ELEC 309
Signals and Systems

Background:
Complex Analysis
Appendices D and E,
Schaum's Outline of Signals and Systems

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Cartesian or Rectangular Form of Complex Numbers

The complex number z in Cartesian or rectangular form is expressed as

$$z = a + jb$$

where $j = \sqrt{-1}$, a is a real number referred to as the *real part* of z, and b is a real number referred to as the *imaginary part* of z.

a and b are often expressed as

$$a = \text{Re}\{z\} \qquad\qquad b = \text{Im}\{z\}$$

where "Re" denotes the "real part of" and "Im" denotes the "imaginary part of".

Polar or Exponential Form of Complex Numbers

The complex number z in polar or exponential form is expressed as

$$z = re^{j\theta} = r\angle\theta$$

where $r > 0$ is the *magnitude* of z and θ is the *angle* or *phase* of z. These quantities are often written as

$$r = |z| \qquad\qquad \theta = \angle z$$

The units for the angle or phase θ are in degrees ($^\circ$) or radians.

WARNING: Units of angle or phase without the degree symbol $^\circ$ are assumed to be in radians.

Representation of a Number in the Complex Plane

From Polar to Rectangular: Euler's Formulas

Euler's formulas are given by

$$\begin{aligned} e^{\pm j\theta} &= \cos(\theta) \pm j\sin(\theta) \\ \cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{j2} \end{aligned}$$

From Polar to Rectangular

We can convert the complex number $z = re^{j\theta} = r\angle\theta$ from polar form to rectangular form $z = a + jb$ via the relationships

$$\begin{aligned} a &= r\cos(\theta) \\ b &= r\sin(\theta). \end{aligned}$$

Example:

Convert $z_1 = 2\sqrt{2}e^{j\pi/4}$ to rectangular form.

From Rectangular to Polar

We can convert the complex number $z = a + jb$ from rectangular form to polar form $z = re^{j\theta} = r\angle\theta$ via the relationships

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right). \end{aligned}$$

WARNING: Using the $\tan^{-1}$ Function on Electronic Calculators

Example:

Convert $z_2 = -3 - j3$ to polar form.

Complex Addition

We can add the complex numbers

$$z_1 = a_1 + jb_1 = r_1 e^{j\theta_1} = r_1 \angle \theta_1$$

and

$$z_2 = a_2 + jb_2 = r_2 e^{j\theta_2} = r_2 \angle \theta_2$$

easily as long as both complex quantities are in **rectangular form** using

$$z_1 + z_2 = (a_1 + a_2) + j\,(b_1 + b_2)\,.$$

Example:

Determine $z_1 + z_2$ if $z_1 = 2\sqrt{2}e^{j\pi/4}$ and $z_2 = -3 - j3$.

Complex Subtraction

We can subtract the complex numbers

$$z_1 = a_1 + jb_1 = r_1 e^{j\theta_1} = r_1 \angle \theta_1$$

and

$$z_2 = a_2 + jb_2 = r_2 e^{j\theta_2} = r_2 \angle \theta_2$$

easily as long as both complex quantities are in **rectangular form** using

$$z_1 - z_2 = (a_1 - a_2) + j\,(b_1 - b_2)\,.$$

Example:

Determine $z_1 - z_2$ if $z_1 = 2\sqrt{2}e^{j\pi/4}$ and $z_2 = -3 - j3$.

Complex Multiplication

We can multiply the complex numbers

$$z_1 = a_1 + jb_1 = r_1 e^{j\theta_1} = r_1 \angle \theta_1$$

and

$$z_2 = a_2 + jb_2 = r_2 e^{j\theta_2} = r_2 \angle \theta_2$$

easily in **polar form** using

$$z_1 z_2 = (r_1 r_2)\, e^{j(\theta_1 + \theta_2)}$$

and (not-so-easily) in rectangular form using

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + j\,(a_1 b_2 + b_1 a_2)\,.$$

Example:

Determine $z_1 z_2$ if $z_1 = 2\sqrt{2}e^{j\pi/4}$ and $z_2 = -3 - j3$.

Complex Division

We can divide the complex numbers

$$z_1 = a_1 + jb_1 = r_1 e^{j\theta_1} = r_1 \angle \theta_1$$

and

$$z_2 = a_2 + jb_2 = r_2 e^{j\theta_2} = r_2 \angle \theta_2$$

easily in **polar form** using

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1 - \theta_2)}$$

and (not-so-easily) in rectangular form using

$$\frac{z_1}{z_2} = \frac{(a_1 a_2 + b_1 b_2) + j\left(-a_1 b_2 + b_1 a_2\right)}{a_2^2 + b_2^2}.$$

Example:

Determine $\frac{z_1}{z_2}$ if $z_1 = 2\sqrt{2}e^{j\pi/4}$ and $z_2 = -3 - j3$.

Complex Conjugate

The *complex conjugate* of

$$z = a + jb = re^{j\theta} = r\angle\theta$$

is denoted as z^* and is given by

$$z^* = a - jb = re^{-j\theta} = r\angle - \theta.$$

Example:

Determine z_1^* and z_2^* if $z_1 = 2\sqrt{2}e^{j\pi/4}$ to $z_2 = -3 - j3$.

Useful Relationships of Complex Conjugates

1. $zz^* = r^2$
2. $\frac{z}{z^*} = e^{j2\theta}$
3. $z + z^* = 2\text{Re}\left\{z\right\}$
4. $z - z^* = j2\text{Im}\left\{z\right\}$
5. $(z_1 + z_2)^* = z_1^* + z_2^*$
6. $(z_1 z_2)^* = z_1^* z_2^*$
7. $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

Powers of Complex Numbers

The n^{th} power of the complex number $z = re^{j\theta}$ is given by

$$z^n = r^n e^{jn\theta} = r^n \left(\cos\left(n\theta\right) + j\sin\left(n\theta\right)\right).$$

This gives us the relationship know as **DeMoivre's relation**:

$$\left(\cos\left(\theta\right) + j\sin\left(\theta\right)\right)^n = \cos\left(n\theta\right) + j\sin\left(n\theta\right).$$

Example:

Determine z_1^4 if $z_1 = 2\sqrt{2}e^{j\pi/4}$.

Roots of Complex Numbers

The n^{th} root of the complex number z is the number w such that

$$w^n = z = re^{j\theta}.$$

Thus, to find the n^{th} root of the complex number z, we must solve

$$w^n - re^{j\theta} = 0,$$

which is an equation of degree n and hence has n roots. These roots are given by

$$w_k = \sqrt[n]{r}\exp\left(j\frac{\theta + 2(k-1)\pi}{n}\right) \text{ for } k = 1, 2, \ldots, n.$$

Example:

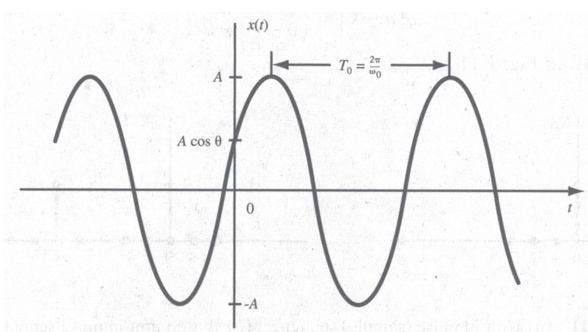
Determine $\sqrt{z_1}$ if $z_1 = 2\sqrt{2}e^{j\pi/4}$.

Sinusoidal Signals

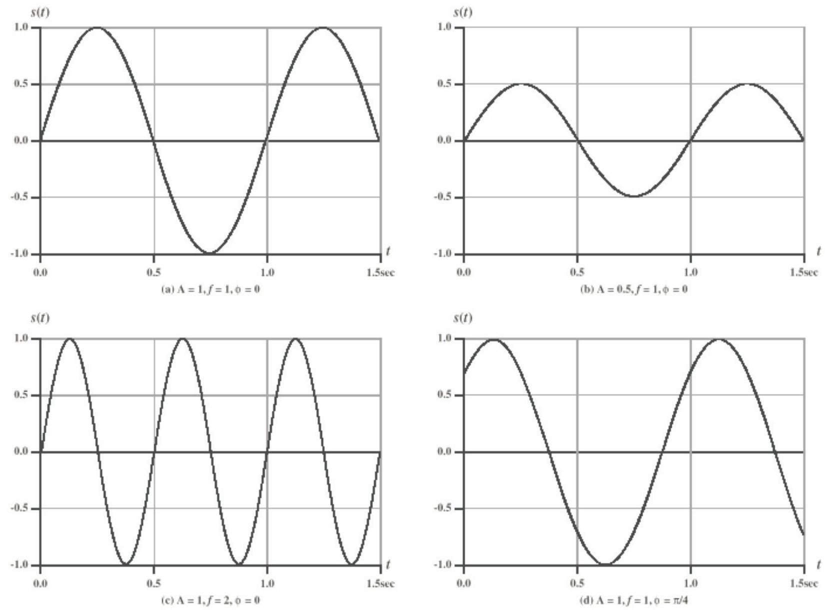
A *sinusoidal* signal can be expressed as

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A is the (real) *amplitude*, ω_0 is the *radian* or *angular frequency* in radians per second, and θ is the *phase angle* in radians.



Sinusoidal Signals



Adding Sinusoidal Signals

Convert $43\cos(61t) + 47\sin(61t)$ into the form $A\cos(\omega_0 t + \phi)$.

Adding Sinusoidal Signals

Convert $43\cos(61t) + 47\sin(61t)$ into the form $A\sin(\omega_0 t + \phi)$.

Sinusoidal Signals

The **sinusoidal** signal $x(t) = A\cos(\omega_0 t + \theta)$ is periodic with *fundamental period*

$$T_0 = \frac{2\pi}{\omega_0} \text{ seconds.}$$

The reciprocal of the fundamental period T_0 is called the *fundamental frequency* f_0 given by

$$f_0 = \frac{1}{T_0} \text{ hertz (Hz).}$$

The *fundamental angular frequency* ω_0 is given by

$$\omega_0 = 2\pi f_0 \text{ radians per second.}$$

Real or Monotonic Exponential Signals

A *real* or *monotonic exponential* signal can be expressed as

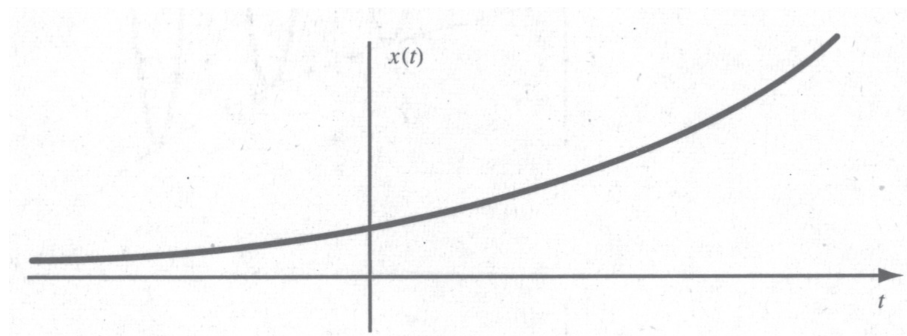
$$x(t) = e^{\sigma t},$$

where σ is a real number.

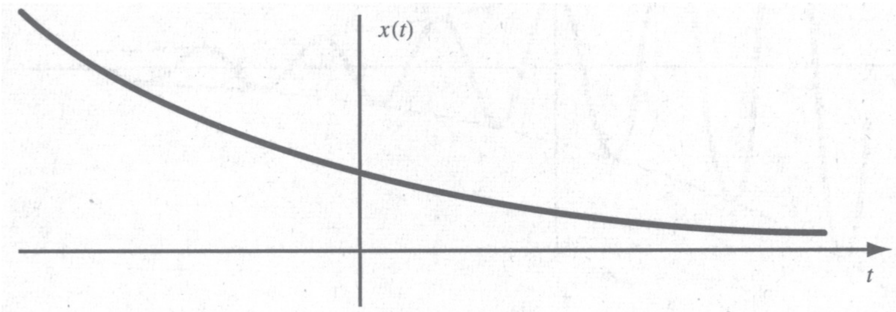
If $\sigma > 0$, then $x(t)$ is a *growing* exponential.

If $\sigma < 0$, then $x(t)$ is a *decaying* exponential.

Sketching Real or Monotonic Exponentials for $\sigma > 0$



Sketching Real or Monotonic Exponentials for $\sigma < 0$



Complex Exponentials

A *complex exponential* signal can be expressed as

$$x(t) = e^{j\omega_0 t}.$$

Using Euler's formula, this signal can be defined as A *complex exponential* signal can be expressed as

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

The *fundamental period* for a complex exponential comes from the underlying sinusoidal signals and is given by

$$T_0 = \frac{2\pi}{\omega_0} \text{ seconds.}$$

Sinusoids vs. Complex Exponentials

Using Euler's formula, the **sinusoidal** signal $x(t) = A\cos(\omega_0 t + \theta)$ can be written as

$$x(t) = A\cos(\omega_0 t + \theta) = A\text{Re}\left\{e^{j(\omega_0 t + \theta)}\right\}.$$

Notice also that Euler's formula gives us the relationship:

$$A\text{Im}\left\{e^{j(\omega_0 t + \theta)}\right\} = A\sin(\omega_0 t + \theta).$$

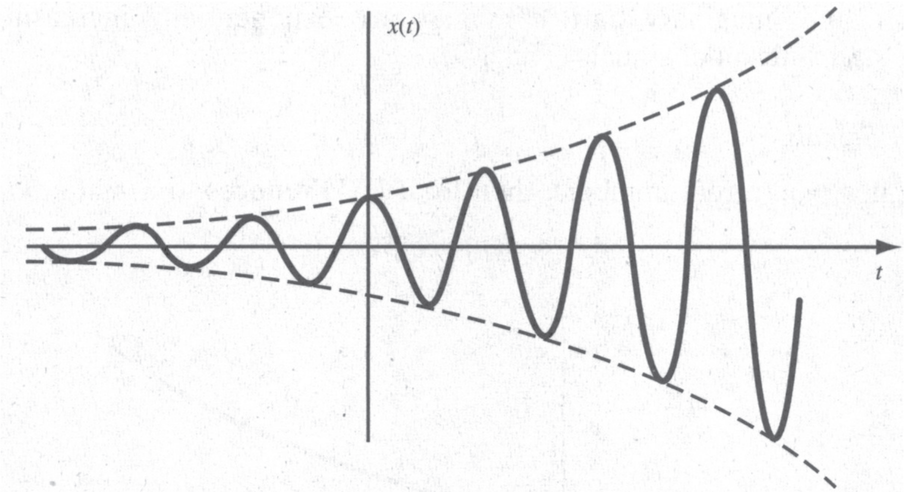
General Complex Exponentials

A *general complex exponential* signal can be expressed as

$$x(t) = e^{(\sigma + j\omega)t} = e^{\sigma t}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

where the real part $e^{\sigma t}\cos(\omega t)$ and imaginary part $e^{\sigma t}\sin(\omega t)$ are exponentially growing $\sigma > 0$ or exponentially decaying $\sigma < 0$ sinusoidal signals.

Sketching General Complex Exponentials for $\sigma > 0$



Sketching General Complex Exponentials for $\sigma < 0$