Math 335 Fall 2013, Exam 2 Key

1.) *[4 points]* Compute the gradient of $f(x, y, z) = x^2z + 3y^4 + 10$.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \boxed{\langle 2xz, \ 12y^3, \ x^2 \rangle}$$

2.) *[6 points]* Let $\vec{F}(x, y, z) = \langle 2xz, \ x^2 + y^2, -3x \rangle$.

a.) Compute the divergence of F.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(x^2 + y^2) + \cancel{\frac{\partial}{\partial z}(-3x)} \to 0$$

$$= \boxed{2z + 2y}$$

c.) Compute the curl of F.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & x^2 + y^2 & -3x \end{vmatrix}$$

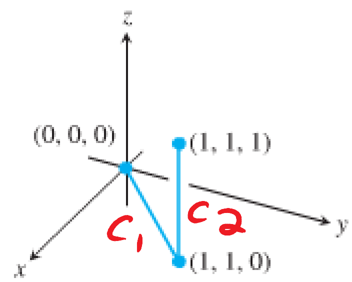
$$= i(0 - 0) - j(-3 - 2x) + k(2x - 0)$$

$$= \boxed{\langle 0, \ 3 + 2x, \ 2x \rangle}$$

3.) *[10 points]* Charmeleon used his immense strength to bend a straight length of wire at a 90 degree angle. The wire consists of straight line paths from (0,0,0) to (1,1,0) and then (1,1,0) to (1,1,1). If the linear density of the wire in g/cm is

$$\rho(x, y, z) = x^2 + 2y - z$$

then find the total mass of the wire.



C_1: $(0,0,0) \to (1,1,0)$ $\quad \vec{r}(t) = \langle t, t, 0 \rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 1, 0 \rangle \quad |\vec{r}\,'(t)| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$$\int_{C_1} x^2 + 2y - z \, ds = \int_0^1 [t^2 + 2t - 0]\sqrt{2}\, dt$$

$$= \sqrt{2}\left[\tfrac{1}{3}t^3 + t^2\right]_0^1$$

$$= \sqrt{2}\left[\tfrac{1}{3} + 1\right]$$

$$= \frac{4\sqrt{2}}{3}$$

C_2: $(1,1,0) \to (1,1,1)$ $\quad \vec{r}(t) = \langle 1, 1, t \rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, 0, 1 \rangle \quad |\vec{r}\,'(t)| = \sqrt{0^2+0^2+1^2} = 1$

$$\int_{C_2} x^2 + 2y - z \, ds = \int_0^1 [1^2 + 2(1) - t](1)\, dt$$

$$= \int_0^1 [3 - t]\, dt = 3t - \tfrac{1}{2}t^2 \Big|_0^1$$

$$= 3 - \tfrac{1}{2} = \tfrac{5}{2}$$

Total $\quad$ Total Mass = Mass on C_1 + Mass on C_2 = $\boxed{\dfrac{4\sqrt{2}}{3} + \dfrac{5}{2}}$

4.) *[10 points]* Charmeleon launches a 3-dimensional fire-based attack with force given by

$$\vec{F}(x, y, z) = \langle 2xy,\ x^2 + 1,\ -3 \rangle.$$

a.) Prove F is conservative.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+1 & -3 \end{vmatrix} = i(0-0) - j(0-0) + k(2x - 2x)$$

$$= \boxed{\langle 0, 0, 0 \rangle} \Rightarrow \text{Conservative}$$

b.) Find a potential function f(x,y,z) that corresponds to F.

1. $\int 2xy\, dx = x^2 y + g_1(y, z)$
2. $\int x^2+1\, dy = x^2 y + y + g_2(x, z)$
3. $\int -3\, dz = -3z + g_3(x, y)$

$$\boxed{f(x,y,z) = x^2 y + y - 3z}$$

c.) Compute the work done by Charmeleon's field on a Squirtle running from the point (0,0,1) to the point (1,2,3).

$$\int_C \vec{F} \cdot \vec{T}\, ds = f(1,2,3) - f(0,0,1)$$

$$= \left[1^2(2) + 2 - 3(3)\right] - \left[0^2(0) + 0 - 3(1)\right]$$

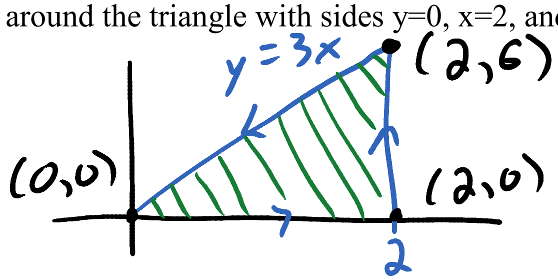
$$= 2 + 2 - 9 - 0 - 0 + 3$$

$$= \boxed{-2}$$

5.) *[10 points]* Find the counterclockwise circulation of

$$\vec{F}(x,y) = \langle \underbrace{x^2y - 2}_{M}, \underbrace{4x - 3y + 1}_{N} \rangle$$

around the triangle with sides y=0, x=2, and y=3x.



Closed 2D curve ⇒ Green's Theorem!

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

$$= \int_0^2 \int_0^{3x} 4 - x^2 \,dy\,dx$$

$$= \int_0^2 4y - x^2 y \Big|_{y=0}^{y=3x} dx$$

$$= \int_0^2 4(3x) - x^2(3x)\,dx$$

$$= \int_0^2 12x - 3x^3\,dx$$

$$= 6x^2 - \frac{3}{4}x^4 \Big|_{x=0}^{x=2}$$

$$= 6(2)^2 - \frac{3}{4}(2)^4$$

$$= 24 - 12$$

$$= \boxed{12}$$

6.) *[10 points]* The surface Q is the portion of the surface $z = 10 - x^2 + 2y$ that is over the rectangle $0 \le x \le 3,\ 0 \le y \le 2$. Calculate the surface integral

$$\iint_Q 2x\ dS.$$

Jacobian to $z = f(x,y)$ is $\sqrt{1+f_x^2+f_y^2}$

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+(-2x)^2+2^2} = \sqrt{5+4x^2}$$

$$\iint_Q 2x\,dS = \int_0^3 \int_0^2 2x\sqrt{5+4x^2}\,dy\,dx$$

$$= \int_0^3 2xy\sqrt{5+4x^2}\Big|_{y=0}^{y=2}\,dx$$

$$= \int_0^3 4x\sqrt{5+4x^2}\,dx$$

$$= \frac{1}{2}\int_0^3 8x\sqrt{5+4x^2}\,dx$$

$u = 5+4x^2$
$du = 8x\,dx$
$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2}$

$$= \frac{1}{2}\cdot\frac{2}{3}(5+4x^2)^{3/2}\Big|_0^3$$

$$= \frac{1}{3}(5+4\cdot 3^2)^{3/2} - \frac{1}{3}(5+4\cdot 0^2)^{3/2}$$

$$= \boxed{\frac{1}{3}(41)^{3/2} - \frac{1}{3}(5)^{3/2}}$$