Math 335 HW 10 NAME: KEY
Due Wednesday 11/5 5:15pm

Practice Problems *(Do not turn in.)*
Sec 5.3 #27, 33, 41
Sec 12.2 #1, 5, 9

Print out this page and write all answers directly on this worksheet. Show all work. Your answers must be clear and legible. All pages must be stapled.

1.) *[4 points]* Let J and Y represent the Bessel functions of the first and second kind, respectively. Write the solution of each ODE below. Part (a) is done for you as an example.

a.) $$x^2y'' + xy' + (x^2 - 1)y = 0$$

$$v = 1 \quad \Rightarrow \quad y = C_1J_1(x) + C_2Y_1(x)$$

b.) $$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

$$v = \frac{1}{3}$$

$$\boxed{y = C_1 J_{\frac{1}{3}}(x) + C_2 Y_{\frac{1}{3}}(x)}$$

c.) $$4x^2y'' + 4xy' + (4x^2 - 25)y = 0$$

$$x^2y'' + xy' + \left(x^2 - \frac{25}{4}\right)y = 0$$

$$v = \frac{5}{2}$$

$$\boxed{y = C_1 J_{5/2}(x) + C_2 Y_{5/2}(x)}$$

2.) *[6 points]* The Bessel Function of the First Kind is

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\ 2^{2n+v}\ \Gamma(1+v+n)} x^{2n+v}$$

Prove by direct calculation that the derivative of J_0 is J_{-1}:

$$\frac{d}{dx} J_0(x) = J_{-1}(x)$$

$$J_{-1}(x) = \sum_{n=0} \frac{(-1)^n}{n!\, 2^{2n-1}\, \Gamma(n)} x^{2n-1}$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n}\, \Gamma(1+n)} x^{2n}$$

$$\frac{d}{dx} J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n}\, \Gamma(1+n)} 2n\, x^{2n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-1)!\ 2^{2n-1}\ \underbrace{\Gamma(1+n)}}\, x^{2n-1}$$

Since $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1+n) = n\Gamma(n)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\ 2^{2n-1}\ \Gamma(n)} x^{2n-1}$$

$$= J_{-1}(x)$$

$\square$

3.) *[10 points]* We want to establish the orthogonality of sine functions on the general interval $(-p, p)$. Let m,n be integers and fix a constant $p > 0$.

a.) Suppose $m \neq n$. Compute $\int_{-p}^{p} \sin\frac{m\pi x}{p} \sin\frac{n\pi x}{p}\, dx$.

<u>*Hint*</u>*: Use the trig identity* $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.

$$\int_{-p}^{p} \sin\frac{m\pi x}{p} \sin\frac{n\pi x}{p}\, dx$$

$$= \int_{-p}^{p} \frac{1}{2}\left[\cos\frac{(m-n)\pi x}{p} - \cos\frac{(m+n)\pi x}{p}\right] dx$$

$$= \frac{1}{2}\left[\frac{p}{(m-n)\pi}\sin\frac{(m-n)\pi x}{p} - \frac{p}{(m+n)\pi}\sin\frac{(m+n)\pi x}{p}\right]_{-p}^{p}$$

$$= \frac{1}{2}\left[\frac{p}{(m-n)\pi}\cancelto{0}{\sin(m-n)\pi} - \frac{p}{(m-n)\pi}\cancelto{0}{\sin(m-n)\pi}\right.$$

$$\left. - \frac{p}{(m+n)\pi}\cancelto{0}{\sin(m+n)\pi} + \frac{p}{(m+n)\pi}\cancelto{0}{\sin -(m+n)\pi}\right]$$

$$= \boxed{0}$$

#2 continued...

b.) Suppose $m = n$. Compute

$$\int_{-p}^{p} \sin\frac{m\pi x}{p}\,\sin\frac{n\pi x}{p}\,dx = \int_{-p}^{p}\left(\sin\frac{n\pi x}{p}\right)^2 dx.$$

<u>Hint</u>: *Use the integration formula* $\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

$$\int_{-p}^{p}\left(\sin\frac{n\pi x}{p}\right)^2 dx$$

$$\boxed{u = \frac{n\pi x}{p} \qquad du = \frac{n\pi}{p}}$$

$$= \frac{p}{n\pi}\int_{-p}^{p} \frac{n\pi}{p}\left(\sin\frac{n\pi x}{p}\right)^2 dx$$

$$= \frac{p}{n\pi}\int_{-n\pi}^{n\pi} (\sin u)^2\, du$$

$$= \frac{p}{n\pi}\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_{-n\pi}^{n\pi}$$

$$= \frac{p}{n\pi}\left[\frac{1}{2}n\pi - \frac{1}{4}\cancelto{0}{\sin 2n\pi} + \frac{1}{2}n\pi + \frac{1}{4}\cancelto{0}{\sin(-2n\pi)}\right]$$

$$= \frac{p}{n\pi}\left[n\pi\right] = \boxed{p}$$