

# ELEC 309

## Signals and Systems

### Homework 4 Assignment

#### Time-Domain Analysis of LTI Systems

1. A continuous-time system is described by the input-output equation

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$$

with initial condition  $y(0) = 1$  and input  $x(t) = u(t)$ .

Using the classical approach to solving LCCDEs:

- (a) Determine the output  $y(t)$  of the system.

The output  $y(t)$  consists of two parts, or  $y(t) = y_p(t) + y_h(t)$ .

The particular solution  $y_p(t) = A$  satisfies

$$5y_p'(t) + 10y_p(t) = 2x(t)$$

For  $t < 0$ ,  $x(t) = 0$  yields

$$5y_p'(t) + 10y_p(t) = 2x(t) \Rightarrow 5(0) + 10A = 2(0) \Rightarrow A = 0.$$

For  $t \geq 0$ ,  $x(t) = 1$  yields

$$5y_p'(t) + 10y_p(t) = 2x(t) \Rightarrow 5(0) + 10A = 2(1) \Rightarrow A = 0.2.$$

Therefore, the particular solution is given by

$$y_p(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.2 & \text{for } t \geq 0 \end{cases} = 0.2u(t).$$

Since this is a first-order differential equation, the complete homogeneous solution is given as  $y_h(t) = K_1 e^{s_1 t}$ . To find  $s_1$ , we let  $y_h(t) = K e^{st}$  in the homogeneous form of the LCCDE. Therefore,

$$\begin{aligned} 5y_h'(t) + 10y_h(t) &= 5sK e^{st} + 10K e^{st} = 0 \\ &= (5s + 10)K e^{st} = 0 \\ &= 5K e^{st}(s + 2) = 0. \end{aligned}$$

Therefore, the characteristic equation is  $s + 2 = 0$ , and  $s_1 = -2$ . Therefore, the complete homogeneous solution is  $y_h(t) = K_1 e^{-2t}$ . Now, we have

$$y(t) = y_p(t) + y_h(t) = 0.2u(t) + K_1 e^{-2t}.$$

Plugging in initial conditions for  $t < 0$ , we have

$$y(0^-) = 0.2u(0^-) + K_1 e^{-2(0^-)} = K_1 = 1.$$

Plugging in initial conditions for  $t \geq 0$ , we have

$$y(0^+) = 0.2u(0^+) + K_1 e^{-2(0^+)} = 0.2 + K_1 = 1.$$

Therefore,  $K_1 = 1$  for  $t < 0$  and  $K_1 = 0.8$  for  $t \geq 0$ , and the complete solution is given by

$$y(t) = y_p(t) + y_h(t) = \begin{cases} e^{-2t} & \text{for } t < 0 \\ 0.2 + 0.8e^{-2t} & \text{for } t \geq 0 \end{cases} = e^{-2t} + [0.2 - 0.2e^{-2t}] u(t).$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Part (a)
% Find y(t) for t<0
syms y(t)
x = 0;
y(t) = dsolve(5*diff(y)+10*y==2*x,y(0)==1);
disp('Problem 1, Part (a): y(t) for t<0:')
pretty(y(t))
% Find y(t) for t>=0
syms y(t)
x = 1;
y(t) = dsolve(5*diff(y)+10*y==2*x,y(0)==1);
disp('Problem 1, Part (a): y(t) for t>=0:')
pretty(y(t))
```

MATLAB Output:

Problem 1, Part (a): y(t) for t<0:  
exp(-2 t)

Problem 1, Part (a): y(t) for t>=0:  
exp(-2 t) 4 1  
----- + -  
5 5

(b) Determine the zero-input output  $y_{zi}(t)$  of the system.

The zero-input output is the response when the input is  $x(t) = 0$ . Therefore,  $y_p(t) = 0$ , and the zero-input output  $y_{zi}(t)$  consists of only one part, or  $y_{zi}(t) = y_h(t)$ .

As we observed in Part (a), the complete homogeneous solution is  $y_h(t) = K_1 e^{-2t}$ . Plugging in initial conditions, we have

$$y_{zi}(0) = K_1 e^{-2(0)} = K_1 = 1.$$

Therefore,  $K_1 = 1$ , and the zero-input output is given by

$$y_{zi}(t) = e^{-2t}.$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Part (b)
% Find y_zi(t)
syms y_zi(t)
y_zi(t) = dsolve(5*diff(y_zi)+10*y_zi==0,y_zi(0)==1);
disp('Problem 1, Part (b): y_zi(t) for all t:')
pretty(y_zi(t))
```

MATLAB Output:

```
Problem 1, Part (b): y_zi(t) for all t:
exp(-2 t)
```

(c) Determine the zero-state output  $y_{zs}(t)$  of the system.

The zero-state output is the response when initial conditions are zero, or  $y(0) = 0$ . Therefore, the zero-state output  $y_{zs}(t)$  consists of two parts, or  $y_{zs}(t) = y_p(t) + y_h(t)$ .

As we observed in Part (a), the particular solution is given by  $y_p(t) = 0.2u(t)$ , and the complete homogeneous solution is  $y_h(t) = K_1 e^{-2t}$ . Now, we have

$$y_{zs}(t) = y_p(t) + y_h(t) = 0.2u(t) + K_1 e^{-2t}.$$

Plugging in initial conditions for  $t < 0$ , we have

$$y_{zs}(0^-) = 0.2u(0^-) + K_1 e^{-2(0^-)} = K_1 = 0.$$

Plugging in initial conditions for  $t \geq 0$ , we have

$$y_{zs}(0^+) = 0.2u(0^+) + K_1 e^{-2(0^+)} = 0.2 + K_1 = 0.$$

Therefore,  $K_1 = 0$  for  $t < 0$  and  $K_1 = -0.2$  for  $t \geq 0$ , and the complete solution is given by

$$y_{zs}(t) = y_p(t) + y_h(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.2 - 0.2e^{-2t} & \text{for } t \geq 0 \end{cases} = \boxed{[0.2 - 0.2e^{-2t}] u(t)}.$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Part (c)
% Find y_zs(t) for t<0
syms y_zs(t)
x = 0;
y_zs(t) = dsolve(5*diff(y_zs)+10*y_zs==2*x,y_zs(0)==0);
disp('Problem 1, Part (c): y_zs(t) for t<0:')
pretty(y_zs(t))
% Find y_zs(t) for t>=0
syms y_zs(t)
x = 1;
y_zs(t) = dsolve(5*diff(y_zs)+10*y_zs==2*x,y_zs(0)==0);
disp('Problem 1, Part (c): y_zs(t) for t>=0:')
pretty(y_zs(t))
```

MATLAB Output:

Problem 1, Part (c): y\_zs(t) for t<0:  
0

Problem 1, Part (c): y\_zs(t) for t>=0:  
1 exp(-2 t)  
- - -  
5 5

(d) If  $y(0) = 0$ , determine the impulse response  $h(t)$  of the system.

The step response is the zero-state output when the input is  $x(t) = u(t)$ . Therefore, the step response is given by

$$y_s(t) = y_{zs}(t) = [0.2 - 0.2e^{-2t}] u(t).$$

The impulse response is the derivative of the step response. Therefore, the impulse response is given by

$$h(t) = \frac{dy_s(t)}{dt} = [0.4e^{-2t}] u(t) + \underbrace{[0.2 - 0.2e^{-2t}]}_0 \delta(t) = \boxed{0.4e^{-2t}u(t)}.$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Parts (d)(e)
% Find y_s(t) for t<0
syms h(t) y_s(t)
x = 0;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d): h(t) for t<0:')
pretty(h(t))
disp('Problem 1, Part (e): y_s(t) for t<0:')
pretty(y_s(t))
% Find y_s(t) for t>=0
syms y_s(t)
x = 1;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d): h(t) for t>=0:')
pretty(h(t))
disp('Problem 1, Part (e): y_s(t) for t>=0:')
pretty(y_s(t))
```

MATLAB Output:

```
Problem 1, Part (d): h(t) for t<0:
0
```

```
Problem 1, Part (d): h(t) for t>=0:
exp(-2 t) 2
-----
```

(e) If  $y(0) = 0$ , determine the step response  $y_s(t)$  of the system.

The step response is the zero-state output when the input is  $x(t) = u(t)$ . Therefore, the step response is given by

$$y_s(t) = y_{zs}(t) = \boxed{[0.2 - 0.2e^{-2t}] u(t)}.$$

Problem 1, Part (a) solution using MATLAB code:

```
% Homework 4 Problem 1
% Parts (d)(e)
% Find y_s(t) for t<0
syms h(t) y_s(t)
x = 0;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d): h(t) for t<0:')
pretty(h(t))
disp('Problem 1, Part (e): y_s(t) for t<0:')
pretty(y_s(t))
% Find y_s(t) for t>=0
syms y_s(t)
x = 1;
y_s(t) = dsolve(5*diff(y_s)+10*y_s==2*x,y_s(0)==0);
h(t) = diff(y_s(t));
disp('Problem 1, Part (d): h(t) for t>=0:')
pretty(h(t))
disp('Problem 1, Part (e): y_s(t) for t>=0:')
pretty(y_s(t))
```

MATLAB Output:

```
Problem 1, Part (e): y_s(t) for t<0:
0
```

```
Problem 1, Part (e): y_s(t) for t>=0:
1   exp(-2 t)
- - -
5       5
```

2. A discrete-time system is described by the input-output equation

$$y[n] - \frac{1}{2}y[n-1] = 2x[n]$$

with initial condition  $y[-1] = 3$  and input  $x[n] = \left(-\frac{1}{2}\right)^n u[n]$ .

Using the classical approach to solving LCCDEs:

(a) Determine the output  $y[n]$  of the system.

The output  $y[n]$  consists of two parts, or  $y[n] = y_p[n] + y_h[n]$ .

The particular solution  $y_p[n] = A \left(-\frac{1}{2}\right)^n$  satisfies

$$y_p[n] - \frac{1}{2}y_p[n-1] = 2x[n]$$

For  $n < 0$ ,  $x[n] = 0$  yields

$$\begin{aligned} y_p[n] - \frac{1}{2}y_p[n-1] &= 2(0) \\ \Rightarrow A \left(-\frac{1}{2}\right)^n - \frac{1}{2}A \left(-\frac{1}{2}\right)^{n-1} &= 0 \\ \Rightarrow A \left(-\frac{1}{2}\right)^n + A \left(-\frac{1}{2}\right)^n &= 0 \Rightarrow 2A = 0 \Rightarrow A = 0. \end{aligned}$$

For  $n \geq 0$ ,  $x[n] = \left(-\frac{1}{2}\right)^n$  yields

$$\begin{aligned} y_p[n] - \frac{1}{2}y_p[n-1] &= 2 \left(-\frac{1}{2}\right)^n \\ \Rightarrow A \left(-\frac{1}{2}\right)^n - \frac{1}{2}A \left(-\frac{1}{2}\right)^{n-1} &= 2 \left(-\frac{1}{2}\right)^n \\ \Rightarrow A \left(-\frac{1}{2}\right)^n + A \left(-\frac{1}{2}\right)^n &= 2 \left(-\frac{1}{2}\right)^n \\ \Rightarrow 2A &= 2 \Rightarrow A = 1. \end{aligned}$$

Therefore, the particular solution is given by

$$y_p[n] = \begin{cases} 0 & \text{for } n < 0 \\ \left(-\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = \left(-\frac{1}{2}\right)^n u[n].$$

Since this is a first-order difference equation, the complete homogeneous solution is given as  $y_h[n] = K_1 z_1^n$ . To find  $s_1$ , we let  $y_h[n] = K z^n$  in the homogeneous form of the LCCDE. Therefore,

$$\begin{aligned} y_h[n] - \frac{1}{2}y_h[n-1] &= 0 \\ K z^n - \frac{1}{2}K z^{n-1} &= 0 \\ K z^{n-1} \left(z - \frac{1}{2}\right) &= 0. \end{aligned}$$

Therefore, the characteristic equation is  $z - \frac{1}{2} = 0$ , and  $z_1 = \frac{1}{2}$ . Therefore, the complete homogeneous solution is  $y_h[n] = K_1 \left(\frac{1}{2}\right)^n$ . Now, we have

$$y[n] = y_p[n] + y_h[n] = \left(-\frac{1}{2}\right)^n u[n] + K_1 \left(\frac{1}{2}\right)^n.$$

Plugging in the initial condition for  $n < 0$ , we have

$$y[-1] = \left(-\frac{1}{2}\right)^{-1} u[-1] + K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 3 \Rightarrow K_1 = \frac{3}{2} = 1.5.$$

To plug in initial conditions for  $n \geq 0$ , we need  $y[0]$ , which is given as

$$y[0] = \frac{1}{2}y[-1] + 2x[0] = \frac{1}{2}(3) + 2(1) = \frac{7}{2} = 3.5.$$

Plugging in the initial condition for  $n \geq 0$ , we have

$$y[0] = \left(-\frac{1}{2}\right)^0 u[0] + K_1 \left(\frac{1}{2}\right)^0 = 1 + K_1 = 3.5 \Rightarrow K_1 = 2.5.$$

Therefore,  $K_1 = 1.5$  for  $n < 0$  and  $K_1 = 2.5$  for  $n \geq 0$ , and the complete solution is given by

$$y[n] = y_p[n] + y_h[n] = \begin{cases} 1.5 \left(\frac{1}{2}\right)^n & \text{for } n < 0 \\ \left(-\frac{1}{2}\right)^n + 2.5 \left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = 1.5 \left(\frac{1}{2}\right)^n + \left[\left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right] u[n].$$

**(b)** Determine the zero-input output  $y_{zi}[n]$  of the system.

The zero-input output is the response when the input is  $x[n] = 0$ . Therefore,  $y_p[n] = 0$ , and the zero-input output  $y_{zi}[n]$  consists of only one part, or  $y_{zi}[n] = y_h[n]$ .

As we observed in Part (a), the complete homogeneous solution is given as  $y_h[n] = K_1 z_1^n$ . Plugging in the initial condition, we have

$$y_{zi}[-1] = K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 3 \Rightarrow K_1 = \frac{3}{2} = 1.5.$$

Therefore,  $K_1 = 1.5$ , and the complete solution is given by

$$y_{zi}[n] = 1.5 \left(\frac{1}{2}\right)^n.$$



(c) Determine the zero-state output  $y_{zs}[n]$  of the system.

The zero-state output is the response when initial conditions are zero, or  $y[-1] = 0$ . Therefore, the zero-state output  $y_{zs}[n]$  consists of two parts, or  $y_{zs}[n] = y_p[n] + y_h[n]$ .

As we observed in Part (a), the particular solution is given by  $y_p[n] = \left(-\frac{1}{2}\right)^n u[n]$ , and the complete homogeneous solution is  $y_h[n] = K_1 z_1^n$ . Now, we have

$$y_{zs}[n] = y_p[n] + y_h[n] = \left(-\frac{1}{2}\right)^n u[n] + K_1 \left(\frac{1}{2}\right)^n.$$

Plugging in the initial condition for  $n < 0$ , we have

$$y_{zs}[-1] = \left(-\frac{1}{2}\right)^{-1} u[-1] + K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 0 \Rightarrow K_1 = 0.$$

To plug in initial conditions for  $n \geq 0$ , we need  $y[0]$ , which is given as

$$y_{zs}[0] = \frac{1}{2}y[-1] + 2x[0] = \frac{1}{2}(0) + 2(1) = 2.$$

Plugging in the initial condition for  $n \geq 0$ , we have

$$y_{zs}[0] = \left(-\frac{1}{2}\right)^0 u[0] + K_1 \left(\frac{1}{2}\right)^0 = 1 + K_1 = 2 \Rightarrow K_1 = 1.$$

Therefore,  $K_1 = 0$  for  $n < 0$  and  $K_1 = 1$  for  $n \geq 0$ , and the complete solution is given by

$$y_{zs}[n] = y_p[n] + y_h[n] = \begin{cases} \left(-\frac{1}{2}\right)^n & \text{for } n < 0 \\ \left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = \left[ \left(-\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n \right] u[n].$$

(d) If  $y[-1] = 0$ , determine the impulse response  $h[n]$  of the system.

To determine the impulse response we will solve for the step response first (see Part (e) below), and determine the impulse response as

$$h[n] = y_s[n] - y_s[n-1] = \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1]$$

For  $n < 0$ ,

$$\begin{aligned} h[n] &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1] \\ &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] (0) - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] (0) = 0. \end{aligned}$$

For  $n = 0$ ,

$$\begin{aligned} h[n] &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1] \\ &= \left[4 - 2\left(\frac{1}{2}\right)^0\right] (1) - \left[4 - 2\left(\frac{1}{2}\right)^{-1}\right] (0) = 2. \end{aligned}$$

For  $n > 0$ ,

$$\begin{aligned} h[n] &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n] - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] u[n-1] \\ &= \left[4 - 2\left(\frac{1}{2}\right)^n\right] (1) - \left[4 - 2\left(\frac{1}{2}\right)^{n-1}\right] (1) \\ &= 2\left(\frac{1}{2}\right)^{n-1} - 2\left(\frac{1}{2}\right)^n = 4\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{2}\right)^n = 2\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

Therefore, the impulse response of the system is given by

$$h[n] = \begin{cases} 0 & \text{for } n < 0 \\ 2\left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = 2\left(\frac{1}{2}\right)^n u[n] = \left(\frac{1}{2}\right)^{n-1} u[n].$$

(e) If  $y[-1] = 0$ , determine the step response  $y_s[n]$  of the system.

Note that the input  $x[n] = u[n]$  and the output  $y[n] = y_s[n]$ . The impulse response  $h[n]$  consists of two parts, or  $y_s[n] = y_{sp}[n] + y_{sh}[n]$ .

The particular solution  $y_p[n] = A$  satisfies

$$y_{sp}[n] - \frac{1}{2}y_{sp}[n-1] = 2x[n]$$

For  $n < 0$ ,  $x[n] = 0$  yields

$$y_{sp}[n] - \frac{1}{2}y_{sp}[n-1] = 2(0) \Rightarrow A - \frac{1}{2}A = 0 \Rightarrow \frac{1}{2}A = 0 \Rightarrow A = 0.$$

For  $n \geq 0$ ,  $x[n] = 1$  yields

$$y_{sp}[n] - \frac{1}{2}y_{sp}[n-1] = 2(1) \Rightarrow A - \frac{1}{2}A = 2 \Rightarrow \frac{1}{2}A = 2 \Rightarrow A = 4.$$

Therefore, the particular solution is given by

$$y_{sp}[n] = \begin{cases} 0 & \text{for } n < 0 \\ 4 & \text{for } n \geq 0 \end{cases} = 4u[n].$$

As we observed in Part (a), the complete homogeneous solution is  $y_{sh}[n] = K_1 \left(\frac{1}{2}\right)^n$ . Now, we have

$$y_s[n] = y_{sp}[n] + y_{sh}[n] = 4u[n] + K_1 \left(\frac{1}{2}\right)^n.$$

Plugging in the initial condition for  $n < 0$ , we have

$$y[-1] = 4u[-1] + K_1 \left(\frac{1}{2}\right)^{-1} = 2K_1 = 0 \Rightarrow K_1 = 0.$$

To plug in initial conditions for  $n \geq 0$ , we need  $y[0]$ , which is given as

$$y[0] = \frac{1}{2}y[-1] + 2x[0] = \frac{1}{2}(0) + 2(1) = 2.$$

Plugging in the initial condition for  $n \geq 0$ , we have

$$y[0] = 4u[0] + K_1 \left(\frac{1}{2}\right)^0 = 4 + K_1 = 2 \Rightarrow K_1 = -2.$$

Therefore,  $K_1 = 0$  for  $n < 0$  and  $K_1 = -2$  for  $n \geq 0$ , and the complete solution is given by

$$y_s[n] = y_{sp}[n] + y_{sh}[n] = \begin{cases} 0 & \text{for } n < 0 \\ 4 - 2\left(\frac{1}{2}\right)^n & \text{for } n \geq 0 \end{cases} = \left[4 - 2\left(\frac{1}{2}\right)^n\right] u[n].$$