ELEC 309
Signals and Systems
Homework 1 Solutions

Time-Domain Analysis of Signals

1. Find the even and odd components of each of the following signals:

   Note that:

$$\begin{aligned}\textbf{an even signal} \times \textbf{an even signal} &= \textbf{an even signal,}\\ \textbf{an odd signal} \times \textbf{an odd signal} &= \textbf{an even signal, and}\\ \textbf{an even signal} \times \textbf{an odd signal} &= \textbf{an odd signal.}\end{aligned}$$

(a) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

Note that

$$\begin{aligned}x(-t) &= \cos(-t) + \sin(-t) + \sin(-t)\cos(-t)\\ &= \cos(t) - \sin(t) - \sin(t)\cos(t)\end{aligned}$$

and

$$\begin{aligned}x_e(t) &= \frac{1}{2}\left[x(t) + x(-t)\right] = \cos(t)\\ x_o(t) &= \frac{1}{2}\left[x(t) - x(-t)\right] = \sin(t) + \sin(t)\cos(t).\end{aligned}$$

(b) $x[n] = 1 + n + 3n^2 + 5n^3 + 9n^4$

Note that

$$x[-n] = 1 - n + 3n^2 - 5n^3 + 9n^4$$

and

$$\begin{aligned}x_e[n] &= \frac{1}{2}\left(x[n] + x[-n]\right) = 1 + 3n^2 + 9n^4\\ x_o[n] &= \frac{1}{2}\left(x[n] - x[-n]\right) = n + 5n^3\end{aligned}$$

(c) $x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$

Note that

$$\begin{aligned} x(-t) &= 1 - t\cos(-t) + t^2\sin(-t) - t^3\sin(-t)\cos(-t) \\ &= 1 - t\cos(t) - t^2\sin(t) + t^3\sin(t)\cos(t) \end{aligned}$$

and

$$\begin{aligned} x_e(t) &= \frac{1}{2}\left[x(t) + x(-t)\right) = 1 + t^3\sin(t)\cos(t) \\ x_o(t) &= \frac{1}{2}\left[x(t) - x(-t)\right) = t\cos(t) + t^2\sin(t) \end{aligned}$$

(d) $x[n] = (1 + n^3)\cos^3(10n)$

Note that

$$\begin{aligned} x[-n] &= \left(1 - n^3\right)\cos^3(-10n) \\ &= \left(1 - n^3\right)\cos^3(10n) \end{aligned}$$

and

$$\begin{aligned} x_e[n] &= \frac{1}{2}\left(x[n] + x[-n]\right) = \cos^3(10n) \\ x_o[n] &= \frac{1}{2}\left(x[n] - x[-n]\right) = n^3\cos^3(10n) \end{aligned}$$

2. For each of the following signals, determine whether it is periodic, and if it is, find the fundamental period:

(a) $x(t) = \cos^2(2\pi t)$

$$x(t) = \cos^2(2\pi t) = \frac{1}{2} + \frac{1}{2}\cos(4\pi t) \Longrightarrow \text{PERIODIC with fundamental period } T_0 = \frac{1}{2} = 0.5.$$

(b) $x(t) = \sin^3(2t)$

$$\begin{aligned} x(t) &= \sin^3(2t) = \sin(2t)\sin^2(2t) = \sin(2t)\left[\frac{1}{2} - \frac{1}{2}\cos(4t)\right] \\ &= \frac{1}{2}\sin(2t) - \frac{1}{2}\sin(2t)\cos(4t) = \frac{1}{2}\sin(2t) - \frac{1}{4}\left[\sin(6t) - \sin(2t)\right] \\ &= \frac{3}{4}\sin(2t) - \frac{1}{4}\sin(6t) \Longrightarrow \text{PERIODIC with fundamental period } T_0 = \pi. \end{aligned}$$

(c) $x(t) = e^{-2t} \cos(2\pi t)$

Due to the aperiodic term e^{-2t}, $x(t) = e^{-2t} \cos(2\pi t)$ is APERIODIC.

(d) $x[n] = (-1)^n$

$$x[n] = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases} \Longrightarrow \text{PERIODIC with fundamental period } N_0 = 2.$$

(e) $x[n] = (-1)^{n^2}$

$$x[n] = (-1)^{n^2} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases} \Longrightarrow \text{PERIODIC with fundamental period } N_0 = 2.$$

(f) $x[n] = \cos(2n)$

$$\frac{\Omega_0}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi} \neq \frac{m}{N_0} \Longrightarrow x[n] = \cos(2n) \text{ is APERIODIC.}$$

(g) $x[n] = \cos(2\pi n)$

$$\frac{\Omega_0}{2\pi} = \frac{2\pi}{2\pi} = \frac{1}{1} = \frac{m}{N_0} \Longrightarrow x[n] = \cos(2\pi n) \text{ is PERIODIC with fundamental period } N_0 = 1.$$

3. Categorize each of the following signals as an energy signal or a power signal, and find the energy or time-averaged power of the signal:

(a) $x(t) = 5\cos(\pi t) + \sin(5\pi t)$

It is easy to see that $x(t)$ is a periodic signal with finite energy content per period. Therefore, the energy of $x(t)$ is given by

$$E_x = \infty, \text{ and } x(t) \text{ is a power signal.}$$

The fundamental period of the cosine term is $T_1 = 2$, and the fundamental period of the sine term is $T_2 = \frac{2}{5}$. Therefore,

$$\frac{T_1}{T_2} = \frac{2}{\frac{2}{5}} = 5.$$

Cross-multiplying, we see that the fundamental period is given by

$$T_0 = T_1 = 5T_2 = 2.$$

Note that

$$\begin{aligned}
|x(t)|^2 = x^2(t) &= [5\cos(\pi t) + \sin(5\pi t)]^2 \\
&= 25\cos^2(\pi t) + 10\cos(\pi t)\sin(5\pi t) + \sin^2(5\pi t) \\
&= 25\left[\frac{1}{2} + \frac{1}{2}\cos(2\pi t)\right] + 10\left[\frac{1}{2}\sin(4\pi t) + \frac{1}{2}\sin(6\pi t)\right] + \frac{1}{2} - \frac{1}{2}\sin(10\pi t) \\
&= 13 + \frac{25}{2}\cos(2\pi t) + 5\sin(4\pi t) + 5\sin(6\pi t) - \frac{1}{2}\sin(10\pi t)
\end{aligned}$$

Since $x(t)$ is periodic, the power of $x(t)$ is given by

$$\begin{aligned}
P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}|x(t)|^2 dt = \frac{1}{2}\int_{-1}^{1} x^2(t)dt \\
&= \frac{1}{2}\int_{-1}^{1} 13 + \frac{25}{2}\cos(2\pi t) + 5\sin(4\pi t) + 5\sin(6\pi t) - \frac{1}{2}\sin(10\pi t)\,dt \\
&= \frac{13}{2}\cancelto{2}{\int_{-1}^{1} dt} + \frac{25}{4}\cancelto{0}{\int_{-1}^{1}\cos(2\pi t)\,dt} + \frac{5}{2}\cancelto{0}{\int_{-1}^{1}\sin(4\pi t)\,dt} + \frac{5}{2}\cancelto{0}{\int_{-1}^{1}\sin(6\pi t)\,dt} - \frac{1}{4}\cancelto{0}{\int_{-1}^{1}\sin(10\pi t)\,dt} \\
&= 13.
\end{aligned}$$

(b) $x(t) = \begin{cases} 5\cos(\pi t) & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

It is easy to see that $x(t)$ is an aperiodic signal with finite energy. Therefore, the power of $x(t)$ is given by

$$P_x = 0, \text{ and } x(t) \text{ is an energy signal.}$$

Note that

$$|x(t)|^2 = x^2(t) = 25\cos^2(\pi t) = 25\left[\frac{1}{2} + \frac{1}{2}\cos(2\pi t)\right] = \frac{25}{2} + \frac{25}{2}\cos(2\pi t).$$

The energy of $x(t)$ is given by

$$E_x = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \int_{-1}^{1} x^2(t) dt = \int_{-1}^{1} \frac{25}{2} + \frac{25}{2}\cos(2\pi t)\, dt = \frac{25}{2}\cancelto{2}{\int_{-1}^{1} dt} + \frac{25}{2}\cancelto{0}{\int_{-1}^{1}\cos(2\pi t)\, dt} = 25.$$

(c) $x[n] = \begin{cases} \sin(\pi n) & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$

Note that

$$x[n] = \begin{cases} \sin(\pi n) & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \cancelto{0}{\sin(-4\pi)} & n = -4 \\ \cancelto{0}{\sin(-3\pi)} & n = -3 \\ \cancelto{0}{\sin(-2\pi)} & n = -2 \\ \cancelto{0}{\sin(\pi)} & n = -1 \\ \cancelto{0}{\sin(0)} & n = 0 \\ \cancelto{0}{\sin(\pi)} & n = 1 \\ \cancelto{0}{\sin(2\pi)} & n = 2 \\ \cancelto{0}{\sin(3\pi)} & n = 3 \\ \cancelto{0}{\sin(4\pi)} & n = 4 \\ 0 & \text{otherwise} \end{cases} = 0.$$

Therefore, $x[n]$ is a zero signal ($E_x = 0$ and $P_x = 0$).

(d) $x[n] = \begin{cases} \cos(\pi n) & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$

Note that

$$x[n] = \begin{cases} \cos(\pi n) & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \cancelto{1}{\cos(-4\pi)} & n = -4 \\ \cancelto{-1}{\cos(-3\pi)} & n = -3 \\ \cancelto{1}{\cos(-2\pi)} & n = -2 \\ \cancelto{-1}{\cos(\pi)} & n = -1 \\ \cancelto{1}{\cos(0)} & n = 0 \\ \cancelto{-1}{\cos(\pi)} & n = 1 \\ \cancelto{1}{\cos(2\pi)} & n = 2 \\ \cancelto{-1}{\cos(3\pi)} & n = 3 \\ \cancelto{1}{\cos(4\pi)} & n = 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (-1)^n & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $x[n]$ is an aperiodic signal with finite energy. Therefore, the power of $x[n]$ is given by

$$P_x = 0, \text{ and } x[n] \text{ is an energy signal.}$$

Note that

$$|x[n]|^2 = x^2[n] = \begin{cases} 1 & -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

The energy of $x(t)$ is given by

$$E_x = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \sum_{n=-4}^{4} x^2[n] = \sum_{n=-4}^{4} 1 = 9.$$

(e) $x[n] = \begin{cases} \cos(\pi n) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

It is easy to see that $x[n]$ is an aperiodic signal generated by removing the negative-time portion of a periodic signal with finite energy content per period. Therefore, the energy of $x[n]$ is given by

$$E_x = \infty, \text{ and } x[n] \text{ is a power signal.}$$

The fundamental period of the generating periodic signal is $T_0 = 2$. Note that

$$|x[n]|^2 = x^2[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The power of $x[n]$ is given by

$$\begin{aligned} P_x &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x[n]\right|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n] = \lim_{N\to\infty} \frac{1}{2N+1} \cancelto{N+1}{\sum_{n=0}^{N} 1} \\ &= \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2} = 0.5. \end{aligned}$$

4. Let

$$x[n] = \begin{cases} n & \text{for } n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Determine $y[n] = x[2n]$.

Note that:

$$\begin{aligned} &\vdots \\ y[-2] &= x[-4] = 0 \\ y[-1] &= x[-2] = 0 \\ y[0] &= x[0] = 0 \\ y[1] &= x[2] = 0 \\ y[2] &= x[4] = 0 \\ &\vdots \end{aligned}$$

Therefore,

$$y[n] = 0 \text{ for all } n.$$

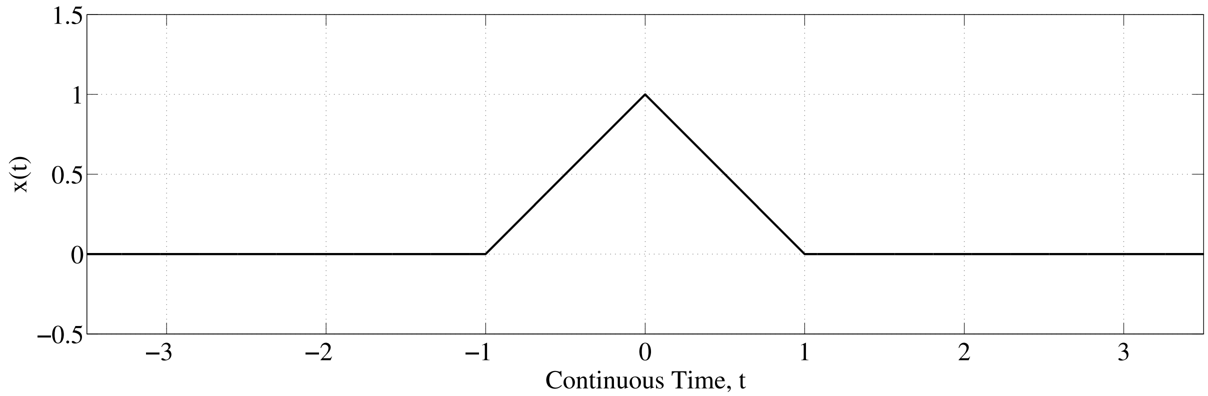
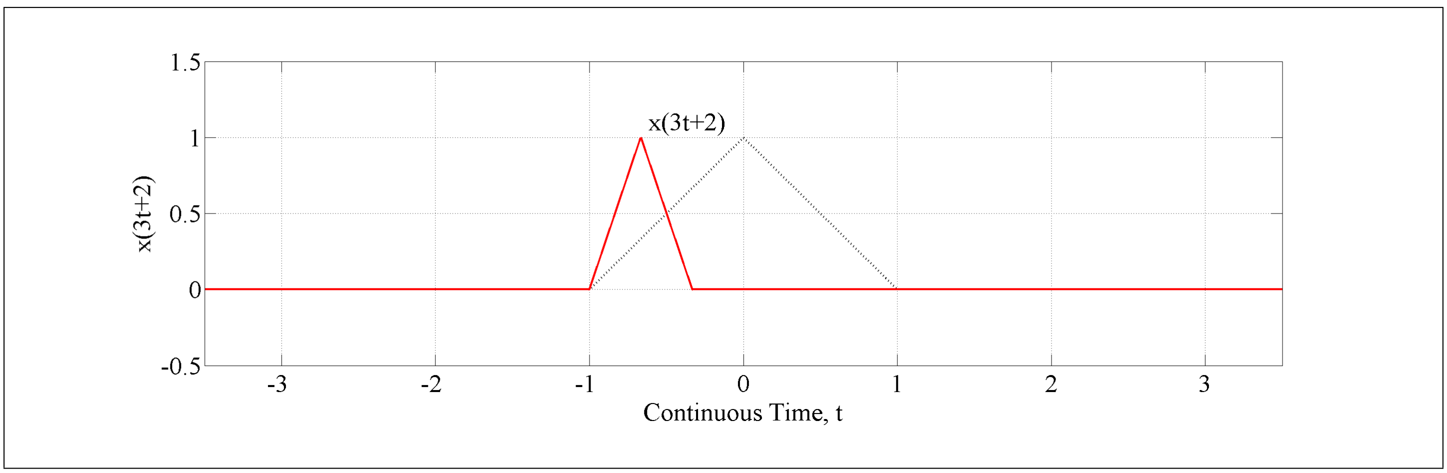


Figure 1: Triangular Pulse Signal

5. A triangular pulse signal $x(t)$ is depicted in Figure 1. Sketch each of the following signals derived from $x(t)$:

(a) $x(3t+2)$



(b) $x(-2t-1)$