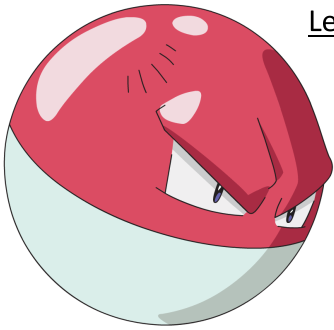


Lecture 4: The Gradient

Voltorb's Goals for the Day

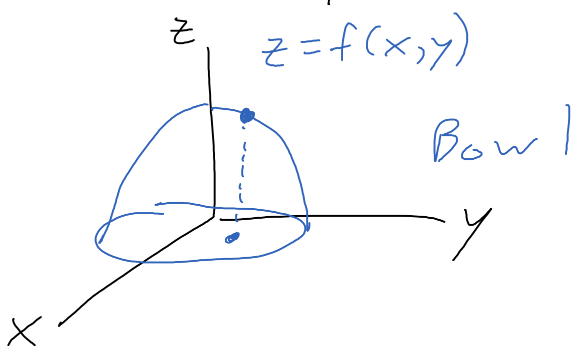
- Discuss the geometric meaning of partial derivatives
- Extend the Chain Rule to multiple variables
- Introduce the gradient and directional derivatives

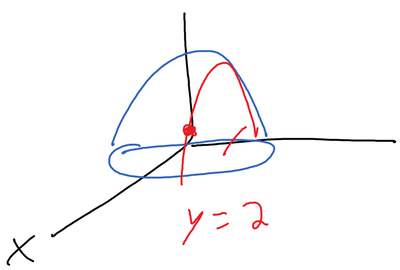
9.4 Partial Derivatives

f_x measures the rate of change in x direction

f_{xx} measures the concavity in the x direction

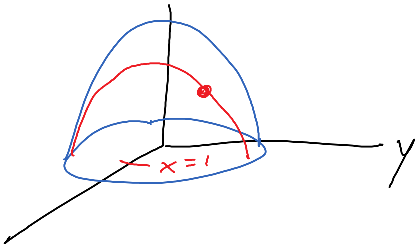
Ex Discuss derivatives for the surface $z=f(x,y)$ at the point $(1,2)$.





$f_x < 0$ Decreasing

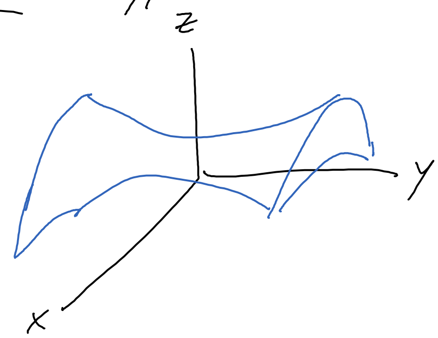
$f_{xx} < 0$ Concave Down



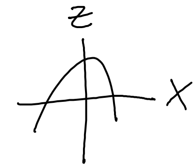
$f_y < 0$ Decreasing

$f_{yy} < 0$ Concave Down

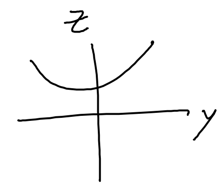
Ex Hyperbolic Paraboloid (Saddle)



$$f_{xx} < 0$$



$$f_{yy} > 0$$



Recall Calc I: The Chain Rule

$f(x) \qquad x(t)$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Extend Chain Rule for Multivariable Function

$f(x,y) \qquad x(s,t) \qquad y(s,t)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

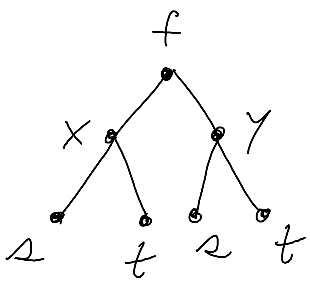
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Tree Diagram

Draw all variables in a tree structure.
Look at all paths that end in the desired variable and multiply by

derivatives along the path.
Then add up all paths.

$f(x,y) \quad x(s,t) \quad y(s,t)$



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

__Ex__ Profit $P(s, C, \ldots)$

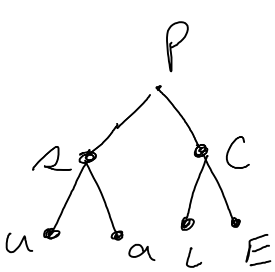
s ↑ sales, C ↑ cost

Sales $s(u, a, \ldots)$

u ↑ unit price, a ↑ advertising

Cost $C(L, E, \ldots)$

L ↑ labor, E ↑ equipment



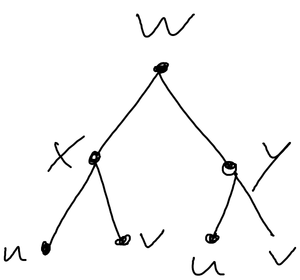
$$\frac{\partial P}{\partial L} = \frac{\partial P}{\partial C}\frac{\partial C}{\partial L}$$

Rate of change in profit when labor costs increase by \$1

Ex Find $\frac{\partial w}{\partial u}$ for

$$w = x^2 \cos y, \quad y = u + 3v, \quad x = \frac{u}{v}.$$

Express your answer in terms of u and v.



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$= (2x\cos y)\left(\frac{1}{v}\right) + (-x^2 \sin y)(1)$$

$$= 2\frac{u}{v}\cos(u+3v)\left(\frac{1}{v}\right) - \left(\frac{u}{v}\right)^2 \sin(u+3v)$$

$$= \boxed{\frac{2u}{v^2}\cos(u+3v) - \frac{u^2}{v^2}\sin(u+3v)}$$

9.5 Directional Derivatives

Def The gradient of a function is a vector that lists the partial derivatives.

$$f(x_1, x_2, \ldots, x_N)$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_N} \right\rangle$$

↳ "grad f"

Ex Compute the gradient of $f(x,y) = x^2y - 3y + 2$.

$$\nabla f = \langle f_x, f_y \rangle = \boxed{\langle 2xy, x^2 - 3 \rangle}$$

Compute gradient of f at the point $(1,2)$.

$$\nabla f(1,2) = \langle 2(1)(2), 1^2 - 3 \rangle = \boxed{\langle 4, -2 \rangle}$$

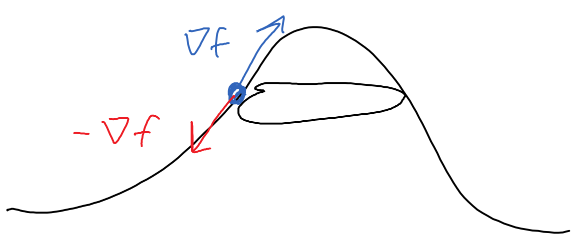
Facts About The Gradient

① ∇f points in the direction of maximum positive rate of change (steepest ascent). The rate of change is $\|\nabla f\|$.

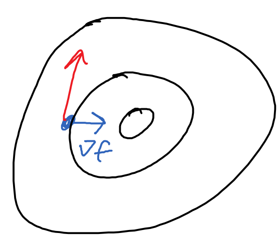
② $-\nabla f$ points in the direction of maximum negative change (steepest descent).

③ ∇f is perpendicular to the level curves of f (f = constant)

$z = f(x, y)$



Contour Plot



Ex Profit $P(x_1, x_2, \ldots, x_n)$

cost sales

∇P tells the company the fastest way to increase profit

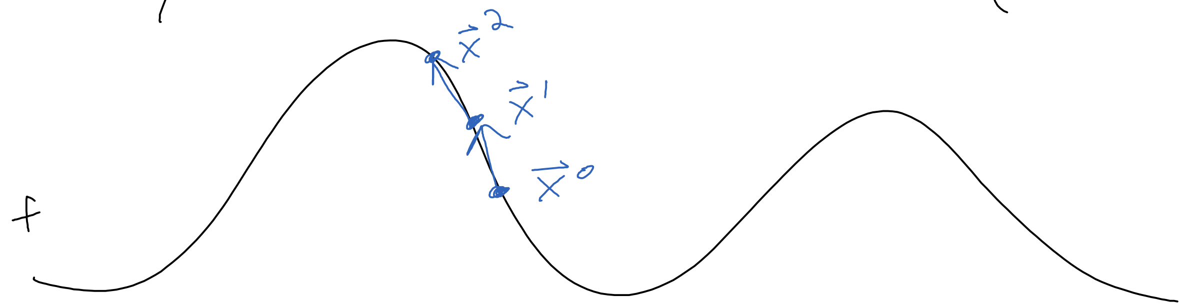
Application Numerical Optimization

Maximize $f(x_1, x_2, \ldots, x_n)$

Idea: Take small steps in the direction of ∇f

Initial Guess $\vec{x}^0$

Update $\vec{x}^{n+1} = \vec{x}^n + \Delta t \left(\nabla f(\vec{x}^n) \right)$



Ex Voltorb is standing on a mountain at the point $(1,2)$. The height of the mountain is given by

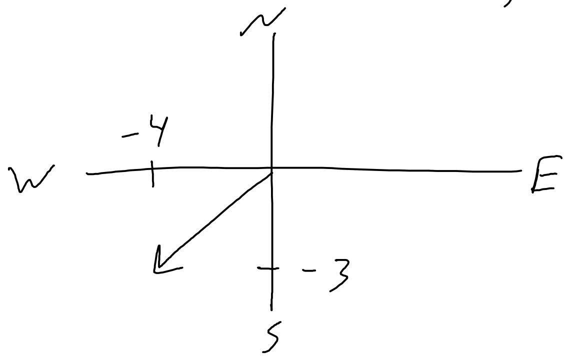
$$f(x,y) = x^2 - 3xy + 2$$

where x-y are aligned to the NESW map directions.

a.) What direction should Voltorb go to go up the mountain most rapidly?

$$\nabla f = \langle 2x - 3y, -3x \rangle$$

$$\nabla f(1,2) = \langle 2(1) - 3(2), -3(1) \rangle = \boxed{\langle -4, -3 \rangle}$$



b.) What direction gives the most rapid descent?

$$-\nabla f(1,2) = \boxed{\langle 4, 3 \rangle}$$

c.) What direction(s) will keep Voltorb at the same elevation?

Go perpendicular to gradient.

$$\nabla f = \langle -4, -3 \rangle$$

$$\langle -4, -3 \rangle \cdot \langle 3, -4 \rangle = 0$$

$$\langle -4, -3 \rangle \cdot \langle -3, 4 \rangle = 0$$

$$\boxed{\langle 3, -4 \rangle, \ \langle -3, 4 \rangle}$$

Trick: A vector $\perp$ to $\langle a, b \rangle$ is

$$\langle -b, a \rangle \quad \text{or} \quad \langle b, -a \rangle.$$

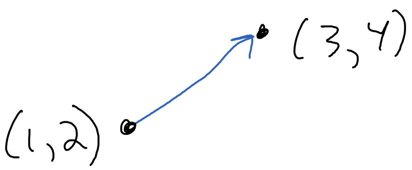
Def Directional Derivative

For a unit vector $\vec{u}$, the rate of change of a function f in the direction of $\vec{u}$ is

$$D_u f = \nabla f \cdot \vec{u}$$

Ex Continue last example

d) If Voltorb starts moving towards the point $(3,4)$, what is the rate of change in elevation?



$$\vec{v} = \text{End} - \text{Start}$$
$$= (3,4) - (1,2) = \langle 2, 2 \rangle$$

Make $\vec{v}$ into a unit vector.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2, 2 \rangle}{\sqrt{2^2 + 2^2}} = \frac{\langle 2, 2 \rangle}{\sqrt{8}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

From part (a), $\nabla f(1,2) = \langle -4, -3 \rangle$.

$$D_u f = \nabla f \cdot \vec{u} = \langle -4, -3 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= -\frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}} = \boxed{-\frac{7}{\sqrt{2}}}$$ Downhill

e.) What direction(s) give $D_u f = 0$?

Same as (c).

$$\boxed{\langle 4, -3 \rangle \text{ and } \langle -4, 3 \rangle}$$