Math 335, Fall 2014 Exam 1 Key

1.) *[5 points]* Compute the curl of $\vec{F} = \langle 4x, z^3, x\cos y\rangle$.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x & z^3 & x\cos y \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y} x\cos y - \frac{\partial}{\partial z} z^3\right) - j\left(\frac{\partial}{\partial x} x\cos y - \frac{\partial}{\partial z} 4x\right) + k\left(\frac{\partial}{\partial x} z^3 - \frac{\partial}{\partial y} 4x\right)$$

$$= i(-x\sin y - 3z^2) - j(\cos y - 0) + k(0 - 0)$$

$$= \boxed{\langle -x\sin y - 3z^2, -\cos y, 0\rangle}$$

2.) *[5 points]* Charmander starts at the point (10,1,4) and walks with velocity function

$$\vec{v}(t) = \langle 3t, \sin 4t, e^{2t}\rangle.$$

Find Charmander's position function $\vec{r}(t)$.

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \frac{3}{2}t^2 + C_1, -\frac{1}{4}\cos 4t + C_2, \frac{1}{2}e^{2t} + C_3\right\rangle$$

At time $t = 0$

$$\vec{r}(0) = (10, 1, 4) = \left\langle \frac{3}{2}0^2 + C_1, -\frac{1}{4}\cos 0 + C_2, \frac{1}{2}e^0 + C_3\right\rangle$$

$$(10, 1, 4) = \left\langle C_1, -\frac{1}{4} + C_2, \frac{1}{2} + C_3\right\rangle$$

$$10 = C_1,\quad 1 = -\frac{1}{4} + C_2 \Rightarrow C_2 = \frac{5}{4},\quad 4 = \frac{1}{2} + C_3 \Rightarrow C_3 = \frac{7}{2}$$

$$\boxed{\vec{r}(t) = \left\langle \frac{3}{2}t^2 + 10, -\frac{1}{4}\cos 4t + \frac{5}{4}, \frac{1}{2}e^{2t} + \frac{7}{2}\right\rangle}$$

3.) *[10 points]* A straight piece of wire extends from the point (2,2,3) to (−3,2,5). The linear density in kg/m of the wire is given by

$$\rho(x, y, z) = x + y^2 z.$$

Calculate the mass of the wire.

$\vec{r}(t) = \langle 2-5t,\ 2,\ 3+2t \rangle \qquad 0 \le t \le 1$

$\vec{v}(t) = \vec{r}\,'(t) = \langle -5,\ 0,\ 2 \rangle$

$\|\vec{v}(t)\| = \sqrt{(-5)^2 + 0^2 + 2^2} = \sqrt{29}$

$\text{Mass} = \int_C x + y^2 z \, ds$

$= \int_0^1 (2-5t) + (2)^2(3+2t)\sqrt{29}\, dt$

$= \sqrt{29} \int_0^1 2 - 5t + 12 + 8t \, dt$

$= \sqrt{29} \int_0^1 3t + 14 \, dt$

$= \sqrt{29} \left[\frac{3}{2} t^2 + 14t \right]_0^1$

$= \sqrt{29} \left[\frac{3}{2} + 14 \right]$

$= \boxed{\dfrac{31\sqrt{29}}{2}}$

4.) *[10 points]I* Let S be the surface composed of all 6 sides of the box

$$0 \le x \le 1, \quad 0 \le y \le 2, \quad 0 \le z \le 3.$$

Compute the outward flux through S of the vector field

$$\vec{F} = \langle x^2 y,\ 4x,\ 2yz \rangle.$$

Closed 3D surface $\Rightarrow$ Divergence Theorem!

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(4x) + \frac{\partial}{\partial z}(2yz)$$

$$= 2xy + 0 + 2y = 2y(x+1)$$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_R \nabla \cdot \vec{F}\, dV$$

$$= \int_0^1 \int_0^2 \int_0^3 2y(x+1)\, dz\, dy\, dx$$

$$= \left[\int_0^1 x+1\, dx\right]\left[\int_0^2 2y\, dy\right]\left[\int_0^3 dz\right]$$

$$= \left[\frac{1}{2}x^2 + x \Big|_0^1\right]\left[y^2 \Big|_0^2\right]\left[z \Big|_0^3\right]$$

$$= \left[\frac{1}{2} + 1\right]\left[2^2\right]\left[3\right]$$

$$= \left[\frac{3}{2}\right]\left[4\right]\left[3\right]$$

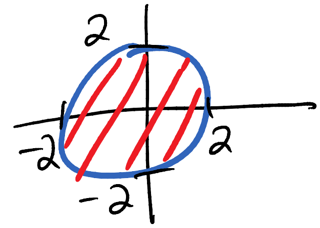
$$= 18$$

5.) *[10 points]* Compute the surface area of the portion of the plane

$$4x - 10y + 2z = 5$$

that is inside the cylinder $x^2 + y^2 \leq 4$.

Shadow R



Surface $2z = 5 - 4x + 10y$

$$z = \underbrace{\frac{5}{2} - 2x + 5y}_{f(x,y)}$$

Jacobian $\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + (-2)^2 + (5)^2} = \sqrt{30}$

$$S.A. = \iint_S dS = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \sqrt{30} \iint_R dA$$

$= \sqrt{30}$ [Area of circle of radius 2]

$$= \sqrt{30}\left[\pi (2)^2\right]$$

$$= \boxed{4\pi\sqrt{30}}$$

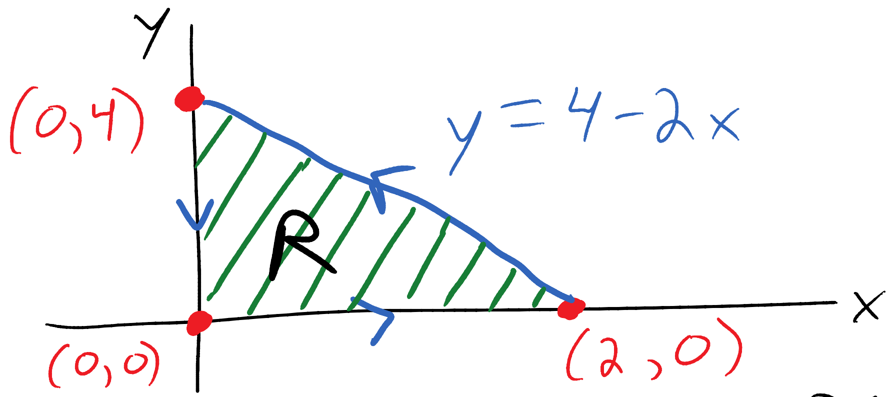
6.) *[10 points]* Charmander runs in a triangular path from the point (0,0) to (2,0) to (0,4) and then back to (0,0). A hurricane starts up with velocity field

$$\vec{F} = \langle 3y^2, 2xy \rangle.$$

Calculate the circulation of air around Charmander's path.

Closed 2D Path ⇒ Green's Theorem!

$$\vec{F} = \langle \underbrace{3y^2}_{M}, \underbrace{2xy}_{N} \rangle$$



$$\text{Circulation} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \int_0^2 \int_0^{4-2x} (2y - 6y)\, dy\, dx$$

$$= -4 \int_0^2 \int_0^{4-2x} y\, dy\, dx$$

$$= -4 \int_0^2 \frac{1}{2} y^2 \Big|_{y=0}^{y=4-2x} dx$$

$$= -2 \int_0^2 (4-2x)^2\, dx$$

$$= -2 \int_0^2 16 - 16x + 4x^2\, dx$$

$$= -2 \left[16x - 8x^2 + \frac{4}{3}x^3 \right]_0^2$$

$$= -2 \left[32 - 32 + \frac{32}{3} \right] = \boxed{-\frac{64}{3}}$$