4

Convolution

In Lecture 3 we introduced and defined a variety of system properties to which we will make frequent reference throughout the course. Of particular importance are the properties of linearity and time invariance, both because systems with these properties represent a very broad and useful class and because with just these two properties it is possible to develop some extremely powerful tools for system analysis and design.

A linear system has the property that the response to a linear combination of inputs is the same linear combination of the individual responses. The property of time invariance states that, in effect, the system is not sensitive to the time origin. More specifically, if the input is shifted in time by some amount, then the output is simply shifted by the same amount.

The importance of linearity derives from the basic notion that for a linear system if the system inputs can be decomposed as a linear combination of some basic inputs and the system response is known for each of the basic inputs, then the response can be constructed as the same linear combination of the responses to each of the basic inputs. Signals (or functions) can be decomposed as a linear combination of basic signals in a wide variety of ways. For example, we might consider a Taylor series expansion that expresses a function in polynomial form. However, in the context of our treatment of signals and systems, it is particularly important to choose the basic signals in the expansion so that in some sense the response is easy to compute. For systems that are both linear and time-invariant, there are two particularly useful choices for these basic signals: delayed impulses and complex exponentials. In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems. The resulting representation is referred to as *convolution*. Later in this series of lectures we develop in detail the decomposition of signals as linear combinations of complex exponentials (referred to as *Fourier analysis*) and the consequence of that representation for linear, time-invariant systems.

In developing convolution in this lecture we begin with the representation of discrete-time signals and linear combinations of delayed impulses. As we discuss, since arbitrary sequences can be expressed as linear combinations of delayed impulses, the output for linear, time-invariant systems can be expressed as the same linear combination of the system response to a delayed impulse. Specifically, because of time invariance, once the response to one impulse at any time position is known, then the response to an impulse at any other arbitrary time position is also known.

In developing convolution for continuous time, the procedure is much the same as in discrete time although in the continuous-time case the signal is represented first as a linear combination of narrow rectangles (basically a staircase approximation to the time function). As the width of these rectangles becomes infinitesimally small, they behave like impulses. The superposition of these rectangles to form the original time function in its limiting form becomes an integral, and the representation of the output of a linear, time-invariant system as a linear combination of delayed impulse responses also becomes an integral. The resulting integral is referred to as the *convolution integral* and is similar in its properties to the convolution sum for discrete-time signals and systems. A number of the important properties of convolution that have interpretations and consequences for linear, time-invariant systems are developed in Lecture 5. In the current lecture, we focus on some examples of the evaluation of the convolution sum and the convolution integral.

Suggested Reading

Section 3.0, Introduction, pages 69–70

Section 3.1, The Representation of Signals in Terms of Impulses, pages 70–75

Section 3.2, Discrete-Time LTI Systems: The Convolution Sum, pages 75–84

Section 3.3, Continuous-Time LTI Systems: The Convolution Integral, pages 88 to mid-90

MARKERBOARD 4.1

System Properties

$x(t) \longrightarrow \boxed{} \longrightarrow y(t)$

$x[n] \qquad\qquad\qquad y[n]$

- Memory
- Invertibility
- Causality
- Stability
- Time Invariance
- Linearity

Time-Invariance

C-T:

$x(t) \rightarrow y(t)$

Then

$x(t-t_0) \rightarrow y(t-t_0)$ any t_0

D-T:

$x[n] \rightarrow y[n]$

$x[n-n_0] \rightarrow y[n-n_0]$ any n_0

Linearity

$\phi_k \rightarrow \psi_k$

Then

$a_1\phi_1 + a_2\phi_2 + \dots \rightarrow a_1\psi_1 + a_2\psi_2 + \dots$

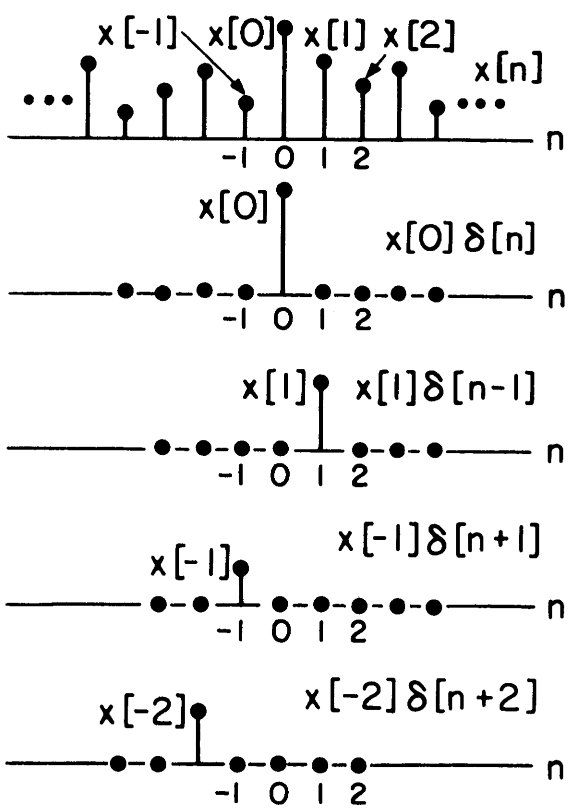
STRATEGY:

- decompose input signal into a linear combination of basic signals
- choose basic signals so that response easy to compute

LTI Systems:

delayed impulses $\Longleftrightarrow$ Convolution

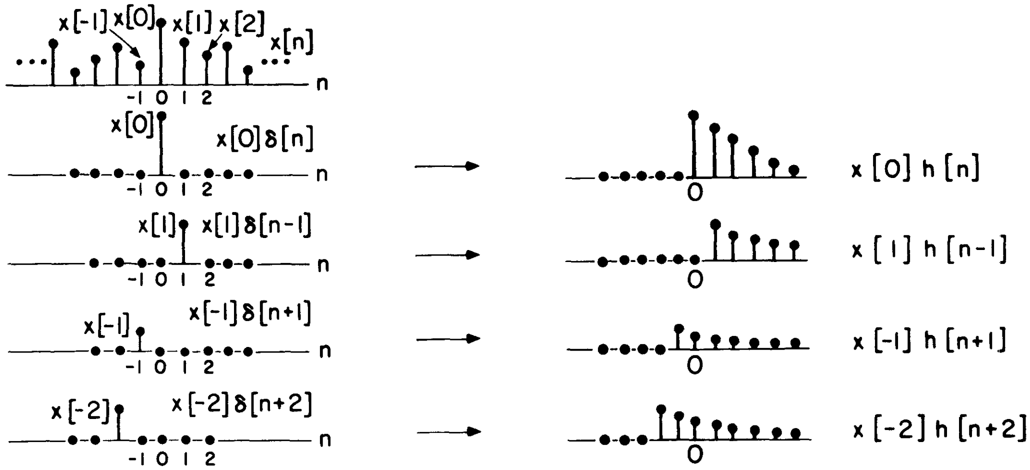
complex exponentials $\Longleftrightarrow$ Fourier Analysis



$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[-1]\delta[n+1] + \cdots = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

TRANSPARENCY 4.1
A general discrete-time signal expressed as a superposition of weighted, delayed unit impulses.

TRANSPARENCY 4.2
The convolution sum for linear, time-invariant discrete-time systems expressing the system output as a weighted sum of delayed unit impulse responses.



TRANSPARENCY 4.3
One interpretation of the convolution sum for an LTI system. Each individual sequence value can be viewed as triggering a response; all the responses are added to form the total output.

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

Linear System:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h_k[n]$$

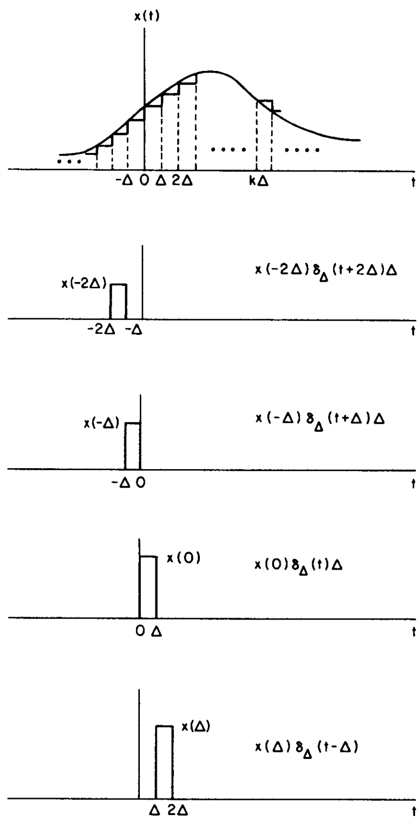
$$\delta[n-k] \rightarrow h_k[n]$$

If Time-Invariant:

$$h_k[n] = h_o[n-k]$$

$$\text{LTI: } \boxed{y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]}$$

Convolution Sum



TRANSPARENCY 4.4
Approximation of a continuous-time signal as a linear combination of weighted, delayed, rectangular pulses. [The amplitude of the fourth graph has been corrected to read $x(0)$.]

$$x(t) \cong x(o)\,\delta_\Delta(t)\,\Delta + x(\Delta)\,\delta_\Delta(t-\Delta)\,\Delta + x(-\Delta)\,\delta_\Delta(t+\Delta)\,\Delta + \dots$$

$$x(t) \cong \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

TRANSPARENCY 4.5
As the rectangular pulses in Transparency 4.4 become increasingly narrow, the representation approaches an integral, often referred to as the sifting integral.

TRANSPARENCY 4.6
Derivation of the convolution integral representation for continuous-time LTI systems.

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \delta_{\Delta}(t - k\Delta)\, \Delta$$

Linear System:

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\; h_{k\Delta}(t)\, \Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, h_{\tau}(t)\, d\tau$$

If Time-Invariant:

$$h_{k\Delta}(t) = h_o(t - k\Delta)$$

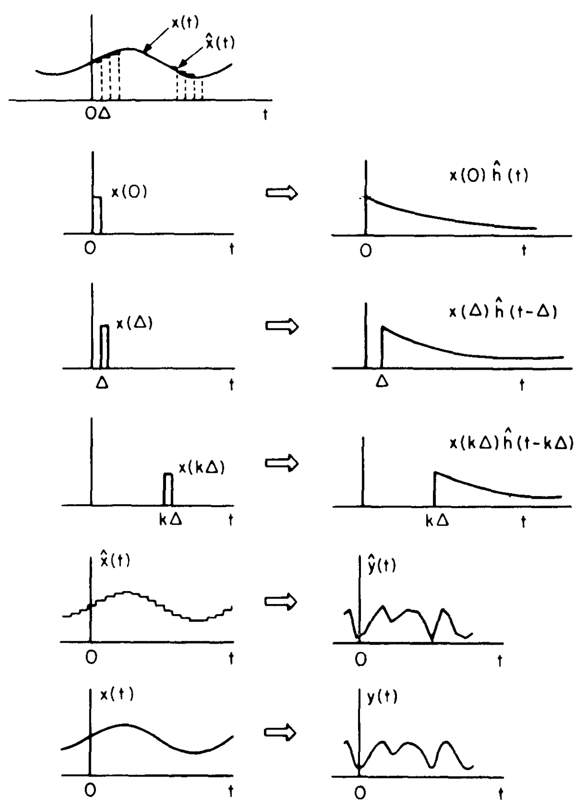
$$h_{\tau}(t) = h_o(t - \tau)$$

LTI:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau}$$

Convolution Integral

TRANSPARENCY 4.7
Interpretation of the convolution integral as a superposition of the responses from each of the rectangular pulses in the representation of the input.



Convolution Sum:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k] = x[n] * h[n]$$

Convolution Integral:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

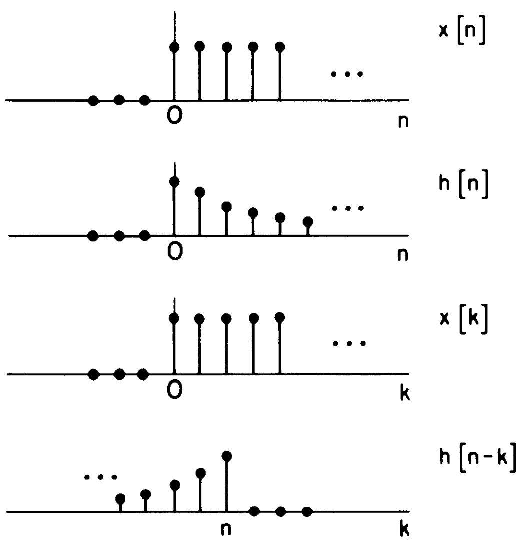
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

TRANSPARENCY 4.8
Comparison of the convolution sum for discrete-time LTI systems and the convolution integral for continuous-time LTI systems.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

$x[n] = u[n]$

$h[n] = \alpha^n u[n]$



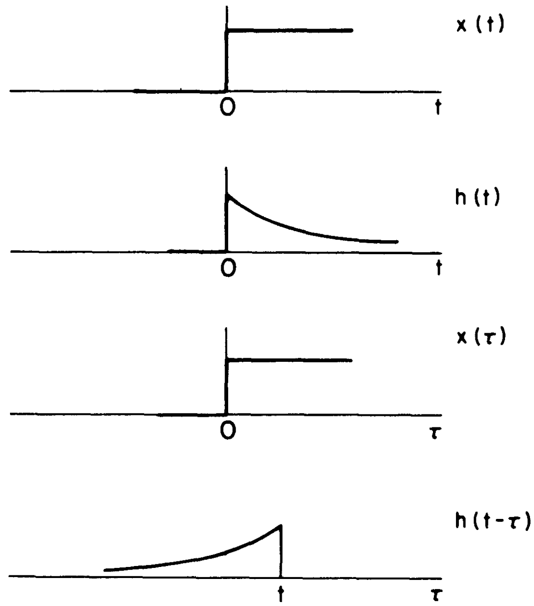
TRANSPARENCY 4.9
Evaluation of the convolution sum for an input that is a unit step and a system impulse response that is a decaying exponential for $n > 0$.

TRANSPARENCY 4.10
Evaluation of the convolution integral for an input that is a unit step and a system impulse response that is a decaying exponential for $t > 0$.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$x(t) = u(t)$

$h(t) = e^{-at}\, u(t)$

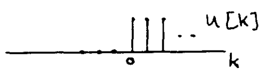


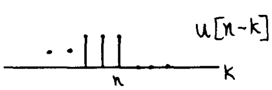
MARKERBOARD 4.2

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} u[k]\, \alpha^{n-k}\, u[n-k]$$

Interval 1: $n<0$

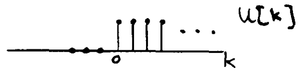




No overlap $\Longrightarrow$

$y[n]=0 \quad n<0$

Interval 2: $n>0$



overlap for $k=0,1,\dots n$

$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

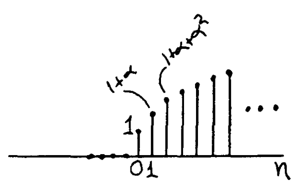
$$= \alpha^{n} \sum_{k=0}^{n} (\alpha^{-1})^{k}$$

$$\sum_{k=0}^{r} \beta^{k} = \frac{1-\beta^{r+1}}{1-\beta}$$

$$y[n] = \alpha^{n} \underbrace{\sum_{k=0}^{n} (\alpha^{-1})^{k}}_{\frac{1-(\alpha^{-1})^{n+1}}{1-\alpha^{-1}}}$$

$\vdots$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} \quad n>0$$



MARKERBOARD 4.3

$$y(t)=\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$=\int_{-\infty}^{+\infty} u(\tau)\, e^{-a(t-\tau)}\, u(t-\tau)\, d\tau$$

Interval 1: $t<0$

No overlap between $u(\tau)$ & $u(t-\tau) \Longrightarrow$

$$y(t)=0 \qquad t<0$$

Interval 2: $t>0$

$$y(t)=\int_{-\infty}^{+\infty} u(\tau)\, e^{-a(t-\tau)}\, u(t-\tau)\, d\tau$$

$u(\tau)u(t-\tau) = 1$

for $\quad 0 \le \tau \le t$

$$y(t)=\int_{0}^{t} e^{-a(t-\tau)}\, d\tau$$

$$=e^{-at}\underbrace{\int_{0}^{t} e^{a\tau}\, d\tau}_{\frac{1}{a}\left[e^{at}-1\right]}$$

$$=\frac{1}{a}\left[1-e^{-at}\right] \qquad t>0$$

$$y(t)=\begin{cases} 0 & t<0 \\ \frac{1}{a}\left[1-e^{-at}\right] & t>0 \end{cases}$$