Lecture 17: Frobenius Solutions

Eevee's Goals for the Day

- Outline the Frobenius Method for finding series solutions at regular singular points
- Define the Gamma and Bessel functions
- Derive the Frobenius solution to Bessel's Equation

Method of Frobenius

Find power series solution at $x=0$ for a DE where $x=0$ is a regular singular point.

1. Plug in $y=\sum_{n=0}^{\infty} c_n x^{n+r}$ into DE.
2. Find indicial equation to get roots r_1, r_2.

   You should also get a general recurrence relation

   $$c_{n+1} = f(n,r)\, c_n$$

3. Plug r_1 into general recurrence relation to get a specific recurrence relation.

   $$a_{n+1} = f(n)\, a_n$$

   First Frobenius solution

   $$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$

④ If $r_1 - r_2$ is not an integer, then plug r_2 into the general recurrence relation.

$$b_{n+1} = f(n)\, b_n$$

Second Frobenius solution

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

If $r_1 - r_2$ is an integer, then we cannot find the solution by hand (Example 5, Sec 5.2).

⑤ Form general solution

$$y = y_1 + y_2$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r_1} + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

$$= x^{r_1}\left[\underline{a_0} + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]$$

$$+ x^{r_2}\left[\underline{b_0} + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right]$$

a_0 and b_0 should be the only 2 unknowns

5.3 Special functions

Gamma function $\quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

Ex Calculate $\Gamma(1)$.

$$\Gamma(1) = \int_0^\infty \underbrace{t^{1-1}}_{1} e^{-t}\, dt$$

$$= \int_0^\infty e^{-t}\, dt$$

$$= -e^{-t} \Big|_0^\infty$$

$$= -\overset{0}{\cancel{e^{-\infty}}} + \overset{1}{\cancel{e^{0}}}$$

$$= 1$$

$$\boxed{\text{Property: } \Gamma(x+1) = x\,\Gamma(x)}$$

<u>Proof</u> $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$

Integration by Parts

$u = t^x$ $\qquad v = -e^{-t}$

$du = x t^{x-1}\,dt$ $\qquad dv = e^{-t}\,dt$

$\int u\,dv = uv - \int v\,du$

$$\int_0^\infty t^x e^{-t}\,dt = t^x\left(-e^{-t}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-t}\right) x\, t^{x-1}\,dt$$

$$= \underbrace{-\underset{\infty}{\infty^x}\,\underset{0}{e^{-\infty}}}_{0} + \underset{0}{0^x}\,e^0 + x\underbrace{\int_0^\infty t^{x-1} e^{-t}\,dt}_{\Gamma(x)}$$

$$= x\,\Gamma(x)$$

$\square$

<u>Note</u> If x is a positive integer n

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$\vdots$$

$$= n(n-1)(n-2)\cdots\cancelto{1}{\Gamma(1)}$$

$$= n!$$

The Gamma function extends the factorial to non-integer values.

Bessel Functions

Bessel Function of the first kind

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+v}\, n!\, \Gamma(1+v+n)} x^{2n+v}$$

Bessel Function of the second kind

$$Y_v(x) = \frac{\cos(v\pi)\, J_v(x) - J_{-v}(x)}{\sin(v\pi)}$$

Bessel's Equation

$$x^2 y'' + x y' + (x^2 - v^2) y = 0$$

The solution of Bessel's Equation is

$$y = C_1 J_v(x) + C_2 Y_v(x)$$

Ex Solve $x^2 y'' + x y' + (x^2 - 36) y = 0$

Bessel's Equation with $v = 6$

$$\boxed{y = C_1 J_6(x) + C_2 Y_6(x)}$$

Ex Find the first Frobenius solution to Bessel's equation

$$x^2 y'' + xy' + (x^2 - v^2)\, y = 0$$

Note $x=0$ is a regular singular point.

Apply Method of Frobenius.

① Plug in $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$y' = \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2}$$

$$x^2 y'' + xy' + (x^2 - v^2)\, y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2}$$
$$+\, x \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1}$$
$$+\, (x^2 - v^2) \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r}$$
$$+ \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r}$$

$$+ \sum_{n=0} (n+r) c_n \cdots$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} v^2 c_n x^{n+r} = 0$$

shift $n+r+2 = k+r$

$n+2 = k$

$n = k-2$

$$\sum_{k=2}^{\infty} c_{k-2} x^{k+r}$$

$n=0 \Rightarrow k=2$

Pull out $n=0$ and $n=1$ terms so starting index is 2.

$$(r-1)r c_0 x^r + r(r+1) c_1 x^{r+1} + \sum_{n=2}^{\infty} (n+r-1)(n+r) c_n x^{n+r}$$

$$+ r c_0 x^r + (r+1) c_1 x^{r+1} + \sum_{n=2}^{\infty} (n+r) c_n x^{n+r}$$

$$+ \sum_{n=2}^{\infty} c_{n-2} x^{n+r}$$

$$- v^2 c_0 x^r - v^2 c_1 x^{r+1} - \sum_{n=2}^{\infty} v^2 c_n x^{n+r} = 0$$

$$\left[(r-1)r + r - v^2\right] c_0 x^r + \left[r(r+1) + r + 1 - v^2\right] c_1 x^{r+1}$$

$$+ \sum_{n=2}^{\infty} \left[(n+r-1)(n+r) c_n + (n+r) c_n + c_{n-2} - v^2 c_n\right] x^{n+r} = 0$$

(2) Find roots and general recurrence relation.

$\underline{x^r}$: $(r-1)r + r - v^2 = 0$

$$r^2 - r + r - v^2 = 0$$

$$r^2 - v^2 = 0$$

$$\boxed{r = \pm v}$$ Indicial roots

$\underline{x^{r+1}}$: $[r(r+1) + r + 1 - v^2]c_1 = 0$

$$[r^2 + r + r + 1 - v^2]c_1 = 0$$

$$[r^2 + 2r + 1 - v^2]c_1 = 0$$

$\underline{r = v}$ $$[v^2 + 2v + 1 - v^2]c_1 = 0$$

$$[2v + 1]c_1 = 0$$

$$c_1 = 0$$

$\underline{x^{n+r}}$: $(n+r-1)(n+r)c_n + (n+r)c_n + c_{n-2} - v^2 c_n = 0$

$$[(n+r-1)(n+r) + n + r - v^2]c_n = -c_{n-2}$$

$$\boxed{c_n = \frac{-c_{n-2}}{(n+r-1)(n+r)+n+r-v^2}}$$

General Recurrence Relation

(3) Find first Frobenius solution by plugging in $r=v$,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+v}$$

a_0 unknown constant

$a_1 = 0$

$$a_n = \frac{-a_{n-2}}{(n+v-1)(n+v)+n+v-v^2}$$

$\underline{n=2}$ $\quad a_2 = -\frac{a_0}{2^2(1+v)}$

$\underline{n=4}$ $\quad a_4 = -\frac{a_2}{2^2 \cdot 2(2+v)} = \frac{a_0}{2^4 \cdot 2 \cdot 1 \cdot (1+v)(2+v)}$

$\vdots$

$$a_{2n} = \frac{a_0(-1)^n}{2^{2n}\, n!\, (1+v)(2+v)\cdots(n+v)}$$

Let $a_0 = C \dfrac{1}{2^{\nu}\Gamma(1+\nu)}$

$$a_{2n} = -C \frac{(-1)^n}{2^n \Gamma(1+\nu)\, 2^{2n}\, n!\, (1+\nu)(2+\nu)\cdots(n+\nu)}$$

$\Gamma(1+\nu+n)$

$$= -C \frac{(-1)^n}{2^{2n+\nu}\, n!\, \Gamma(1+\nu+n)}$$

Series Solution

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu}\, n!\, \Gamma(1+\nu+n)}\, x^{2n+\nu}$$

Bessel Function of first kind $J_\nu(x)$

I know that sucked.

I was just trying to show you that the Bessel functions can be derived as Frobenius solutions to Bessel's Equation. It takes a lot of algebra to do that though.

The Bessel Function of the second kind Y is even more painful to derive.

Bessel's Equation has many applications including heat conduction, electromagnetic waves, and fluid flow. You may see it come up in physics applications and I want you to realize that you can look up the two crazy functions that form the solution.