Lecture 23: The Heat Equation

Meowth's Goals for the Day

- Introduce product solutions for BVPs
- Derive Fourier's solution to the Heat Equation
- Give an automatic F to any slackers who skipped class today to study for their Signals exam

13.3 The Heat Equation

Superposition Principle

For any linear DE, if u_1 and u_2 are both solutions, then so is the sum

$$u_1 + u_2.$$

Last week, we showed how to obtain product solutions to a PDE.

3 Possible Solutions: u_1, u_2, u_3

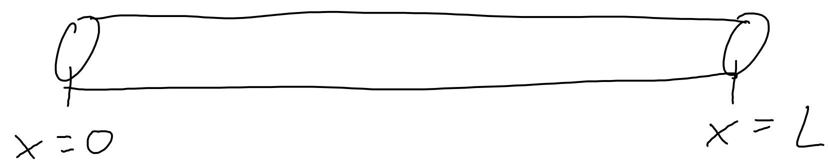
General Solution: $u_1 + u_2 + u_3$

When we impose conditions (ICs/BCs), we will see that 2/3 solutions disappear.

$$u = \underset{0}{\cancel{u_1}} + \underset{0}{\cancel{u_2}} + u_3 = u_3$$

Heat Equation

A metal bar of length L with thermal diffusivity constant k.



Temperature $u(x,t)$

$$u_t = k\, u_{xx}$$

IC : Assume initial temperature is $f(x)$

$$u(x,0) = f(x) \qquad 0 < x < L$$

BC : Assume ends are fixed at 0°C

$$u(0,t) = 0$$
$$u(L,t) = 0$$

Dirichlet BCs

Seek a separable solution

$$u(x,t) = w(x)\, v(t)$$

Plug this into the PDE.

$$u_t = k\, u_{xx}$$

$$(wv)_t = k\,(wv)_{xx}$$

$$w\, v_t = k v\, w_{xx}$$

$$\underbrace{\frac{v_t}{kv}}_{\text{Function of } t} = \underbrace{\frac{w_{xx}}{w}}_{\text{Function of } x} = -\lambda$$

λ → separation constant

3 Cases: λ is zero, negative, or positive

① $\lambda = 0$

$$\frac{v_t}{kv} = 0 \qquad\qquad \frac{w_{xx}}{w} = 0$$

$$\int v_t = \int 0 \qquad\qquad \int w_{xx} = \int 0$$

$$\int w_x = \int C_2$$

$$v = C_1 \qquad\qquad w = C_2 x + C_3$$

$$u_1 = vw = (C_1)(C_2 x + C_3)$$

$$= Ax + B$$

BC: $u(0,t) = 0$, $u(L,t) = 0$

$x=0$: $u_1(0,t) = B = 0 \Rightarrow B = 0$

$x=L$: $u_1(L,t) = AL + \underbrace{B}_{0} = 0 \Rightarrow A = 0$

$u_1 = 0$ Trivial!

② $\lambda = -\alpha^2$

$$\frac{v_t}{kv} = \alpha^2$$

$$v_t = \alpha^2 k v$$

$$\frac{dv}{dt} = \alpha^2 k v$$

$$\int \frac{1}{v}\, dv = \int \alpha^2 k\, dt$$

$$\ln v = \alpha^2 k t + C_1$$

$$v = e^{\alpha^2 k t + C_1}$$

$$v = C_2 e^{\alpha^2 k t}$$

$$\frac{w_{xx}}{w} = \alpha^2$$

$$w_{xx} = \alpha^2 w$$

$$w_{xx} - \alpha^2 w = 0$$

$$\Downarrow$$

$$r^2 - \alpha^2 = 0$$

$$r^2 = \alpha^2$$

$$r = \pm \alpha$$

$$w = C_3 e^{\alpha x} + C_4 e^{-\alpha x}$$

$$u_2 = vw = \left(C_2 e^{\alpha^2 kt}\right)\left(C_3 e^{\alpha x} + C_4 e^{-\alpha x}\right)$$

$$= e^{\alpha^2 kt}\left[A e^{\alpha x} + B e^{-\alpha x}\right]$$

$\underline{BC}$: $u(0,t) = 0$, $u(L,t) = 0$

$x = 0$: $u_2(0,t) = e^{\alpha^2 kt}\left[A \cancelto{1}{e^0} + B \cancelto{1}{e^0}\right] = 0$

$$A + B = 0$$

$$A = -B$$

$x = L$: $u_2(L,t) = e^{\alpha^2 kt}\left[A e^{\alpha L} + B e^{-\alpha L}\right] = 0$

$A = -B \rightarrow$ $-B e^{\alpha L} + B e^{-\alpha L} = 0$

$$B\left(-e^{\alpha L} + e^{-\alpha L}\right) = 0$$

$$B = 0$$

$$\Rightarrow A = 0$$

$u_2(x,t) = 0$ Trivial!

(3) $\lambda = \alpha^2$

$$\frac{v_t}{kv} = -\alpha^2$$

$$v_t = -\alpha^2 k v$$

$$\frac{dv}{dt} = -\alpha^2 k v$$

$$\int \frac{1}{v}\, dv = \int -\alpha^2 k \, dt$$

$$\ln v = -\alpha^2 k t + C_1$$

$$v = e^{-\alpha^2 k t + C_1}$$

$$v = C_2 e^{-\alpha^2 k t}$$

$$\frac{w_{xx}}{w} = -\alpha^2$$

$$w_{xx} = -\alpha^2 w$$

$$w_{xx} + \alpha^2 w = 0$$

$$\Downarrow$$

$$r^2 + \alpha^2 = 0$$

$$r^2 = -\alpha^2$$

$$r = \pm \alpha i$$

$$w = C_3 \cos(\alpha x) + C_4 \sin(\alpha x)$$

$$u_3(x,t) = vw = \left(C_2 e^{-\alpha^2 k t}\right)\left(C_3 \cos(\alpha x) + C_4 \sin(\alpha x)\right)$$

$$= e^{-\alpha^2 k t}\left[A \cos(\alpha x) + B \sin(\alpha x)\right]$$

<u>BC:</u> $u(0,t)=0, \quad u(L,t)=0$

$$x=0: \quad u_3(0,t)=e^{-\alpha^2 kt}\left[A\cancelto{1}{\cos(0)}+B\cancelto{0}{\sin(0)}\right]=0$$

$$e^{-\alpha^2 kt}\left[A\right]=0$$

$$A=0$$

$$x=L: \quad u_3(L,t)=e^{-\alpha^2 kt}\left[B\sin(\alpha L)\right]=0$$

$B=0$ or $\sin(\alpha L)=0$

$\downarrow$

Trivial

$$\alpha L = n\pi$$

$$\alpha=\frac{n\pi}{L} \quad \leftarrow \text{Eigenvalues}$$

$$u_n = vw = e^{-\left(\frac{n\pi}{L}\right)^2 kt} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$\nwarrow$ Eigenfunctions

By the Superposition Principle, the general solution is

$$u=\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt}\sin\left(\frac{n\pi x}{L}\right)$$

$\underline{\text{IC:}}$ $u(x,0) = f(x)$

$t=0$: $u(x,0) = \sum_{n=1}^{\infty} B_n \cancel{e}^{0} \sin\left(\frac{n\pi x}{L}\right) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{on } (0,L)$$

Fourier Sine Series on $(0,L)$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier's Solution to the Heat Equation

$$u_t = k u_{xx}$$

$$u(x,0) = f(x) \qquad \text{for } 0<x<L$$

$$u(0,t) = u(L,t) = 0$$

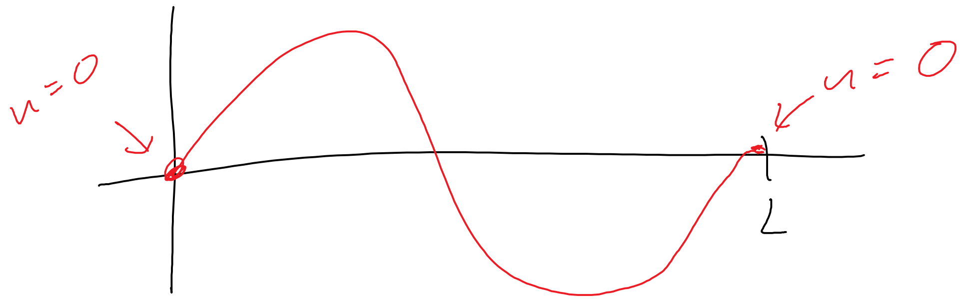
The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

where $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Why did we end up with a sine series?

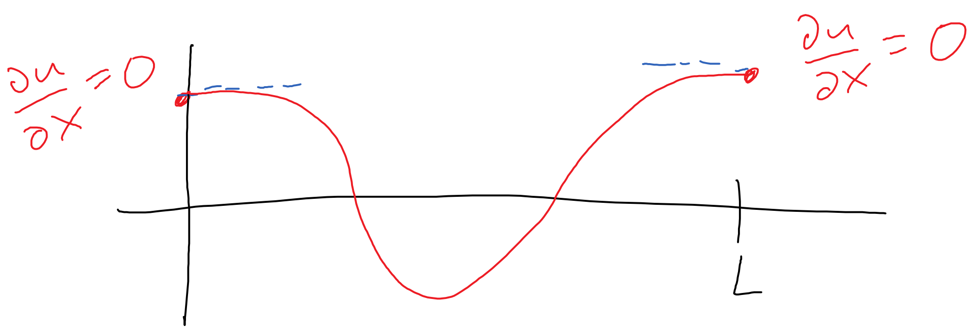
BC: $u(0,t) = u(L,t) = 0$



What if we change to Neumann BCs?

BC: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

The ends of the bar are insulated.



For Neumann BCs, use a Fourier Cosine Series.

Look at Practice Exam 3 #4