ELEC 309
Signals and Systems
TEST 2

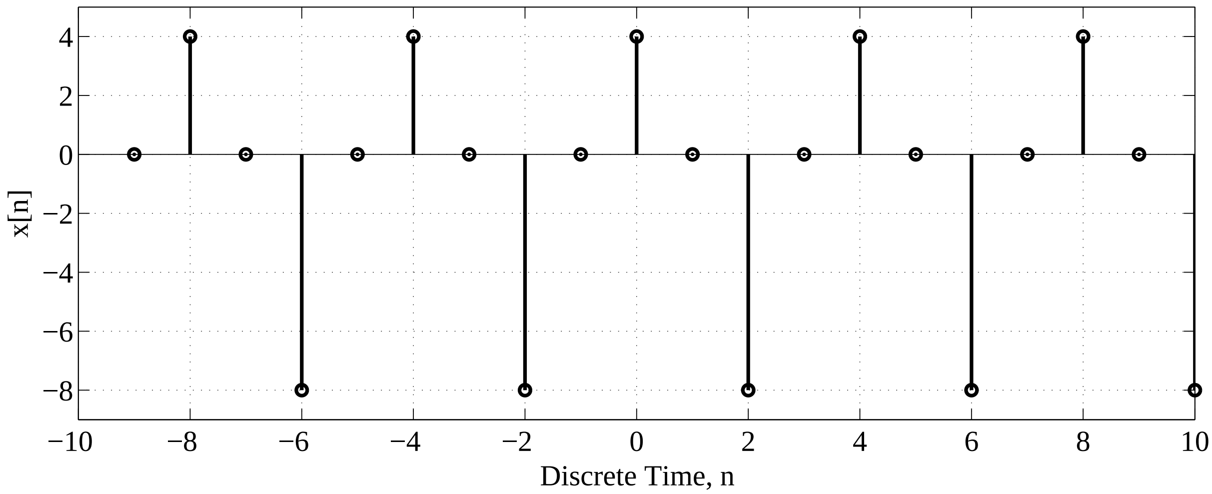
November 2013

Name: $\underbrace{\text{ANSWER KEY}}_{\text{By writing my name, I understand that I am bound by The Citadel Honor Code.}}$

Read all of the following information before starting the test:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. When you do use your calculator, explain all relevant mathematics.
- **Box, circle, or otherwise indicate your final answers.**
- This test has 3 problems and is worth 50 points.
- Check to ensure that you have all pages. It is your responsibility to make sure that you have all of the pages!
- If you remove the staple, you must re-staple your pages IN ORDER. Failure to do so will result in a deduction of 5 points from your final score.
- Good luck!

1. Consider the periodic sequence $x[n]$ shown below:



(a) (4 points) Determine the fundamental period N_0 and fundamental angular frequency Ω_0.

$$N_0 = \boxed{4 \text{ s}}$$

and

$$\Omega_0 = \frac{2\pi}{N_0} = \boxed{\frac{\pi}{2} \text{ rad/s.}}$$

(b) (5 points) Determine the Fourier series coefficients $\mathcal{D}_k$.

$$\mathcal{D}_k = \frac{1}{N_0} \sum_{n=<N_0>} x[n]e^{-jk\Omega_0 n} = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\pi n/2}$$

$$= \frac{1}{4}\left[4e^0 - 8e^{-jk\pi}\right] = 1 - 2e^{-jk\pi} = 1 - 2\left(e^{-j\pi}\right)^k$$

$$= \boxed{1 - 2(-1)^k = \begin{cases} -1 & k = 0 \\ 3 & k = 1 \\ -1 & k = 2 \\ 3 & k = 3 \end{cases}}$$

(Problem 1 continued)

(c) (10 points) Verify Parseval's theorem.

In the time domain,

$$\begin{aligned} P_x &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x[n]\right|^2 \\ &= \frac{1}{N_0} \sum_{n=<N_0>} \left|x[n]\right|^2 \\ &= \frac{1}{4} \sum_{n=0}^{3} x^2[n] = \frac{(4)^2 + (-8)^2}{4} = \boxed{20} \end{aligned}$$

In the frequency domain,

$$\begin{aligned} P_x &= \sum_{k=<N_0>} \left|\mathcal{D}_k\right|^2 \\ &= \sum_{k=0}^{3} \left|\mathcal{D}_k\right|^2 \\ &= (-1)^2 + (3)^2 + (-1)^2 + (3)^2 = \boxed{20} \end{aligned}$$

2. Consider a filter with impulse response given by

$$h(t) = \frac{61}{\pi}\text{sinc}\,(61t)\,.$$

(a) (5 points) Determine the filter frequency response $H(\omega)$.

Using row 18 of Table 7.1 with $W = 61$, we have

$$H(\omega) = \mathcal{F}\{h(t)\} = \text{rect}\left(\frac{\omega}{2\cdot 61}\right) = \boxed{\text{rect}\left(\frac{\omega}{122}\right)}$$

(b) (5 points) Determine the output signal $y(t)$ of this filter if the input signal $x(t)$ is given by

$$x(t) = \cos\,(61\pi t)\,.$$

Using row 9 of Table 7.1 with $\omega_0 = 61\pi$, we have

$$X(\omega) = \mathcal{F}\{x(t)\} = \pi\left[\delta\left(\omega - 61\pi\right) + \delta\left(\omega + 61\pi\right)\right].$$

Therefore,

$$\begin{aligned} Y(\omega) = X(\omega)H(\omega) &= \pi\left[\delta\left(\omega - 61\pi\right) + \delta\left(\omega + 61\pi\right)\right]\cdot\text{rect}\left(\frac{\omega}{2\cdot 61}\right) \\ &= \pi\text{rect}\left(\frac{61\pi}{122}\right)\delta\left(\omega - 61\pi\right) + \pi\text{rect}\left(\frac{-61\pi}{122}\right)\delta\left(\omega + 61\pi\right) = 0, \end{aligned}$$

and

$$y(t) = \boxed{0.}$$

(Problem 2 continued)

(c) (5 points) Determine the output signal $y(t)$ of this filter if the input signal $x(t)$ is given by

$$x(t) = \mathrm{sinc}^2(\pi t).$$

Using row 20 of Table 7.1 with $W = 2\pi$, we have

$$X(\omega) = \mathcal{F}\{x(t)\} = \Delta\left(\frac{\omega}{2 \cdot 2\pi}\right) = \Delta\left(\frac{\omega}{4\pi}\right).$$

Therefore,

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) = \Delta\left(\frac{\omega}{4\pi}\right) \cdot \mathrm{rect}\left(\frac{\omega}{122}\right) \\ &= \Delta\left(\frac{\omega}{4\pi}\right) = X(\omega), \end{aligned}$$

and

$$y(t) = \boxed{x(t) = \mathrm{sinc}^2(\pi t).}$$

(d) (5 points) Determine the output signal $y(t)$ of this filter if the input signal $x(t)$ is given by

$$x(t) = \frac{2015}{\pi}\mathrm{sinc}(2015t).$$

Using row 18 of Table 7.1 with $W = 2015$, we have

$$X(\omega) = \mathcal{F}\{x(t)\} = \mathrm{rect}\left(\frac{\omega}{2 \cdot 2015}\right) = \boxed{\mathrm{rect}\left(\frac{\omega}{4030}\right)}$$

Therefore,

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) = \mathrm{rect}\left(\frac{\omega}{4030}\right) \cdot \mathrm{rect}\left(\frac{\omega}{122}\right) \\ &= \mathrm{rect}\left(\frac{\omega}{122}\right) = H(\omega), \end{aligned}$$

and

$$y(t) = \boxed{h(t) = \frac{61}{\pi}\mathrm{sinc}(61t).}$$

3. Consider a continuous-time LTI system whose input $x(t)$ and output $y(t)$ are related by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 2x(t)$$

(a) (5 points) Determine the system frequency response $H(\omega)$.

Taking the Fourier transform of both sides, we have

$$(j\omega)^2 Y(\omega) + 4j\omega Y(\omega) + 4Y(\omega) = 2X(\omega)$$

The frequency response is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{(j\omega)^2 + 4j\omega + 4} = \boxed{\frac{2}{(2 + j\omega)^2}}$$

(b) (5 points) Determine the system impulse response $h(t)$.

From row 4 of Table 7.1, we have

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \boxed{2te^{-2t}u(t)}$$

(c) (1 point) This system is

A. $\boxed{\textbf{causal.}}$

B. noncausal.