9 Fourier Transform Properties

Solutions to Recommended Problems

S9.1

The Fourier transform of $x(t)$ is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-t/2}u(t)e^{-j\omega t}\,dt \qquad \text{(S9.1-1)}$$

Since $u(t) = 0$ for $t < 0$, eq. (S9.1-1) can be rewritten as

$$X(\omega) = \int_{0}^{\infty} e^{-(1/2+j\omega)t}\,dt = \frac{+2}{1 + j2\omega}$$

It is convenient to write $X(\omega)$ in terms of its real and imaginary parts:

$$X(\omega) = \frac{2}{1 + j2\omega}\left(\frac{1 - j2\omega}{1 - j2\omega}\right) = \frac{2 - j4\omega}{1 + 4\omega^2} = \frac{2}{1 + 4\omega^2} - j\frac{4\omega}{1 + 4\omega^2}$$

$$\text{Magnitude of } X(\omega) = \frac{2}{\sqrt{1 + 4\omega^2}}$$

$$X(\omega) = \tan^{-1}(-2\omega) = -\tan^{-1}(2\omega)$$

$$Re\{X(\omega)\} = \frac{+2}{1 + 4\omega^2}, \qquad Im\{X(\omega)\} = \frac{-4\omega}{1 + 4\omega^2}$$

(a)

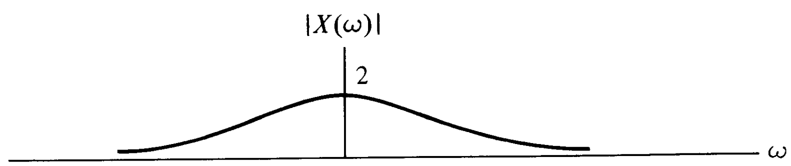

Figure S9.1-1

(b)

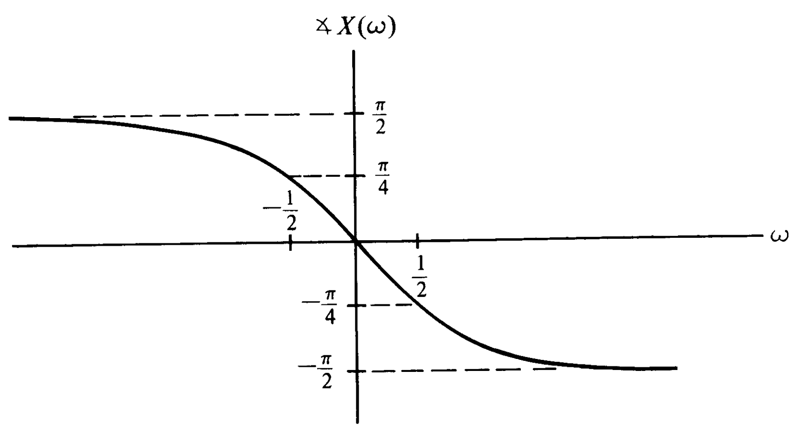

Figure S9.1-2

(c)

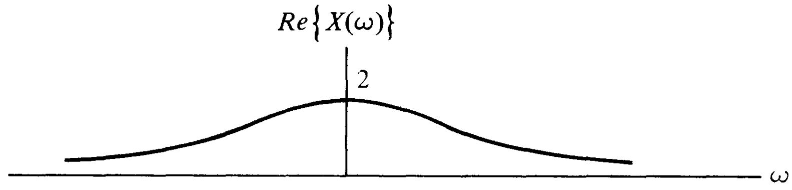

Figure S9.1-3

(d)

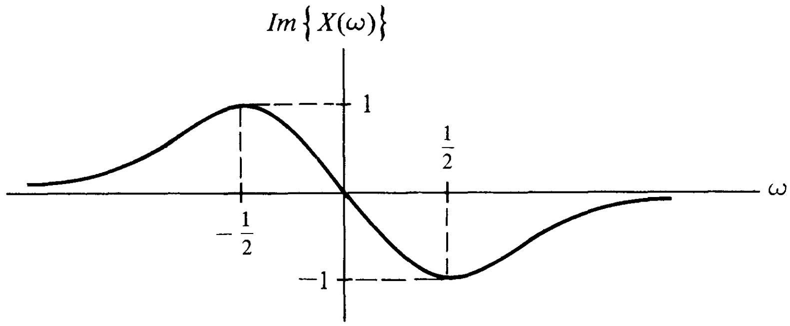

Figure S9.1-4

S9.2

(a) The magnitude of $X(\omega)$ is given by

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)},$$

where $X_R(\omega)$ is the real part of $X(\omega)$ and $X_I(\omega)$ is the imaginary part of $X(\omega)$. It follows that

$$|X(\omega)| = \begin{cases} \sqrt{2}, & |\omega| < W, \\ 0, & |\omega| > W \end{cases}$$

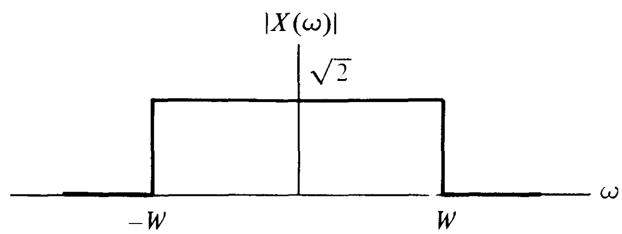

Figure S9.2-1

The phase of $X(\omega)$ is given by

$$\sphericalangle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right) = \tan^{-1}(1), \qquad |\omega| < W$$

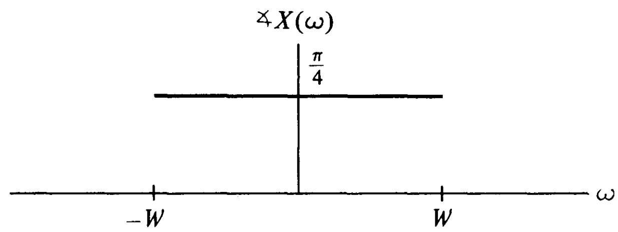

Figure S9.2-2

(b)
$$X(\omega) = \begin{cases} 1+j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$
$$X(-\omega) = \begin{cases} 1+j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$
$$X^*(\omega) = \begin{cases} 1-j, & |\omega| < W \\ 0, & \text{otherwise} \end{cases}$$

Hence, the signal is not real.

S9.3

For $x(t)$ to be real-valued, $X(\omega)$ is conjugate symmetric:

$$X(-\omega) = X^*(\omega)$$

(a) $X(\omega) = |X(\omega)|e^{j\measuredangle X(\omega)}$
$= |X(\omega)|\cos(\measuredangle X(\omega)) + j|X(\omega)|\sin(\measuredangle X(\omega))$

Therefore,

$$\begin{aligned} X(-\omega) &= |X(-\omega)|\cos(\measuredangle X(-\omega)) + j|X(-\omega)|\sin(\measuredangle X(-\omega)) \\ &= |X(\omega)|\cos(\measuredangle X(\omega)) - j|X(\omega)|\sin(\measuredangle X(\omega)) \\ &= X^*(\omega) \end{aligned}$$

Hence, $x(t)$ is real-valued.

(b)
$$\begin{aligned} X(\omega) &= X_R(\omega) + jX_I(\omega) \\ X(-\omega) &= X_R(-\omega) + jX_I(-\omega) \\ &= X_R(\omega) + j[-X_I(\omega) + 2\pi] \qquad \text{for } \omega > 0 \\ X^*(\omega) &= X_R(\omega) - jX_I(\omega) \end{aligned}$$

Therefore,

$$X^*(\omega) \neq X(-\omega)$$

Hence, $x(t)$ is not real-valued.

S9.4

(a) (i) $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

We take the complex conjugate of both sides to get

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt$$

Since $x(t)$ is real-valued,

$$X^*(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt$$

Therefore,

$$\begin{aligned} X^*(-\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= X(\omega) \end{aligned}$$

(ii) $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

Taking the complex conjugate of both sides, we have

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}\,d\omega$$

Therefore,

$$x^*(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{j\omega t}\,d\omega$$

Since $x(t) = x^*(-t)$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{j\omega t}\,d\omega$$

This shows that $X(\omega)$ must be real-valued.

(b) (i) Since $x(t)$ is real, $X(\omega) = X^*(-\omega)$. Since $x(t)$ is real and even, it satisfies $x(t) = x^*(-t)$ and, therefore, $X(\omega)$ is real. Hence, $X(\omega) = X^*(-\omega) = X(-\omega)$. It follows that $X(\omega)$ is real and even.

(ii) If $x(t)$ is real, $X(\omega) = X^*(-\omega)$. Since $x(t)$ is real and odd, $x(t) = -x^*(-t)$; an analysis similar to part (a)(ii) proves that $X(\omega)$ must be imaginary. Hence, $X(\omega) = X^*(-\omega) = -X(-\omega)$. It follows that $X(\omega)$ is also odd.

S9.5

(a)
$$\begin{aligned} \mathcal{F}\{e^{-\alpha|t|}\} &= \mathcal{F}\{e^{-\alpha t}u(t) + e^{\alpha t}u(-t)\} \\ &= \frac{1}{\alpha + j\omega} + \frac{1}{\alpha - j\omega} \\ &= \frac{2\alpha}{\alpha^2 + \omega^2} \end{aligned}$$

(b) Duality states that

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(\omega)$$
$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega)$$

Since

$$e^{-\alpha|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\alpha}{\alpha^2 + \omega^2},$$

we have

$$\frac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} \pi e^{-|\omega|}$$

(c) $\dfrac{1}{1+(3t)^2} \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{3}\pi e^{-|\omega/3|}$ since $x(at) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

(d) We are given Figure S9.5-1.

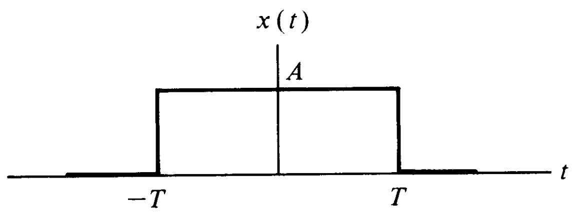

Figure S9.5-1

$$X(\omega) = A\int_{-T}^{T} e^{-j\omega t}\,dt = \frac{A}{-j\omega}(e^{-j\omega T} - e^{j\omega T})$$
$$= A\,\frac{-2j\sin\omega T}{-j\omega}$$
$$= 2TA\,\frac{\sin(\omega T)}{\omega T}$$

Sketches of $y(t)$, $Y(\omega)$, and $X(\omega)$ are given in Figure S9.5-2.

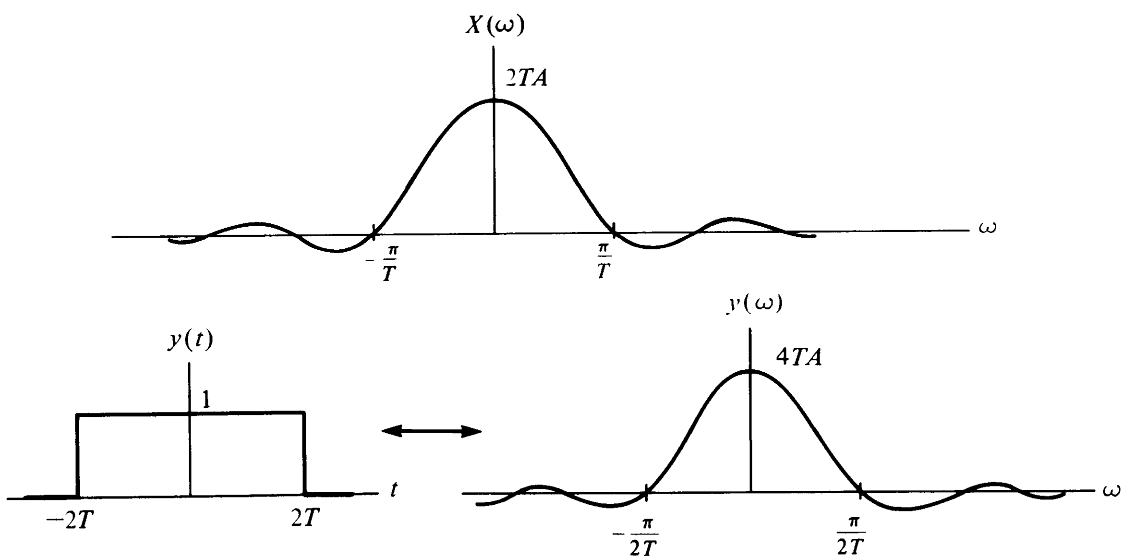

Figure S9.5-2

Substituting $2T$ for T in $X(\omega)$, we have

$$Y(\omega) = 2(2T)\frac{\sin(\omega 2T)}{\omega 2T}$$

The zero crossings are at

$$\omega_z 2T = n\pi, \qquad \text{or} \qquad \omega_z = n\frac{\pi}{2T}$$

S9.6

(a) $x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

Substituting $t = 0$ in the preceding equation, we get

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$$

(b) $X(\omega) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$

Substituting $\omega = 0$ in the preceding equation, we get

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$

S9.7

(a) We are given the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{S9.7-1}$$

Taking the Fourier transform of eq. (S9.7-1), we have

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

Hence,

$$Y(\omega)[2 + j\omega] = X(\omega)$$

and

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2 + j\omega},$$

$$H(\omega) = \frac{1}{2 + j\omega} = \frac{1}{2 + j\omega}\left(\frac{2 - j\omega}{2 - j\omega}\right) = \frac{2 - j\omega}{4 + \omega^2}$$

$$= \frac{2}{4 + \omega^2} - j\frac{\omega}{4 + \omega^2},$$

$$|H(\omega)|^2 = \frac{4}{(4 + \omega^2)^2} + \frac{\omega^2}{(4 + \omega^2)^2} = \frac{4 + \omega^2}{(4 + \omega^2)^2},$$

$$|H(\omega)| = \frac{1}{\sqrt{4 + \omega^2}}$$

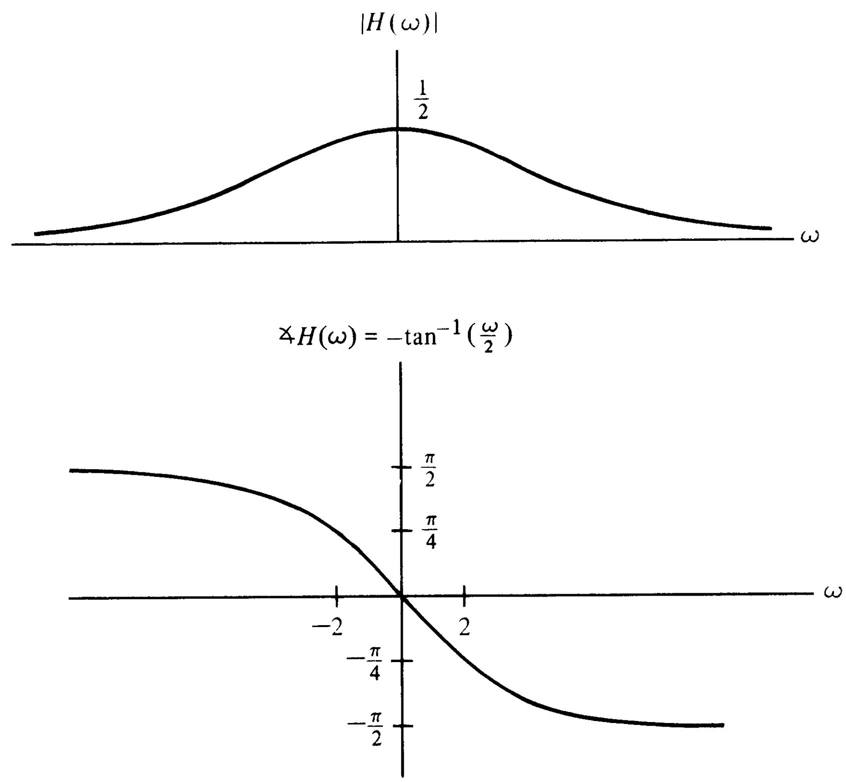

Figure S9.7

(b) We are given $x(t) = e^{-t}u(t)$. Taking the Fourier transform, we obtain

$$X(\omega) = \frac{1}{1 + j\omega}, \qquad H(\omega) = \frac{1}{2 + j\omega}$$

Hence,

$$Y(\omega) = \frac{1}{(1 + j\omega)(2 + j\omega)} = \frac{1}{1 + j\omega} - \frac{1}{2 + j\omega}$$

(c) Taking the inverse transform of $Y(\omega)$, we get

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

S9.8

A triangular signal can be represented as the convolution of two rectangular pulses, as indicated in Figure S9.8.

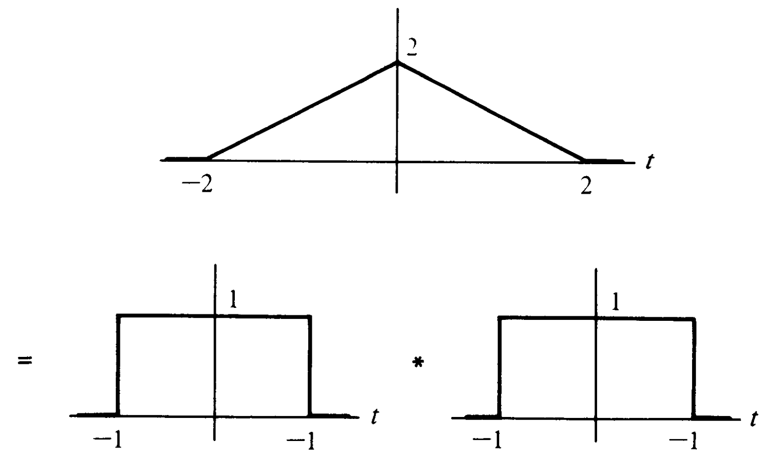

Figure S9.8

Since each of the rectangular pulses on the right has a Fourier transform given by $(2 \sin \omega)/\omega$, the convolution property tells us that the triangular function will have a Fourier transform given by the square of $(2 \sin \omega)/\omega$:

$$X(\omega) = \frac{4 \sin^2 \omega}{\omega^2}$$

Solutions to Optional Problems

S9.9

We can compute the function $x(t)$ by taking the inverse Fourier transform of $X(\omega)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} \pi e^{j\omega t}\, d\omega \\ &= \frac{1}{2}\left(\frac{1}{jt}\right)(e^{j\omega_0 t} - e^{-j\omega_0 t}) \\ &= \frac{\sin \omega_0 t}{t} \end{aligned}$$

Therefore,

$$y(t) = \cos(\omega_c t)\left[\frac{\sin(\omega_0 t)}{t}\right]$$

From the multiplicative property, we have

$$Y(\omega) = X(\omega) * [\pi\delta(\omega - \omega_c) - \pi\delta(\omega + \omega_c)]$$

$Y(\omega)$ is sketched in Figure S9.9.

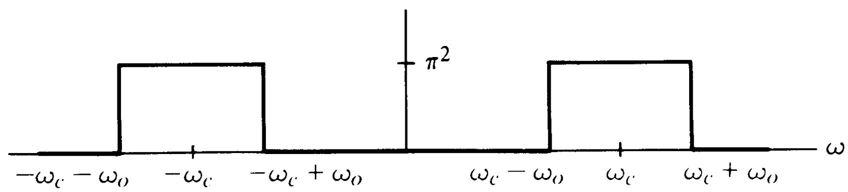

Figure S9.9

S9.10

(a) $x(t) = e^{-\alpha t} \cos \omega_0 t u(t), \qquad \alpha > 0$
$\qquad = e^{-\alpha t}u(t)\cos(\omega_0 t)$

Therefore,

$$X(\omega) = \frac{1}{2\pi}\frac{1}{\alpha + j\omega} * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$
$$= \frac{1/2}{\alpha + j(\omega - \omega_0)} + \frac{1/2}{\alpha + j(\omega + \omega_0)}$$

(b) $x(t) = e^{-3|t|} \sin 2t$

$$e^{-3|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{6}{9 + \omega^2}$$

$$\sin 2t \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)], \qquad \omega_0 = 2$$

Therefore,

$$X(\omega) = \frac{1}{2\pi}\left(\frac{6}{9 + \omega^2}\right) * \left\{\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]\right\}$$
$$= \frac{j3}{9 + (\omega + 2)^2} - \frac{j3}{9 + (\omega - 2)^2}$$

(c) $x(t) = \dfrac{\sin \pi t}{\pi t}\left(\dfrac{\sin 2\pi t}{\pi t}\right),$

$$X(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega),$$

where

$$X_1(\omega) = \begin{cases} 1, & |\omega| < \pi, \\ 0, & \text{otherwise} \end{cases}$$

$$X_2(\omega) = \begin{cases} 1, & |\omega| < 2\pi, \\ 0, & \text{otherwise} \end{cases}$$

Hence, $X(\omega)$ is given by the convolution shown in Figure S9.10.

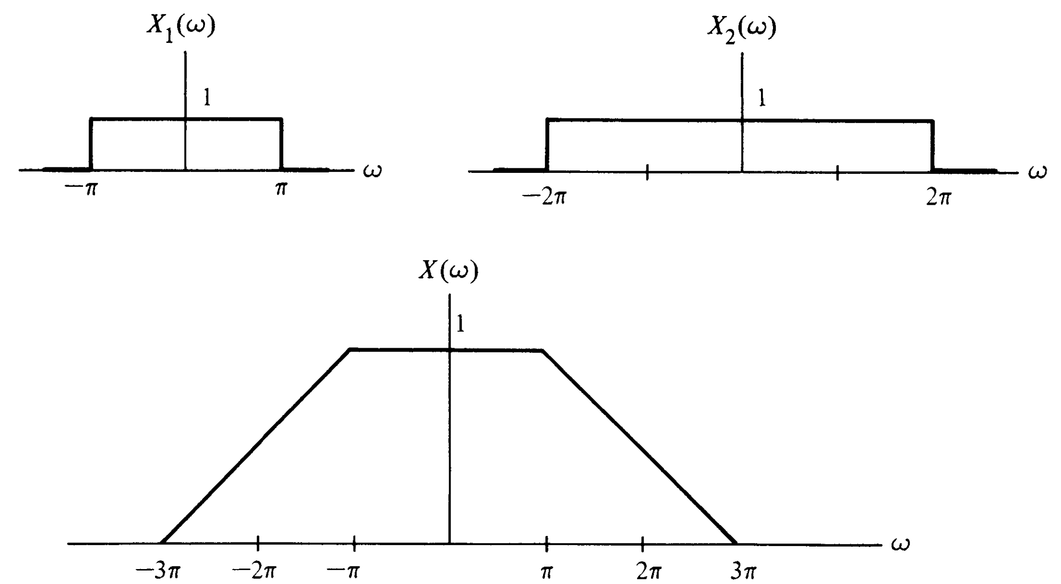

Figure S9.10

S9.11

We are given the LCCDE

$$\frac{dy(t)}{dt} + 2y(t) = A \cos \omega_0 t$$

We can view the LCCDE as

$$\frac{dy(t)}{dt} + 2y(t) = x(t),$$

the transfer function of which is given by

$$H(\omega) = \frac{1}{2 + j\omega} \qquad \text{and} \qquad x(t) = A \cos \omega_0 t$$

We have already seen that for LTI systems,

$$y(t) = |H(\omega_0)|\, A \cos(\omega_0 t + \phi), \qquad \text{where } \phi = \sphericalangle H(\omega_0)$$

$$= \frac{1}{\sqrt{4 + \omega_0^2}} A \cos(\omega_0 t + \phi)$$

For the maximum value of $y(t)$ to be $A/3$, we require

$$\frac{1}{4 + \omega_0^2} = \frac{1}{9}$$

Therefore, $\omega_0 = \pm\sqrt{5}$.

S9.12

(a) $$\mathcal{F}\left\{\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + 3y(t)\right\} = -\omega^2 Y(\omega) + 2j\omega Y(\omega) + 3Y(\omega)$$
$$= (-\omega^2 + j2\omega + 3)Y(\omega),$$
$$A(\omega) = -\omega^2 + j2\omega + 3$$

(b) $$\mathcal{F}\left\{\frac{4dx(t)}{dt} - x(t)\right\} = 4j\omega X(\omega) - X(\omega)$$
$$= (j4\omega - 1)X(\omega),$$
$$B(\omega) = j4\omega - 1,$$
$$A(\omega)Y(\omega) = B(\omega)X(\omega),$$
$$Y(\omega) = \frac{B(\omega)}{A(\omega)}X(\omega)$$
$$= H(\omega)X(\omega)$$

Therefore,

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{-1 + j4\omega}{-\omega^2 + 3 + j2\omega}$$
$$= \frac{1 - j4\omega}{\omega^2 - 3 - j2\omega}$$

S9.13

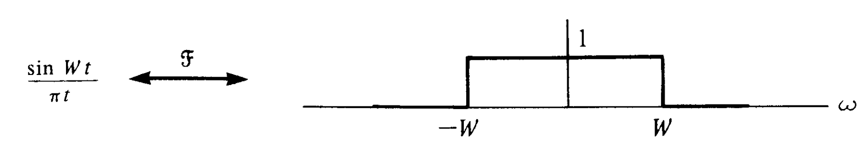

Figure S9.13-1

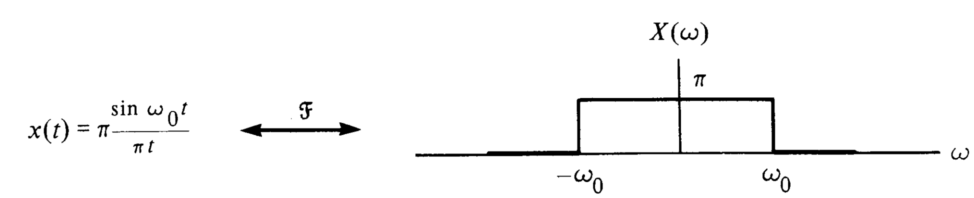

Figure S9.13-2

$$h(t) = \pi \frac{\sin(2\omega_0 t)}{\pi t} \quad \overset{\mathfrak{F}}{\longleftrightarrow}$$

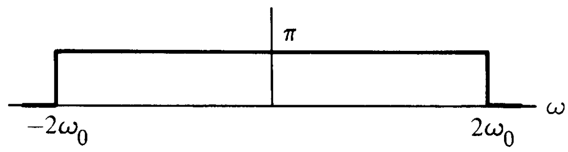

Figure S9.13-3

$$Y(\omega) = X(\omega)H(\omega) =$$

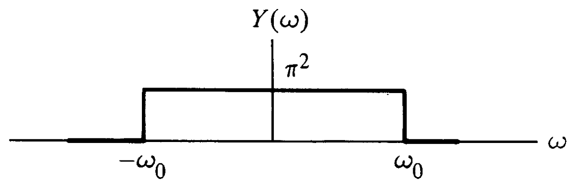

Figure S9.13-4

Therefore, $y(t) = \pi \dfrac{\sin(\omega_0 t)}{t}$.

S9.14

(a) Energy $= \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$

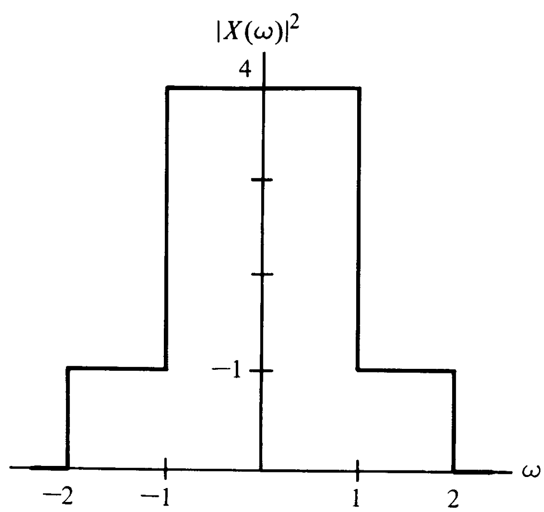

Figure S9.14-1

$$\text{Area} = (4)(2) + (2)(1)(1)$$
$$= 10$$
$$\text{Energy} = \frac{5}{\pi}$$

(b)

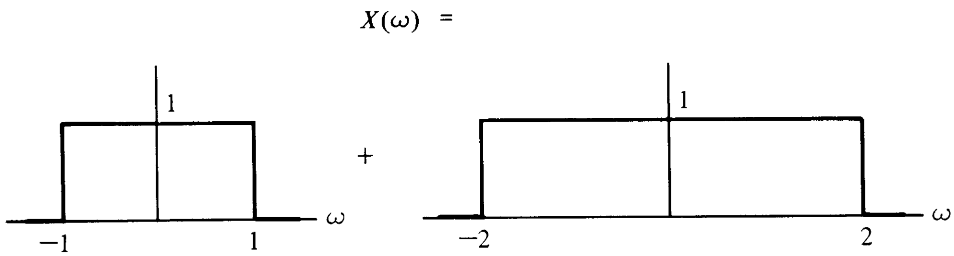

Figure S9.14-2

$$x(t) = \frac{\sin t}{\pi t} + \frac{\sin 2t}{\pi t}$$

S9.15

Given that

$$y_1(t) = 2\pi X(-\omega)|_{\omega=t}$$

we have

$$y_1(t) = 2\pi \int_{u=-\infty}^{\infty} x(u)e^{jtu}\,du$$

Similarly, let $y_2(t)$ be the output due to passing $x(t)$ through F twice.

$$\begin{aligned}
y_2(t) &= 2\pi \int_{v=-\infty}^{\infty} 2\pi \int_{u=-\infty}^{\infty} x(u)e^{jvu}\,du\, e^{jtv}\,dv \\
&= (2\pi)^2 \int_{u=-\infty}^{\infty} x(u) \int_{v=-\infty}^{\infty} e^{j(t+u)v}\,dv\,du \\
&= (2\pi)^2 \int_{u=-\infty}^{\infty} x(u)(2\pi)\delta(t+u)\,du \\
&= (2\pi)^3\, x(-t)
\end{aligned}$$

Finally, let $y_3(t)$ be the output due to passing $x(t)$ through F three times.

$$\begin{aligned}
y_3(t) = w(t) &= 2\pi \int_{u=-\infty}^{\infty} (2\pi)^3 x(-u)e^{jtu}\,du \\
&= (2\pi)^4 \int_{-\infty}^{\infty} e^{-jtu}\,x(u)\,du \\
&= (2\pi)^4 X(t)
\end{aligned}$$

S9.16

We are given

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \qquad a > 0$$

Let $n = 1$:

$$x(t) = e^{-at}u(t), \qquad a > 0,$$
$$X(\omega) = \frac{1}{a + j\omega}$$

Let $n = 2$:

$$x(t) = te^{-at}u(t),$$
$$X(\omega) = j\frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right) \qquad \text{since} \quad tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega)$$
$$= \frac{1}{(a + j\omega)^2}$$

Assume it is true for n:

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t),$$
$$X(\omega) = \frac{1}{(a + j\omega)^n}$$

We consider the case for $n + 1$:

$$x(t) = \frac{t^n}{n!}e^{-at}u(t),$$
$$X(\omega) = \frac{j}{n}\frac{d}{d\omega}\left[\frac{1}{(a + j\omega)^n}\right]$$
$$= \frac{j}{n}\frac{d}{d\omega}[(a + j\omega)^{-n}]$$
$$= \frac{j}{n}(-n)(a + j\omega)^{-n-1}j$$
$$= \frac{1}{(a + j\omega)^{n+1}}$$

Therefore, it is true for all n.