ELEC 312
Systems I

Steady-State Response Analysis
(Adapted from Notes by Dr. Robert Barsanti)

(Images from Nise, 7^{th} Edition)

Required Reading: Chapter 7,
Control Systems Engineering

March 5, 2015

Steady-State Response

The **steady-state** response of a **stable** LTI system is the eventual system output after all transients have essentially faded to zero.

> Thus, the **steady-state** response can only exist if the system is **stable**.

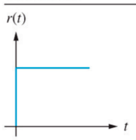
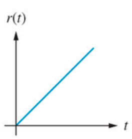
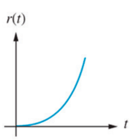
So we now turn our attention away from the short-term (transient) behavior of our systems, and focus on the long-term (**steady-state**) behavior.

> **Steady-state error** is the difference between the input and the output for a prescribed test input as $t \to \infty$.

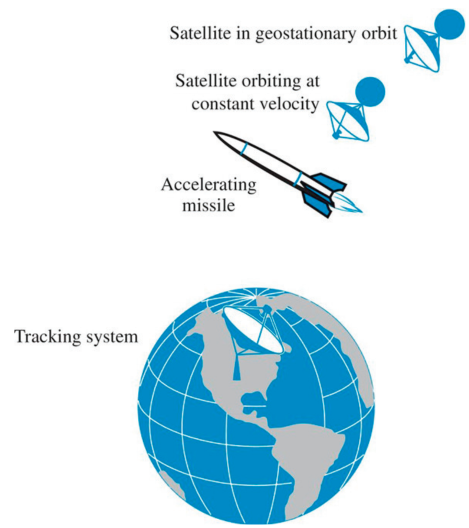
Test inputs used for steady-state error analysis and design are summarized following.

Steady-State Response to Test Inputs

In practice, we are concerned with the system response to inputs of a **step**, **ramp**, and **parabola**.

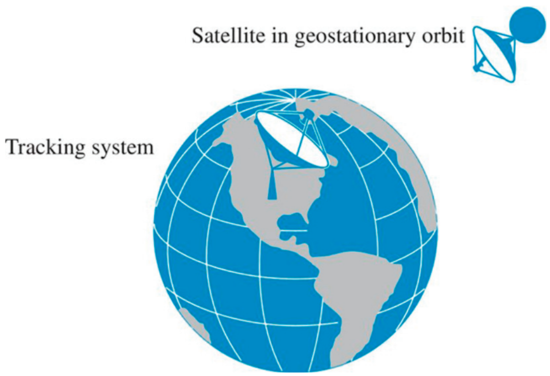
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Steady-State Response to Test Inputs



Assume a position control system, where the output position follows the input commanded position.

Steady-State Response to Step Inputs



Step inputs represent **constant position** and thus are useful in determining the ability of the control system to position itself with respect to a stationary target, such as a **satellite in geostationary orbit**.

An **antenna position control** is an example of a system that can be tested for accuracy using **step** inputs.

Steady-State Response to Step Inputs

Consider a system with closed-loop transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}.$$

If we let $r(t) = u(t) \overset{\mathcal{L}}{\longleftrightarrow} R(s) = \frac{1}{s}$ (a unit step input), then

$$\begin{aligned} C(s) = T(s)R(s) = T(s) \cdot \frac{1}{s} &= \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \cdot \frac{1}{s} \\ &= \underbrace{\frac{K_1}{s}}_{\text{steady-state response } C_{SS}(s)} + \underbrace{\text{terms due to poles of } T(s)}_{\text{tend to zero if } T(s) \text{ is stable system}}. \end{aligned}$$

Solving for K_1, we have

$$K_1 = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \cdot \frac{1}{s} \cdot s \Bigg|_{s=0} = T(s)\Bigg|_{s=0} = T(0) = \frac{b_0}{a_0}.$$

Steady-State Response to Step Inputs

Therefore, if $T(s)$ is stable, then the response due to the poles of $T(s)$ will tend to zero as $t \to \infty$, and the steady-state response $c_{SS}(t)$ due to a unit step input is

$$C_{SS}(s) = \frac{b_0}{a_0} \cdot \frac{1}{s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} c_{SS}(t) = \frac{b_0}{a_0} u(t).$$

The final value of the steady-state response is given by

$$c(\infty) = \lim_{t \to \infty} c(t) = \lim_{t \to \infty} c_{SS}(t) = T(0) = \frac{b_0}{a_0}.$$

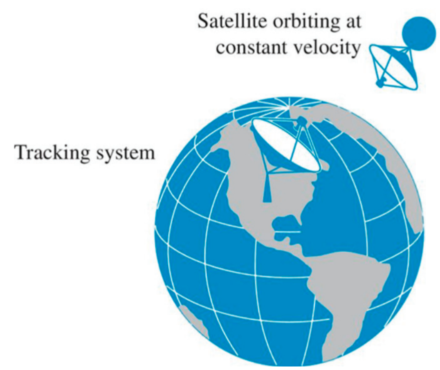
Similarly, by the Laplace transform final value theorem, we have

$$\boxed{c(\infty) = \lim_{t \to \infty} c(t) = \lim_{s \to 0} sC(s) = \lim_{s \to 0} s\left[T(s) \cdot \frac{1}{s}\right] = \lim_{s \to 0} T(s) = T(0) = \frac{b_0}{a_0}}$$

Note: If $T(0) = 1$ (which requires that $a_0 = b_0$), then

$$c_{SS}(t) = u(t) = r(t) \text{ and } c(\infty) = 1.$$

Steady-State Response to Ramp Inputs



Ramp inputs can be used to test the ability of a position control system to follow a **constant-velocity** target.

For example, a position control system that tracks a satellite that moves across the sky at a **constant angular velocity** would be tested with a **ramp** input to evaluate the steady-state error between the satellite's angular position and that of the control system.

Steady-State Response to Ramp Inputs

Consider a system with transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

If we let $r(t) = tu(t) \overset{\mathcal{L}}{\longleftrightarrow} R(s) = \frac{1}{s^2}$ (a ramp input), then

$$\begin{aligned} C(s) = T(s)R(s) = T(s)\cdot\frac{1}{s^2} &= \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}\cdot\frac{1}{s^2} \\ &= \underbrace{\frac{K_1}{s^2} + \frac{K_2}{s}}_{\text{steady-state response } C_{SS}(s)} + \underbrace{\text{terms due to poles of } T(s)}_{\text{tend to zero if } T(s) \text{ is stable system}}. \end{aligned}$$

Solving for K_1, we have

$$K_1 = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}\cdot\frac{1}{s^2}\cdot s^2\Bigg|_{s=0} = T(s)\Bigg|_{s=0} = T(0) = \frac{b_0}{a_0}.$$

Steady-State Response to Ramp Inputs

Multiplying our partial-fraction expansion by s^2 to solve for K_2, we have

$$\begin{aligned} T(s)\cdot\frac{1}{s^2}\cdot s^2 &= \frac{K_1}{s^2}\cdot s^2 + \frac{K_2}{s}\cdot s^2 + s^2\cdot[\text{terms due to poles of } T(s)] \\ \Rightarrow T(s) &= K_1 + K_2 s + s^2\cdot[\text{terms due to poles of } T(s)] \\ \Rightarrow K_2 s &= T(s) - K_1 - s^2\cdot[\text{terms due to poles of } T(s)] \end{aligned}$$

Taking the derivative $\frac{d}{ds}$ of both sides and evaluating at $s = 0$, we have

$$\begin{aligned} K_2 &= T'(s) - 0 - \left(2s\cdot[\text{terms due to poles of } T(s)] + s^2\cdot\tfrac{d}{ds}[\text{terms due to poles of } T(s)]\right)\Big|_{s=0} \\ &= T'(s)\Big|_{s=0} \\ &= \frac{(a_n s^n + \ldots + a_0)\left(m b_m s^{m-1} + \ldots + b_1\right) - \left(n a_n s^{n-1} + \ldots + a_1\right)(b_m s^m + \ldots + b_0)}{\left(a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0\right)^2}\Bigg|_{s=0} \\ &= T'(0) = \frac{a_0 b_1 - b_0 a_1}{a_0^2}. \end{aligned}$$

Steady-State Response to Ramp Inputs

Therefore, if $T(s)$ is stable, then the response due to the poles of $T(s)$ will tend to zero as $t \to \infty$, and the steady-state response $c_{SS}(t)$ due to a ramp input is

$$C_{SS}(s) = \frac{b_0}{a_0}\cdot\frac{1}{s^2} + \frac{a_0 b_1 - b_0 a_1}{a_0^2}\cdot\frac{1}{s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} c_{SS}(t) = \left[\frac{b_0}{a_0}t + \frac{a_0 b_1 - b_0 a_1}{a_0^2}\right]u(t).$$

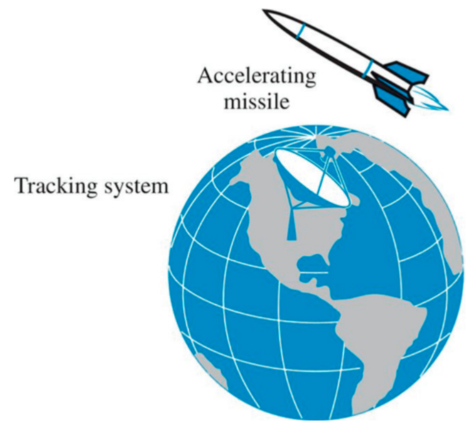
The steady-state response $c_{SS}(t)$ due to a ramp input is also given by

$$c_{SS}(t) = \left[T(0)t + T'(0)\right]u(t).$$

Note: If $T(0) = 1$ and $T'(0) = 0$ (which requires that $a_0 = b_0$ and $a_1 = b_1$), then

$$c_{SS}(t) = tu(t) = r(t).$$

Steady-State Response to Parabolic Inputs



Parabolic inputs, whose second derivatives are constant, represent **constant-acceleration** inputs to position control systems and can be used to represent accelerating targets, such as the missile shown above, to determine the steady-state error performance.

Steady-State Response to Parabolic Inputs

Consider a system with transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

If we let $r(t) = \frac{1}{2}t^2 u(t) \overset{\mathcal{L}}{\longleftrightarrow} R(s) = \frac{1}{s^3}$ (a parabolic input), then

$$C(s) = T(s)R(s) = T(s)\cdot\frac{1}{s^3} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}\cdot\frac{1}{s^3}$$
$$= \underbrace{\frac{K_1}{s^3} + \frac{K_2}{s^2} + \frac{K_3}{s}}_{\text{steady-state response } C_{SS}(s)} + \underbrace{\text{terms due to poles of } T(s)}_{\text{tend to zero if } T(s) \text{ is stable system}}.$$

Solving for K_1, we have

$$K_1 = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}\cdot\frac{1}{s^3}\cdot s^3\bigg|_{s=0} = T(s)\bigg|_{s=0} = T(0) = \frac{b_0}{a_0}.$$

Steady-State Response to Parabolic Inputs

Multiplying our partial-fraction expansion by s^3 to solve for K_2, we have

$$T(s)\cdot\frac{1}{s^3}\cdot s^3 = \frac{K_1}{s^3}\cdot s^3 + \frac{K_2}{s^2}\cdot s^3 + \frac{K_3}{s}\cdot s^3 + s^3\cdot[\text{terms due to poles of } T(s)]$$
$$\Rightarrow T(s) = K_1 + K_2 s + K_3 s^2 + s^3\cdot[\text{terms due to poles of } T(s)]$$
$$\Rightarrow K_2 s = T(s) - K_1 - K_3 s^2 - s^3\cdot[\text{terms due to poles of } T(s)]$$

Taking the derivative $\frac{d}{ds}$ of both sides and evaluating at $s=0$, we have

$$K_2 = T'(s) - 0 - 2K_3 s - \left(3s^2\cdot[\text{terms due to poles of } T(s)] + s^3\cdot\tfrac{d}{ds}[\text{terms due to poles of } T(s)]\right)\bigg|_{s=0}$$
$$= T'(s)\Big|_{s=0}$$
$$= \frac{(a_n s^n + \ldots + a_0)\left(m b_m s^{m-1} + \ldots + b_1\right) - \left(n a_n s^{n-1} + \ldots + a_1\right)(b_m s^m + \ldots + b_0)}{\left(a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0\right)^2}\Bigg|_{s=0}$$
$$= T'(0) = \frac{a_0 b_1 - b_0 a_1}{a_0^2}.$$

Steady-State Response to Parabolic Inputs

To determine K_3, we must take the derivative $\frac{d}{ds}$ of both sides of

$$K_2 = T'(s) - 0 - 2K_3 s - \left(3s^2\cdot[\text{terms due to poles of } T(s)] + s^3\cdot\tfrac{d}{ds}[\text{terms due to poles of } T(s)]\right)$$

and evaluate at $s=0$. Therefore, we have

$$2K_3 = T''(s) - \left(6s\cdot[\text{terms due to poles of } T(s)] + 3s^2\cdot\tfrac{d}{ds}[\text{terms due to poles of } T(s)]\right)$$
$$- \left(3s^2\cdot[\text{terms due to poles of } T(s)] + s^3\cdot\tfrac{d}{ds}[\text{terms due to poles of } T(s)]\right)\bigg|_{s=0}$$
$$= T''(s)\Big|_{s=0} = T''(0) = \frac{a_0^2\left[2a_0b_2 - 2a_2b_0\right] - 2a_0a_1\left[a_0b_1 - a_1b_0\right]}{a_0^4},$$

and

$$K_3 = \frac{1}{2}T''(0) = \frac{a_0^2\left[a_0b_2 - a_2b_0\right] - a_0a_1\left[a_0b_1 - a_1b_0\right]}{a_0^4}.$$

Steady-State Response to Parabolic Inputs

Therefore, if $T(s)$ is stable, then the response due to the poles of $T(s)$ will tend to zero as $t\to\infty$, and the steady-state response $c_{SS}(t)$ due to a parabolic input is

$$C_{SS}(s) = \frac{b_0}{a_0}\cdot\frac{1}{s^3} + \frac{a_0b_1 - b_0a_1}{a_0^2}\cdot\frac{1}{s^2} + \frac{a_0^2\left[a_0b_2 - a_2b_0\right] - a_0a_1\left[a_0b_1 - a_1b_0\right]}{a_0^4}\cdot\frac{1}{s}$$
$$\overset{\mathcal{L}^{-1}}{\longleftrightarrow} c_{SS}(t) = \left[\frac{b_0}{2a_0}t^2 + \frac{a_0b_1 - b_0a_1}{a_0^2}t + \frac{a_0^2\left[a_0b_2 - a_2b_0\right] - a_0a_1\left[a_0b_1 - a_1b_0\right]}{a_0^4}\right]u(t).$$

The steady-state response $c_{SS}(t)$ due to a parabolic input is also given by

$$c_{SS}(t) = \left[\frac{1}{2}T(0)t^2 + T'(0)t + \frac{1}{2}T''(0)\right]u(t).$$

Note: If $T(0)=1$ and $T'(0)=T''(0)=0$ (which requires that $a_0=b_0$, $a_1=b_1$, and $a_2=b_2$), then

$$c_{SS}(t) = \frac{1}{2}t^2u(t) = r(t).$$

Steady-State Response to Test Inputs: Example 1

Given $T(s) = \frac{2}{s+1}$ and $r(t) = 5u(t)$, find $c_{SS}(t)$.

Steady-State Response to Test Inputs: Example 2

Given $T(s) = \frac{3s+2}{s^2+3s+2}$ and $r(t) = (2+t)u(t)$, find $c_{SS}(t)$.

Steady-State Response to Test Inputs: Example 3

Given $T(s) = \frac{-1}{s^2-1}$ and $r(t) = -u(t)$, find $c_{SS}(t)$.

Steady-State Response to Test Inputs: Example 4

Given $T(s) = \frac{2}{s+1}$ and $r(t) = 3tu(t)$, find $c_{SS}(t)$.

Steady-State Response to Test Inputs: Example 5

Given $T(s) = \frac{9s^2+9s+68}{s^3+9s^2+9s+68}$ and $r(t) = \left[61t^2 - 20t + 16\right] u(t)$, find $c_{SS}(t)$.

Steady-State Response to Test Inputs: Summary

Given a **stable** transfer function

$$T(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0},$$

the steady-state system response can:

- Track a step input for $b_0 = a_0$.
- Track a ramp input for $b_0 = a_0$ and $b_1 = a_1$.
- Track a parabolic input for $b_0 = a_0$, $b_1 = a_1$, and $b_2 = a_2$.

Steady-State Performance: Accuracy using Steady-State Error

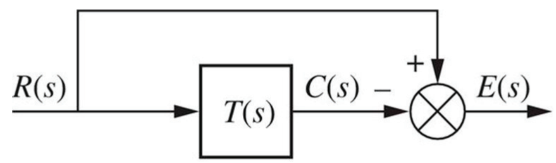
The **steady-state performance** is concerned with the response of our system as $t \to \infty$.

The difference between the final steady-state value of the system and the input (or desired output) is known as the system **steady-state error**.

The steady-state error is a measure of system accuracy.

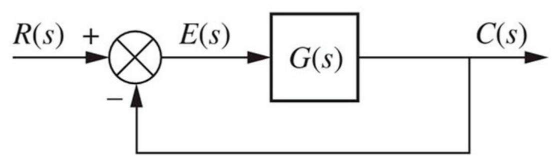
We are NOT concerned with speed of response–only with the long time eventual system output.

Steady-State Performance: Accuracy using Steady-State Error



Assuming a closed-loop transfer function, $T(s)$, the error, $E(s)$, is formed by taking the difference between the input and the output, as shown above.

Here we are interested in the steady-state, or final, value of $e(t)$.



For unity feedback systems, $E(s)$ appears as shown above.

Sources of Steady-State Errors

Many steady-state errors in control systems result from the **nonlinear** behavior of the system or its components.

Examples of this are:

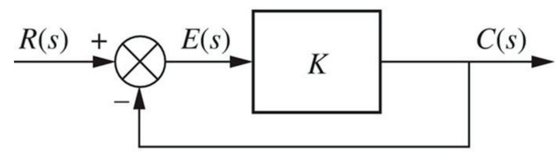
- backlash in gears,
- frictional effects, or
- amplifier saturations.

These nonlinear behaviors and resulting errors are **beyond our studies**.

The type of errors we will concern ourselves with arise from the **configuration of the system itself** and the **type of applied input.**

Sources of Steady-State Errors: Demonstrative Example 1

Consider:



If the input is a unit step function, then an error will exist unless the output $C(s) = R(s) = \frac{1}{s}$.

However, with a constant gain K in the forward path, then $C(s) = KE(s)$ and the error is given by

$$E(s) = R(s) - C(s) = R(s) - KE(s) \Rightarrow E(s) = \frac{1}{K+1}R(s).$$

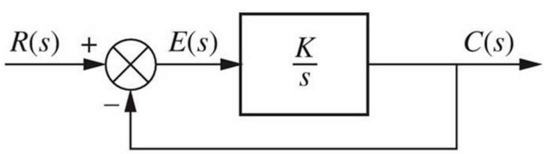
For $r(t) = u(t)$, the error will be constant and given by $e(t) = \frac{1}{K+1}u(t) \neq 0$.

Due to the system configuration (a pure forward path gain), an error must exist.

Thus, there will always be a steady-state error for a step input, and this error will diminish as K increases.

Sources of Steady-State Errors: Demonstrative Example 2

Now consider:



If the input is a unit step function, then an error will exist unless the output $C(s) = R(s) = \frac{1}{s}$.

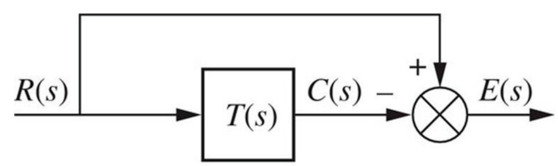
Now, with a constant gain K and pure integrator in the forward path, then $C(s) = \frac{K}{s}E(s)$ and the error is given by

$$E(s) = R(s) - C(s) = R(s) - \frac{K}{s}E(s) \Rightarrow E(s) = \frac{s}{s+K}R(s).$$

Therefore, for $r(t) = u(t)$, the error will be $e(t) = e^{-Kt}u(t)$, which approaches zero as $t \to \infty$.

Thus, by virtue of the system configuration (a pure integrator with gain K), the steady-state error is zero.

Steady-State Errors for Test Inputs



Given a stable system represented by a closed-loop transfer function of the form

$$T(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

The error term $E(s)$ is determined using

$$E(s) = R(s) - C(s) = R(s) - T(s)R(s) = R(s)\left[1 - T(s)\right].$$

Using the Laplace transform final value theorem, the steady-state error is given by

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)\left[1 - T(s)\right].$$

Steady-State Errors for Step Inputs

The steady-state output of a stable system with closed-loop transfer function $T(s)$ due to a reference step input $r(t) = u(t)$ is

$$c_{SS}(t) = T(0) = \frac{b_0}{a_0}.$$

Therefore, the **steady-state error response** of a stable system with closed-loop transfer function $T(s)$ due to a reference step input $r(t) = u(t)$ is given by $e_{\text{step}}(t) = r(t) - c_{SS}(t)$, and the **steady-state error** is given by

$$e_{\text{step}}(\infty) = \lim_{t\to\infty} e_{\text{step}}(t) = \lim_{t\to\infty} r(t) - c_{SS}(t) = \boxed{1 - T(0) = 1 - \frac{b_0}{a_0}.}$$

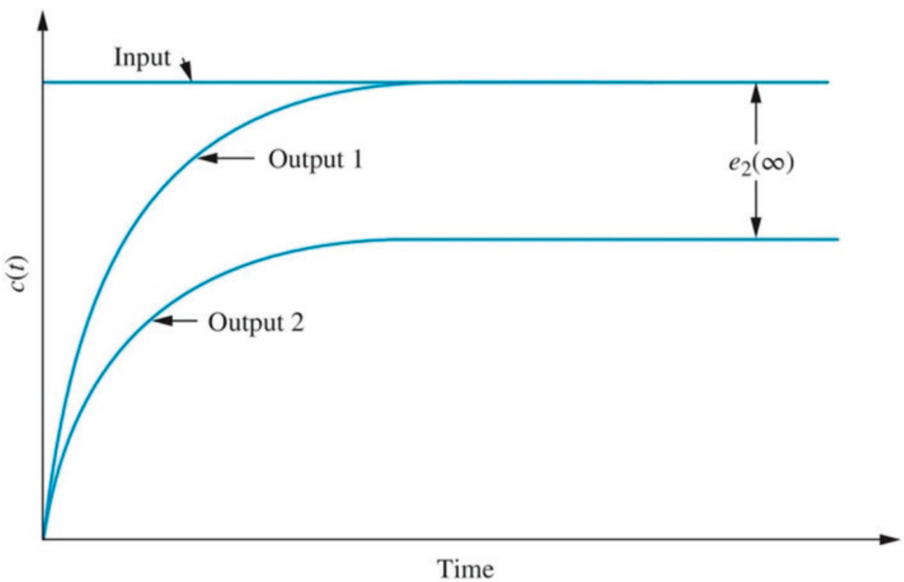
Steady-State Errors for Step Inputs

Consider the following cases for $T(0)$:

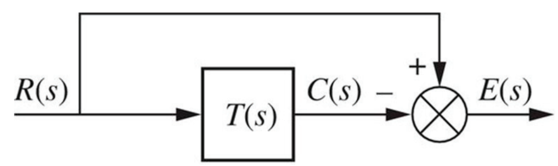
- $T(0) = 1$ (which implies that $b_0 = a_0$):
  - Implies $c_{SS}(t) = r(t)$.
  - Steady-state error is given by $e_{\text{step}}(\infty) = 0$.
  - An example of this case is shown by Output 1 on the following slide.

- $T(0) \neq 1$ (which implies that $b_0 \neq a_0$):
  - Implies $c_{SS}(t) \neq r(t)$.
  - Steady-state error is given by $e_{\text{step}}(\infty) =$ a nonzero constant.
  - An example of this case is shown by Output 2 on the following slide.

Steady-State Errors for Step Inputs



Steady-State Errors for Step Inputs: Example 1



Find the steady-state error for the system above if $T(s) = \frac{5}{s^2+7s+10}$ and the input is a unit step.

Steady-State Errors for Ramp Inputs

The steady-state output of a stable system with closed-loop transfer function $T(s)$ due to a reference ramp input $r(t) = tu(t)$ is

$$c_{SS}(t) = T(0)t + T'(0) = \frac{b_0}{a_0}t + \frac{a_0 b_1 - b_0 a_1}{a_0^2}.$$

Therefore, the **steady-state error response** of a stable system with closed-loop transfer function $T(s)$ due to a reference ramp input $r(t) = tu(t)$ is given by $e_{\text{ramp}}(t) = r(t) - c_{SS}(t)$, and the **steady-state error** is given by

$$e_{\text{ramp}}(\infty) = \lim_{t\to\infty} e_{\text{ramp}}(t) = \lim_{t\to\infty} r(t) - c_{SS}(t)$$

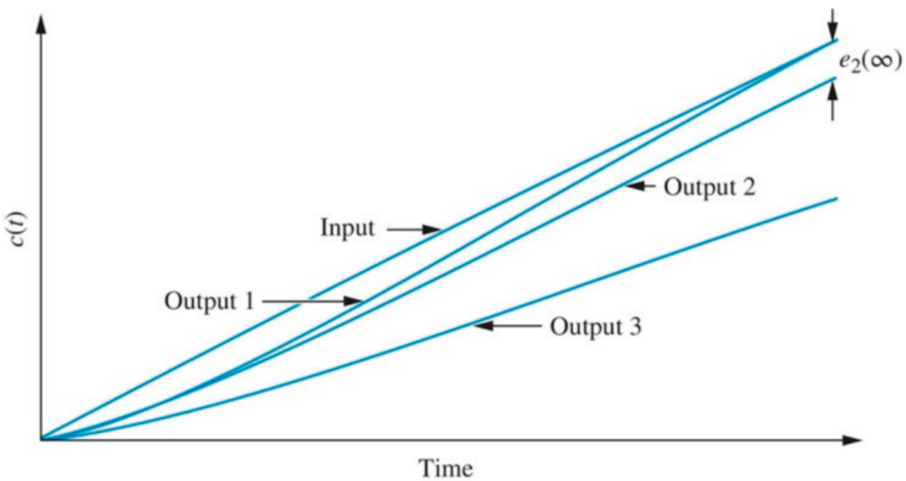
$$= \boxed{[1 - T(0)]\, t - T'(0) = \left(1 - \frac{b_0}{a_0}\right) t - \frac{a_0 b_1 - b_0 a_1}{a_0^2}.}$$

Steady-State Errors for Ramp Inputs

Consider the following cases for $T(0)$ and $T'(0)$:

- $T(0) = 1$ and $T'(0) = 0$ (which implies that $b_0 = a_0$ and $b_1 = a_1$):
  - Implies $c_{SS}(t) = r(t)$.
    ($c_{SS}(t)$ and $r(t)$ have the same slope and vertical-axis intercept).
  - Steady-state error is given by $e_{\text{ramp}}(\infty) = 0$.
  - An example of this case is shown by Output 1 on the following slide.
- $T(0) = 1$ and $T'(0) \neq 0$ (which implies that $b_0 = a_0$ and $b_1 \neq a_1$):
  - Implies $c_{SS}(t) \neq r(t)$, where $c_{SS}(t)$ and $r(t)$ have the same slope but different vertical-axis intercept.
  - Steady-state error is given by $e_{\text{ramp}}(\infty) =$ a nonzero constant.
  - An example of this case is shown by Output 2 on the following slide.
- $T(0) \neq 1$ (which implies that $b_0 \neq a_0$):
  - Implies $c_{SS}(t) \neq r(t)$, where $c_{SS}(t)$ and $r(t)$ have different slopes.
  - Steady-state error is given by $e_{\text{ramp}}(\infty) = \pm\infty$, depending on the sign of $1 - T(0)$.
  - An example of this case is shown by Output 3 on the following slide.

Steady-State Errors for Ramp Inputs



Steady-State Errors for Parabolic Inputs

The steady-state output of a stable system with closed-loop transfer function $T(s)$ due to a reference parabolic input $r(t) = \frac{1}{2}t^2 u(t)$ is

$$c_{SS}(t) = \frac{1}{2}T(0)t^2 + T'(0)t + \frac{1}{2}T''(0)$$
$$= \frac{b_0}{2a_0}t^2 + \frac{a_0 b_1 - b_0 a_1}{a_0^2}t + \frac{a_0^2\left[a_0 b_2 - a_2 b_0\right] - a_0 a_1\left[a_0 b_1 - a_1 b_0\right]}{a_0^4}.$$

Therefore, the **steady-state error response** of a stable system with closed-loop transfer function $T(s)$ due to a reference parabolic input $r(t) = \frac{1}{2}t^2 u(t)$ is given by $e_{\text{parabola}}(t) = r(t) - c_{SS}(t)$, and the **steady-state error** is given by

$$e_{\text{parabola}}(\infty) = \lim_{t\to\infty} e_{\text{parabola}}(t) = \lim_{t\to\infty} r(t) - c_{SS}(t)$$

$$= \boxed{\frac{1}{2}\left(1 - T(0)\right)t^2 - T'(0)t - \frac{1}{2}T''(0)}$$

$$= \frac{1}{2}\left(1 - \frac{b_0}{a_0}\right)t^2 - \frac{a_0 b_1 - b_0 a_1}{a_0^2}t - \frac{a_0^2\left[a_0 b_2 - a_2 b_0\right] - a_0 a_1\left[a_0 b_1 - a_1 b_0\right]}{a_0^4}.$$

Steady-State Errors for Parabolic Inputs

Consider the following cases for $T(0)$, $T'(0)$, and $T''(0)$:

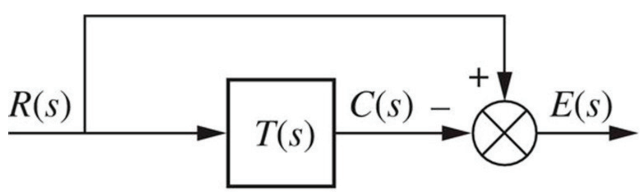
- $T(0) = 1$, $T'(0) = 0$, and $T''(0) = 0$ (implies $b_0 = a_0$, $b_1 = a_1$, and $b_2 = a_2$):
  - Implies $c_{SS}(t) = r(t)$.
  - Steady-state error is given by $e_{\text{parabola}}(\infty) = 0$.
- $T(0) = 1$, $T'(0) = 0$, and $T''(0) \neq 0$ (implies $b_0 = a_0$, $b_1 = a_1$, and $b_2 \neq a_2$):
  - Implies $c_{SS}(t) \neq r(t)$, where $c_{SS}(t)$ and $r(t)$ have a different vertical-axis intercept.
  - Steady-state error is given by $e_{\text{parabola}}(\infty) =$ a nonzero constant.
- $T(0) = 1$, $T'(0) \neq 0$ (which implies that $b_0 = a_0$ and $b_1 \neq a_1$):
  - Implies $c_{SS}(t) \neq r(t)$, where $c_{SS}(t)$ and $r(t)$ have different slopes.
  - Steady-state error is given by $e_{\text{parabola}}(\infty) = \pm\infty$ (depends on sign of $T'(0)$).
- $T(0) \neq 1$ (which implies that $b_0 \neq a_0$):
  - Implies $c_{SS}(t) \neq r(t)$
  - Steady-state error is given by $e_{\text{parabola}}(\infty) = \pm\infty$, depending on the sign of $1 - T(0)$.

Tracking Test Inputs

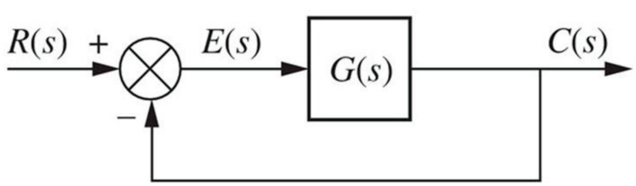
Consider a stable system with closed-loop transfer function $T(s)$.

1. Since $T(s)$ is stable, the system is capable of tracking any **zero** reference input. In other words, the steady-state error to a zero reference input is zero. Thus, the system will **recover** (or return to zero output) for finite duration disturbances. This is sometimes called **regulation**.
2. If $T(0) = 1$ (which implies that $a_0 = b_0$), then $e_{\text{step}}(\infty) = 0$, and the system will track any **step** reference input.
3. If $T(0) = 1$ and $T'(0) = 0$ (which implies that $a_0 = b_0$ and $a_1 = b_1$), then $e_{\text{ramp}}(\infty) = 0$, and the system will track any **ramp** reference input.
4. If $T(0) = 1$, $T'(0) = 0$, and $T''(0) = 0$ (which implies that $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$), then $e_{\text{parabola}}(\infty) = 0$, and the system will track any **parabolic** reference input.

Steady-State Errors for Unity-Feedback Systems

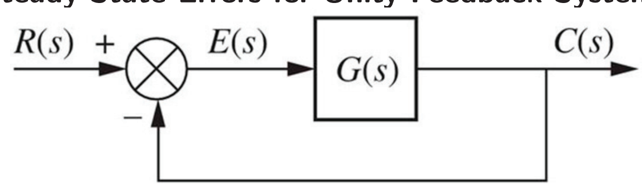


Steady-state errors can be calculated from the system closed-loop transfer function $T(s)$ as we have seen, or from the open-loop transfer function $G(s)$ for unity-feedback systems (see below).



Studying the unity-feedback open-loop transfer function leads to insight into the factors affecting steady-state errors.

Steady-State Errors for Unity-Feedback Systems



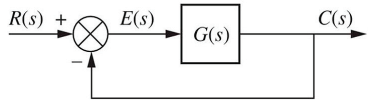
The error term is given in the complex domain as

$$E(s) = R(s) - C(s)$$
$$C(s) = E(s)G(s)$$
$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)},$$

and the steady-state error can be determined by the Laplace transform final value theorem as

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \boxed{\lim_{s\to 0} \frac{sR(s)}{1 + G(s)}}.$$

Steady-State Errors for Unity-Feedback Systems: Step Inputs



If the input signal is a unit step, then $r(t) = u(t) \xleftrightarrow{\mathcal{L}} R(s) = \frac{1}{s}$ and the error term is given in the complex domain as

$$E(s) = \frac{R(s)}{1+G(s)} = \frac{1}{s+sG(s)}.$$

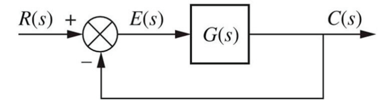
The steady-state error can be determined by the Laplace transform final value theorem as

$$e_{\text{step}}(\infty) = \lim_{t\to\infty} e_{\text{step}}(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+\lim\limits_{s\to 0} G(s)}.$$

In order to have $e_{\text{step}}(\infty) = 0$, then $\lim\limits_{s\to 0} G(s) = \infty$.

Thus, $G(s)$ must have at least 1 pole at $s = 0$ for $e_{\text{step}}(\infty) = 0$. In other words, $G(s)$ must have at least 1 pure integrator $\left(\frac{1}{s}\right)$ term.

Steady-State Errors for Unity-Feedback Systems: Ramp Inputs



If the input signal is a ramp, then $r(t) = tu(t) \xleftrightarrow{\mathcal{L}} R(s) = \frac{1}{s^2}$ and the error term is given in the complex domain as

$$E(s) = \frac{R(s)}{1+G(s)} = \frac{1}{s^2+s^2G(s)}.$$

The steady-state error can be determined by the Laplace transform final value theorem as

$$e_{\text{ramp}}(\infty) = \lim_{t\to\infty} e_{\text{ramp}}(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim\limits_{s\to 0} sG(s)}.$$

In order to have $e_{\text{ramp}}(\infty) = 0$, then $\lim\limits_{s\to 0} sG(s) = \infty$.

Thus, $G(s)$ must have at least 2 poles at $s = 0$ for $e_{\text{ramp}}(\infty) = 0$. In other words, $G(s)$ must have at least 2 pure integrator $\left(\frac{1}{s}\right)$ terms.

Steady-State Errors for Unity-Feedback Systems: Parabola Inputs

If the input signal is a parabola, then $r(t) = \frac{1}{2}t^2u(t) \xleftrightarrow{\mathcal{L}} R(s) = \frac{1}{s^3}$ and the error term is given in the complex domain as

$$E(s) = \frac{R(s)}{1+G(s)} = \frac{1}{s^3+s^3G(s)}.$$

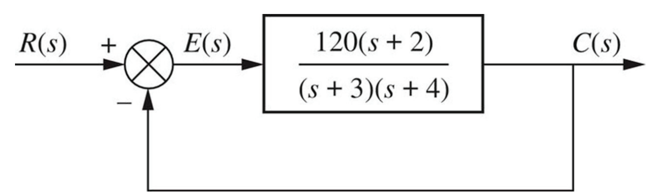
The steady-state error can be determined by the Laplace transform final value theorem as

$$e_{\text{parabola}}(\infty) = \lim_{t\to\infty} e_{\text{parabola}}(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim\limits_{s\to 0} s^2G(s)}.$$

In order to have $e_{\text{parabola}}(\infty) = 0$, then $\lim\limits_{s\to 0} s^2G(s) = \infty$.

Thus, $G(s)$ must have at least 3 poles at $s = 0$ for $e_{\text{parabola}}(\infty) = 0$. In other words, $G(s)$ must have at least 3 pure integrator $\left(\frac{1}{s}\right)$ terms.

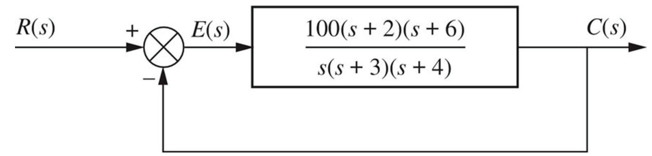
Steady-State Errors for Unity-Feedback Systems: Example 1 (No Pure Integrators)



Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ for the system above.

Steady-State Errors for Unity-Feedback Systems: Example 1 (Continued)

Steady-State Errors for Unity-Feedback Systems: Example 2 (One Pure Integrator)

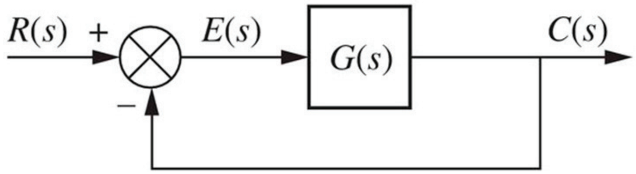


Find the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ for the system above.

Steady-State Errors for Unity-Feedback Systems: Example 2 (Continued)

Steady-State Errors for Unity-Feedback Systems: Example 2 (Continued)

Steady-State Response Specifications: Static Error Constants



We next define parameters that we can use as steady-state error performance specifications, just as we used ζ, ω_n, σ_d, ω_d, T_p, $\%OS$, T_s, or T_r for transient response performance specifications.

These **steady-state error performance specifications** are called **static error constants**.

Steady-State Response Specifications: Position Constant

Recall for a step input $r(t) = u(t)$:

$$e_{\mathsf{step}}(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

We define the **position constant** as

$$K_p = \lim_{s \to 0} G(s)$$

so that the steady-state error for a step input is given by

$$e_{\mathsf{step}}(\infty) = \frac{1}{1 + K_p}.$$

Notice that increasing the position constant, K_p, decreases the steady-error for a step input, $e_{\mathsf{step}}(\infty)$.

Steady-State Response Specifications: Velocity Constant

Recall for a ramp input $r(t) = tu(t)$:

$$e_{\mathsf{ramp}}(\infty) = \frac{1}{\lim_{s \to 0} sG(s)}$$

We define the **velocity constant** as

$$K_v = \lim_{s \to 0} sG(s)$$

so that the steady-state error for a ramp input is given by

$$e_{\mathsf{ramp}}(\infty) = \frac{1}{K_v}.$$

Notice that increasing the velocity constant, K_v, decreases the steady-error for a ramp input, $e_{\mathsf{ramp}}(\infty)$.

Steady-State Response Specifications: Acceleration Constant

Recall for a parabolic input $r(t) = \frac{1}{2}t^2u(t)$:

$$e_{\mathsf{parabola}}(\infty) = \frac{1}{\lim_{s \to 0} s^2G(s)}$$

We define the **acceleration constant** as

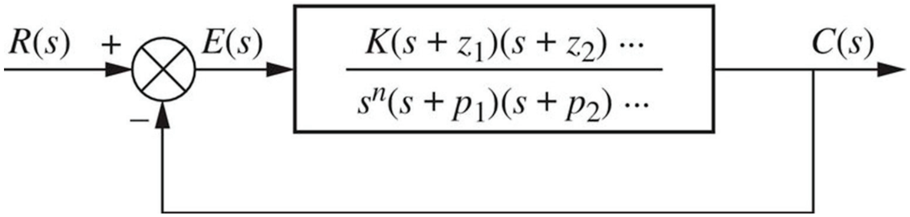
$$K_a = \lim_{s \to 0} s^2G(s)$$

so that the steady-state error for a parabola input is given by

$$e_{\mathsf{parabola}}(\infty) = \frac{1}{K_a}.$$

Notice that increasing the acceleration constant, K_a, decreases the steady-error for a parabola input, $e_{\mathsf{parabola}}(\infty)$.

Steady-State Response Specifications: System Type



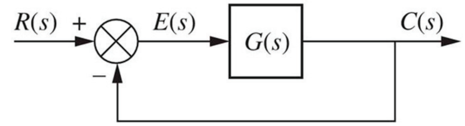
We continue with **unity-feedback** systems, and define **system type** to be the number of pure integrator terms in the forward path $G(s)$.

This corresponds to the number n shown above in the forward-path transfer function,

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}.$$

Note that this applies **only** to unity-feedback systems!

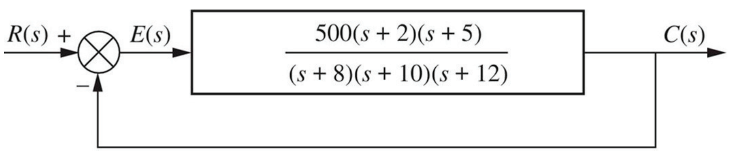
Steady-State Response Specifications: Summary



$$K_p = \lim_{s\to 0} G(s) \qquad K_v = \lim_{s\to 0} sG(s) \qquad K_a = \lim_{s\to 0} s^2G(s)$$

		Type 0		**Type 1**		**Type 2**	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Steady-State Response Specifications: Example 1



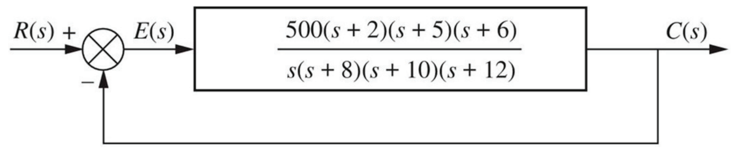
TYPE ___ SYSTEM:

Find the **static error constants** and the **steady-state error** of the above system for step, ramp, and parabolic inputs.

Steady-State Response Specifications: Example 1 (continued)

Steady-State Response Specifications: Example 1 (continued)

Steady-State Response Specifications: Example 2



TYPE ___ SYSTEM:

Find the **static error constants** and the **steady-state error** of the above system for step, ramp, and parabolic inputs.

Steady-State Response Specifications: Example 2 (continued)

Steady-State Response Specifications: Example 2 (continued)

Steady-State Response Specifications: Interpretations

The static error constants provide us with a lot of information about our system.

Suppose we are told a control system has specification $K_v = 1000$. We can draw several conclusions:

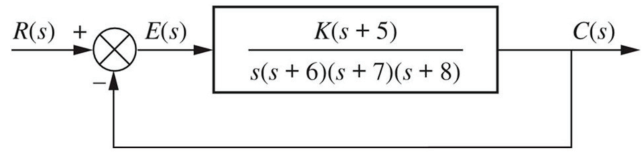
1. The system is stable.

2. The system is of Type 1, since only Type 1 systems have K_v's that are finite nonzero constants. Recall that $K_v = 0$ for Type 0 systems, whereas $K_v = \infty$ for Type 2 systems.

3. A ramp input is the test signal. Since K_v is specified as a finite constant, and the steady-state error for a ramp input is inversely proportional to K_v, we know the test input is a ramp.

4. The steady-state error between the input ramp and the output ramp is $\frac{1}{K_v} = 0.001$ per unit of input slope.

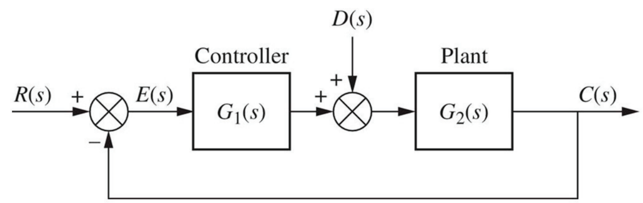
Steady-State Response Specifications: Gain Design Example 1



Given the control system above, find the value of K so that there is 10% error in the steady state.

Steady-State Response Specifications: Gain Design Example 1 (continued)

Steady-State Error for Disturbances



The output in the complex domain is given by

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s).$$

Note that $C(s) = R(s) - E(s)$.

Substituting the second equation into the first equation and solving for $E(s)$ yields

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s).$$

Steady-State Error for Disturbances

To find the steady-state value of the error, use the final value theorem:

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$
$$= \underbrace{\lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}X(s)}_{\text{Steady-State Error Due to Input}} - \underbrace{\lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)}_{\text{Steady-State Error Due to Disturbance}} = e_R(\infty)+e_D(\infty),$$

where

$$e_R(\infty) = \lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}X(s),$$

which is the steady-state error due to the input $R(s)$, and

$$e_D(\infty) = -\lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s),$$

which is the steady-state error due to the disturbance $D(s)$.

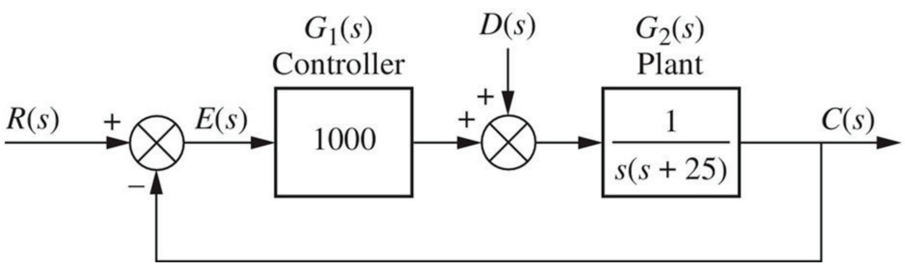
Steady-State Error for Disturbances

To investigate the conditions on $e_D(\infty)$ that must exist to reduce the disturbance, we assume $D(s) = \frac{1}{s}$, a unit step input. So,

$$e_D(\infty) = -\lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s) = -\lim_{s\to 0}\frac{G_2(s)}{1+G_1(s)G_2(s)}$$
$$= -\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)} + \lim_{s\to 0}G_1(s)}$$

Therefore, the steady-state error produced by a step disturbance can be reduced by increasing the DC gain of $G_1(s)$ or decreasing the DC gain of $G_2(s)$.

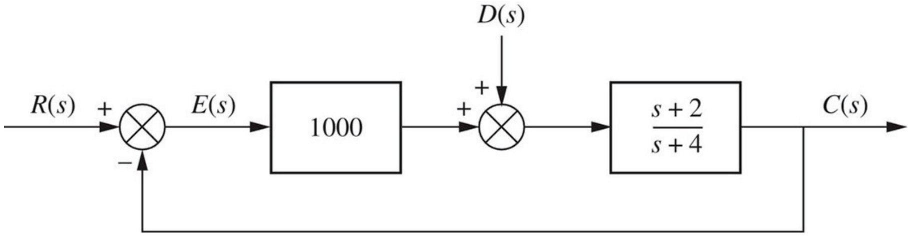
Steady-State Error for Disturbances: Example 1



Find the steady-state error component due to a step disturbance for the system above.

Steady-State Error for Disturbances: Example 1 (continued)

Steady-State Error for Disturbances: Example 2



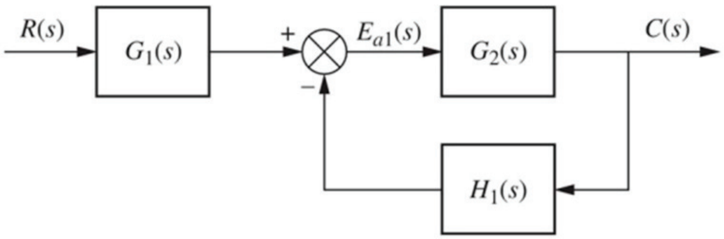
Find the steady-state error component due to a step disturbance for the system above.

Steady-State Error for Disturbances: Example 2 (continued)

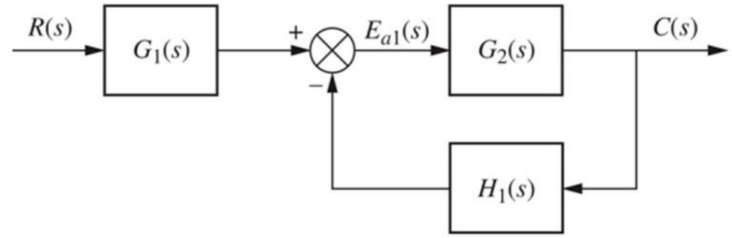
Steady-State Error for Non-Unity Feedback Systems

Recall that **system types** and **static error constants** were defined **only** for **unity feedback systems**.

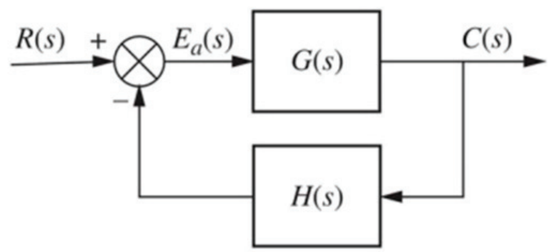
In order to derive a method for handling steady-state errors for non-unity feedback systems, we take a general non-unity feedback system and convert it to a unity feedback configuration.



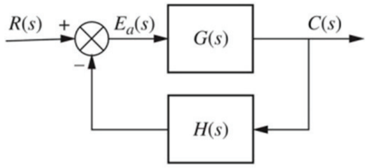
Steady-State Error for Non-Unity Feedback Systems



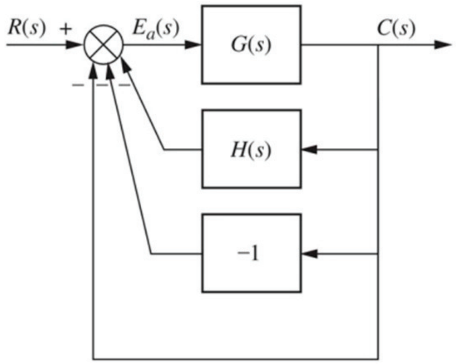
Use block diagram reduction techniques (move system $G_1(s)$ to the right of the summing junction) to convert the system above to the general non-unity feedback system below. Note that $G(s) = G_1(s)G_2(s)$ and $H(s) = H_1(s)/G_1(s)$.



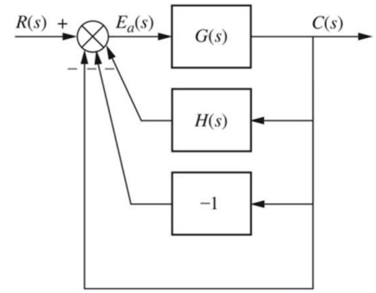
Steady-State Error for Non-Unity Feedback Systems



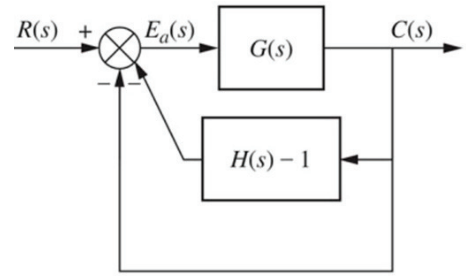
Add in a unity-feedback path. We also have to subtract a unity-feedback path to not change the overall system.



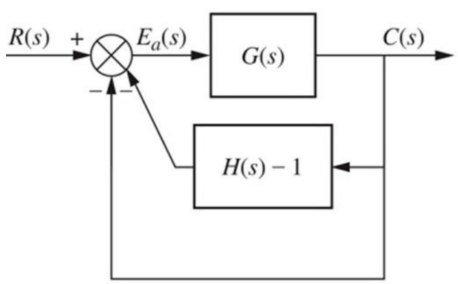
Steady-State Error for Non-Unity Feedback Systems



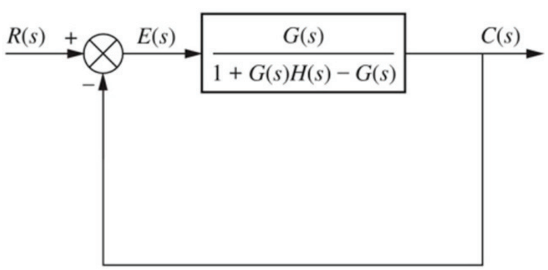
Combine the two parallel feedback paths in the middle.



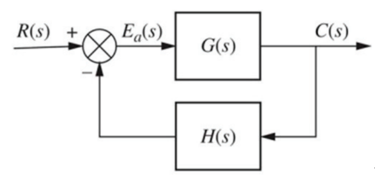
Steady-State Error for Non-Unity Feedback Systems



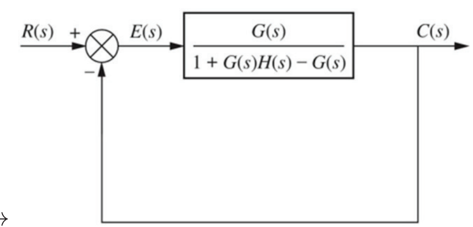
Reduce the inner feedback system to its closed-loop transfer function.



Steady-State Error for Non-Unity Feedback Systems



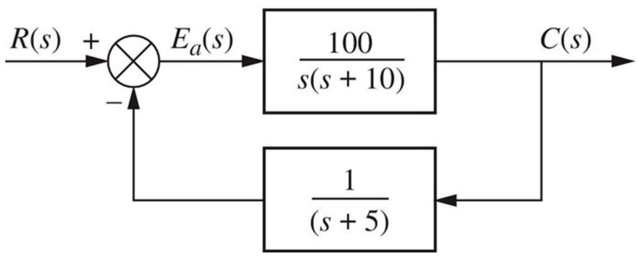
$\longleftrightarrow$



Therefore, we can convert a non-unity feedback system with forward-path transfer function $G(s)$ and feedback-path transfer function $H(s)$ to a unity feedback system with forward-path transfer function given by

$$G_{\text{new}}(s) = \frac{G(s)}{1 + G(s)\,[H(s) - 1]}.$$

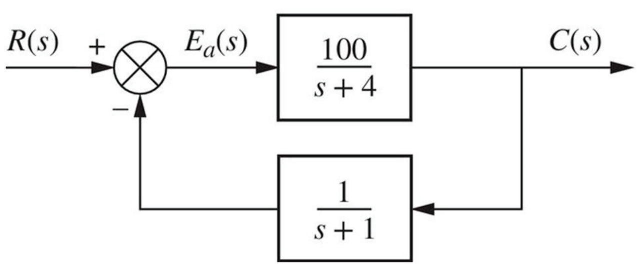
Steady-State Error for Non-Unity Feedback Systems: Example 1



For the system shown above, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input.

Steady-State Error for Non-Unity Feedback Systems: Example 1 (continued)

Steady-State Error for Non-Unity Feedback Systems: Example 2



For the system shown above, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input.

Steady-State Error for Non-Unity Feedback Systems: Example 2 (continued)