

# Examples: Cartesian Coordinates

Sketch the following vector fields:

$$\mathbf{H} = \frac{1}{x} \hat{\mathbf{y}} \left( \frac{\text{A}}{\text{m}} \right)$$

$$\mathbf{E} = x \hat{\mathbf{x}} - y \hat{\mathbf{y}} \left( \frac{\text{V}}{\text{m}} \right)$$

# Examples: Cylindrical Coordinates

Sketch the following vector fields:

$$\mathbf{H} = -1 \mathbf{a}_{\phi} \left( \frac{\text{A}}{\text{m}} \right)$$

$$\mathbf{E} = \sin \phi \hat{\mathbf{r}} \left( \frac{\text{V}}{\text{m}} \right)$$



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**ELEC 318 – *Electromagnetic Fields***

**Lecture 3(c)**

**Review of Vector Calculus:  
Differential Length/Area/Volume, Integrals**

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# Diff. Length, Area, Volume: Cartesian

## differential length, area, volume:

- useful for integration along a path, open/closed surface
- allows us to answer questions like

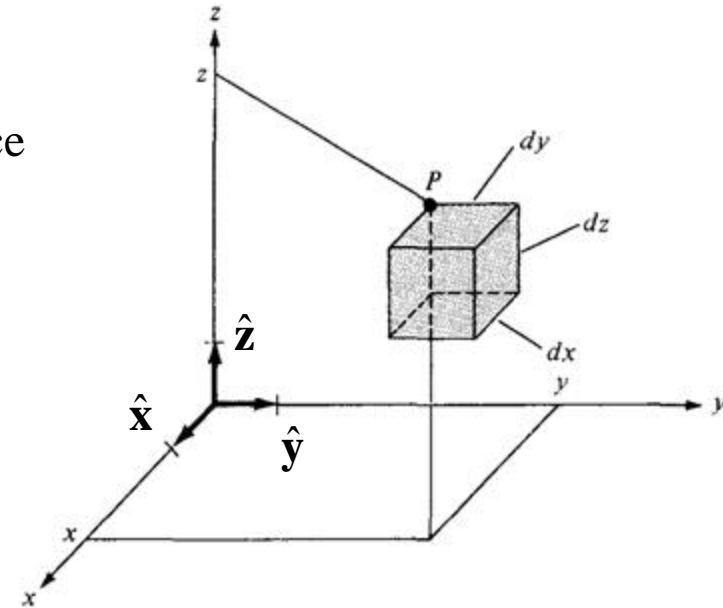
“What is the strength of  $\mathbf{E}$  along this path?”

“What is the density of  $\mathbf{H}$  across this surface?”

“How much of  $\mathbf{q}$  is contained within this shape?”

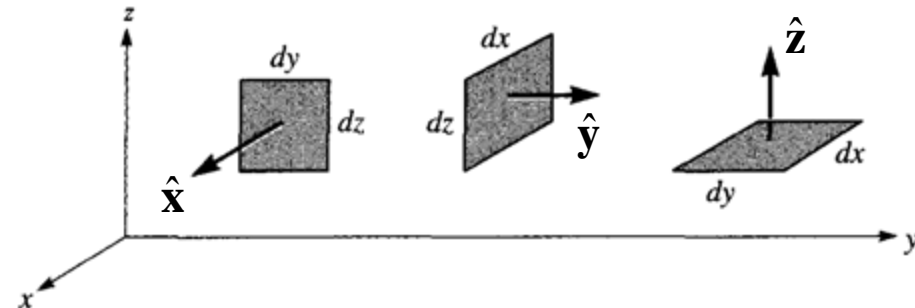
differential length (displacement, distance):

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$



differential area:

$$\begin{aligned} d\mathbf{S} &= dS \hat{\mathbf{n}} \\ &= dx dy \hat{\mathbf{z}} \quad \text{or} \quad dy dz \hat{\mathbf{x}} \quad \text{or} \quad dz dx \hat{\mathbf{y}} \end{aligned}$$



differential volume:  $dv = dx dy dz$

# Diff. Length, Area, Volume: Cylindrical

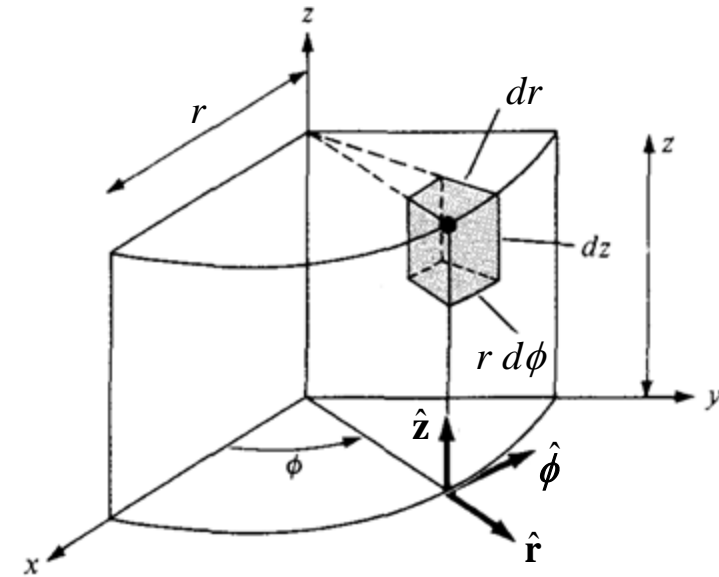
differential length (displacement, distance):

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$$

differential area:

$$\begin{aligned} d\mathbf{S} &= dS \hat{\mathbf{n}} \\ &= r d\phi dz \hat{\mathbf{r}} \quad \text{or} \quad dr dz \hat{\boldsymbol{\phi}} \quad \text{or} \quad r dr d\phi \hat{\mathbf{z}} \end{aligned}$$

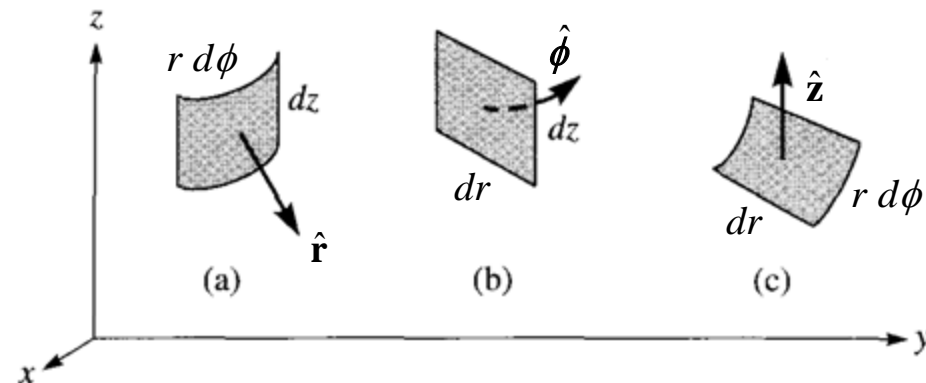
differential volume:  $dv = r dr d\phi dz$



Example:

If the charge density in the wedge (above, right), is  $\rho_v = 2r - 3z$  (C/cm<sup>3</sup>), then the amount of charge contained in the tiny curved cube is

$$(2r^2 - 3rz) dr d\phi dz \quad (\text{coulombs})$$



## Example: Cylindrical Area

Find the area of a cylindrical surface described by  $r = 5$ ,  $30^\circ \leq \phi \leq 60^\circ$ , and  $0 \leq z \leq 3$ .

# Diff. Length, Area, Volume: Spherical

differential length (displacement, distance):

$$d\mathbf{l} = dR \hat{\mathbf{R}} + R d\theta \hat{\boldsymbol{\theta}} + R \sin \theta d\phi \hat{\boldsymbol{\phi}}$$

differential area:  $d\mathbf{S} = dS \hat{\mathbf{n}}$

$$= R^2 \sin \theta d\theta d\phi \hat{\mathbf{R}} \quad \text{or}$$

$$= R \sin \theta dR d\phi \hat{\boldsymbol{\theta}} \quad \text{or}$$

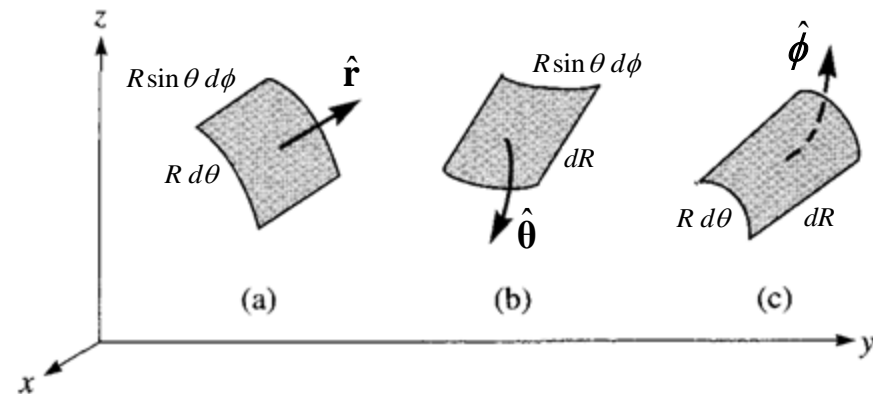
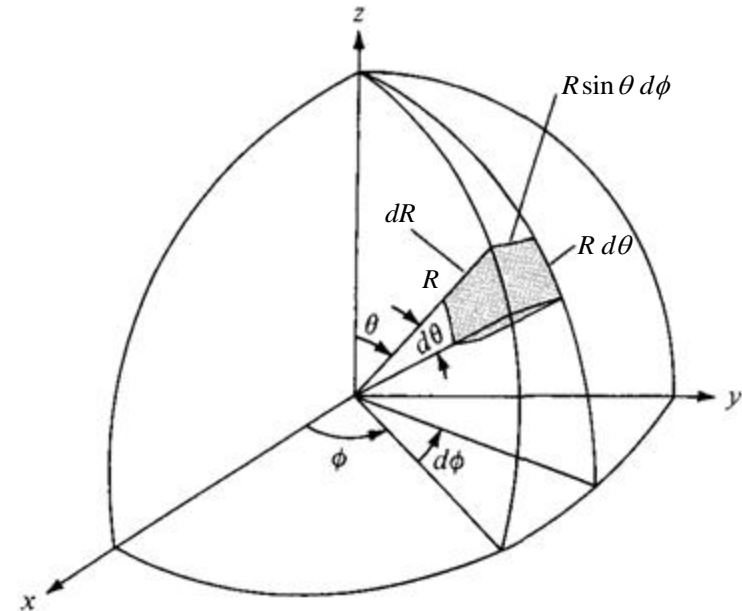
$$= R dR d\theta \hat{\boldsymbol{\phi}}$$

differential volume:  $dv = R^2 \sin \theta dR d\theta d\phi$

Example:

If the magnetic flux density crossing the sphere (above, right), is  $\mathbf{B} = 4 \sin \phi \mathbf{R}$  (Wb/m<sup>2</sup>), then the amount of flux through the tiny curved square (on the outside of the sphere) is

$$4 R^2 \sin \phi \sin \theta d\theta d\phi \quad (\text{Wb})$$



# Line Integrals

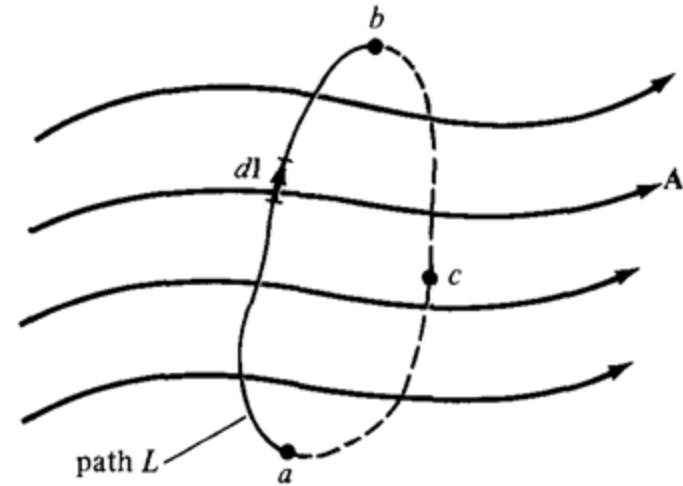
## line (single) integral:

- integration of the tangential (parallel) component of a vector ( $\mathbf{A}$ ) along a path ( $L$ ):

$$\int_L \mathbf{A} \cdot d\mathbf{l}$$

- allows us to answer the question  
“How much of  $\mathbf{A}$  is projected along a path?”

If the path ( $L$ ) is closed (forms a *surface*),  
then the integral becomes  $\oint_L \mathbf{A} \cdot d\mathbf{l}$



Examples:

$$V_{21} = -\int_{P1}^{P2} \mathbf{E} \cdot d\mathbf{l}$$

$$I_{\text{enc}} = \oint \mathbf{H} \cdot d\mathbf{l}$$

As before, for rectangular coordinates,  
for cylindrical coordinates,  
and for spherical coordinates,

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$$

$$d\mathbf{l} = dR \hat{\mathbf{R}} + R d\theta \hat{\boldsymbol{\theta}} + R \sin \theta d\phi \hat{\boldsymbol{\phi}}$$



## Example: Mixed Coordinates

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \frac{y}{x}, \quad z = z$$

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z$$

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$$

$$\hat{\mathbf{x}} = \cos \phi \hat{\mathbf{r}} - \sin \phi \hat{\boldsymbol{\phi}} \quad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

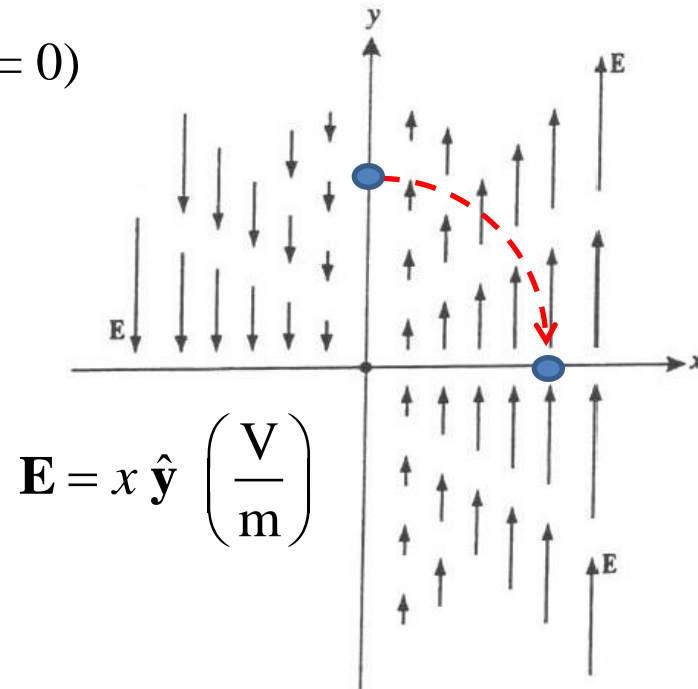
$$\hat{\mathbf{y}} = \sin \phi \hat{\mathbf{r}} + \cos \phi \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{r}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \quad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

For the path from  $P_1$  ( $x = 0$ ,  $y = 2$  m) to  $P_2$  ( $x = 2$  m,  $y = 0$ ) and  $\mathbf{E}$  illustrated, compute

$$-\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$



# Surface Integrals

## surface (double) integral:

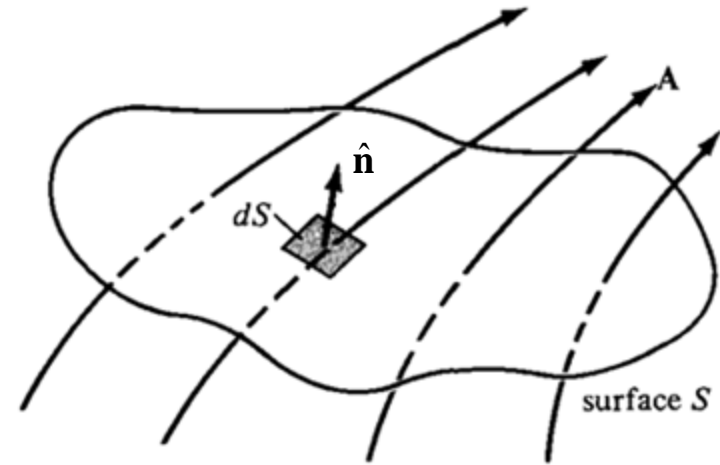
- integration of the normal component of a vector ( $\mathbf{A}$ ) across a surface ( $S$ ):

$$\int_S \mathbf{A} \cdot d\mathbf{S}$$

(“flux”)

- allows us to answer the question

“How much of  $\mathbf{A}$  crosses a given surface?”



If the surface ( $S$ ) is closed (forms a *volume*), then the integral becomes

$$\oint_S \mathbf{A} \cdot d\mathbf{S} \quad (\text{“net outward flux”})$$

Examples:

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$Q_{\text{enc}} = \oint \mathbf{D} \cdot d\mathbf{S}$$

As before, for rectangular coordinates,  $d\mathbf{S} = dx dy \hat{\mathbf{z}}$  or  $dy dz \hat{\mathbf{x}}$  or  $dz dx \hat{\mathbf{y}}$

for cylindrical coordinates,  $d\mathbf{S} = r d\phi dz \hat{\mathbf{r}}$  or  $d\rho dz \hat{\boldsymbol{\phi}}$  or  $r dr d\phi \hat{\mathbf{z}}$

for spherical coordinates,  $d\mathbf{S} = R^2 \sin \theta d\theta d\phi \hat{\mathbf{R}}$  or  $R \sin \theta dR d\phi \hat{\boldsymbol{\theta}}$  or  $R dR d\theta \hat{\boldsymbol{\phi}}$

# Example: Surface Integral

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \frac{y}{x}, \quad z = z$$
$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z$$

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\hat{\mathbf{x}} = \cos \phi \hat{\mathbf{r}} - \sin \phi \hat{\phi}$$

$$\hat{\mathbf{y}} = \sin \phi \hat{\mathbf{r}} + \cos \phi \hat{\phi}$$

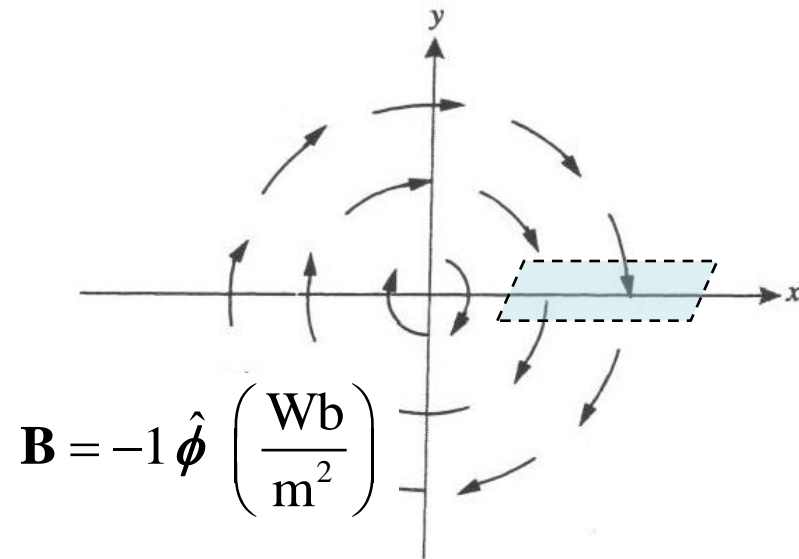
$$\hat{\mathbf{r}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}$$

$$\hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

$$d\mathbf{S} = dx dy \hat{\mathbf{z}} \quad \text{or} \quad dy dz \hat{\mathbf{x}} \quad \text{or} \quad dz dx \hat{\mathbf{y}}$$

$$d\mathbf{S} = r d\phi dz \hat{\mathbf{r}} \quad \text{or} \quad d\phi dz \hat{\phi} \quad \text{or} \quad r dr d\phi \hat{\mathbf{z}}$$

Determine the total flux that crosses over the surface defined by  $y = 0$ ,  
 $z = -1$  to  $1$  m,  $x = 2$  to  $5$  m for the flux density depicted.



# Volume Integrals

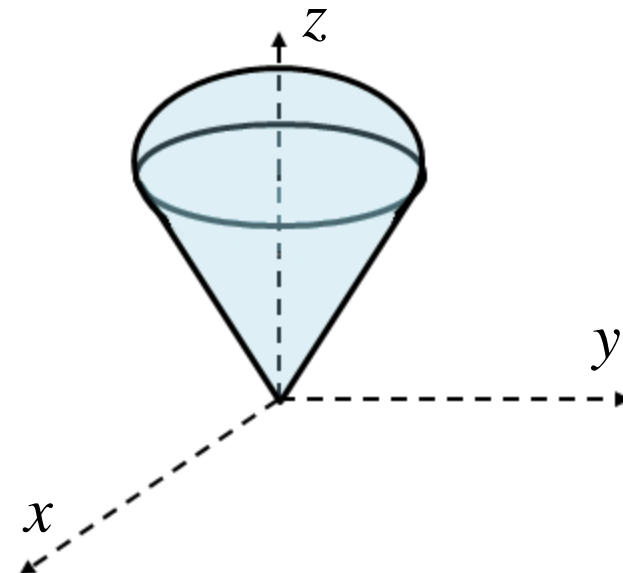
## volume (triple) integral:

- integration of a scalar *density* throughout a volume (V):

$$\int_v A_v \cdot dv$$

- allows us to answer the question

“How much of [a particular quantity] is contained within [a given volume] ?”



Examples:

$$Q = \int_v \rho_v dv$$

$$W_E = \frac{1}{2} \epsilon \int_v |\mathbf{E}|^2 dv$$

As before, for rectangular coordinates,  $dv = dx dy dz$

for cylindrical coordinates,  $dv = r dr d\phi dz$

for spherical coordinates,  $dv = R^2 \sin \theta dR d\theta d\phi$

## Example: Volume Integral

A sphere of radius 2 cm contains a volume charge density  $\rho_v$  given by

$$\rho_v = 4 \cos^2 \theta \quad (\text{C/m}^3).$$

Find the total charge  $Q$  contained in the sphere.

## Example: Unit Vector, Cylindrical

Find an expression for the unit vector of vector **A** in cylindrical coordinates.

