2 Signals and Systems: Part I

Solutions to Recommended Problems

S2.1

(a) We need to use the relations $\omega = 2\pi f$, where f is frequency in hertz, and $T = 2\pi/\omega$, where T is the fundamental period. Thus, $T = 1/f$.

(i) $f = \dfrac{\omega}{2\pi} = \dfrac{\pi/3}{2\pi} = \dfrac{1}{6}\text{ Hz},\quad T = \dfrac{1}{f} = 6\text{ s}$

(ii) $f = \dfrac{3\pi/4}{2\pi} = \dfrac{3}{8}\text{ Hz},\quad T = \dfrac{8}{3}\text{ s}$

(iii) $f = \dfrac{3/4}{2\pi} = \dfrac{3}{8\pi}\text{ Hz},\quad T = \dfrac{8\pi}{3}\text{ s}$

Note that the frequency and period are independent of the delay τ_x and the phase θ_x.

(b) We first simplify:

$$\cos(\omega(t + \tau) + \theta) = \cos(\omega t + \omega\tau + \theta)$$

Note that $\omega\tau + \theta$ could also be considered a phase term for a delay of zero. Thus, if $\omega_x = \omega_y$ and $\omega_x\tau_x + \theta_x = \omega_y\tau_y + \theta_y + 2\pi k$ for any integer k, $y(t) = x(t)$ for all t.

(i) $\omega_x = \omega_y,\quad \omega_x\tau_x + \theta_x = 2\pi,\quad \omega_y\tau_y + \theta_y = \dfrac{\pi}{3}(1) - \dfrac{\pi}{3} = 0 + 2\pi k$

Thus, $x(t) = y(t)$ for all t.

(ii) Since $\omega_x \neq \omega_y$, we conclude that $x(t) \neq y(t)$.

(iii) $\omega_x = \omega_y,\quad \omega_x\tau_x + \theta_x = \frac{3}{4}(\frac{1}{2}) + \frac{1}{4} \neq \frac{3}{4}(1) + \frac{3}{8} + 2\pi k$

Thus, $x(t) \neq y(t)$.

S2.2

(a) To find the period of a discrete-time signal is more complicated. We need the smallest N such that $\Omega N = 2\pi k$ for some integer $k > 0$.

(i) $\dfrac{\pi}{3}N = 2\pi k \Rightarrow N = 6,\quad k = 1$

(ii) $\dfrac{3\pi}{4}N = 2\pi k \Rightarrow N = 8,\quad k = 2$

(iii) $\frac{3}{4}N = 2\pi k \Rightarrow$ There is no N such that $\frac{3}{4}N = 2\pi k$, so $x[n]$ is *not* periodic.

(b) For discrete-time signals, if $\Omega_x = \Omega_y + 2\pi k$ and $\Omega_x\tau_x + \theta_x = \Omega_y\tau_y + \theta_y + 2\pi k$, then $x[n] = y[n]$.

(i) $\dfrac{\pi}{3} \neq \dfrac{8\pi}{3} + 2\pi k$ (the closest is $k = -1$), so $x[n] \neq y[n]$

(ii) $\Omega_x = \Omega_y,\quad \dfrac{3\pi}{4}(2) + \dfrac{\pi}{4} = \dfrac{3\pi}{4} - \pi + 2\pi k,\quad k = 1$, so $x[n] = y[n]$

(iii) $\Omega_x = \Omega_y,\quad \frac{3}{4}(1) + \frac{1}{4} = \frac{3}{4}(0) + 1 + 2\pi k,\quad k = 0,\quad x[n] = y[n]$

S2.3

(a) (i) This is just a shift to the right by two units.

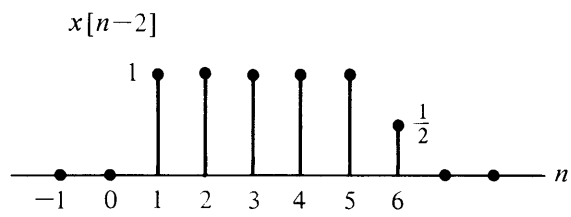

Figure S2.3-1

(ii) $x[4 - n] = x[-(n - 4)]$, so we flip about the $n = 0$ axis and then shift to the right by 4.

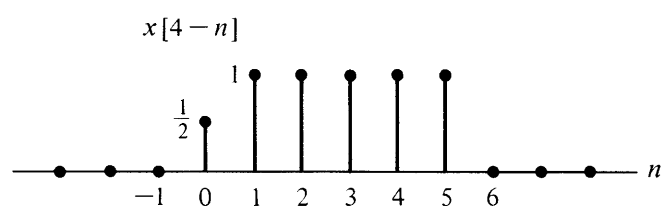

Figure S2.3-2

(iii) $x[2n]$ generates a new signal with $x[n]$ for even values of n.

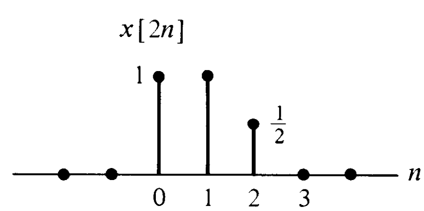

Figure S2.3-3

(b) The difficulty arises when we try to evaluate $x[n/2]$ at $n = 1$, for example (or generally for n an odd integer). Since $x[\frac{1}{2}]$ is not defined, the signal $x[n/2]$ does not exist.

S2.4

By definition a signal is even if and only if $x(t) = x(-t)$ or $x[n] = x[-n]$, while a signal is odd if and only if $x(t) = -x(-t)$ or $x[n] = -x[-n]$.

(a) Since $x(t)$ is symmetric about $t = 0$, $x(t)$ is even.

(b) It is readily seen that $x(t) \neq x(-t)$ for all t, and $x(t) \neq -x(-t)$ for all t; thus $x(t)$ is neither even nor odd.

(c) Since $x(t) = -x(-t)$, $x(t)$ is odd in this case.

(d) Here $x[n]$ seems like an odd signal at first glance. However, note that $x[n] = -x[-n]$ evaluated at $n = 0$ implies that $x[0] = -x[0]$ or $x[0] = 0$. The analogous result applies to continuous-time signals. The signal is therefore neither even nor odd.

(e) In similar manner to part (a), we deduce that $x[n]$ is even.

(f) $x[n]$ is odd.

S2.5

(a) Let $Ev\{x[n]\} = x_e[n]$ and $Od\{x[n]\} = x_o[n]$. Since $x_e[n] = y[n]$ for $n \geq 0$ and $x_e[n] = x_e[-n]$, $x_e[n]$ must be as shown in Figure S2.5-1.

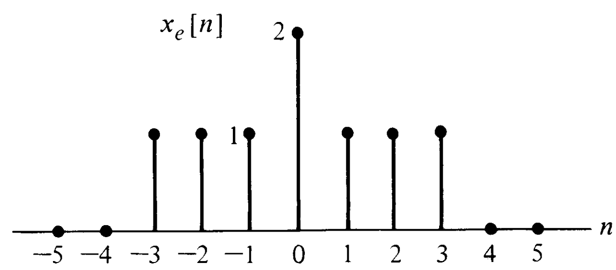

Figure S2.5-1

Since $x_o[n] = y[n]$ for $n < 0$ and $x_o[n] = -x_o[-n]$, along with the property that $x_o[0] = 0$, $x_o[n]$ is as shown in Figure S2.5-2.

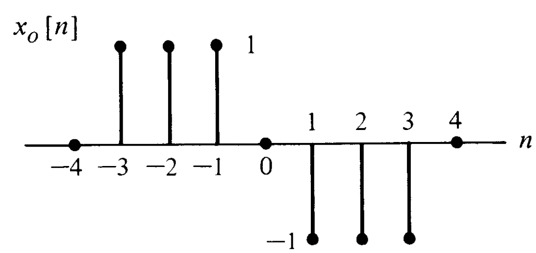

Figure S2.5-2

Finally, from the definition of $Ev\{x[n]\}$ and $Od\{x[n]\}$, we see that $x[n] = x_e[n] + x_o[n]$. Thus, $x[n]$ is as shown in Figure S2.5-3.

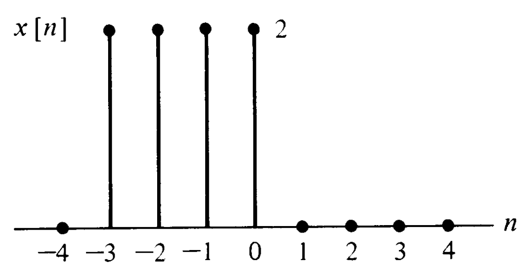

Figure S2.5-3

(b) In order for $w[n]$ to equal 0 for $n < 0$, $Od\{w[n]\}$ must be given as in Figure S2.5-4.

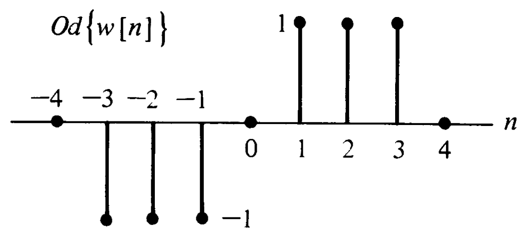

Figure S2.5-4

Thus, $w[n]$ is as in Figure S2.5-5.

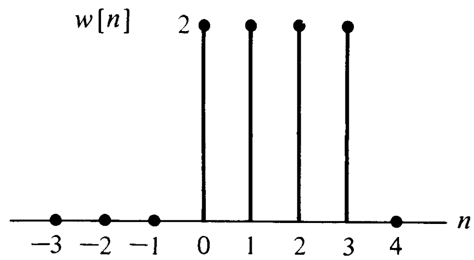

Figure S2.5-5

S2.6

(a) For $\alpha = -\frac{1}{2}$, α^n is as shown in Figure S2.6-1.

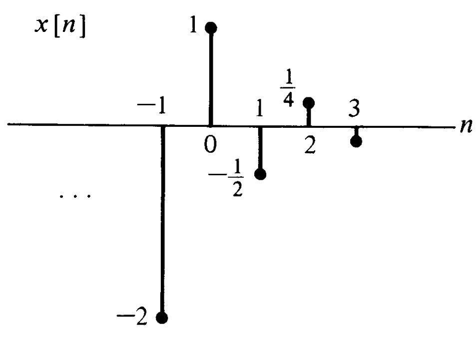

Figure S2.6-1

(b) We need to find a β such that $e^{\beta n} = (-e^{-1})^n$. Expressing -1 as $e^{j\pi}$, we find

$$e^{\beta n} = (e^{j\pi}e^{-1})^n \qquad \text{or} \qquad \beta = -1 + j\pi$$

Note that any $\beta = -1 + j\pi + j2\pi k$ for k an integer will also satisfy the preceding equation.

(c) $Re\{e^{(-1+j\pi)t}\}\Big|_{t=n} = e^{-n}Re\{e^{j\pi n}\} = e^{-n}\cos \pi n,$

$Im\{e^{(-1+j\pi)t}\}\Big|_{t=n} = e^{-n}Im\{e^{j\pi n}\} = e^{-n}\sin \pi n$

Since $\cos \pi n = (-1)^n$ and $\sin \pi n = 0$, $Re\{x(t)\}$ and $Im\{y(t)\}$ for t an integer are shown in Figures S2.6-2 and S2.6-3, respectively.

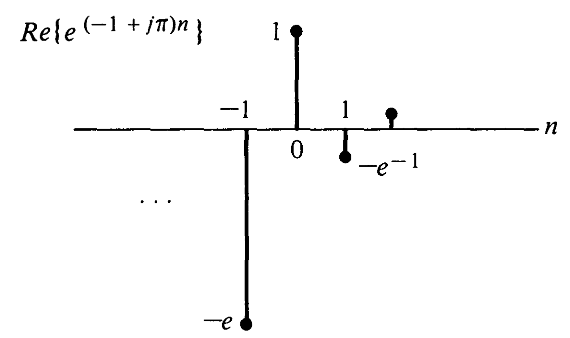

Figure S2.6-2

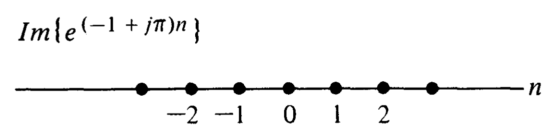

Figure S2.6-3

S2.7

First we use the relation $(1 + j) = \sqrt{2}e^{j\pi/4}$ to yield

$$x(t) = \sqrt{2} \cdot \sqrt{2}e^{j\pi/4}e^{j\pi/4}e^{(-1+j2\pi)t} = 2e^{j\pi/2}e^{(-1+j2\pi)t}$$

(a) $Re\{x(t)\} = 2e^{-t}Re\{e^{j\pi/2}e^{j2\pi t}\} = 2e^{-t}\cos\left(2\pi t + \frac{\pi}{2}\right)$

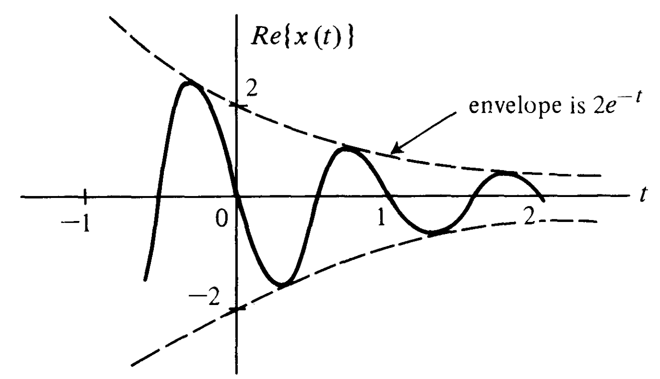

Figure S2.7-1

(b) $Im\{x(t)\} = 2e^{-t}Im\{e^{j\pi/2}e^{j2\pi t}\} = 2e^{-t}\sin\left(2\pi t + \frac{\pi}{2}\right)$

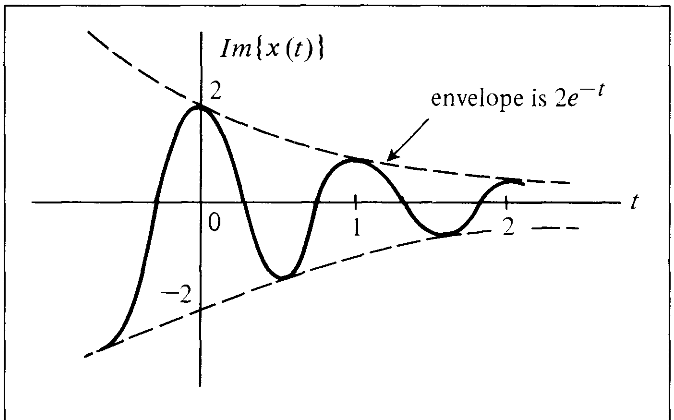

Figure S2.7-2

(c) *Note* that $x(t + 2) + x^*(t + 2) = 2Re\{x(t + 2)\}$. So the signal is a shifted version of the signal in part (a).

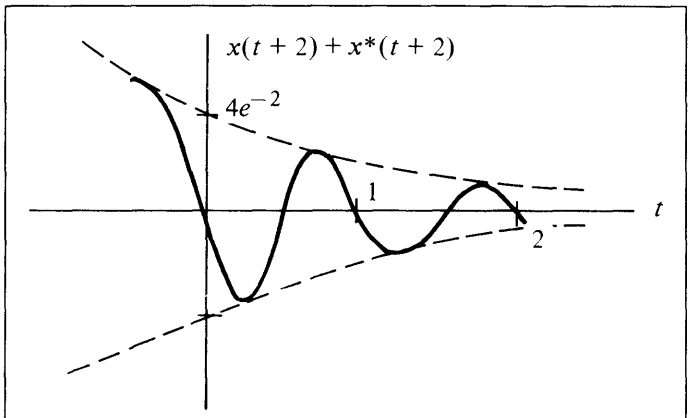

Figure S2.7-3

S2.8

(a) We just need to recognize that $\alpha = 3/a$ and $C = 2$ and use the formula for S_N, $N = 6$.

$$\sum_{n=0}^{5} 2\left(\frac{3}{a}\right)^n = 2\,\frac{1 - \left(\frac{3}{a}\right)^6}{1 - \left(\frac{3}{a}\right)}$$

(b) This requires a little manipulation. Let $m = n - 2$. Then

$$\sum_{n=2}^{6} b^n = \sum_{m=0}^{4} b^{m+2} = b^2 \sum_{m=0}^{4} b^m = b^2\,\frac{1 - b^5}{1 - b}$$

(c) We need to recognize that $(\frac{2}{3})^{2n} = (\frac{4}{9})^n$. Thus,

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{2n} = \sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^{n} = \frac{1}{1-\frac{4}{9}} \qquad \text{since } \left|\frac{4}{9}\right| < 1$$

S2.9

(a) The sum $x(t) + y(t)$ will be periodic if there exist integers n and k such that $nT_1 = kT_2$, that is, if $x(t)$ and $y(t)$ have a common (possibly not fundamental) period. The fundamental period of the combined signal will be nT_1 for the smallest allowable n.

(b) Similarly, $x[n] + y[n]$ will be periodic if there exist integers n and k such that $nN_1 = kN_2$. But such integers always exist, a trivial example being $n = N_2$ and $k = N_1$. So the sum is always periodic with period nN_1 for n the smallest allowable integer.

(c) We first decompose $x(t)$ and $y(t)$ into sums of exponentials. Thus,

$$x(t) = \frac{1}{2}e^{j(2\pi t/3)} + \frac{1}{2}e^{-j(2\pi t/3)} + \frac{e^{j(16\pi t/3)}}{j} - \frac{e^{-j(16\pi t/3)}}{j},$$

$$y(t) = \frac{e^{j\pi t}}{2j} - \frac{e^{-j\pi t}}{2j}$$

Multiplying $x(t)$ and $y(t)$, we get

$$\begin{aligned} z(t) = {} & \frac{1}{4j}e^{j(5\pi/3)t} - \frac{1}{4j}e^{-j(\pi/3)t} + \frac{1}{4j}e^{j(\pi/3)t} - \frac{1}{4j}e^{-j(5\pi/3)t} \\ & - \frac{1}{2}e^{j(19\pi/3)t} + \frac{1}{2}e^{j(13\pi/3)t} + \frac{1}{2}e^{-j(13\pi/3)t} - \frac{1}{2}e^{-j(19\pi/3)t} \end{aligned}$$

We see that all complex exponentials are powers of $e^{j(\pi/3)t}$. Thus, the fundamental period is $2\pi/(\pi/3) = 6$ s.

S2.10

(a) Let $\sum_{n=-\infty}^{\infty} x[n] = S$. Define $m = -n$ and substitute

$$\sum_{m=-\infty}^{\infty} x[-m] = -\sum_{m=-\infty}^{\infty} x[m]$$

since $x[m]$ is odd. But the preceding sum equals $-S$. Thus, $S = -S$, or $S = 0$.

(b) Let $y[n] = x_1[n]x_2[n]$. Then $y[-n] = x_1[-n]x_2[-n]$. But $x_1[-n] = -x_1[n]$ and $x_2[-n] = x_2[n]$. Thus, $y[-n] = -x_1[n]x_2[n] = -y[n]$. So $y[n]$ is odd.

(c) Recall that $x[n] = x_e[n] + x_o[n]$. Then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x^2[n] &= \sum_{n=-\infty}^{\infty} (x_e[n] + x_o[n])^2 \\ &= \sum_{n=-\infty}^{\infty} x_e^2[n] + 2\sum_{n=-\infty}^{\infty} x_e[n]x_o[n] + \sum_{n=-\infty}^{\infty} x_o^2[n] \end{aligned}$$

But from part (b), $x_e[n]x_o[n]$ is an odd signal. Thus, using part (a) we find that the second sum is zero, proving the assertion.

(d) The steps are analogous to parts (a)–(c). Briefly,

(i) $$S = \int_{t=-\infty}^{\infty} x_o(t)\,dt = \int_{r=-\infty}^{\infty} x_o(-r)\,dr$$
$$= -\int_{r=-\infty}^{\infty} x_o(r)\,dr = -S, \quad \text{or} \quad S = 0, \qquad \text{where } r = -t$$

(ii) $$y(t) = x_o(t)x_e(t),$$
$$y(-t) = x_o(-t)x_e(-t) = -x_o(t)x_e(t)$$
$$= -\,y(t), \qquad y(t) \text{ is odd}$$

(iii) $$\int_{t=-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} (x_e(t) + x_o(t))^2\,dt$$
$$= \int_{-\infty}^{\infty} x_e^2(t)\,dt + 2\int_{-\infty}^{\infty} x_e(t)x_o(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt,$$

while $2\int_{-\infty}^{\infty} x_e(t)x_o(t)\,dt = 0$

S2.11

(a) $x[n] = e^{j\omega_o nT} = e^{j2\pi nT/T_o}$. For $x[n] = x[n + N]$, we need

$$x[n + N] = e^{j2\pi(n+N)T/T_o} = e^{j[2\pi n(T/T_o) + 2\pi N(T/T_o)]} = e^{j2\pi nT/T_o}$$

The two sides of the equation will be equal only if $2\pi N(T/T_o) = 2\pi k$ for some integer k. Therefore, T/T_o must be a rational number.

(b) The fundamental period of $x[n]$ is the smallest N such that $N(T/T_o) = N(p/q) = k$. The smallest N such that Np has a divisor q is the least common multiple (LCM) of p and q, divided by p. Thus,

$$N = \frac{\text{LCM}(p, q)}{p}; \qquad \text{note that } k = \frac{\text{LCM}(p, q)}{q}$$

The fundamental frequency is $2\pi/N$, but $n = (kT_o)/T$. Thus,

$$\Omega = \frac{2\pi}{N} = \frac{2\pi T}{kT_o} = \frac{1}{k}\,\omega_o T = \frac{q}{\text{LCM}(p, q)}\,\omega_o T$$

(c) We need to find a value of m such that $x[n + N] = x(nT + mT_o)$. Therefore, $N = m(T_o/T)$, where $m(T_o/T)$ must be an integer, or $m(q/p)$ must be an integer. Thus, $mq = \text{LCM}(p, q)$, $m = \text{LCM}(p, q)/q$.