Lecture 19: Properties of Fourier Series

Teddiursa's Goals for the Day

- Establish the Odd-Even properties of Fourier Series
- Discuss the graph of a Fourier Series, especially at jumps
- Practice computing Fourier Series

12.2 Fourier Series

Fourier Series of $f(x)$ on $[-L, L]$

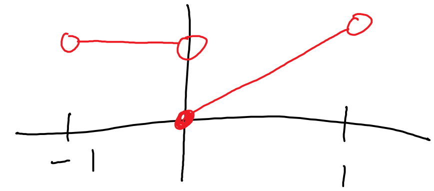
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad \left(a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Ex Last Lecture

$$f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \le x < 1 \end{cases}$$



Since there is a jump discontinuity at $x=0$, we cannot represent $f(x)$ with a power series.

But there does exist a Fourier series!

$$f(x) = \underbrace{\frac{3}{4}}_{a_0 = \frac{3}{2}} + \sum_{n=1}^{\infty} \left[\underbrace{\frac{(-1)^n - 1}{n^2\pi^2}}_{a_n} \cos(n\pi x) - \underbrace{\frac{1}{n\pi}}_{b_n} \sin(n\pi x) \right]$$

Properties of Fourier Series

1) Average Value

$$\frac{1}{2} a_0 = \frac{1}{2} \frac{1}{L} \int_{-L}^{L} f(x) \underbrace{\cos \overbrace{\frac{0\pi x}{L}}^{n=0}}_{1} \, dx$$

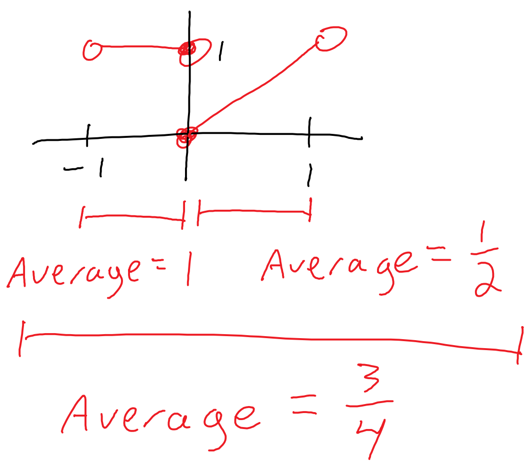
$$= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$= \frac{\text{Sum of values of } f(x) \text{ on } (-L, L)}{\text{Length of interval } (-L, L)}$$

$= $ Average of $f(x)$ on $(-L, L)$

The first term of the Fourier Series $\frac{1}{2}a_0$ tells us the average value of the function.

Ex Average of $f(x)$

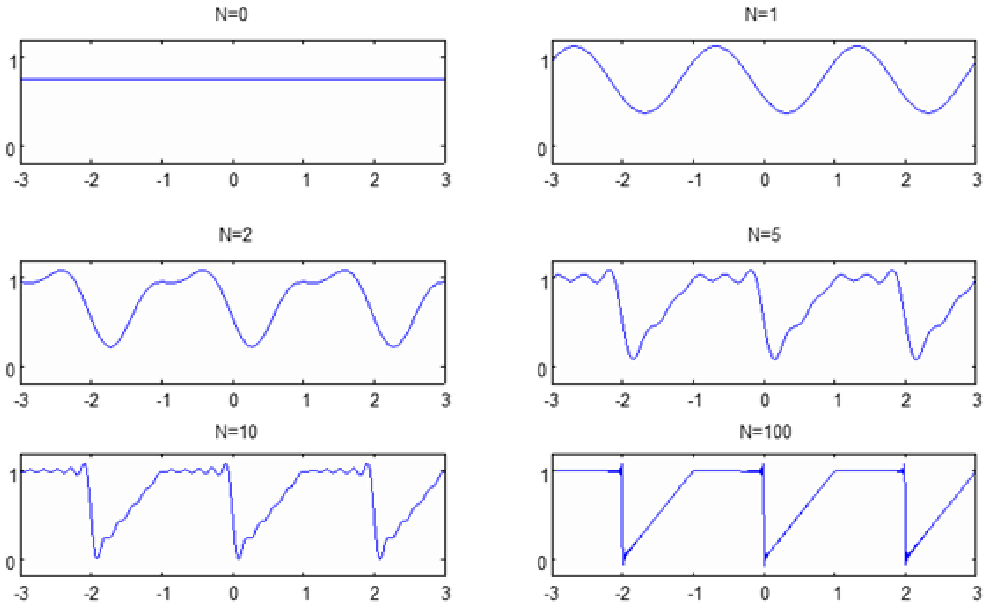


② Approximation

In practice, we approximate a function with a partial sum.

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

The more terms N we add to a Fourier Series, the better the approximation becomes.



Ringing

Near a jump discontinuity, the Fourier series will oscillate $\Rightarrow$ Gibbs Phenomenon

This causes echo in audio signals and "ringing" artifacts in images.

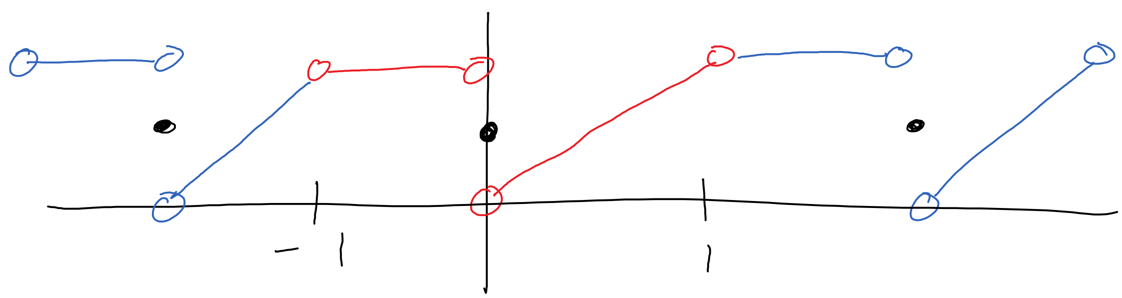
③ Periodicity

Sine and cosine are periodic functions

$\sin x, \cos x$ Period $= 2\pi$

$\sin \frac{\pi x}{L}, \cos \frac{\pi x}{L}$ Period $= 2L$

The Fourier Series is $2L$-periodic.



(4) Existence

For what functions can we build a Fourier Series?

Dirichlet's Theorem

If $f(x)$ is bounded and has a finite number of discontinuities and extrema on $(-L,L)$, then the Fourier Series will converge at every point in $(-L,L)$ to the value

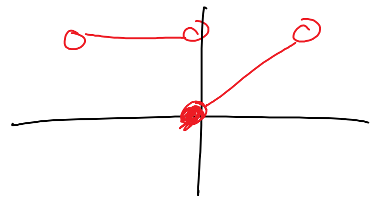
$$\frac{1}{2}\left[f(x^-) + f(x^+)\right]$$

where

$$f(x^-) = \lim_{x \to x^-} f(x)$$ Left-hand limit

$$f(x^+) = \lim_{x \to x^+} f(x)$$ Right-hand limit

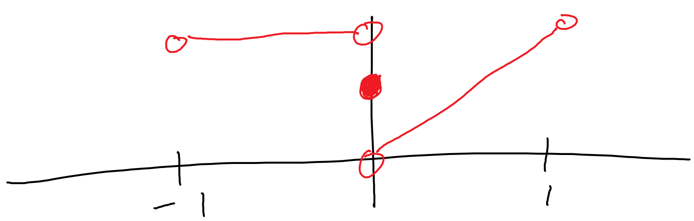
Ex $f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \le x < 1 \end{cases}$



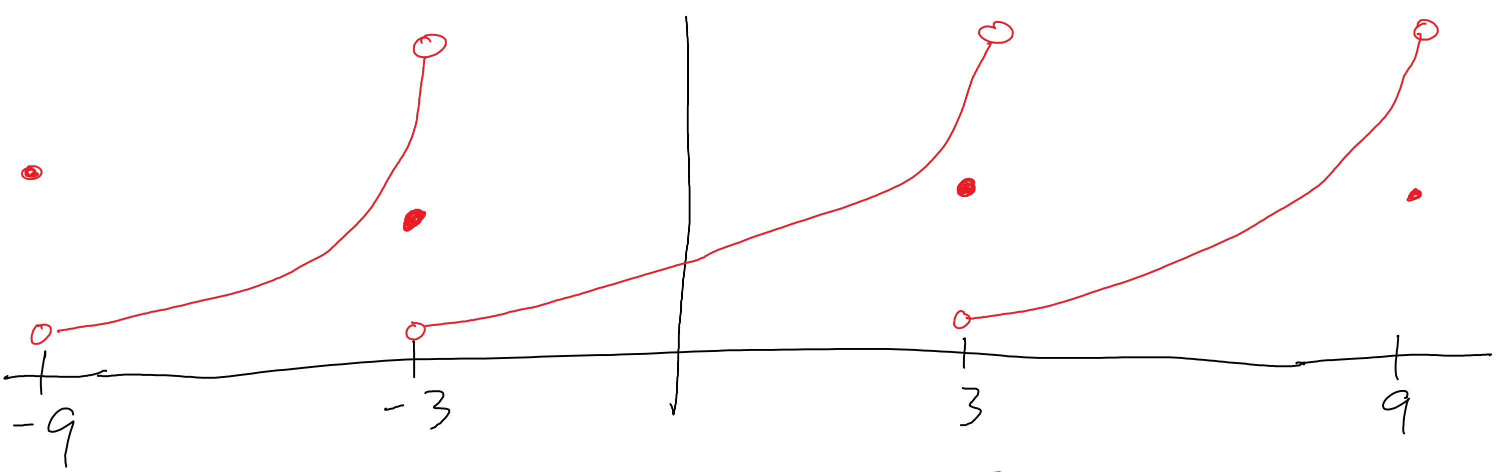
There is a jump discontinuity at $x=0$.

The Fourier series at $x=0$ converges to

$$\frac{1}{2}\left[f(0^-) + f(0^+)\right] = \frac{1}{2}\left[1 + 0\right] = \frac{1}{2}$$



Ex Graph the Fourier series of e^x on $(-3,3)$.



What is the value at $x=3$?

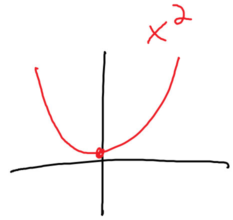
$$\frac{1}{2}\left[f(3^-) + f(3^+)\right] = \frac{1}{2}\left[e^3 + e^{-3}\right]$$

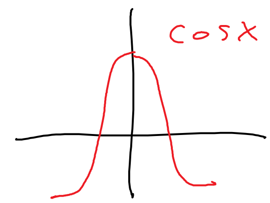
⑤ Odd / Even

An <u>even</u> function satisfies $f(-x) = f(x)$

<u>Ex</u> $x^2, x^4, x^6, \cos x$

Symmetric about y-axis

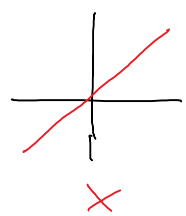


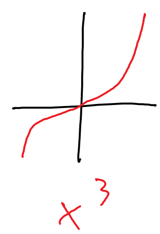


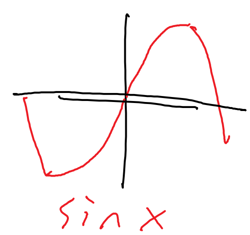
An <u>odd</u> function satisfies $f(-x) = -f(x)$

<u>Ex</u> $x, x^3, x^5, x^7, \sin x$

Symmetric about origin

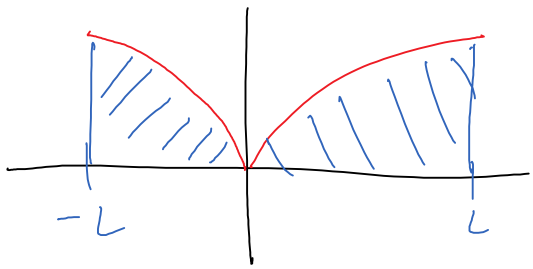






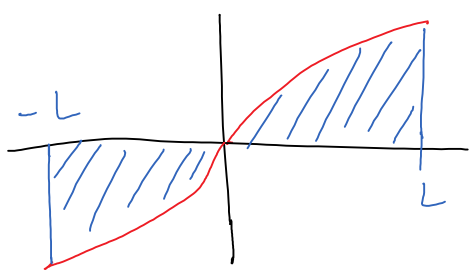
Even $f(x)$

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$



Odd $f(x)$

$$\int_{-L}^{L} f(x)\,dx = 0$$



Suppose $f(x)$ is odd.

Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\ \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\text{odd}}\,dx = 0$$

Suppose $f(x)$ is even.

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}}\;\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{odd}}\,dx = 0$$

Theorem (Odd-Even Theorem)

Fourier series of $f(x)$ on $(-L, L)$

i.) If $f(x)$ is odd, then $a_n = 0$.

The Fourier series for $f(x)$ consists of only sine term.

ii.) If $f(x)$ is even, then $b_n = 0$.

The Fourier series for $f(x)$ consists of only cosine terms (plus the constant).

Identifying a function as odd/even will immediately tell you the coefficients a_n/b_n are zero.

This will save you A LOT of time and calculation.