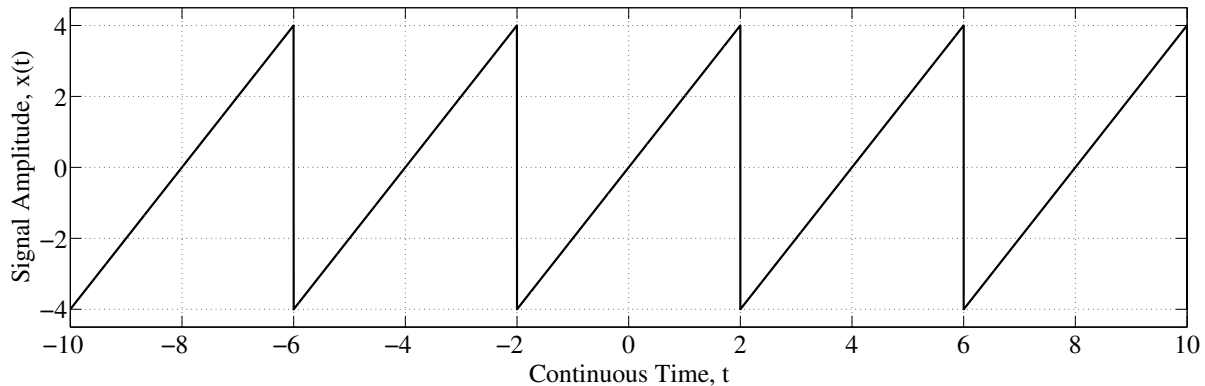# ELEC 309
Signals and Systems
Homework 5 Solutions

Frequency-Domain Analysis of Signals

1. For the periodic continuous-time signal $x(t)$ shown above:

We can write the periodic continuous-time signal above as

$$x(t) = 2t - 4k \text{ for } k - 2 \leq t < k + 2$$

for all integers k. The fundamental period is $T_0 = 4$ seconds, and the fundamental angular frequency is $\omega_0 = 2\pi/T_0 = \pi/2$ rad/sec.

(a) Find the exponential Fourier series representation of $x(t)$.

The average value or dc component is given by

$$D_0 = \frac{1}{T_0} \int_{T_0} x(t)dt = \frac{1}{4} \int_{-2}^{2} 2t\, dt = \frac{1}{2} \left[\frac{t^2}{2} \right]_{-2}^{2} = \boxed{0,}$$

To find the rest of the exponential Fourier series coefficients, we will using the following identities:

$$\int_{a}^{b} x e^{cx} dx = \frac{1}{c^2} e^{cx} (cx - 1) \Big|_{a}^{b} \quad \text{with } c = -jn\pi/2$$

$$\cos(\theta) = \frac{e^{+\theta} + e^{-\theta}}{2} \qquad \text{(Euler's)}$$

$$\sin(\theta) = \frac{e^{+\theta} - e^{-\theta}}{j2} \qquad \text{(Euler's)}$$

The rest of the exponential Fourier series coefficients are given by

$$D_n = \frac{1}{T_0} \int_{T_0} x(t)e^{-jn\omega_0 t} dt = \frac{1}{4} \int_{-2}^{2} 2te^{-jn\pi t/2} dt = \frac{1}{2} \int_{-2}^{2} te^{-jn\pi t/2} dt$$

$$= \frac{1}{2} \left(\frac{4}{-n^2\pi^2} \right) \left[e^{-jn\pi t/2} \left(-jn\pi t/2 - 1 \right) \right]_{-2}^{2}$$

$$= \left(\frac{2}{-n^2\pi^2} \right) \left[e^{-jn\pi} \left(-jn\pi - 1 \right) - e^{jn\pi} \left(jn\pi - 1 \right) \right]$$

$$= \left(\frac{j2}{n\pi} \right) \left[e^{jn\pi} + e^{-jn\pi} \right] - \left(\frac{2}{n^2\pi^2} \right) \left[e^{jn\pi} - e^{-jn\pi} \right]$$

$$= \left(\frac{j4}{n\pi} \right) \left[\frac{e^{jn\pi} + e^{-jn\pi}}{2} \right] - \left(\frac{j4}{n^2\pi^2} \right) \left[\frac{e^{jn\pi} - e^{-jn\pi}}{j2} \right]$$

$$= \left(\frac{j4}{n\pi} \right) \underbrace{\cos(n\pi)}_{(-1)^n} - \left(\frac{j4}{n^2\pi^2} \right) \underbrace{\sin(n\pi)}_{0} = \boxed{\frac{j4\cos(n\pi)}{n\pi} = \frac{j4(-1)^n}{n\pi}}.$$

Therefore, we can represent $x(t)$ using the exponential Fourier series representation given by

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} = \boxed{\sum_{n=-\infty, n\neq 0}^{\infty} \frac{j4\cos(n\pi)}{n\pi} e^{jn\pi t/2} = \sum_{n=-\infty, n\neq 0}^{\infty} \frac{j4(-1)^n}{n\pi} e^{jn\pi t/2}}.$$

(b) Find the trigonometric Fourier series representation of $x(t)$.

Using our identities from the class notes, we have

$$a_0 = D_0 = \boxed{0,}$$

$$a_n = 2\text{Re}\{D_n\} = \boxed{0,} \text{ and}$$

$$b_n = -2\text{Im}\{D_n\} = \boxed{\frac{-8\cos(n\pi)}{n\pi} = \frac{-8(-1)^n}{n\pi}}.$$

Therefore, we can represent $x(t)$ using the trigonometric Fourier series representation given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{-8\cos(n\pi)}{n\pi} \sin(n\pi t/2) = \sum_{n=1}^{\infty} \frac{-8(-1)^n}{n\pi} \sin(n\pi t/2).}$$

(c) Find the harmonic/compact Fourier series representation of $x(t)$.

Using our identities from the class notes, we have

$$C_0 = D_0 = \boxed{0,}$$

$$C_n = 2|D_n| = \boxed{\frac{8}{n\pi},} \text{ and}$$

$$\theta_n = \angle D_n = \begin{cases} -\pi/2 & \text{for } n \text{ even} \\ \pi/2 & \text{for } n \text{ odd} \end{cases} = \boxed{-\frac{\pi}{2}(-1)^n = \frac{\pi}{2}(-1)^{n+1}.}$$

Therefore, we can represent $x(t)$ using the harmonic/compact Fourier series representation given by

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right)$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{8}{n\pi} \cos\left(n\pi t/2 - \frac{\pi}{2}(-1)^n\right) = \sum_{n=1}^{\infty} \frac{-8(-1)^n}{n\pi} \cos\left(n\pi t/2 - \pi/2\right).}$$

(d) Verify Parseval's theorem for $x(t)$, using the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The power in the time domain is given by

$$P_x = \frac{1}{T_0} \int_{T_0} x^2(t) dt = \frac{1}{4} \int_{-2}^{2} 4t^2 dt = \left[\frac{t^3}{3}\right]_{-2}^{2} = \boxed{\frac{16}{3} = 5.3333.}$$

The power in the frequency domain is given by

$$P_x = \sum_{n=-\infty}^{\infty} |D_n|^2 = 2\sum_{n=1}^{\infty} \frac{16}{n^2\pi^2} = \frac{32}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{32}{\pi^2}\left(\frac{\pi^2}{6}\right) = \boxed{\frac{16}{3} = 5.3333.}$$

$$= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{64}{n^2\pi^2} = \frac{32}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{32}{\pi^2}\left(\frac{\pi^2}{6}\right) = \boxed{\frac{16}{3} = 5.3333.}$$

$$= C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 = \frac{1}{2}\sum_{n=1}^{\infty} \frac{64}{n^2\pi^2} = \frac{32}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{32}{\pi^2}\left(\frac{\pi^2}{6}\right) = \boxed{\frac{16}{3} = 5.3333.}$$

Thus, Parseval's theorem has been verified.

(e) Using MATLAB, write a script m-file to plot the Fourier spectra for the signal. (Plot $|D_n|$ vs. ω and $\angle D_n$ vs. ω (where $\omega = n\omega_0$) on a single figure by using the *subplot* command.) Upload a copy of your MATLAB script m-file to the course website.

MATLAB code to plot the Fourier spectra:

```matlab
% Fundamental period and angular frequency
T0 = 4;
w0 = 2*pi/T0;

% Range of n and omega values
n = -15:15;
w = n*w0;

% Exponential Fourier coefficients determined by hand
Dn = 1i*4*(-1).^n./(n*pi);
% Need to explicitly specify D_0
Dn(n==0) = 0;

% Generate magnitude and phase of complex Fourier coefficients
magDn = abs(Dn);
phaseDn = angle(Dn);

% Plot Fourier spectra
figure(1)
% Plot magnitude spectrum
subplot(2,1,1), stem(w,magDn), grid on
xlabel('Angular Frequency, $$\omega$$ (rad/sec)','Interpreter','LaTeX');
ylabel('$$|D_n|$$','Interpreter','LaTeX');
title('Amplitude Spectrum','Interpreter','LaTeX');
% Plot phase spectrum
subplot(2,1,2), stem(w,phaseDn), grid on
xlabel('Angular Frequency, $$\omega$$ (rad/sec)','Interpreter','LaTeX');
ylabel('$$\angle D_n$$','Interpreter','LaTeX');
title('Phase Spectrum','Interpreter','LaTeX');
```