ELEC 309
Signals and Systems

Complex-Domain Analysis of Continuous-Time (Chapter 3) and Discrete-Time (Chapter 4) Systems

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December 1, 2014

System Representation in the Continuous-Time Domain

Previously, we showed that the response $y(t)$ of a continuous-time LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$, or

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)$$

where

$$y(t) = x(t) * h(t).$$

System Representation in the Complex Domain

Applying the convolution property to the previous convolution equation, we have

$$X(s) \longrightarrow \boxed{\quad H(s) \quad} \longrightarrow Y(s)$$

where

$$Y(s) = X(s)H(s),$$

and $X(s)$, $Y(s)$, and $H(s)$ are the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.

The System Transfer Function

Solving for $H(s)$, we have

$$\mathcal{L}\{h(t)\} = \boxed{H(s) = \frac{Y(s)}{X(s)}} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}}.$$

The Laplace transform $H(s)$ of $h(t)$ is called the **transfer function** (or **system function**) of the system.

Note that the transfer function $H(s)$ **completely characterizes** the system because the impulse response $h(t)$ completely characterizes the system.

The System Transfer Function: Example 1

Suppose the step response of a continuous-time LTI system is given by

$$y_s(t) = \left(1 - e^{-t}\right) u(t).$$

Determine the transfer function.

The System Transfer Function: Example 1

The System Transfer Function: Example 2

Suppose the step response of a continuous-time LTI system is given by

$$y_s(t) = \left(1 - e^{-t}\right) u(t).$$

Determine the impulse response.

The System Transfer Function: Example 2

System Representation in the Discrete-Time Domain

Previously, we showed that the response $y[n]$ of a discrete-time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$, or

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

where

$$y[n] = x[n] * h[n].$$

System Representation in the Complex Domain

Applying the convolution property of the z transform to the previous convolution equation, we have

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

where

$$Y(z) = X(z)H(z),$$

and $X(z)$, $Y(z)$, and $H(z)$ are the z transforms of $x[n]$, $y[n]$, and $h[n]$, respectively.

The System Transfer Function

Solving for $H(z)$, we have

$$\mathcal{Z}\{h[n]\} = \boxed{H(z) = \frac{Y(z)}{X(z)}} = \frac{\mathcal{Z}\{y[n]\}}{\mathcal{Z}\{x[n]\}}.$$

The z transform $H(z)$ of $h[n]$ is called the **transfer function** (or **system function**) of the system.

Note that the transfer function $H(z)$ **completely characterizes** the system because the impulse response $h[n]$ completely characterizes the system.

The System Transfer Function: Example 1

Suppose the step response of a discrete-time LTI system is given by

$$y_s[n] = \left(1 - \left(\frac{1}{2}\right)^n\right) u[n].$$

Determine the transfer function.

The System Transfer Function: Example 1

The System Transfer Function: Example 2

Suppose the step response of a discrete-time LTI system is given by

$$y_s[n] = \left(1 - \left(\frac{1}{2}\right)^n\right) u[n].$$

Determine the impulse response.

The System Transfer Function: Example 2

Properties of Continuous-Time LTI Systems in Terms of the System Transfer Function

Many of the properties of continuous-time LTI systems can be closely associated with the characteristics of the system transfer function $H(s)$ in the complex plane.

Specifically, many of the properties of continuous-time LTI systems are related to the **pole locations** and the **ROC** of the system transfer function $H(s)$.

Properties of Continuous-Time LTI Systems: Causality and the System Transfer Function

Time Domain: For a causal continuous-time LTI system with impulse response $h(t)$, we have

$$h(t) = 0 \text{ for } t < 0.$$

Complex Domain: Since $h(t)$ is a **right-sided** signal, the corresponding requirement on $H(s)$ is that the ROC of $H(s)$ must be of the form

$$\text{Re}\,(s) > \sigma_{\text{max}}.$$

In other words, the ROC is the region in the complex plane to the right of all of the system poles.

Properties of Continuous-Time LTI Systems: Causality and the System Transfer Function - Example

The transfer function of a system is given by

$$H(s) = \frac{1}{s+1} \text{ with ROC} = \text{Re}(s) > -1.$$

Is this system causal?

Properties of Continuous-Time LTI Systems: Stability and the System Transfer Function

Time Domain: For a BIBO-stable continuous-time LTI system with impulse response $h(t)$, we have

$$\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty.$$

Complex Domain: The corresponding requirement on $H(s)$ is that the ROC of $H(s)$ contain the imaginary axis.

In other words, the ROC of $H(s)$ must contain the axis line corresponding to $s = j\omega$.

Properties of Continuous-Time LTI Systems: Stability and the System Transfer Function - Example

The transfer function of a system is given by

$$H(s) = \frac{1}{s+1} \text{ with ROC} = \text{Re}(s) > -1.$$

Is this system BIBO-stable?

Properties of Continuous-Time LTI Systems: Causality and Stability in the Complex Domain

For a continuous-time LTI system with transfer function $H(s)$ to be both causal and BIBO-stable, then the ROC of $H(s)$ must be of the form

$$\text{Re}\,(s) > \sigma_{\text{max}}.$$

and must contain the imaginary axis.

Therefore, $\sigma_{\text{max}} < 0$.

Therefore, for a continuous-time LTI system with transfer function $H(s)$ to be both causal and BIBO-stable, then all poles of $H(s)$ must lie outside of the ROC of $H(s)$.

Therefore, for a continuous-time LTI system with transfer function $H(s)$ to be both causal and BIBO-stable, then all poles of $H(s)$ must have Re$(s) < 0$.

In other words, for a continuous-time LTI system with transfer function $H(s)$ to be both causal and BIBO-stable, then **ALL** of the poles of $H(s)$ must lie in the **left-half of the complex plane**.

Properties of Continuous-Time LTI Systems: Causality and Stability in the Complex Domain - Example

The transfer function of a system is given by

$$H(s) = \frac{1}{s+1} \text{ with ROC} = \text{Re}(s) > -1.$$

Is this system causal and BIBO-stable?

Properties of Discrete-Time LTI Systems in Terms of the System Transfer Function

Many of the properties of discrete-time LTI systems can be closely associated with the characteristics of the system transfer function $H(z)$ in the complex plane.

Specifically, many of the properties of discrete-time LTI systems are related to the **pole locations** and the **ROC** of the system transfer function $H(z)$.

Properties of Discrete-Time LTI Systems: Causality and the System Transfer Function

Time Domain: For a causal discrete-time LTI system with impulse response $h[n]$, we have

$$h[n] = 0 \text{ for } n < 0.$$

Complex Domain: Since $h[n]$ is a **right-sided** signal, the corresponding requirement on $H(z)$ is that the ROC of $H(z)$ must be of the form

$$|z| > r_{\mathsf{max}}.$$

In other words, the ROC is the exterior of a circle containing all of the poles of $H(z)$ in the complex z-plane.

Properties of Discrete-Time LTI Systems: Causality and the System Transfer Function - Example

The transfer function of a system is given by

$$H(z) = \frac{z}{2z^2 - 3z + 1} \text{ with ROC} = |z| > 1.$$

Is this system causal?

Properties of Discrete-Time LTI Systems: Stability and the System Transfer Function

Time Domain: For a BIBO-stable discrete-time LTI system with impulse response $h[n]$, we have

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty.$$

Complex Domain: The corresponding requirement on $H(z)$ is that the ROC of $H(z)$ must contain the unit circle.

In other words, the ROC of $H(z)$ must contain the circle corresponding to $|z| = 1$.

Properties of Discrete-Time LTI Systems: Stability and the System Transfer Function - Example

The transfer function of a system is given by

$$H(z) = \frac{z}{2z^2 - 3z + 1} \text{ with ROC} = |z| > 1.$$

Is this system BIBO-stable?

Properties of Discrete-Time LTI Systems: Causality and Stability in the Complex Domain

For a discrete-time LTI system with transfer function $H(z)$ to be both causal and BIBO-stable, then the ROC of $H(z)$ must be of the form

$$|z| > r_{\text{max}}.$$

and must contain the unit circle. Therefore, $r_{\text{max}} < 1$.

In other words, for a discrete-time LTI system with transfer function $H(z)$ to be both causal and BIBO-stable, then **ALL** of the poles of $H(z)$ must lie in the **unit circle in the complex z-plane**.

Properties of Discrete-Time LTI Systems: Causality and Stability in the Complex Domain - Example

The transfer function of a system is given by

$$H(z) = \frac{z}{2z^2 - 3z + 1} \text{ with ROC} = |z| > 1.$$

Is this system causal and BIBO-stable?

Transfer Functions for Continuous-Time LTI Systems Described by LCCDEs

Previously, we considered a continuous-time LTI system for which input $x(t)$ and output $y(t)$ satisfy a general N^{th}-order linear constant-coefficient differential equation (LCCDE) given by

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

Transfer Functions for Continuous-Time LTI Systems Described by LCCDEs

Taking the Laplace transform of both sides and applying the differentiation property of the Laplace transform, we have

$$\sum_{n=0}^{N} a_n s^n Y(s) = \sum_{m=0}^{M} b_m s^m X(s).$$

or

$$Y(s) \sum_{n=0}^{N} a_n s^n = X(s) \sum_{m=0}^{M} b_m s^m.$$

Transfer Functions for Continuous-Time LTI Systems Described by LCCDEs

Therefore, a continuous-time LTI system whose input $x(t)$ and output $y(t)$ satisfy a general N^{th}-order LCCDE has a **transfer function** given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n}. \tag{1}$$

Hence, $H(s)$ is always rational. Note that the ROC of $H(s)$ is not specified by Equation 1 but must be inferred with additional requirements on the system such as the causality or the stability.

Transfer Functions for Continuous-Time LTI Systems Described by LCCDEs: Example

Determine the LCCDE of an LTI system with impulse response

$$h(t) = e^{-t}u(t).$$

Transfer Functions for Discrete-Time LTI Systems Described by Linear Constant-Coefficient Difference Equations

Previously, we considered a discrete-time LTI system for which input $x[n]$ and output $y[n]$ satisfy a general N^{th}-order linear constant-coefficient difference equation given by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

Transfer Functions for Discrete-Time LTI Systems Described by Linear Constant-Coefficient Difference Equations

Taking the z transform of both sides and applying the time-shifting and linearity properties of the z transform, we have

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z).$$

or

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{m=0}^{M} b_m z^{-m}.$$

Transfer Functions for Discrete-Time LTI Systems Described by Linear Constant-Coefficient Difference Equations

Therefore, a discrete-time LTI system whose input $x[n]$ and output $y[n]$ satisfy a general N^{th}-order linear constant-coefficient difference equation has a **transfer function** given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{2}$$

Hence, $H(z)$ is always rational. Note that the ROC of $H(z)$ is not specified by Equation 2 but must be inferred with additional requirements on the system such as the causality or the stability.

Transfer Functions for Discrete-Time LTI Systems Described by LCCDEs: Example

Determine the LCCDE of an LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

Properties of the Unilateral Laplace Transform: Example

Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by

$$y''(t) + 7y'(t) + 12y(t) = x'(t) + 2x(t)$$

with the initial conditions $y(0) = 0$ and $y'(0) = 1$.
Find $y(t)$ when $x(t) = e^{-2t}u(t)$.

Properties of the Unilateral Laplace Transform: Example

Properties of the Unilateral Laplace Transform: Example

The Unilateral Laplace Transform: The System Transfer Function

With the unilateral Laplace transform, the system transfer function

$$H(s) = \frac{Y(s)}{X(s)}$$

is defined under the condition that the LTI system meet the condition of **initial rest** (or an **initially-relaxed condition**).

In other words, the system transfer function determined using the unilateral Laplace transform is only defined for an LTI system with **all initial conditions equal to zero.**

The Unilateral Laplace Transform: The System Transfer Function - Example

Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by

$$y''(t) + 7y'(t) + 12y(t) = x'(t) + 2x(t)$$

with the initial conditions $y(0) = 0$ and $y'(0) = 1$. Determine the transfer function $H(s)$ and the impulse response $h(t)$.

The Unilateral Laplace Transform: The System Transfer Function - Example

The Unilateral Laplace Transform: The System Transfer Function - Example

Properties of the Unilateral z Transform: Example

Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] + 7y[n-1] + 12y[n-2] = x[n] + 2x[n-1]$$

with the initial conditions $y[-1] = 0$ and $y[-2] = 1$. Find $y[n]$ when $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

Properties of the Unilateral z Transform: Example

Properties of the Unilateral z Transform: Example

The Unilateral z Transform: The System Transfer Function

With the unilateral z transform, the system transfer function

$$H(z) = \frac{Y(z)}{X(z)}$$

is defined under the condition that the LTI system meet the condition of **initial rest** (or an **initially-relaxed condition**).

In other words, the system transfer function determined using the unilateral z transform is only defined for an LTI system with **all initial conditions equal to zero.**

The Unilateral z Transform: The System Transfer Function - Example

Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] + 7y[n-1] + 12y[n-2] = x[n] + 2x[n-1]$$

with the initial conditions $y[-1] = 0$ and $y[-2] = 1$. Determine the transfer function $H(z)$ and the impulse response $h[n]$.

The Unilateral z Transform: The System Transfer Function - Example

The Unilateral z Transform: The System Transfer Function - Example

The Unilateral Laplace Transform: Circuit Analysis

The solution for signals in an electric circuit can be found without writing an integrodifferential equation if the circuit operations and signals are represented with their unilateral Laplace transform equivalents.

In order to use this technique, we require the unilateral Laplace transform models for individual circuit elements.

The Unilateral Laplace Transform: Circuit Analysis - Signal Sources

A voltage signal $v(t)$ can written in terms of its unilateral Laplace transform, or

$$\boxed{v(t) \longleftrightarrow V(s).}$$

A current signal $i(t)$ can written in terms of its unilateral Laplace transform, or

$$\boxed{i(t) \longleftrightarrow I(s).}$$

The Unilateral Laplace Transform: Circuit Analysis - Complex Impedance for a Resistance

Using Ohm's Law, we have

$$\boxed{v(t) = Ri(t) \longleftrightarrow V(s) = RI(s).}$$

The **complex impedance** (with respect to the unilateral Laplace transform) for a resistance is given by

$$\boxed{\frac{V(s)}{I(s)} = R.}$$

The Unilateral Laplace Transform: Circuit Analysis - Complex Impedance for an Inductance

Using the current-voltage relationship for inductances, we have

$$\boxed{v(t) = L\frac{di(t)}{dt} \longleftrightarrow V(s) = sLI(s) - Li(0).}$$

If the current through the inductance is initially zero ($i(0) = 0$), then the **complex impedance** (with respect to the unilateral Laplace transform) for an inductance is given by

$$\boxed{\frac{V(s)}{I(s)} = sL.}$$

The Unilateral Laplace Transform: Circuit Analysis - Complex Impedance for a Capacitance

Using the current-voltage relationship for capacitances, we have

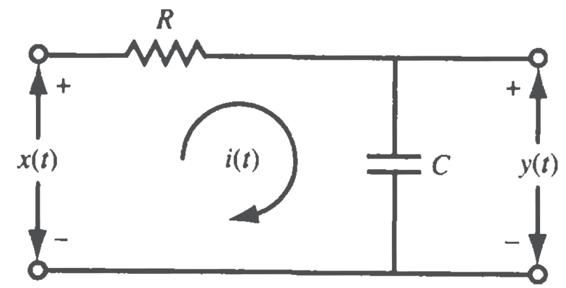
$$\boxed{i(t) = C\frac{dv(t)}{dt} \longleftrightarrow I(s) = sCV(s) - Cv(0).}$$

If the voltage across the capacitance is initially zero ($v(0) = 0$), then the **complex impedance** (with respect to the unilateral Laplace transform) for a capacitance is given by

$$\boxed{\frac{V(s)}{I(s)} = \frac{1}{sC}.}$$

The Unilateral Laplace Transform: Circuit Analysis - Example

Find the output $y(t)$, given $R = 500\text{k}\Omega$, $C = 2\mu$F, an initial capacitor voltage of $y(0) = 2$ V, and an input $x(t) = u(t)$.



The Unilateral Laplace Transform: Circuit Analysis - Example

The Unilateral Laplace Transform: Circuit Analysis - Example

Transfer Functions for Continuous-Time LTI Systems in Series

For two continuous-time LTI systems with impulse response $h_1(t)$ and $h_2(t)$ in **series** (or **cascade**),

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

the overall impulse response $h(t)$ is given by

$$h(t) = h_1(t) * h_2(t).$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Transfer Functions for Continuous-Time LTI Systems in Series

For two continuous-time LTI systems with transfer functions $H_1(s)$ and $H_2(s)$ in **series** (or **cascade**),

$$X(s) \longrightarrow \boxed{H_1(s)} \longrightarrow \boxed{H_2(s)} \longrightarrow Y(s)$$

the overall transfer function $H(s)$ is given by

$$H(s) = H_1(s)H_2(s).$$

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

Transfer Functions for Continuous-Time LTI Systems in Series: Example

Determine the overall impulse response of two cascaded continuous-time LTI systems with impulse responses

$$h_1(t) = e^{-t}u(t) \text{ and } h_2(t) = e^{-2t}u(t).$$

Transfer Functions for Continuous-Time LTI Systems in Parallel

For two continuous-time LTI systems with impulse response $h_1(t)$ and $h_2(t)$ in **parallel**,

the overall impulse response $h(t)$ is given by

$$h(t) = h_1(t) + h_2(t).$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Transfer Functions for Continuous-Time LTI Systems in Parallel

For two continuous-time LTI systems with transfer functions $H_1(s)$ and $H_2(s)$ in **parallel**,

the overall transfer function $H(s)$ is given by

$$H(s) = H_1(s) + H_2(s).$$

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

Transfer Functions for Continuous-Time LTI Systems in Parallel: Example

Determine the overall impulse response of a parallel combination of two continuous-time LTI systems with impulse responses

$$h_1(t) = e^{-t}u(t) \text{ and } h_2(t) = e^{-2t}u(t).$$

Transfer Functions for Discrete-Time LTI Systems in Series

For two discrete-time LTI systems with impulse response $h_1[n]$ and $h_2[n]$ in **series** (or **cascade**),

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$

the overall impulse response $h[n]$ is given by

$$h[n] = h_1[n] * h_2[n].$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Transfer Functions for Discrete-Time LTI Systems in Series

For two discrete-time LTI systems with transfer functions $H_1(z)$ and $H_2(z)$ in **series** (or **cascade**),

$$X(z) \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow Y(z)$$

the overall transfer function $H(z)$ is given by

$$H(z) = H_1(z)H_2(z) \text{ with ROC} = R \supset R_1 \cap R_2.$$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

Transfer Functions for Discrete-Time LTI Systems in Series: Example

Determine the overall impulse response of two cascaded discrete-time LTI systems with impulse responses

$$h_1[n] = u[n] \text{ and } h_2[n] = \left(\frac{1}{2}\right)^n u[n]$$

Transfer Functions for Discrete-Time LTI Systems in Parallel

For two discrete-time LTI systems with impulse response $h_1[n]$ and $h_2[n]$ in **parallel**,

the overall impulse response $h[n]$ is given by

$$h[n] = h_1[n] + h_2[n].$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Transfer Functions for Discrete-Time LTI Systems in Parallel

For two discrete-time LTI systems with transfer functions $H_1(z)$ and $H_2(z)$ in **parallel**,

the overall transfer function $H(z)$ is given by

$$H(z) = H_1(z) + H_2(z) \text{ with ROC} = R \supset R_1 \cap R_2.$$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

Transfer Functions for Discrete-Time LTI Systems in Parallel: Example

Determine the overall impulse response of a parallel combination of two discrete-time LTI systems with impulse responses

$$h_1[n] = u[n] \text{ and } h_2[n] = \left(\frac{1}{2}\right)^n u[n]$$