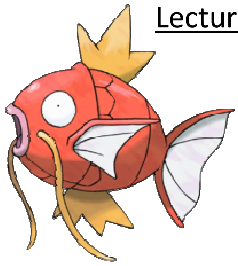

Lecture 15: The Power Series Method

Magikarp's Goals for the Day

- Practice solving linear ODEs using the Power Series Method
- Discuss ordinary vs. singular points

5.1 Ordinary + Singular Points

We can solve any linear ODE using the Power Series Method:

1. Plug in power series expansion for y and all its derivatives.

$$y = \sum_{n=0}^{\infty} a_n (x-c)^n$$

2. Combine all terms into one series.
3. Find the recurrence relation for the series.

Fact When a series equals zero, then all coefficients have to equal zero.

$$\sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow a_n = 0$$

We can use this fact to find the recurrence relation for the series.

Ex Find the recurrence relation for

$$\sum_{n=0}^{\infty} \underbrace{\left[3a_n - 2na_{n+1}\right]} x^n = 0$$

$$3a_n - 2na_{n+1} = 0$$

$$-2na_{n+1} = -3a_n$$

$$\boxed{a_{n+1} = \frac{3a_n}{2n}}$$

Always solve for the highest index.

Ex Solve Airy's Equation

$$y'' - xy = 0$$

Find power series solution about $x=0$. Write out the terms through x^5.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Plug this into the original DE.

$$y'' - xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Let $k+1 = n-2$

$k+3 = n$

$k+2 = n-1$

$n=2 \Rightarrow k=-1$

$$\sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Pull out $k=-1$ term

$$(-1+3)(-1+2) a_{-1+3} x^{-1+1} + \sum_{k=0}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2a_2 + \sum_{n=0}^{\infty} \Big[(n+3)(n+2) a_{n+3} - a_n \Big] x^{n+1} = 0$$

$a_2 = 0$ $\qquad$ $(n+3)(n+2) a_{n+3} - a_n = 0$

$a_2 = 0 \qquad (n+3)(n+2)a_{n+3} - a_n = 0$

$$(n+3)(n+2)\,a_{n+3} = a_n$$

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)}$$

Recurrence Relation

$$y = \underbrace{a_0}_{\text{unknowns}} + \underbrace{a_1}_{} x + \underbrace{a_2}_{a_2=0} x^2 + \underbrace{a_3}_{\frac{1}{6}a_0} x^3 + \underbrace{a_4}_{\frac{1}{12}a_1} x^4 + \underbrace{a_5}_{0} x^5 + \dots$$

$\underline{n=0:}$ $\quad a_3 = \dfrac{a_0}{(0+3)(0+2)} = \dfrac{1}{6}a_0$

$\underline{n=1:}$ $\quad a_4 = \dfrac{a_1}{(1+3)(1+2)} = \dfrac{1}{12}a_1$

$\underline{n=2:}$ $\quad a_5 = \dfrac{a_2}{(2+3)(2+2)} = \dfrac{1}{20}\cancel{a_2}^{\,0} = 0$

$$\boxed{y = a_0 + a_1 x + 0x^2 + \frac{1}{6}a_0 x^3 + \frac{1}{12}a_1 x^4 + 0x^5 + \dots}$$

Note In general, a DE of order n will have n unknown constants.

In the last example, the DE was order 2 so we got 2 unknowns a_0 and a_1.

Your textbook writes the series by pulling out the unknown constants.

$$y = a_0\left[1 + \frac{1}{6}x^3 + \dots\right]$$
$$+ a_1\left[x + \frac{1}{12}x^4 + \dots\right]$$

If we were given 2 initial conditions, we could solve for the 2 unknowns.

Ex Solve $y'' - xy = 0, \quad y(0) = 3, \quad y'(0) = 4.$

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \dots$$

$y(0) = 3$ $\qquad 3 = a_0$

$$y' = a_1 + \frac{3}{6} a_0 x^2 + \frac{4}{12} a_1 x^3 + \dots$$

$y'(0) = 4$ $\qquad 4 = a_1$

$$\boxed{y = 3 + 4x + \frac{1}{2} x^3 + \frac{1}{3} x^4 + \dots}$$

Ex Find the power series solution about $x = 0$ to

$$2y'' - 3xy' + (x+3)y = 0$$

Write the terms through x^4.

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + (x+3) \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 3 a_n x^n = 0$$

$$\underbrace{\sum_{n=2} 2n(n-1)a_n x^{n-2}}_{k=n-2} - \sum_{n=1} 3n a_n x^n + \underbrace{\sum_{n=0} a_n x^{n+1}}_{m=n+1} + \sum_{n=0} 3a_n x^n = 0$$

$k = n-2$ $\qquad\qquad$ $m = n+1$

$k+2 = n$ $\qquad\qquad$ $m-1 = n$

$k+1 = n-1$ $\qquad\qquad$ $n=0 \Rightarrow m=1$

$n=2 \Rightarrow k=0$

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k - \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{m=1}^{\infty} a_{m-1}x^m + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Pull out zero terms from 1st and 4th series. Combine.

$$\underbrace{2(2)(1)a_2 x^0 + 3a_0 x^0}_{4a_2 + 3a_0 = 0} + \sum_{n=1}^{\infty} \underbrace{\left[2(n+2)(n+1)a_{n+2} - 3n a_n + a_{n-1} + 3a_n \right]}_{2(n+2)(n+1)a_{n+2} + 3(1-n)a_n + a_{n-1} = 0} x^n = 0$$

$$4a_2 + 3a_0 = 0$$

$$a_2 = -\frac{3}{4}a_0$$

$$2(n+2)(n+1)a_{n+2} + 3(1-n)a_n + a_{n-1} = 0$$

$$2(n+2)(n+1)a_{n+2} = 3(n-1)a_n - a_{n-1}$$

For $n \geq 1$ $\rightarrow$ $$a_{n+2} = \frac{3(n-1)a_n - a_{n-1}}{2(n+2)(n+1)}$$

$\underline{n=1:}$ $$a_3 = \frac{3\cancelto{0}{(1-1)}a_1 - a_0}{2(1+2)(1+1)} = \frac{-a_0}{2(3)(2)} = -\frac{1}{12}a_0$$

$\underline{n=2:}$ $$a_4 = \frac{3(2-1)a_2 - a_1}{2(2+2)(2+1)} = \frac{3a_2 - a_1}{2(4)(3)} = \frac{1}{8}a_2 - \frac{1}{24}a_1$$

$$= \frac{1}{8}\left(-\frac{3}{4}a_0\right) - \frac{1}{24}a_1 = -\frac{3}{32}a_0 - \frac{1}{24}a_1$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$\boxed{y = a_0 + a_1 x - \frac{3}{4}a_0 x^2 - \frac{1}{12}a_0 x^3 + \left(-\frac{3}{32}a_0 - \frac{1}{24}a_1\right)x^4 + \ldots}$$