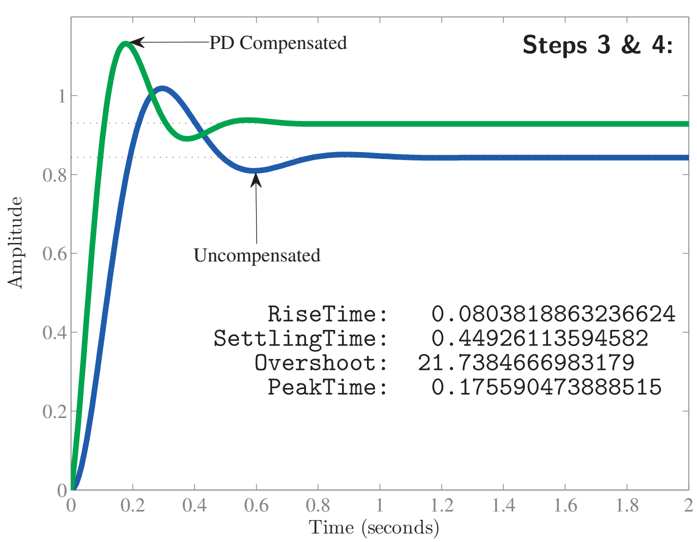
Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

$G_{\mathrm{PD}}(s) = s + 55.92$



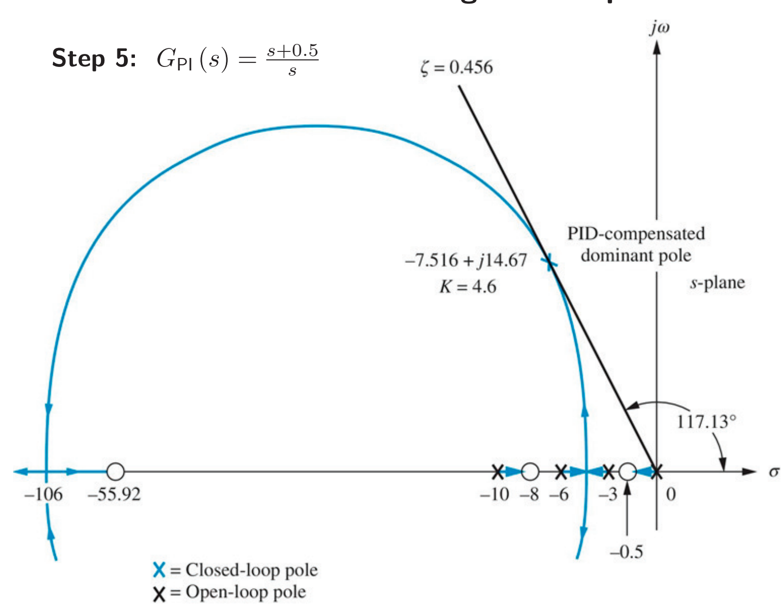
Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

PD Step Response for $G(s) = \frac{5.34(s+8)(s+55.92)}{(s+3)(s+6)(s+10)}$

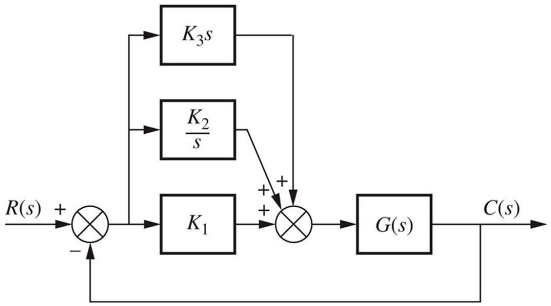


Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

Step 5: $G_{\mathrm{PI}}(s) = \frac{s+0.5}{s}$



Improving Steady-State Error and Transient Response: PID Controller Design: Example 1



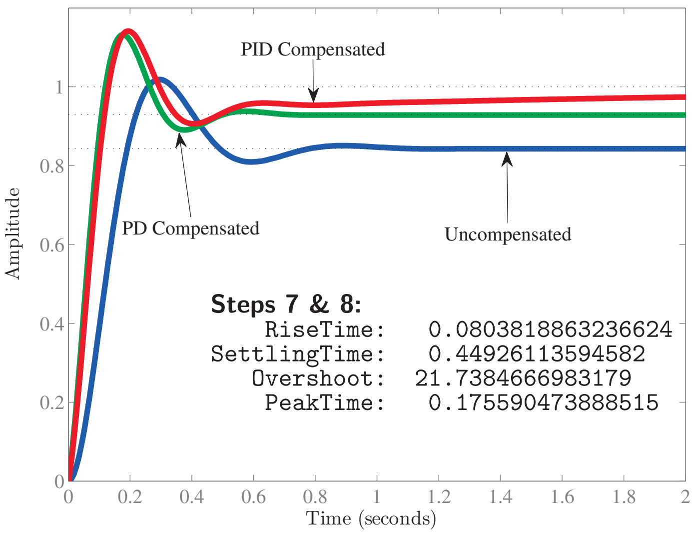
Step 6:

$$G_{\text{PID}}(s) = \frac{K(s+55.92)(s+0.5)}{s} = \frac{4.6(s+55.92)(s+0.5)}{s} =$$
$$= \frac{4.6(s^2+56.42s+27.96)}{s} = \frac{4.6s^2+259.5s+128.6}{s} = \frac{K_1 s + K_2 + K_3 s^2}{s}$$

Therefore, $K_1 = 259.5$, $K_2 = 128.6$, and $K_3 = 4.6$.

Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

PID Step Response for $G(s) = \frac{4.6(s+8)(s+55.92)(s+0.5)}{(s+3)(s+6)(s+10)s}$



Improving Steady-State Error and Transient Response: PID Controller Design: Example 1

	Uncompensated	PD-compensated	PID-compensated
Plant and compensator	$\frac{K(s+8)}{(s+3)(s+6)(s+10)}$	$\frac{K(s+8)(s+55.92)}{(s+3)(s+6)(s+10)}$	$\frac{K(s+8)(s+55.92)(s+0.5)}{(s+3)(s+6)(s+10)s}$
Dominant poles	$-5.415 \pm j10.57$	$-8.13 \pm j15.87$	$-7.516 \pm j14.67$
K	121.5	5.34	4.6
ζ	0.456	0.456	0.456
ω_n	11.88	17.83	16.49
$\%OS$	20	20	20
T_s	0.739	0.492	0.532
T_p	0.297	0.198	0.214
K_p	5.4	13.27	∞
$e(\infty)$	0.156	0.070	0
Other poles	−8.169	−8.079	−8.099, −0.468
Zeros	−8	−8, −55.92	−8, −55.92, −0.5
Comments	Second-order approx. OK	Second-order approx. OK	Zeros at −55.92 and −0.5 not canceled