15 Discrete-Time Modulation

Solutions to Recommended Problems

S15.1

Recall that the Fourier transform of a train of impulses $p(t)$ is $P(\omega)$, as shown in Figure S15.1-1.

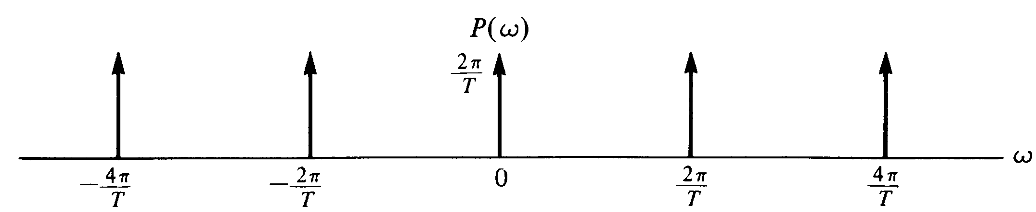

Figure S15.1-1

Since $x_p(t) = x(t)p(t)$,

$$X_p(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)P(\omega - \theta)\,d\theta$$

by the modulation property. Thus, $X_p(\omega)$ is composed of repeated versions of $X(\omega)$ centered at $2\pi k/T$ for an integer k and scaled by $1/T$, as shown in Figure S15.1-2.

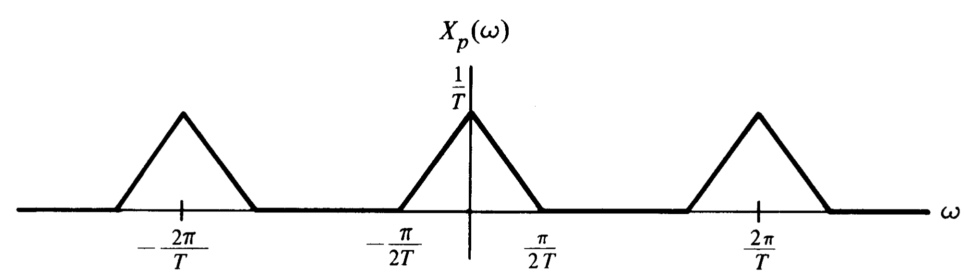

Figure S15.1-2

Since $X_r(\omega) = X_p(\omega)H(\omega)$, it is as indicated in Figure S15.1-3.

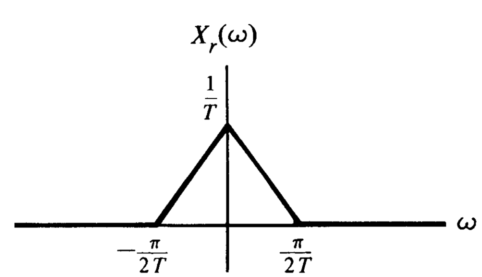

Figure S15.1-3

Thus

$$X_r(\omega) = \frac{1}{T}X(\omega) \qquad \text{or} \qquad x_p = \frac{1}{T}x(t)$$

S15.2

For $\Omega_0 = \pi/2$, $C(\Omega)$ is given as in Figure S15.2-1.

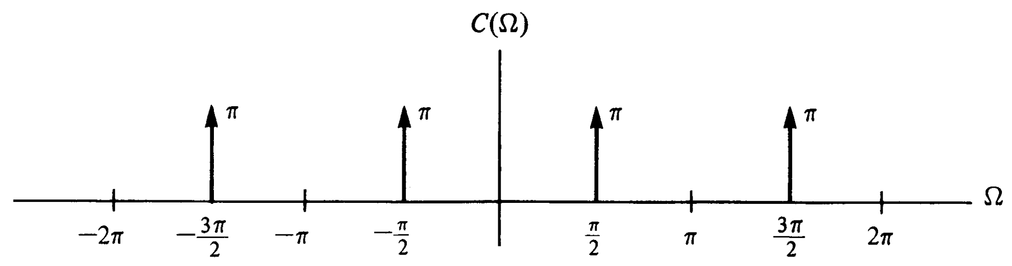

Figure S15.2-1

By the modulation theorem,

$$\mathcal{F}\{x[n]c[n]\} = \mathcal{F}\{y[n]\} = Y(\Omega) = \frac{1}{2\pi}\int_{2\pi} C(\theta)\, X(\Omega - \theta)\, d\theta$$

Thus, $Y(\Omega)$ is $X(\Omega)$ centered on each impulse in Figure S15.2-1 and scaled by $\frac{1}{2}$, as shown in Figure S15.2-2.

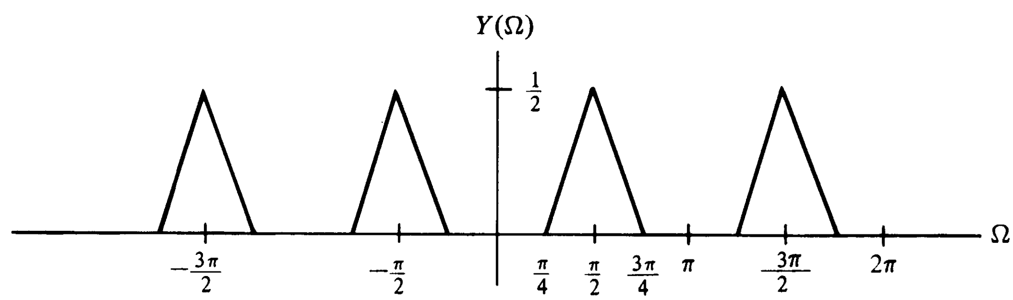

Figure S15.2-2

For Ω_0, $= \pi/4$, $C(\Omega)$ is given as in Figure S15.2-3.

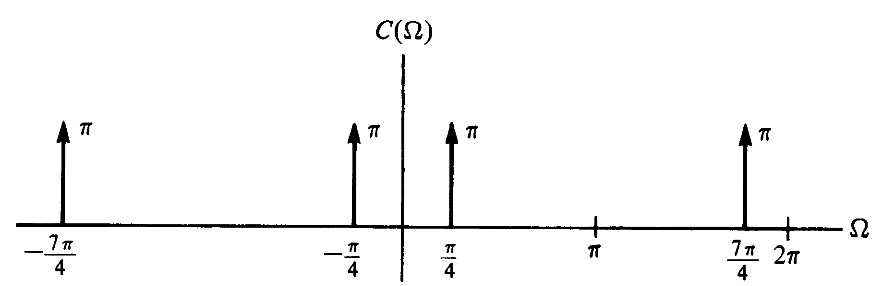

Figure S15.2-3

Thus, $Y(\Omega)$ in this case is as shown in Figure S15.2-4.

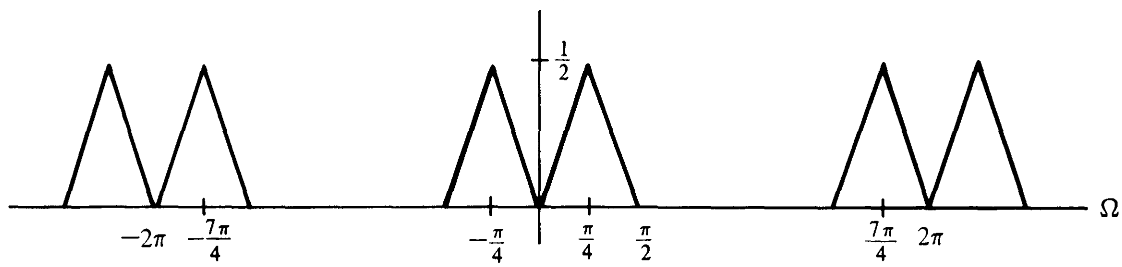

Figure S15.2-4

S15.3

From the lecture we know that the system in Figure S15.3-1 is equivalent to a filter with response centered at $\Omega = \pi$, as shown in Figure S15.3-2.

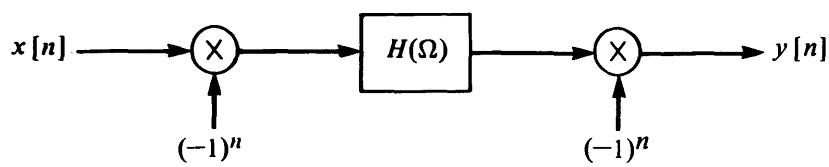

Figure S15.3-1

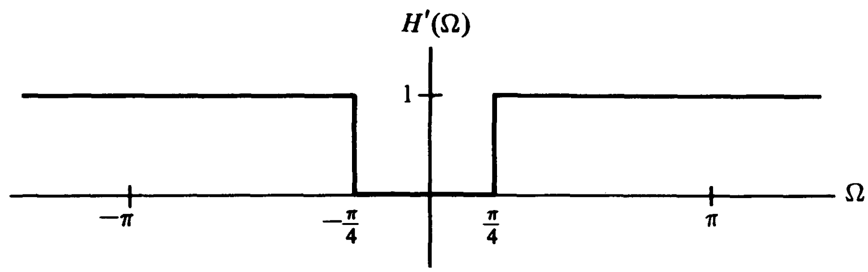

Figure S15.3-2

Therefore, the total response is the sum of $H'(\Omega)$ and $H(\Omega)$, shown in Figure S15.3-3.

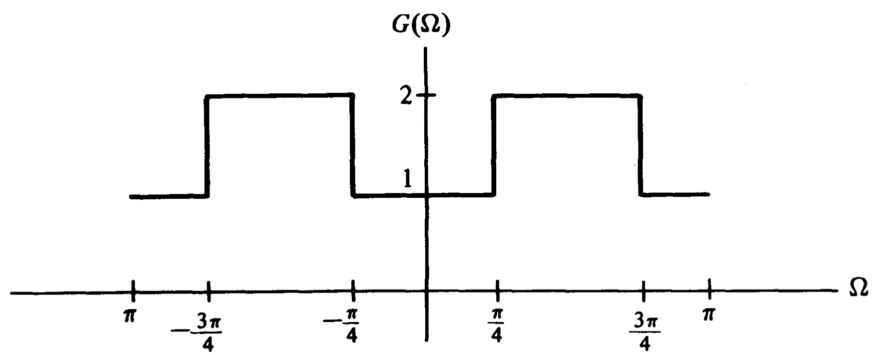

Figure S15.3-3

As an example, consider $x[n]$ with Fourier transform $X(\Omega)$ as in Figure S15.3-4.

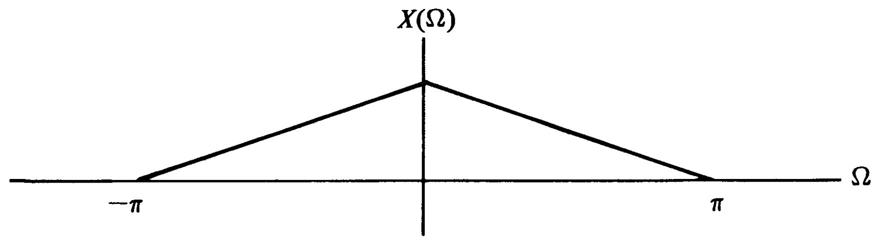

Figure S15.3-4

Then, after multiplication by $(-1)^n$, the resulting signal has the Fourier transform given in Figure S15.3-5.

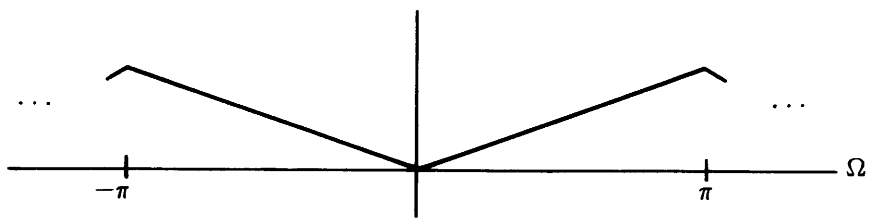

Figure S15.3-5

After filtering by $H(\Omega)$, the resulting signal has the spectrum given in Figure S15.3-6.

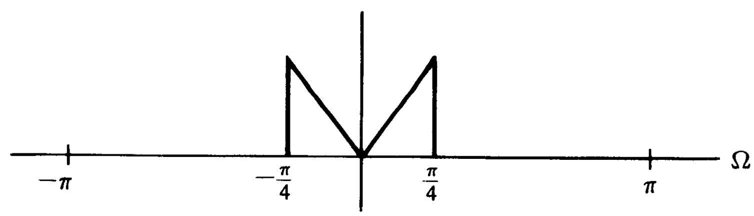

Figure S15.3-6

Finally, multiplying by $(-1)^n$ again yields the spectrum in Figure S15.3-7.

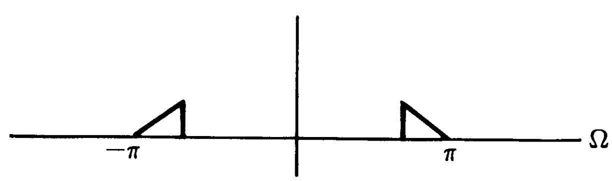

Figure S15.3-7

Thus, the spectrum of $y[n]$ is given by the sum of the spectrum in Figure S15.3-8 and $X(\Omega)$, as shown in Figure S15.3-8.

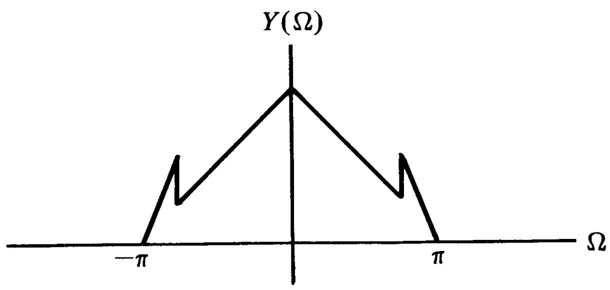

Figure S15.3-8

S15.4

(a) $P(\Omega)$ is composed of impulses spaced at $2\pi/N$, where N is the period of the sequence. In this case $N = 2$. The amplitude is $2\pi a_k$:

$$\begin{aligned} a_k &= \tfrac{1}{2} \sum_{n=0}^{1} p[n] e^{-j(2\pi k n/2)} \\ &= \tfrac{1}{2} [1 e^{-j(2\pi k 0/2)} + 0 e^{-j(2\pi k 1/2)}] = \tfrac{1}{2} \end{aligned}$$

Thus, $P(\Omega)$ is as shown in Figure S15.4-1.

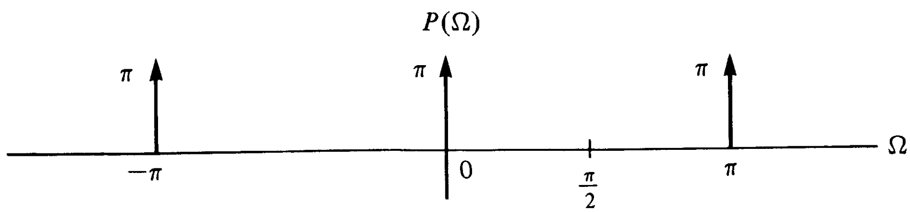

Figure S15.4-1

We now perform the periodic convolution of $X(\Omega)$ with $P(\Omega)$ and scale by $1/(2\pi)$ to obtain the spectrum in Figure S15.4-2.

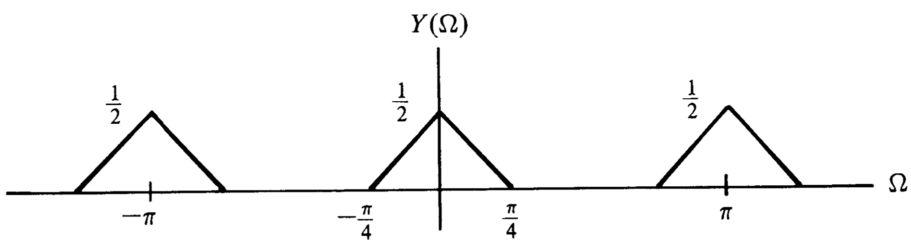

Figure S15.4-2

(b) To recover $x[n]$ from $y[n]$, we can filter $y[n]$ with $H(\Omega)$ given as in Figure S15.4-3.

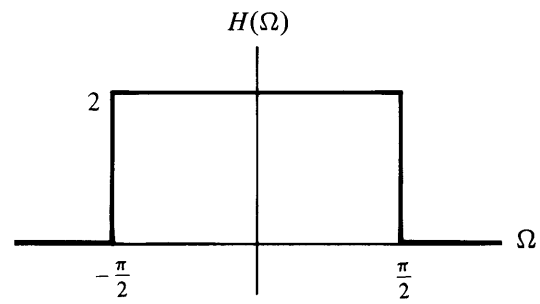

Figure S15.4-3

(c) Using $p[n]$ we can send only every other sample of $x_1[n]$. Similarly, we can send every other sample of $x_2[n]$ and interleave them over one channel. Note, however, that we can do this only because $X(\Omega)$ is bandlimited to less than $\pi/2$.

S15.5

We note that $s(t)$ is a periodic signal. Therefore, $S(\omega)$ is composed of impulses centered at $(2\pi k)/T$ for integer k. The impulse at $\omega = 0$ has area given by $2\pi a_0$, where a_0 is the zeroth Fourier series coefficient of $s(t)$:

$$a_0 = \frac{1}{T}\int_T s(t)\,dt = \int_{-\Delta/2}^{\Delta/2} 1dt = \frac{\Delta}{T}$$

Thus, $S(\omega)$ is as shown in Figure S15.5-1.

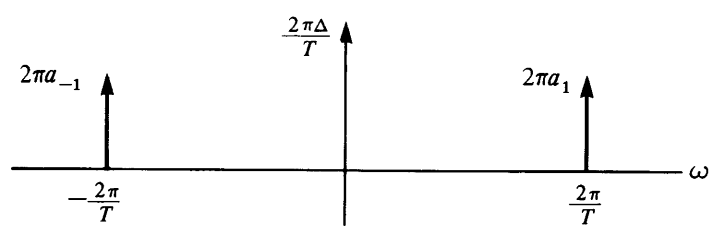

Figure S15.5-1

The Fourier transform of $x(t)s(t)$, denoted by $R(\omega)$, is given by

$$R(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)S(\omega - \theta)\,d\theta = \sum_{n=-\infty}^{\infty} a_n X\left(\omega - \frac{2\pi n}{T}\right)$$

If $X(\omega) = 0$ for $|\omega| > \pi/T$, then $R(\omega)$ will equal $(\Delta/T)X(\omega)$ in the region $|\omega| < \pi/T$. Therefore, for $H(\omega)$ as in Figure S15.5-2, the signal $y(t) = x(t)$.

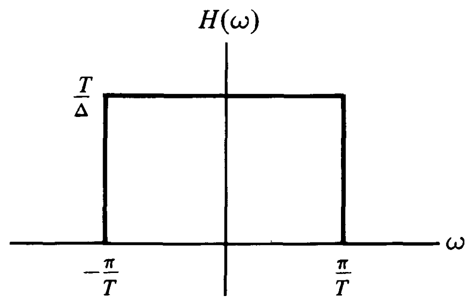

Figure S15.5-2

Solutions to Optional Problems

S15.6

(a) Consider the labeling of the system in Figure S15.6.

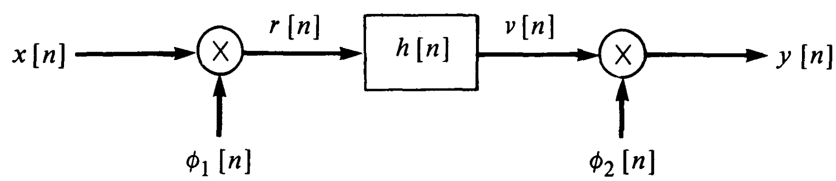

Figure S15.6

$$r[n] = \phi_1[n]x[n]$$

$$v[n] = \sum_{k=-\infty}^{\infty} r[k]h[n-k] = \sum_{k=-\infty}^{\infty} \phi_1[k]x[k]h[n-k]$$

$$y[n] = v[n]\phi_2[n] = \phi_2[n]\sum_{k=-\infty}^{\infty} h[n-k]\phi_1[k]x[k]$$

Suppose $x_1[n] = \alpha x[n]$. Then

$$y_1[n] = \phi_2[n]\sum_{k=-\infty}^{\infty} h[n-k]\phi_1[k]\alpha x[k] = \alpha y[n]$$

Now let $x_2[n] = x_1[n] + x_0[n]$. Then

$$y_2[n] = \phi_2[n]\sum_{k=-\infty}^{\infty} h[n-k]\phi_1[k](x_1[k] + x_0[k]) = y_1[n] + y_0[n]$$

and the system is linear.

If $\phi_1[n] = \delta[n]$, then

$$y[n] = \phi_2[n] \sum_{k=-\infty}^{\infty} h[n-k]\delta[k]x[k] = \phi_2[n]h[n]x[0]$$

If $x[n]$ is shifted so $x_1[n] = x[n-1]$, then

$$y_1[n] = \phi_2[n]h[n]x_1[0] = \phi_2[n]h[n]x[-1] \neq y[n-1]$$

and the system is not time-invariant.

(b) From part (a),

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[n-k]z^{-k}x[k]$$

Let $x[n-m] = x_1[n]$. Then

$$y_1[n] = z^n \sum_{k=-\infty}^{\infty} h[n-k]z^{-k}x_1[k] = z^n \sum_{k=-\infty}^{\infty} h[n-k]z^{-k}x[k-m]$$

Let $p = k - m$, $k = p + m$. Then

$$\begin{aligned} y_1[n] &= z^n \sum_{p=-\infty}^{\infty} h[(n-m)-p]z^{-p-m}x[p] \\ &= z^{n-m} \sum_{p=-\infty}^{\infty} h[(n-m)-p]z^{-p}x[p] \\ &= y[n-m] \end{aligned}$$

Therefore, the system is time-invariant.

S15.7

In general, $w(t)$ is recoverable from $w_p(t)$ if $W_p(\omega)$ contains repeated versions of $W(\omega)$ that do not overlap, i.e., that have no aliasing, as shown in Figure S15.7.

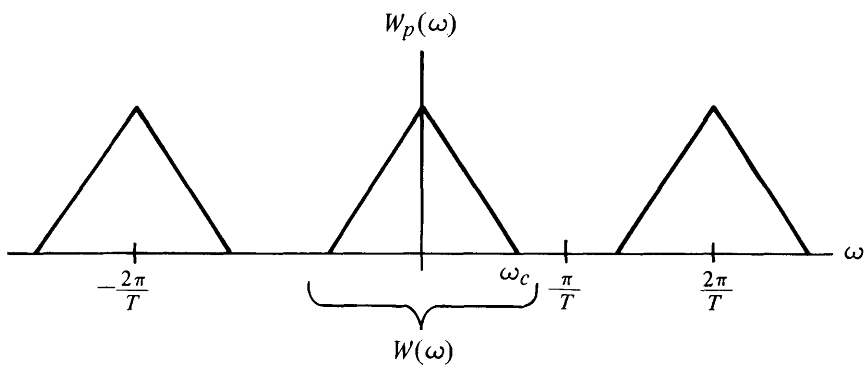

Figure S15.7

Since $W(\omega)$ is repeated with period $2\pi/T$, the largest frequency component of $W(\omega)$, ω_c, must be less than or equal to π/T. From the modulation property,

$$W(\omega) = \frac{1}{2\pi} X(\omega) * X_2(\omega)$$

Thus, since the length of a convolution of two signals is the sum of the individual lengths,

$$\omega_c = \omega_1 + \omega_2$$

From the preceding observations,

$$\frac{\pi}{T} > \omega_1 + \omega_2 \qquad \text{or} \qquad T < \frac{\pi}{\omega_1 + \omega_2}$$

S15.8

(a) If $\alpha_1 = -\Omega_i/2\pi$, then the portion of $X(\Omega)$ around Ω_i will be modulated down to about $\Omega = 0$ and then filtered by $H(\Omega)$. We now need to reshift the spectrum back to its original position. Therefore, we need to modulate by $e^{j\Omega_i n}$, or $\beta = +\Omega_i/2\pi$.

(b) Consider $i = 0, 1$. Then the corresponding filters are as given in Figure S15.8.

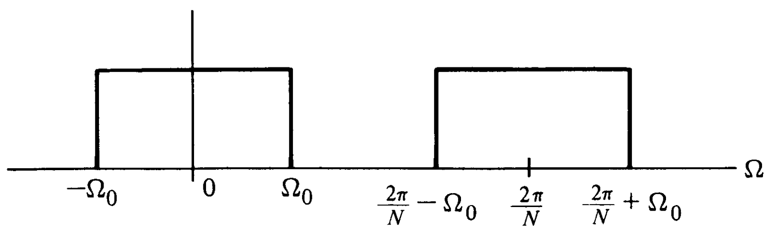

Figure S15.8

For no overlap and complete coverage of the frequency band, we need

$$\Omega_0 = \frac{2\pi}{N} - \Omega_0, \qquad \text{or} \qquad \Omega_0 = \frac{\pi}{N}$$

S15.9

(a) Since $s(t)$ is periodic in T, $S(\omega)$ will consist of impulses located at $2\pi k/T$. See Figure S15.9-1.

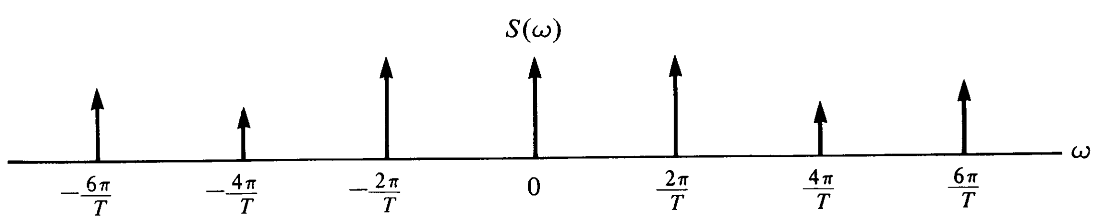

Figure S15.9-1

If $\int_{-\infty}^{\infty} s(t) = 0$, then the spectrum looks like Figure S15.9-2.

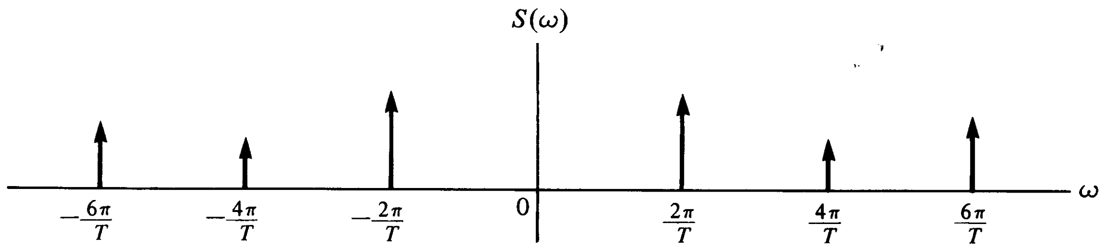

Figure S15.9-2

Of course, other impulses may also be zero.

(b) $Y(\omega)$ will be equal to a sum of the shifted and scaled versions of $X(\omega)$. Specifically,

$$Y(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)S(\omega-\theta)\,d\theta = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} S\left(\frac{2\pi n}{T}\right)X\left(\omega-\frac{2\pi n}{T}\right) \qquad \text{(S15.9-1)}$$
$$= \sum_{n=-\infty}^{\infty} a_n X\left(\omega - \frac{2\pi n}{T}\right),$$

where a_n is the nth Fourier series coefficient of one period of $s(t)$. For some region $Y(\omega)$ to be zero, successive terms in the sum in eq. (S15.9-1) cannot overlap. Thus, the maximum T is such that $\pi/T = \omega_c$, or $T = \pi/\omega_c$.

(c) In general, we need to find some n such that $a_n \neq 0$. Then we use an ideal real bandpass filter to isolate the nth term of the sum in eq. (S15.9-1). The resulting signal $r(t)$ has Fourier transform $R(\omega)$ given by

$$R(\omega) = a_n X\left(\omega - \frac{2\pi n}{T}\right) + a_{-n}X\left(\omega + \frac{2\pi n}{T}\right)$$

Let $a_n = r_n e^{j\theta_n}$. Then $r(t)$ can be thought of as

$$r(t) = x(t)\left[2r_n \cos\left(\frac{2\pi n t}{T} + \theta_n\right)\right]$$

(remember the effect of modulating by a cosine signal). Suppose we multiply $r(t)$ by

$$\frac{1}{r_n}\cos\left(\frac{2\pi n t}{T} + \theta_n\right)$$

Then

$$q(t) = r(t)\frac{1}{r_n}\cos\left(\frac{2\pi n t}{T} + \theta_n\right) = x(t)\,2\cos^2\left(\frac{2\pi n t}{T} + \theta_n\right)$$
$$= x(t)\left[1 + \cos\left(\frac{4\pi n t}{T} + 2\theta_n\right)\right]$$

If we now use a lowpass filter with cutoff π/T, we get $x(t)$. If we had picked the *smallest* n such that $a_n \neq 0$, we could have avoided the bandpass filtering because higher harmonics are eliminated by the lowpass filter.

S15.10

(a) $Y(\Omega)$ will consist of repeated versions of $X(\Omega)$ centered at $(2\pi/5) + 2\pi k$ and scaled by $\frac{1}{2}$. Thus, $Y(\Omega)$ is as shown in Figure S15.10-1.

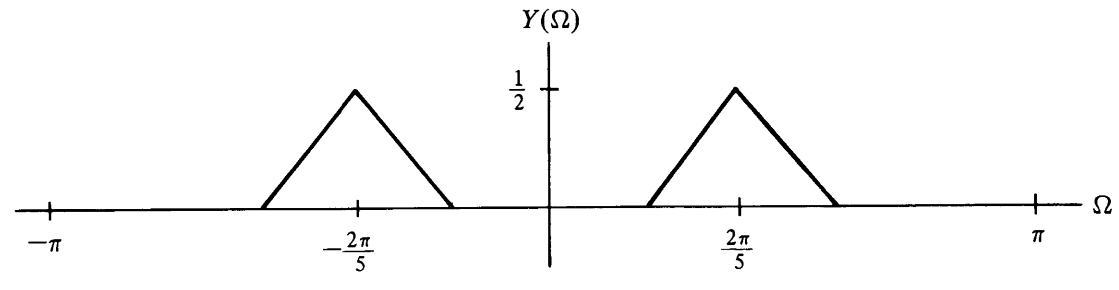

Figure S15.10-1

(b) $Z(\Omega)$ will consist, in turn, of repeated versions of $Y(\Omega)$, centered at $(4\pi/5) + 2\pi k$ and scaled by $\frac{1}{2}$, as shown in Figure S15.10-2.

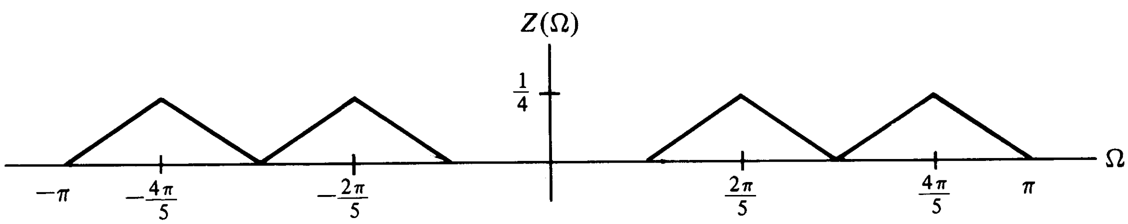

Figure S15.10-2

Note that the version of $Y(\Omega)$ centered at $6\pi/5$ contributes to the spectrum between $-3\pi/5$ and π.

(c) Two possible choices are given in Figures S15.10-3 and S15.10-4.

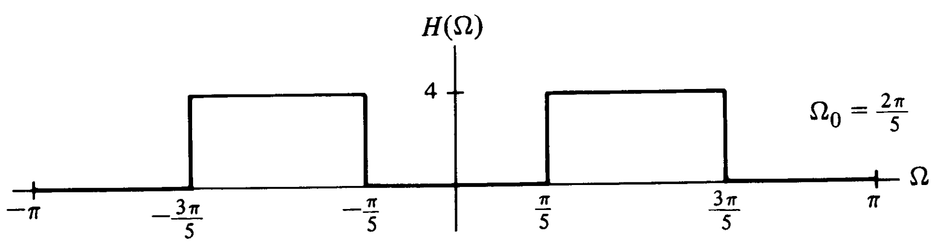

Figure S15.10-3

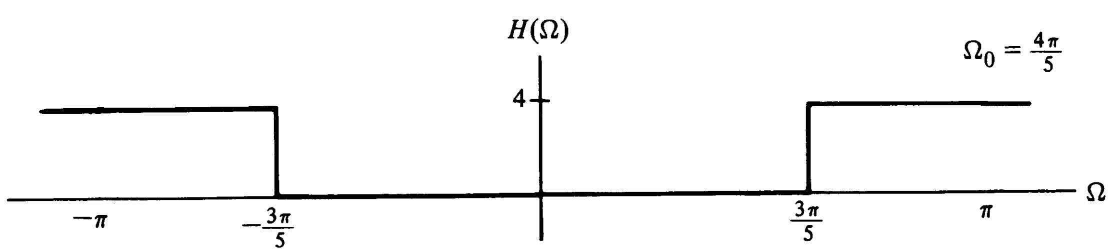

Figure S15.10-4