Lecture 14: Power Series

Squirtle's Goals for the Day

- Review sigma notation and series arithmetic
- Review Taylor series approximations
- Introduce the Power Series Method for solving linear ODEs

5.1 Solutions about Ordinary Points

Def A power series about $x=c$ has the form

$$y=\sum_{n=0}^{\infty} a_n (x-c)^n$$

In particular, the power series about $x=0$ is

$$y=\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

The Taylor series is an example of power series.

$$f(x)=\sum_{n=0}^{\infty} f^{(n)}(c)\,\frac{(x-c)^n}{n!}$$

Ex Find Taylor series for e^x about $x=0$.

$f(x)=e^x \qquad f(0)=e^0=1$

$f'(x)=e^x \qquad f'(0)=1$

$f''(x)=e^x \qquad f''(0)=1$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \cancel{f^{(n)}(0)}^{\,1} \frac{x^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

<u>Recall</u> Factorial

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

By definition, $0! = 1$

<u>Review</u> Series

① Sigma Notation

ending index →

$$\sum_{n=1}^{N} a_n x^n$$

← n^{th} coefficient

↑ starting index

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

(2) We can add power series if they have the same starting and power on x.

$$\sum_{k=0}^{\infty} 3a_k x^k \quad + \sum_{m=0}^{\infty} 4a_{m+1} x^m$$

$$\sum_{n=0}^{\infty} \left[3a_n + 4a_{n+1} \right] x^n$$

(3) We can "shift" a series by substituting variables. We do this to control the power on x.

Ex Shift the series so the exponent is x^n.

$$\sum_{k=0}^{\infty} 2k(k-1) a_k x^{k+2}$$

Let $k+2 = n$

$k = n-2$

$k-1 = n-3$

$$k=0 \Rightarrow n=2$$

$$\sum_{n=2}^{\infty} 2(n-2)(n-3)\, a_{n-2}\, x^{n}$$

<u>Ex</u> Combine as one series:

$$\sum_{n=0}^{\infty} n\, a_n x^n + \sum_{n=1}^{\infty} (n+2)\, a_n\, x^{n-1}$$

Let $k = n-1$

$$k+1 = n$$

$$k+3 = n+2$$

$$n=1 \Rightarrow k = 1-1 = 0$$

$$\sum_{n=0}^{\infty} n\, a_n x^n + \sum_{k=0}^{\infty} (k+3)\, a_{k+1}\, x^{k}$$

$$\sum_{n=0}^{\infty} \left[n\, a_n + (n+3)\, a_{n+1} \right] x^{n}$$

4) We can pull out terms to change the starting index.

Ex start $\sum_{n=0}^{\infty} a_n n x^{n+1}$ at $n=2$.

$$\overbrace{\underbrace{a_0(0)x^{0+1}}_{0}}^{n=0} + \overbrace{a_1(1)x^{1+1}}^{n=1} + \overbrace{\sum_{n=2}^{\infty} a_n n x^{n+1}}^{n\geq 2}$$

$$a_1 x^2 + \sum_{n=2}^{\infty} a_n n x^{n+1}$$

(5) We can differentiate series.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \ldots$$

$$y''' = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

Power series Method

To solve a linear ODE, plug in the power series for y and all its derivatives. Then match up coefficients of x^n.

Ex Solve $y' = y$.

a.) Separation of Variables

$$\frac{dy}{dx} = y$$

$$\int \frac{1}{y}\,dy = \int dx$$

$$\ln y = x + C$$

$$\boxed{y = Ke^x}$$

b.) Power Series Method about $x = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Plug this into the original DE.

$$y' = y$$

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

set one side to zero.

Let $n-1 = k$

$n = k+1$

$n=1 \Rightarrow k=0$

Combine series.

$$\sum_{n=0}^{\infty} (k+1)\, a_{k+1} x^{k} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\underline{(n+1)\, a_{n+1} - a_n} \right] x^n = 0$$

Since the series equals zero for all values of x, the coefficients must equal zero.

$$(n+1)\, a_{n+1} - a_n = 0$$

$$(n+1)\, a_{n+1} = a_n$$

$$a_{n+1} = \frac{a_n}{n+1}$$

$\leftarrow$ This is called a recurrence relation.

Plug in a few values for n.

$\underline{n=0}$ $\quad a_1 = \dfrac{a_0}{0+1} = a_0$

$\underline{n=1}$ $\quad a_2 = \dfrac{a_1}{1+1} = \frac{1}{2}a_1 = \frac{1}{2}a_0$

$\underline{n=2}$ $\quad a_3 = \dfrac{a_2}{2+1} = \frac{1}{3}a_2 = \frac{1}{3}\left(\frac{1}{2}a_0\right) = \frac{1}{6}a_0$

$\underline{n=3}$ $\quad a_4 = \dfrac{a_3}{3+1} = \frac{1}{4}a_3 = \frac{1}{4}\left(\frac{1}{6}a_0\right) = \frac{1}{24}a_0$

And so on. Let's write our series.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$\boxed{y = a_0 + a_0 x + \frac{1}{2}a_0 x^2 + \frac{1}{6}a_0 x^3 + \frac{1}{24}a_0 x^4 + \ldots}$$

On an exam or homework problem, I would have to specify how many terms to write out.

Our answer showed the first 5 terms (through x^4).

Note every term has a_0. We could factor it out if you want. (Your book does this.)

$$y = a_0\left[1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right]$$

Now notice the part in brackets is the Taylor series for e^x.

So our solution is a constant times e^x.

So this answer does coincide with the simple answer we got using Separation of Variables: $y = Ke^x$.

We just happened to recognize our answer as the Taylor series for e^x.

In general, the Power Series Method produces a series that you won't be able to match up to a function.

The idea is that the finite series gives an approximation of the solution. The more terms you add on, the more accurate the approximation will become.