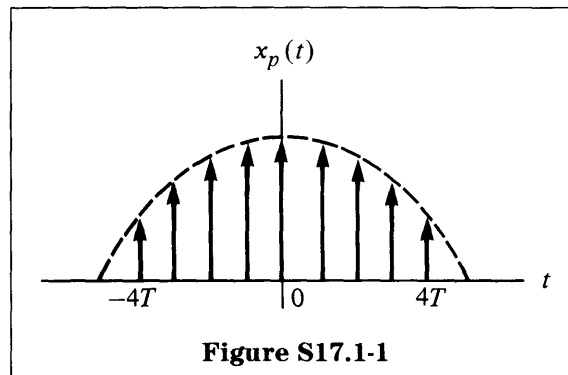


17 Interpolation

Solutions to Recommended Problems

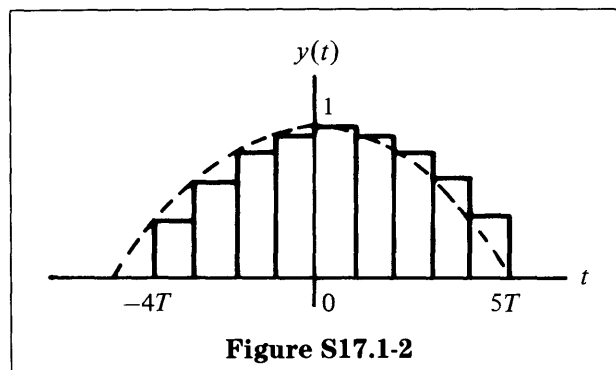
S17.1

It is more convenient to solve this problem in the time domain than in the frequency domain. Since $x_p(t) = x(t)p(t)$ and $p(t)$ is an impulse train, $x_p(t)$ is a sampled version of $x(t)$, as shown in Figure S17.1-1.



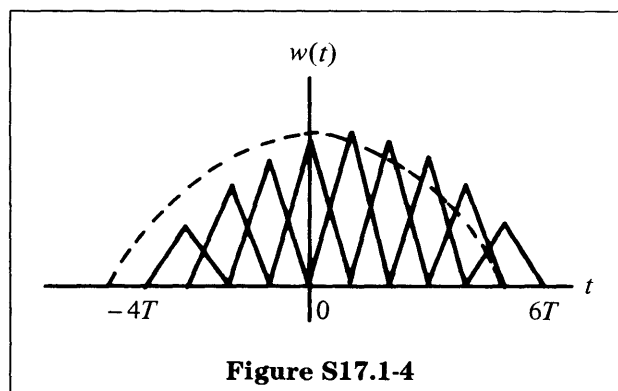
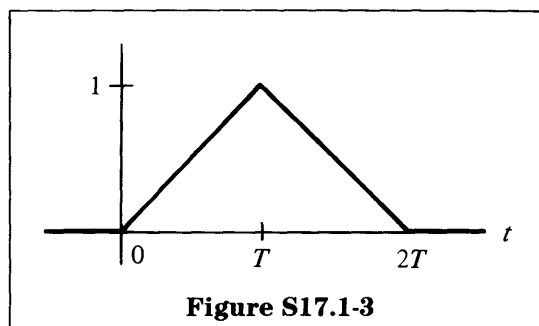
$$\begin{aligned}x_p(t) &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\&= \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)\end{aligned}$$

Since $y(t) = x_p(t) * h(t)$ and $x_p(t)$ is impulsive, the convolution carried out in the time domain is as shown in Figure S17.1-2.

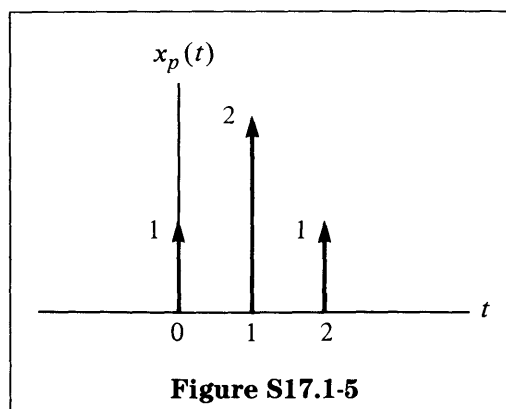


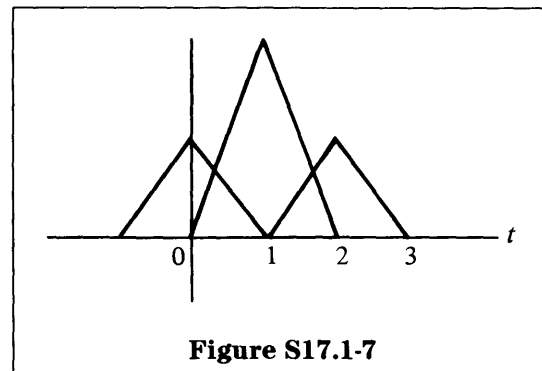
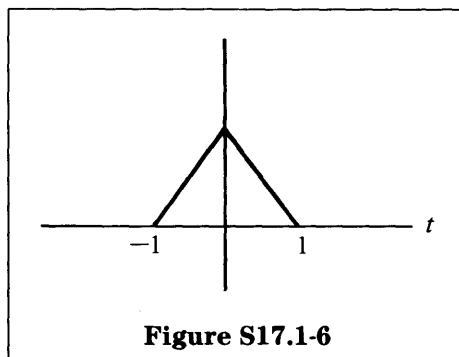
Here we have that $x(t)$ is sampled by a rectangular pulse train as opposed to an impulse train.

Since $h(t) * (1/T)h(t)$, shown in Figure S17.1-3, is wider than the sampling period T , the resultant $w(t)$ is not a triangularly sampled version of $x(t)$. $w(t)$ consists of the superposition of waveforms shown in Figure S17.1-4.

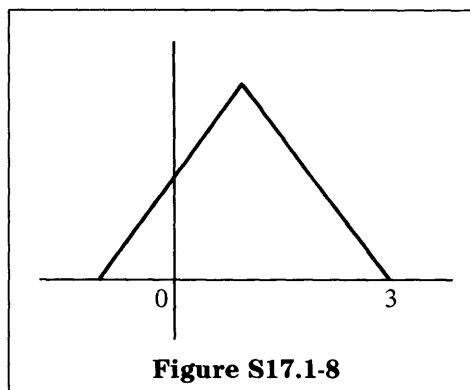


We note that this superposition is actually a linear interpolation between the samples of $x(t)$. For example, Figure S17.1-5 convolved with Figure S17.1-6 equals Figure S17.1-7.

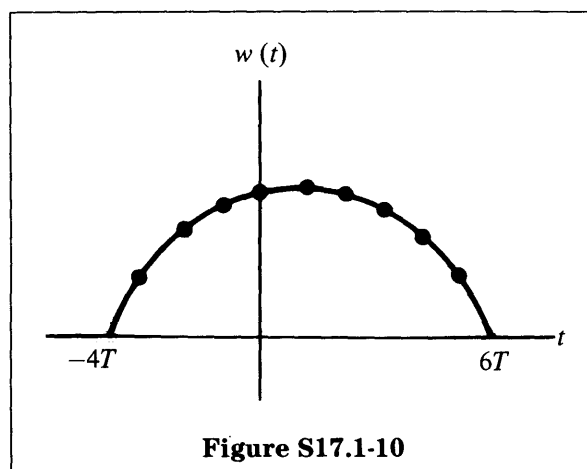
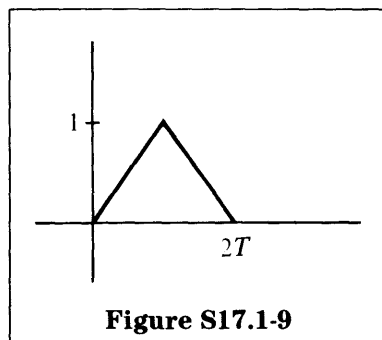




Now adding the shifted and scaled triangles yields Figure S17.1-8, which we see is the linear interpolation between samples of $x_p(t)$.

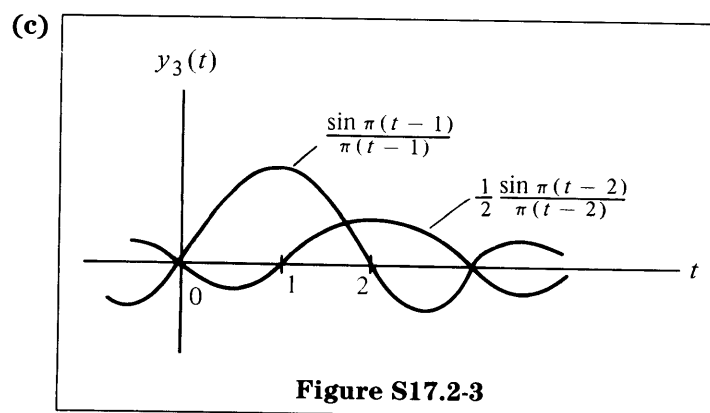
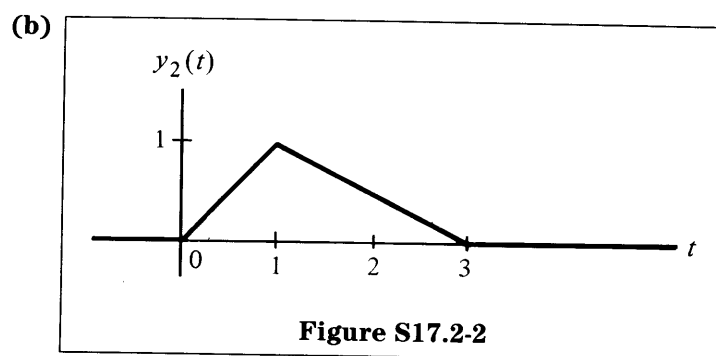
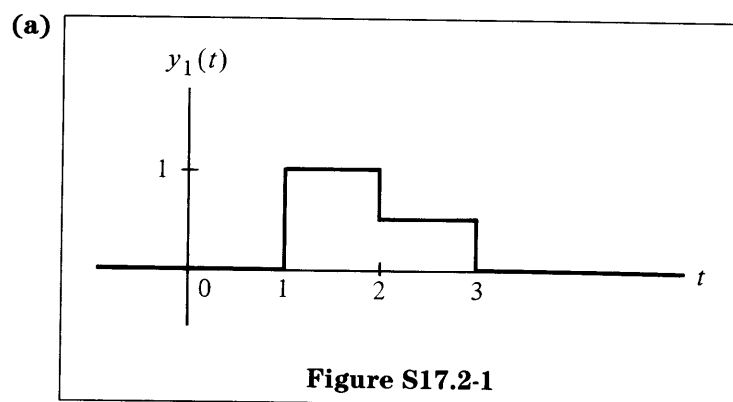


Now since $(1/T)h(t) * h(t)$ is Figure S17.1-9, we expect that $w(t)$ is the linear interpolation of $x_p(t)$ shifted right by T , shown in Figure S17.1-10.

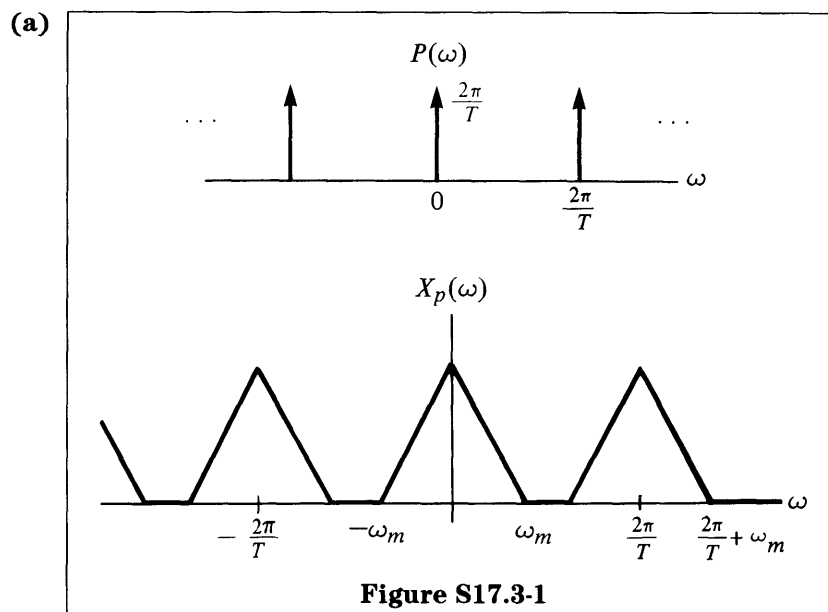


S17.2

$y(t)$ in all cases is the superposition of two signals.



S17.3

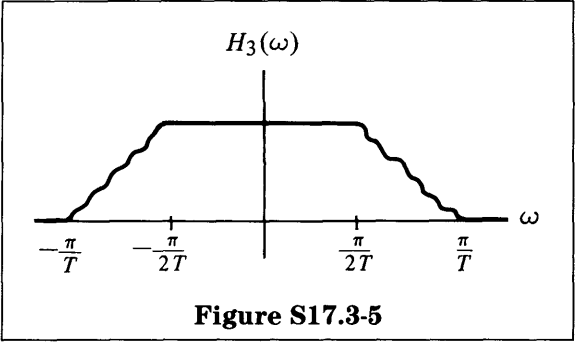
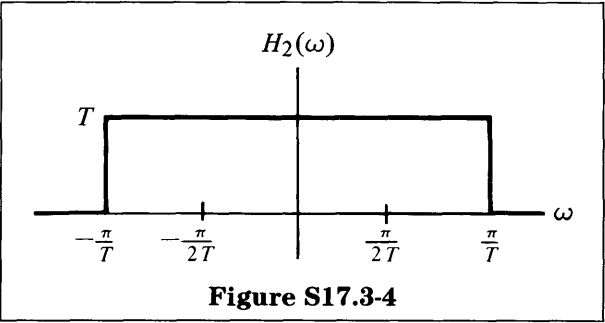
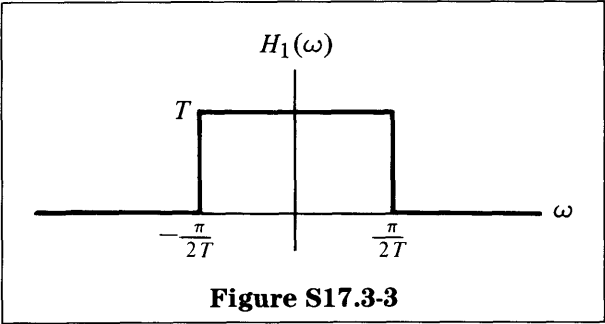
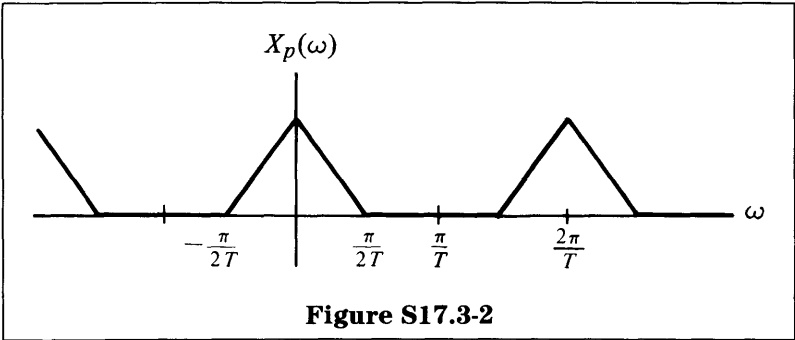


We have

$$\begin{aligned}\frac{2\pi}{T} - \omega_m &> \omega_m, \\ \frac{2\pi}{T} &> 2\omega_m, \\ \omega_s &> 2\omega_m,\end{aligned}$$

where ω_s is the sampling frequency. To ensure no aliasing we require that $T < \pi/\omega_m$.

- (b) To recover $x(t)$ from $x_p(t)$, we must interpolate. We have previously shown that by lowpass filtering, the spectrum is recovered, assuming that sampling has been performed at a sufficiently high rate. The interpolation may be done in many different ways, however, depending on the cutoff frequency we choose for the lowpass filter. For example, any of the filters $H_1(\omega)$, $H_2(\omega)$, and $H_3(\omega)$ in Figures S17.3-3 to S17.3-5 may be used to interpolate $x_p(t)$, whose Fourier transform is shown in Figure S17.3-2, to yield $x(t)$.



(c) If

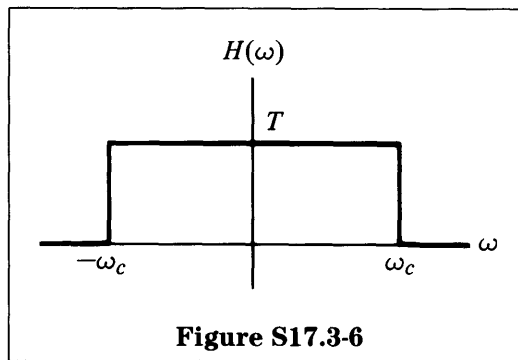
$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)$$

and $H(\omega)$ is an ideal lowpass filter whose Fourier transform is shown in Figure S17.3-6, then

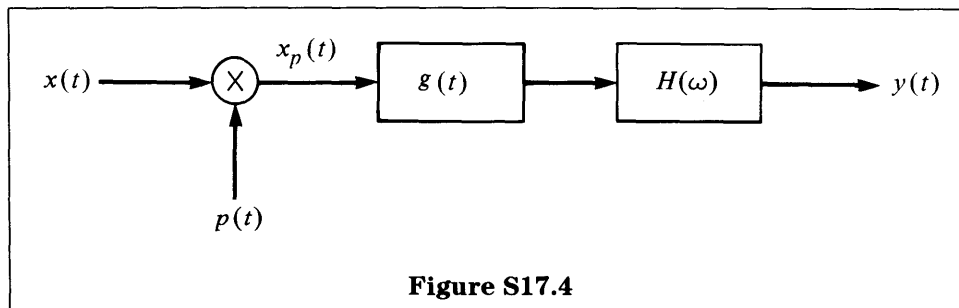
$$h(t) = \frac{T \sin \omega_c t}{\pi t} = \frac{T \omega_c}{\pi} \operatorname{sinc} \frac{\omega_c t}{\pi}$$

and

$$x(t) = x_p(t) * h(t) = \frac{T \omega_c}{\pi} \sum_{n=-\infty}^{\infty} x(nT) \operatorname{sinc} \frac{(t - nT)}{\pi}$$



S17.4



We want to choose $H(\omega)$ so that the cascade of the two filters is an ideal lowpass filter. In this example,

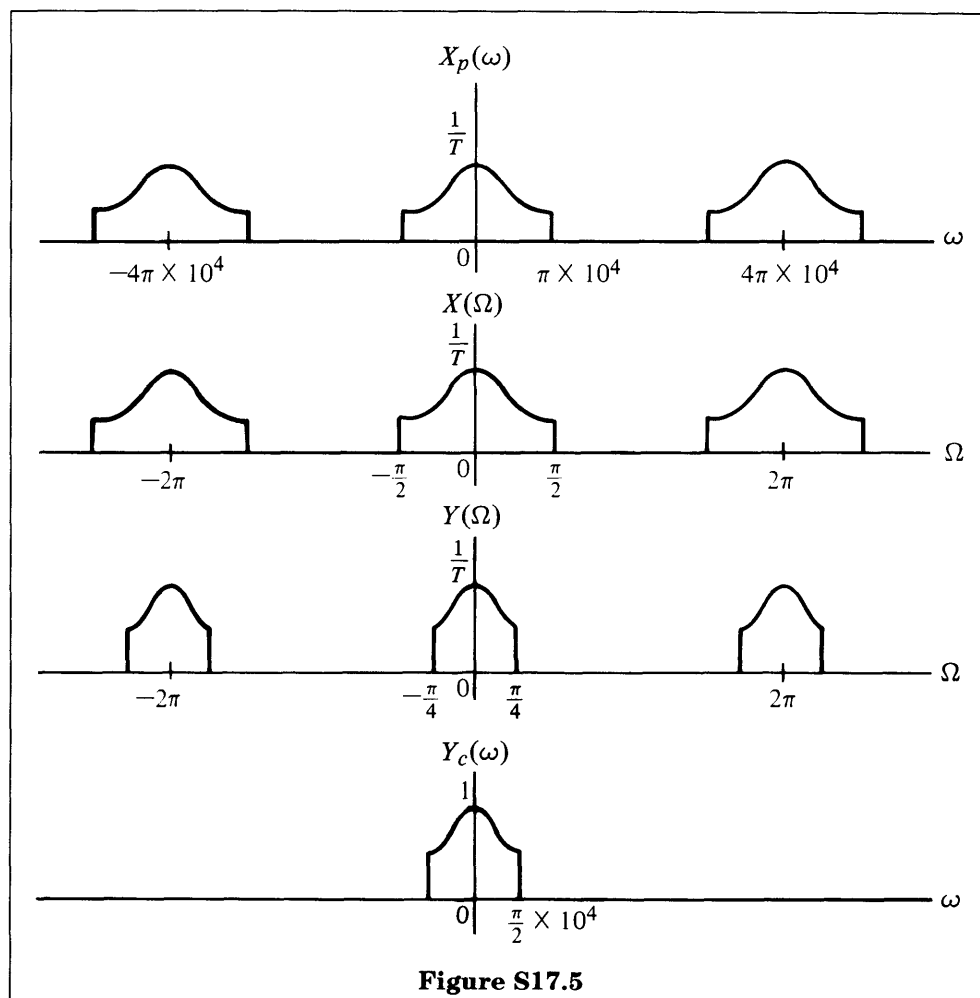
$$G(\omega) = 2 \sin \frac{\omega \Delta / 2}{\omega} e^{-j\omega \Delta / 2}$$

so that

$$H(\omega) = \frac{\omega}{2 \sin \omega \Delta / 2} e^{j\omega \Delta / 2}, \quad |\omega| < \pi / \Delta,$$

$$H(\omega) = 0 \quad \text{otherwise}$$

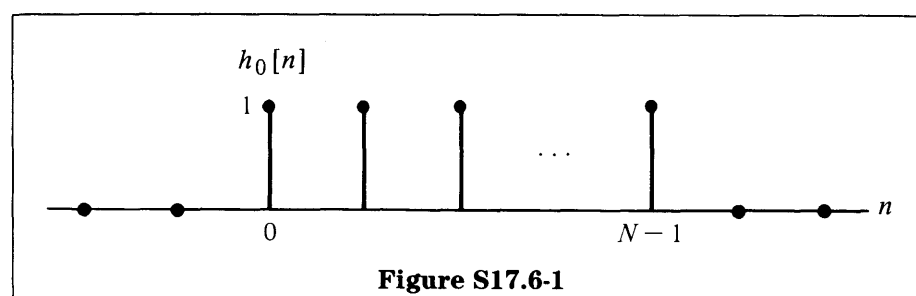
S17.5



Solutions to Optional Problems

S17.6

- (a) We want $h_0[n]$ such that $x_p[n] * h_0[n] = x_0[n]$. We sketch $h_0[n]$ in Figure S17.6-1.



(b) $x_0[n] = x_p[n] * h_0[n]$

If there is no aliasing, we can recover $x[n]$ from $x_0[n]$ by proper filtering. Since

$$H_0(\Omega) = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = e^{-j\Omega(N-1)/2} \frac{\sin \Omega N/2}{\sin \Omega/2},$$

we require

$$H(\Omega) = \begin{cases} N/H_0(\Omega), & |\Omega| \leq \frac{\pi}{N}, \\ 0, & \frac{\pi}{N} < |\Omega| \leq \pi \end{cases}$$

$$= \begin{cases} Ne^{j\Omega(N-1)/2} \frac{\sin \Omega/2}{\sin \Omega N/2}, & |\Omega| \leq \frac{\pi}{N} \\ 0, & \frac{\pi}{N} < |\Omega| \leq \pi \end{cases}$$

(c) $h_1[n] = (1/N)(h_0[n] * h_0[-n])$, so $h_1[n]$ is a triangular discrete-time pulse, as shown in Figure S17.6-2.

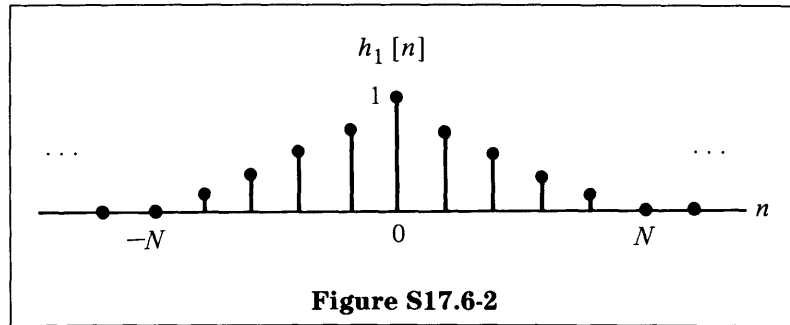


Figure S17.6-2

(d) We require that

$$H(\Omega) = \begin{cases} \frac{N}{H_1(\Omega)}, & \text{for } |\Omega| \leq \frac{\pi}{N}, \\ 0, & \text{for } \frac{\pi}{N} < |\Omega| < \pi \end{cases}$$

From part (c),

$$H(\Omega) = N \left[\frac{N}{|H_0(\Omega)|^2} \right] = \begin{cases} N^2 \left(\frac{\sin \Omega/2}{\sin \Omega N/2} \right)^2, & |\Omega| \leq \frac{\pi}{N}, \\ 0, & \frac{\pi}{N} < |\Omega| < \pi \end{cases}$$

S17.7

(a) Taking Fourier transforms of both sides of the LCCDE yields

$$j\omega Y_c(\omega) + Y_c(\omega) = 1,$$

$$Y_c(\omega) = \frac{1}{1 + j\omega},$$

so $y_c(t) = e^{-t}u(t)$. By examining the sampler followed by conversion to an impulse train, we note that

$$y[n] = y_c(nT) = e^{-nT}u[n]$$

(b) Since $y[n] = e^{-nT}u[n]$, we can take the Fourier transform to yield

$$\begin{aligned} Y(\Omega) &= \sum_{n=0}^{\infty} e^{-nT} e^{-j\Omega n} \\ &= \frac{1}{1 - e^{-(T+j\Omega)}} \end{aligned}$$

In order for $w[n] = \delta[n]$, we require $W(\Omega) = 1$ for all Ω . Thus,

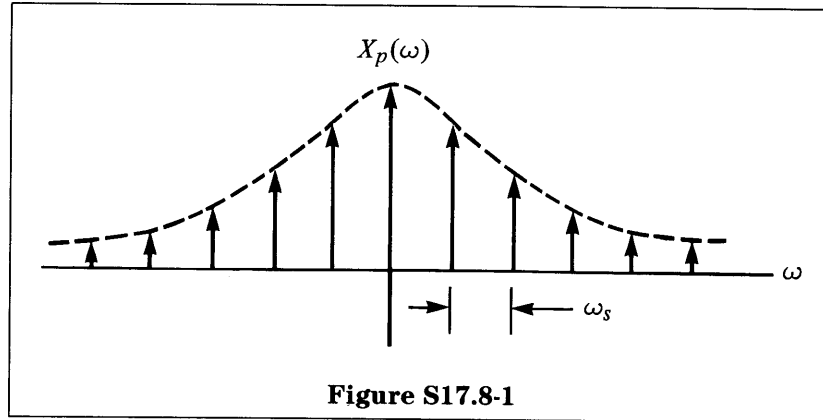
$$\begin{aligned} Y(\Omega)H(\Omega) &= W(\Omega) = 1, \\ H(\Omega) &= 1 - e^{-(T+j\Omega)} \end{aligned}$$

Now since $H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$, we see by inspection that

$$\begin{aligned} h[0] &= 1, \\ h[1] &= -e^{-T}, \\ h[n] &= 0, \quad n \neq 0, 1 \end{aligned}$$

S17.8

(a) Since $X_p(\omega) = X(\omega)P(\omega)$, we conclude that $X_p(\omega)$ is as shown in Figure S17.8-1.



$$\begin{aligned} X_p(\omega) &= X(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \\ &= \sum_{n=-\infty}^{\infty} X(n\omega_s) \delta(\omega - n\omega_s) \end{aligned}$$

(b) From the convolution theorem,

$$y(t) = x(t) * p(t)$$

and

$$p(t) = \frac{1}{\omega_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{2\pi n}{\omega_s}\right)$$

The fact that

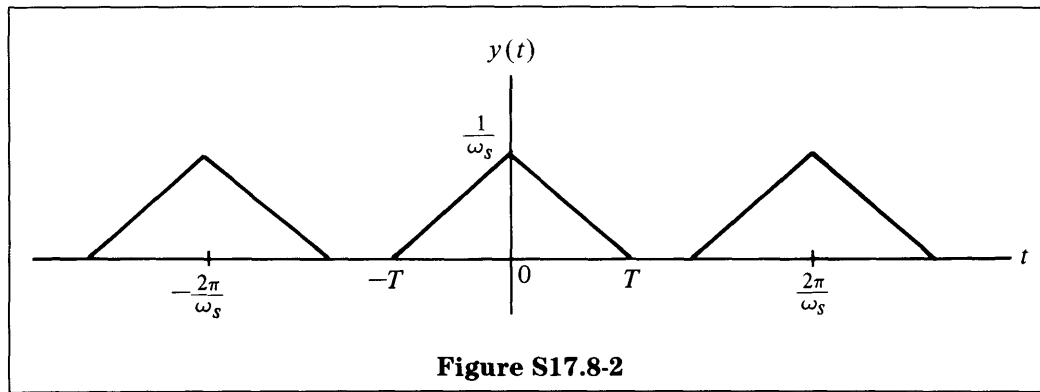
$$\mathcal{F}\{p(t)\} = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

can be easily verified. Therefore,

$$y(t) = x(t) * \frac{1}{\omega_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{2\pi n}{\omega_s}\right)$$

(c) We see from the sketch in Figure S17.8-2 that for no time-domain aliasing,

$$T \leq \frac{2\pi}{\omega_s} - T, \quad \text{so} \quad \omega_s \leq \frac{\pi}{T}$$



(d) $x(t)$ may be recovered from $y(t)$, assuming that no time-domain aliasing has occurred, by low-time filtering $y(t)$ from $t = -T$ to T and applying a gain of ω_s .

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