Convolution - Analog*

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Abstract

Introduces convolution for analog signals.

In this module we examine convolution for continuous time signals. This will result in the convolution integral and its properties[1]. These concepts are very important in Electrical Engineering and will make any engineer's life a lot easier if the time is spent now to truly understand what is going on.

In order to fully understand convolution, you may find it useful to look at the discrete-time convolution[2] as well. Accompanied to this module there is a fully worked out example[3] with mathematics and figures. It will also be helpful to experiment with the Convolution - Continuous time[4] applet available from Johns Hopkins University[5] . These resources offers different approaches to this crucial concept.

1 Derivation of the convolution integral

The derivation used here closely follows the one discussed in the motivation section above. To begin this, it is necessary to state the assumptions we will be making. In this instance, the only constraints on our system are that it be linear and time-invariant.

Brief Overview of Derivation Steps:

1. An impulse input leads to an impulse response output.
2. A shifted impulse input leads to a shifted impulse response output. This is due to the time-invariance of the system.
3. We now scale the impulse input to get a scaled impulse output. This is using the scalar multiplication property of linearity.
4. We can now "sum up" an infinite number of these scaled impulses to get a sum of an infinite number of scaled impulse responses. This is using the additivity attribute of linearity.
5. Now we recognize that this infinite sum is nothing more than an integral, so we convert both sides into integrals.
6. Recognizing that the input is the function $f(t)$, we also recognize that the output is exactly the convolution integral.

*Version 1.7: Jan 9, 2004 12:49 am +0000

[1]"Properties of Continuous Time Convolution" <http://cnx.org/content/m10088/latest/>
[2]"Discrete Time Convolution" <http://cnx.org/content/m10087/latest/>
[3]"Convolution - Complete example" <http://cnx.org/content/m11541/latest/>
[4]http://www.jhu.edu/~signals/convolve/
[5]http://www.jhu.edu

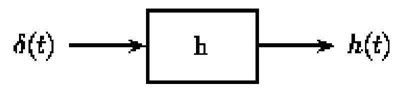

Figure 1: We begin with a system defined by its impulse response, $h(t)$.

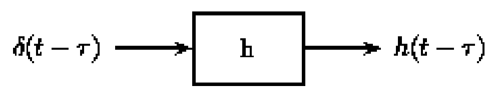

Figure 2: We then consider a shifted version of the input impulse. Due to the time invariance of the system, we obtain a shifted version of the output impulse response.

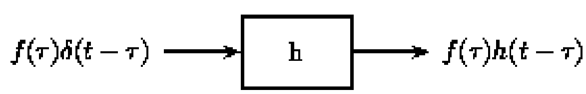

Figure 3: Now we use the scaling part of linearity by scaling the system by a value, $f(\tau)$, that is constant with respect to the system variable, t.

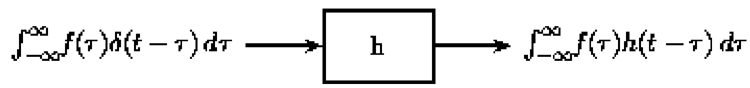

Figure 4: We can now use the additivity aspect of linearity to add an infinite number of these, one for each possible τ. Since an infinite sum is exactly an integral, we end up with the integration known as the Convolution Integral. Using the sampling property[6], we recognize the left-hand side simply as the input $f(t)$.

2 Convolution Integral

As mentioned above, the convolution integral provides an easy mathematical way to express the output of an LTI system based on an arbitrary signal, $x(t)$, and the system's impulse response, $h(t)$. The **convolution integral** is expressed as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad (1)$$

Convolution is such an important tool that it is represented by the symbol *, and can be written as

$$y(t) = x(t) * h(t) \qquad (2)$$

By making a simple change of variables into the convolution integral, $\tau = t - \tau$, we can easily show that convolution is **commutative**:

$$x(t) * h(t) = h(t) * x(t) \qquad (3)$$

which gives an equivivalent form of (1)

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \qquad (4)$$

For more information on the characteristics of the convolution integral, read about the Properties of Convolution[7].

3 Implementation of Convolution

Taking a closer look at the convolution integral, we find that we are multiplying the input signal by the time-reversed impulse response and integrating. This will give us the value of the output at one given value of t. If we then shift the time-reversed impulse response by a small amount, we get the output for another value of t. Repeating this for every possible value of t, yields the total output function. While we would never actually do this computation by hand in this fashion, it does provide us with some insight into what is actually happening. We find that we are essentially reversing the impulse response function and sliding it across the input function, integrating as we go. This method, referred to as the **graphical method**, provides us with a much simpler way to solve for the output for simple (contrived) signals, while improving our intuition for the more complex cases where we rely on computers. In fact Texas Instruments[8] develops Digital Signal Processors[9] which have special instruction sets for computations such as convolution.

[6] "Analog signals": Section The (Dirac) delta function <http://cnx.org/content/m11478/latest/#s2s1>
[7] "Properties of Continuous Time Convolution" <http://cnx.org/content/m10088/latest/>
[8] http://www.ti.com
[9] http://dspvillage.ti.com/docs/toolssoftwarehome.jhtml

4 Summary

Convolution is a truly important concept, which **must** be well understood.

NOTE: $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

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Go to? Introduction[10]; Convolution - Full example[11]; Convolution - Discrete time[12]; Properties of convolution[13]

[10]"Introduction to Convolution" <http://cnx.org/content/m11542/latest/>
[11]"Convolution - Complete example" <http://cnx.org/content/m11541/latest/>
[12]"Discrete Time Convolution" <http://cnx.org/content/m10087/latest/>
[13]"Properties of Continuous Time Convolution" <http://cnx.org/content/m10088/latest/>