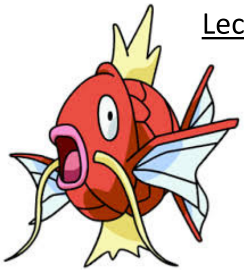


Lecture 16: Ordinary vs Singular Points

Magikarp's Goals for the Day

- Discuss ordinary and singular points
- Define regular singular points
- Introduce the Frobenius method for finding power series solutions around singular points

5.2 Solutions about Singular Points

Def A function that represented locally as a power series is called analytic.

Ex Analytic Functions

Polynomial $3x^2+2x+1$

Exponential $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Sine/Cosine

Ex. Not analytic functions

$\frac{1}{x}$ is not analytic at $x=0$

$\sqrt[3]{x}$ is not analytic at $x=0$

Taylor series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$ ← not defined at $x=0$

<u>Def</u> Rewrite the DE so the leading coefficient is 1.

$$y'' + P(x)y' + Q(x)y = 0$$

If both $P(x)$ and $Q(x)$ are analytic at x_0,
then x_0 is called an <u>ordinary point</u> of the DE.

Otherwise, is called a <u>singular point</u> of the DE.

<u>Ex</u> Find singular points of

$$(x+2)y'' + 2xy' - 3y = 0.$$

Rewrite so leading coefficient is one.

$$y'' + \underbrace{\frac{2x}{x+2}}_{P(x)} y' - \underbrace{\frac{3}{x+2}}_{Q(x)} y = 0$$

$\boxed{x = -2 \text{ is a singular point}}$

Ex Find singular points of

$$(x^2+1)y'' - 3y = 0$$

Rewrite $$y'' \underbrace{}_{P(x)=0} - \underbrace{\frac{3}{x^2+1}}_{Q(x)} y = 0$$

"Trouble" when $x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=\pm i$

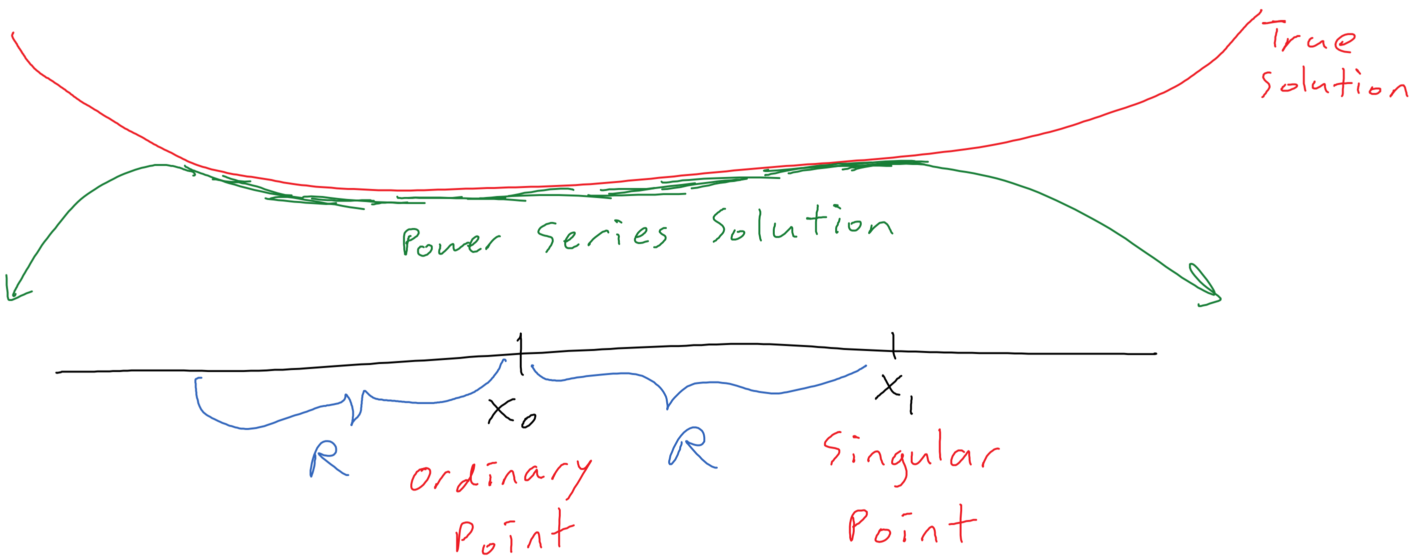
$\boxed{x = \pm i \text{ are singular points}}$

Theorem 5.1.1 $y'' + P(x)y' + Q(x)y = 0$

If x_0 is an ordinary point, then there exists a power series solution centered at x_0:

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

The series converges at least on the interval $|x-x_0| < R$ where R is the distance to the closest singular point.



How do we find a solution at a singular point?

Def If $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are both analytic at x_0, then x_0 is called a regular point.

Ex $x^2 y'' - 3xy' + 2y = 0$
Classify the singular points.

$$y'' - \underbrace{\frac{3}{x}}_{P(x)} y' + \underbrace{\frac{2}{x^2}}_{Q(x)} y = 0$$

$x=0$ is a singular point

$$xP(x) = x\left[-\frac{3}{x}\right] = -3 \quad \text{analytic}$$
$$x^2 Q(x) = x^2\left[\frac{2}{x^2}\right] = 2 \quad \text{analytic}$$
$\Rightarrow$ $\boxed{x=0 \text{ is a regular singular point}}$

Theorem 5.2.1 (Frobenius' Theorem)

If x_0 is a regular singular point of the DE
$$y'' + P(x)y' + Q(x)y = 0,$$
then there exists at least one solution of the form
$$y = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$
where r is a constant.

Note: The types of solutions are often called Frobenius solutions. The procedure for finding solutions is called the Method of Frobenius.

Ex Solve $8xy'' + y' + 2y = 0$ around $x=0$.

Rewrite $y'' + \underbrace{\frac{1}{8x}}_{P(x)} y' + \underbrace{\frac{1}{4x}}_{Q(x)} y = 0$

$x=0$ is a singular point

$\Rightarrow$ Standard power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ does not exist

$$\left.\begin{aligned} xP(x) &= x\left[\frac{1}{8x}\right] = \frac{1}{8} \quad \text{analytic} \\ x^2 Q(x) &= x^2\left[\frac{1}{4x}\right] = \frac{x}{4} \quad \text{analytic} \end{aligned}\right] \Rightarrow$$ $x=0$ is a regular singular point

$\Rightarrow$ There exists a Frobenius solution at $x=0$

Seek a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \leftarrow$$ Unlike power series, still start at $n=0$.

$$y'' = \sum^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$\qquad n=0$

Plug y, y', and y'' into original DE,

$$8xy'' + y' + 2y = 0$$

$$8x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$
$$+ \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$
$$+ 2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 8(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$
$$+ \underbrace{\sum_{n=0}^{\infty} 2 c_n x^{n+r}}_{\text{Shift to } x^{k+r-1}} = 0$$

$n+r = k+r-1$

$n = k-1$

$n=0 \Rightarrow k=1$

$$\sum_{k=1}^{\infty} 2 c_{k-1} x^{k+r-1}$$

Pull out $n=0$ term of the first 2 series.

$$8(0+r)(0+r-1)c_0 x^{0+r-1} + (0+r)c_0 x^{0+r-1}$$

$$+ \sum_{n=1}^{\infty} 8(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r)c_n x^{n+r-1}$$

$$+ \sum_{k=1}^{\infty} 2c_{k-1} x^{k+r-1} = 0$$

$$\underline{8r(r-1)c_0} x^{r-1} + \underline{r c_0} x^{r-1}$$ ← Coefficients of x^{r-1} give zero.

$$+ \sum_{n=1}^{\infty} \Big[\underbrace{8(n+r)(n+r-1)c_n + (n+r)c_n + 2c_{n-1}}_{=0} \Big] x^{n+r-1} = 0$$

Match terms and set to zero.

$\underline{x^{r-1}}$:

$$8r(r-1)c_0 + rc_0 = 0$$ ← Do not assume $c_0 = 0$.

$$8r(r-1) + r = 0$$

$$8r^2 - 8r + r = 0$$

$$8r^2 - 7r = 0$$

$$r(8r-7) = 0$$

$$r = 0, \frac{7}{8}$$ ← We call these values the indicial roots.

$\underline{x^{n+r-1}}$: $8c_n(n+r)(n+r-1)+c_n(n+r)+2c_{n-1}=0$

$$c_n=\frac{-2}{8(n+r)(n+r-1)+(n+r)}\,c_{n-1}$$

↳ Recurrence Relation

Find first Frobenius solution at $r=0$.

$$y_1=\sum_{n=0}^{\infty}a_n x^{n+0}$$

Plug $r=0$ into recurrence relation.

$$a_n=\frac{-2}{8(n)(n-1)+n}\,a_{n-1}$$

$\underline{n=1}$: $a_1=\dfrac{-2}{8(1)(1-1)+1}a_0=-2a_0$

$\underline{n=2}$: $a_2=\dfrac{-2}{8(2)(2-1)+2}=\dfrac{-2}{18}(-2a_0)=\dfrac{2}{9}a_0$

$$y_1=a_0+a_1x+a_2x^2+\dots$$

$$y_1=a_0-2a_0x+\frac{2}{9}a_0x^2+\dots$$

Find second Frobenius solution at $r = \frac{7}{8}$.

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+\frac{7}{8}}$$

$$c_n = \frac{-2}{8(n+r)(n+r-1) + (n+r)} c_{n-1}$$

$\downarrow$ Plug in $r = \frac{7}{8}$

$$b_n = \frac{-2}{8(n+\frac{7}{8})(n+\frac{7}{8}-1) + (n+\frac{7}{8})} b_{n-1}$$

$$= \frac{-2}{8(n+\frac{7}{8})(n-\frac{1}{8}) + n + \frac{7}{8}} b_{n-1}$$

$$= \frac{-2}{8(n^2 + \frac{6}{8}n - \frac{7}{64}) + n + \frac{7}{8}} b_{n-1}$$

$$= \frac{-2}{8n^2 + 6n - \cancel{\frac{7}{8}} + n + \cancel{\frac{7}{8}}} b_{n-1}$$

$$= \frac{-2}{8n^2 + 7n} b_{n-1}$$

$\underline{n=1}$ $b_1 = -\frac{2}{15} b_0$

$\underline{n=2}$ $b_2 = -\frac{1}{23} b_1 = -\frac{1}{23}\left(-\frac{2}{15} b_0\right) = \frac{2}{345} b_0$

$$y_2 = b_0 x^{7/8} + b_1 x^{15/8} + b_2 x^{23/8} + \dots$$

$$= b_0 x^{7/8} - \frac{2}{15} b_0 x^{15/8} + \frac{2}{345} b_0 x^{23/8} + \dots$$

The solution of the DE is

$$y = y_1 + y_2$$

$$= \boxed{\begin{aligned} &a_0 - 2a_0 x + \frac{2}{9} a_0 x^2 + \dots \\ &+ b_0 x^{7/8} - \frac{2}{15} b_0 x^{15/8} + \frac{2}{345} b_0 x^{23/8} + \dots \end{aligned}}$$

Your book writes the solution as:

$$y = a_0\left[1 - 2x + \frac{2}{9} x^2 + \dots\right]$$

$$+ b_0 x^{7/8}\left[1 - \frac{2}{15} x + \frac{2}{345} x^2 + \dots\right]$$