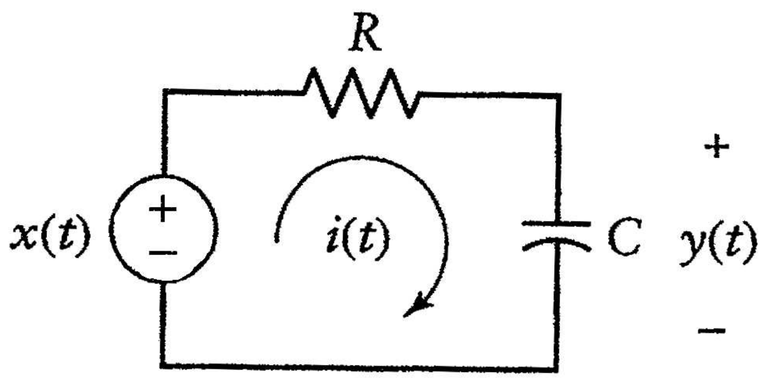
ELEC 309
Signals and Systems
Homework 3 Solutions

Time-Domain Analysis of LTI Systems



1. The RC-circuit above (with $R = 100$ kΩ and $C = 10\mu$F) is an LTI system with input signal $x(t)$, output signal $y(t)$, and impulse response given by

$$h(t) = e^{-t/RC}u(t).$$

(a) Using convolution, determine the voltage across the capacitor if $x(t) = u(t) - u(t-2)$.

The RC time constant is given by $RC = 10^5\,(10^{-5}) = 1$. Therefore, the impulse response is given by

$$h(t) = e^{-t}u(t),$$

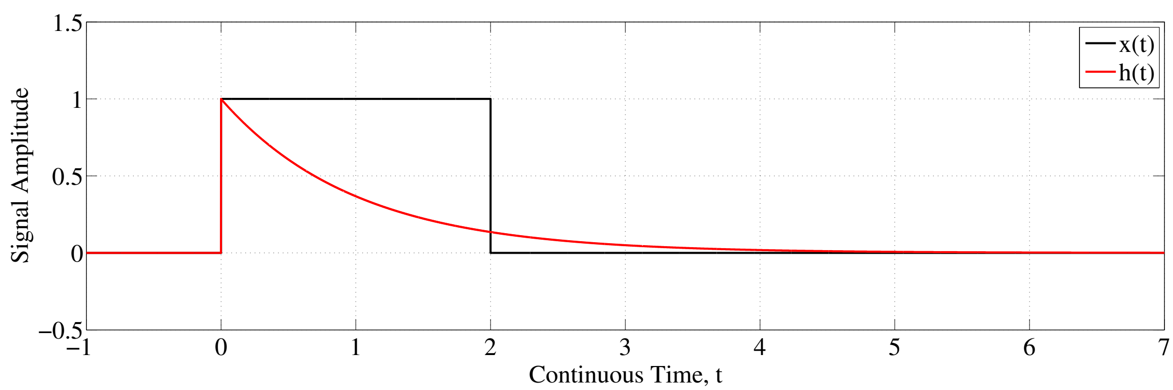


Figure 1: Input Signal $x(t)$ and Impulse Response $h(t)$

Using Equation 2 from the class notes, the voltage across the capacitor is given by

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} [u(\tau) - u(\tau-2)]\, e^{-(t-\tau)} u(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-(t-\tau)} u(\tau)u(t-\tau)d\tau - \int_{-\infty}^{\infty} e^{-(t-\tau)} u(\tau-2)u(t-\tau)d\tau \end{aligned}$$

Note that

$$u(\tau)u(t-\tau) = \begin{cases} 1 & \text{for } 0 \le \tau \le t \text{ and } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$u(\tau-2)u(t-\tau) = \begin{cases} 1 & \text{for } 2 \le \tau \le t \text{ and } t \ge 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{-(t-\tau)} u(\tau)u(t-\tau)d\tau - \int_{-\infty}^{\infty} e^{-(t-\tau)} u(\tau-2)u(t-\tau)d\tau \\ &= \begin{cases} 0 & \text{for } t < 0 \\ \int_0^t e^{-(t-\tau)}d\tau & \text{for } 0 \le t < 2 \\ \int_0^t e^{-(t-\tau)}d\tau - \int_2^t e^{-(t-\tau)}d\tau & \text{for } t \ge 2 \end{cases} = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } 0 \le t < 2 \\ 1 - e^{-t} - \left[1 - e^{-(t-2)}\right] & \text{for } t \ge 2 \end{cases} \\ &= \boxed{\begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } 0 \le t < 2 \\ e^{-(t-2)} - e^{-t} & \text{for } t \ge 2 \end{cases} = \left[1 - e^{-t}\right] u(t) - \left[1 - e^{-(t-2)}\right] u(t-2)} \end{aligned}$$

(b) Using MATLAB, plot $y(t)$ from part (a).

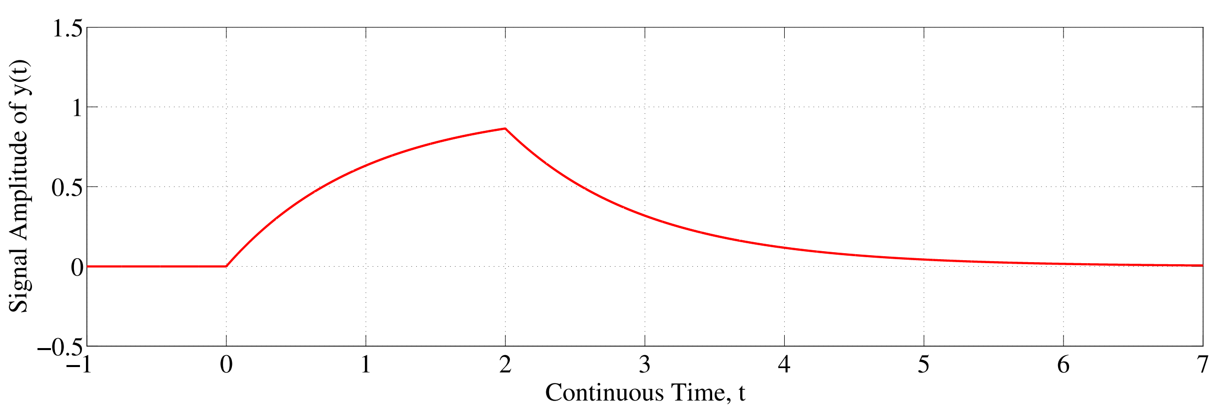


Figure 2: Output Signal $y(t)$

Alternate Solution for Part (a)

Using Equation 3 from the class notes, the voltage across the capacitor is given by

$$\begin{aligned} y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)\left[u(t-\tau) - u(t-\tau-2)\right]d\tau \\ &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau - \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau-2)d\tau \end{aligned}$$

Note that

$$u(\tau)u(t-\tau) = \begin{cases} 1 & \text{for } 0 \le \tau \le t \text{ and } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$u(\tau)u(t-\tau-2) = \begin{cases} 1 & \text{for } 0 \le \tau \le t-2 \text{ and } t \ge 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau - \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau-2)d\tau \\ &= \begin{cases} 0 & \text{for } t < 0 \\ \int_0^t e^{-\tau}d\tau & \text{for } 0 \le t < 2 \\ \int_0^t e^{-\tau}d\tau - \int_0^{t-2} e^{-\tau}d\tau & \text{for } t \ge 2 \end{cases} = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } 0 \le t < 2 \\ 1 - e^{-t} - \left[1 - e^{-(t-2)}\right] & \text{for } t \ge 2 \end{cases} \\ &= \boxed{\begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } 0 \le t < 2 \\ e^{-(t-2)} - e^{-t} & \text{for } t \ge 2 \end{cases} = \left[1 - e^{-t}\right]u(t) - \left[1 - e^{-(t-2)}\right]u(t-2)} \end{aligned}$$

2. The input-output relationship for the LTI system that is a four-point moving-average system is given by

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k].$$

(a) Determine the impulse response $h[n]$ of this LTI system.

Since $\delta[n] \Longrightarrow h[n]$, the input-output can be rewritten as

$$\boxed{h[n] = \frac{1}{4}\sum_{k=0}^{3} \delta[n-k] = \frac{1}{4}\left(u[n] - u[n-4]\right)}$$

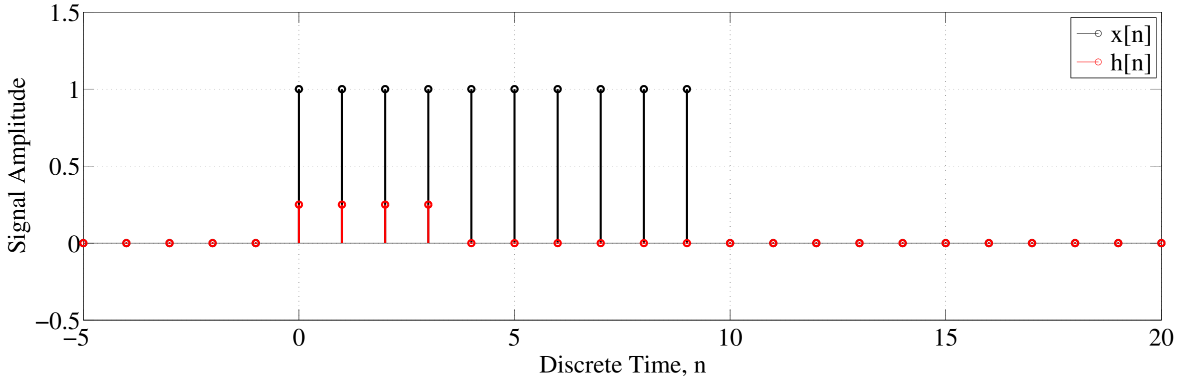


Figure 3: Input Signal $x[n]$ and Impulse Response $h[n]$

(b) Using convolution, determine the response of the system when the input is the rectangular pulse given by

$$x[n] = u[n] - u[n-10].$$

Using Equation 5 from the class notes, the output of the four-point moving-average system is given by

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(u[k] - u[k-10]\right) \frac{1}{4}\left(u[n-k] - u[n-k-4]\right) \\ &= \frac{1}{4}\left[\sum_{k=-\infty}^{\infty} u[k]u[n-k] - u[k]u[n-4-k] - u[k-10]u[n-k] + u[k-10]u[n-4-k]\right] \end{aligned}$$

Note that

$$
\begin{aligned}
u[k]u[n-k] &= \begin{cases} 1 & \text{for } 0 \le k \le n \text{ and } n \ge 0 \\ 0 & \text{otherwise,} \end{cases} \\
u[k]u[n-4-k] &= \begin{cases} 1 & \text{for } 0 \le k \le n-4 \text{ and } n \ge 4 \\ 0 & \text{otherwise,} \end{cases} \\
u[k-10]u[n-k] &= \begin{cases} 1 & \text{for } 10 \le k \le n \text{ and } n \ge 10 \\ 0 & \text{otherwise, and} \end{cases} \\
u[k-10]u[n-4-k] &= \begin{cases} 1 & \text{for } 10 \le k \le n-4 \text{ and } n \ge 14 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y([n] &= \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k]u[n-k] - \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k]u[n-4-k] \\
&\quad - \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k-10]u[n-k] + \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k-10]u[n-4-k] \\
&= \begin{cases} 0 & \text{for } n < 0 \\ \frac{1}{4}\sum\limits_{k=0}^{n} 1 & \text{for } 0 \le n < 4 \\ \frac{1}{4}\left[\sum\limits_{k=0}^{n} 1 - \sum\limits_{k=0}^{n-4} 1\right] & \text{for } 4 \le n < 10 \\ \frac{1}{4}\left[\sum\limits_{k=0}^{n} 1 - \sum\limits_{k=0}^{n-4} 1 - \sum\limits_{k=10}^{n} 1\right] & \text{for } 10 \le n < 14 \\ \frac{1}{4}\left[\sum\limits_{k=0}^{n} 1 - \sum\limits_{k=0}^{n-4} 1 - \sum\limits_{k=10}^{n} 1 + \sum\limits_{k=10}^{n-4} 1\right] & \text{for } n \ge 14 \end{cases} \\
&= \begin{cases} 0 & \text{for } n < 0 \\ \frac{1}{4}[n+1] & \text{for } 0 \le n < 4 \\ \frac{1}{4}[n+1-(n-4+1)] & \text{for } 4 \le n < 10 \\ \frac{1}{4}[n+1-(n-4+1)-(n-10+1)] & \text{for } 10 \le n < 14 \\ \frac{1}{4}[n+1-(n-4+1)-(n-10+1)+(n-4-10+1)] & \text{for } n \ge 14 \end{cases} \\
&= \begin{cases} 0 & \text{for } n < 0 \\ \frac{n+1}{4} & \text{for } 0 \le n < 4 \\ 1 & \text{for } 4 \le n < 10 \\ \frac{13-n}{4} & \text{for } 10 \le n < 14 \\ 0 & \text{for } n \ge 14 \end{cases} \\
&= \left[\frac{n+1}{4}\right](u[n]-u[n-4]) + (u[n-4]-u[n-10]) + \left[\frac{13-n}{4}\right](u[n-10]-u[n-14]) \\
&= \left(\frac{n+1}{4}\right)u[n] + \left(\frac{3-n}{4}\right)u[n-4] + \left(\frac{9-n}{4}\right)u[n-10] + \left(\frac{n-13}{4}\right)u[n-14]
\end{aligned}
$$

(c) Using MATLAB, plot $y[n]$ from part (b).

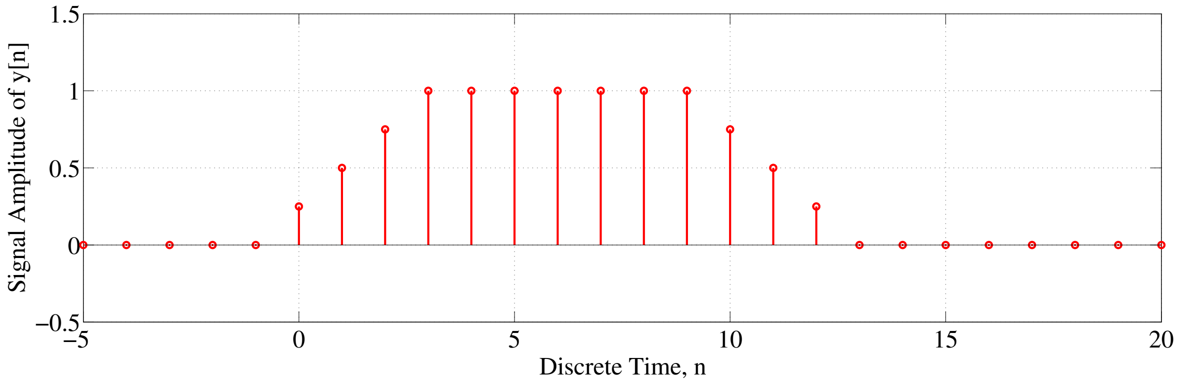


Figure 4: Output Signal $y[n]$

Alternate Solution for Part (b)

Using Equation 6 from the class notes, the output of the four-point moving-average system is given by

$$\begin{aligned} y[n] = h[n] * x[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{4}\left(u[k]-u[k-4]\right)\left(u[n-k]-u[n-k-10]\right) \\ &= \frac{1}{4}\left(\sum_{k=-\infty}^{\infty} u[k]u[n-k] - u[k]u[n-k-10] - u[k-4]u[n-k] + u[k-4]u[n-10-k]\right] \end{aligned}$$

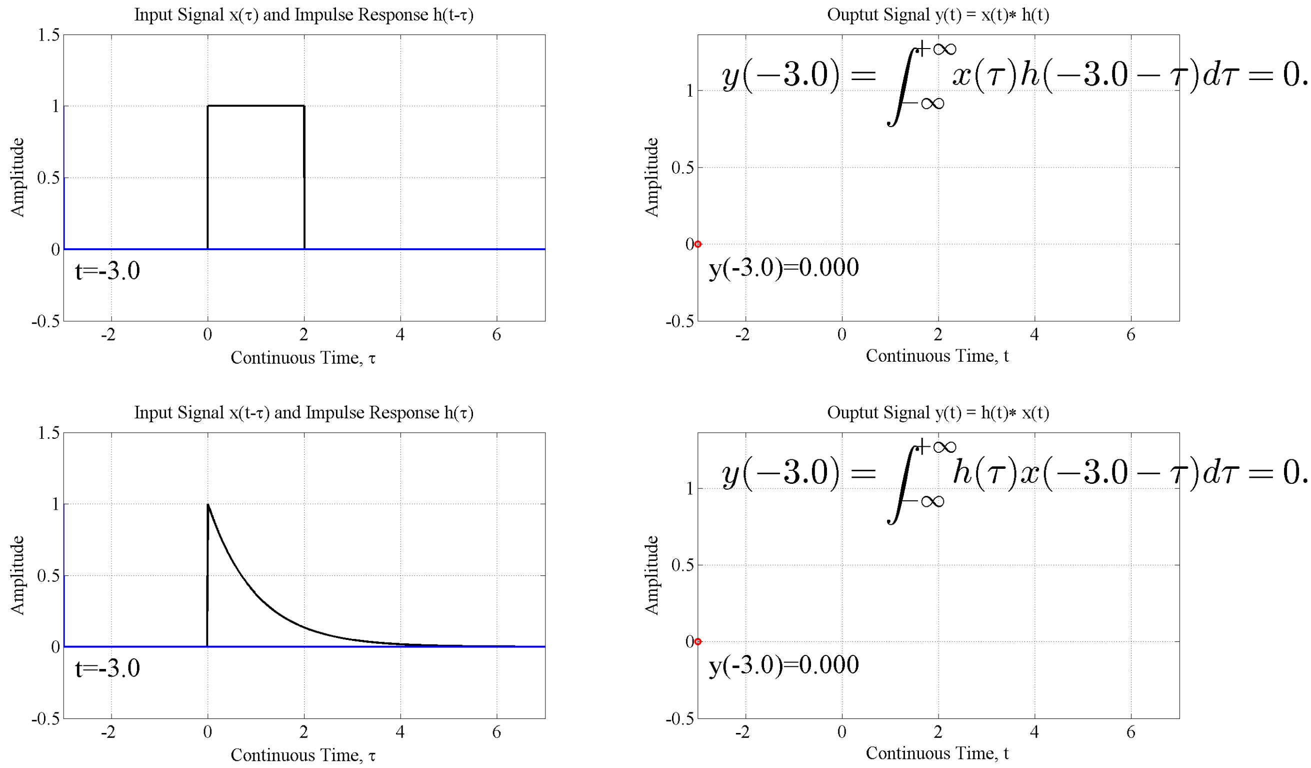
Note that

$$\begin{aligned} u[k]u[n-k] &= \begin{cases} 1 & \text{for } 0 \leq k \leq n \text{ and } n \geq 0 \\ 0 & \text{otherwise,} \end{cases} \\ u[k-4]u[n-k] &= \begin{cases} 1 & \text{for } 4 \leq k \leq n \text{ and } n \geq 4 \\ 0 & \text{otherwise,} \end{cases} \\ u[k]u[n-10-k] &= \begin{cases} 1 & \text{for } 0 \leq k \leq n-10 \text{ and } n \geq 10 \\ 0 & \text{otherwise, and} \end{cases} \\ u[k-4]u[n-10-k] &= \begin{cases} 1 & \text{for } 4 \leq k \leq n-10 \text{ and } n \geq 14 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$
\begin{aligned}
y[n] &= \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k]u[n-k] - \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k]u[n-10-k] \\
&\quad - \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k-4]u[n-k] + \frac{1}{4}\sum_{k=-\infty}^{\infty} u[k-4]u[n-10-k] \\
&= \begin{cases}
0 & \text{for } n < 0 \\
\frac{1}{4}\sum\limits_{k=0}^{n} 1 & \text{for } 0 \le n < 4 \\
\frac{1}{4}\left[\sum\limits_{k=0}^{n} 1 - \sum\limits_{k=4}^{n} 1\right] & \text{for } 4 \le n < 10 \\
\frac{1}{4}\left[\sum\limits_{k=0}^{n} 1 - \sum\limits_{k=4}^{n} 1 - \sum\limits_{k=0}^{n-10} 1\right] & \text{for } 10 \le n < 14 \\
\frac{1}{4}\left[\sum\limits_{k=0}^{n} 1 - \sum\limits_{k=4}^{n} 1 - \sum\limits_{k=0}^{n-10} 1 + \sum\limits_{k=4}^{n-10} 1\right] & \text{for } n \ge 14
\end{cases} \\
&= \begin{cases}
0 & \text{for } n < 0 \\
\frac{1}{4}\left[n+1\right] & \text{for } 0 \le n < 4 \\
\frac{1}{4}\left[n+1-(n-4+1)\right] & \text{for } 4 \le n < 10 \\
\frac{1}{4}\left[n+1-(n-4+1)-(n-10+1)\right] & \text{for } 10 \le n < 14 \\
\frac{1}{4}\left[n+1-(n-4+1)-(n-10+1)+(n-10-4+1)\right] & \text{for } n \ge 14
\end{cases} \\
&= \begin{cases}
0 & \text{for } n < 0 \\
\frac{n+1}{4} & \text{for } 0 \le n < 4 \\
1 & \text{for } 4 \le n < 10 \\
\frac{13-n}{4} & \text{for } 10 \le n < 14 \\
0 & \text{for } n \ge 14
\end{cases} \\
&= \left[\frac{n+1}{4}\right](u[n]-u[n-4]) + (u[n-4]-u[n-10]) + \left[\frac{13-n}{4}\right](u[n-10]-u[n-14]) \\
&= \left(\frac{n+1}{4}\right)u[n] + \left(\frac{3-n}{4}\right)u[n-4] + \left(\frac{9-n}{4}\right)u[n-10] + \left(\frac{n-13}{4}\right)u[n-14]
\end{aligned}
$$

Animation for Problem 1



Animation for Problem 2