Properties of Continuous Time Convolution*

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Abstract

This module discusses the properties of continuous time convolution.

1 Introduction

We have already shown the important role that continuous time convolution plays in signal processing. This section provides discussion and proof of some of the important properties of continuous time convolution. Analogous properties can be shown for continuous time circular convolution with trivial modification of the proofs provided except where explicitly noted otherwise.

2 Continuous Time Convolution Properties

2.1 Associativity

The operation of convolution is associative. That is, for all continuous time signals f_1, f_2, f_3 the following relationship holds.

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3 \tag{1}$$

In order to show this, note that

$$\begin{aligned} (f_1 * (f_2 * f_3))(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau_1) f_2(\tau_2) f_3((t-\tau_1)-\tau_2)\, d\tau_2 d\tau_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau_1) f_2((\tau_1+\tau_2)-\tau_1) f_3(t-(\tau_1+\tau_2))\, d\tau_2 d\tau_1 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau_1) f_2(\tau_3-\tau_1) f_3(t-\tau_3)\, d\tau_1 d\tau_3 \\ &= ((f_1 * f_2) * f_3)(t) \end{aligned} \tag{2}$$

proving the relationship as desired through the substitution $\tau_3 = \tau_1 + \tau_2$.

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2.2 Commutativity

The operation of convolution is commutative. That is, for all continuous time signals f_1, f_2 the following relationship holds.

$$f_1 * f_2 = f_2 * f_1 \tag{3}$$

In order to show this, note that

$$\begin{aligned}(f_1 * f_2)(t) &= \int_{-\infty}^{\infty} f_1(\tau_1) f_2(t-\tau_1)\, d\tau_1 \\ &= \int_{-\infty}^{\infty} f_1(t-\tau_2) f_2(\tau_2)\, d\tau_2 \\ &= (f_2 * f_1)(t)\end{aligned} \tag{4}$$

proving the relationship as desired through the substitution $\tau_2 = t - \tau_1$.

2.3 Distribitivity

The operation of convolution is distributive over the operation of addition. That is, for all continuous time signals f_1, f_2, f_3 the following relationship holds.

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3 \tag{5}$$

In order to show this, note that

$$\begin{aligned}(f_1 * (f_2 + f_3))(t) &= \int_{-\infty}^{\infty} f_1(\tau)(f_2(t-\tau) + f_3(t-\tau))\, d\tau \\ &= \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau + \int_{-\infty}^{\infty} f_1(\tau) f_3(t-\tau)\, d\tau \\ &= (f_1 * f_2 + f_1 * f_3)(t)\end{aligned} \tag{6}$$

proving the relationship as desired.

2.4 Multilinearity

The operation of convolution is linear in each of the two function variables. Additivity in each variable results from distributivity of convolution over addition. Homogenity of order one in each varible results from the fact that for all continuous time signals f_1, f_2 and scalars a the following relationship holds.

$$a(f_1 * f_2) = (af_1) * f_2 = f_1 * (af_2) \tag{7}$$

In order to show this, note that

$$\begin{aligned}(a(f_1 * f_2))(t) &= a\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} (af_1(\tau)) f_2(t-\tau)\, d\tau \\ &= ((af_1) * f_2)(t) \\ &= \int_{-\infty}^{\infty} f_1(\tau)(af_2(t-\tau))\, d\tau \\ &= (f_1 * (af_2))(t)\end{aligned} \tag{8}$$

proving the relationship as desired.

2.5 Conjugation

The operation of convolution has the following property for all continuous time signals f_1, f_2.

$$\overline{f_1 * f_2} = \overline{f_1} * \overline{f_2} \tag{9}$$

In order to show this, note that

$$\begin{array}{rl} \left(\overline{f_1 * f_2}\right)(t) & = \overline{\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau} \\ & = \int_{-\infty}^{\infty} \overline{f_1(\tau) f_2(t-\tau)} d\tau \\ & = \int_{-\infty}^{\infty} \overline{f_1}(\tau) \overline{f_2}(t-\tau)\, d\tau \\ & = \left(\overline{f_1} * \overline{f_2}\right)(t) \end{array} \tag{10}$$

proving the relationship as desired.

2.6 Time Shift

The operation of convolution has the following property for all continuous time signals f_1, f_2 where S_T is the time shift operator.

$$S_T(f_1 * f_2) = (S_T f_1) * f_2 = f_1 * (S_T f_2) \tag{11}$$

In order to show this, note that

$$\begin{array}{rl} S_T(f_1 * f_2)(t) & = \int_{-\infty}^{\infty} f_2(\tau) f_1((t-T)-\tau)\, d\tau \\ & = \int_{-\infty}^{\infty} f_2(\tau) S_T f_1(t-\tau)\, d\tau \\ & = ((S_T f_1) * f_2)(t) \\ & = \int_{-\infty}^{\infty} f_1(\tau) f_2((t-T)-\tau)\, d\tau \\ & = \int_{-\infty}^{\infty} f_1(\tau) S_T f_2(t-\tau)\, d\tau \\ & = f_1 * (S_T f_2)(t) \end{array} \tag{12}$$

proving the relationship as desired.

2.7 Differentiation

The operation of convolution has the following property for all continuous time signals f_1, f_2.

$$\frac{d}{dt}(f_1 * f_2)(t) = \left(\frac{df_1}{dt} * f_2\right)(t) = \left(f_1 * \frac{df_2}{dt}\right)(t) \tag{13}$$

In order to show this, note that

$$\begin{array}{rl} \frac{d}{dt}(f_1 * f_2)(t) & = \int_{-\infty}^{\infty} f_2(\tau) \frac{d}{dt} f_1(t-\tau)\, d\tau \\ & = \left(\frac{df_1}{dt} * f_2\right)(t) \\ & = \int_{-\infty}^{\infty} f_1(\tau) \frac{d}{dt} f_2(t-\tau)\, d\tau \\ & = \left(f_1 * \frac{df_2}{dt}\right)(t) \end{array} \tag{14}$$

proving the relationship as desired.

2.8 Impulse Convolution

The operation of convolution has the following property for all continuous time signals f where δ is the Dirac delta funciton.

$$f * \delta = f \tag{15}$$

In order to show this, note that

$$\begin{aligned} (f * \delta)(t) &= \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau \\ &= f(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau \\ &= f(t) \end{aligned} \tag{16}$$

proving the relationship as desired.

2.9 Width

The operation of convolution has the following property for all continuous time signals f_1, f_2 where $Duration\,(f)$ gives the duration of a signal f.

$$Duration\,(f_1 * f_2) = Duration\,(f_1) + Duration\,(f_2) \tag{17}$$

. In order to show this informally, note that $(f_1 * f_2)(t)$ is nonzero for all t for which there is a τ such that $f_1(\tau)\, f_2(t-\tau)$ is nonzero. When viewing one function as reversed and sliding past the other, it is easy to see that such a τ exists for all t on an interval of length $Duration\,(f_1) + Duration\,(f_2)$. Note that this is not always true of circular convolution of finite length and periodic signals as there is then a maximum possible duration within a period.

3 Convolution Properties Summary

As can be seen the operation of continuous time convolution has several important properties that have been listed and proven in this module. With slight modifications to proofs, most of these also extend to continuous time circular convolution as well and the cases in which exceptions occur have been noted above. These identities will be useful to keep in mind as the reader continues to study signals and systems.