# Math 335 Practice Exam 1 Key

**1.)** *[10 points]* Compute the line integral

$$\int_C y^2 + 2xz\ ds$$

where C is the straight line path from (1,2,3) to (-2,0,1).

Parametrize the path C: $\vec{x} = (1-t)\vec{x}_0 + \vec{x}_1 t,\ 0 \le t \le 1$

$x = (1-t)1 - 2t = 1 - 3t$

$y = (1-t)2 + 0t = 2 - 2t \qquad 0 \le t \le 1$

$z = (1-t)3 + 1t = 3 - 2t$

$\vec{r}(t) = \langle 1-3t,\ 2-2t,\ 3-2t \rangle$

$\vec{r}\,'(t) = \langle -3, -2, -2 \rangle$

$|\vec{v}| = \sqrt{(-3)^2 + (-2)^2 + (-2)^2} = \sqrt{17}$

$\int_C f(x,y,z)\,ds = \int_a^b f(t)\,|v(t)|\,dt$

$\int_C (y^2 + 2xz)\,ds = \int_0^1 \left[(2-2t)^2 + 2(1-3t)(3-2t)\right]\sqrt{17}\,dt$

$= \sqrt{17}\int_0^1 4 - 8t + 4t^2 + 2(3 - 2t - 9t + 6t^2)\,dt$

$= \sqrt{17}\int_0^1 16t^2 - 30t + 10\,dt$

$= \sqrt{17}\left[\frac{16}{3}t^3 - 15t^2 + 10t\right]_0^1$

$= \sqrt{17}\left[\frac{16}{3} - 15 + 10\right]$

$= \boxed{\frac{\sqrt{17}}{3}}$

**2.)** *[10 points]* Charmander projects a heat field given by

$$\vec{F} = \langle -3 + 2x\sin y\,, 2y + x^2\cos y \rangle.$$

**a.)** Prove F is conservative.

$$\frac{\partial M}{\partial y} = 2x\cos y$$

$$\frac{\partial N}{\partial x} = 2x\cos y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow F \text{ conservative}$$

**b.)** Find a potential function f(x,y) that corresponds to F.

Find $f(x,y)$ s.t $\frac{\partial f}{\partial x} = -3 + 2x\sin y$, $\frac{\partial f}{\partial y} = 2y + x^2\cos y$

Integrate M

$$f(x,y) = \int -3 + 2x\sin y \, dx = -3x + x^2\sin y + g(y)$$

Differentiate this w.r.t. y and set equal to N.

$$\frac{\partial f}{\partial y} = x^2\cos y + g'(y) = 2y + x^2\cos y \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$$

$$f(x,y) = -3x + x^2\sin y + y^2$$

**c.)** Compute the work done by Charmander's field on a Pikachu running from the point (1,3) to the point (-2,0).

$$\int_C F \cdot T \, ds = f(-2,0) - f(1,3)$$

$$= 6 - (-3 + \sin 3 + 9)$$

$$= \boxed{-\sin 3}$$

**3.)** *[10 points]* Charmander runs along the curve C parametrized by

$$\vec{r}(t) = (e^t \cos t)i + (e^t \sin t)j, \quad 0 \le t \le 2\pi.$$

Find the work done on Charmander by the vector field $\vec{F} = \frac{xi+yj}{(x^2+y^2)^{3/2}}$ in two ways.

**a.)** Use the parametrization to evaluate the line integral.

**b.)** Prove F is a conservative vector field and find a corresponding potential function.

a.) $\int_C F \cdot T\, ds = \int_0^{2\pi} F(x(t), y(t)) \cdot \frac{d\vec{r}}{dt}\, dt$

$= \int_0^{2\pi} \left\langle \frac{e^t\cos t}{e^{3t}}, \frac{e^t \sin t}{e^{3t}} \right\rangle \cdot \langle e^t\cos t - e^t \sin t, e^t \sin t + e^t \cos t \rangle dt$

$= \int_0^{2\pi} \frac{1}{e^{3t}} \left( e^{2t}\cos^2 t - e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \cos t \sin t \right) dt$

$= \int_0^{2\pi} e^{-t}\, dt$

$= -e^{-t} \Big|_0^{2\pi}$

$= \boxed{-e^{-2\pi} + 1}$

b.) $F = \left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right\rangle = \langle M, N \rangle \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Find potential function $f(x,y)$ s.t. $\nabla f = F$.

$f(x,y) = \int \frac{x}{(x^2+y^2)^{3/2}}\, dx = -(x^2+y^2)^{-1/2} + g(y)$

$\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y) = C$

$f(x,y) = -(x^2+y^2)^{-1/2} + C$

$\int_C F \cdot T\, ds = f(e^{2\pi}, 0) - f(1,0)$

$= -\left((e^{2\pi})^2 + 0\right)^{-1/2} + 1$

$= \boxed{-e^{-2\pi} + 1}$

**4.)** *[10 points]* Suppose C is the counterclockwise path around the square bounded by x=0, x=1, y=0, and y=1. The vector field F is given by $\vec{F} = \langle \underbrace{2xy + x}_{M}, \underbrace{xy - y}_{N} \rangle$.

**a.)** Find the circulation of F around C.

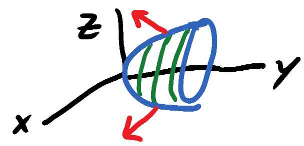
Closed 2D Path $\Rightarrow$ Green's Theorem!

$$\text{Circulation} = \oint_C F \cdot T\, ds$$
$$= \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$
$$= \int_0^1 \int_0^1 ( y - 2x )\, dx\, dy$$
$$= \int_0^1 xy - x^2 \Big|_{x=0}^{x=1} dy$$
$$= \int_0^1 y - 1\, dy$$
$$= \frac{1}{2}y^2 - y \Big|_0^1$$
$$= \frac{1}{2} - 1$$
$$= \boxed{-\frac{1}{2}}$$

**b.)** Find the outward flux of F around C.

$$\text{Flux} = \oint_C F \cdot n\, ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$
$$= \int_0^1 \int_0^1 (2y + 1 + x - 1)\, dx\, dy$$
$$= \int_0^1 2xy + \frac{1}{2}x^2 \Big|_{x=0}^{x=1} dy$$
$$= \int_0^1 2y + \frac{1}{2}\, dy$$
$$= y^2 + \frac{1}{2}y \Big|_0^1$$
$$= 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$

**5.)** *[10 points]* Compute the outward flux of $\vec{F} = \langle 0, y, -z\rangle$ through the surface of the paraboloid $y = x^2 + z^2,\ \ 0 \le y \le 1$.




Write $\underbrace{y - x^2 - z^2}_{G(x,y,z)} = 0.$

A perpendicular to surface is $\nabla G = \langle -2x, 1, -2z\rangle$

But this points inside so choose $-\nabla G = \langle 2x, -1, 2z\rangle$

To make it a unit vector, $\vec{n} = \dfrac{\langle 2x,-1,2z\rangle}{|\langle 2x,-1,2z\rangle|} = \dfrac{\langle 2x,-1,2z\rangle}{\sqrt{1+4x^2+4z^2}}$

$$\text{Flux} = \iint_Q \vec{F}\cdot\vec{n}\, dS$$

$$= \iint_{0\le x^2+z^2\le 1} \langle 0, y, -z\rangle \cdot \frac{\langle 2x,-1,2z\rangle}{\cancel{\sqrt{1+4x^2+4z^2}}}\, \cancel{\sqrt{1+(2x)^2+(2z)^2}}\, dx\,dz$$

$$= \iint_{0\le x^2+z^2\le 1} -x^2 - 3z^2\, dx\,dz \quad \leftarrow \ y = x^2+z^2$$

Go To Polar!

$$= \int_0^{2\pi}\int_0^1 \left(-r^2\cos^2\theta - 3r^2\sin^2\theta\right) r\,dr\,d\theta$$

Limits are constants, so can break up.

$$= \left[\int_0^{2\pi} -\cos^2\theta - 3\sin^2\theta\, d\theta\right]\left[\int_0^1 r^3\, dr\right]$$

$$= \left[-2\theta + \frac{1}{2}\sin 2\theta \,\Big|_0^{2\pi}\right]\left[\frac{1}{4}r^4\,\Big|_0^1\right]$$

$$= [-4\pi]\left[\frac{1}{4}\right]$$

$$= \boxed{-\pi}$$

So fluid is flowing in.