24

Butterworth Filters

To illustrate some of the ideas developed in Lecture 23, we introduce in this lecture a simple and particularly useful class of filters referred to as *Butterworth filters*. Filters in this class are specified by two parameters, the cutoff frequency and the filter order. The frequency response of these filters is monotonic, and the sharpness of the transition from the passband to the stopband is dictated by the filter order. For continuous-time Butterworth filters, the poles associated with the square of the magnitude of the frequency response are equally distributed in angle on a circle in the *s*-plane, concentric with the origin and having a radius equal to the cutoff frequency. When the cutoff frequency and the filter order have been specified, the poles characterizing the system function are readily obtained. Once the poles are specified, it is straightforward to obtain the differential equation characterizing the filter.

In this lecture, we illustrate the design of a discrete-time filter through the use of the impulse-invariant design procedure applied to a Butterworth filter. The filter specifications are given in terms of the discrete-time frequency variable and then mapped to a corresponding set of specifications for the continuous-time filter. A Butterworth filter meeting these specifications is determined. The resulting continuous-time system function is then mapped to the desired discrete-time system function.

A limitation on the use of impulse invariance as a design procedure for discrete-time systems is the requirement that the continuous-time filter be bandlimited to avoid aliasing. An alternative procedure, called the *bilinear transformation*, corresponds to a mapping of the entire imaginary axis in the *s*-plane to once around the unit circle. Consequently, there is no aliasing introduced by this procedure. However, since the imaginary axis has infinite length and the unit circle has a finite circumference, by necessity there must be a nonlinear distortion in mapping between the two frequency axes. The use of the bilinear transformation is therefore limited to mapping filters that are approximately piecewise constant. For filters of this type, the inherent nonlinear distortion in the frequency axis is easily accommodated by prewarping the critical frequencies (e.g., passband edge and stopband edge) prior to carrying out the design of the associated continuous-time system. This procedure is illustrated with the design of a Butterworth filter.

Suggested Reading

Section 6.5, The Class of Butterworth Frequency-Selective Filters, pages 422–428

Section 9.7.3, Butterworth Filters, pages 611–614

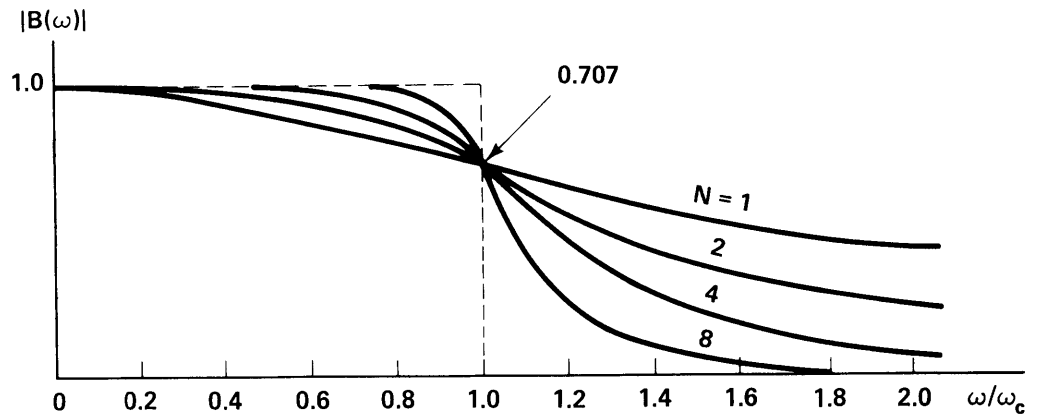Section 10.8.3, The Bilinear Transformation, pages 665–667

BUTTERWORTH FILTERS

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$

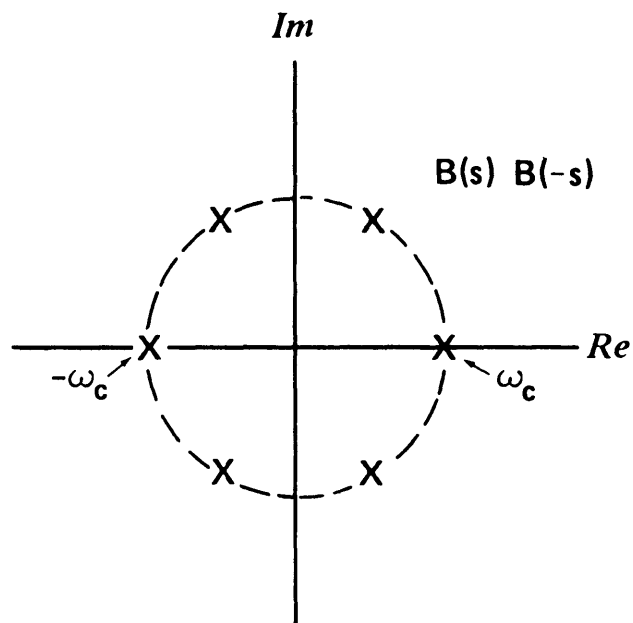$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$

$$B(j\omega)\, B(-j\omega) \longrightarrow B(s)\, B(-s)$$

$$B(s)\, B(-s) = \frac{1}{1 + (s/j\omega_c)^{2N}}$$

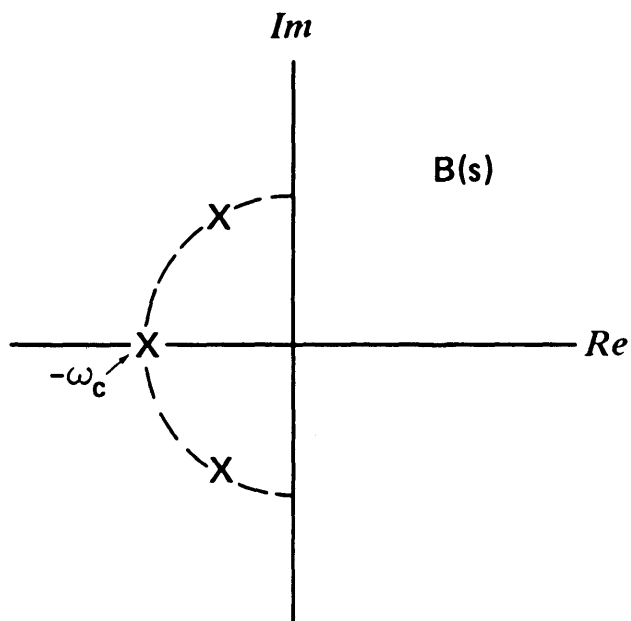poles of B(s) B(-s) at

$$s = (-1)^{1/2N}\, j\omega_c$$

$$B(s) \, B(-s) \; = \; \frac{1}{1 + (s/j\omega_c)^{2N}}$$

B(s) B(-s)

Im / Re axis pole-zero plot

$-\omega_c$ ω_c

$$B(s) \, B(-s) \; = \; \frac{1}{1 + (s/j\omega_c)^{2N}}$$

B(s)

Im / Re axis pole-zero plot

$-\omega_c$

MARKERBOARD
24.1 (a)

Sampling Rate = 10 kHz

Desired Specifications on $H_c(j\omega)$:

- $20 \log_{10} |H_c(j\omega)| \geq -1$ (db)

  at $\omega = 2\pi \cdot 1 \text{kHz}$

- $20 \log_{10} |H_c(j\omega)| \leq -15$ (db)

  at $\omega = 2\pi \cdot 1.5 \text{kHz}$

$x_c(t) \rightarrow \boxed{\begin{array}{c} CT \rightarrow DT \\ f_s \end{array}} \rightarrow x_d[n]$

$x_d[n] \rightarrow \boxed{\begin{array}{c} DT \\ Filter \end{array}} \rightarrow y_d[n]$

$y_d[n] \rightarrow \boxed{\begin{array}{c} DT \rightarrow CT \\ f_s \end{array}} \rightarrow y_c(t)$

$f_s = 10 \text{kHz}$

Corresponding Specifications on $H_d(e^{j\Omega})$

$10 \text{kHz} \rightarrow 2\pi$

$1 \text{kHz} \rightarrow 0.2\pi$

$1.5 \text{kHz} \rightarrow 0.3\pi$

- $20 \log_{10} |H_d(e^{j\Omega})| \geq -1$

  at $\Omega = 0.2\pi$

- $20 \log_{10} |H_d(e^{j\Omega})| \leq -15$

  at $\Omega = 0.3\pi$

$h_d[n] = h_c(nT)$

$T = ?$

$H_c(j\omega)$

1kHz 1.5kHz frequency

TRANSPARENCY
24.5
Spectra illustrating
impulse invariance.

$H_c(j\omega)$

1

$-\omega_c$ ω_c ω

$H_c\left(\dfrac{j\Omega}{T}\right)$

$-\omega_c T$ $\omega_c T$ Ω

$1/T$

$$H_d(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} H_c\left[j\left(\frac{\Omega}{T} - \frac{2\pi k}{T} \right) \right]$$

-2π π π 2π Ω

$-(\omega_c T)$ $(\omega_c T)$
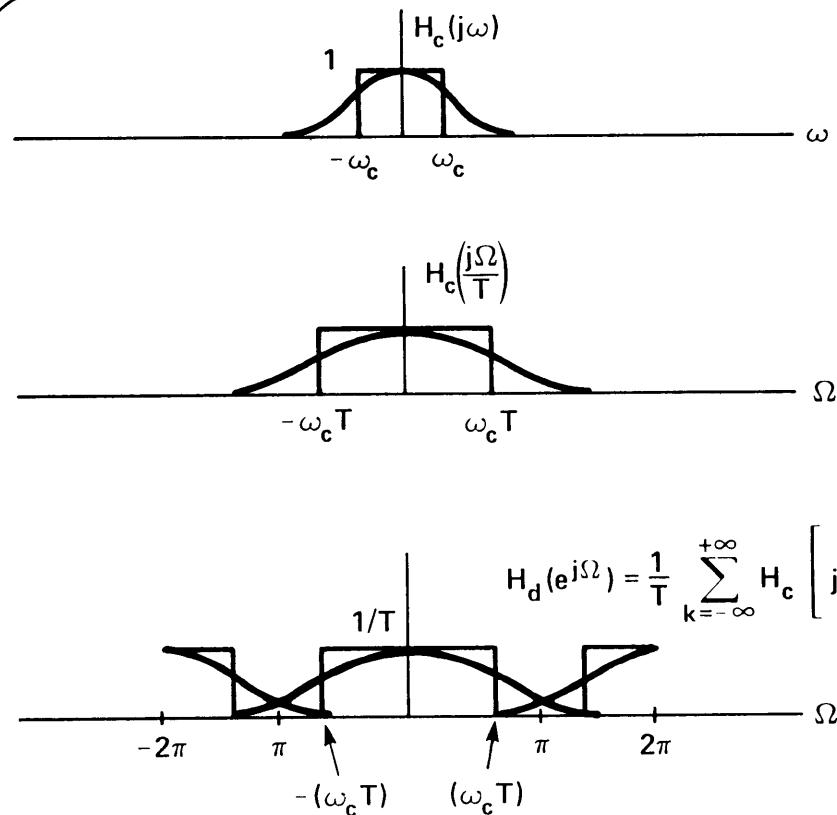
$$h_d[n] = h_b(nT) \qquad T = 1$$

$$20 \log_{10} |B(j\, 0.2\pi)| \geqslant -1$$

$$20 \log_{10} |B(j\, 0.3\pi)| \leqslant -15$$

$$|B(j\omega)|^2 = \frac{1}{1 + (j\omega/j\omega_c)^{2N}}$$
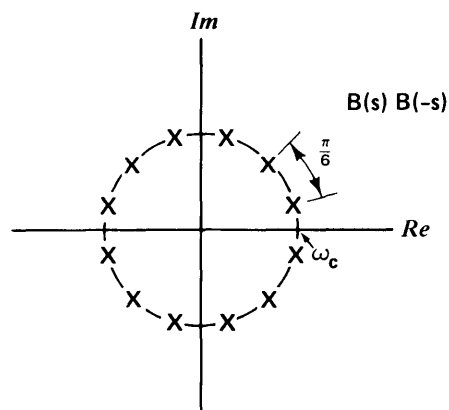
equality for $\quad$ N = 5.88
$\qquad\qquad\quad \omega_c = 0.7047$

choose $\qquad$ N = 6
$\qquad\qquad\quad \omega_c = 0.703$

N=6 $\qquad\qquad \omega_c = 0.703$

1. Determine B(s)

2. Partial fraction expansion

$$B(s) = \sum_k \frac{A_k}{s - s_k}$$

3. $H_d(z) = \sum_k \dfrac{A_k}{1 - e^{s_k} z^{-1}}$
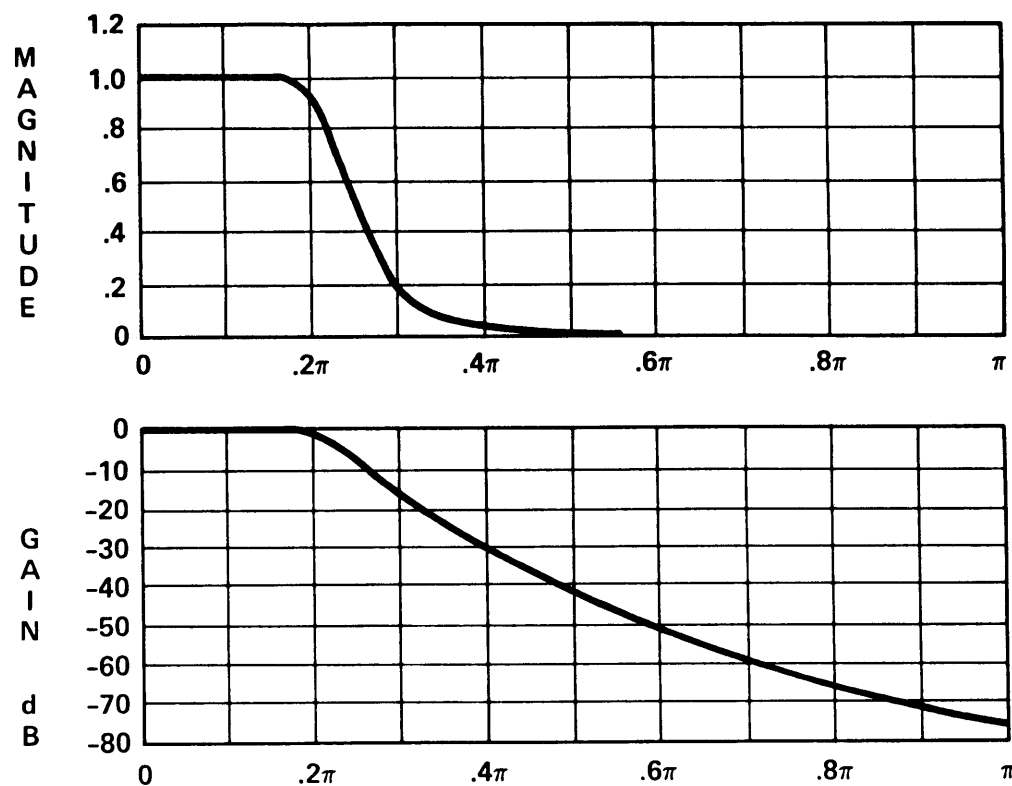
BILINEAR TRANSFORMATION

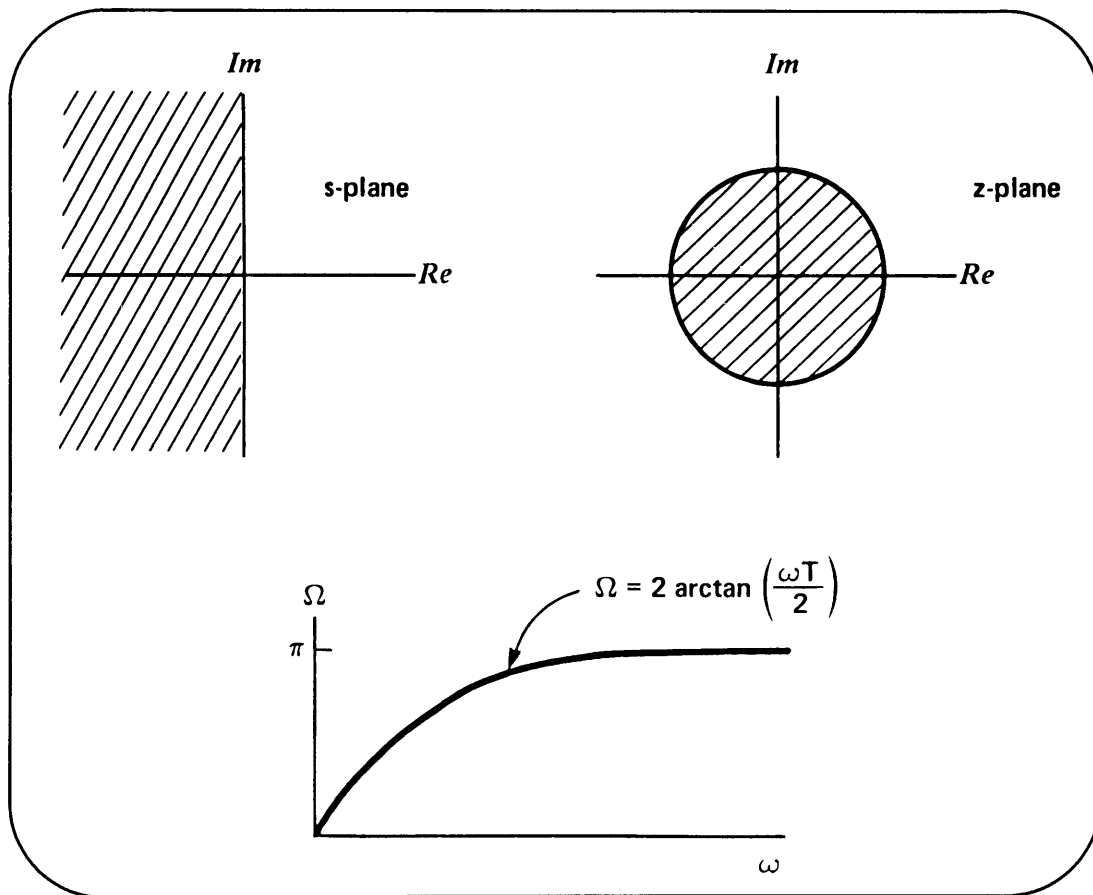$$H_c(s) \longrightarrow H_d(z)$$

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$z = \frac{1 + (T/2)\, s}{1 - (T/2)\, s}$$

For $z = e^{j\Omega}$

$$s = \frac{2}{T} \left[\frac{1 - e^{-j\Omega}}{1 + e^{-j\Omega}} \right] = \frac{2}{T}\, j \tan \frac{\Omega}{2} = j\omega$$

$$\omega = \frac{2}{T} \tan \frac{\Omega}{2}$$

$$\Omega = 2\ arctan\left(\frac{\omega T}{2}\right)$$

$$\Omega = 2\ arctan\left(\frac{\omega T}{2}\right)$$

$$\omega_p = \frac{2}{T}\ tan\left(\frac{\Omega_p}{2}\right)$$

$$\omega_s = \frac{2}{T}\ tan\left(\frac{\Omega_s}{2}\right)$$
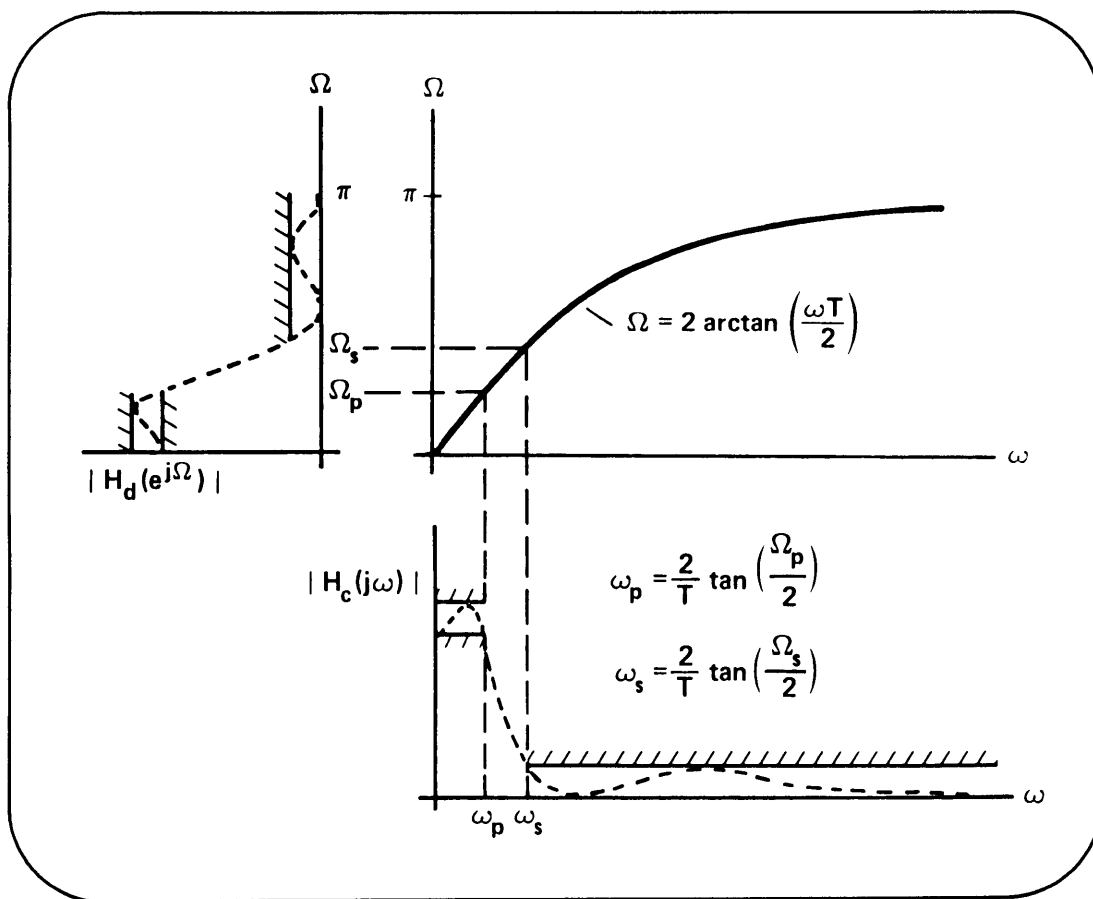
MARKERBOARD
24.1 (b)
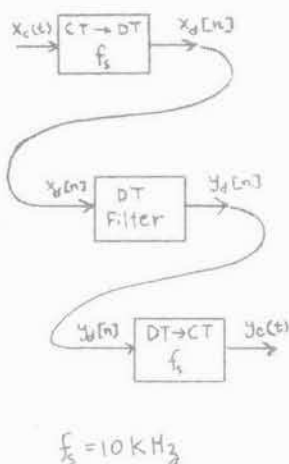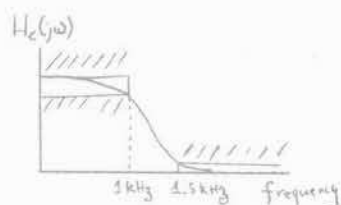
Sampling Rate = 10 kHz

Desired Specifications
on $H_c(j\omega)$:

$\bullet\ 20\ \log_{10} |H_c(j\omega)| \geq -1\ (db)$
at $\omega = 2\pi \cdot 1 kHz$

$\bullet\ 20\ \log_{10} |H_c(j\omega)| \leq -15\ (db)$
at $\omega = 2\pi \cdot 1.5 kHz$

Corresponding
Specifications on $H_d(e^{j\Omega})$

$10 kHz \rightarrow 2\pi$

$1 kHz \rightarrow 0.2\pi$

$1.5 kHz \rightarrow 0.3\pi$

$\bullet\ 20\ \log_{10} |H_d(e^{j\Omega})| \geq -1$
at $\Omega = 0.2\pi$

$\bullet\ 20\ \log_{10} |H_d(e^{j\Omega})| \leq -15$
at $\Omega = 0.3\pi$

$h_d[n] = h_c(nT)$

$T = ?$

$x_c(t) \rightarrow$ [CT → DT, f_s] $\rightarrow x_d[n]$

$x_d[n] \rightarrow$ [DT Filter] $\rightarrow y_d[n]$

$y_d[n] \rightarrow$ [DT → CT, f_s] $\rightarrow y_c(t)$

$f_s = 10 KHz$

$H_c(j\omega)$

1 kHz   1.5 kHz   frequency

TRANSPARENCY
24.12
Determination of the parameters for a Butterworth filter to be mapped to a discrete-time filter with the specifications in Transparency 24.6.

$$\Omega = 2\ \text{arctan}\left(\frac{\omega T}{2}\right)\quad T=1$$

$$\omega = 2\ \tan\left(\frac{\Omega}{2}\right)$$

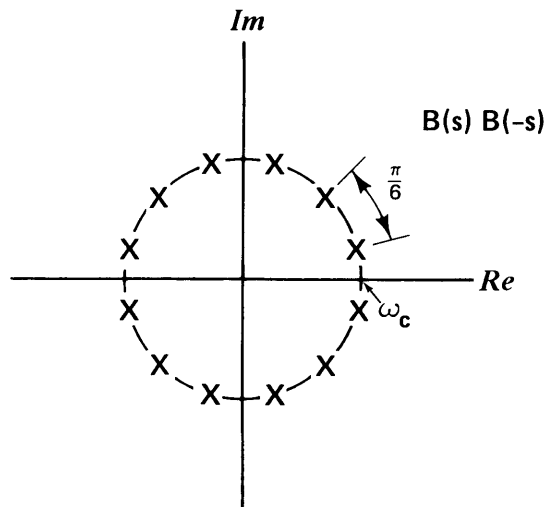$$20\ \log_{10}\left|G\left(j\ 2\ \tan\left(\frac{0.2\pi}{2}\right)\right)\right| \geq -1$$

$$20\ \log_{10}\left|G\left(j\ 2\ \tan\left(\frac{0.3\pi}{2}\right)\right)\right| \leq -15$$

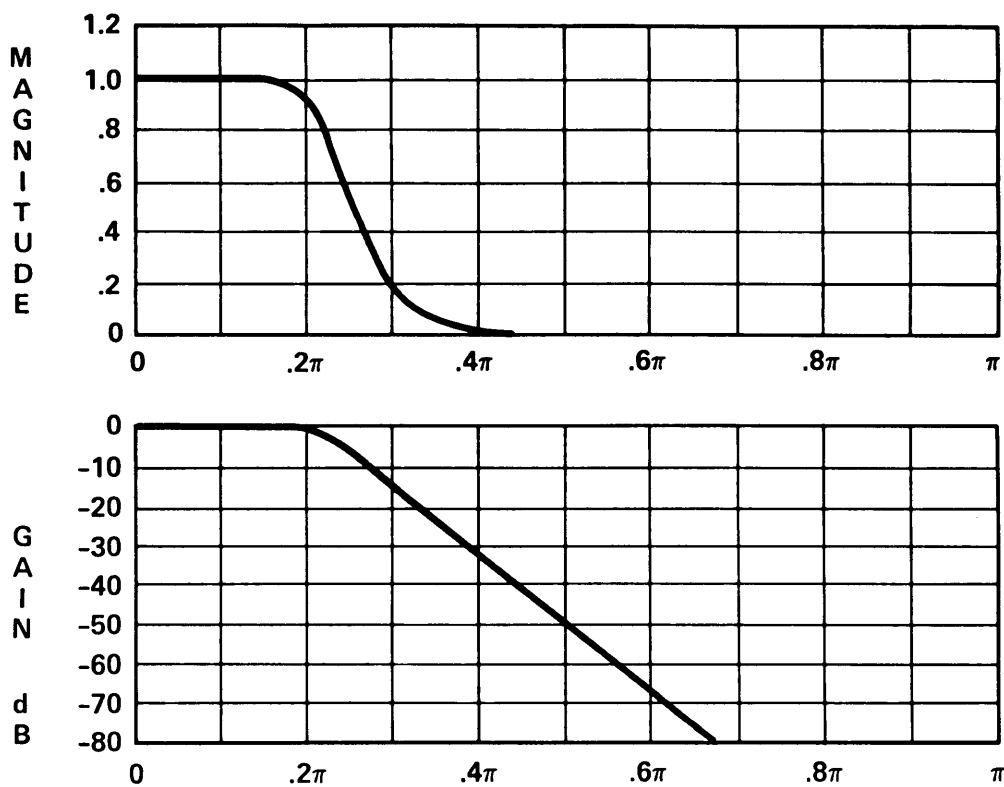**equality for   N = 5.3**

choose   N = 6

$$\omega_c = 0.76622$$

N=6 ω_c = 0.76622

Im

B(s) B(-s)

$\frac{\pi}{6}$

Re

ω_c

1. Determine B(s)

2. $s \rightarrow 2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$

TRANSPARENCY 24.13
The pole-zero plot associated with the squared magnitude function for the desired Butterworth filter and the steps involved in the determination of the discrete-time filter.

MAGNITUDE

1.2
1.0
.8
.6
.4
.2
0

0   .2π   .4π   .6π   .8π   π

TRANSPARENCY 24.14
Frequency response for the discrete-time filter obtained by mapping a Butterworth filter to a digital filter through the bilinear transformation.

GAIN dB

0
-10
-20
-30
-40
-50
-60
-70
-80

0   .2π   .4π   .6π   .8π   π

TRANSPARENCY 24.15
Comparison of the frequency responses of digital filters obtained through impulse invariance and through the bilinear transformation.
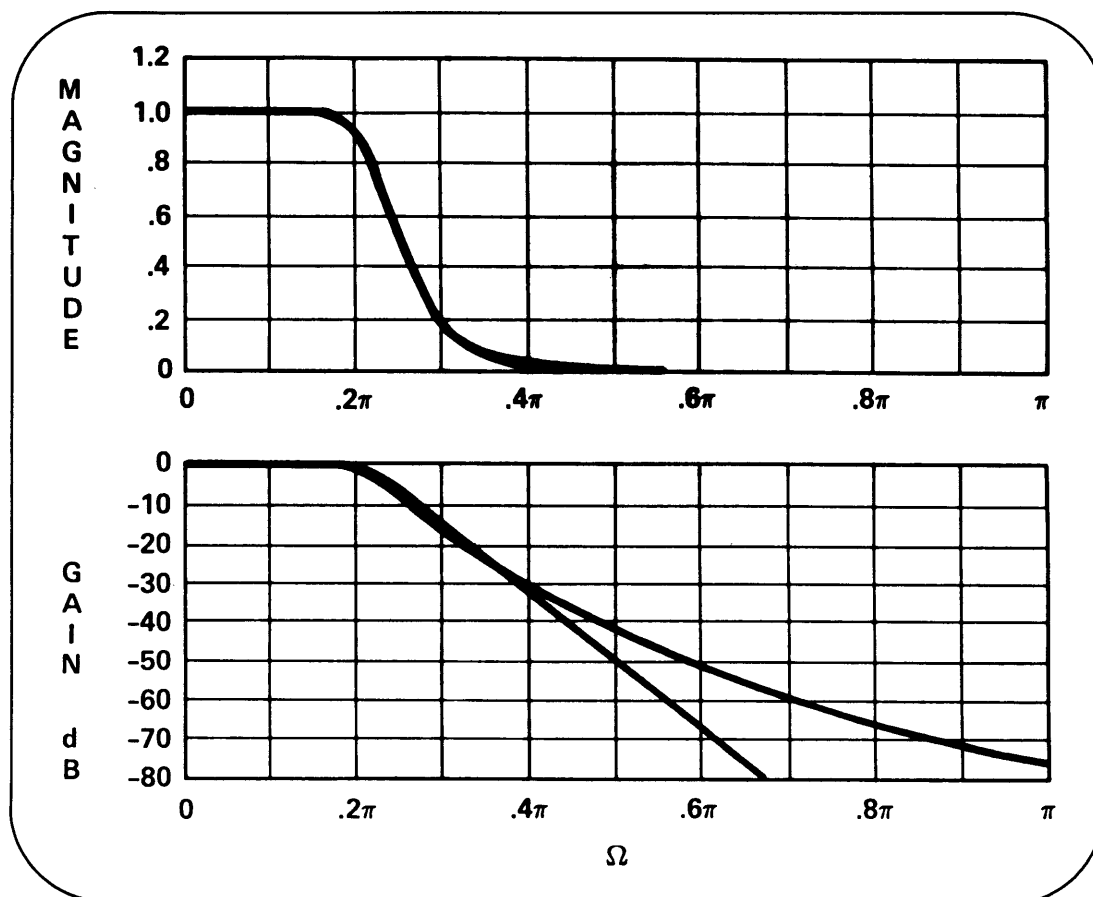
MIT OpenCourseWare
http://ocw.mit.edu

Resource: Signals and Systems
Professor Alan V. Oppenheim

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.