ELEC 312
Systems I

Laplace Transform Review

(Derived from Notes by Dr. Robert Barsanti)

(Images from Nise, 7^{th} Edition)

Required Reading: Chapter 2,
Control Systems Engineering

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The Laplace Transform

The **Laplace Transform** of a general function of time $f(t)$ is defined as

$$\boxed{\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty f(t)e^{-st}dt.}$$

Some texts refer to this definition at the **unilateral** (or one-sided) Laplace transform to differentiate it from the two-sided or *bilateral* Laplace transform, which is defined as $F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$.

The result of this transform is a function of the independent variable s, corresponding to the variable in the exponent e^{-st}.

In general, s is a complex variable. Mathematically, $s = \sigma + j\omega$.

Notation:

$$f(t) \xleftrightarrow{\mathcal{L}} F(s) = \mathcal{L}\{f(t)\}$$

Convergence of the Laplace Integral

Notice that we integrate over all values of t in the range $(0, \infty)$ and that, in general, it will be necessary to restrict the values of s to some range, in order that the Laplace Transform integral will **converge**, or give a finite result.

The region of values of s in the two-dimensional complex plane for which the Laplace transform converges is known as the region of convergence (ROC).

Laplace Transform of Basic Signals: Unit Impulse Function $\delta(t)$

The Laplace transform of a unit impulse function is given by

$$\mathcal{L}\left\{\delta(t)\right\} = \int_0^\infty \delta(t)e^{-st}dt = 1 \text{ for all } s.$$

Therefore, the Laplace transform pair for a unit impulse function is

$$\delta(t) \longleftrightarrow 1 \text{ with ROC} = \text{all } s.$$

Laplace Transform of Basic Signals: Unit Step Function $u(t)$

The Laplace transform of a unit step function is given by

$$\mathcal{L}\left\{u(t)\right\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt =$$
$$= -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \text{ for Re}\left\{s\right\} > 0.$$

Therefore, the Laplace transform pair for a unit step function is

$$u(t) \longleftrightarrow \frac{1}{s} \text{ with ROC} = \text{Re}\left\{s\right\} > 0.$$

Laplace Transform of Basic Signals: Ramp Function $tu(t)$

The Laplace transform of a ramp function is given by

$$\mathcal{L}\left\{tu(t)\right\} = \int_0^\infty tu(t)e^{-st}dt = \int_0^\infty te^{-st}dt =$$
$$= \frac{e^{-st}}{(-s)^2}\left(-st-1\right)\Big|_0^\infty = \frac{1}{s^2} - \frac{1}{s^2}\left[\lim_{t\to\infty} e^{-st}\left(-st-1\right)\right]$$
$$= \frac{1}{s^2} - \frac{1}{s^2}\left[\lim_{t\to\infty}\frac{-1}{e^{st}}\right] = \frac{1}{s^2} \text{ for Re}\left\{s\right\} > 0.$$

Therefore, the Laplace transform pair for a ramp function is

$$tu(t) \longleftrightarrow \frac{1}{s^2} \text{ with ROC} = \text{Re}\left\{s\right\} > 0.$$

Laplace Transform of Basic Signals: Parabola Function $\frac{1}{2}t^2u(t)$

The Laplace transform of a parabola function is given by

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t)\right\} = \int_0^\infty \frac{1}{2}t^2u(t)e^{-st}dt = \int_0^\infty \frac{1}{2}t^2e^{-st}dt =$$
$$= \frac{e^{-st}}{2(-s)^3}\left(s^2t^2+2st+2\right)\Big|_0^\infty$$
$$= \frac{1}{s^3} - \frac{1}{s^3}\left[\lim_{t\to\infty} e^{-st}\left(\frac{1}{2}s^2t^2+st+1\right)\right]$$
$$= \frac{1}{s^3} - \frac{1}{s^3}\left[\lim_{t\to\infty}\frac{1}{e^{st}}\right] = \frac{1}{s^3} \text{ for Re}\left\{s\right\} > 0.$$

Therefore, the Laplace transform pair for a parabola function is

$$\frac{1}{2}t^2u(t) \longleftrightarrow \frac{1}{s^3} \text{ with ROC} = \text{Re}\left\{s\right\} > 0.$$

Laplace Transform of Basic Signals: Complex Exponential

The Laplace transform of a complex exponential function

$$f(t) = e^{-at} \text{ where } a = \text{Re}\{a\} + j\text{Im}\{a\}$$

is given by

$$F(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = -\frac{e^{-(s+a)t}}{s+a}\Big|_0^\infty$$
$$= \frac{1}{s+a} - \frac{1}{s+a}\left[\lim_{t\to\infty} e^{-(s+a)t}\right] = \frac{1}{s+a} \text{ for Re}\{s\} > -\text{Re}\{a\}.$$

This gives the Laplace Transform pair

$$\boxed{e^{at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \text{ for Re}\{s\} > -\text{Re}\{a\}.}$$

Laplace Transform of Basic Signals: Sinusoidal Function

The Laplace transform of a sinusoidal function

$$f(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

is given by

$$\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \xleftrightarrow{\mathcal{L}} \frac{1}{2}\left(\frac{1}{s-j\omega_0}\right) + \frac{1}{2}\left(\frac{1}{s+j\omega_0}\right) = \frac{s}{s^2+\omega_0^2} \text{ for } \mathrm{Re}\{s\} > 0.$$

This gives the Laplace Transform pair

$$\boxed{\cos(\omega_0 t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_0^2} \text{ for } \mathrm{Re}\{s\} > 0.}$$

Laplace Transform of Basic Signals: Sinusoidal Function

The Laplace transform of a sinusoidal function

$$f(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

is given by

$$\frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} \xleftrightarrow{\mathcal{L}} \frac{1}{2j}\left(\frac{1}{s-j\omega_0}\right) - \frac{1}{2j}\left(\frac{1}{s+j\omega_0}\right) = \frac{\omega_0}{s^2+\omega_0^2} \text{ for } \mathrm{Re}\{s\} > 0.$$

This gives the Laplace Transform pair

$$\boxed{\sin(\omega_0 t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2+\omega_0^2} \text{ for } \mathrm{Re}\{s\} > 0.}$$

Example:

Find the Laplace Transform of $f(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$.

Example (continued):

Example:

Find the Laplace Transform of $f(t) = e^{-2t}u(t) + e^{-t}\cos(3t)u(t)$.

Example (continued):

Properties of the Laplace Transform: Linearity

If

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \text{ with ROC} = R_1 \text{ and}$$
$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s) \text{ with ROC} = R_2,$$

then

$$\boxed{\alpha_1 x_1(t) + \alpha_2 x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \alpha_1 X_1(s) + \alpha_2 X_2(s)}$$

with ROC$= R' \supset R_1 \cap R_2$.

(The ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually, the ROC will simply be $R' = R_1 \cap R_2$.)

Properties of the Laplace Transform: Time Shifting

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \text{ with ROC} = R,$$

then

$$\boxed{x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s)}$$

with ROC$= R' = R$.

(The ROC of the resultant Laplace transform is unaffected by the time-shifting operation.)

Example:

Since $\delta(t) \xleftrightarrow{\mathcal{L}} 1$, show $\delta(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}$.

Properties of the Laplace Transform: Time Scaling

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \text{ with ROC} = R,$$

then

$$\boxed{x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)}$$

with ROC$= R' = aR$.

(The ROC of the resultant Laplace transform is the original ROC scaled by the constant a.)

Properties of the Laplace Transform: Convolution

If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \text{ with ROC} = R_1 \text{ and}$$
$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \text{ with ROC} = R_2,$$

then

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s)}$$

with ROC$= R' \supset R_1 \cap R_2$.

(The ROC of the resultant Laplace transform is at least as large as the region in common between R_1 and R_2. Usually, the ROC will simply be $R' = R_1 \cap R_2$.)

Properties of the Laplace Transform: Shifting in the s-Domain or Exponential Multiplication:

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \text{ with ROC} = R,$$

then

$$e^{s_0 t}x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)$$

with ROC$= R' = R + \text{Re}\,(s_0)$.

(The ROC of the resultant Laplace transform is shifted by an amount equal to Re$\,(s_0)$.)

Properties of the Laplace Transform: Differentiation in the Time Domain

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \text{ with ROC} = R,$$

then

$$\boxed{\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s) - x(0)}$$

with ROC$= R' \supset R$.

(The ROC of the resultant Laplace transform is the original ROC unless there is a pole-zero cancellation at $s = 0$.)

Properties of the Laplace Transform: Differentiation in the Time Domain

This property can be extended by repeated application. For example,

$$\frac{d^2x(t)}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2X(s) - sx(0) - x'(0),$$
$$\frac{d^3x(t)}{dt^3} \overset{\mathcal{L}}{\longleftrightarrow} s^3X(s) - s^2x(0) - sx'(0) - x''(0), \text{ or}$$
$$\frac{d^4x(t)}{dt^4} \overset{\mathcal{L}}{\longleftrightarrow} s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0).$$

Extension:

$$\boxed{\mathcal{L}\left[\frac{d^n}{dt^n}x(t)\right] = s^nX(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \ldots - sx^{(n-2)}(0) - x^{(n-1)}(0).}$$

Example:

Find $Y(s)$ given $y''(t) + 3y(t) = 0$ for $t \geq 0$, $y(0) = 2$, and $y'(0) = 1$.

Example (continued):

Properties of the Laplace Transform: Integration in the Time Domain

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC $= R$, then

$$\boxed{\int_0^t x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)}$$

with ROC$= R' = R \cap \{\mathrm{Re}(s) > 0\}$.

(The ROC of the resultant Laplace transform follows from the possible introduction of an additional pole at $s = 0$ by the multiplication of $1/s$.)

Properties of the Laplace Transform: Final Value Theorem

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, then

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).}$$

Proof: Recall that $\displaystyle\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt = sX(s) - x(0)$.

Taking $\displaystyle\lim_{s\to 0}$ of both sides $\Rightarrow \displaystyle\lim_{s\to 0}\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt = \lim_{s\to 0}[sX(s) - x(0)]$

$$\Rightarrow \int_0^\infty \frac{d}{dt}x(t)\lim_{s\to 0} e^{-st}dt = \int_0^\infty dx(t) = \lim_{b\to\infty} x(b) - x(0) = \left[\lim_{s\to 0} sX(s)\right] - x(0)$$

$$\Rightarrow \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Note: Both $\mathcal{L}\{x(t)\}$ and $\mathcal{L}\{x'(t)\}$ must exist. Also, $\displaystyle\lim_{t\to\infty} x(t)$ must exist, which means that all the poles for $sX(s)$ must be in the left-half plane.

Example:

Given $\mathcal{L}[f(t)] = F(s) = \frac{1}{s(s+1)}$, what is $\displaystyle\lim_{t\to\infty} f(t)$?

Properties of the Laplace Transform: Initial Value Theorem

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, then

$$\boxed{\lim_{t\to 0} x(t) = x(0) = \lim_{s\to\infty} sX(s).}$$

Proof: Recall that $\displaystyle\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt = sX(s) - x(0)$.

Taking $\displaystyle\lim_{s\to\infty}$ of both sides $\Rightarrow \displaystyle\lim_{s\to\infty}\int_0^\infty \frac{d}{dt}x(t)e^{-st}dt = \lim_{s\to\infty}[sX(s) - x(0)]$

$$\Rightarrow \int_0^\infty \frac{d}{dt}x(t)\lim_{s\to\infty} e^{-st}dt = 0 = \left[\lim_{s\to\infty} sX(s)\right] - x(0)$$

$$\Rightarrow x(0) = \lim_{s\to\infty} sX(s)$$

Example:

Given $x(t) = \cos(\omega_0 t)$, what is $x(0)$?

Inverse Laplace Transform

The Inverse Laplace Transform is defined as

$$\boxed{x(t) = \mathcal{L}^{-1}\left[X(s)\right] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds,}$$

where c is the abscissa of convergence for $X(s)$. We SELDOM use the above integral to evaluate an Inverse Laplace Transform. Instead, we make use of table of known transform pairs. With a good table and the technique of partial fraction expansion, we can find the inverse transform via table lookup.

Partial Fraction Expansion

In general, $X(s)$ will be a fraction of polynomials in the variable s. The method is to expand $F(s)$ into fractional terms that can be equated to those in our table of transform pairs.

Partial Fraction Expansion
Case 1: Real, Non-Repeated Poles

Consider $X(s) = \frac{1}{(s+1)(s+2)}$.

Partial Fraction Expansion
Case 1: Real, Non-Repeated Poles (continued)

Partial Fraction Expansion
Case 2: Real Repeated Poles

Consider $X(s) = \frac{4s^2}{(s-0.5)(s+1)^2}$.

Partial Fraction Expansion
Case 2: Real Repeated Poles (continued)

Partial Fraction Expansion
Case 3: Complex Poles

Consider $X(s) = \frac{s}{(s^2+2s+2)(s^2+1)}$.

Partial Fraction Expansion
Case 3: Complex Poles (continued)

Partial Fraction Expansion
Case 3: Complex Poles (continued)