ELEC 309
Signals and Systems
Homework 2 Solutions

Time-Domain Analysis of Signals

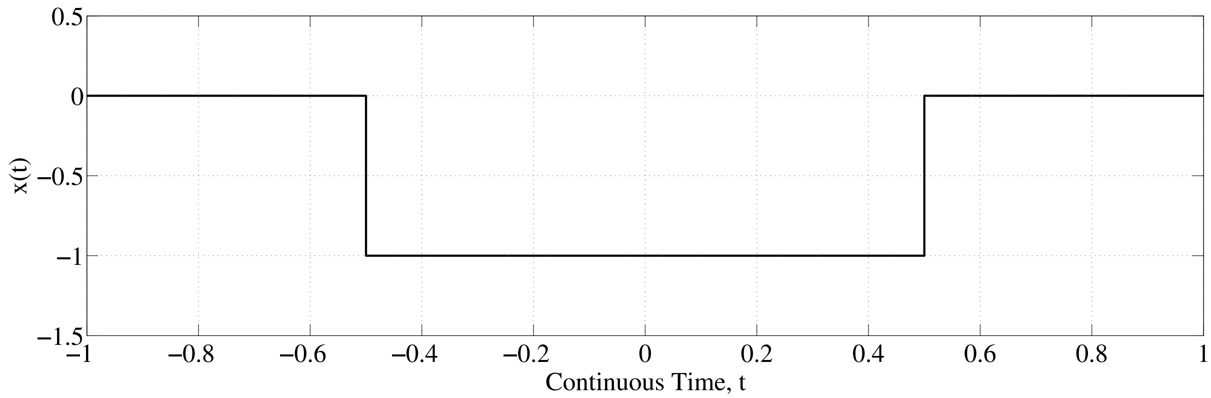

Figure 1: Rectangular Pulse Signal

1. A rectangular pulse signal $x(t)$ is depicted in Figure 1. Express $x(t)$ as a weighted sum of unit step functions.

The rectangular pulse signal $x(t)$ can be written as

$$x(t) = -u(t+0.5) + u(t-0.5) = \boxed{u(t-0.5) - u(t+0.5)}$$

2. A discrete-time signal $x[n]$ is given by

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Express $x[n]$ as a weighted sum of unit step functions.

The discrete-time signal $x[n]$ can be written as

$$x[n] = \boxed{u[n] - u[n-10]}$$

(b) Express $x[n]$ as a weighted sum of unit impulse functions.

The discrete-time signal $x[n]$ can be written as

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \ldots + \delta[n-8] + \delta[n-9] = \boxed{\sum_{k=0}^{9} \delta[n-k]}$$

3. Simplify

(a) $\int_{-\pi}^{\pi} e^{-(t-\pi)^2}\delta(t)dt$

$$\int_{-\pi}^{\pi} e^{-(t-\pi)^2}\delta(t)dt = e^{-(t-\pi)^2}\Big|_{t=0} = \boxed{e^{-\pi^2} = 5.1723 \times 10^{-5}}$$

(b) $\int_{-\pi}^{\pi} e^{-(t-\pi)^2}\delta(t-2\pi)dt$

$$\int_{-\pi}^{\pi} e^{-(t-\pi)^2}\delta(t-2\pi)dt = \boxed{0} \text{ since } \delta(t-2\pi) = 0 \text{ for } -\pi \leq t \leq \pi.$$

(c) $\cos(2\pi t)\,\delta(-2t)$

$$\cos(2\pi t)\,\delta(-2t) = \cos(2\pi t)\Big|_{t=0} \cdot \frac{1}{|-2|}\delta(t) = \boxed{\frac{1}{2}\delta(t).}$$

Time-Domain Analysis of Systems

4. The systems that follow have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$. For each system determine whether it is (i) memoryless, (ii) causal, (iii) linear, (iv) time-invariant, (v) invertible, and (vi) stable.

   (a) $y(t) = \cos(x(t))$

(i) By inspection, we see that the output $y(t)$ depends only on the current value of the input signal $x(t)$. Therefore, the system is $\boxed{\text{memoryless.}}$

(ii) By inspection, we see that the output $y(t)$ depends only on the current value and not future values of the input signal $x(t)$. Therefore, the system is $\boxed{\text{causal.}}$

(iii) If $x(t) = \pi$, then the output is $y(t) = -1$. To satisfy the scaling property, we should have $0 \cdot x(t) = 0 \Longrightarrow 0 \cdot y(t) = 0$. But $0 \cdot x(t) = 0 \Longrightarrow 1 \neq 0 \cdot y(t) = 0$. Therefore, the system is $\boxed{\text{nonlinear.}}$

(iv) For an arbitrary input signal $x(t)$, we have $x(t) \longrightarrow \cos(x(t)) = y(t)$. For an arbitrary time shift t_0, we have $x(t - t_0) \longrightarrow \cos(x(t - t_0)) = y(t - t_0)$. Therefore, this system is $\boxed{\text{time-invariant.}}$

(v) The system with input-output equation $y(t) = \cos(x(t))$ is $\boxed{\text{noninvertible}}$ since cosine is a multiple-valued function. If $y(t) = 1$, then it is impossible to determine whether $x(t) = 0$, $x(t) = 2\pi$, $x(t) = 4\pi$, etc.

(vi) Consider an arbitrary bounded input $x(t)$. At any time t, $-1 \leq y(t) \leq 1$. Therefore, for any bounded input, we also have a bounded output. Therefore, the system is $\boxed{\text{BIBO stable.}}$

(b) $y[n] = 2x[n]u[n]$

(i) By inspection, we see that the output $y[n]$ depends only on the current value of the input signal $x[n]$. Therefore, the system is $\boxed{\text{memoryless.}}$

(ii) By inspection, we see that the output $y[n]$ depends only on the current value and not future values of the input signal $x[n]$. Therefore, the system is $\boxed{\text{causal.}}$

(iii) Let $x_1[n] \implies 2x_1[n]u[n] = y_1[n]$ and $x_2[n] \implies 2x_2[n]u[n] = y_2[n]$. Also, let α_1 and α_2 be arbitrary. Then,

$$\begin{aligned}\alpha_1 x_1[n] + \alpha_2 x_2[n] \implies & 2\left(\alpha_1 x_1[n] + \alpha_2 x_2[n]\right) u[n] \\ & = \alpha_1 \left(2x_1[n]u[n]\right) + \alpha_2 \left(2x_2[n]u[n]\right) \\ & = \alpha_1 y_1[n] + \alpha_2 y_2[n].\end{aligned}$$

Therefore, the system is $\boxed{\text{linear.}}$

(iv) Let

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \implies 2\left(\frac{1}{2}\right)^n u[n]u[n] = 2\left(\frac{1}{2}\right)^n u[n] = y[n].$$

Now, consider $x[n]$ shifted left, or

$$\begin{aligned}x[n+1] = \left(\frac{1}{2}\right)^{n+1} u[n+1] \implies & 2\left(\frac{1}{2}\right)^{n+1} u[n+1]u[n] = 2\left(\frac{1}{2}\right)^{n+1} u[n] \\ & \neq y[n+1] = 2\left(\frac{1}{2}\right)^{n+1} u[n+1].\end{aligned}$$

Therefore, this system is $\boxed{\text{time-varying.}}$

(v) Consider

$$\begin{aligned}x_1[n] = u[n] &\implies 2u[n]u[n] = 2u[n] = y[n] \text{ and} \\ x_2[n] = u[n+1] &\implies 2u[n+1]u[n] = 2u[n] = y[n].\end{aligned}$$

Since two different input signals produce the same output signal, the system is $\boxed{\text{noninvertible}}$.

(vi) Consider an arbitrary bounded input $x[n]$ such that $|x[n]| \leq k_1$. If $y[n] = 2x[n]u[n]$, then $|y[n]| \leq 2k_1$. Therefore, for any bounded input, we also have a bounded output. Therefore, the system is $\boxed{\text{BIBO stable.}}$

(c) $y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau$

(i) By inspection, we see that the output $y(t)$ does not depend only on the current value of the input signal $x(t)$. Therefore, the system $\boxed{\text{has memory.}}$

(ii) Consider $t = -2$. Then

$$y(-2) = \int_{-\infty}^{-1} x(\tau)d\tau,$$

which means $y(-2)$ depends on future values of $x(t)$, specifically $x(t)$ for $-2 < t \leq -1$. Therefore, the system is $\boxed{\text{non-causal.}}$

(iii) Let $x_1(t) \Longrightarrow \int_{-\infty}^{t/2} x_1(\tau)d\tau = y_1(t)$ and $x_2(t) \Longrightarrow \int_{-\infty}^{t/2} x_2(\tau)d\tau = y_2(t)$. Also, let α_1 and α_2 be arbitrary. Then,

$$\begin{aligned} \alpha_1 x_1(t) + \alpha_2 x_2(t) &\Longrightarrow \int_{-\infty}^{t/2} \alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)d\tau \\ &= \alpha_1 \int_{-\infty}^{t/2} x_1(\tau)d\tau + \alpha_2 \int_{-\infty}^{t/2} x_2(\tau)d\tau \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t). \end{aligned}$$

Therefore, the system is $\boxed{\text{linear.}}$

(iv) Let $x(t) \Longrightarrow \int_{-\infty}^{t/2} x(\tau)d\tau = y(t)$ and shift t_0 be arbitrary. Then

$$x(t - t_0) \Longrightarrow \int_{-\infty}^{(t-t_0)/2} x(\tau)d\tau = y(t - t_0).$$

Therefore, this system is $\boxed{\text{time-invariant.}}$

(v) Using $u = 2\tau$ and $du = 2d\tau$, we have

$$y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau = 2\int_{-\infty}^{t} x(u/2)du.$$

Taking the derivative of both sides with respect to t, we have

$$\frac{dy(t)}{dt} = 2x\left(\frac{t}{2}\right) \Longrightarrow x\left(\frac{t}{2}\right) = \frac{1}{2}\frac{dy(t)}{dt} \Longrightarrow x\left(t\right) = \frac{dy(2t)}{dt}\cdot.$$

Since we have a formula to calculate $x(t)$ given $y(t)$, the system is $\boxed{\text{invertible}}$.

(vi) Consider bounded input $x(t) = u(t)$. Then

$$y(t) = \int_{-\infty}^{t/2} x(\tau)d\tau = \begin{cases} \int_0^{t/2} d\tau & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} t/2 & t \geq 0 \\ 0 & t < 0 \end{cases} = (t/2)u(t),$$

which is unbounded as $t \to \infty$. Therefore, the system is $\boxed{\text{BIBO unstable.}}$

(d) $y[n] = \cos(2[n+1]) + x[n]$

(i) By inspection, we see that the output $y[n]$ depends only on the current value of the input signal $x[n]$. Therefore, the system is $\boxed{\text{memoryless.}}$

(ii) By inspection, we see that the output $y[n]$ depends only on the current value and not future values of the input signal $x[n]$. Therefore, the system is $\boxed{\text{causal.}}$

(iii) Consider

$$x[n] = 0 \Longrightarrow y[n] = \cos(2[n+1]) + x[n] = \cos(2[n+1]) + 0 = \cos(2[n+1]) \neq 0.$$

Therefore, the scaling property is not satisfied, and the system is $\boxed{\text{nonlinear.}}$

(iv) Let $x[n] \Longrightarrow \cos(2[n+1]) + x[n] = y[n]$ and shift k be arbitrary. Then

$$x[n-k] \Longrightarrow \cos(2[n+1]) + x[n-k] \neq y[n-k] = \cos(2[n-k+1]) + x[n-k].$$

Therefore, this system is $\boxed{\text{time-varying.}}$

(v) Note that this system simply adds $\cos(2[n+1])$ to the input. Therefore, given the signal $y[n]$, it is easy to determine $x[n]$. Therefore, this system is $\boxed{\text{invertible}}$.

(vi) Consider an arbitrary bounded input $x[n]$ such that $|x[n]| \leq k_1$. If $y[n] = \cos(2[n+1]) + x[n]$, then $|y[n]| \leq 1 + k_1$. Therefore, for any bounded input, we also have a bounded output. Therefore, the system is $\boxed{\text{BIBO stable.}}$