ELEC 309
Signals and Systems
Homework 4 Assignment

Time-Domain Analysis of LTI Systems

1. A continuous-time system is described by the input-output equation

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$$

with initial condition $y(0) = 1$ and input $x(t) = u(t)$.

Using the classical approach to solving LCCDEs:

(a) Determine the output $y(t)$ of the system.

The output $y(t)$ consists of two parts, or $y(t) = y_p(t) + y_h(t)$.
The particular solution $y_p(t) = A$ satisfies

$$5y_p'(t) + 10y_p(t) = 2x(t)$$

For $t < 0$, $x(t) = 0$ yields

$$5y_p'(t) + 10y_p(t) = 2x(t) \Rightarrow 5(0) + 10A = 2(0) \Rightarrow A = 0.$$

For $t \geq 0$, $x(t) = 1$ yields

$$5y_p'(t) + 10y_p(t) = 2x(t) \Rightarrow 5(0) + 10A = 2(1) \Rightarrow A = 0.2.$$

Therefore, the particular solution is given by

$$y_p(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.2 & \text{for } t \geq 0 \end{cases} = 0.2u(t).$$

Since this is a first-order differential equation, the complete homogeneous solution is given as $y_h(t) = K_1 e^{s_1 t}$. To find s_1, we let $y_h(t) = Ke^{st}$ in the homogeneous form of the LCCDE. Therefore,

$$\begin{aligned} 5y_h'(t) + 10y_h(t) &= 5sKe^{st} + 10Ke^{st} = 0 \\ &= (5s + 10)Ke^{st} = 0 \\ &= 5Ke^{st}(s + 2) = 0. \end{aligned}$$

Therefore, the characteristic equation is $s+2=0$, and $s_1=-2$. Therefore, the complete homogeneous solution is $y_h(t)=K_1e^{-2t}$. Now, we have

$$y(t)=y_p(t)+y_h(t)=0.2u(t)+K_1e^{-2t}.$$

Plugging in initial conditions for $t<0$, we have

$$y(0^-)=0.2u(0^-)+K_1e^{-2(0^-)}=K_1=1.$$

Plugging in initial conditions for $t\geq 0$, we have

$$y(0^+)=0.2u(0^+)+K_1e^{-2(0^+)}=0.2+K_1=1.$$

Therefore, $K_1=1$ for $t<0$ and $K_1=0.8$ for $t\geq 0$, and the complete solution is given by

$$\boxed{y(t)=y_p(t)+y_h(t)=\begin{cases}e^{-2t} & \text{for } t<0\\ 0.2+0.8e^{-2t} & \text{for } t\geq 0\end{cases}=e^{-2t}+\left[0.2-0.2e^{-2t}\right]u(t).}$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Part (a)
% Find y(t) for t<0
syms y(t)
x = 0;
y(t) = dsolve(5*diff(y)+10*y==2*x,y(0)==1);
disp('Problem 1, Part (a):  y(t) for t<0:')
pretty(y(t))
% Find y(t) for t>=0
syms y(t)
x = 1;
y(t) = dsolve(5*diff(y)+10*y==2*x,y(0)==1);
disp('Problem 1, Part (a):  y(t) for t>=0:')
pretty(y(t))
```

<u>MATLAB Output:</u>

```
Problem 1, Part (a):  y(t) for t<0:
exp(-2 t)

Problem 1, Part (a):  y(t) for t>=0:
exp(-2 t) 4   1
----------- + -
     5        5
```

(b) Determine the zero-input output $y_{zi}(t)$ of the system.

The zero-input output is the response when the input is $x(t) = 0$. Therefore, $y_p(t) = 0$, and the zero-input output $y_{zi}(t)$ consists of only one part, or $y_{zi}(t) = y_h(t)$.
As we observed in Part (a), the complete homogeneous solution is $y_h(t) = K_1 e^{-2t}$. Plugging in initial conditions, we have

$$y_{zi}(0) = K_1 e^{-2(0)} = K_1 = 1.$$

Therefore, $K_1 = 1$, and the zero-input output is given by

$$\boxed{y_{zi}(t) = e^{-2t}.}$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Part (b)
% Find y_zi(t)
syms y_zi(t)
y_zi(t) = dsolve(5*diff(y_zi)+10*y_zi==0,y_zi(0)==1);
disp('Problem 1, Part (b):  y_zi(t) for all t:')
pretty(y_zi(t))
```

MATLAB Output:

```
Problem 1, Part (b):  y_zi(t) for all t:
exp(-2 t)
```

(c) Determine the zero-state output $y_{zs}(t)$ of the system.

The zero-state output is the response when initial conditions are zero, or $y(0) = 0$. Therefore, the zero-state output $y_{zs}(t)$ consists of two parts, or $y_{zs}(t) = y_p(t) + y_h(t)$.
As we observed in Part (a), the particular solution is given by $y_p(t) = 0.2u(t)$, and the complete homogeneous solution is $y_h(t) = K_1 e^{-2t}$. Now, we have

$$y_{zs}(t) = y_p(t) + y_h(t) = 0.2u(t) + K_1 e^{-2t}.$$

Plugging in initial conditions for $t < 0$, we have

$$y_{zs}(0^-) = 0.2u(0^-) + K_1 e^{-2(0^-)} = K_1 = 0.$$

Plugging in initial conditions for $t \geq 0$, we have

$$y_{zs}(0^+) = 0.2u(0^+) + K_1 e^{-2(0^+)} = 0.2 + K_1 = 0.$$

Therefore, $K_1 = 0$ for $t < 0$ and $K_1 = -0.2$ for $t \geq 0$, and the complete solution is given by

$$y_{zs}(t) = y_p(t) + y_h(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.2 - 0.2e^{-2t} & \text{for } t \geq 0 \end{cases} = \boxed{\left[0.2 - 0.2e^{-2t}\right] u(t).}$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Part (c)
% Find y_zs(t) for t<0
syms y_zs(t)
x = 0;
y_zs(t) = dsolve(5*diff(y_zs)+10*y_zs==2*x,y_zs(0)==0);
disp('Problem 1, Part (c):  y_zs(t) for t<0:')
pretty(y_zs(t))
% Find y_zs(t) for t>=0
syms y_zs(t)
x = 1;
y_zs(t) = dsolve(5*diff(y_zs)+10*y_zs==2*x,y_zs(0)==0);
disp('Problem 1, Part (c):  y_zs(t) for t>=0:')
pretty(y_zs(t))
```

MATLAB Output:

```
Problem 1, Part (c):  y_zs(t) for t<0:
0

Problem 1, Part (c):  y_zs(t) for t>=0:
1   exp(-2 t)
- - ---------
5       5
```

(d) If $y(0) = 0$, determine the impulse response $h(t)$ of the system.

The step response is the zero-state output when the input is $x(t) = u(t)$. Therefore, the step response is given by

$$y_s(t) = y_{zs}(t) = \left[0.2 - 0.2e^{-2t}\right] u(t).$$

The impulse response is the derivative of the step response. Therefore, the impulse response is given by

$$h(t) = \frac{dy_s(t)}{dt} = \left[0.4e^{-2t}\right] u(t) + \cancelto{0}{\left[0.2 - 0.2e^{-2t}\right]} \delta(t) = \boxed{0.4e^{-2t} u(t).}$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Parts (d)(e)
% Find y_s(t) for t<0
syms h(t) y_s(t)
x = 0;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d):  h(t) for t<0:')
pretty(h(t))
disp('Problem 1, Part (e):  y_s(t) for t<0:')
pretty(y_s(t))
% Find y_s(t) for t>=0
syms y_s(t)
x = 1;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d):  h(t) for t>=0:')
pretty(h(t))
disp('Problem 1, Part (e):  y_s(t) for t>=0:')
pretty(y_s(t))
```

MATLAB Output:

```
Problem 1, Part (d):  h(t) for t<0:
0

Problem 1, Part (d):  h(t) for t>=0:
exp(-2 t) 2
-----------
     5
```

(e) If $y(0) = 0$, determine the step response $y_s(t)$ of the system.

The step response is the zero-state output when the input is $x(t) = u(t)$. Therefore, the step response is given by

$$y_s(t) = y_{zs}(t) = \boxed{\left[0.2 - 0.2e^{-2t}\right] u(t).}$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Parts (d)(e)
% Find y_s(t) for t<0
syms h(t) y_s(t)
x = 0;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d):  h(t) for t<0:')
pretty(h(t))
disp('Problem 1, Part (e):  y_s(t) for t<0:')
pretty(y_s(t))
% Find y_s(t) for t>=0
syms y_s(t)
x = 1;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d):  h(t) for t>=0:')
pretty(h(t))
disp('Problem 1, Part (e):  y_s(t) for t>=0:')
pretty(y_s(t))
```

MATLAB Output:

```
Problem 1, Part (e):  y_s(t) for t<0:
0

Problem 1, Part (e):  y_s(t) for t>=0:
1   exp(-2 t)
- - ---------
5       5
```

2. A discrete-time system is described by the input-output equation

$$y[n] - \frac{1}{2}y[n-1] = 2x[n]$$

with initial condition $y[-1] = 3$ and input $x[n] = \left(-\frac{1}{2}\right)^n u[n]$.
Using the classical approach to solving LCCDEs:

(a) Determine the output $y[n]$ of the system.

The output $y[n]$ consists of two parts, or $y[n] = y_p[n] + y_h[n]$.
The particular solution $y_p[n] = A\left(-\frac{1}{2}\right)^n$ satisfies

$$y_p[n] - \frac{1}{2}y_p[n-1] = 2x[n]$$

For $n < 0$, $x[n] = 0$ yields

$$\begin{aligned}
y_p[n] - \frac{1}{2}y_p[n-1] &= 2(0) \\
\Rightarrow A\left(-\frac{1}{2}\right)^n - \frac{1}{2}A\left(-\frac{1}{2}\right)^{n-1} &= 0 \\
\Rightarrow A\left(-\frac{1}{2}\right)^n + A\left(-\frac{1}{2}\right)^n &= 0 \Rightarrow 2A = 0 \Rightarrow A = 0.
\end{aligned}$$

For $n \geq 0$, $x[n] = \left(-\frac{1}{2}\right)^n$ yields

$$\begin{aligned}
y_p[n] - \frac{1}{2}y_p[n-1] &= 2\left(-\frac{1}{2}\right)^n \\
\Rightarrow A\left(-\frac{1}{2}\right)^n - \frac{1}{2}A\left(-\frac{1}{2}\right)^{n-1} &= 2\left(-\frac{1}{2}\right)^n \\
\Rightarrow A\left(-\frac{1}{2}\right)^n + A\left(-\frac{1}{2}\right)^n &= 2\left(-\frac{1}{2}\right)^n \\
\Rightarrow 2A = 2 \Rightarrow A = 1.
\end{aligned}$$

Therefore, the particular solution is given by

$$y_p[n] = \begin{cases} 0 & \text{for } n < 0 \\ \left(-\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = \left(-\frac{1}{2}\right)^n u[n].$$

Since this is a first-order difference equation, the complete homogeneous solution is given as $y_h[n] = K_1 z_1^n$. To find s_1, we let $y_h[n] = Kz^n$ in the homogeneous form of the LCCDE. Therefore,

$$\begin{aligned}
y_h[n] - \frac{1}{2}y_h[n-1] &= 0 \\
Kz^n - \frac{1}{2}Kz^{n-1} &= 0 \\
Kz^{n-1}\left(z - \frac{1}{2}\right) &= 0.
\end{aligned}$$

Therefore, the characteristic equation is $z - \frac{1}{2} = 0$, and $z_1 = \frac{1}{2}$. Therefore, the complete homogeneous solution is $y_h[n] = K_1 \left(\frac{1}{2}\right)^n$. Now, we have

$$y[n] = y_p[n] + y_h[n] = \left(-\frac{1}{2}\right)^n u[n] + K_1 \left(\frac{1}{2}\right)^n.$$

Plugging in the initial condition for $n < 0$, we have

$$y[-1] = \left(-\frac{1}{2}\right)^{-1} u[-1] + K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 3 \Rightarrow K_1 = \frac{3}{2} = 1.5.$$

To plug in initial conditions for $n \geq 0$, we need $y[0]$, which is given as

$$y[0] = \frac{1}{2}y[-1] + 2x[0] = \frac{1}{2}(3) + 2(1) = \frac{7}{2} = 3.5.$$

Plugging in the initial condition for $n \geq 0$, we have

$$y[0] = \left(-\frac{1}{2}\right)^0 u[0] + K_1 \left(\frac{1}{2}\right)^0 = 1 + K_1 = 3.5 \Rightarrow K_1 = 2.5.$$

Therefore, $K_1 = 1.5$ for $n < 0$ and $K_1 = 2.5$ for $n \geq 0$, and the complete solution is given by

$$\boxed{y[n] = y_p[n] + y_h[n] = \begin{cases} 1.5\left(\frac{1}{2}\right)^n & \text{for } n < 0 \\ \left(-\frac{1}{2}\right)^n + 2.5\left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = 1.5\left(\frac{1}{2}\right)^n + \left[\left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right] u[n].}$$

(b) Determine the zero-input output $y_{zi}[n]$ of the system.

The zero-input output is the response when the input is $x[n] = 0$. Therefore, $y_p[n] = 0$, and the zero-input output $y_{zi}[n]$ consists of only one part, or $y_{zi}[n] = y_h[n]$.
As we observed in Part (a), the complete homogeneous solution is given as $y_h[n] = K_1 z_1^n$. Plugging in the initial condition, we have

$$y_{zi}[-1] = K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 3 \Rightarrow K_1 = \frac{3}{2} = 1.5.$$

Therefore, $K_1 = 1.5$, and the complete solution is given by

$$\boxed{y_{zi}[n] = 1.5\left(\frac{1}{2}\right)^n.}$$

(c) Determine the zero-state output $y_{zs}[n]$ of the system.

The zero-state output is the response when initial conditions are zero, or $y[-1] = 0$. Therefore, the zero-state output $y_{zs}[n]$ consists of two parts, or $y_{zs}[n] = y_p[n] + y_h[n]$.
As we observed in Part (a), the particular solution is given by $y_p[n] = \left(-\frac{1}{2}\right)^n u[n]$, and the complete homogeneous solution is $y_h[n] = K_1 z_1^n$. Now, we have

$$y_{zs}[n] = y_p[n] + y_h[n] = \left(-\frac{1}{2}\right)^n u[n] + K_1 \left(\frac{1}{2}\right)^n.$$

Plugging in the initial condition for $n < 0$, we have

$$y_{zs}[-1] = \left(-\frac{1}{2}\right)^{-1} u[-1] + K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 0 \Rightarrow K_1 = 0.$$

To plug in initial conditions for $n \geq 0$, we need $y[0]$, which is given as

$$y_{zs}[0] = \frac{1}{2} y[-1] + 2x[0] = \frac{1}{2}(0) + 2(1) = 2.$$

Plugging in the initial condition for $n \geq 0$, we have

$$y_{zs}[0] = \left(-\frac{1}{2}\right)^0 u[0] + K_1 \left(\frac{1}{2}\right)^0 = 1 + K_1 = 2 \Rightarrow K_1 = 1.$$

Therefore, $K_1 = 0$ for $n < 0$ and $K_1 = 1$ for $n \geq 0$, and the complete solution is given by

$$\boxed{y_{zs}[n] = y_p[n] + y_h[n] = \begin{cases} & \text{for } n < 0 \\ \left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = \left[\left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right] u[n].}$$

(d) If $y[-1] = 0$, determine the impulse response $h[n]$ of the system.

To determine the impulse response we will solve for the step response first (see Part (e) below), and determine the impulse response as

$$h[n] = y_s[n] - y_s[n-1] = \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1]$$

For $n < 0$,

$$\begin{aligned} h[n] &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1] \\ &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] (0) - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] (0) = 0. \end{aligned}$$

For $n = 0$,

$$\begin{aligned} h[n] &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1] \\ &= \left[4 - 2\left(\frac{1}{2}\right)^0\right] (1) - \left[4 - 2\left(\frac{1}{2}\right)^{-1}\right] (0) = 2. \end{aligned}$$

For $n > 0$,

$$\begin{aligned} h[n] &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1] \\ &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] (1) - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] (1) \\ &= 2\left(\frac{1}{2}\right)^{n-1} - 2\left(\frac{1}{2}\right)^n = 4\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{2}\right)^n = 2\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

Therefore, the impulse response of the system is given by

$$\boxed{h[n] = \begin{cases} 0 & \text{for } n < 0 \\ 2\left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = 2\left(\frac{1}{2}\right)^n u[n] = \left(\frac{1}{2}\right)^{n-1} u[n].}$$

(e) If $y[-1] = 0$, determine the step response $y_s[n]$ of the system.

Note that the input $x[n] = u[n]$ and the output $y[n] = y_s[n]$. The impulse response $h[n]$ consists of two parts, or $y_s[n] = y_{sp}[n] + y_{sh}[n]$.
The particular solution $y_p[n] = A$ satisfies

$$y_{sp}[n] - \frac{1}{2}y_{sp}[n-1] = 2x[n]$$

For $n < 0$, $x[n] = 0$ yields

$$y_{sp}[n] - \frac{1}{2}y_{sp}[n-1] = 2(0) \Rightarrow A - \frac{1}{2}A = 0 \Rightarrow \frac{1}{2}A = 0 \Rightarrow A = 0.$$

For $n \geq 0$, $x[n] = 1$ yields

$$y_{sp}[n] - \frac{1}{2}y_{sp}[n-1] = 2(1) \Rightarrow A - \frac{1}{2}A = 2 \Rightarrow \frac{1}{2}A = 2 \Rightarrow A = 4.$$

Therefore, the particular solution is given by

$$y_{sp}[n] = \begin{cases} 0 & \text{for } n < 0 \\ 4 & \text{for } n \geq 0 \end{cases} = 4u[n].$$

As we observed in Part (a), the complete homogeneous solution is $y_{sh}[n] = K_1\left(\frac{1}{2}\right)^n$. Now, we have

$$y_s[n] = y_{sp}[n] + y_{sh}[n] = 4u[n] + K_1\left(\frac{1}{2}\right)^n.$$

Plugging in the initial condition for $n < 0$, we have

$$y[-1] = 4u[-1] + K_1\left(\frac{1}{2}\right)^{-1} = 2K_1 = 0 \Rightarrow K_1 = 0.$$

To plug in initial conditions for $n \geq 0$, we need $y[0]$, which is given as

$$y[0] = \frac{1}{2}y[-1] + 2x[0] = \frac{1}{2}(0) + 2(1) = 2.$$

Plugging in the initial condition for $n \geq 0$, we have

$$y[0] = 4u[0] + K_1\left(\frac{1}{2}\right)^0 = 4 + K_1 = 2 \Rightarrow K_1 = -2.$$

Therefore, $K_1 = 0$ for $n < 0$ and $K_1 = -2$ for $n \geq 0$, and the complete solution is given by

$$\boxed{y_s[n] = y_{sp}[n] + y_{sh}[n] = \begin{cases} 0 & \text{for } n < 0 \\ 4 - 2\left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = \left[4 - 2\left(\frac{1}{2}\right)^n\right]u[n].}$$