23 Mapping Continuous-Time Filters to Discrete-Time Filters

Solutions to Recommended Problems

S23.1

(a) $X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad \left|\frac{1}{2}z^{-1}\right| < 1,$

so the ROC is $|z| > \frac{1}{2}$.

(b)
$$X_2(z) = \sum_{n=-\infty}^{0} (-3)^n z^{-n}$$
$$= \sum_{n=0}^{\infty} (-3)^{-n} z^n = \frac{1}{1 + \frac{1}{3}z}, \qquad |-3^{-1}z| < 1,$$

so the ROC is $|z| < 3$.

We can also show this by using the property that

$$x[-n] \overset{Z}{\longleftrightarrow} X(z^{-1})$$

Letting $x[-n] = x_2[n]$, we have

$$x[-n] = (-3)^n u[-n],$$
$$x[+n] = (-\tfrac{1}{3})^n u[n],$$
$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}}$$

Therefore,

$$X_2(z) = \frac{1}{1 + \frac{1}{3}z},$$

and the ROC is $|z| < 3$.

(c) Using linearity we see that

$$X_3(z) = X_1(z) + X_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z}$$

The ROC is the common ROC for $X_1(z)$ and $X_2(z)$, which is $\frac{1}{2} < |z| < 3$, as shown in Figure S23.1.

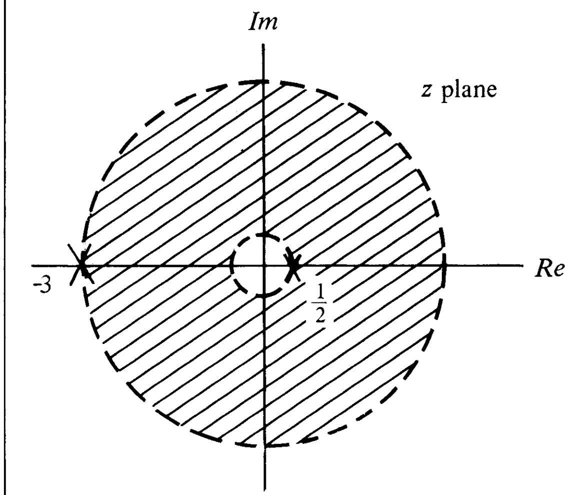

Figure S23.1

(d) Using the time-shifting property

$$x[n - n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z),$$

we have

$$X_4(z) = z^{-5}X_1(z) = \frac{z^{-5}}{1 - \frac{1}{2}z^{-1}}$$

Delaying the sequence does not affect the ROC of the corresponding z-transform, so the ROC is $|z| > \frac{1}{2}$.

(e) Using the time-shifting property, we have

$$X_5(z) = \frac{z^5}{1 - \frac{1}{2}z^{-1}},$$

and the ROC is $|z| > \frac{1}{2}$.

(f) $X_6(z) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{3}z^{-1}},$

and the ROC is $|\frac{1}{3}z^{-1}| < 1$, or $|z| > \frac{1}{3}$.

(g) Using the convolution property, we have

$$X_7(z) = X_1(z)X_6(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right),$$

and the ROC is $|z| > \frac{1}{2}$, corresponding to $\mathrm{ROC}_1 \cap \mathrm{ROC}_6$.

S23.2

(a) We have

$$y[n] - 3y[n - 1] + 2y[n - 2] = x[n]$$

Taking the z-transform of both sides, we obtain

$$Y(z)[1 - 3z^{-1} + 2z^{-2}] = X(z),$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^2}{z^2 - 3z + 2} = \frac{z^2}{(z - 2)(z - 1)},$$

and the ROC is outside the outermost pole for the causal (and therefore right-sided) system, as shown in Figure S23.2.

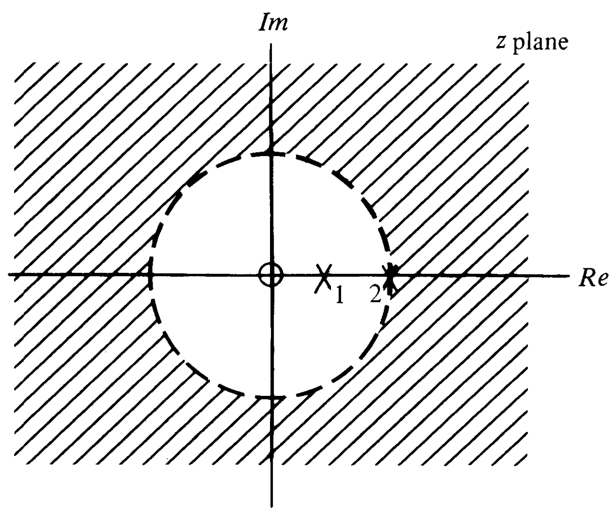

Figure S23.2

(b) Using partial fractions, we have

$$H(z) = \frac{1}{(1 - 2z^{-1})(1 - z^{-1})} = \frac{2}{1 - 2z^{-1}} + \frac{-1}{1 - z^{-1}}$$

By inspection we recognize that the corresponding causal $h[n]$ is the sum of two terms:

$$\begin{aligned} h[n] &= (2)2^n u[n] + (-1)1^n u[n] \\ &= 2^{n+1}u[n] - u[n] \\ &= (2^{n+1} - 1)u[n]. \end{aligned}$$

The system is not stable because the ROC does not include the unit circle. We can also conclude this from the fact that

$$\sum_{n=0}^{\infty} |2^{n+1}| = \infty$$

(c) Since $x[n] = 3^n u[n]$,

$$X(z) = \frac{1}{1 - 3z^{-1}}, \qquad |z| > 3,$$

so

$$Y(z) = H(z)X(z) = \frac{1}{(1 - 3z^{-1})} \frac{1}{(1 - 2z^{-1})(1 - z^{-1})}$$

Using partial fractions, we have

$$Y(z) = \frac{\frac{9}{2}}{1 - 3z^{-1}} + \frac{-4}{1 - 2z^{-1}} + \frac{\frac{1}{2}}{1 - z^{-1}}, \qquad |z| > 3$$

since the output is also causal. Therefore,

$$y[n] = (\tfrac{9}{2})3^n u[n] - (4)2^n u[n] + \tfrac{1}{2}u[n]$$

(d) There are two other possible impulse responses for the same

$$H(z) = \frac{1}{1 - 3z^{-1} + 2z^{-2}}$$

corresponding to different ROCs. For the ROC $|z| < 1$ the system impulse response is left-sided. Therefore, since

$$H(z) = \frac{2}{1 - 2z^{-1}} + \frac{-1}{1 - z^{-1}},$$

then

$$\begin{aligned} h[n] &= -(2)2^n u[-n-1] + (1)u[-n-1] \\ &= -2^{n+1}u[-n-1] + u[-n-1] \end{aligned}$$

For the ROC $1 < |z| < 2$, which yields a two-sided impulse response, we have

$$h[n] = -2^{n+1}u[-n-1] - u[n]$$

since the second term corresponding to $-1/(1 - z^{-1})$ has the ROC $1 < |z|$. Neither system is stable since the ROCs do not include the unit circle.

S23.3

(a) Consider

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n}$$

Letting $m = n - n_0$, we have

$$\begin{aligned} X_1(z) &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \\ &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} \\ &= z^{-n_0}X(z) \end{aligned}$$

It is clear that the ROC of $X_1(z)$ is identical to that of $X(z)$ since both require that $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converge in the ROC.

(b) Property 10.5.3 corresponds to multiplication of $x[n]$ by a real or complex exponential. There are three cases listed in the text, which we consider separately here.

(i)
$$\begin{aligned} X_1(z) &= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\,(ze^{-j\Omega_0})^{-n} \\ &= X(ze^{-j\Omega_0}), \end{aligned}$$

with the same ROC as for $X(z)$.

(ii) Now suppose that

$$\begin{aligned} X_2(z) &= \sum_{n=-\infty}^{\infty} z_0^n x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} \\ &= X\left(\frac{z}{z_0}\right) \end{aligned}$$

Letting $z' = z/z_0$, we see that the ROC for $X_2(z)$ are those values of z such that z' is in the ROC of $X(z')$. If the ROC of $X(z)$ is $R_0 < |z| < R_1$, then the ROC of $X_2(z)$ is $R_0|z_0| < |z| < R_1|z_0|$.

(iii) This proof is the same as that for part (ii), with $a = z_0$.

(c) We want to show that

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

Consider

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Then

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} -nx[n]z^{-n-1}$$

$$= -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n},$$

so

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n},$$

which is what we wanted to show. The ROC is the same as for $X(z)$ except for possible trouble due to the presence of the z^{-1} term.

S23.4

(a)

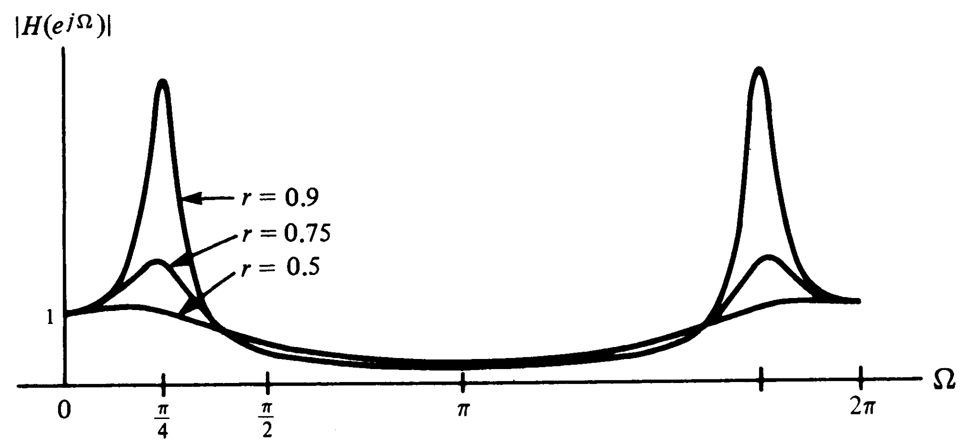

Figure S23.4-1

(b)

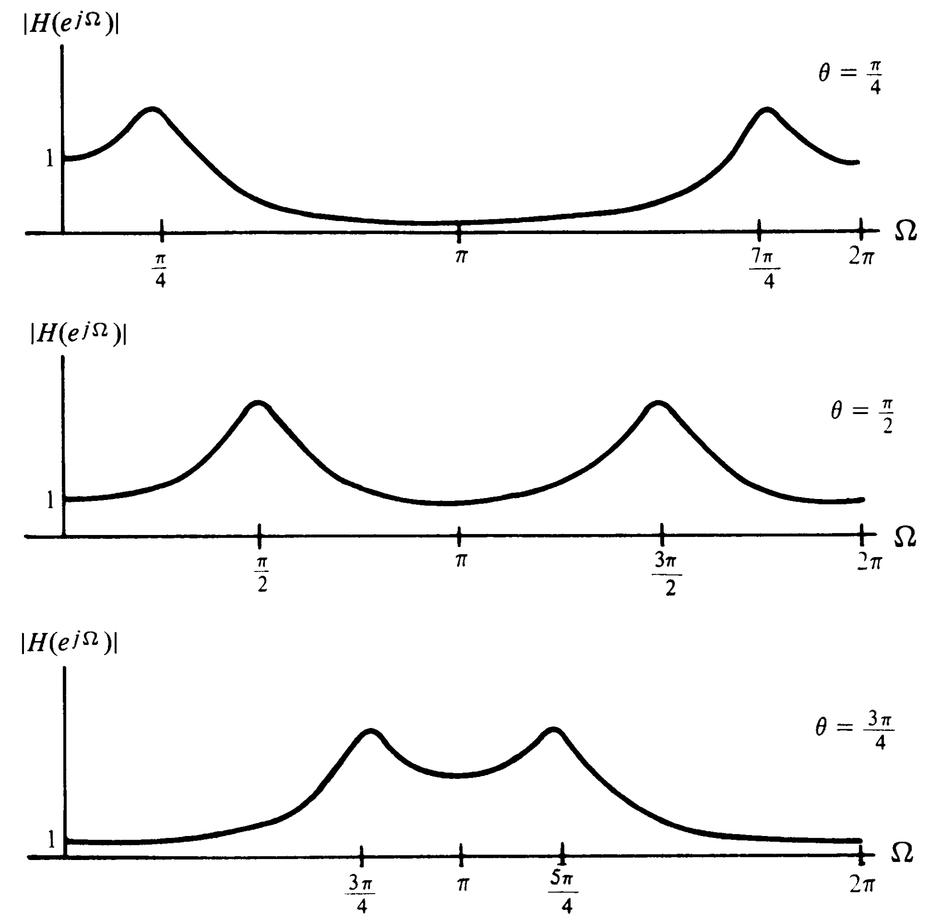

Figure S23.4-2

S23.5

(a)

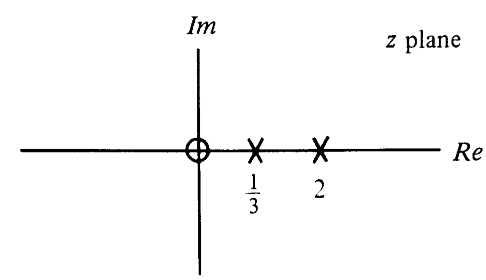

Figure S23.5-1

(b)

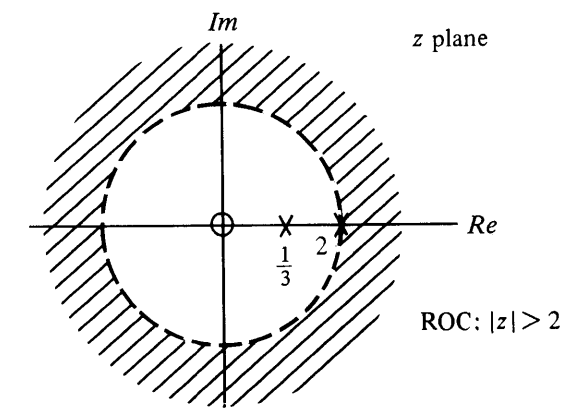

Figure S23.5-2

The ROC is $|z| > 2$. The system is not stable because the ROC does not include the unit circle.

(c)

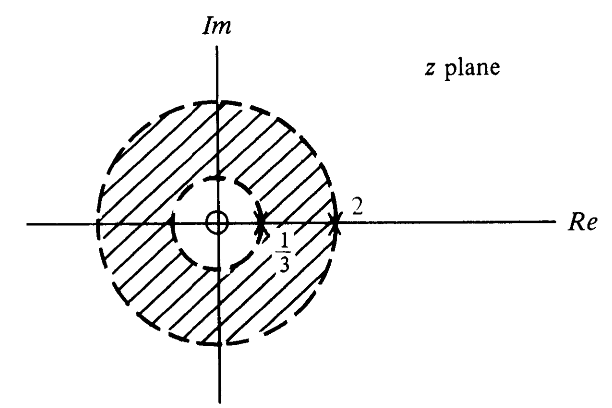

Figure S23.5-3

The ROC is $2 > |z| > \frac{1}{3}$, which for this case includes the unit circle. The corresponding impulse response is two-sided because the ROC is annular. Therefore, the system is not causal.

(d)

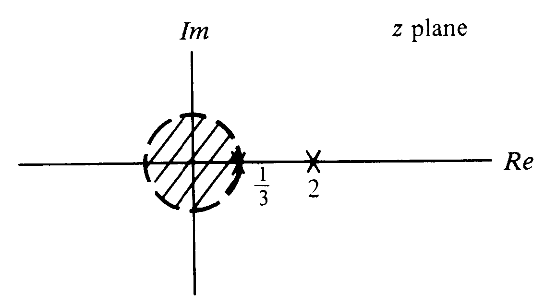

Figure S23.5-4

The remaining ROC does not include the unit circle and is not outside the outermost pole. Therefore, the system is not stable and not causal.

S23.6

(a) $\dfrac{d^2y(t)}{dt^2} + 5\dfrac{dy(t)}{dt} + 6y(t) = x(t) + 2\dfrac{dx(t)}{dt}$

is the system differential equation. Taking Laplace transforms of both sides, we have

$$Y(s)(s^2 + 5s + 6) = X(s)(1 + 2s),$$

so

$$H_c(s) = \frac{Y(s)}{X(s)} = \frac{1 + 2s}{s^2 + 5s + 6}$$
$$= \frac{1 + 2s}{(s + 3)(s + 2)} = \frac{5}{s + 3} + \frac{-3}{s + 2}$$

Assuming the system is causal, we obtain by inspection

$$h_c(t) = 5e^{-3t}u(t) - 3e^{-2t}u(t)$$

(b) Using the fact that the continuous-time system function $A_k/(s - s_k)$ maps to the discrete-time system function $A_k/(1 - e^{s_kT}z^{-1})$ (see page 662 of the text), we have

$$H_d(z) = \frac{5}{1 - e^{-3T}z^{-1}} - \frac{3}{1 - e^{-2T}z^{-1}}$$

(c) Suppose $T = 0.01$. Then

$$H_d(z) = \frac{5}{1 - e^{-0.03}z^{-1}} - \frac{3}{1 - e^{-0.02}z^{-1}}$$

Letting $a = e^{-0.03}$, $b = e^{-0.02}$, we have

$$H_d(z) = \frac{5}{1 - az^{-1}} - \frac{3}{1 - bz^{-1}}$$

So by inspection, assuming causality,

$$h_d[n] = 5a^nu[n] - 3b^nu[n]$$

(d) From part (a), we have

$$h_c(t) = 5e^{-3t}u(t) - 3e^{-2t}u(t)$$

Replacing t by $nT = 0.01n$, we have

$$h_c(nT) = 5e^{-0.03n}u(0.01n) - 3e^{-0.02n}u(0.02n)$$

Letting $a = e^{-0.03}$ and $b = e^{-0.02}$ yields

$$h_c(nT) = 5a^nu[n] - 3b^nu[n],$$

which agrees with the result in part (c).

Solutions to Optional Problems

S23.7

(a) The differential equation is

$$\frac{dy(t)}{dt} + 0.5y(t) = x(t)$$

Taking the Laplace transform yields

$$Y(s)[s + 0.5] = X(s),$$

so

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s + 0.5},$$

$$H(\omega) = \frac{1}{j\omega + 0.5},$$

which is sketched in Figure S23.7-1.

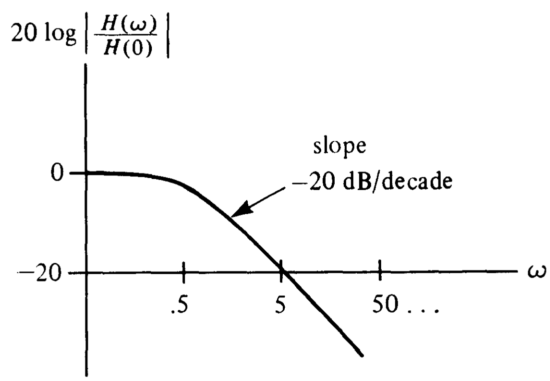

Figure S23.7-1

(b) $\dfrac{y[n + 1] - y[n]}{T} + 0.5y[n] = x[n]$

Taking the z-transform of both sides yields

$$\frac{1}{T}(z - 1)Y(z) + 0.5Y(z) = X(z),$$

$$Y(z)\left(0.5 + \frac{z - 1}{T}\right) = X(z)$$

Letting $T = 2$ yields

$$\frac{Y(z)}{Z(z)} = H_d(z) = \frac{2}{z}$$

Now since

$$|H_d(e^{j\Omega})| = |H_d(z)|\,\Big|_{z=e^{j\Omega}}$$

we have

$$|H_d(e^{j\Omega})| = 2, \qquad \text{for all } \Omega,$$

which is an all-pass filter and is sketched in Figure S23.7-2.

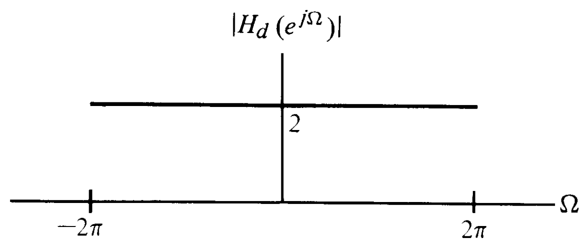

Figure S23.7-2

(c) $$H_d(z) = \frac{1}{0.5 - \frac{1}{T} + \frac{1}{T}z} = \frac{T}{(0.5T - 1) + z}$$

The pole is located at $z_0 = -(0.5T - 1)$ and, since we assume causality, we require that the ROC be outside this pole. When the pole moves onto or outside the unit circle, stability does not exist. The filter is unstable for

$$|z_0| \geq 1 \quad \text{or} \quad |-(0.5T - 1)| \geq 1,$$
$$|0.5T - 1| \geq 1,$$
$$T \geq 4$$

Therefore, for $T \geq 4$, the system is not stable.

S23.8

(a) $$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X(z^{-1}) = \sum_{n=-\infty}^{\infty} x[n]z^{n}$$

Letting $m = -n$, we have

$$X(z^{-1}) = \sum_{m=-\infty}^{\infty} x[-m]z^{-m} = \sum_{m=-\infty}^{\infty} x[m]z^{-m} = X(z)$$

(b) $$X(z) = A\frac{\prod_k (z - a_k)}{\prod_k (z - b_k)}$$

from the definition of a rational z-transform. Now

$$X(z^{-1}) = A\frac{\prod_k (z^{-1} - a_k)}{\prod_k (z^{-1} - b_k)}$$

Each pole (or zero) at z_0 in $X(z)$ goes to a pole (or zero) z_0^{-1} in $X(z^{-1})$. This implies that $z_0 = 1$ or that $X(z)$ must have another pole (or zero) at z_0^{-1}.

(c) (i) $x[n] = \delta[n+1] + \delta[n-1]$,

$$X(z) = z + z^{-1} = \frac{z^2+1}{z} = \frac{(z+j)(z-j)}{z}$$

The zeros are at $z = j$, $1/j$, and the poles are at $z = 0$, $z = \infty$.

(ii) $x[n] = \delta[n+1] - \frac{5}{2}\delta[n] + \delta[n-1]$,

$$X(z) = z - \tfrac{5}{2} + z^{-1}$$
$$= \frac{z^2 - \frac{5}{2}z + 1}{z} = \frac{(z-\frac{1}{2})(z-2)}{z}$$

The zeros are at $z = \frac{1}{2}$, 2, and the poles are at 0, ∞.

(d) (i)

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n},$$
$$Y^*(z) = \left(\sum_{n=-\infty}^{\infty} y[n]z^{-n}\right)^* = \sum_{n=-\infty}^{\infty} y[n](z^*)^{-n} = Y(z^*),$$

so

$$Y^*(z^*) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = Y(z)$$

(ii) Since $Y(z)$ is rational,

$$Y(z) = A\frac{\prod_k (z-a_k)}{\prod_k (z-b_k)}$$

Now if a term such as $(z - a_k)$ appears in $Y(z)$, a term such as $(z^* - a_k)^*$ must also appear in $Y(z)$. For example,

$$Y(z) = (z-a_k)(z^*-a_k)^*,$$
$$Y^*(z^*) = [(z^*-a_k)(z-a_k)^*]^*$$
$$= (z-a_k^*)(z-a_k) = Y(z)$$

So if a pole (or zero) appears at $z = a_k$, a pole (or zero) must also appear at $z = a_k^*$ because

$$(z-a_k^*) = 0 \Rightarrow z = a_k^*$$

(e) Both conditions discussed in parts (b) and (d) hold, i.e., a real, even sequence is considered. A pole at $z = z_p$ implies a pole at $1/z_p$ from part (b). The poles at $z = z_p$ and $z = 1/z_p$ imply poles at $z = z_p^*$ and $z = (1/z_p)^*$ from part (d). Therefore, if $z_p = \rho e^{j\theta}$, poles exist at

$$\frac{1}{\rho e^{j\theta}} = \frac{1}{\rho}e^{-j\theta}, \qquad (\rho e^{j\theta})^* = \rho e^{-j\theta}, \qquad \left(\frac{1}{\rho e^{j\theta}}\right)^* = \frac{1}{\rho}e^{j\theta}$$

S23.9

(a)

$$X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} x_1[k]\hat{X}_2(z),$$

where

$$\hat{X}_2(z) = \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$$
$$= Z\{x_2[n-k]\}$$

(b) $Z\{x_2[n-k]\} = z^{-k}X_2(z)$ from the time-shifting property of the z-transform, so

$$X_3(z) = \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}X_2(z)$$

(c) $X_3(z) = X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}$
$= X_1(z)X_2(z)$

S23.10

Consider $x[n]$ to be composed of a causal and an anticausal part:

$$x[n] = x[n]u[-n-1] + x[n]u[n]$$

Let

$$x_1[n] = x[n]u[-n-1],$$
$$x_2[n] = x[n]u[n],$$

so that

$$x[n] = x_1[n] + x_2[n]$$

and

$$X(z) = X_1(z) + X_2(z)$$

It is clear that every pole of $X_2(z)$ is also a pole of $X(z)$. The only way for this not to be true is by pole cancellation from $X_1(z)$. But pole cancellation cannot happen because a pole a_k that appears in $X_2(z)$ yields a contribution $(a_k)^n u[n]$, which cannot be canceled by terms of $x_1[n]$ that are of the form $(b_k)^n u[-n-1]$.

From the linearity property of z-transforms, if

$$y[n] = y_1[n] + y_2[n]$$

then

$$Y(z) = Y_1(z) + Y_2(z),$$

with the ROC of $Y(z)$ being at least the intersection of the ROC of $Y_1(z)$ and the ROC of $Y_2(z)$. The "at least" specification is required because of possible pole cancellation. In our case, pole cancellation cannot occur, so the ROC of $X(z)$ is exactly the intersection of the ROC of $X_1(z)$ and the ROC of $X_2(z)$.

Now suppose $X_2(z)$ has a pole outside the unit circle. Since $x_2[n]$ is causal, the ROC of $X_2(z)$ must be outside the unit circle, which implies that the ROC of $X(z)$ must be outside the unit circle. This is a contradiction, however, because $x[n]$ is assumed to be absolutely summable, which implies that $X(z)$ has an ROC that includes the unit circle.

Therefore, all poles of the z-transform of $x[n]u[n]$ must be within the unit circle.

S23.11

(a) If $h[n] = h_c(nT)$, then

$$s_d[n] = \sum_{k=-\infty}^{n} h_c(kT)$$

The proof follows.

$$s_d[n] = \sum_{k=-\infty}^{\infty} u[n-k]h_d[k]$$
$$= \sum_{k=-\infty}^{n} h_d[k],$$

but $h_d[k] = h_c(kT)$, so

$$s_d[n] = \sum_{k=-\infty}^{n} h_c(kT)$$

(b) If $s_d[n] = s_c(nT)$, then $h_d[n]$ does not necessarily equal $h_c(nT)$. For example,

$$h_c(t) = e^{-at}u(t),$$
$$s_c(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)\,d\tau$$
$$= \int_0^t e^{-a\tau}d\tau = \frac{1}{a}(1 - e^{-at}), \qquad t \geq 0$$
$$s_d[n] = s_c(nT) = \frac{1}{a}(1 - e^{-anT}), \qquad n \geq 0$$

However,

$$s_d[n] = \sum_{k=-\infty}^{n} h_d[k],$$

so

$$s_d[n] - s_d[n-1] = h_d[n]$$

and, in our case, for $n \geq 0$,

$$h_d[n] = \frac{1}{a}(1 - e^{-anT}) - \frac{1}{a}(1 - e^{-a(n-1)T})$$
$$= \frac{1}{a}e^{-anT}(e^{aT} - 1)$$

But, for $n \geq 0$,

$$h_c(nT) = e^{-anT}u(nT)$$
$$\neq \frac{1}{a}e^{-anT}(e^{aT} - 1)$$

S23.12

(a) From the differential equation

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s),$$

we have

$$\frac{Y(s)}{X(s)} = H_c(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Now consider

$$y_1[n] \equiv \nabla^{(1)}\{x[n]\} \equiv \frac{x[n+1] - x[n-1]}{2},$$

$$Y_1(z) = Z\{y_1[n]\} = Z\{\nabla^{(1)}\{x[n]\}\} = \left(\frac{z - z^{-1}}{2}\right) X(z),$$

$$y_2[n] = \nabla^{(2)}\{x[n]\} = \nabla^{(1)}\{y_1[n]\} = \frac{y_1[n+1] - y_1[n-1]}{2},$$

$$Y_2(z) = \frac{z - z^{-1}}{2} Y_1(z) = \left(\frac{z - z^{-1}}{2}\right)^2 X(z)$$

By induction,

$$Z\{\nabla^{(k)}\{x[n]\}\} = \left(\frac{z - z^{-1}}{2}\right)^k X(z)$$

Therefore,

$$\sum_{k=0}^{N} a_k \left(\frac{z - z^{-1}}{2}\right)^k Y(z) = \sum_{k=0}^{M} b_k \left(\frac{z - z^{-1}}{2}\right)^k X(z),$$

$$H_d(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \left(\frac{z - z^{-1}}{2}\right)^k}{\sum_{k=0}^{N} a_k \left(\frac{z - z^{-1}}{2}\right)^k}$$

$$= H_c(s)\Big|_{s=(z-z^{-1})/2}$$

(b) $H_c(s)\Big|_{s=(z-z^{-1})/2} = H_d(z)$

from part (a). Consider $s = j\omega$, $z = e^{j\Omega}$. So

$$j\omega = \frac{e^{j\Omega} - e^{-j\Omega}}{2},$$

and, thus, $\omega = \sin \Omega$ is the mapping between discrete-time and continuous-time frequencies. Since $H(\omega) = \omega$ for $|\omega| < 1$, $H_d(e^{j\Omega})$ is as indicated in Figure S23.12.

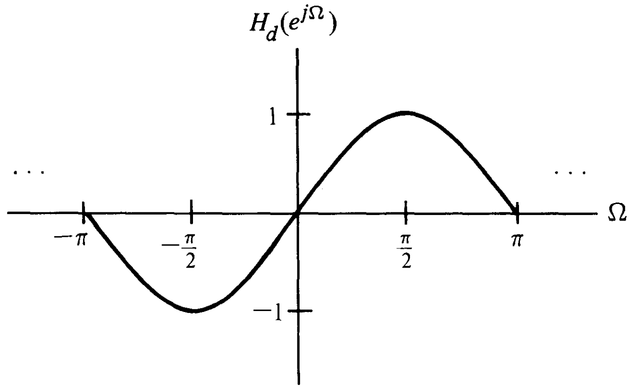

Figure S23.12

(c) From part (a) we see that

$$H_d(z) = H_d(z^{-1})$$

and that $H_d(z)$ is a rational z-transform.

$$H_d(z) = \frac{A\prod_{i=1}^{P} (z - z_{0_i})^{M_i}}{\prod_{i=1}^{Q} N_i(z - z_{p_i})^{N_i}}$$

Therefore, if a term such as $(z - z_{0_i})$ appears, $(z^{-1} - z_{0_i})$ must also appear. If $H_d(z)$ has a pole within the unit circle, it must also have a pole outside the unit circle. If the ROC includes the unit circle, it is therefore not outside the outermost pole (which lies outside the unit circle) and, therefore, $H_d(z)$ does not correspond to a causal filter.

Consider

$$H_c(s) = \frac{1}{s + \frac{1}{2}}$$

corresponding to a stable, causal $h_c(t)$.

$$H_d(z) = H_c(s)\Big|_{s=(z-z^{-1})/2} = \frac{1}{\dfrac{z - z^{-1}}{2} + \dfrac{1}{2}}$$

$$= \frac{2z}{z^2 + z - 1} = \frac{2z}{\left[z - \dfrac{(-1 + \sqrt{5})}{2}\right]\left[z - \dfrac{(-1 - \sqrt{5})}{2}\right]},$$

so poles of z are at 0.618, -1.618. Therefore, $H_d(z)$ is not causal if it is assumed stable because stability and causality require that all poles be inside the unit circle.

S23.13

(a) We are given that

$$H_c(s) = \frac{A}{(s - s_0)^2}$$

From Table 9.2 of the text (page 604), we see that $h_c(t) = Ate^{s_0 t}u(t)$. To verify, consider

$$\frac{1}{s - s_0} = \int_{-\infty}^{\infty} e^{s_0 t}u(t)e^{-st}dt,$$

$$\frac{d}{ds}\left(\frac{1}{s - s_0}\right) = \frac{d}{ds}\left[\int_{-\infty}^{\infty} e^{s_0 t}u(t)e^{-st}dt\right],$$

$$\frac{1}{(s - s_0)^2} = \int_{-\infty}^{\infty} te^{s_0 t}u(t)e^{-st}dt$$

Therefore,

$$te^{s_0 t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s - s_0)^2}$$

(b) $h_d[n] = h_c(nT) = AnTe^{s_0 nT}u[n]$

(c) $H_d(z) = \sum_{n=-\infty}^{\infty} h_d[n]z^{-n} = AT\sum_{n=0}^{\infty} ne^{s_0 nT}z^{-n}$

From Table 10.2 of the text (page 655),

$$n\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$$

This can be verified:

$$\frac{1}{1 - \alpha z^{-1}} = \sum_{n=-\infty}^{\infty} \alpha^n u[n]z^{-n}$$

$$\frac{d}{dz}\left(\frac{1}{1 - \alpha z^{-1}}\right) = \frac{d}{dz}\sum_{n=-\infty}^{\infty} \alpha^n u[n]z^{-n}$$

$$\frac{-\alpha z^{-2}}{(1 - \alpha z^{-1})^2} = \sum_{n=-\infty}^{\infty} (-n)\alpha^n u[n]z^{-n-1}$$

$$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \sum_{n=-\infty}^{\infty} n\alpha^n u[n]z^{n}$$

In our case, $\alpha = e^{s_0 T}$, so

$$H_d(z) = \frac{ATe^{s_0 T}z^{-1}}{(1 - e^{s_0 T}z^{-1})^2}$$

(d) $H_c(s) = \dfrac{1}{(s + 1)(s + 2)^2} = \dfrac{1}{s + 1} + \dfrac{-1}{s + 2} + \dfrac{-1}{(s + 2)^2}$

Using the first-order pole result for $1/(s + 1)$ and $-1/(s + 2)$ and the second-order pole result for $-1/(s + 2)^2$, we have

$$H_d(z) = \frac{1}{1 - e^{-T}z^{-1}} - \frac{1}{1 - e^{-2T}z^{-1}} - \frac{Te^{-2T}z^{-1}}{(1 - e^{-2T}z^{-1})^2}$$

After some algebra, we obtain

$$H_d(z) = \frac{z[z(-e^{-2T} + e^{-T} - Te^{-2T}) + e^{-4T} - e^{-3T} + Te^{-3T}]}{(z - e^{-T})(z - e^{-2T})^2}$$

The corresponding pole-zero pattern is shown in Figure S23.13.

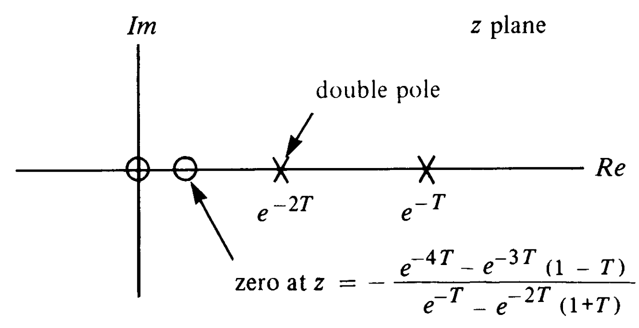

Figure S23.13