26 Feedback Example: The Inverted Pendulum

Solutions to Recommended Problems

S26.1

(a) $$\frac{Ld^2\theta(t)}{dt^2} = g\theta(t) - a(t) + Lx(t),$$

$$\frac{Ld^2\theta(t)}{dt^2} - g\theta(t) = Lx(t)$$

Taking the Laplace transform of both sides yields

$$s^2L\theta(s) - g\theta(s) = LX(s),$$

$$\theta(s) = \frac{X(s)}{s^2 - g/L},$$

$$\frac{\theta(s)}{X(s)} = \frac{1}{s^2 - g/L} = \frac{1}{(s + \sqrt{g/L})(s - \sqrt{g/L})},$$

The pole at $\sqrt{g/L}$ is in the right half-plane and therefore the system is unstable.

(b) We are given that $a(t) = K\theta(t)$. See Figure S26.1-1.

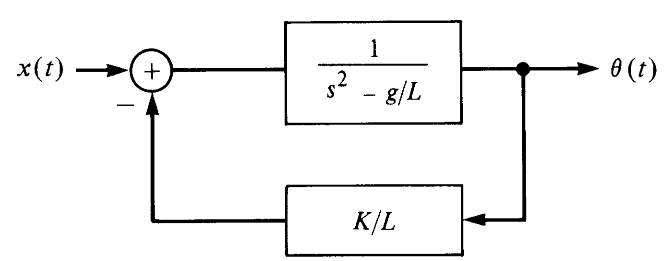

Figure S26.1-1

$$\frac{\theta(s)}{X(s)} = \frac{H}{1 + GH},$$

so, with

$$H = \frac{1}{s^2 - g/L} \quad \text{and} \quad G = \frac{K}{L},$$

$\theta(s)/X(s)$ is given by

$$\frac{\theta(s)}{X(s)} = \frac{1}{s^2 - (g/L) + (K/L)}$$

The poles of the system are at

$$s = \pm\sqrt{\frac{K - g}{L}},$$

which implies that the system is unstable. Any $K < g$ will cause the system poles to be pure imaginary, thereby causing an oscillatory impulse response.

(c) Now the system is as indicated in Figure S26.1-2.

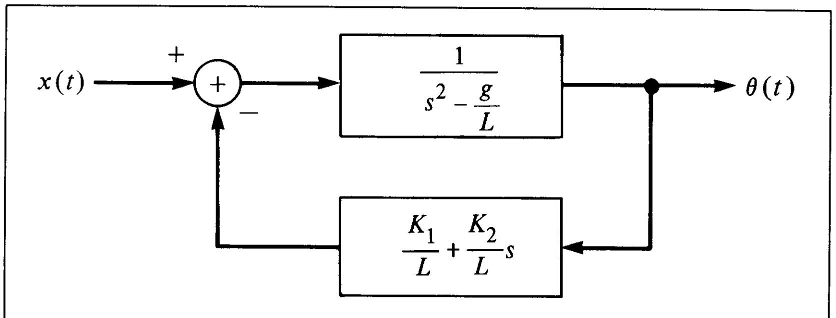

Figure S26.1-2

$$H(s) = \frac{1}{s^2 - \frac{g}{L} + \frac{K_1}{L} + \frac{K_2}{L}s}$$
$$= \frac{1}{s^2 + \frac{K_2}{L}s + \frac{K_1 - g}{L}}$$

The poles are at

$$\frac{-K_2}{2L} \pm \sqrt{\left(\frac{K_2}{2L}\right)^2 - \frac{(K_1 - g)}{L}},$$

which can be adjusted to yield a stable system. A general second-order system can be expressed as

$$H_g(s) = \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

so, for our case,

$$\omega_n^2 = \frac{K_1 - g}{L} \quad \text{and} \quad 2\zeta\omega_n = \frac{K_2}{L},$$
$$g = 9.8 \text{ m/s}^2$$
$$L = 0.5 \text{ m}$$
$$\zeta = 1$$
$$\omega_n = 3 \text{ rad/s}$$
$$K_1 = 14.3 \text{ m/s}^2$$
$$K_2 = 3 \text{ m/s}$$

S26.2

(a) Here

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$
$$G(s) = K$$

The closed-loop transfer function $H_c(s)$ is

$$H_c(s) = \frac{H}{1 + GH} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 + K\omega_n^2}$$

$$= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2(1 + K)}$$

$$= \frac{\omega_n^2}{s^2 + 2\left(\frac{\zeta\omega_n}{\hat{\omega}_n}\right)\hat{\omega}_n s + \hat{\omega}_n^2}, \qquad \text{where } \hat{\omega}_n = \omega_n(1 + K)^{1/2}$$

$$= \frac{(\omega_n^2/\hat{\omega}_n^2)\,\hat{\omega}_n^2}{s^2 + 2\hat{\zeta}\hat{\omega}_n + \hat{\omega}_n^2}, \qquad \text{where } \hat{\zeta} = \zeta\frac{\omega_n}{\hat{\omega}_n}$$

Therefore,

$$\hat{\omega}_n = \omega_n(1 + K)^{1/2},$$

$$\hat{\zeta} = \frac{\zeta}{(1 + K)^{1/2}},$$

$$A = \frac{\omega_n^2}{\hat{\omega}_n^2} = \frac{1}{1 + K},$$

for $K = 1$, $\hat{\omega}_n = \sqrt{2}\omega_n$, and $\hat{\zeta} = \zeta/\sqrt{2}$.

(b) Now we want to determine the poles of the closed-loop system

$$H_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2(1 + K)}$$

The poles are at

$$-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2(1 + K)}$$

(c)

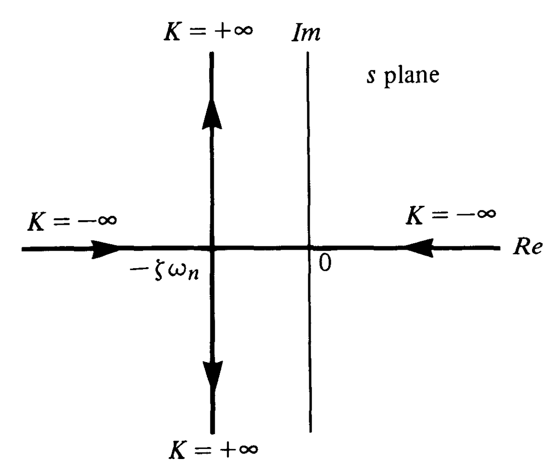

Figure S26.2

The poles start out at $\pm\infty$, approach each other and touch at $K = \zeta^2 - 1$, and then proceed to $-\zeta\omega_n \pm j\infty$.

S26.3

(a) $$\frac{Y(s)}{X(s)} = H_1(s) = \frac{K_1K_2}{1 + \dfrac{K_1K_2\alpha}{\beta s + r}} = \frac{(\beta s + r)K_1K_2}{\beta s + r + K_1K_2\alpha}$$

(b) $$\frac{Y(s)}{W(s)} = H_2(s) = \frac{K_2}{1 + \dfrac{K_1K_2\alpha}{\beta s + r}} = \frac{(\beta s + r)K_2}{\beta s + r + K_1K_2\alpha}$$

(c) For stability we require the pole to be in the left half-plane.

$$s_p = -\left(\frac{r + K_1K_2\alpha}{\beta}\right) < 0$$

$$\Rightarrow \frac{r + K_1K_2\alpha}{\beta} > 0$$

If $\beta > 0$, then $r/\alpha > -K_1K_2$; if $\beta < 0$, then $r/\alpha < -K_1K_2$.

S26.4

$$H(s) = \frac{K}{1 + \dfrac{K(s+1)}{s + 100}} = \frac{K(s + 100)}{s + 100 + Ks + K}$$

$$= \frac{K(s + 100)}{(K + 1)\left(s + \dfrac{100 + K}{K + 1}\right)}$$

(a) $K = 0.01$,

$$H(s) = \frac{0.01(s + 100)}{1.01(s + 99.0198)}$$

The zero is at $s = -100$, and the pole is at $s = -99.0198$.

(b) $K = 1$,

$$H(s) = \frac{s + 100}{2\left(s + \dfrac{101}{2}\right)}$$

The zero is at $s = -100$; the pole is at $s = -50.5$.

(c) $K = 10$,

$$H(s) = \frac{10(s + 100)}{11\left(s + \dfrac{110}{11}\right)}$$

The zero is at $s = -100$; the pole is at $s = -10$.

(d) $K = 100$,

$$H(s) = \frac{100(s + 100)}{101\left(s + \dfrac{200}{101}\right)}$$

The zero is at $s = -100$; the pole is at $s = -1.9802$.

S26.5

(a) $$H(s) = \frac{\frac{1}{s+1}}{1 + \frac{K}{s+1}} = \frac{1}{s+1+K}$$

The pole is at $s = -1 - K$, as shown in Figure S26.5-1.

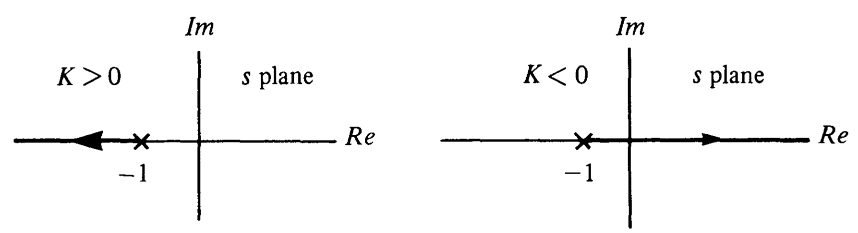

Figure S26.5-1

The pole moves from infinity to negative infinity as K changes from negative infinity to infinity.

(b) $$H(s) = \frac{\frac{1}{s-1}}{1 + \left(\frac{K}{s+3}\frac{1}{s-1}\right)} = \frac{s+3}{(s+3)(s-1)+K}$$

$$= \frac{s+3}{s^2+2s+K-3}$$

The poles are at $s_p = -1 \pm \sqrt{1-(K-3)}$, as shown in Figure S26.5-2.

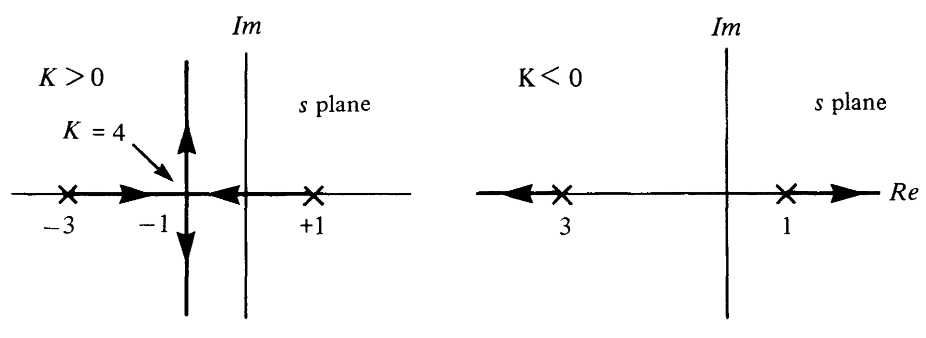

Figure S26.5-2

The poles start at $\pm\infty$ when $K = -\infty$, move toward -1, touch when $K = 4$, and proceed to $-1 \pm j\infty$ as K approaches positive infinity.

Solutions to Optional Problems

S26.6

(a) The poles for the closed-loop system are determined by the denominator of the closed-loop transfer function

$$1 + \frac{Kz}{(z - \frac{1}{2})(z - \frac{1}{4})} = 0,$$

so

$$(z - \tfrac{1}{2})(z - \tfrac{1}{4}) + Kz = 0$$

Since we are told a pole occurs when $z = -1$, we want to solve the equation for K:

$$K = \left. \frac{-(z - \frac{1}{2})(z - \frac{1}{4})}{z} \right|_{z=-1} = \frac{15}{8}$$

(b) In a similar manner to that in part (a),

$$K = \left. \frac{-(z - \frac{1}{2})(z - \frac{1}{4})}{z} \right|_{z=1} = \frac{-3}{8}$$

(c) From the root locus diagram in Figure P26.6, we see that for $K > 0$ when K exceeds a critical value of $K = \frac{15}{8}$, as determined in part (a), one root remains outside the unit circle. Similarly, when $K < -\frac{3}{8}$, one root is outside the unit circle. Therefore, to ensure stability, we need

$$-\tfrac{3}{8} < K < \tfrac{15}{8}$$

S26.7

(a) The closed-loop transfer function is

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$$

and, therefore, from the given $H_c(s)$ and $H_p(s)$, we have

$$\frac{Y(s)}{X(s)} = \frac{\dfrac{K\alpha}{s + \alpha}}{1 + \dfrac{K\alpha}{s + \alpha}} = \frac{K\alpha}{s + \alpha + K\alpha} = \frac{K\alpha}{s + (K + 1)\alpha}$$

The system is stable for denominator roots in the left half of the s plane; therefore $-(K + 1)\alpha < 0$ implies that the system is stable.

Now since $E(s)H_c(s)H_p(s) = Y(s)$, we have

$$\frac{E(s)}{X(s)} = \frac{1}{1 + H_c(s)H_p(s)} = \frac{s + \alpha}{s + \alpha + K\alpha} = \frac{s + \alpha}{s + (K + 1)\alpha}$$

The final value theorem, $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$, shows that

$$\lim_{s\to 0} \frac{s(s + \alpha)}{s + (K + 1)\alpha} = 0 \qquad \text{for } -(K + 1)\alpha < 0$$

Note that if $x(t) = u(t)$, then

$$E(s) = \left(\frac{1}{s}\right)\frac{s+\alpha}{s+(K+1)\alpha}$$

and

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{s+\alpha}{s+(K+1)\alpha} = \frac{1}{K+1} \neq 0, \text{ for } -(K+1)\alpha < 0$$

so $\lim_{t\to\infty} e(t) \neq 0$.

(b)
$$\frac{Y(s)}{X(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$$

$$= \frac{\left(K_1 + \dfrac{K_2}{s}\right)\dfrac{\alpha}{s+\alpha}}{1 + \left(K_1 + \dfrac{K_2}{s}\right)\dfrac{\alpha}{s+\alpha}}$$

$$= \frac{(sK_1 + K_2)\alpha}{s(s+\alpha) + (K_1 s + K_2)\alpha} = \frac{\left(s + \dfrac{K_2}{K_1}\right)K_1\alpha}{s^2 + s\alpha(K_1+1) + K_2\alpha}$$

The poles for this system occur at

$$s = \frac{-\alpha(K_1+1)}{2} \pm \sqrt{\left(\frac{\alpha(K_1+1)}{2}\right)^2 - K_2\alpha}$$

Note that if $\alpha(K_1 + 1) > 0$ and if $K_2\alpha > 0$, we are assured that both poles are in the left half-plane. Therefore, $\alpha(K_1 + 1) > 0$ and $K_2\alpha > 0$ are the conditions for stability. Now since

$$E(s) = X(s)\frac{1}{1 + H_c(s)H_p(s)}$$
$$= \frac{1}{s}\,\frac{s(s+\alpha)}{s^2 + \alpha(K_1+1)s + K_2\alpha},$$

then

$$\lim_{s\to 0} sE(s) = 0 \qquad \text{implies that} \qquad \lim_{t\to\infty} e(t) = 0,$$

for $\alpha(K_1 + 1) > 0$ and $K_2\alpha > 0$, so we can track a step with this stable system.

S26.8

(a)
$$\frac{Y(s)}{X(s)} = H(s)C(s)$$
$$= \frac{1}{(s+1)(s-2)}\left(\frac{s-2}{s+3}\right)$$

We can see from this expression that the overall transfer function for the system is

$$\frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+3)},$$

a stable system. In effect, the system was made stable by canceling a pole of $H(s)$ with a zero of $C(s)$. In practice, if this is not done exactly, i.e., if any component tolerances cause the zero to be slightly off from $s = 2$, the resultant system will still be unstable.

(b) $$\frac{Y(s)}{X(s)} = \frac{C(s)H(s)}{1 + C(s)H(s)} = \frac{K}{(s+1)(s-2)+K}$$
$$= \frac{K}{s^2 - s + K - 2}$$

The poles are at

$$\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - (K-2)}$$

We see from this that at least one pole is in the right half-plane, i.e., there is instability for all values of K.

(c) $$\frac{Y(s)}{X(s)} = \frac{K(s+a)\dfrac{1}{(s+1)(s-2)}}{1 + K(s+a)\dfrac{1}{(s+1)(s-2)}}$$
$$= \frac{K(s+a)}{(s+1)(s-2) + K(s+a)}$$
$$= \frac{K(s+a)}{s^2 - s - 2 + Ks + Ka} = \frac{K(s+a)}{s^2 + (K-1)s + (Ka-2)}$$

The poles are at

$$-\frac{(K-1)}{2} \pm \sqrt{\left(\frac{K-1}{2}\right)^2 - (Ka-2)}$$

Now, if $Ka - 2 > 0$, the system is stable. $K > 2/a$ because $a > 0$ is assumed. This is true for $1 > a > 0$ and $2 > a > 1$. For $a \geq 2$, the system is stable for $K > 1$.

(d) $$\frac{Y(s)}{X(s)} = \frac{K(s+a)}{s^2 + (K-1)s + (Ka-2)}, \qquad a = 2$$

We want $K - 1 = \omega_n$, $2K - 2 = \omega_n^2$. So

$$(K-1)^2 = 2K - 2,$$
$$K = 3 \qquad \text{or} \qquad K = 1$$

If $K = 1$, then $\omega_n = 0$, so $K = 3$ implies that $\omega_n = 2$.

S26.9

(a) $$\frac{E(s)}{X(s)} = \frac{1}{1 + H(s)} = \frac{s^l}{s^l + G(s)}, \text{ where}$$

$$G(s) = \frac{K \prod_{k=1}^{m} (s - \beta_K)}{\prod_{k=1}^{n-l} (s - \alpha_K)}$$

For $s = 0$, $G(s)$ constant $\equiv g$.

$$E(s) = \frac{(1/s)s^l}{s^l + g} \qquad \text{and} \qquad \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s^l}{s^l + g} = 0$$

Thus, $\lim_{t \to \infty} e(t) = 0$.

(b) $E(s) = \dfrac{s^{-1}}{s + G(s)}$ for $l = 1$, $x(t) = u_{-2}(t)$

So

$$\lim_{s\to 0} sE(s) = \left.\frac{1}{s + G(s)}\right|_{s=0} = \frac{1}{g} = \text{Constant}$$

(c) $E(s) = \dfrac{s^{1-k}}{s + G(s)}$, $\quad sE(s) = \dfrac{s^{2-k}}{s + G(s)}$

For $k > 2$,

$$\lim_{s\to 0} sE(s) = \infty, \qquad \lim_{t\to\infty} e(t) = \infty$$

(d) (i) $E(s) = \dfrac{s^{l-k}}{s^l + G(s)}$, $\quad sE(s) = \dfrac{s^{l-k+1}}{s^l + G(s)}$

If $k \le l$, then

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^{l-k+1}}{s^l + G(s)} = \frac{0}{0 + g} = 0,$$

so $\lim_{t\to\infty} e(t) = 0$.

(ii) If $k = l + 1$ and since

$$E(s) = \frac{s^{l-k}}{s^l + G(s)},$$

then

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{1}{s^l + G(s)} = \frac{1}{g} = \text{Constant}$$

Thus, $\lim_{t\to\infty} e(t) = \text{Constant}$.

(iii) If $k > l + 1$, then since

$$E(s) = \frac{s^{l-k}}{s^l + G(s)}, \qquad sE(s) = \frac{s^{l-k+1}}{s^l + G(s)}$$

$\lim_{s\to 0} sE(s) = \infty$ implies $\lim_{t\to\infty} e(t) = \infty$.

S26.10

(a) $\dfrac{E(z)}{X(z)} = \dfrac{1}{1 + H(z)}$,

$$E(z) = \frac{X(z)}{1 + H(z)} = \frac{\dfrac{z}{z-1}}{1 + \dfrac{1}{(z-1)(z+\frac{1}{2})}} = \frac{z(z+\frac{1}{2})}{(z-1)(z+\frac{1}{2}) + 1}$$

$$= \frac{z^2 + \frac{1}{2}z}{z^2 - \frac{1}{2}z + \frac{1}{2}} = 1 + \frac{z - \frac{1}{2}}{z^2 - \frac{1}{2}z + \frac{1}{2}}$$

The poles are at $\frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{1}{2}}$. These poles are inside the unit circle and therefore yield stable inverse z-transforms, so $e[n] = \delta[n] + (2 \text{ stable sequences})$. So $\lim_{n\to\infty} e[n] = 0$.

(b) $H(z) = \dfrac{A(z)}{(z-1)B(z)}$

since $H(z)$ has a pole at $z = 1$. Now

$$\frac{E(z)}{X(z)} = \frac{1}{1 + H(z)} = \frac{(z-1)B(z)}{(z-1)B(z) + A(z)},$$

$$E(z) = \frac{\left(\dfrac{z}{z-1}\right)(z-1)B(z)}{(z-1)B(z) + A(z)} \qquad \text{for } x[n] = u[n]$$

$$= \frac{zB(z)}{(z-1)B(z) + A(z)}$$

Furthermore, we know that

$$\frac{Y(z)}{X(z)} = \frac{H(z)}{1 + H(z)} = \frac{(z-1)B(z)}{(z-1)B(z) + A(z)}$$

There are no poles for $|z| > 1$ because $h[n]$ is stable. Therefore,

$$E(z) = \frac{zB(z)}{(z-1)B(z) + A(z)}$$

has no poles for $|z| > 1$, and $\lim_{n\to\infty} e[n] = 0$.

(c) $H(z) = \dfrac{z^{-1}}{1 - z^{-1}} = \dfrac{1}{z-1}$,

$$\frac{E(z)}{X(z)} = \frac{1}{1 + H(z)} = \frac{z-1}{z},$$

$$E(z) = \frac{z-1}{z}X(z) = \left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right) \qquad \text{for } x[n] = u[n]$$

$$= 1 \Rightarrow e[n] = \delta[n],$$

so $e[n] = 0,\ n \geq 1$

(d) $H(z) = \dfrac{\frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1 + \frac{1}{4}z^{-1})(1 - z^{-1})}$,

$$\frac{E(z)}{X(z)} = \frac{1}{1 + H(z)} = \frac{(1 + \frac{1}{4}z^{-1})(1 - z^{-1})}{(1 + \frac{1}{4}z^{-1})(1 - z^{-1}) + \frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}},$$

$$E(z) = \frac{(1 + \frac{1}{4}z^{-1})}{(1 + \frac{1}{4}z^{-1})(1 - z^{-1}) + \frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}}$$

$$= 1 + \tfrac{1}{4}z^{-1}$$

Therefore,

$$e[n] = \delta[n] + \tfrac{1}{4}\delta[n-1]$$
$$= 0, \qquad n \geq 2$$

(e) $\dfrac{E(z)}{X(z)} = \dfrac{1}{1 + H(z)}, \qquad H(z) = \dfrac{X(z)}{E(z)} - 1$

For $x[n] = u[n]$, we have

$$X(z) = \frac{1}{1 - z^{-1}}$$

We would like

$$e[n] = \sum_{k=0}^{N-1} a_k \delta[n - k],$$

so

$$E(z) = \sum_{k=0}^{N-1} a_k z^{-k}$$

Therefore,

$$H(z) = \frac{1 - (1 - z^{-1})\left(\sum_{k=0}^{N-1} a_k z^{-k}\right)}{(1 - z^{-1})\left(\sum_{k=0}^{N-1} a_k z^{-k}\right)}$$

(f) $H(z) = \dfrac{z^{-1} + z^{-2} - z^{-3}}{(1 + z^{-1})(1 - z^{-1})^2}, \qquad \dfrac{E(z)}{X(z)} = \dfrac{1}{1 + H(z)}$

Now $x[n] = (n + 1)u[n]$ and

$$X(z) = \frac{1}{(1 - z^{-1})^2},$$

so

$$\begin{aligned} E(z) &= \frac{(1 + z^{-1})(1 - z^{-1})^2 \dfrac{1}{(1 - z^{-1})^2}}{(1 + z^{-1})(1 - z^{-1})^2 + z^{-1} + z^{-2} - z^{-3}} \\ &= \frac{1 + z^{-1}}{1} \end{aligned}$$

and

$$\begin{aligned} e[n] &= \delta[n] + \delta[n - 1] \\ &= 0, \qquad n \geq 2 \end{aligned}$$