

# CALCULUS III

## Surface Integrals

Paul Dawkins

*Table of Contents*

|   |           |
|---|-----------|
| <b>Preface .....</b>                    | <b>ii</b> |
| <b>Surface Integrals.....</b>           | <b>3</b>  |
| Introduction.....                       | 3         |
| Parametric Surfaces.....                | 4         |
| Surface Integrals .....                 | 10        |
| Surface Integrals of Vector Fields..... | 19        |
| Stokes' Theorem .....                   | 29        |
| Divergence Theorem.....                 | 34        |

## Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.
2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.
3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.
4. This is somewhat related to the previous three items, but is important enough to merit its own item. **THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!!** Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

## Surface Integrals

### *Introduction*

---

In the previous chapter we looked at evaluating integrals of functions or vector fields where the points came from a curve in two- or three-dimensional space. We now want to extend this idea and integrate functions and vector fields where the points come from a surface in three-dimensional space. These integrals are called surface integrals.

Here is a list of the topics covered in this chapter.

[Parametric Surfaces](#) – In this section we will take a look at the basics of representing a surface with parametric equations. We will also take a look at a couple of applications.

[Surface Integrals](#) – Here we will introduce the topic of surface integrals. We will be working with surface integrals of functions in this section.

[Surface Integrals of Vector Fields](#) – We will look at surface integrals of vector fields in this section.

[Stokes' Theorem](#) – We will look at Stokes' Theorem in this section.

[Divergence Theorem](#) – Here we will take a look at the Divergence Theorem.

## Parametric Surfaces

Before we get into surface integrals we first need to talk about how to parameterize a surface. When we parameterized a curve we took values of  $t$  from some interval  $[a, b]$  and plugged them into

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

and the resulting set of vectors will be the position vectors for the points on the curve.

With surfaces we'll do something similar. We will take points,  $(u, v)$ , out of some two-dimensional space  $D$  and plug them into

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

and the resulting set of vectors will be the position vectors for the points on the surface  $S$  that we are trying to parameterize. This is often called the **parametric representation** of the **parametric surface**  $S$ .

We will sometimes need to write the **parametric equations** for a surface. There are really nothing more than the components of the parametric representation explicitly written down.

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

**Example 1** Determine the surface given by the parametric representation

$$\vec{r}(u, v) = u\vec{i} + u \cos v \vec{j} + u \sin v \vec{k}$$

**Solution**

Let's first write down the parametric equations.

$$x = u \qquad y = u \cos v \qquad z = u \sin v$$

Now if we square  $y$  and  $z$  and then add them together we get,

$$y^2 + z^2 = u^2 \cos^2 v + u^2 \sin^2 v = u^2 (\cos^2 v + \sin^2 v) = u^2 = x^2$$

So, we were able to eliminate the parameters and the equation in  $x$ ,  $y$ , and  $z$  is given by,

$$x^2 = y^2 + z^2$$

From the [Quadric Surfaces](#) section notes we can see that this is a cone that opens along the  $x$ -axis.

We are much more likely to need to be able to write down the parametric equations of a surface than identify the surface from the parametric representation so let's take a look at some examples of this.

**Example 2** Give parametric representations for each of the following surfaces.

(a) The elliptic paraboloid  $x = 5y^2 + 2z^2 - 10$ . [\[Solution\]](#)

(b) The elliptic paraboloid  $x = 5y^2 + 2z^2 - 10$  that is in front of the  $yz$ -plane.  
[\[Solution\]](#)

(c) The sphere  $x^2 + y^2 + z^2 = 30$ . [\[Solution\]](#)

(d) The cylinder  $y^2 + z^2 = 25$ . [\[Solution\]](#)

**Solution**

**(a) The elliptic paraboloid**  $x = 5y^2 + 2z^2 - 10$ .

This one is probably the easiest one of the four to see how to do. Since the surface is in the form  $x = f(y, z)$  we can quickly write down a set of parametric equations as follows,

$$x = 5y^2 + 2z^2 - 10 \quad y = y \quad z = z$$

The last two equations are just there to acknowledge that we can choose  $y$  and  $z$  to be anything we want them to be. The parametric representation is then,

$$\vec{r}(y, z) = (5y^2 + 2z^2 - 10)\vec{i} + y\vec{j} + z\vec{k}$$

[\[Return to Problems\]](#)

**(b) The elliptic paraboloid**  $x = 5y^2 + 2z^2 - 10$  **that is in front of the  $yz$ -plane.**

This is really a restriction on the previous parametric representation. The parametric representation stays the same.

$$\vec{r}(y, z) = (5y^2 + 2z^2 - 10)\vec{i} + y\vec{j} + z\vec{k}$$

However, since we only want the surface that lies in front of the  $yz$ -plane we also need to require that  $x \geq 0$ . This is equivalent to requiring,

$$5y^2 + 2z^2 - 10 \geq 0 \quad \text{or} \quad 5y^2 + 2z^2 \geq 10$$

[\[Return to Problems\]](#)

**(c) The sphere**  $x^2 + y^2 + z^2 = 30$ .

This one can be a little tricky until you see how to do it. In spherical coordinates we know that the equation of a sphere of radius  $a$  is given by,

$$\rho = a$$

and so the equation of this sphere (in spherical coordinates) is  $\rho = \sqrt{30}$ . Now, we also have the following conversion formulas for converting Cartesian coordinates into spherical coordinates.

$$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$$

However, we know what  $\rho$  is for our sphere and so if we plug this into these conversion formulas we will arrive at a parametric representation for the sphere. Therefore, the parametric representation is,

$$\vec{r}(\theta, \varphi) = \sqrt{30} \sin \varphi \cos \theta \vec{i} + \sqrt{30} \sin \varphi \sin \theta \vec{j} + \sqrt{30} \cos \varphi \vec{k}$$

All we need to do now is come up with some restriction on the variables. First we know that we have the following restriction.

$$0 \leq \varphi \leq \pi$$

This is enforced upon us by choosing to use spherical coordinates. Also, to make sure that we only trace out the sphere once we will also have the following restriction.

$$0 \leq \theta \leq 2\pi$$

[\[Return to Problems\]](#)

**(d) The cylinder**  $y^2 + z^2 = 25$ .

As with the last one this can be tricky until you see how to do it. In this case it makes some sense to use cylindrical coordinates since they can be easily used to write down the equation of a cylinder.

In cylindrical coordinates the equation of a cylinder of radius  $a$  is given by

$$r = a$$

and so the equation of the cylinder in this problem is  $r = 5$ .

Next, we have the following conversion formulas.

$$x = x \qquad y = r \sin \theta \qquad z = r \cos \theta$$

Notice that they are slightly different from those that we are used to seeing. We needed to change them up here since the cylinder was centered upon the  $x$ -axis.

Finally, we know what  $r$  is so we can easily write down a parametric representation for this cylinder.

$$\vec{r}(x, \theta) = x\vec{i} + 5\sin\theta\vec{j} + 5\cos\theta\vec{k}$$

We will also need the restriction  $0 \leq \theta \leq 2\pi$  to make sure that we don't retrace any portion of the cylinder. Since we haven't put any restrictions on the "height" of the cylinder there won't be any restriction on  $x$ .

[\[Return to Problems\]](#)

In the first part of this example we used the fact that the function was in the form  $x = f(y, z)$  to quickly write down a parametric representation. This can always be done for functions that are in this basic form.

$$\begin{aligned} z = f(x, y) &\Rightarrow \vec{r}(x, y) = x\vec{i} + y\vec{j} + f(x, y)\vec{k} \\ x = f(y, z) &\Rightarrow \vec{r}(y, z) = f(y, z)\vec{i} + y\vec{j} + z\vec{k} \\ y = f(x, z) &\Rightarrow \vec{r}(x, z) = x\vec{i} + f(x, z)\vec{j} + z\vec{k} \end{aligned}$$

Okay, now that we have practice writing down some parametric representations for some surfaces let's take a quick look at a couple of applications.

Let's take a look at finding the tangent plane to the parametric surface  $S$  given by,

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

First, define

$$\begin{aligned} \vec{r}_u(u, v) &= \frac{\partial x}{\partial u}(u, v)\vec{i} + \frac{\partial y}{\partial u}(u, v)\vec{j} + \frac{\partial z}{\partial u}(u, v)\vec{k} \\ \vec{r}_v(u, v) &= \frac{\partial x}{\partial v}(u, v)\vec{i} + \frac{\partial y}{\partial v}(u, v)\vec{j} + \frac{\partial z}{\partial v}(u, v)\vec{k} \end{aligned}$$

Now, provided  $\vec{r}_u \times \vec{r}_v \neq \vec{0}$  it can be shown that the vector  $\vec{r}_u \times \vec{r}_v$  will be orthogonal to the surface  $S$ . This means that it can be used for the normal vector that we need in order to write down the equation of a tangent plane. This is an important idea that will be used many times throughout the next couple of sections.

Let's take a look at an example.

**Example 3** Find the equation of the tangent plane to the surface given by

$$\vec{r}(u, v) = u\vec{i} + 2v^2\vec{j} + (u^2 + v)\vec{k}$$

at the point  $(2, 2, 3)$ .

**Solution**

Let's first compute  $\vec{r}_u \times \vec{r}_v$ . Here are the two individual vectors.

$$\vec{r}_u(u, v) = \vec{i} + 2u\vec{k} \qquad \vec{r}_v(u, v) = 4v\vec{j} + \vec{k}$$

Now the cross product (which will give us the normal vector  $\vec{n}$ ) is,

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 4v & 1 \end{vmatrix} = -8uv\vec{i} - \vec{j} + 4v\vec{k}$$

Now, this is all fine, but in order to use it we will need to determine the value of  $u$  and  $v$  that will give us the point in question. We can easily do this by setting the individual components of the parametric representation equal to the coordinates of the point in question. Doing this gives,

$$\begin{aligned} 2 &= u & \Rightarrow & u = 2 \\ 2 &= 2v^2 & \Rightarrow & v = \pm 1 \\ 3 &= u^2 + v \end{aligned}$$

Now, as shown, we have the value of  $u$ , but there are two possible values of  $v$ . To determine the correct value of  $v$  let's plug  $u$  into the third equation and solve for  $v$ . This should tell us what the correct value is.

$$3 = 4 + v \quad \Rightarrow \quad v = -1$$

Okay so we now know that we'll be at the point in question when  $u = 2$  and  $v = -1$ . At this point the normal vector is,

$$\vec{n} = 16\vec{i} - \vec{j} - 4\vec{k}$$

The tangent plane is then,

$$\begin{aligned} 16(x - 2) - (y - 2) - 4(z - 3) &= 0 \\ 16x - y - 4z &= 18 \end{aligned}$$

You do [remember](#) how to write down the equation of a plane, right?

The second application that we want to take a quick look at is the surface area of the parametric surface  $S$  given by,



$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

and as we will see it again comes down to needing the vector  $\vec{r}_u \times \vec{r}_v$ .

So, provided  $S$  is traced out exactly once as  $(u, v)$  ranges over the points in  $D$  the surface area of  $S$  is given by,

$$A = \iint_D \|\vec{r}_u \times \vec{r}_v\| dA$$

Let's take a look at an example.

**Example 4** Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder  $x^2 + y^2 = 12$  and above the  $xy$ -plane.

**Solution**

Okay we've got a couple of things to do here. First we need the parameterization of the sphere. We parameterized a sphere earlier in this section so there isn't too much to do at this point. Here is the parameterization.

$$\vec{r}(\theta, \varphi) = 4 \sin \varphi \cos \theta \vec{i} + 4 \sin \varphi \sin \theta \vec{j} + 4 \cos \varphi \vec{k}$$

Next we need to determine  $D$ . Since we are not restricting how far around the  $z$ -axis we are rotating with the sphere we can take the following range for  $\theta$ .

$$0 \leq \theta \leq 2\pi$$

Now, we need to determine a range for  $\varphi$ . This will take a little work, although it's not too bad. First, let's start with the equation of the sphere.

$$x^2 + y^2 + z^2 = 16$$

Now, if we substitute the equation for the cylinder into this equation we can find the value of  $z$  where the sphere and the cylinder intersect.

$$x^2 + y^2 + z^2 = 16$$

$$12 + z^2 = 16$$

$$z^2 = 4 \quad \Rightarrow \quad z = \pm 2$$

Now, since we also specified that we only want the portion of the sphere that lies above the  $xy$ -plane we know that we need  $z = 2$ . We also know that  $\rho = 4$ . Plugging this into the following conversion formula we get,

$$z = \rho \cos \varphi$$

$$2 = 4 \cos \varphi$$

$$\cos \varphi = \frac{1}{2} \quad \Rightarrow \quad \varphi = \frac{\pi}{3}$$

So, it looks like the range of  $\varphi$  will be,

$$0 \leq \varphi \leq \frac{\pi}{3}$$

Finally, we need to determine  $\vec{r}_\theta \times \vec{r}_\varphi$ . Here are the two individual vectors.

$$\vec{r}_\theta(\theta, \varphi) = -4 \sin \varphi \sin \theta \vec{i} + 4 \sin \varphi \cos \theta \vec{j}$$

$$\vec{r}_\varphi(\theta, \varphi) = 4 \cos \varphi \cos \theta \vec{i} + 4 \cos \varphi \sin \theta \vec{j} - 4 \sin \varphi \vec{k}$$

Now let's take the cross product.

$$\begin{aligned} \vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 \sin \varphi \sin \theta & 4 \sin \varphi \cos \theta & 0 \\ 4 \cos \varphi \cos \theta & 4 \cos \varphi \sin \theta & -4 \sin \varphi \end{vmatrix} \\ &= -16 \sin^2 \varphi \cos \theta \vec{i} - 16 \sin \varphi \cos \varphi \sin^2 \theta \vec{k} - 16 \sin^2 \varphi \sin \theta \vec{j} - 16 \sin \varphi \cos \varphi \cos^2 \theta \vec{k} \\ &= -16 \sin^2 \varphi \cos \theta \vec{i} - 16 \sin^2 \varphi \sin \theta \vec{j} - 16 \sin \varphi \cos \varphi (\sin^2 \theta + \cos^2 \theta) \vec{k} \\ &= -16 \sin^2 \varphi \cos \theta \vec{i} - 16 \sin^2 \varphi \sin \theta \vec{j} - 16 \sin \varphi \cos \varphi \vec{k} \end{aligned}$$

We now need the magnitude of this,

$$\begin{aligned} \|\vec{r}_\theta \times \vec{r}_\varphi\| &= \sqrt{256 \sin^4 \varphi \cos^2 \theta + 256 \sin^4 \varphi \sin^2 \theta + 256 \sin^2 \varphi \cos^2 \varphi} \\ &= \sqrt{256 \sin^4 \varphi (\cos^2 \theta + \sin^2 \theta) + 256 \sin^2 \varphi \cos^2 \varphi} \\ &= \sqrt{256 \sin^2 \varphi (\sin^2 \varphi + \cos^2 \varphi)} \\ &= 16 \sqrt{\sin^2 \varphi} \\ &= 16 |\sin \varphi| \\ &= 16 \sin \varphi \end{aligned}$$

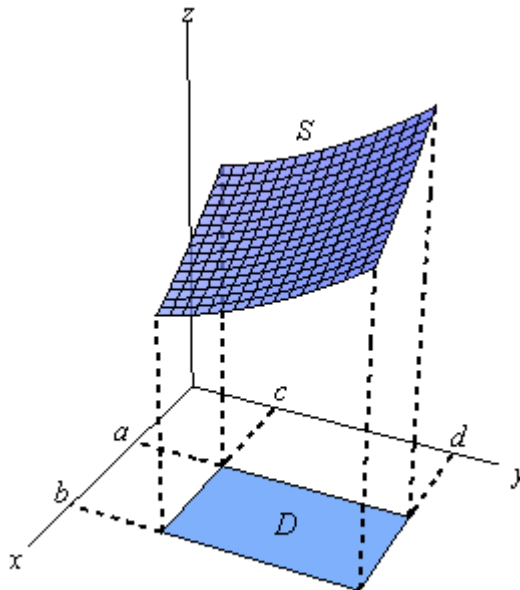
We can drop the absolute value bars in the sine because sine is positive in the range of  $\varphi$  that we are working with.

We can finally get the surface area.

$$\begin{aligned} A &= \iint_D 16 \sin \varphi \, dA \\ &= \int_0^{2\pi} \int_0^{\pi/3} 16 \sin \varphi \, d\varphi \, d\theta \\ &= \int_0^{2\pi} -16 \cos \varphi \Big|_0^{\pi/3} \, d\theta \\ &= \int_0^{2\pi} 8 \, d\theta \\ &= 16\pi \end{aligned}$$

## Surface Integrals

It is now time to think about integrating functions over some surface,  $S$ , in three-dimensional space. Let's start off with a sketch of the surface  $S$  since the notation can get a little confusing once we get into it. Here is a sketch of some surface  $S$ .



The region  $S$  will lie above (in this case) some region  $D$  that lies in the  $xy$ -plane. We used a rectangle here, but it doesn't have to be of course. Also note that we could just as easily look at a surface  $S$  that was in front of some region  $D$  in the  $yz$ -plane or the  $xz$ -plane. Do not get so locked into the  $xy$ -plane that you can't do problems that have regions in the other two planes.

Now, how we evaluate the surface integral will depend upon how the surface is given to us. There are essentially two separate methods here, although as we will see they are really the same.

First, let's look at the surface integral in which the surface  $S$  is given by  $z = g(x, y)$ . In this case the surface integral is,

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} dA$$

Now, we need to be careful here as both of these look like standard double integrals. In fact the integral on the right is a standard double integral. The integral on the left however is a surface integral. The way to tell them apart is by looking at the differentials. The surface integral will have a  $dS$  while the standard double integral will have a  $dA$ .

In order to evaluate a surface integral we will substitute the equation of the surface in for  $z$  in the integrand and then add on the often messy square root. After that the integral is a standard double integral and by this point we should be able to deal with that.

Note as well that there are similar formulas for surfaces given by  $y = g(x, z)$  (with  $D$  in the  $xz$ -plane) and  $x = g(y, z)$  (with  $D$  in the  $yz$ -plane). We will see one of these formulas in the examples and we'll leave the other to you to write down.

The second method for evaluating a surface integral is for those surfaces that are given by the parameterization,

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

In these cases the surface integral is,

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) \|\vec{r}_u \times \vec{r}_v\| dA$$

where  $D$  is the range of the parameters that trace out the surface  $S$ .

Before we work some examples let's notice that since we can parameterize a surface given by  $z = g(x, y)$  as,

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + g(x, y)\vec{k}$$

we can always use this form for these kinds of surfaces as well. In fact it can be shown that,

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

for these kinds of surfaces. You might want to verify this for the practice of computing these cross products.

Let's work some examples.

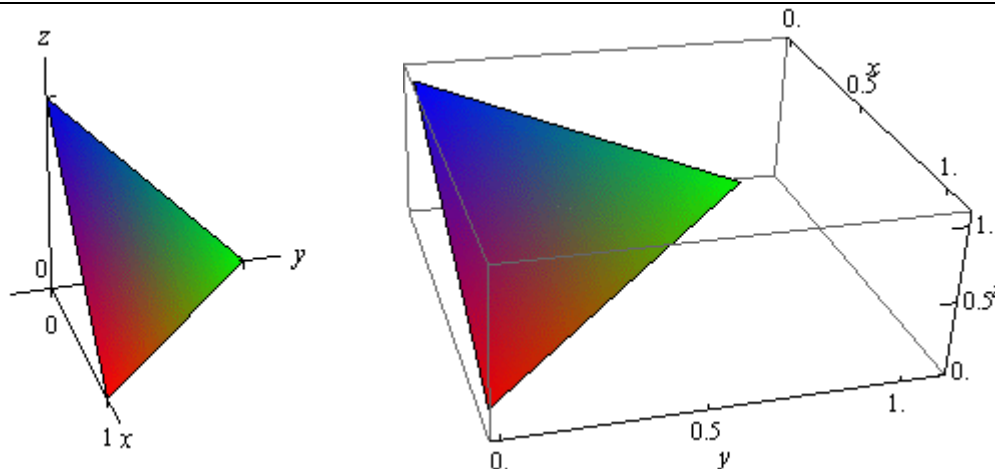
**Example 1** Evaluate  $\iint_S 6xy dS$  where  $S$  is the portion of the plane  $x + y + z = 1$  that lies in front of the  $yz$ -plane.

**Solution**

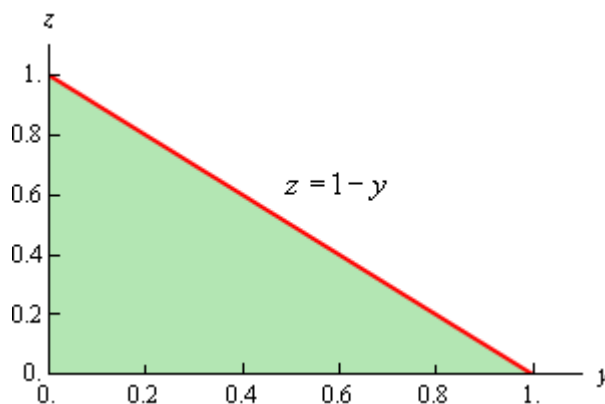
Okay, since we are looking for the portion of the plane that lies in front of the  $yz$ -plane we are going to need to write the equation of the surface in the form  $x = g(y, z)$ . This is easy enough to do.

$$x = 1 - y - z$$

Next we need to determine just what  $D$  is. Here is a sketch of the surface  $S$ .



Here is a sketch of the region  $D$ .



Notice that the axes are labeled differently than we are used to seeing in the sketch of  $D$ . This was to keep the sketch consistent with the sketch of the surface. We arrived at the equation of the hypotenuse by setting  $x$  equal to zero in the equation of the plane and solving for  $z$ . Here are the ranges for  $y$  and  $z$ .

$$0 \leq y \leq 1 \quad 0 \leq z \leq 1 - y$$

Now, because the surface is not in the form  $z = g(x, y)$  we can't use the formula above.

However, as noted above we can modify this formula to get one that will work for us. Here it is,

$$\iint_S f(x, y, z) dS = \iint_D f(g(y, z), y, z) \sqrt{1 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2} dA$$

The changes made to the formula should be the somewhat obvious changes. So, let's do the integral.

$$\iint_S 6xy dS = \iint_D 6(1 - y - z)y \sqrt{1 + (-1)^2 + (-1)^2} dA$$

Notice that we plugged in the equation of the plane for the  $x$  in the integrand. At this point we've got a fairly simple double integral to do. Here is that work.

$$\begin{aligned}
\iint_S 6xy \, dS &= \sqrt{3} \iint_D 6(y - y^2 - zy) \, dA \\
&= 6\sqrt{3} \int_0^1 \int_0^{1-y} (y - y^2 - zy) \, dz \, dy \\
&= 6\sqrt{3} \int_0^1 \left( yz - zy^2 - \frac{1}{2} z^2 y \right) \bigg|_0^{1-y} dy \\
&= 6\sqrt{3} \int_0^1 \frac{1}{2} y - y^2 + \frac{1}{2} y^3 \, dy \\
&= 6\sqrt{3} \left( \frac{1}{4} y^2 - \frac{1}{3} y^3 + \frac{1}{8} y^4 \right) \bigg|_0^1 = \frac{\sqrt{3}}{4}
\end{aligned}$$

**Example 2** Evaluate  $\iint_S z \, dS$  where  $S$  is the upper half of a sphere of radius 2.

**Solution**

We gave the parameterization of a sphere in the previous [section](#). Here is the parameterization for this sphere.

$$\vec{r}(\theta, \varphi) = 2 \sin \varphi \cos \theta \vec{i} + 2 \sin \varphi \sin \theta \vec{j} + 2 \cos \varphi \vec{k}$$

Since we are working on the upper half of the sphere here are the limits on the parameters.

$$0 \leq \theta \leq 2\pi \qquad 0 \leq \varphi \leq \frac{\pi}{2}$$

Next, we need to determine  $\vec{r}_\theta \times \vec{r}_\varphi$ . Here are the two individual vectors.

$$\vec{r}_\theta(\theta, \varphi) = -2 \sin \varphi \sin \theta \vec{i} + 2 \sin \varphi \cos \theta \vec{j}$$

$$\vec{r}_\varphi(\theta, \varphi) = 2 \cos \varphi \cos \theta \vec{i} + 2 \cos \varphi \sin \theta \vec{j} - 2 \sin \varphi \vec{k}$$

Now let's take the cross product.

$$\begin{aligned}
\vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 \sin \varphi \sin \theta & 2 \sin \varphi \cos \theta & 0 \\ 2 \cos \varphi \cos \theta & 2 \cos \varphi \sin \theta & -2 \sin \varphi \end{vmatrix} \\
&= -4 \sin^2 \varphi \cos \theta \vec{i} - 4 \sin \varphi \cos \varphi \sin^2 \theta \vec{k} - 4 \sin^2 \varphi \sin \theta \vec{j} - 4 \sin \varphi \cos \varphi \cos^2 \theta \vec{k} \\
&= -4 \sin^2 \varphi \cos \theta \vec{i} - 4 \sin^2 \varphi \sin \theta \vec{j} - 4 \sin \varphi \cos \varphi (\sin^2 \theta + \cos^2 \theta) \vec{k} \\
&= -4 \sin^2 \varphi \cos \theta \vec{i} - 4 \sin^2 \varphi \sin \theta \vec{j} - 4 \sin \varphi \cos \varphi \vec{k}
\end{aligned}$$

Finally, we need the magnitude of this,

$$\begin{aligned}
\|\vec{r}_\theta \times \vec{r}_\varphi\| &= \sqrt{16 \sin^4 \varphi \cos^2 \theta + 16 \sin^4 \varphi \sin^2 \theta + 16 \sin^2 \varphi \cos^2 \varphi} \\
&= \sqrt{16 \sin^4 \varphi (\cos^2 \theta + \sin^2 \theta) + 16 \sin^2 \varphi \cos^2 \varphi} \\
&= \sqrt{16 \sin^2 \varphi (\sin^2 \varphi + \cos^2 \varphi)} \\
&= 4 \sqrt{\sin^2 \varphi} \\
&= 4 |\sin \varphi| \\
&= 4 \sin \varphi
\end{aligned}$$

We can drop the absolute value bars in the sine because sine is positive in the range of  $\varphi$  that we are working with. The surface integral is then,

$$\iint_S z \, dS = \iint_D 2 \cos \varphi (4 \sin \varphi) \, dA$$

Don't forget that we need to plug in for  $x$ ,  $y$  and/or  $z$  in these as well, although in this case we just needed to plug in  $z$ . Here is the evaluation for the double integral.

$$\begin{aligned}
\iint_S z \, dS &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 4 \sin(2\varphi) \, d\varphi \, d\theta \\
&= \int_0^{2\pi} \left( -2 \cos(2\varphi) \right) \Big|_0^{\frac{\pi}{2}} d\theta \\
&= \int_0^{2\pi} 4 \, d\theta \\
&= 8\pi
\end{aligned}$$

**Example 3** Evaluate  $\iint_S y \, dS$  where  $S$  is the portion of the cylinder  $x^2 + y^2 = 3$  that lies between  $z = 0$  and  $z = 6$ .

**Solution**

We parameterized up a cylinder in the previous [section](#). Here is the parameterization of this cylinder.

$$\vec{r}(z, \theta) = \sqrt{3} \cos \theta \vec{i} + \sqrt{3} \sin \theta \vec{j} + z \vec{k}$$

The ranges of the parameters are,

$$0 \leq z \leq 6 \quad 0 \leq \theta \leq 2\pi$$

Now we need  $\vec{r}_z \times \vec{r}_\theta$ . Here are the two vectors.

$$\vec{r}_z(z, \theta) = \vec{k}$$

$$\vec{r}_\theta(z, \theta) = -\sqrt{3} \sin \theta \vec{i} + \sqrt{3} \cos \theta \vec{j}$$

Here is the cross product.

$$\begin{aligned}\vec{r}_z \times \vec{r}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\sqrt{3} \sin \theta & \sqrt{3} \cos \theta & 0 \end{vmatrix} \\ &= -\sqrt{3} \cos \theta \vec{i} - \sqrt{3} \sin \theta \vec{j}\end{aligned}$$

The magnitude of this vector is,

$$\|\vec{r}_z \times \vec{r}_\theta\| = \sqrt{3 \cos^2 \theta + 3 \sin^2 \theta} = \sqrt{3}$$

The surface integral is then,

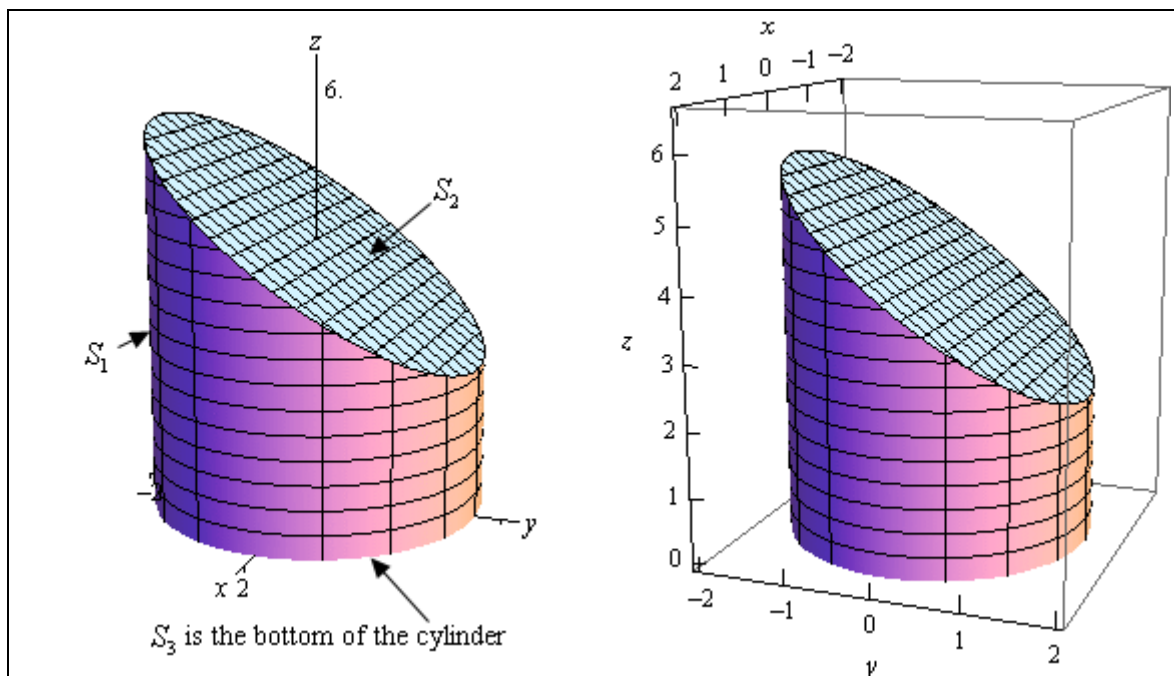
$$\begin{aligned}\iint_S y \, dS &= \iint_D \sqrt{3} \sin \theta (\sqrt{3}) \, dA \\ &= 3 \int_0^{2\pi} \int_0^6 \sin \theta \, dz \, d\theta \\ &= 3 \int_0^{2\pi} 6 \sin \theta \, d\theta \\ &= (-18 \cos \theta) \Big|_0^{2\pi} \\ &= 0\end{aligned}$$

**Example 4** Evaluate  $\iint_S y + z \, dS$  where  $S$  is the surface whose side is the cylinder  $x^2 + y^2 = 3$ , whose bottom is the disk  $x^2 + y^2 \leq 3$  in the  $xy$ -plane and whose top is the plane  $z = 4 - y$ .

**Solution**

There is a lot of information that we need to keep track of here. First, we are using pretty much the same surface (the integrand is different however) as the previous example. However, unlike the previous example we are putting a top and bottom on the surface this time. Let's first start out with a sketch of the surface.





Actually we need to be careful here. There is more to this sketch than the actual surface itself. We're going to let  $S_1$  be the portion of the cylinder that goes from the  $xy$ -plane to the plane. In other words, the top of the cylinder will be at an angle. We'll call the portion of the plane that lies inside (*i.e.* the cap on the cylinder)  $S_2$ . Finally, the bottom of the cylinder (not shown here) is the disk of radius  $\sqrt{3}$  in the  $xy$ -plane and is denoted by  $S_3$ .

In order to do this integral we'll need to note that just like the standard double integral, if the surface is split up into pieces we can also split up the surface integral. So, for our example we will have,

$$\iint_S y + z \, dS = \iint_{S_1} y + z \, dS + \iint_{S_2} y + z \, dS + \iint_{S_3} y + z \, dS$$

We're going to need to do three integrals here. However, we've done most of the work for the first one in the previous example so let's start with that.

### $S_1$ : The Cylinder

The parameterization of the cylinder and  $\|\vec{r}_z \times \vec{r}_\theta\|$  is,

$$\vec{r}(z, \theta) = \sqrt{3} \cos \theta \vec{i} + \sqrt{3} \sin \theta \vec{j} + z \vec{k} \quad \|\vec{r}_z \times \vec{r}_\theta\| = \sqrt{3}$$

The difference between this problem and the previous one is the limits on the parameters. Here they are.

$$0 \leq \theta \leq 2\pi$$

$$0 \leq z \leq 4 - y = 4 - \sqrt{3} \sin \theta$$

The upper limit for the  $z$ 's is the plane so we can just plug that in. However, since we are on the cylinder we know what  $y$  is from the parameterization so we will also need to plug that in.

Here is the integral for the cylinder.

$$\begin{aligned}
 \iint_{S_1} y + z \, dS &= \iint_D (\sqrt{3} \sin \theta + z)(\sqrt{3}) \, dA \\
 &= \sqrt{3} \int_0^{2\pi} \int_0^{4-\sqrt{3} \sin \theta} \sqrt{3} \sin \theta + z \, dz \, d\theta \\
 &= \sqrt{3} \int_0^{2\pi} \sqrt{3} \sin \theta (4 - \sqrt{3} \sin \theta) + \frac{1}{2} (4 - \sqrt{3} \sin \theta)^2 \, d\theta \\
 &= \sqrt{3} \int_0^{2\pi} 8 - \frac{3}{2} \sin^2 \theta \, d\theta \\
 &= \sqrt{3} \int_0^{2\pi} 8 - \frac{3}{4} (1 - \cos(2\theta)) \, d\theta \\
 &= \sqrt{3} \left( \frac{29}{4} \theta + \frac{3}{8} \sin(2\theta) \right) \Big|_0^{2\pi} \\
 &= \frac{29\sqrt{3}\pi}{2}
 \end{aligned}$$

$S_2$  : Plane on Top of the Cylinder

In this case we don't need to do any parameterization since it is set up to use the formula that we gave at the start of this section. Remember that the plane is given by  $z = 4 - y$ . Also note that, for this surface,  $D$  is the disk of radius  $\sqrt{3}$  centered at the origin.

Here is the integral for the plane.

$$\begin{aligned}
 \iint_{S_2} y + z \, dS &= \iint_D (y + 4 - y) \sqrt{(0)^2 + (-1)^2 + 1} \, dA \\
 &= \sqrt{2} \iint_D 4 \, dA
 \end{aligned}$$

Don't forget that we need to plug in for  $z$ ! Now at this point we can proceed in one of two ways. Either we can proceed with the integral or we can recall that  $\iint_D dA$  is nothing more than the area of  $D$  and we know that  $D$  is the disk of radius  $\sqrt{3}$  and so there is no reason to do the integral.

Here is the remainder of the work for this problem.

$$\begin{aligned}
 \iint_{S_2} y + z \, dS &= 4\sqrt{2} \iint_D dA \\
 &= 4\sqrt{2} \left( \pi (\sqrt{3})^2 \right) \\
 &= 12\sqrt{2} \pi
 \end{aligned}$$

$S_3$  : Bottom of the Cylinder

Again, this is set up to use the initial formula we gave in this section once we realize that the equation for the bottom is given by  $g(x, y) = 0$  and  $D$  is the disk of radius  $\sqrt{3}$  centered at the origin. Also, don't forget to plug in for  $z$ .

Here is the work for this integral.

$$\begin{aligned}
 \iint_{S_3} y + z \, dS &= \iint_D (y + 0) \sqrt{(0)^2 + (0)^2 + (1)^2} \, dA \\
 &= \iint_D y \, dA \\
 &= \int_0^{2\pi} \int_0^{\sqrt{3}} r^2 \sin \theta \, dr \, d\theta \\
 &= \int_0^{2\pi} \left( \frac{1}{3} r^3 \sin \theta \right) \Big|_0^{\sqrt{3}} d\theta \\
 &= \int_0^{2\pi} \sqrt{3} \sin \theta \, d\theta \\
 &= -\sqrt{3} \cos \theta \Big|_0^{2\pi} \\
 &= 0
 \end{aligned}$$

We can now get the value of the integral that we are after.

$$\begin{aligned}
 \iint_S y + z \, dS &= \iint_{S_1} y + z \, dS + \iint_{S_2} y + z \, dS + \iint_{S_3} y + z \, dS \\
 &= \frac{29\sqrt{3} \pi}{2} + 12\sqrt{2} \pi + 0 \\
 &= \frac{\pi}{2} (29\sqrt{3} + 24\sqrt{2})
 \end{aligned}$$

## Surface Integrals of Vector Fields

---

Just as we did with line integrals we now need to move on to surface integrals of vector fields. Recall that in line integrals the orientation of the curve we were integrating along could change the answer. The same thing will hold true with surface integrals. So, before we really get into doing surface integrals of vector fields we first need to introduce the idea of an **oriented surface**.

Let's start off with a surface that has two sides (while this may seem strange, recall that the [Möbius Strip](#) is a surface that only has one side!) that has a tangent plane at every point (except possibly along the boundary). Making this assumption means that every point will have two unit normal vectors,  $\vec{n}_1$  and  $\vec{n}_2 = -\vec{n}_1$ . This means that every surface will have two sets of normal vectors. The set that we choose will give the surface an orientation.

There is one convention that we will make in regards to certain kinds of oriented surfaces. First we need to define a **closed surface**. A surface  $S$  is closed if it is the boundary of some solid region  $E$ . A good example of a closed surface is the surface of a sphere. We say that the closed surface  $S$  has a **positive** orientation if we choose the set of unit normal vectors that point outward from the region  $E$  while the **negative** orientation will be the set of unit normal vectors that point in towards the region  $E$ .

Note that this convention is only used for closed surfaces.

In order to work with surface integrals of vector fields we will need to be able to write down a formula for the unit normal vector corresponding to the orientation that we've chosen to work with. We have two ways of doing this depending on how the surface has been given to us.

First, let's suppose that the function is given by  $z = g(x, y)$ . In this case we first define a new function,

$$f(x, y, z) = z - g(x, y)$$

In terms of our new function the surface is then given by the equation  $f(x, y, z) = 0$ . Now, [recall](#) that  $\nabla f$  will be orthogonal (or normal) to the surface given by  $f(x, y, z) = 0$ . This means that we have a normal vector to the surface. The only potential problem is that it might not be a unit normal vector. That isn't a problem since we also know that we can turn any vector into a unit vector by dividing the vector by its length. In or case this is,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|}$$

In this case it will be convenient to actually compute the gradient vector and plug this into the formula for the normal vector. Doing this gives,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{-g_x \vec{i} - g_y \vec{j} + \vec{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}$$

Now, from a notational standpoint this might not have been so convenient, but it does allow us to make a couple of additional comments.

First, notice that the component of the normal vector in the  $z$ -direction (identified by the  $\vec{k}$  in the normal vector) is always positive and so this normal vector will generally point upwards. It may not point directly up, but it will have an upwards component to it.

This will be important when we are working with a closed surface and we want the positive orientation. If we know that we can then look at the normal vector and determine if the “positive” orientation should point upwards or downwards. Remember that the “positive” orientation must point out of the region and this may mean downwards in places. Of course if it turns out that we need the downward orientation we can always take the negative of this unit vector and we’ll get the one that we need. Again, remember that we always have that option when choosing the unit normal vector.

Before we move onto the second method of giving the surface we should point out that we only did this for surfaces in the form  $z = g(x, y)$ . We could just as easily done the above work for surfaces in the form  $y = g(x, z)$  (so  $f(x, y, z) = y - g(x, z)$ ) or for surfaces in the form  $x = g(y, z)$  (so  $f(x, y, z) = x - g(y, z)$ ).

Now, we need to discuss how to find the unit normal vector if the surface is given parametrically as,

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

In this case [recall](#) that the vector  $\vec{r}_u \times \vec{r}_v$  will be normal to the tangent plane at a particular point.

But if the vector is normal to the tangent plane at a point then it will also be normal to the surface at that point. So, this is a normal vector. In order to guarantee that it is a unit normal vector we will also need to divide it by its magnitude.

So, in the case of parametric surfaces one of the unit normal vectors will be,

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

As with the first case we will need to look at this once it’s computed and determine if it points in the correct direction or not. If it doesn’t then we can always take the negative of this vector and that will point in the correct direction.

Finally, remember that we can always parameterize any surface given by  $z = g(x, y)$  (or  $y = g(x, z)$  or  $x = g(y, z)$ ) easily enough and so if we want to we can always use the parameterization formula to find the unit normal vector.

Okay, now that we’ve looked at oriented surfaces and their associated unit normal vectors we can actually give a formula for evaluating surface integrals of vector fields.

Given a vector field  $\vec{F}$  with unit normal vector  $\vec{n}$  then the surface integral of  $\vec{F}$  over the surface  $S$  is given by,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

where the right hand integral is a standard surface integral. This is sometimes called the **flux of  $\vec{F}$  across  $S$** .

Before we work any examples let's notice that we can substitute in for the unit normal vector to get a *somewhat* easier formula to use. We will need to be careful with each of the following formulas however as each will assume a certain orientation and we may have to change the normal vector to match the given orientation.

Let's first start by assuming that the surface is given by  $z = g(x, y)$ . In this case let's also assume that the vector field is given by  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$  and that the orientation that we are after is the "upwards" orientation. Under all of these assumptions the surface integral of  $\vec{F}$  over  $S$  is,

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \vec{n} \, dS \\ &= \iint_D (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot \left( \frac{-g_x\vec{i} - g_y\vec{j} + \vec{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \right) \sqrt{(g_x)^2 + (g_y)^2 + 1} \, dA \\ &= \iint_D (P\vec{i} + Q\vec{j} + R\vec{k}) \cdot (-g_x\vec{i} - g_y\vec{j} + \vec{k}) \, dA \\ &= \iint_D -Pg_x - Qg_y + R \, dA\end{aligned}$$

Now, remember that this assumed the "upward" orientation. If we'd needed the "downward" orientation then we would need to change the signs on the normal vector. This would in turn change the signs on the integrand as well. So, we really need to be careful here when using this formula. In general it is best to rederive this formula as you need it.

When we've been given a surface that is not in parametric form there are in fact 6 possible integrals here. Two for each form of the surface  $z = g(x, y)$ ,  $y = g(x, z)$  and  $x = g(y, z)$ . Given each form of the surface there will be two possible unit normal vectors and we'll need to choose the correct one to match the given orientation of the surface. However, the derivation of each formula is similar to that given here and so shouldn't be too bad to do as you need to.

Notice as well that because we are using the unit normal vector the messy square root will always drop out. This means that when we do need to derive the formula we won't really need to put this in. All we'll need to work with is the numerator of the unit vector. We will see at least one more of these derived in the examples below. It should also be noted that the square root is nothing more than,

$$\sqrt{(g_x)^2 + (g_y)^2 + 1} = \|\nabla f\|$$

so in the following work we will probably just use this notation in place of the square root when we can to make things a little simpler.

Let's now take a quick look at the formula for the surface integral when the surface is given parametrically by  $\vec{r}(u, v)$ . In this case the surface integral is,

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \vec{n} \, dS \\ &= \iint_D \vec{F} \cdot \left( \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \right) \|\vec{r}_u \times \vec{r}_v\| \, dA \\ &= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA\end{aligned}$$

Again note that we may have to change the sign on  $\vec{r}_u \times \vec{r}_v$  to match the orientation of the surface and so there is once again really two formulas here. Also note that again the magnitude cancels in this case and so we won't need to worry that in these problems either.

Note as well that there are even times when we will use the definition,  $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS$ , directly. We will see an example of this below.

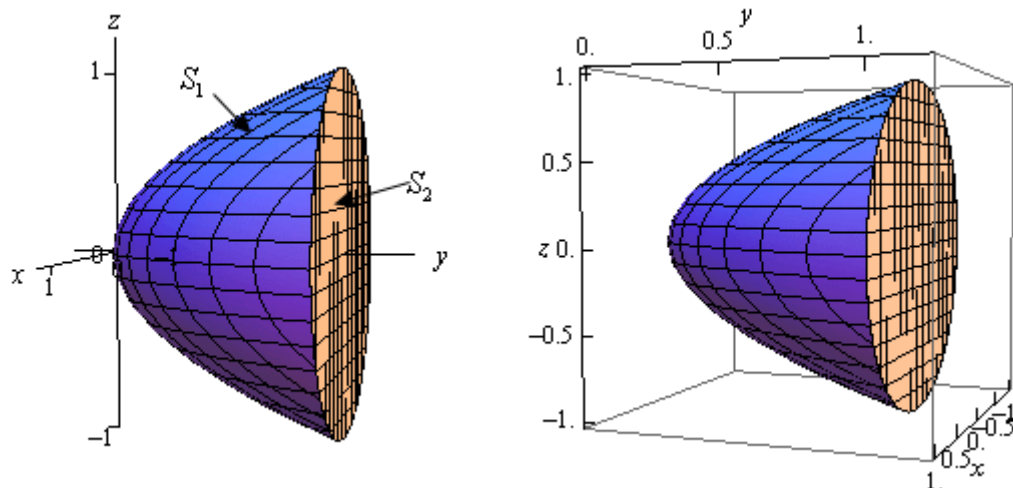
Let's now work a couple of examples.

**Example 1** Evaluate  $\iint_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = y\vec{j} - z\vec{k}$  and  $S$  is the surface given by the paraboloid  $y = x^2 + z^2$ ,  $0 \leq y \leq 1$  and the disk  $x^2 + z^2 \leq 1$  at  $y = 1$ . Assume that  $S$  has positive orientation.

**Solution**

Okay, first let's notice that the disk is really nothing more than the cap on the paraboloid. This means that we have a closed surface. This is important because we've been told that the surface has a positive orientation and by convention this means that all the unit normal vectors will need to point outwards from the region enclosed by  $S$ .

Let's first get a sketch of  $S$  so we can get a feel for what is going on and in which direction we will need to unit normal vectors to point.



As noted in the sketch we will denote the paraboloid by  $S_1$  and the disk by  $S_2$ . Also note that in order for unit normal vectors on the paraboloid to point away from the region they will all need to point generally in the negative y direction. On the other hand, unit normal vectors on the disk will need to point in the positive y direction in order to point away from the region.

Since  $S$  is composed of the two surfaces we'll need to do the surface integral on each and then add the results to get the overall surface integral. Let's start with the paraboloid. In this case we have the surface in the form  $y = g(x, z)$  so we will need to derive the correct formula since the one given initially wasn't for this kind of function. This is easy enough to do however. First define,

$$f(x, y, z) = y - g(x, z) = y - x^2 - z^2$$

We will next need the gradient vector of this function.

$$\nabla f = \langle -2x, 1, -2z \rangle$$

Now, the y component of the gradient is positive and so this vector will generally point in the positive y direction. However, as noted above we need the normal vector point in the negative y direction to make sure that it will be pointing away from the enclosed region. This means that we will need to use

$$\vec{n} = \frac{-\nabla f}{\|-\nabla f\|} = \frac{\langle 2x, -1, 2z \rangle}{\|\nabla f\|}$$

Let's note a couple of things here before we proceed. We don't really need to divide this by the magnitude of the gradient since this will just cancel out once we actually do the integral. So, because of this we didn't bother computing it. Also, the dropping of the minus sign is not a typo. When we compute the magnitude we are going to square each of the components and so the minus sign will drop out.

$S_1$  : The Paraboloid

Okay, here is the surface integral in this case.



$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S} &= \iint_D (y \vec{j} - z \vec{k}) \cdot \left( \frac{\langle 2x, -1, 2z \rangle}{\|\nabla f\|} \right) \|\nabla f\| dA \\
&= \iint_D -y - 2z^2 dA \\
&= \iint_D -(x^2 + z^2) - 2z^2 dA \\
&= -\iint_D x^2 + 3z^2 dA
\end{aligned}$$

Don't forget that we need to plug in the equation of the surface for  $y$  before we actually compute the integral. In this case  $D$  is the disk of radius 1 in the  $xz$ -plane and so it makes sense to use polar coordinates to complete this integral. Here are polar coordinates for this region.

$$\begin{aligned}
x &= r \cos \theta & z &= r \sin \theta \\
0 \leq \theta &\leq 2\pi & 0 \leq r &\leq 1
\end{aligned}$$

Note that we kept the  $x$  conversion formula the same as the one we are used to using for  $x$  and let  $z$  be the formula that used the sine. We could have done it any order, however in this way we are at least working with one of them as we are used to working with.

Here is the evaluation of this integral.

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S} &= -\iint_D x^2 + 3z^2 dA \\
&= -\int_0^{2\pi} \int_0^1 (r^2 \cos^2 \theta + 3r^2 \sin^2 \theta) r dr d\theta \\
&= -\int_0^{2\pi} \int_0^1 (\cos^2 \theta + 3\sin^2 \theta) r^3 dr d\theta \\
&= -\int_0^{2\pi} \left( \frac{1}{2}(1 + \cos(2\theta)) + \frac{3}{2}(1 - \cos(2\theta)) \right) \left( \frac{1}{4} r^4 \right) \Big|_0^1 d\theta \\
&= -\frac{1}{8} \int_0^{2\pi} 4 - 2\cos(2\theta) d\theta \\
&= -\frac{1}{8} (4\theta - \sin(2\theta)) \Big|_0^{2\pi} \\
&= -\pi
\end{aligned}$$

### $S_2$ : The Cap of the Paraboloid

We can now do the surface integral on the disk (cap on the paraboloid). This one is actually fairly easy to do and in fact we can use the definition of the surface integral directly. First let's notice that the disk is really just the portion of the plane  $y = 1$  that is in front of the disk of radius 1 in the  $xz$ -plane.

Now we want the unit normal vector to point away from the enclosed region and since it must

also be orthogonal to the plane  $y = 1$  then it must point in a direction that is parallel to the  $y$ -axis, but we already have a unit vector that does this. Namely,

$$\vec{n} = \vec{j}$$

the standard unit basis vector. It also points in the correct direction for us to use. Because we have the vector field and the normal vector we can plug directly into the definition of the surface integral to get,

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} (y\vec{j} - z\vec{k}) \cdot (\vec{j}) dS = \iint_{S_2} y dS$$

At this point we need to plug in for  $y$  (since  $S_2$  is a portion of the plane  $y = 1$  we do know what it is) and we'll also need the square root this time when we convert the surface integral over to a double integral. In this case since we are using the definition directly we won't get the canceling of the square root that we saw with the first portion. To get the square root we'll need to acknowledge that

$$y = 1 = g(x, z)$$

and so the square root is,

$$\sqrt{(g_x)^2 + 1 + (g_z)^2}$$

The surface integral is then,

$$\begin{aligned} \iint_{S_2} \vec{F} \cdot d\vec{S} &= \iint_{S_2} y dS \\ &= \iint_D 1\sqrt{0+1+0} dA = \iint_D dA \end{aligned}$$

At this point we can acknowledge that  $D$  is a disk of radius 1 and this double integral is nothing more than the double integral that will give the area of the region  $D$  so there is no reason to compute the integral. Here is the value of the surface integral.

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \pi$$

Finally, to finish this off we just need to add the two parts up. Here is the surface integral that we were actually asked to compute.

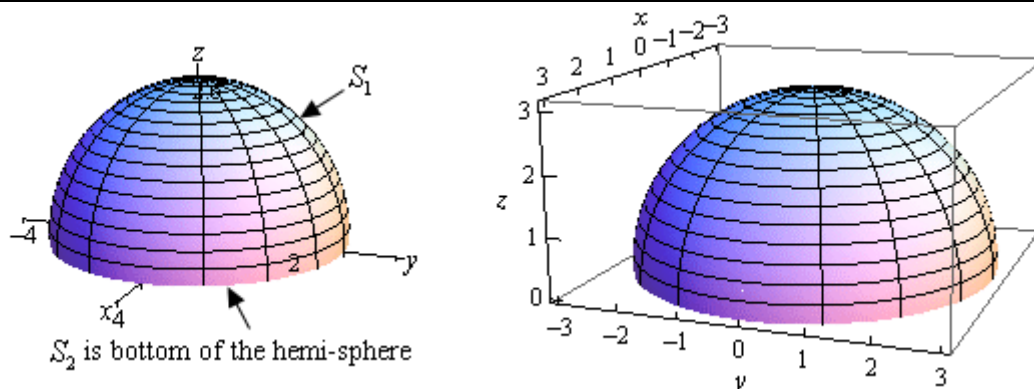
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = -\pi + \pi = 0$$

**Example 2** Evaluate  $\iint_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = x\vec{i} + y\vec{j} + z^4\vec{k}$  and  $S$  is the upper half the sphere

$x^2 + y^2 + z^2 = 9$  and the disk  $x^2 + y^2 \leq 9$  in the plane  $z = 0$ . Assume that  $S$  has the positive orientation.

**Solution**

So, as with the previous problem we have a closed surface and since we are also told that the surface has a positive orientation all the unit normal vectors must point away from the enclosed region. To help us visualize this here is a sketch of the surface.



We will call  $S_1$  the hemisphere and  $S_2$  will be the bottom of the hemisphere (which isn't shown on the sketch). Now, in order for the unit normal vectors on the sphere to point away from enclosed region they will all need to have a positive  $z$  component. Remember that the vector must be normal to the surface and if there is a positive  $z$  component and the vector is normal it will have to be pointing away from the enclosed region.

On the other hand, the unit normal on the bottom of the disk must point in the negative  $z$  direction in order to point away from the enclosed region.

### $S_1$ : The Sphere

Let's do the surface integral on  $S_1$  first. In this case since the surface is a sphere we will need to use the parametric representation of the surface. This is,

$$\vec{r}(\theta, \varphi) = 3 \sin \varphi \cos \theta \vec{i} + 3 \sin \varphi \sin \theta \vec{j} + 3 \cos \varphi \vec{k}$$

Since we are working on the hemisphere here are the limits on the parameters that we'll need to use.

$$0 \leq \theta \leq 2\pi \qquad 0 \leq \varphi \leq \frac{\pi}{2}$$

Next, we need to determine  $\vec{r}_\theta \times \vec{r}_\varphi$ . Here are the two individual vectors and the cross product.

$$\vec{r}_\theta(\theta, \varphi) = -3 \sin \varphi \sin \theta \vec{i} + 3 \sin \varphi \cos \theta \vec{j}$$

$$\vec{r}_\varphi(\theta, \varphi) = 3 \cos \varphi \cos \theta \vec{i} + 3 \cos \varphi \sin \theta \vec{j} - 3 \sin \varphi \vec{k}$$

$$\begin{aligned} \vec{r}_\theta \times \vec{r}_\varphi &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 \sin \varphi \sin \theta & 3 \sin \varphi \cos \theta & 0 \\ 3 \cos \varphi \cos \theta & 3 \cos \varphi \sin \theta & -3 \sin \varphi \end{vmatrix} \\ &= -9 \sin^2 \varphi \cos \theta \vec{i} - 9 \sin \varphi \cos \varphi \sin^2 \theta \vec{k} - 9 \sin^2 \varphi \sin \theta \vec{j} - 9 \sin \varphi \cos \varphi \cos^2 \theta \vec{k} \\ &= -9 \sin^2 \varphi \cos \theta \vec{i} - 9 \sin^2 \varphi \sin \theta \vec{j} - 9 \sin \varphi \cos \varphi (\sin^2 \theta + \cos^2 \theta) \vec{k} \\ &= -9 \sin^2 \varphi \cos \theta \vec{i} - 9 \sin^2 \varphi \sin \theta \vec{j} - 9 \sin \varphi \cos \varphi \vec{k} \end{aligned}$$

Note that we won't need the magnitude of the cross product since that will cancel out once we start doing the integral.

Notice that for the range of  $\varphi$  that we've got both sine and cosine are positive and so this vector will have a negative  $z$  component and as we noted above in order for this to point away from the enclosed area we will need the  $z$  component to be positive. Therefore we will need to use the following vector for the unit normal vector.

$$\vec{n} = -\frac{\vec{r}_\theta \times \vec{r}_\varphi}{\|\vec{r}_\theta \times \vec{r}_\varphi\|} = \frac{9\sin^2 \varphi \cos \theta \vec{i} + 9\sin^2 \varphi \sin \theta \vec{j} + 9\sin \varphi \cos \varphi \vec{k}}{\|\vec{r}_\theta \times \vec{r}_\varphi\|}$$

Again, we will drop the magnitude once we get to actually doing the integral since it will just cancel in the integral.

Okay, next we'll need

$$\vec{F}(\vec{r}(\theta, \varphi)) = 3\sin \varphi \cos \theta \vec{i} + 3\sin \varphi \sin \theta \vec{j} + 81\cos^4 \varphi \vec{k}$$

Remember that in this evaluation we are just plugging in the  $x$  component of  $\vec{r}(\theta, \varphi)$  into the vector field *etc.*

We also may as well get the dot product out of the way that we know we are going to need.

$$\begin{aligned}\vec{F}(\vec{r}(\theta, \varphi)) \cdot (\vec{r}_\theta \times \vec{r}_\varphi) &= 27\sin^3 \varphi \cos^2 \theta + 27\sin^3 \varphi \sin^2 \theta + 729\sin \varphi \cos^5 \varphi \\ &= 27\sin^3 \varphi + 729\sin \varphi \cos^5 \varphi\end{aligned}$$

Now we can do the integral.

$$\begin{aligned}\iint_{S_1} \vec{F} \cdot d\vec{S} &= \iint_D \vec{F} \cdot \left( \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \right) \|\vec{r}_\theta \times \vec{r}_\varphi\| dA \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 27\sin^3 \varphi + 729\sin \varphi \cos^5 \varphi d\varphi d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 27\sin \varphi (1 - \cos^2 \varphi) + 729\sin \varphi \cos^5 \varphi d\varphi d\theta \\ &= - \int_0^{2\pi} \left( 27 \left( \cos \varphi - \frac{1}{3} \cos^3 \varphi \right) + \frac{729}{6} \cos^6 \varphi \right) \Big|_0^{\frac{\pi}{2}} d\theta \\ &= \int_0^{2\pi} \frac{279}{2} d\theta \\ &= 279\pi\end{aligned}$$

$S_1$  : The Bottom of the Hemi-Sphere

Now, we need to do the integral over the bottom of the hemisphere. In this case we are looking at the disk  $x^2 + y^2 \leq 9$  that lies in the plane  $z = 0$  and so the equation of this surface is actually  $z = 0$ . The disk is really the region  $D$  that tells us how much of the surface we are going to use.

This also means that we can use the definition of the surface integral here with

$$\vec{n} = -\vec{k}$$

We need the negative since it must point away from the enclosed region.

The surface integral in this case is,

$$\begin{aligned}\iint_{S_2} \vec{F} \cdot d\vec{S} &= \iint_{S_2} (x\vec{i} + y\vec{j} + z^4\vec{k}) \cdot (-\vec{k}) dS \\ &= \iint_{S_2} -z^4 dS\end{aligned}$$

Remember, however, that we are in the plane given by  $z = 0$  and so the surface integral becomes,

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} -z^4 dS = \iint_{S_2} 0 dS = 0$$

The last step is to then add the two pieces up. Here is surface integral that we were asked to look at.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = 279\pi + 0 = 279\pi$$

We will leave this section with a quick interpretation of a surface integral over a vector field. If  $\vec{v}$  is the velocity field of a fluid then the surface integral

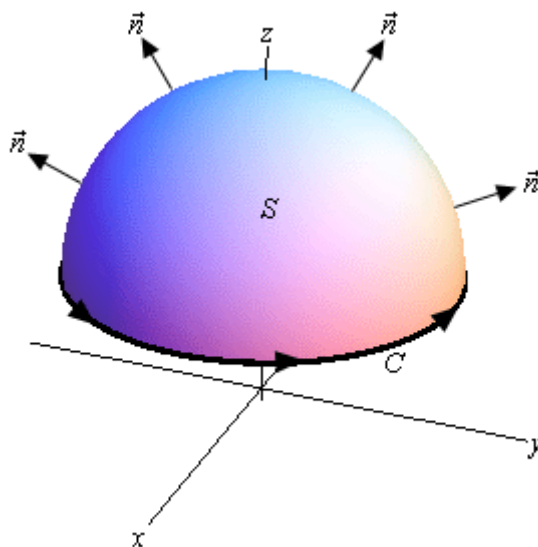
$$\iint_S \vec{v} \cdot d\vec{S}$$

represents the volume of fluid flowing through  $S$  per time unit (*i.e.* per second, per minute, or whatever time unit you are using).

## Stokes' Theorem

In this section we are going to take a look at a theorem that is a higher dimensional version of [Green's Theorem](#). In Green's Theorem we related a line integral to a double integral over some region. In this section we are going to relate a line integral to a surface integral. However, before we give the theorem we first need to define the curve that we're going to use in the line integral.

Let's start off with the following surface with the indicated orientation.



Around the edge of this surface we have a curve  $C$ . This curve is called the **boundary curve**. The orientation of the surface  $S$  will induce the **positive orientation of  $C$** . To get the positive orientation of  $C$  think of yourself as walking along the curve. While you are walking along the curve if your head is pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on  $C$ .

Now that we have this curve definition out of the way we can give Stokes' Theorem.

### Stokes' Theorem

Let  $S$  be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve  $C$  with positive orientation. Also let  $\vec{F}$  be a vector field then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

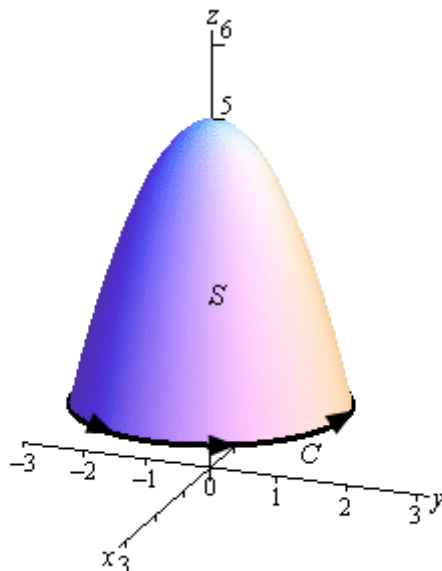
In this theorem note that the surface  $S$  can actually be any surface so long as its boundary curve is given by  $C$ . This is something that can be used to our advantage to simplify the surface integral on occasion.

Let's take a look at a couple of examples.

**Example 1** Use Stokes' Theorem to evaluate  $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$  where  $\vec{F} = z^2 \vec{i} - 3xy \vec{j} + x^3 y^3 \vec{k}$  and  $S$  is the part of  $z = 5 - x^2 - y^2$  above the plane  $z = 1$ . Assume that  $S$  is oriented upwards.

**Solution**

Let's start this off with a sketch of the surface.



In this case the boundary curve  $C$  will be where the surface intersects the plane  $z = 1$  and so will be the curve

$$\begin{aligned} 1 &= 5 - x^2 - y^2 \\ x^2 + y^2 &= 4 \quad \text{at } z = 1 \end{aligned}$$

So, the boundary curve will be the circle of radius 2 that is in the plane  $z = 1$ . The parameterization of this curve is,

$$\vec{r}(t) = 2 \cos t \vec{i} + 2 \sin t \vec{j} + \vec{k}, \quad 0 \leq t \leq 2\pi$$

The first two components give the circle and the third component makes sure that it is in the plane  $z = 1$ .

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

So, it looks like we need a couple of quantities before we do this integral. Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\begin{aligned} \vec{F}(\vec{r}(t)) &= (1)^2 \vec{i} - 3(2 \cos t)(2 \sin t) \vec{j} + (2 \cos t)^3 (2 \sin t)^3 \vec{k} \\ &= \vec{i} - 12 \cos t \sin t \vec{j} + 64 \cos^3 t \sin^3 t \vec{k} \end{aligned}$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\begin{aligned}\vec{r}'(t) &= -2\sin t \vec{i} + 2\cos t \vec{j} \\ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= -2\sin t - 24\sin t \cos^2 t\end{aligned}$$

We can now do the integral.

$$\begin{aligned}\iint_S \text{curl } \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} -2\sin t - 24\sin t \cos^2 t \, dt \\ &= \left(2\cos t + 8\cos^3 t\right)\Big|_0^{2\pi} \\ &= 0\end{aligned}$$

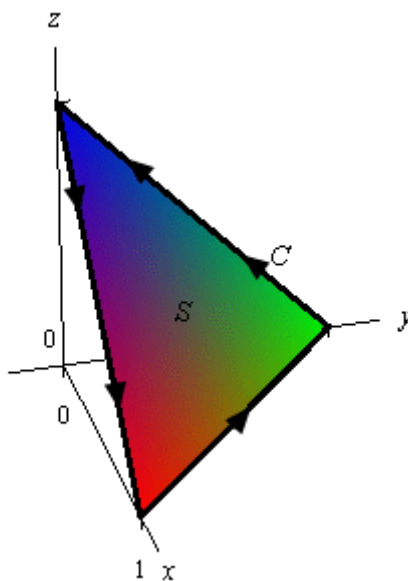
**Example 2** Use Stokes' Theorem to evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = z^2 \vec{i} + y^2 \vec{j} + x \vec{k}$  and  $C$  is the triangle with vertices  $(1,0,0)$ ,  $(0,1,0)$  and  $(0,0,1)$  with counter-clockwise rotation.

**Solution**

We are going to need the curl of the vector field eventually so let's get that out of the way first.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 2z \vec{j} - \vec{j} = (2z-1) \vec{j}$$

Now, all we have is the boundary curve for the surface that we'll need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.





Since the plane is oriented upwards this induces the positive direction on  $C$  as shown. The equation of this plane is,

$$x + y + z = 1 \quad \Rightarrow \quad z = g(x, y) = 1 - x - y$$

Now, let's use Stokes' Theorem and get the surface integral set up.

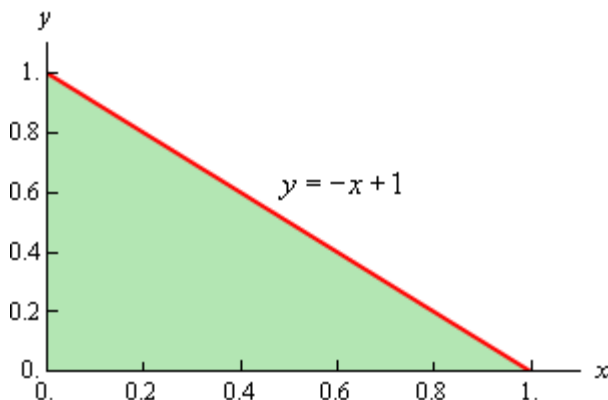
$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl } \vec{F} \cdot d\vec{S} \\ &= \iint_S (2z - 1) \vec{j} \cdot d\vec{S} \\ &= \iint_D (2z - 1) \vec{j} \cdot \frac{\nabla f}{\|\nabla f\|} \|\nabla f\| dA \end{aligned}$$

Okay, we now need to find a couple of quantities. First let's get the gradient. Recall that this comes from the function of the surface.

$$\begin{aligned} f(x, y, z) &= z - g(x, y) = z - 1 + x + y \\ \nabla f &= \vec{i} + \vec{j} + \vec{k} \end{aligned}$$

Note as well that this also points upwards and so we have the correct direction.

Now,  $D$  is the region in the  $xy$ -plane shown below,



We get the equation of the line by plugging in  $z = 0$  into the equation of the plane. So based on this the ranges that define  $D$  are,

$$0 \leq x \leq 1 \quad 0 \leq y \leq -x + 1$$

The integral is then,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_D (2z - 1) \vec{j} \cdot (\vec{i} + \vec{j} + \vec{k}) dA \\ &= \int_0^1 \int_0^{-x+1} 2(1 - x - y) - 1 dy dx \end{aligned}$$

Don't forget to plug in for  $z$  since we are doing the surface integral on the plane. Finishing this out gives,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \int_0^{-x+1} 1 - 2x - 2y \, dy \, dx \\ &= \int_0^1 \left( y - 2xy - y^2 \right) \Big|_0^{-x+1} dx \\ &= \int_0^1 x^2 - x \, dx \\ &= \left( \frac{1}{3} x^3 - \frac{1}{2} x^2 \right) \Big|_0^1 \\ &= -\frac{1}{6}\end{aligned}$$

In both of these examples we were able to take an integral that would have been somewhat unpleasant to deal with and by the use of Stokes' Theorem we were able to convert it into an integral that wasn't too bad.

### ***Divergence Theorem***

In this section we are going to relate surface integrals to triple integrals. We will do this with the Divergence Theorem.

#### **Divergence Theorem**

Let  $E$  be a simple solid region and  $S$  is the boundary surface of  $E$  with positive orientation. Let  $\vec{F}$  be a vector field whose components have continuous first order partial derivatives. Then,

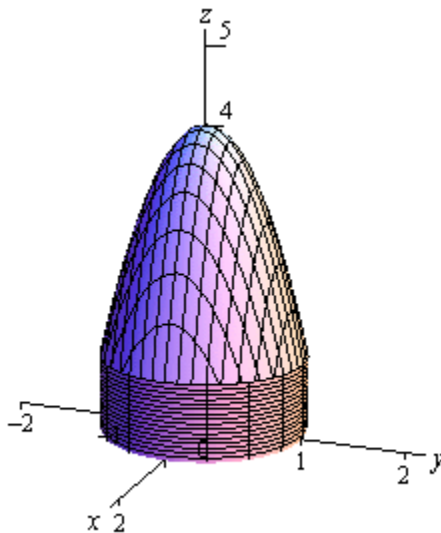
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

Let's see an example of how to use this theorem.

**Example 1** Use the divergence theorem to evaluate  $\iint_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = xy\vec{i} - \frac{1}{2}y^2\vec{j} + z\vec{k}$  and the surface consists of the three surfaces,  $z = 4 - 3x^2 - 3y^2$ ,  $1 \leq z \leq 4$  on the top,  $x^2 + y^2 = 1$ ,  $0 \leq z \leq 1$  on the sides and  $z = 0$  on the bottom.

#### **Solution**

Let's start this off with a sketch of the surface.



The region  $E$  for the triple integral is then the region enclosed by these surfaces. Note that cylindrical coordinates would be a perfect coordinate system for this region. If we do that here are the limits for the ranges.

$$0 \leq z \leq 4 - 3r^2$$

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

We'll also need the divergence of the vector field so let's get that.

$$\operatorname{div} \vec{F} = y - y + 1 = 1$$

The integral is then,

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iiint_E \operatorname{div} \vec{F} \, dV \\&= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r \, dz \, dr \, d\theta \\&= \int_0^{2\pi} \int_0^1 4r - 3r^3 \, dr \, d\theta \\&= \int_0^{2\pi} \left( 2r^2 - \frac{3}{4}r^4 \right) \Big|_0^1 d\theta \\&= \int_0^{2\pi} \frac{5}{4} d\theta \\&= \frac{5}{2}\pi\end{aligned}$$