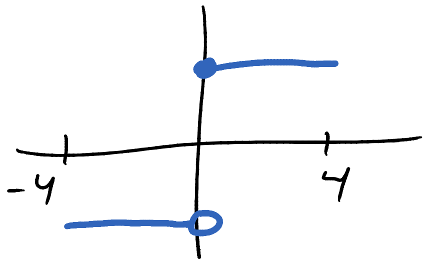
Math 335, Fall 2013 Exam 3 Key

1.) Let f(x) be the square wave

$$f(x) = \begin{cases} -3 & if\ x < 0 \\ 3 & if\ x \geq 0 \end{cases}$$

a.) *[10 points]* Compute the Fourier series of f(x) on the interval $(-4,4)$.



$f(x)$ odd $\Rightarrow a_n = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{4}\left[\int_{-4}^{0} -3\sin\frac{n\pi x}{4}\,dx + \int_0^4 3\sin\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{4}\left[\frac{12}{n\pi}\cos\frac{n\pi x}{4}\Big|_{-4}^{0} - \frac{12}{n\pi}\cos\frac{n\pi x}{4}\Big|_{0}^{4}\right]$$

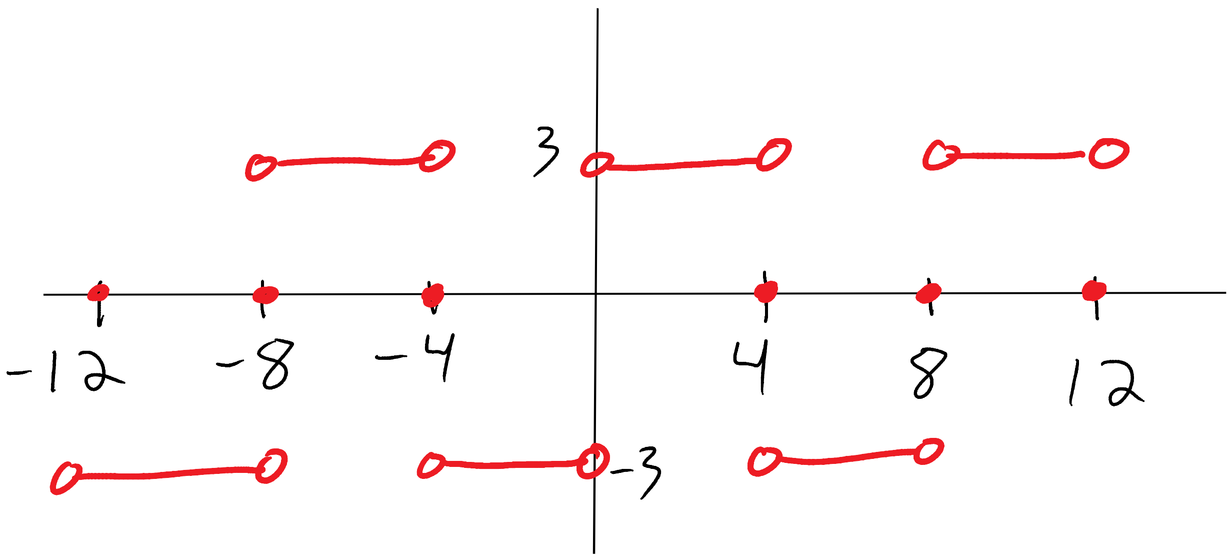
$$= \frac{3}{n\pi}\left[\overset{1}{\cos(0)} - \overset{(-1)^n}{\cos(-n\pi)} - \overset{(-1)^n}{\cos(n\pi)} + \overset{1}{\cos(0)}\right]$$

$$= \frac{3}{n\pi}\left[2 - 2(-1)^n\right] = \frac{6}{n\pi}\left[1-(-1)^n\right]$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{6}{n\pi}\left[1-(-1)^n\right]\sin\frac{n\pi x}{4}}$$

#1 continued...

b.) *[3 points]* Sketch the Fourier series that you computed in part (a) on the axes below for $-12 \le x \le 12$. Label the y-axis.



c.) *[2 points]* What value does the Fourier series converge to at $x = 4$?

$$\frac{1}{2}\left[f(4^-) + f(4^+)\right] = \frac{1}{2}\left[-3 + 3\right] = \boxed{0}$$

2.) *[15 points]* Solve the boundary value problem below.

$$u_t = 5u_{xx}$$
$$u(0,t) = 0, \quad u(4,t) = 0 \qquad \text{for all } t \geq 0$$
$$u(x,0) = \begin{cases} 0 & if\ 0 < x < 2 \\ 3 & if\ 2 \leq x < 4 \end{cases} \longrightarrow f(x)$$

Heat Equation

$k = 5 \quad L = 4$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{4}\int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\int_2^4 3\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{3}{2}\left[-\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\Big|_2^4\right]$$

$$= -\frac{6}{n\pi}\Big[\underbrace{\cos(n\pi)}_{(-1)^n} - \underbrace{\cos\left(\tfrac{n\pi}{2}\right)}_{\substack{0 \text{ if } n \text{ odd} \\ (-1)^{n/2} \text{ if } n \text{ even}}}\Big]$$

$$= -\frac{6}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} -\frac{6}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{4}\right)e^{-5\left(\frac{n\pi}{4}\right)^2 t}}$$

3.) *[20 points]* A vibrating string has length 2 meters and tension constant $a = 3$. The two ends of the string are clamped. The string is in motion and an observer starts recording its motion just as the string is at equilibrium position (a perfectly horizontal line between the clamped ends). At time t=0, the string is at equilibrium position but the velocity of the string is 6 m/sec at all points along the string.

a.) Write a PDE and full set of conditions that describes this situation.

b.) Compute the vertical displacement of the string u(x,t) at time t and position x.

a.) Wave Equation $\quad a=3, L=2$

$$u_{tt} = 9u_{xx}$$

$$u(0,t) = u(2,t) = 0$$

$$u(x,0) = 0 \quad \leftarrow f(x) \quad \text{Initial shape}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 6 \quad \leftarrow g(x) \quad \text{Initial Velocity}$$

b.) $f(x) = 0 \Rightarrow A_n = 0$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3n\pi}\int_0^2 6\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{4}{n\pi}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{8}{n^2\pi^2}\left[\cos(n\pi) - \cos(0)\right] = -\frac{8}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} -\frac{8}{n^2\pi^2}\left[(-1)^n - 1\right]\sin\left(\frac{3n\pi t}{2}\right)\sin\left(\frac{n\pi x}{2}\right)}$$