24 Butterworth Filters

Recommended Problems

P24.1

Do the following for a fifth-order Butterworth filter with cutoff frequency of 1 kHz and transfer function $B(s)$.

(a) Write the expression for the magnitude squared of the frequency response.

(b) Sketch the locations of the poles of $B(s)B(-s)$.

(c) Indicate the locations of the poles of $B(s)$, assuming that $B(s)$ represents a causal and stable filter.

(d) Indicate the locations of the poles of $B(-s)$.

P24.2

Figure P24.2-1 shows the frequency response of a discrete-time filter.

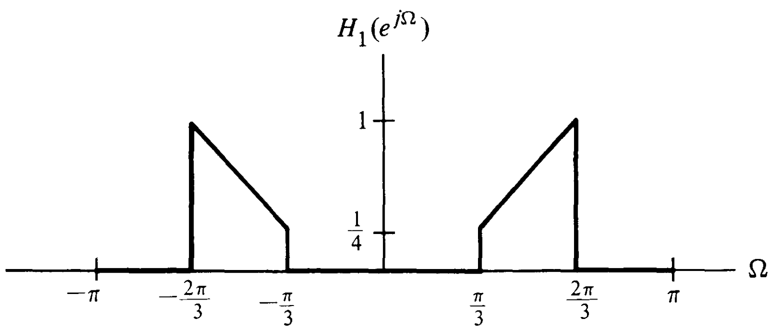

Figure P24.2-1

(a) Determine and sketch the analog frequency response characteristic that (assuming no aliasing) will map to the discrete-time frequency response given in the figure when the impulse invariance method is used.

(b) Sketch the analog frequency response that will map to the discrete-time frequency response in Figure P24.2-1 when the bilinear transformation is applied.

(c) Repeat parts (a) and (b) for the discrete-time frequency response characteristic in Figure P24.2-2.

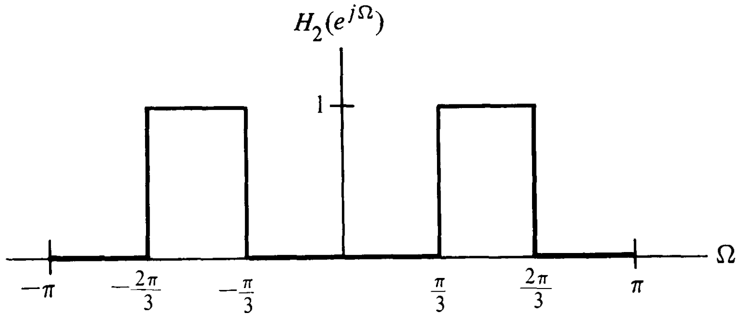

Figure P24.2-2

P24.3

Consider the system function

$$H_c(s) = \frac{1}{s + a}$$

(a) Determine the discrete-time transfer function $H_d(z)$ obtained by mapping $H_c(s)$ to $H_d(z)$ using the bilinear transformation with $T = 2$.

(b) Find the range of the constant a for which $H(s)$ is stable and causal.

(c) Verify that if $H(s)$ is stable and causal, then $H(z)$ is also stable and causal.

P24.4

Consider the following discrete-time filter specifications:

$$1 \geq |H_d(e^{j\Omega})| \geq 0.8 \qquad \text{for } 0 \leq \Omega \leq \frac{\pi}{4},$$

$$0.2 \geq |H_d(e^{j\Omega})| \geq 0 \qquad \text{for } \frac{3\pi}{4} \leq \Omega \leq \pi$$

To design $H_d(e^{j\Omega})$ using the impulse invariance method or the bilinear transformation, we need first to specify a continuous-time filter $B(j\omega)$. Assume that we will use a Butterworth filter.

(a) Set up the proper equations for the order N and the cutoff frequency ω_c of the continuous-time filter $B(j\omega)$ that will map to $H_d(e^{j\Omega})$ when the impulse invariance method is used. Set $T = 1$.

(b) Set up the proper equations for N and ω_c of $B(j\omega)$ when the bilinear transformation is used. Set $T = 1$.

P24.5

Consider the system in Figure P24.5, which implements a continuous-time filter by discrete-time processing.

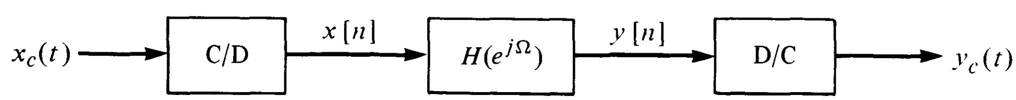

Figure P24.5

The sampling frequency is 15 kHz. The continuous-time filter must satisfy the following specifications:

$$1 \geq |H_c(j\omega)| \geq 0.9 \qquad \text{for } 0 \leq \omega \leq (2\pi)3000,$$
$$0.1 \geq |H_c(j\omega)| \geq 0 \qquad \text{for } (2\pi)4500 \leq \omega < \infty$$

(a) Determine the appropriate specifications for $H(e^{j\Omega})$, the frequency response of the discrete-time filter.

(b) Suppose that to design $H(e^{j\Omega})$, we use the impulse invariance method. We need to introduce a second continuous-time filter, $G(j\omega)$. Using $T = 3$ for the value of the parameter T in the impulse invariance design procedure, determine the filter specifications of $G(j\omega)$.

(c) Suppose that we now use the bilinear transformation to design $H(e^{j\Omega})$. Using $T = 2$ for the value of the parameter T in the bilinear transformation method, determine the filter specification of $G(j\omega)$.

(d) With either the impulse invariance or bilinear design procedure, is $H(e^{j\Omega})$ or $H_c(j\omega)$ dependent in any way on the parameter T?

Optional Problems

P24.6

In this problem we consider more closely the design procedure for continuous-time Butterworth filters.

Suppose that we are to design a filter $B(j\omega)$ such that

$$20 \log_{10}|B(j2\pi)| \geq -1 \text{ dB},$$
$$20 \log_{10}|B(j3\pi)| \leq -15 \text{ dB}$$

(a) There are two unknown parameters, the order N of $B(s)$ and the cutoff frequency ω_c. Set up the two simultaneous equations for N and ω_c and verify that $N = 5.88$ and $\omega_c = 7.047$ satisfy the equations.

(b) Since N is not an integer, we must choose N to be the next higher integer. We can now pick whether to meet exactly the stopband specification and exceed the passband specification or vice versa. Find ω_c such that the passband specification is met exactly, and verify that the stopband specification is exceeded.

(c) What would happen if we picked $N = 5$?

P24.7

We want to design a discrete-time lowpass filter with a passband magnitude characteristic that is constant to within 0.75 dB for frequencies below $\Omega = 0.2613\pi$ and that has a stopband attenuation of at least 20 dB for frequencies between $\Omega = 0.4018\pi$ and π. Determine the poles of the lowest-order Butterworth continuous-time transfer function that, when mapped to a discrete-time filter using the bilinear transformation with $T = 1$, will meet the specifications. If possible, exceed the stopband specifications. Indicate also how you would proceed to obtain the transfer function of the discrete-time filter.

P24.8

Suppose that we want to design a discrete-time filter using the impulse invariance method. The filter specifications are given by

$$1 \geq |H_d(e^{j\Omega})| \geq a, \qquad 0 \leq \Omega \leq 0.2\pi,$$
$$b \geq |H_d(e^{j\Omega})| \geq 0, \qquad 0.3\pi \leq \Omega \leq \pi$$

Using $T = 3$, we obtain the corresponding filter specifications for the associated continuous-time filter $H_b(s)$:

$$3 \geq |H_b(j\omega)| \geq 3a, \qquad 0 \leq \omega \leq 0.2\pi/3,$$
$$3b \geq |H_b(j\omega)| \geq 0 \qquad 0.3\pi/3 \leq \omega < \infty$$

Assume that a filter $H_s(s)$ satisfies the specifications *exactly;* thus,

$$\left| H_s\left(j\frac{0.2\pi}{3}\right) \right| = 3a, \qquad \left| H_s\left(j\frac{0.3\pi}{3}\right) \right| = 3b$$

The designed discrete-time filter is given by

$$H(e^{j\Omega}) = \frac{1}{3}\sum_{k=-\infty}^{\infty} H_s\left[j\left(\frac{\Omega}{3} - \frac{2\pi k}{3}\right)\right]$$

Study the case of $T = 2$.

(a) For $T = 2$, give the filter specifications for the associated continuous-time filter $\hat{H}_b(s)$.

(b) Verify that the continuous-time filter given by

$$\hat{H}_s(s) = \frac{2}{3} H_s\left(\frac{2s}{3}\right)$$

satisfies part (a) exactly.

(c) Substitute $\hat{H}_s(s)$ from part (b) to solve for $\hat{H}(e^{j\Omega})$ and verify that $H(e^{j\Omega}) = \hat{H}(e^{j\Omega})$. Thus, the value of T does not affect the final discrete-time filter designed.

P24.9

As mentioned in Section 10.8.3 of the text, the bilinear transformation map from the s plane to the z plane can be interpreted as arising from the use of the trapezoidal rule in numerically integrating differential equations.

(a) Consider a continuous-time system for which the differential equation is

$$\frac{dy(t)}{dt} = x(t) \tag{P24.9-1}$$

or, equivalently,

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \tag{P24.9-2}$$

Determine the system function $H(s)$ for this continuous-time system.

In numerical analysis the procedure known as the trapezoidal rule for integration proceeds by approximating the continuous-time function as a set of contiguous trapezoids, as illustrated in Figure P24.9(a), and then adding their areas to compute the total integral. The area A of an individual trapezoid, with dimensions shown in Figure P24.9(b), is

$$A = \left(\frac{b + a}{2}\right) h$$

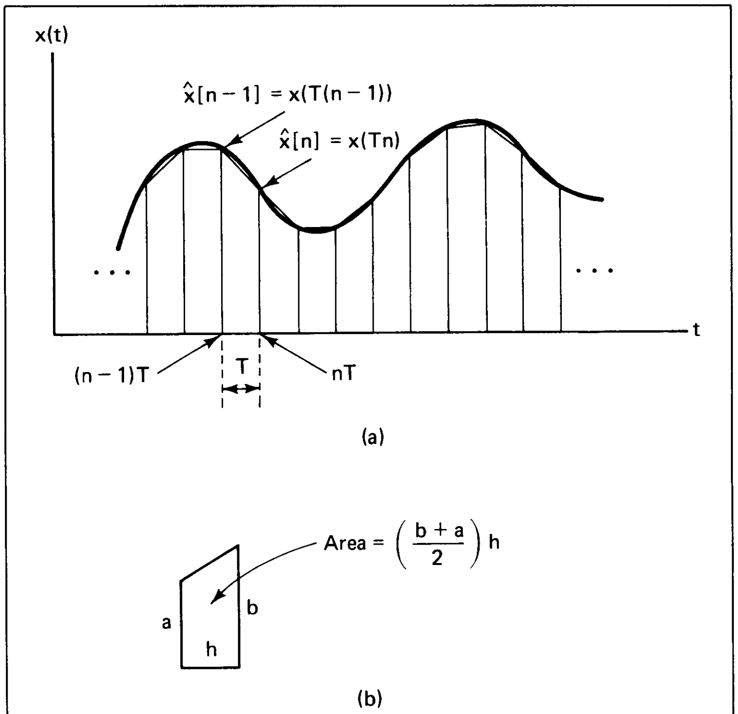

Figure P24.9

(b) What is the area A_n in the trapezoidal approximation between $x[(n-1)T]$ and $x(nT)$?

(c) From eq. (P24.9-2), $y(nT)$ denotes the area under $x(t)$ up to time $t = nT$. Let $\hat{y}[n]$ denote the approximation to $y(nT)$ obtained using the trapezoidal rule for integration, that is,

$$\hat{y}[n] = \sum_{k=-\infty}^{n} A_k$$

Show that

$$\hat{y}[n] = \hat{y}[n-1] + A_n$$

(d) With $\hat{x}[n]$ defined as $\hat{x}[n] = x(nT)$, show that the trapezoidal rule approximation to eq. (P24.9-2) becomes

$$\hat{y}[n] = \hat{y}[n-1] + \frac{T}{2}\{\hat{x}[n-1] + \hat{x}[n]\} \qquad \text{(P24.9-3)}$$

(e) Determine the system function corresponding to the difference equation in part (d). Demonstrate, in particular, that it is the same as would be obtained by applying the bilinear transformation to the continuous-time system function corresponding to eq. (P24.9-1).

Resource: Signals and Systems
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