20
The Laplace Transform

Since we first introduced Fourier analysis in Lecture 7, we have relied heavily on its properties in the analysis and representation of signals and linear, time-invariant systems. The Fourier transform was developed from the concept of representing signals as a linear combination of basic signals that were chosen to be eigenfunctions of linear, time-invariant systems. With the eigenfunctions chosen to be the signals $e^{j\omega t}$, this representation led to the Fourier transform synthesis equation, and a given LTI system could then be represented by the spectrum of eigenvalues as a function of ω, that is, the change in amplitude that the system applies to each of the basic inputs $e^{j\omega t}$.

In this and the next several lectures we introduce a generalization of the Fourier transform, referred to as the *Laplace transform*. In addition to leading to a number of new insights, the use of the Laplace transform removes some of the restrictions encountered with the Fourier transform. Specifically, the Laplace transform converges for a broader class of signals than does the Fourier transform.

The general class of eigenfunctions for LTI systems consists of the complex exponentials e^{st}, where s is a complex number. The use of this more general class in place of the complex exponentials $e^{j\omega t}$ leads to the representation of signals and systems in terms of the Laplace transform. The response of an LTI system to a complex exponential of the form e^{st} is $H(s)e^{st}$, and $H(s)$, which represents the change in amplitude, is referred to as the system function. As developed in the lecture, $H(s)$ is the Laplace transform of the system impulse response.

The Laplace transform and the Fourier transform are closely related in a number of ways. When s is purely imaginary, i.e., when $s = j\omega$, the Laplace transform reduces to the Fourier transform. More generally, the Laplace transform can be viewed as the Fourier transform of a signal after an exponential weighting has been applied. Because of this exponential weighting, the Laplace transform can converge for signals for which the Fourier transform does not converge.

The Laplace transform is a function of a general complex variable s, and for any given signal the Laplace transform converges for a range of values of s.

This range is referred to as the *region of convergence* (ROC) and plays an important role in specifying the Laplace transform associated with a given signal. In particular, two different signals can have Laplace transforms with identical algebraic expressions and differing only in the ROC, i.e., in the range of values of s for which the expression is valid.

For the most part, signals with which we will deal in this and subsequent lectures will be represented by Laplace transforms for which the associated algebraic expression is a ratio of polynomials in the complex variable s. The roots of the numerator polynomial are referred to as the zeros of the Laplace transform, and the roots of the denominator polynomial are referred to as the poles of the Laplace transform. It is typically convenient to represent the Laplace transform graphically in the complex s-plane by marking the location of the poles with $\times$ and the location of the zeros with $\bigcirc$. With the exception of an overall scale factor, this pole-zero diagram specifies the algebraic expression for the Laplace transform. In addition, the ROC must be indicated. As discussed in the lecture, there are a number of properties of the ROC in relation to the poles of the Laplace transform and in relation to certain properties of the signal in the time domain. These properties often permit us to identify the region of convergence from only the pole-zero pattern in the s-plane when some auxiliary information about the signal in the time domain is known, such as whether the signal is a right-sided, left-sided, or two-sided signal.

Suggested Reading

Section 9.0, Introduction, page 573

Section 9.1, The Laplace Transform, pages 573–579

Section 9.2, The Region of Convergence for Laplace Transforms, pages 579–587

Section 9.3, The Inverse Laplace Transform, pages 587–590

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20.1

Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

LTI Systems:

Impulse Response $h(t)$

$$e^{j\omega t} \longrightarrow H(\omega) e^{j\omega t}$$

$$H(\omega) \overset{\mathcal{F}}{\longleftrightarrow} h(t)$$

$$e^{st} \longrightarrow \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$e^{s(t-\tau)} = e^{st} e^{-s\tau}$$

$s = \sigma + j\omega$

$$e^{st} \longrightarrow e^{st}\underbrace{\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau}_{H(s)}$$

$$e^{st} \longrightarrow H(s)\, e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st} dt$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$s = \sigma + j\omega$

$$\underbrace{X(s)\Big|_{s=j\omega}}_{X(j\omega)} = \mathcal{F}\underbrace{\{x(t)\}}_{X(\omega)}$$

New Notation:

$$\mathcal{F}\{x(t)\} = X(j\omega)$$

MARKERBOARD
20.2

$$X(s)\Big|_{s=j\omega} = X(j\omega)$$

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-(\sigma+j\omega)t} dt$$

$$= \int_{-\infty}^{+\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

$$X(s) = \mathcal{F}\{x(t)\, e^{-\sigma t}\}$$

$\mathcal{L}$ may converge when $\mathcal{F}$ doesn't

Example 9.1

$$x(t) = e^{-at} u(t)$$

$$X(j\omega) = \frac{1}{j\omega + a} \qquad a > 0$$

$$a + \sigma > 0 \qquad e^{-at} e^{-\sigma t}$$
$$\sigma > -a$$

$$X(s) = \frac{1}{s+a} \qquad \Re e\{s\} > -a$$

$$e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \Re e\{s\} > -a$$

Example 9.2

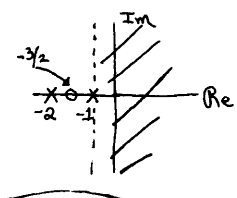
$$-e^{-at} u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

$$\Re e\{s\} < -a$$

Example 9.3

$$e^{-t} u(t) + e^{-2t} u(t)$$

$$\overset{\mathcal{L}}{\longleftrightarrow} \frac{2s+3}{(s+1)(s+2)} \qquad \Re e\{s\} > -1$$



$$X(s) = \frac{N(s)}{D(s)}$$

$N(s) = 0$ zeroes of $X(s)$

$D(s) = 0$ Poles of $X(s)$

TRANSPARENCY 20.1
Properties of the region of convergence of the Laplace transform.

PROPERTIES OF THE REGION OF CONVERGENCE

- The ROC contains no poles

$$X(s) = \frac{N(s)}{D(s)}$$

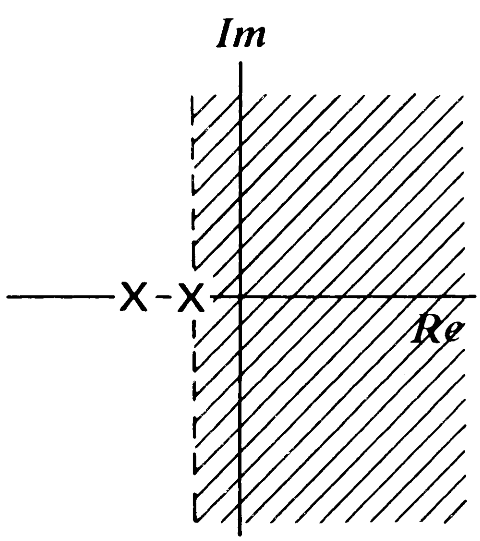
poles of $X(s) \Rightarrow D(s) = 0$

- The ROC of $X(s)$ consists of a strip parallel to the $j\omega$-axis in the s-plane

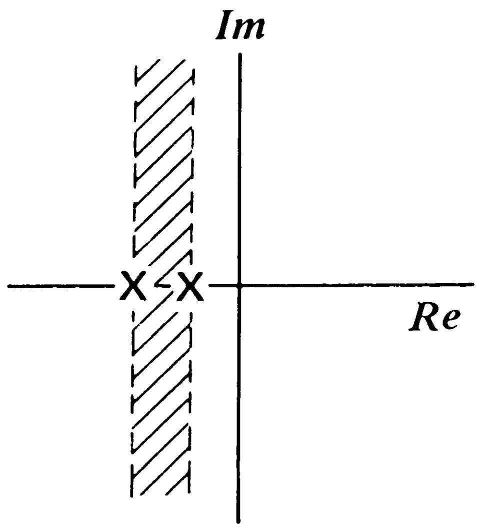
- $\mathcal{F}\{x(t)\}$ converges $\Leftrightarrow$ ROC includes the $j\omega$-axis in the s-plane

TRANSPARENCY 20.2
Three choices for the region of convergence associated with a specified pole-zero plot are shown in Transparencies 20.2–20.4.

$$X(s) = \frac{1}{(s+1)(s+2)}$$

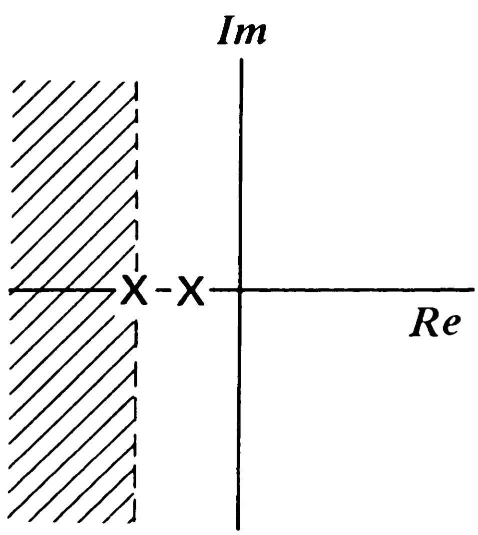


$$X(s) = \frac{1}{(s+1)(s+2)}$$



TRANSPARENCY
20.3

$$X(s) = \frac{1}{(s+1)(s+2)}$$

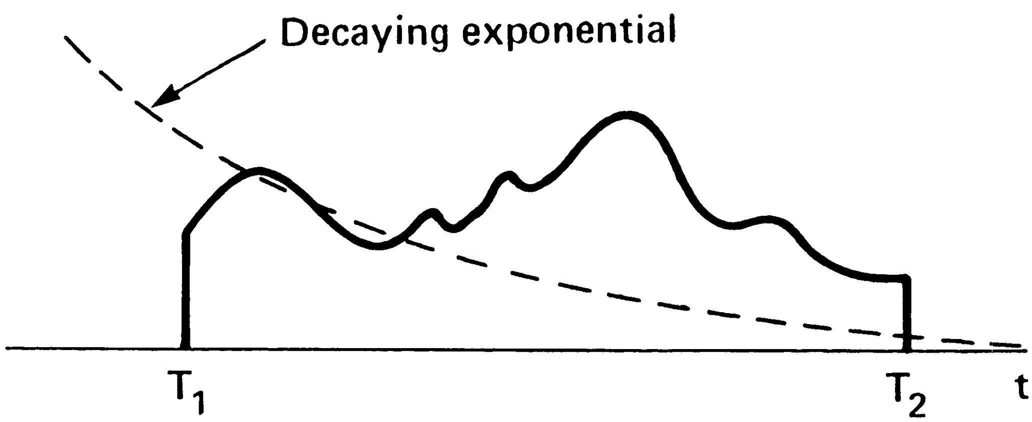


TRANSPARENCY
20.4

TRANSPARENCY 20.5
Transparencies 20.5 and 20.6 illustrate an interpretation of the property that for a finite-duration signal the ROC is the entire s-plane. This transparency demonstrates multiplying by a decaying exponential.

- x(t) finite duration

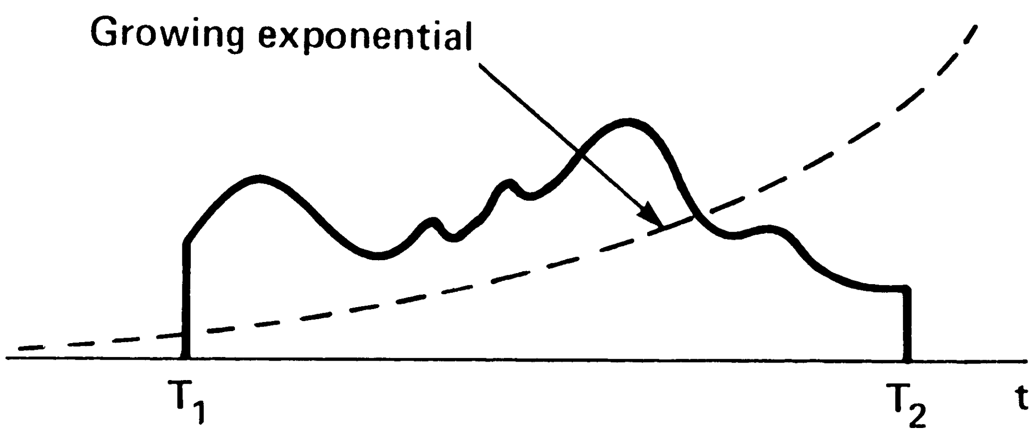
  => ROC is entire s-plane



TRANSPARENCY 20.6
Multiplying by a growing exponential.

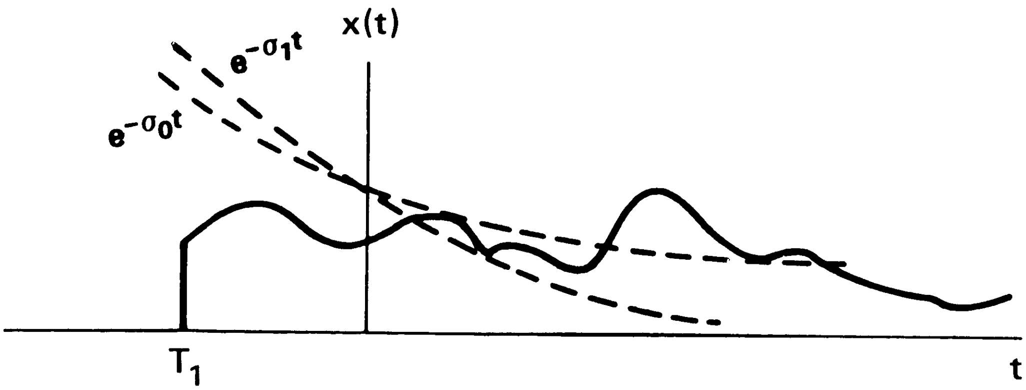
- x(t) finite duration

  => ROC is entire s-plane



$x(t)$ right-sided and $Re\{s\} = \sigma_o$ is in ROC

=> all values for which $Re\{s\} > \sigma_o$ are in ROC



$x(t)$ right-sided and $X(s)$ rational

=> ROC is to the right of the rightmost pole.

TRANSPARENCY 20.7
Interpretation of the property that for a right-sided signal the ROC is to the right of the rightmost pole.

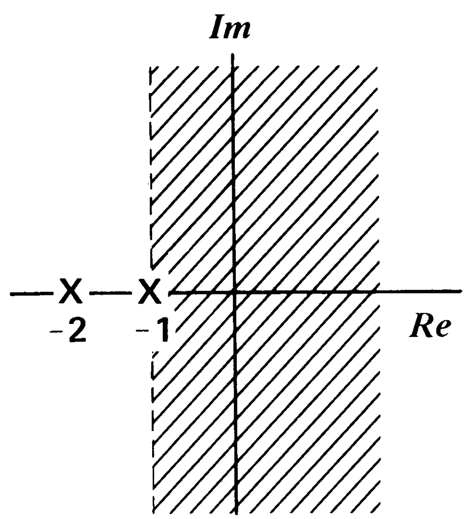
- $x(t)$ left-sided and $Re\{s\} = \sigma_o$ is in ROC
  => all values for which $Re\{s\} < \sigma_o$ are in ROC

- $x(t)$ left-sided and $X(s)$ rational
  => ROC to the left of the leftmost pole.

- $x(t)$ two-sided and $Re\{s\} = \sigma_o$ is in ROC
  => ROC is a strip in the s-plane

TRANSPARENCY 20.8
The ROC for a left-sided sequence and for a two-sided sequence.

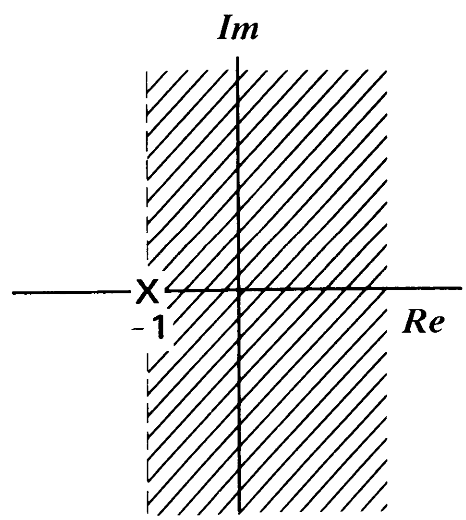
TRANSPARENCY 20.9
Decomposing a specified Laplace transform into a partial fraction expansion.



$$X(s) = \frac{1}{(s+1)(s+2)} \qquad Re\{s\} > -1$$

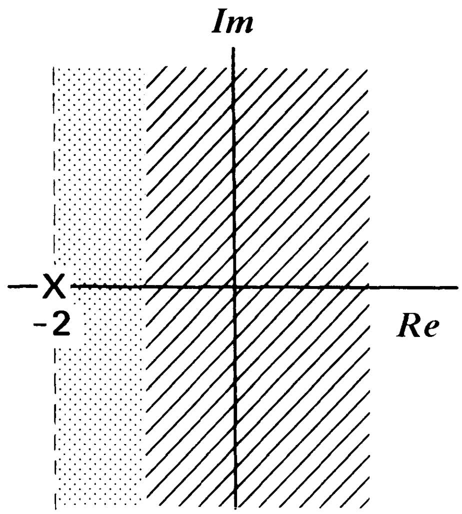
$$= \frac{1}{s+1} - \frac{1}{s+2} \qquad Re\{s\} > -1$$

TRANSPARENCY 20.10
Pole-zero pattern and inverse Laplace transform associated with the first term in the expansion in Transparency 20.9.



$$X_1(s) = \frac{1}{s+1} \qquad Re\{s\} > -1$$

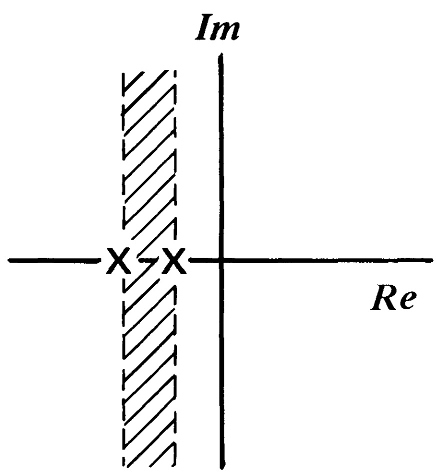
$$x_1(t) = e^{-t}u(t)$$



$$X_2(s) = \frac{-1}{s+2} \qquad Re\{s\} > -1$$

$$x_2(t) = -e^{-2t}u(t)$$

TRANSPARENCY 20.11
Pole-zero pattern and inverse Laplace transform for the second term in the partial fraction expansion in Transparency 20.9.



$$X(s) = \frac{1}{(s+1)(s+2)} \qquad -2 < Re\{s\} < -1$$

$$= \frac{1}{s+1} - \frac{1}{s+2} \qquad -2 < Re\{s\} < -1$$

TRANSPARENCY 20.12
The inverse transform that results from the same pole-zero pattern as in Transparency 20.9, but with a different choice for the ROC.