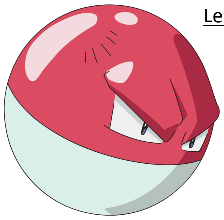


Lecture 5: Vector Fields

Voltorb's Goals for the Day

- Discuss vector fields in 2D and 3D
- Practice drawing 2D vector fields
- Define curl and divergence

9.7 Curl and Divergence

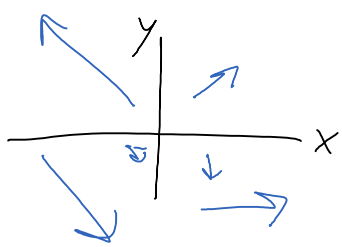
A vector field assigns a vector to each point.

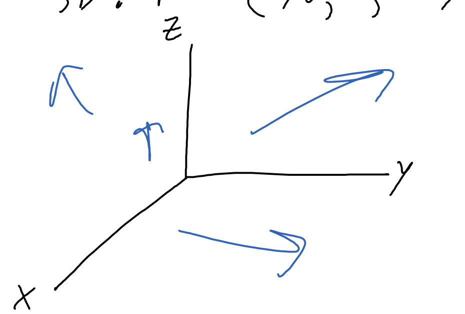
2D: $\vec{F} = \langle x^2 y, x+2y \rangle$

3D: $\vec{F} = \langle x^2 y, x+2y, xz^3 \cos y \rangle$

To visualize a vector field, we draw an arrow indicating the direction at each point.
The length of the arrow reflects the magnitude.

2D: $\vec{F} = \langle N, P \rangle$ 3D: $\vec{F} = \langle N, P, R \rangle$

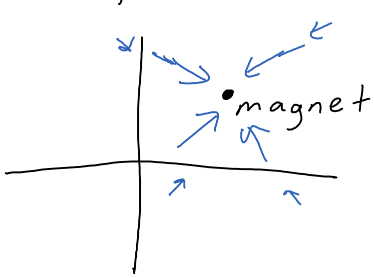


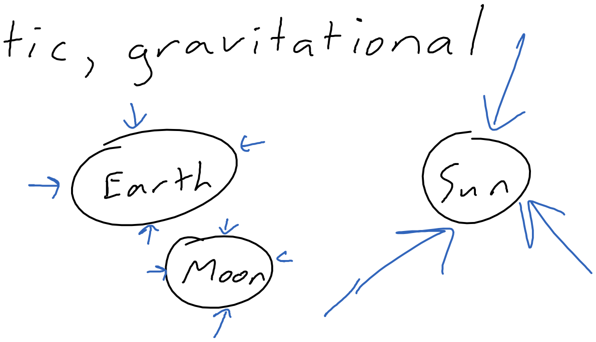


Applications

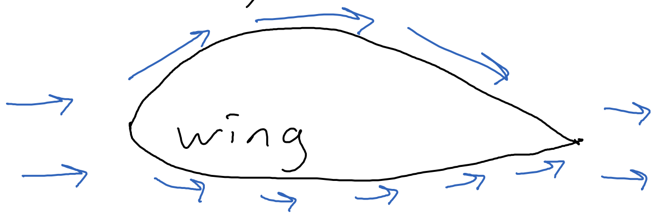
① $\vec{F}$ = force

physical, electromagnetic, gravitational



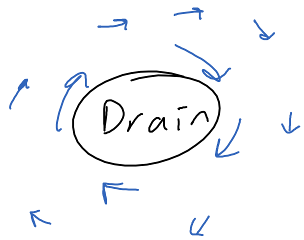


② $\vec{F}$ = velocity field of a fluid (liquid/air)



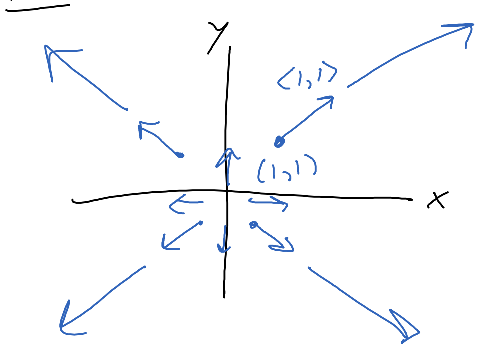
Bernoulli's Principle

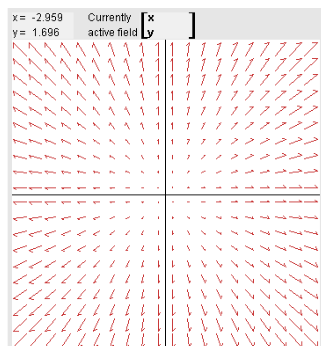
High velocity
$\Rightarrow$ Low pressure
$\Rightarrow$ LIFT



Whirlpool

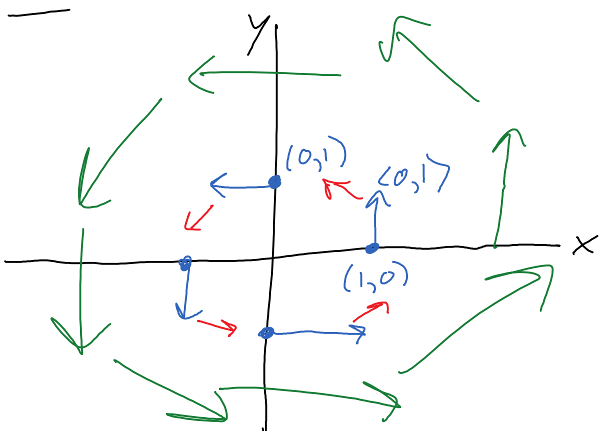
<u>Ex</u> Sketch $F = \langle x, y \rangle$

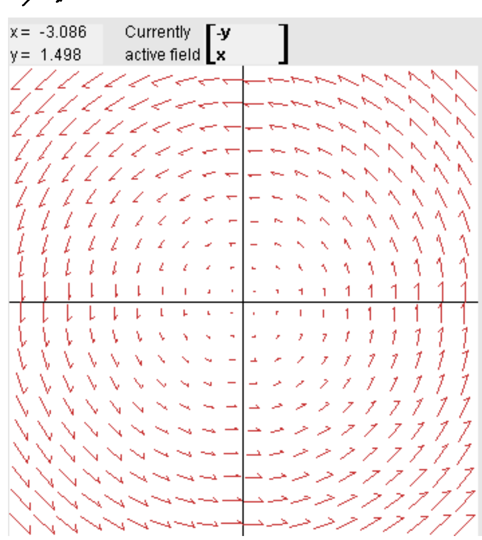




The origin is a repeller (source).

<u>Ex</u> Sketch $\vec{F} = \langle -y, x \rangle$.





Whirlpool (Spin Field)

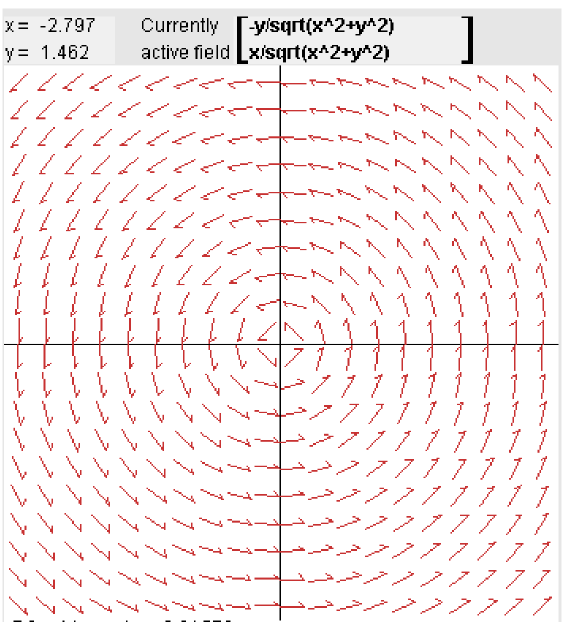
Ex How do we make the whirlpool uniform so that each vector has length one?

$\vec{F} = \langle -y, x \rangle$ whirlpool

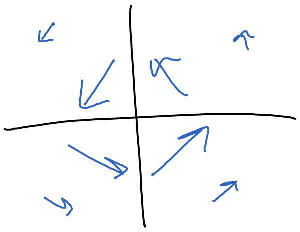
$$\|\vec{F}\| = \sqrt{(-y)^2 + (x^2)} = \sqrt{x^2 + y^2}$$

$$\vec{G} = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$$

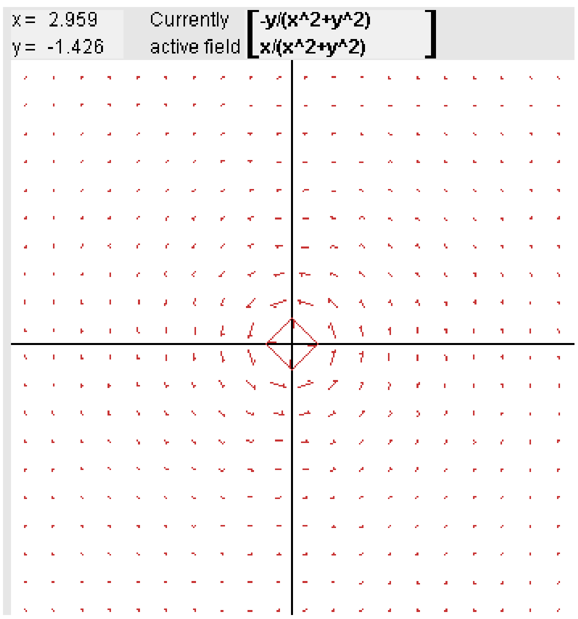
Then every vector has length $\|\vec{G}\| = 1$.



Ex How would you make a hurricane?

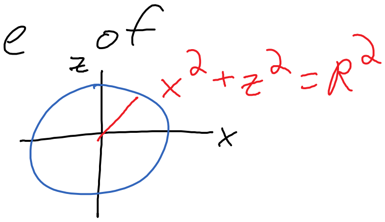


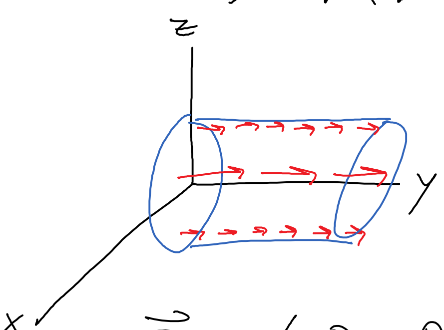
$$\vec{H} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$



Ex Laminar Flow

Model the velocity of a fluid flowing through a cylindrical pipe of radius R.





As fluid gets closer to the wall, it slows down.

$$\vec{F} = \langle 0, R^2-(x^2+z^2), 0 \rangle$$

Def $\vec{F} = \langle M, N, P \rangle$

Divergence of $\vec{F}$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Curl of $\vec{F}$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

Notation Think $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M, N, P \rangle$$

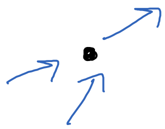
$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

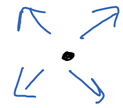
Note divergence is a scalar.

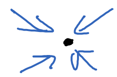
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

Application $\vec{F}$ = velocity field of a fluid

Divergence tells us the net change of fluid at a point.



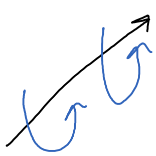
$\nabla \cdot \vec{F} > 0$ expansion  source

$\nabla \cdot \vec{F} < 0$ compression  sink

$\nabla \cdot \vec{F} = 0$ incompressible

(divergence-free, solenoidal)

Curl tells us the rotation of the field.



$\nabla \times \vec{F}$ = axis of rotation

$\|\nabla \times \vec{F}\|$ = speed of rotation

$\nabla \times \vec{F} = \langle 0,0,0 \rangle \Rightarrow$ irrotational

Application Maxwell's Equations

Electric Field $\vec{E}(x,y,z,t)$

Magnetic Field $\vec{H}(x,y,z,t)$

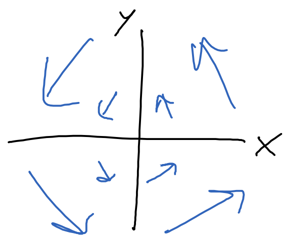
Assume we are in a vacuum.

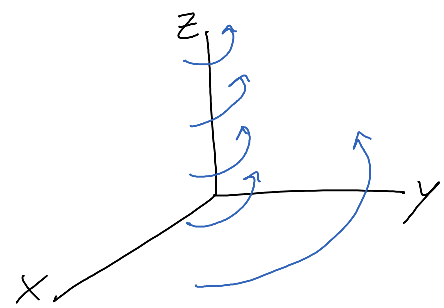
$$\left.\begin{array}{l} \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{H} = 0 \end{array}\right] \text{ incompressible}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad (c = \text{speed of light})$$

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial E}{\partial t}$$

Ex Compute divergence and curl of $\vec{F} = \langle -y, x, 0 \rangle$.





Divergence

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle -y, x, 0 \rangle$$

$$= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0)$$

$$= 0 + 0 + 0$$

$$= \boxed{0}$$

incompressible

Curl

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle -y, x, 0 \rangle$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 \end{vmatrix} - j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ -y & 0 \end{vmatrix} + k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ -y & x \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(1 - -1)$$

$$= \boxed{\langle 0, 0, 2 \rangle}$$

Rotate around z-axis

Angular speed $\|\langle 0,0,2 \rangle\| = 2$