

Example: Line Charge, **Finite**

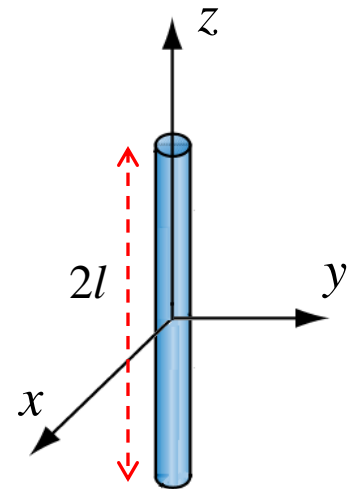
Calculate the electric field \mathbf{E} at any point P in the r - ϕ plane due to a line charge of length $2l$, centered on the origin and extending along the z axis, with a constant charge density ρ_l .

$$\mathbf{R} - \mathbf{R}' = r \hat{\mathbf{r}} - z' \hat{\mathbf{z}}, \quad dl = dz'$$

$$\mathbf{E} = \int_L \frac{\rho_l dl}{4\pi\epsilon_0} \frac{r \hat{\mathbf{r}} - z' \hat{\mathbf{z}}}{(r^2 + z'^2)^{3/2}}$$

$$= \frac{\rho_l dz'}{4\pi\epsilon_0} \left\{ \int_{-l}^{+l} \frac{r \hat{\mathbf{r}}}{(r^2 + z'^2)^{3/2}} - \int_{-l}^{+l} \frac{z' \hat{\mathbf{z}}}{(r^2 + z'^2)^{3/2}} \right\}$$

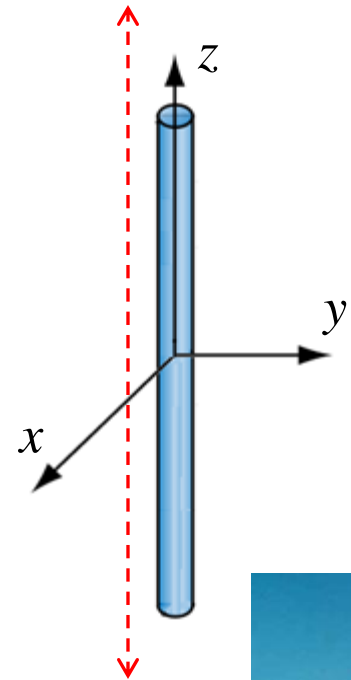
$$= \hat{\mathbf{r}} \frac{\rho_l \cdot r}{4\pi\epsilon_0} \int_{-l}^{+l} \frac{dz'}{(r^2 + z'^2)^{3/2}} = \frac{\rho_l}{2\pi\epsilon_0} \cdot \frac{l}{r\sqrt{r^2 + l^2}} \hat{\mathbf{r}}$$



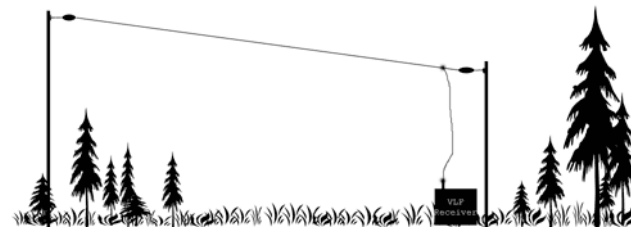
$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \int_L \rho_l dl \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int_S \rho_s dS \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int_V \rho_v dV \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3} \end{aligned}$$

Example: Line Charge, **Infinite**

Calculate the electric field \mathbf{E} at any point $P (r, \phi, z)$
due to a line charge of infinite length along the z axis,
with a constant charge density ρ_l .



Very-Low-Frequency (VLF) antennas





**THE
CITADEL**
THE MILITARY COLLEGE OF SOUTH CAROLINA

Dr. Gregory J. Mazzaro
Spring 2015

ELEC 318 – *Electromagnetic Fields*

Lecture 4(b)

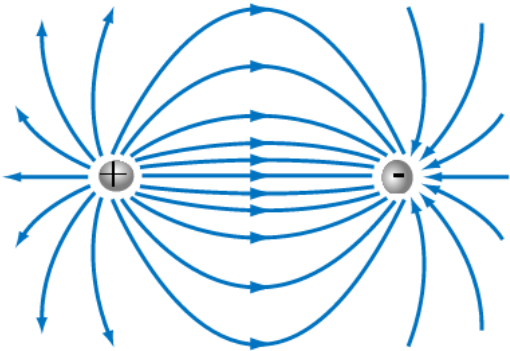
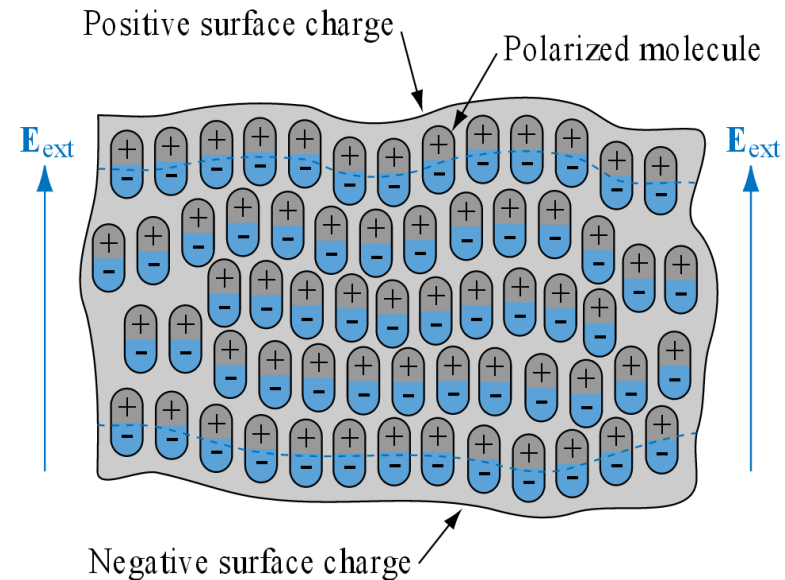
**Electrostatic Fields:
Gauss' Law & Electric Potential**

Electric Flux Density

electric flux density, \mathbf{D} (C/m^2)

$$\mathbf{D} = \epsilon \mathbf{E}$$

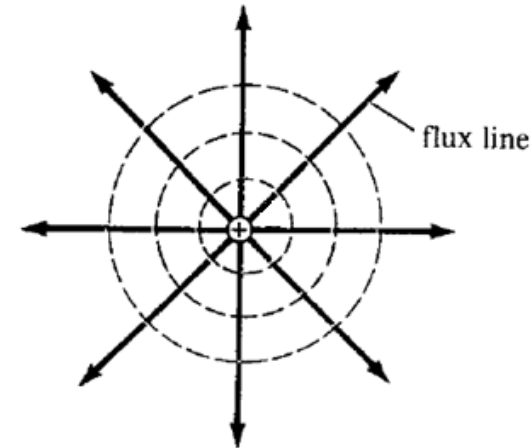
- similar to electric *field intensity*,
but it encompasses permittivity, ϵ
- for a given \mathbf{E} field, a measure of how
polarized a particular material is
- useful for particular electrostatic calculations
(e.g. for analyzing fields at boundaries
between material media)
- 1 line of flux begins on a 1-C positive charge
& terminates on a 1-C negative charge



near a single point
charge q in
free space $\epsilon_0 \dots$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3}$$

$$\mathbf{D} = \frac{q}{4\pi} \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3}$$



Gauss' Law

- Gauss's Law:**
- 1 of 4 of Maxwell's Equations
 - 1 of 2 that govern the behavior of *electric fields*
 - can be written in 2 ways:

$$\nabla \cdot \mathbf{D} = \rho_v$$

differential
form

$$\int_V \rho_v dV = Q$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{D} dV$$

(Divergence Theorem)

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

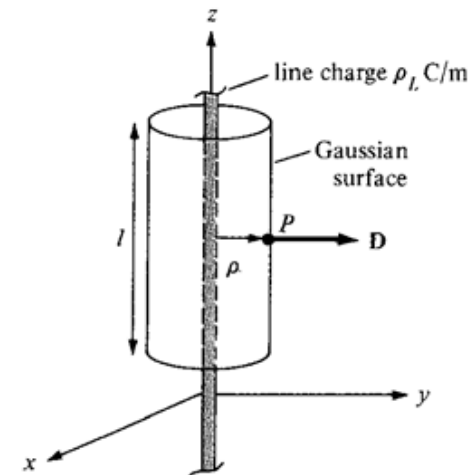
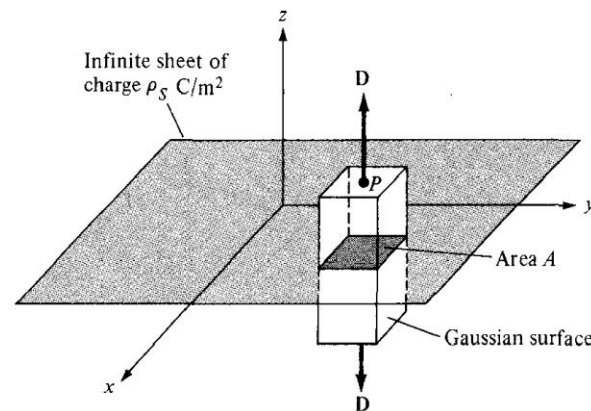
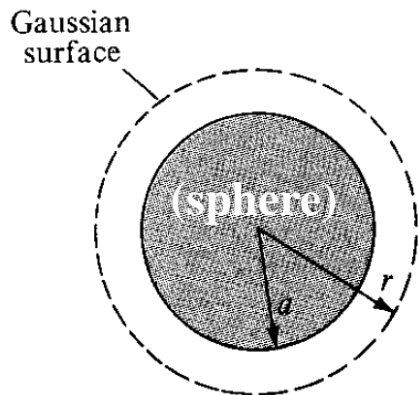
integral
form

Q = charge contained in a
“Gaussian” surface

\mathbf{D} = electric flux density

$d\mathbf{S}$ is *normal* to the surface
and directed *outward*

ρ_v = volume charge
density (C/m^3)

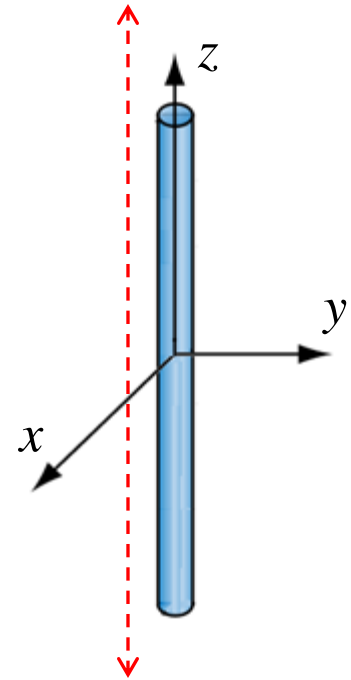


→ an alternative to Coulomb's Law for determining *electric field*, under symmetry

Example: Line Charge, **Infinite**

Calculate the electric field \mathbf{E} at any point $P (r, \phi, z)$
due to a line charge of infinite length along the z axis,
with a constant charge density ρ_l , in free space.

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$



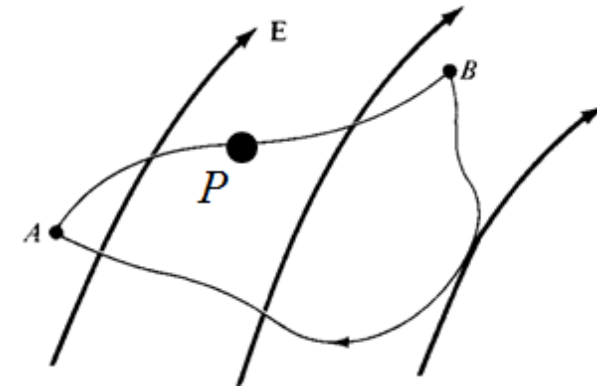
Electric (Scalar) Potential

electric potential (V)

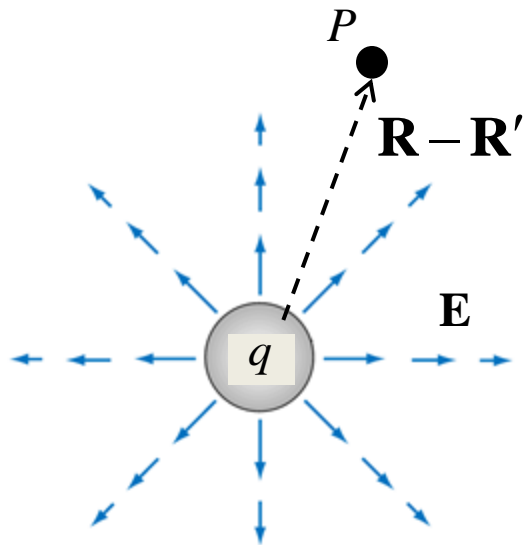
-- a measure of *work* required to move a unit charge through an \mathbf{E} field:

$$W = -\int_A^B \mathbf{F} \cdot d\mathbf{l} = -q \int_A^B \mathbf{E} \cdot d\mathbf{l}$$

$$V = \frac{W}{q} \Rightarrow V_{AB} = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$$



for a point charge q ...



$$\begin{aligned} V_{\text{point}} &= -\int_{\infty}^{|\mathbf{R}-\mathbf{R}'|} \mathbf{E}_{\text{point}} \cdot d\mathbf{l} \\ &= -\int_{\infty}^{|\mathbf{R}-\mathbf{R}'|} \left(\frac{q}{4\pi\epsilon_0} \frac{\mathbf{R}-\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|^3} \right) \cdot \left(\frac{\mathbf{R}-\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|} \right) dr \\ &= \frac{q}{4\pi\epsilon_0} \cdot \left\{ \frac{1}{|\mathbf{R}-\mathbf{R}'|} - \frac{1}{\infty} \right\} \end{aligned}$$

$$V_{\text{point}} = \frac{q}{4\pi\epsilon_0 |\mathbf{R}-\mathbf{R}'|}$$

V for a charge distribution:

$$dV = \frac{dq}{4\pi\epsilon_0 |\mathbf{R}-\mathbf{R}'|}$$

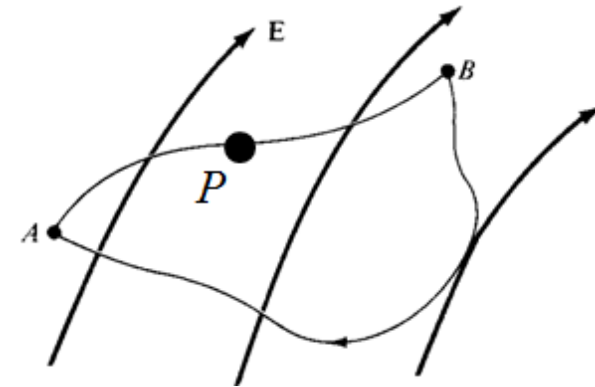
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for a **static** electric field, V_{AB} is independent of the path chosen from A to B

(Faraday's Law)

$$V_{AP} - V_{PB} = V_{AB} \quad \forall P(x, y, z)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

no time variation
(i.e. *static* fields)

$$\nabla \times \mathbf{E} = 0$$

(Stokes' Theorem)

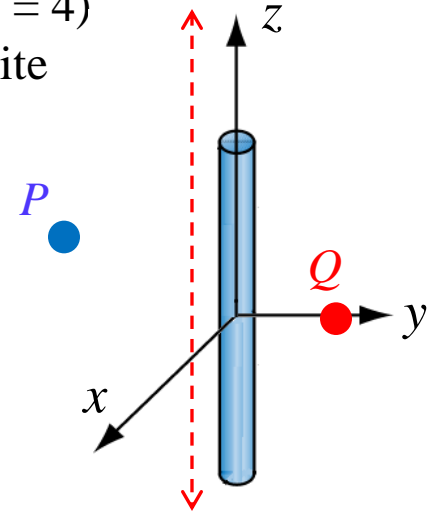
$$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint_L \mathbf{E} \cdot d\mathbf{l}$$

Example: V from E & Work

Calculate the difference in electric potential from point $P(x = 4, y = 0, z = 4)$ to point $Q(x = 0, y = 2, z = 0)$ in the presence of a line charge of infinite length along the z axis with a constant charge density of 16 pC/m .

Determine the work required to move 3 mC from P to Q .



$$V_{PQ} = -\int_P^Q \mathbf{E} \cdot d\mathbf{l}$$

$$V = \frac{W}{q}$$

$$\mathbf{E}_{\text{line charge}}^{\text{infinite}} = \frac{\rho_l}{2\pi\epsilon_0 r} \hat{\mathbf{r}}$$

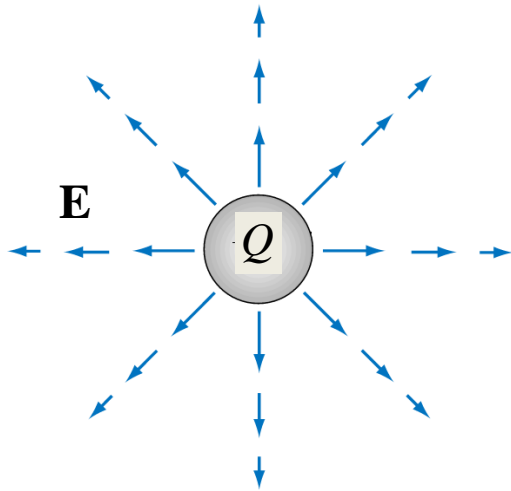
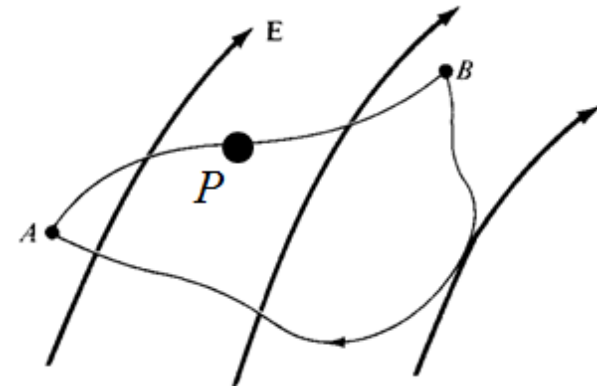
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$$dV = -\mathbf{E} \cdot d\mathbf{l} = -E_x dx - E_y dy - E_z dz$$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

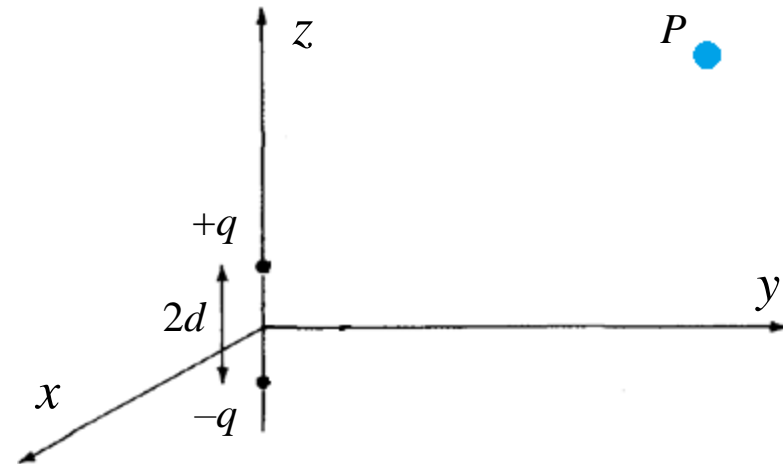
$$\mathbf{E} = -\frac{\partial V}{\partial x} \mathbf{i} - \frac{\partial V}{\partial y} \mathbf{j} - \frac{\partial V}{\partial z} \mathbf{k}$$

$$\mathbf{E} = -\nabla V \quad \text{"E from V"}$$

→ It's usually easier to calculate V , then take its gradient rather than to compute E directly from Coulomb's Law.

Example: E from V , Dipole

For the dipole depicted, determine \mathbf{E} at all points $P(R, \theta, \phi)$. You may assume that $R \gg 2d$.



$$\mathbf{E} = -\nabla V$$

$$\begin{aligned}\nabla V &= \frac{\partial V}{\partial x} \hat{\mathbf{x}} + \frac{\partial V}{\partial y} \hat{\mathbf{y}} + \frac{\partial V}{\partial z} \hat{\mathbf{z}} \\ &= \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial V}{\partial z} \hat{\mathbf{z}} \\ &= \frac{\partial V}{\partial R} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}}\end{aligned}$$

To be studied **outside of class**



- dipole moment
- Poisson's equation & Laplace's equation