21 Continuous-Time Second-Order Systems

Solutions to Recommended Problems

S21.1

(a) $H_2(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt = -\int_{-\infty}^{0} e^{-(a+s)t}\,dt$

Following our previous arguments, we can integrate only if the function dies out as t goes to minus infinity. e^{-xt} will die out as t goes to minus infinity only if $Re\{x\}$ is negative. Thus we need $Re\{a + s\} < 0$ or $Re\{s\} < -a$. For s in this range,

$$H_2(s) = \frac{1}{a + s}$$

(b) (i) $h_1(t)$ has a pole at $-a$ and no zeros. Furthermore, since $a > 0$, the pole must be in the left half-plane. Since $h_1(t)$ is causal, the ROC must be to the right of the rightmost pole, as given in D, Figure P21.1-4.

(ii) $h_2(t)$ is left-sided; hence the ROC is to the left of the leftmost pole. Since a is positive, the pole is in the left half-plane, as shown in A, Figure P21.1-1.

(iii) $h_3(t)$ is right-sided and has a pole in the right half-plane, as given in E, Figure P21.1-5.

(iv) $h_4(t)$ is left-sided and has a pole in the right half-plane, as shown in C, Figure P21.1-3.

For a signal to be stable, its ROC must include the $j\omega$ axis. Thus, C, D, and F qualify. B is an ROC that includes a pole, which is impossible; hence it corresponds to no signal.

S21.2

(a) By definition,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$
$$= \int_{0}^{\infty} e^{-t}e^{-st}\,dt$$

We limit the integral to $(0, \infty)$ because of $u(t)$, so

$$X(s) = \int_{0}^{\infty} e^{-(1+s)t}\,dt = \frac{-1}{1+s}\,e^{-(1+s)t}\Big|_{0}^{\infty}$$

If the real part of $(1 + s)$ is positive, i.e., $Re\{s\} > -1$, then

$$\lim_{t\to\infty} e^{-(1+s)t} = 0$$

Thus

$$X(s) = \frac{0(-1)}{1+s} - \frac{1(-1)}{1+s} = \frac{1}{1+s}, \qquad Re\{s\} > -1$$

The condition on $Re\{s\}$ is the ROC and basically indicates the region for which $1/(1+s)$ is equal to the integral defined originally. Similarly,

$$H(s) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(2+s)t}\,dt = \frac{1}{s+2}, \qquad Re\{s\} > -2$$

(b) By the convolution property of the Laplace transform, $Y(s) = H(s)X(s)$ in a manner similar to the property of the Fourier transform. Thus,

$$Y(s) = \frac{1}{(s+1)(s+2)}, \qquad Re\{s\} > -1,$$

where the ROC is the intersection of individual ROCs.

(c) Here we can use partial fractions:

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2},$$

$$A = Y(s)(s+1)\Big|_{s=-1} = 1,$$

$$B = Y(s)(s+2)\Big|_{s=-2} = -1$$

Thus,

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}, \qquad Re\{s\} > -1$$

Recognizing the individual Laplace transforms, we have

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

S21.3

(a) The property to be derived is

$$x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s),$$

with the same ROC as $X(s)$.

Let $y(t) = x(t - t_0)$. Then

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}\,dt = \int_{-\infty}^{\infty} x(t - t_0)e^{-st}\,dt$$

Let $p = t - t_0$. Then $t = p + t_0$ and $dp = dt$. Substituting

$$Y(s) = \int_{-\infty}^{\infty} x(p)e^{-s(p+t_0)}\,dp$$

Since we are not integrating over s or t_0, we can remove the e^{-st_0} term,

$$Y(s) = e^{-st_0}\int_{-\infty}^{\infty} x(p)e^{-sp}dp = e^{-st_0}X(s)$$

Note that wherever $X(s)$ converges, the integral defining $Y(s)$ also converges; thus the ROC of $X(s)$ is the same as the ROC of $Y(s)$.

(b) Now we study one of the most useful properties of the Laplace transform.

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s),$$

with the ROC containing $R_1 \cap R_2$. Let

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)\, d\tau$$

Then

$$\begin{aligned} Y(s) &= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x_1(\tau)x_2(t - \tau)e^{-st}\, d\tau\, dt \\ &= \int_{\tau=-\infty}^{\infty} x_1(\tau) \int_{t=-\infty}^{\infty} x_2(t - \tau)e^{-st}\, dt\, d\tau \end{aligned}$$

Suppose we are in a region of the s plane where $X_2(s)$ converges. Then using the property shown in part (a), we have

$$\int_{-\infty}^{\infty} x_2(t - \tau)e^{-st}\, dt = e^{-s\tau}X_2(s)$$

Substituting, we have

$$Y(s) = \int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-s\tau}X_2(s)\, d\tau = X_2(s) \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}\, d\tau$$

We can associate this last integral with $X_1(s)$ if we are also in the ROC of $x_1(t)$. Thus $Y(s) = X_2(s)X_1(s)$ for s inside at least the region $R_1 \cap R_2$. It could happen that the ROC is *larger*, but it *must* contain $R_1 \cap R_2$.

S21.4

(a) From the properties of the Laplace transform,

$$Y(s) = X(s)H(s)$$

A second relation occurs due to the differential equation. Since

$$\frac{d^k x(t)}{dt^k} \overset{\mathcal{L}}{\longleftrightarrow} s^k X(s)$$

and using the linearity property of the Laplace transform, we can take the Laplace transform of both sides of the differential equation, yielding

$$s^2Y(s) - sY(s) - 2Y(s) = X(s).$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s - 2)(s + 1)}$$

The pole-zero plot is shown in Figure S21.4-1.

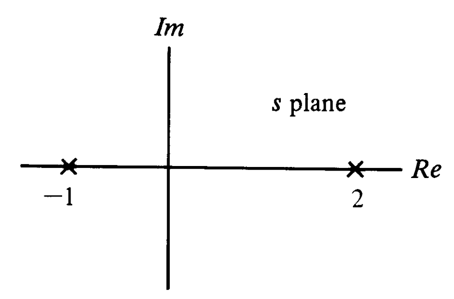

Figure S21.4-1

(b) (i) For a stable system, the ROC must include the $j\omega$ axis. Thus the ROC must be as drawn in Figure S21.4-2.

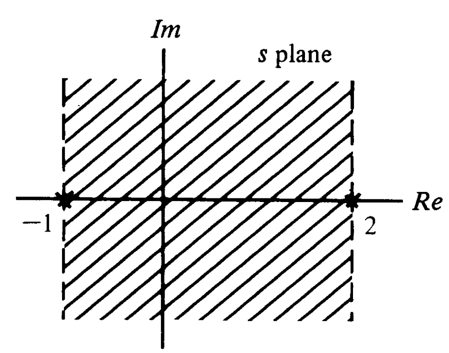

Figure S21.4-2

(ii) For a causal system, the ROC must be to the right of the rightmost pole, as shown in Figure S21.4-3.

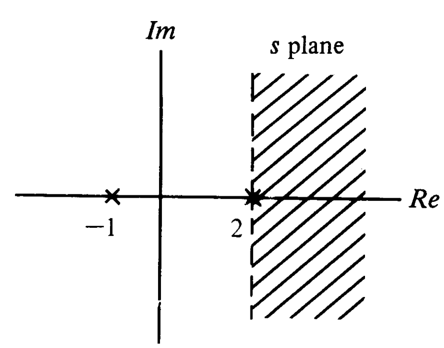

Figure S21.4-3

(iii) For a system that is not causal or stable, we are left with an ROC that is to the left of $s = -1$, as shown in Figure S21.4-4.

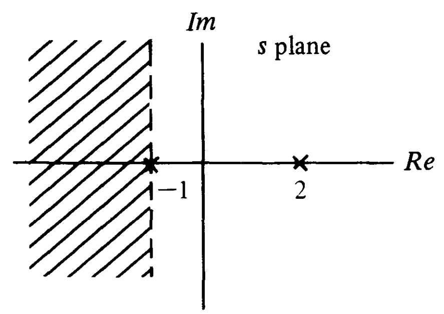

Figure S21.4-4

(c) To take the inverse Laplace transform, we use the partial fraction expansion:

$$H(s) = \frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2}$$

We now take the inverse Laplace transform of each term in the partial fraction expansion. Since the system is causal, we choose right-sided signals in both cases. Thus,

$$h(t) = -\tfrac{1}{3}e^{-t}u(t) + \tfrac{1}{3}e^{+2t}u(t)$$

S21.5

$\omega = 0$: Since there is a zero at $s = 0$, $|H(j0)| = 0$. You may think that the phase is also zero, but if we move slightly on the $j\omega$ axis, $\measuredangle H(j\omega)$ becomes

$$(\text{Angle to } s = 0) - (\text{Angle to } s = -1) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\omega = 1$: The distance to $s = 0$ is 1 and the distance to $s = -1$ is $\sqrt{2}$. Thus

$$|H(j1)| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The phase is

$$(\text{Angle to } s = 0) - (\text{Angle to } s = -1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \measuredangle H(j1)$$

$\omega = \infty$: The distance to $s = 0$ and $s = -1$ is infinite; however, the ratio tends to 1 as ω increases. Thus, $|H(j\infty)| = 1$. The phase is given by

$$\frac{\pi}{2} - \frac{\pi}{2} = 0$$

The magnitude and phase of $H(j\omega)$ are given in Figure S21.5.

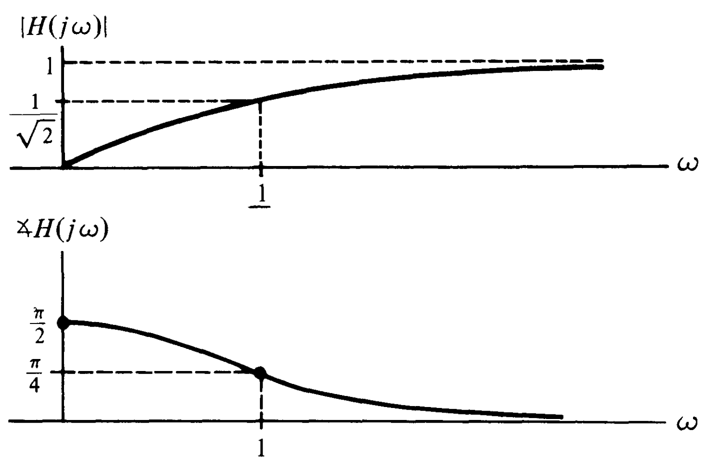

Figure S21.5

S21.6

The pole-zero plot is shown in Figure S21.6.

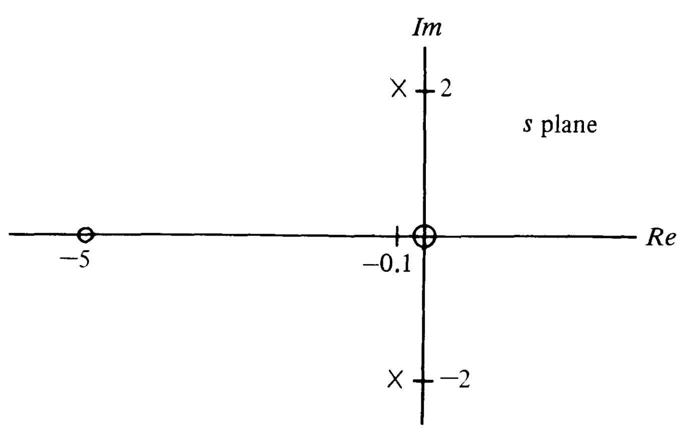

Figure S21.6

Because the zero at $s = -5$ is so far away from the $j\omega$ axis, it will have virtually no effect on $|H(j\omega)|$. Since there is a zero at $\omega = 0$ and poles near $\omega = 2$, we estimate a valley (actually a null) at $\omega = 0$ and a peak at $\omega \simeq \pm 2$.

Solutions to Optional Problems

S21.7

(a) Let $y(t)$ be the system response to the excitation $x(t)$. Then the differential equation relating $y(t)$ to $x(t)$ is

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Integrating twice, we have

$$y(t) + 2\zeta\omega_n \int_{-\infty}^{t} y(\tau)\,d\tau + \omega_n^2 \int_{-\infty}^{t}\int_{-\infty}^{\tau'} y(\tau)\,d\tau\,d\tau' = \omega_n^2 \int_{-\infty}^{t}\int_{-\infty}^{\tau'} x(\tau)\,d\tau\,d\tau',$$

or

$$y(t) = -2\zeta\omega_n \int_{-\infty}^{t} y(\tau)\,d\tau - \omega_n^2 \int_{-\infty}^{t}\int_{-\infty}^{\tau'} y(\tau)\,d\tau\,d\tau' + \omega_n^2 \int_{-\infty}^{t}\int_{-\infty}^{\tau'} x(\tau)\,d\tau\,d\tau',$$

shown in Figure S21.7-1.

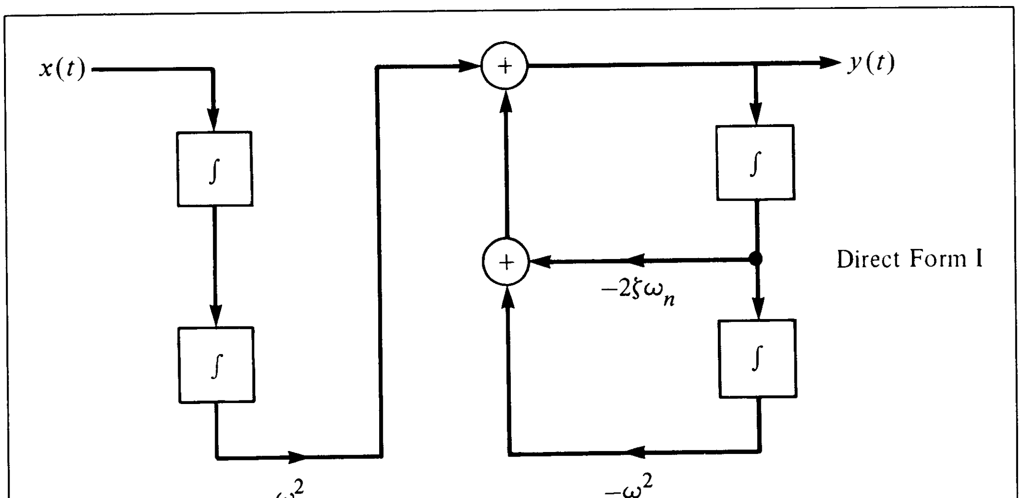

Figure S21.7-1

Recall that Figure S21.7-1 can be simplified as given in Figure S21.7-2.

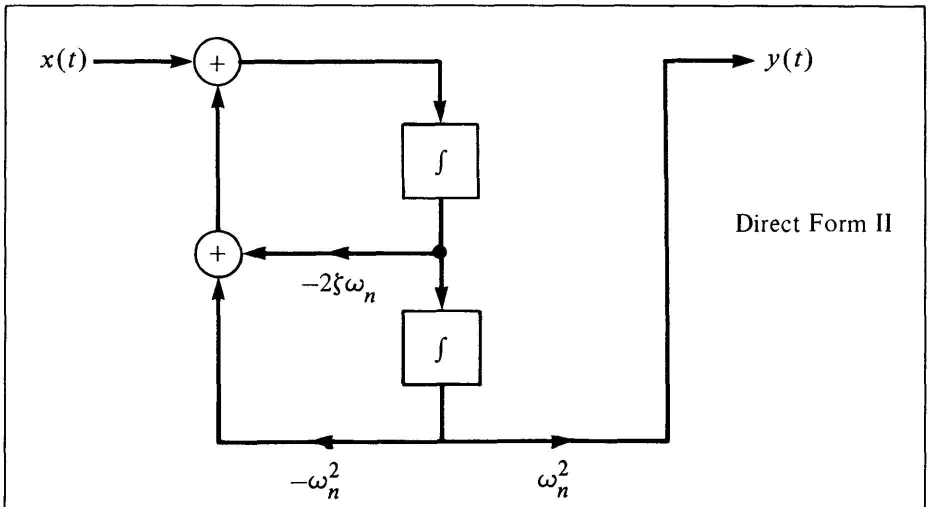

Figure S21.7-2

(b) (i) For a constant ω_n and $0 \leq \zeta < 1$, $H(s)$ has a conjugate pole pair on a circle centered at the origin of radius ω_n. As ζ changes from 0 to 1, the poles move from close to the $j\omega$ axis to $-\omega_n$, as shown in Figures S21.7-3, S21.7-4, and S21.7-5.

Figure S21.7-3 shows that for $\zeta \simeq 0$ the pole is close to the $j\omega$ axis, so $|H(j\omega)|$ has a peak very near ω_n.

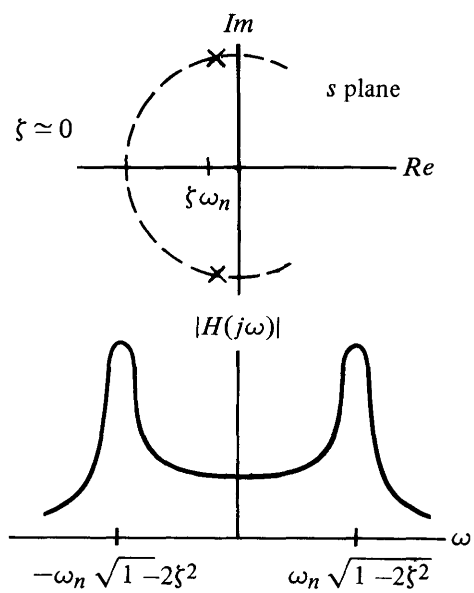

Figure S21.7-3

Figure S21.7-4 shows that the peaks are closer together and more spread out at $\zeta = 0.5$.

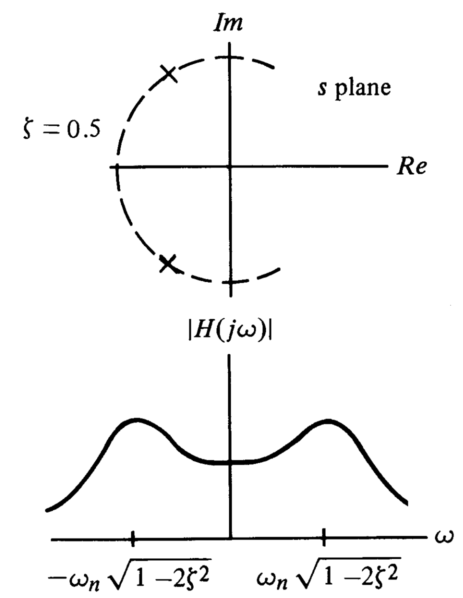

Figure S21.7-4

Figure S21.7-5 shows that at $\zeta \simeq 1$ the poles are so close together and far from the $j\omega$ axis that $|H(j\omega)|$ has a single peak.

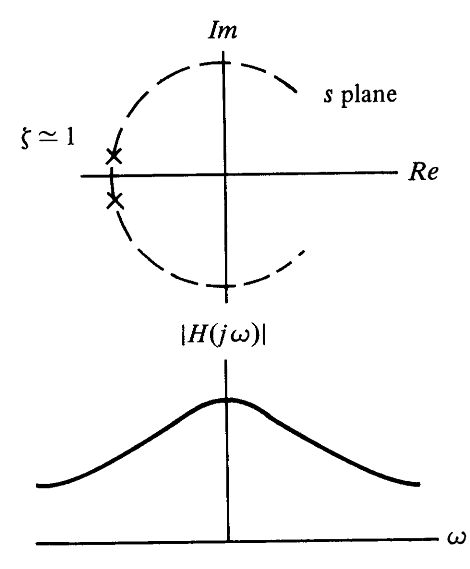

Figure S21.7-5

(ii) For constant ζ between 0 and 1, the poles are located on two straight lines. As ω_n increases, the peak frequency increases as well as the bandwidth, as indicated in Figures S21.7-6 and S21.7-7.

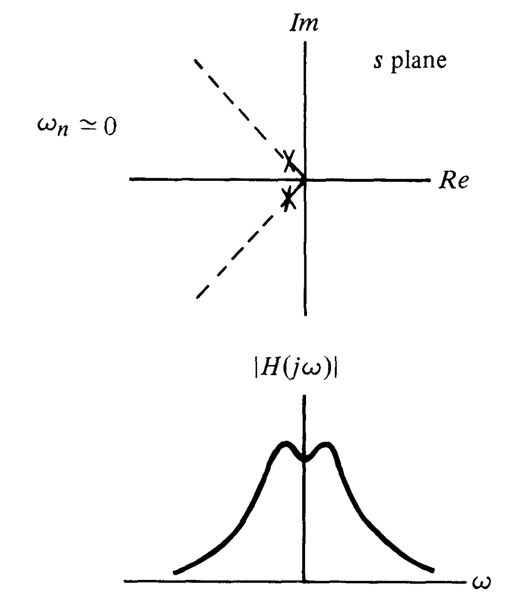

Figure S21.7-6

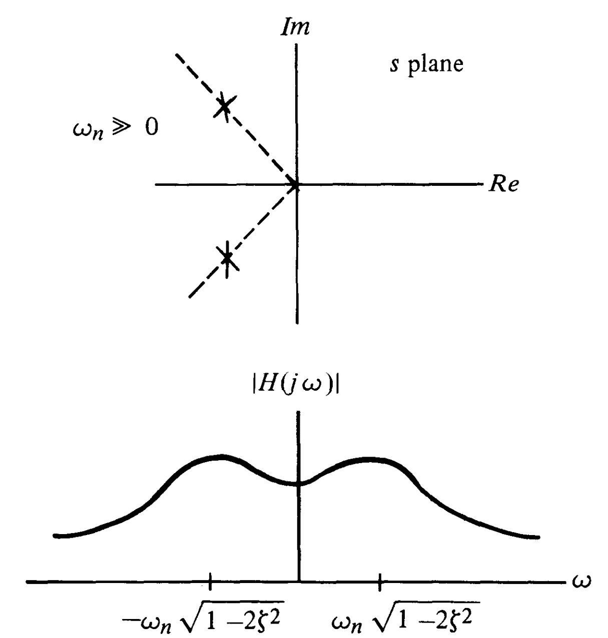

Figure S21.7-7

S21.8

(a) (i) The parallel implementation of $H(s)$, shown in Figure S21.8-1, can be drawn directly from the form for $H(s)$ given in the problem statement. The corresponding differential equations for each section are as follows:

$$\frac{d^2y_1(t)}{dt^2} + \frac{dy_1(t)}{dt} + y_1(t) = \frac{dx(t)}{dt},$$

$$\frac{d^2y_2(t)}{dt^2} + \frac{2dy_2(t)}{dt} + 2y(t) = x(t),$$

$$y(t) = y_1(t) + y_2(t)$$

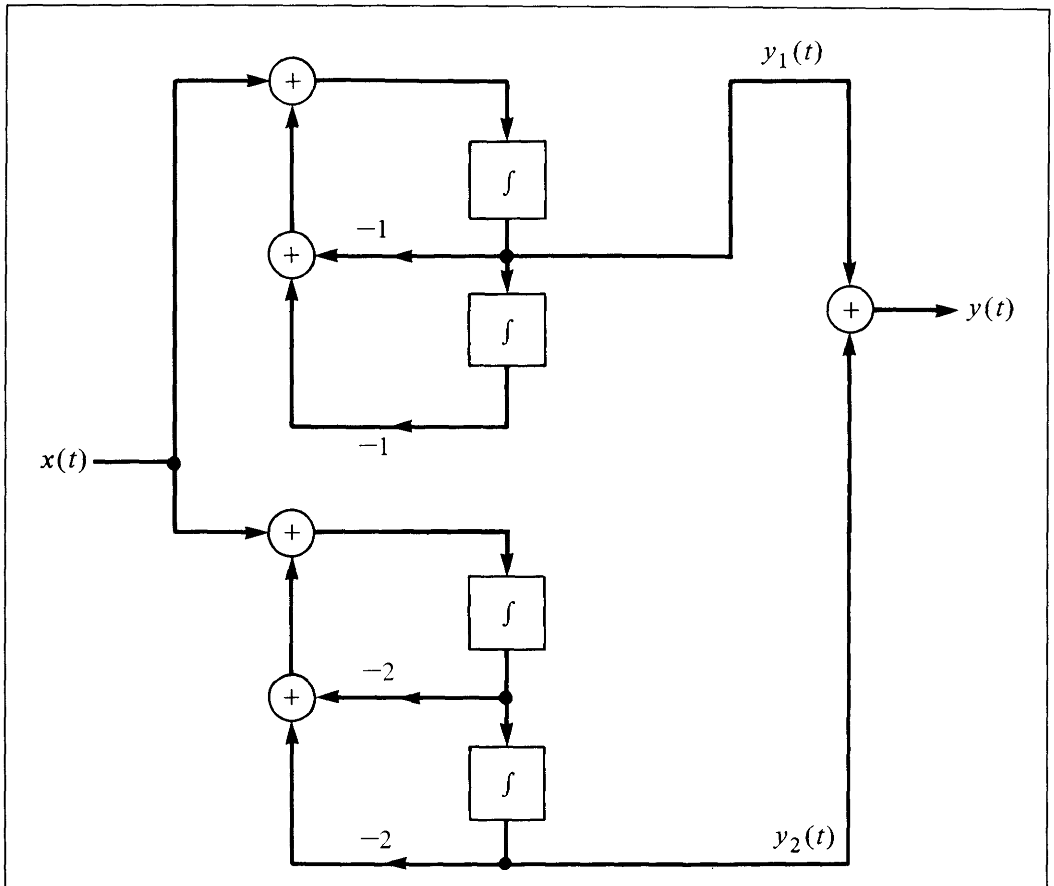

Figure S21.8-1

(ii) To generate the cascade implementation, shown in Figure S21.8-2, we first express $H(s)$ as a product of second-order sections. Thus,

$$H(s) = \frac{s(s^2 + 2s + 2) + (s^2 + s + 1)}{(s^2 + s + 1)(s^2 + 2s + 2)} = \frac{s^3 + 3s^2 + 3s + 1}{(s^2 + s + 1)(s^2 + 2s + 2)}$$

Now we need to separate the numerator into two sections. In this case, the numerator equals $(s + 1)^3$, so an obvious choice is

$$(s + 1)(s^2 + 2s + 1)$$

Thus,

$$H(s) = \left(\frac{s + 1}{s^2 + s + 1}\right)\left(\frac{s^2 + 2s + 1}{s^2 + 2s + 2}\right)$$

The corresponding differential equations are as follows:

$$\frac{d^2r(t)}{dt^2} + \frac{dr(t)}{dt} + r(t) = x(t) + \frac{dx(t)}{dt},$$
$$\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + 2y(t) = \frac{d^2r(t)}{dt^2} + \frac{2dr(t)}{dt} + r(t)$$

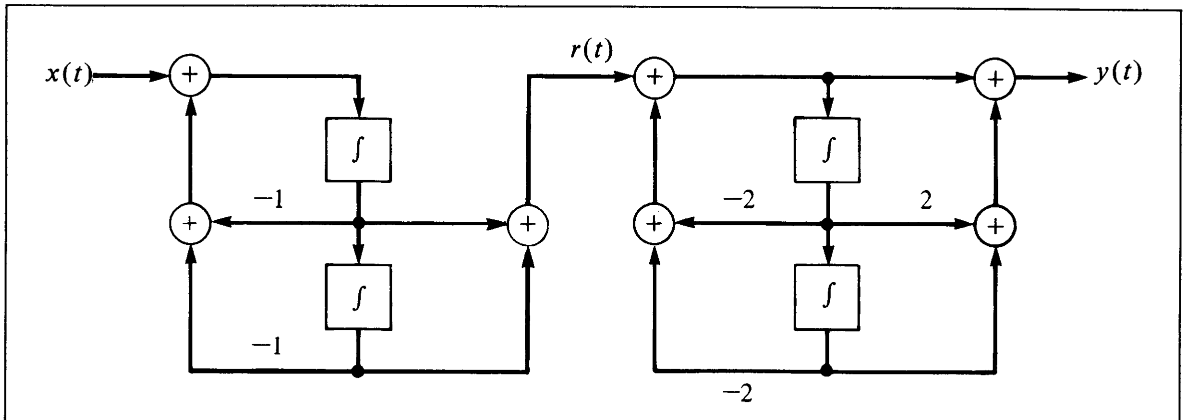

Figure S21.8-2

(b) We see that we could have decomposed $H(s)$ as

$$H(s) = \left(\frac{s^2 + 2s + 1}{s^2 + s + 1}\right)\left(\frac{s + 1}{s^2 + 2s + 2}\right)$$

Thus, the cascade implementation is not unique.

S21.9

(a) Decompose $\sin \omega_0 t$ as

$$\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

Then

$$x_1(t) = \sin(\omega_0 t)u(t) = \frac{e^{j\omega_0 t}}{2j}u(t) - \frac{e^{-j\omega_0 t}}{2j}u(t)$$

Using the transform pair given in the problem statement and the linearity property of the Laplace transform, we have

$$X_1(s) = \frac{1}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right]$$
$$= \frac{1}{2j}\frac{2j\omega_0}{s^2 + \omega_0^2} = \frac{\omega_0}{s^2 + \omega_0^2},$$

with an ROC corresponding to $Re\{s\} > 0$.

(b) $x_2(t) = e^{-2t}\sin(\omega_0 t)u(t)$. Since

$$e^{-2t}\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s + 2),$$

the ROC is shifted by 2. Therefore,

$$e^{-2t}\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s+2)^2+\omega_0^2},$$

and the ROC is $Re\{s\} > -2$. Here we have used our answer to part (a).

(c) Since

$$tx(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{dX(s)}{ds},$$

with the same ROC as $X(s)$, then

$$te^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{(s+2)}\right],$$

Thus

$$te^{-2t}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} -\left[\frac{-1}{(s+2)^2}\right] = \frac{1}{(s+2)^2}$$

with the ROC given by $Re\{s\} > -2$.

(d) Here we use partial fractions:

$$\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3},$$

$$A = \left[\frac{s+1}{s+3}\right]\Bigg|_{s=-2} = -1, \qquad B = \left[\frac{s+1}{s+2}\right]\Bigg|_{s=-3} = \frac{-2}{-1} = 2,$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3} \qquad \text{(S21.9-1)}$$

The ROC associated with the first term of eq. (S21.9-1) is $Re\{s\} > -2$ and the ROC associated with the second term is $Re\{s\} > -3$ to be consistent with the given total ROC. Thus,

$$x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$

(e) From properties of the Laplace transform we know that

$$x(t-T) \overset{\mathcal{L}}{\longleftrightarrow} e^{-sT}X(s),$$

with the same ROC as $X(s)$. Since

$$e^{-3t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3},$$

with an ROC given by $Re\{s\} > -3$, $(1 - e^{-2s})/(s+3)$ must correspond to

$$x(t) = e^{-3t}u(t) - e^{-3(t-2)}u(t-2)$$

S21.10

(a) (1), (2): An impulse has a constant Fourier transform whose magnitude is unaffected by a time shift. Hence, the Fourier transform magnitudes of (1) and (2) are shown in (c).

(3), (5): A decaying exponential corresponds to a lowpass filter; hence, (3) could be (a) or (d). By comparing it with (5), we see that (5) corresponds to $kte^{-at}u(t)$, which has a double pole at $-a$. Thus, (5) is a steeper lowpass filter than (3). Hence, (3) corresponds to (d) and (5) corresponds to (a).

(4), (7): These signals are of the form $e^{-at}\cos(\omega_0 t)u(t)$. For larger a, the poles are farther to the left. Hence $|H(j\omega)|$ for larger a is less peaky. Thus, (4) corresponds to (f) and (7) corresponds to (g).

(6): If we convolve $x(t) = 1$ with $h(t)$ given in (6), we find that the output is zero. Thus (6) corresponds to a null at $\omega = 0$, either (b) or (h). Note that (6) can be thought of as an $h(t)$ given by (1) minus an $h(t)$ given by (3). Thus, the Fourier transform is the difference between a constant and a lowpass filter. Therefore, (6) is a highpass filter, or (b).

(b) (a), (d): These are simple lowpass filters that correspond to (i) or (ii). Since (a) is a steeper lowpass filter, we associate (a) with (ii) and (d) with (i).

(b), (h): These require a null at zero, and thus could correspond to (iii) or (viii). In the case of (iii), as ω increases, one pole-zero pair is canceled so that for large ω, $H(s)$ looks like a lowpass filter. Hence, (b) corresponds to (viii) and (h) corresponds to (iii).

(c): Here we need a pole-zero plot that is an all-pass system. The only possible pole-zero plot is (vi).

(e): Here we need a null on the $j\omega$ axis, but not at $\omega = 0$. The only possibility is (v).

(f), (g): These are resonant second-order systems that could correspond to (iv) or (vii). Since poles closer to the $j\omega$ axis lead to peakier Fourier transforms, (f) must correspond to (iv) and (g) to (vii).