Examples: Cartesian Coordinates

Sketch the following vector fields:

$$\mathbf{H} = \frac{1}{x}\,\hat{\mathbf{y}} \quad \left(\frac{\text{A}}{\text{m}}\right)$$

$$\mathbf{E} = x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}} \quad \left(\frac{\text{V}}{\text{m}}\right)$$

Examples: Cylindrical Coordinates

Sketch the following vector fields:

$$\mathbf{H} = -1\,\hat{\boldsymbol{\phi}} \quad \left(\frac{\text{A}}{\text{m}}\right)$$

$$\mathbf{E} = \sin\phi\;\hat{\mathbf{r}} \quad \left(\frac{\text{V}}{\text{m}}\right)$$

Dr. Gregory J. Mazzaro
Spring 2015

ELEC 318 – *Electromagnetic Fields*

Lecture 3(c)

Review of Vector Calculus: Differential Length/Area/Volume, Integrals

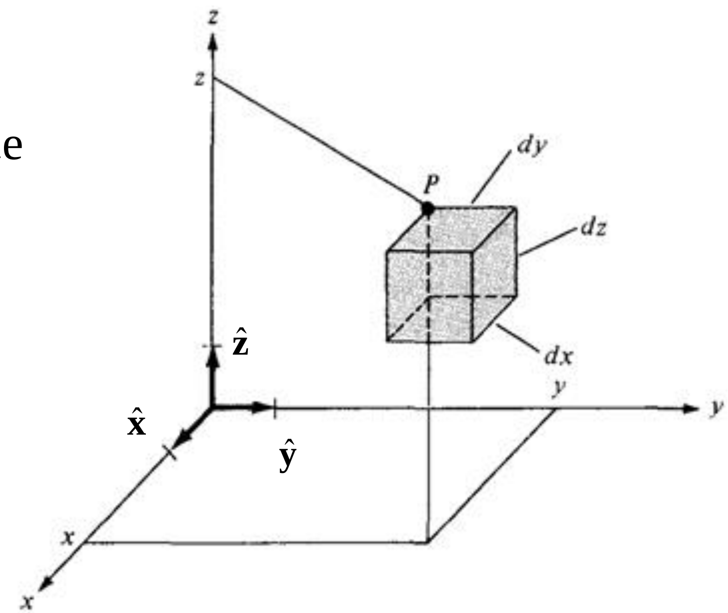
THE CITADEL, THE MILITARY COLLEGE OF SOUTH CAROLINA
171 Moultrie Street, Charleston, SC 29409

The Citadel — Department of Electrical & Computer Engineering

differential length, area, volume:

-- useful for integration along a path, open/closed surface
-- allows us to answer questions like

"What is the strength of **E** along this path?"
"What is the density of **H** across this surface?"
"How much of ***q*** is contained within this shape?"



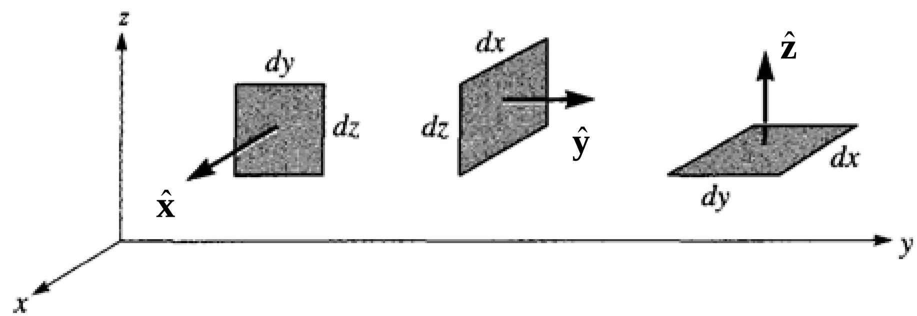
differential length (displacement, distance):

$$d\mathbf{l} = dx\ \hat{\mathbf{x}} + dy\ \hat{\mathbf{y}} + dz\ \hat{\mathbf{z}}$$

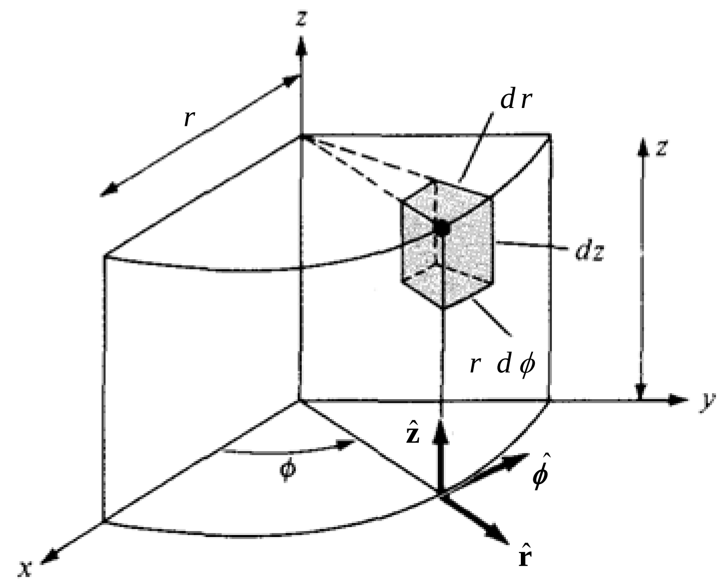
differential area:

$$d\mathbf{S} = dS\ \hat{\mathbf{n}}$$

$$= dx\ dy\ \hat{\mathbf{z}} \quad \text{or} \quad dy\ dz\ \hat{\mathbf{x}} \quad \text{or} \quad dz\ dx\ \hat{\mathbf{y}}$$



differential volume: $dv = dx\ dy\ dz$



<u>differential length</u> (displacement, distance):

$$d\mathbf{l} = dr\ \hat{\mathbf{r}} + r\ d\phi\ \hat{\boldsymbol{\phi}} + dz\ \hat{\mathbf{z}}$$

<u>differential area</u>:

$$d\mathbf{S} = dS\ \hat{\mathbf{n}}$$

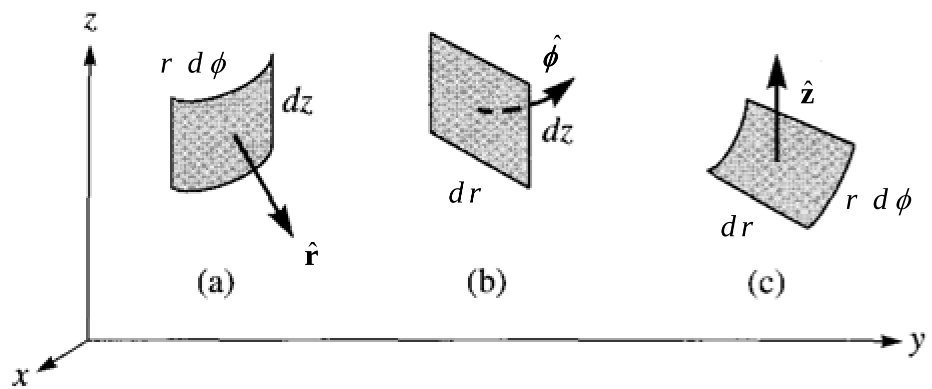
$$= r\ d\phi\ dz\ \hat{\mathbf{r}} \quad \text{or} \quad dr\ dz\ \hat{\boldsymbol{\phi}} \quad \text{or} \quad r\ dr\ d\phi\ \hat{\mathbf{z}}$$

<u>differential volume</u>: $dv = r\ dr\ d\phi\ dz$

<u>Example</u>:

If the charge density in the wedge (above, right), is $\rho_v = 2r - 3z$ (C/cm^3), then the amount of charge contained in the tiny curved cube is

$$\left(2r^2 - 3rz\right) dr\ d\phi\ dz \quad (\text{coulombs})$$



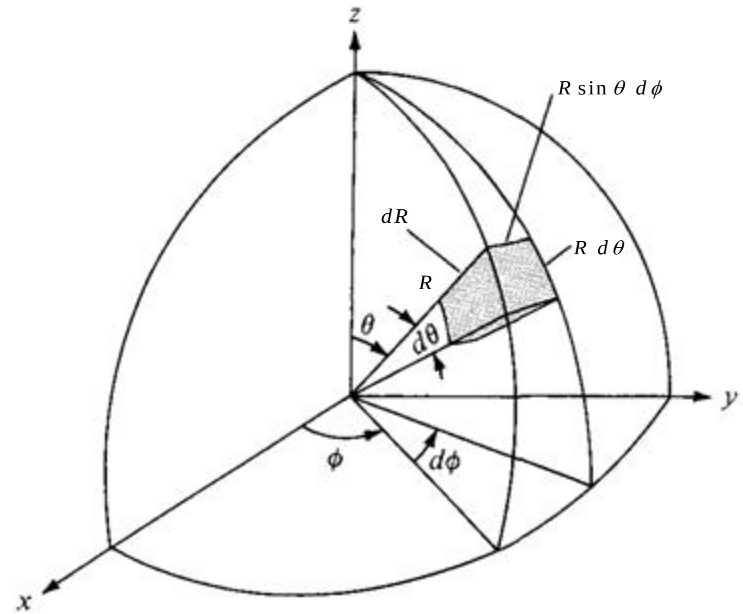
Example: Cylindrical Area

Find the area of a cylindrical surface described by $r = 5$, $30° \le \phi \le 60°$, and $0 \le z \le 3$.

Diff. Length, Area, Volume: Spherical

differential length (displacement, distance):

$$d\mathbf{l} = dR\ \hat{\mathbf{R}} + R\ d\theta\ \hat{\boldsymbol{\theta}} + R\sin\theta\ d\phi\ \hat{\boldsymbol{\phi}}$$



differential area:

$$d\mathbf{S} = dS\ \hat{\mathbf{n}}$$

$$= R^2 \sin\theta\ d\theta\ d\phi\ \hat{\mathbf{R}} \quad \text{or}$$

$$= R\sin\theta\ dR\ d\phi\ \hat{\boldsymbol{\theta}} \quad \text{or}$$

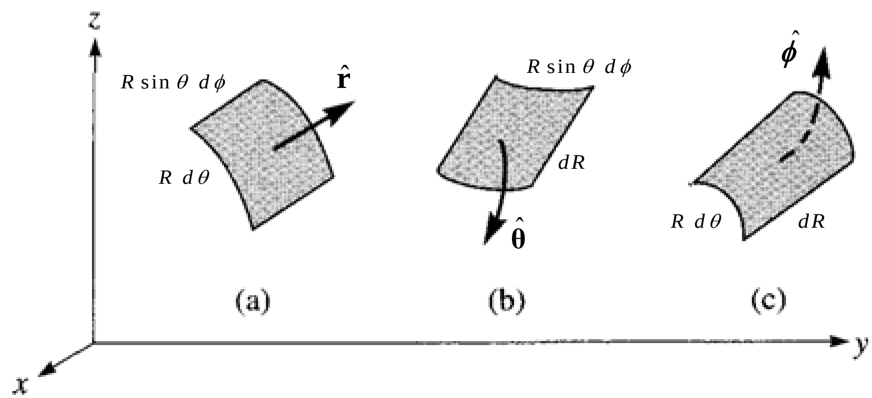
$$= R\ dR\ d\theta\ \hat{\boldsymbol{\phi}}$$

differential volume: $dv = R^2 \sin\theta\ dR\ d\theta\ d\phi$

Example:

If the magnetic flux density crossing the sphere (above, right), is $\boldsymbol{B} = 4\sin\phi\,\mathbf{R}$ (Wb/m^2), then the amount of flux through the tiny curved square (on the outside of the sphere) is

$$4R^2 \sin\phi\,\sin\theta\ d\theta\ d\phi \quad \text{(Wb)}$$



Line Integrals

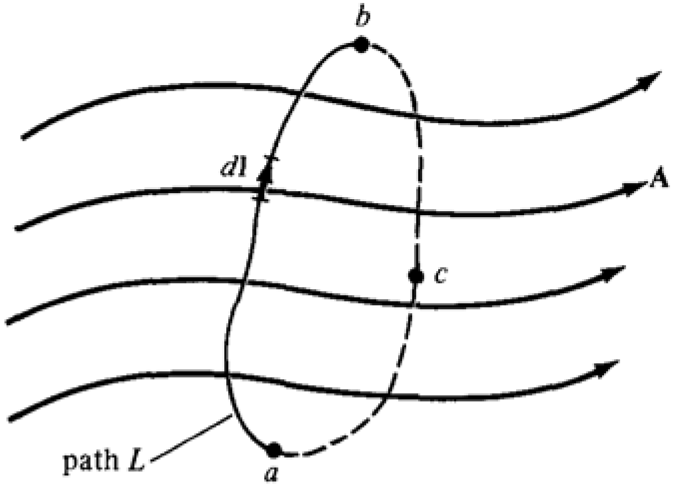
line (single) **integral**:

-- integration of the tangential (parallel) component of a vector (**A**) along a path (*L*):

$$\int_L \mathbf{A} \cdot d\mathbf{l}$$

-- allows us to answer the question

"How much of ***A*** is projected along a path?"



If the path (*L*) is closed (forms a *surface*), then the integral becomes

$$\oint_L \mathbf{A} \cdot d\mathbf{l}$$

Examples:

$$V_{21} = -\int_{P1}^{P2} \mathbf{E} \cdot d\mathbf{l} \qquad I_{\text{enc}} = \oint \mathbf{H} \cdot d\mathbf{l}$$

As before, for rectangular coordinates,

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

for cylindrical coordinates,

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

and for spherical coordinates,

$$d\mathbf{l} = dR\,\hat{\mathbf{R}} + R\,d\theta\,\hat{\boldsymbol{\theta}} + R\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

Example: Mixed Coordinates

$$r = \sqrt{x^2 + y^2}, \ \phi = \tan^{-1}\frac{y}{x}, \ z = z$$

$$x = r\cos\phi, \ \ y = r\sin\phi, \ \ z = z$$

$$\hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{r}} - \sin\phi\,\hat{\boldsymbol{\phi}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{r}} + \cos\phi\,\hat{\boldsymbol{\phi}}$$

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

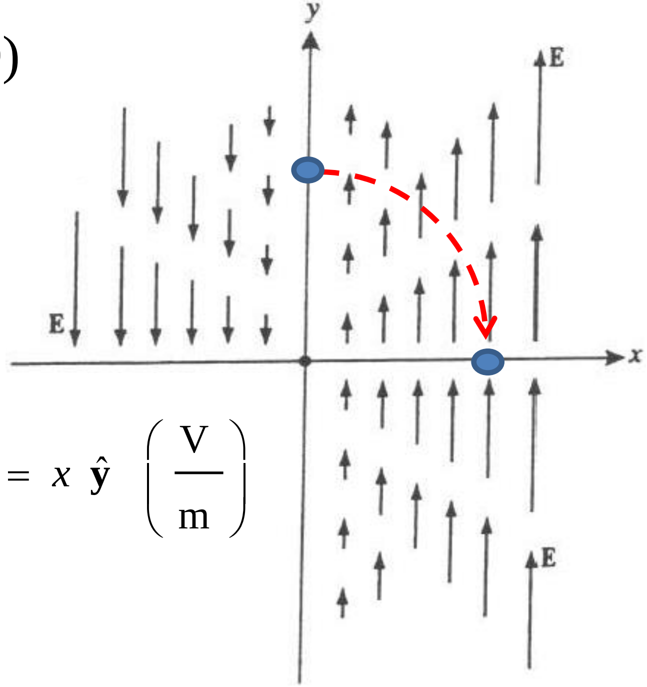
$$\hat{\mathbf{r}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \qquad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

For the path from P_1 ($x = 0$, $y = 2$ m) to P_2 ($x = 2$ m, $y = 0$) and **E** illustrated, compute

$$-\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

$$\mathbf{E} = x\,\hat{\mathbf{y}} \left(\frac{\text{V}}{\text{m}}\right)$$

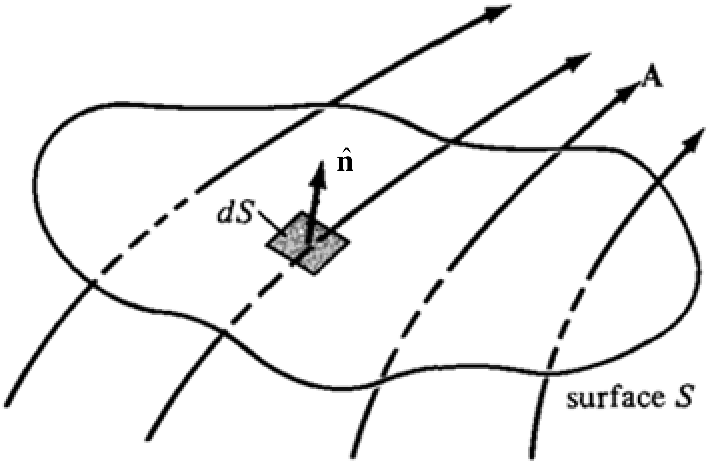


Surface Integrals

surface (double) **integral**:

-- integration of the normal component of a vector (**A**) across a surface (*S*): $\int_S \mathbf{A} \cdot d\mathbf{S}$ ("flux")

-- allows us to answer the question "How much of ***A*** crosses a given surface?"



If the surface (*S*) is closed (forms a *volume*), then the integral becomes $\oint_S \mathbf{A} \cdot d\mathbf{S}$ ("net outward flux")

Examples: $\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$ $\quad Q_{enc} = \oint \mathbf{D} \cdot d\mathbf{S}$

As before, for rectangular coordinates, $d\mathbf{S} = dx\, dy\, \hat{\mathbf{z}}$ or $dy\, dz\, \hat{\mathbf{x}}$ or $dz\, dx\, \hat{\mathbf{y}}$

for cylindrical coordinates, $d\mathbf{S} = r\, d\phi\, dz\, \hat{\mathbf{r}}$ or $dr\, dz\, \hat{\boldsymbol{\phi}}$ or $r\, dr\, d\phi\, \hat{\mathbf{z}}$

for spherical coordinates, $d\mathbf{S} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{R}}$ or $R \sin\theta\, dR\, d\phi\, \hat{\boldsymbol{\theta}}$ or $R\, dR\, d\theta\, \hat{\boldsymbol{\phi}}$

Example: Surface Integral

$$r=\sqrt{x^2+y^2},\quad \phi=\tan^{-1}\frac{y}{x},\quad z=z$$

$$x=r\cos\phi,\quad y=r\sin\phi,\quad z=z$$

$$\Psi=\int_S \mathbf{B}\cdot d\mathbf{S}$$

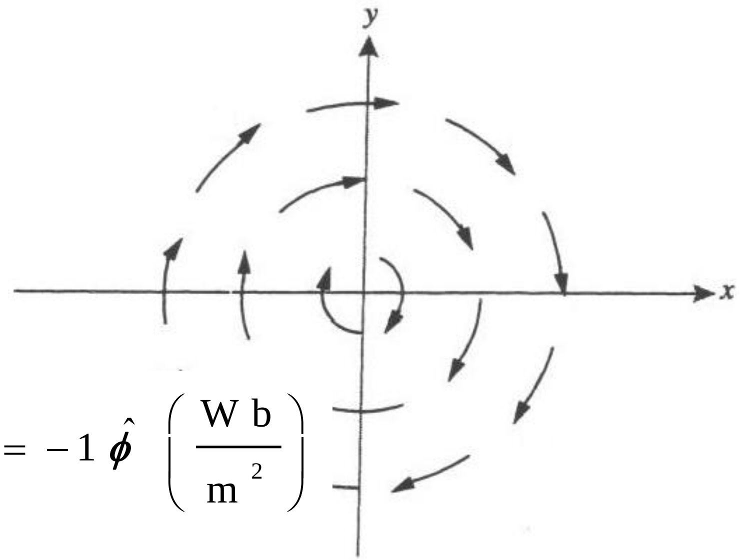
$$\hat{\mathbf{x}}=\cos\phi\,\hat{\mathbf{r}}-\sin\phi\,\hat{\boldsymbol{\phi}}\qquad \hat{\mathbf{r}}=\cos\phi\,\hat{\mathbf{x}}+\sin\phi\,\hat{\mathbf{y}}$$

$$\hat{\mathbf{y}}=\sin\phi\,\hat{\mathbf{r}}+\cos\phi\,\hat{\boldsymbol{\phi}}\qquad \hat{\boldsymbol{\phi}}=-\sin\phi\,\hat{\mathbf{x}}+\cos\phi\,\hat{\mathbf{y}}$$

$$d\mathbf{S}=dx\,dy\,\hat{\mathbf{z}}\quad\text{or}\quad dy\,dz\,\hat{\mathbf{x}}\quad\text{or}\quad dz\,dx\,\hat{\mathbf{y}}$$

$$d\mathbf{S}=r\,d\phi\,dz\,\hat{\mathbf{r}}\quad\text{or}\quad d\phi\,dz\,\hat{\boldsymbol{\phi}}\quad\text{or}\quad r\,dr\,d\phi\,\hat{\mathbf{z}}$$

Determine the total flux (upward) that crosses over the surface defined by $y = 0$, $z = -1$ to 1 m, $x = 2$ to 5 m, for the flux density depicted.



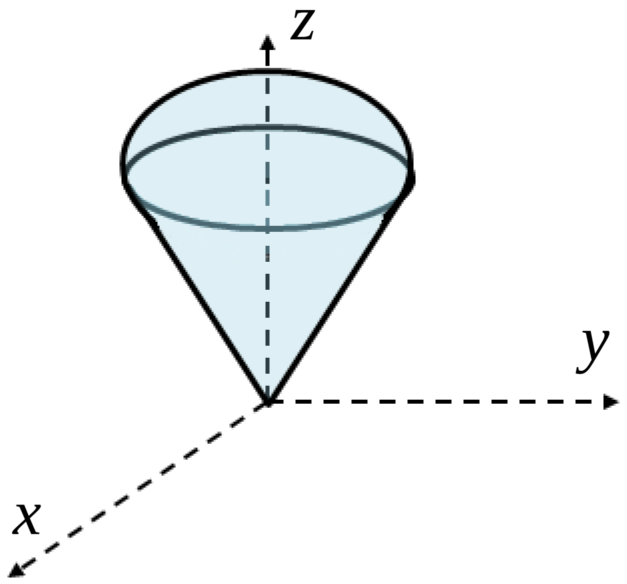
$$\mathbf{B}=-1\,\hat{\boldsymbol{\phi}}\left(\frac{\text{Wb}}{\text{m}^2}\right)$$

Volume Integrals

volume (triple) **integral**:

-- integration of a scalar *density* throughout a volume (*V*): $\int_v A_v \cdot dv$

-- allows us to answer the question "How much of [a particular quantity] is contained within [a given volume] ?"



<u>Examples</u>: $Q = \int_v \rho_v \, dv$ $W_E = \frac{1}{2}\varepsilon \int_v |\mathbf{E}|^2 \, dv$

As before, for rectangular coordinates, $dv = dx\ dy\ dz$

for cylindrical coordinates, $dv = r\ dr\ d\phi\ dz$

for spherical coordinates, $dv = R^2 \sin\theta\ dR\ d\theta\ d\phi$