ELEC 312-81
Systems I
TEST 2

Thursday, April 14, 2010 **Name:** ANSWER KEY

By writing my name, I understand that I am bound by The Citadel Honor Code.

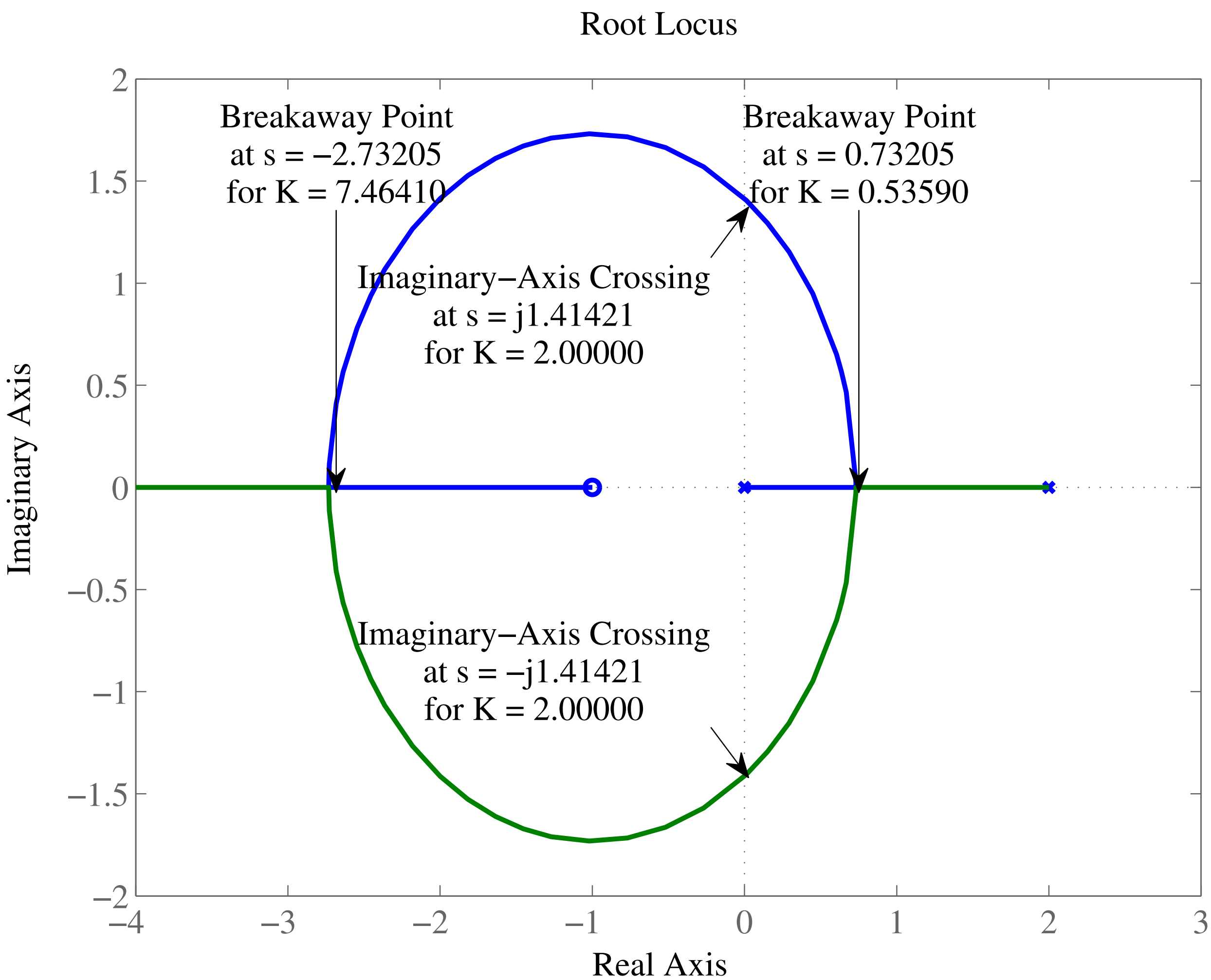
Read all of the following information before starting the test:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. When you do use your calculator, explain all relevant mathematics.
- **Box, circle, or otherwise indicate your final answers.**
- This test has 2 problems and is worth 50 points, plus some extra credit at the end.
- Check to ensure that you have all pages. It is your responsibility to make sure that you have all of the pages!
- If you remove the staple, you must re-staple your pages IN ORDER. Failure to do so will result in a deduction of 5 points from your final score.
- Good luck!

1. Consider a unity-feedback control system with the feedforward transfer function given by

$$G(s) = \frac{K(s+1)}{s(s-2)}.$$

You may use the grid below to sketch your root locus as you determine your answers for each root locus rule. However, you will only be graded on your answers below.



(a) (1 point) (**Circle the correct answer.**) The system has
A. 1 closed-loop pole.
B. 2 closed-loop poles.
C. 3 closed-loop poles.
D. 4 closed-loop poles.

(b) (1 point) (**Circle the correct answer.**) The root locus plot contains
A. 1 branch.
B. 2 branches.
C. 3 branches.
D. 4 branches.

(c) (1 point) (**Circle the correct answer.**) The root locus plot contains

A. 1 asymptote.

B. 2 asymptotes.

C. 3 asymptotes.

D. 4 asymptotes.

(d) (1 point) (**Circle the correct answer.**) The angle(s) for the asymptote(s) is/are

A. $\pm 45°, \pm 135°$.

B. $\pm 60°, 180°$.

C. $\pm 90°$.

D. 180°.

(e) (2 points) Determine the real-axis intercept for the asymptote(s).

$$\boldsymbol{\sigma_0 = \frac{(0+2)-(-1)}{2-1} = \boxed{3.}}$$

(f) (2 points) Determine the interval(s) on the real axis where the root locus exists.

The root locus exists on the real axis only in the interval $(-\infty, -1)$ and the interval $(0, 2)$.

(g) (3 points) Determine the breakaway (not break-in) point.

By inspection, the root locus breaks away from the real axis somewhere in the interval $(0, 2)$. The breakaway point is given by:

$$N(s) = s + 1 \Rightarrow N'(s) = 1 \textbf{ and } D(s) = s^2 - 2s \Rightarrow D'(s) = 2s - 2$$
$$D'(s)N(s) - D(s)N'(s) = (2s - 2)(s + 1) - (s^2 - 2s)(1) = s^2 + 2s - 2 = 0$$

The solutions to the breakaway equation are $s = -2.73205$ and $s = 0.73205$. Thus, the root locus *breaks away* from the real axis at $\boxed{s = 0.73205}$.

(h) (3 points) Determine the break-in (not breakaway) point.

By inspection, the root locus breaks into the real axis somewhere in the interval $(-\infty, -1)$. The break-in point is given by:

$$N(s) = s + 1 \Rightarrow N'(s) = 1 \textbf{ and } D(s) = s^2 - 2s \Rightarrow D'(s) = 2s - 2$$
$$D'(s)N(s) - D(s)N'(s) = (2s - 2)(s + 1) - (s^2 - 2s)(1) = s^2 + 2s - 2 = 0$$

The solutions to the breakaway equation are $s = -2.73205$ and $s = 0.73205$. Thus, the root locus *breaks in* to the real axis at $\boxed{s = -2.73205}$.

(i) (3 points) Determine the gain K at the breakaway (not break-in) point.

$$\boldsymbol{K = \left.\frac{-1}{G(s)H(s)}\right|_{s=0.73205} = \left.\frac{-s(s-2)}{s+1}\right|_{s=0.73205} = \boxed{0.53590}}$$

(j) (3 points) Determine the gain K at the break-in (not breakaway) point.

$$\boldsymbol{K = \left.\frac{-1}{G(s)H(s)}\right|_{s=-2.73205} = \left.\frac{-s(s-2)}{s+1}\right|_{s=-2.73205} = \boxed{7.46410}}$$

(k) (3 points) Determine the imaginary-axis crossings.

$$K = \left.\frac{-1}{G(s)H(s)}\right|_{s=j\omega} = \left.\frac{-s(s-2)}{s+1}\right|_{s=j\omega} = \frac{-j\omega(j\omega-2)}{j\omega+1} = \frac{-j\omega(j\omega-2)}{j\omega+1}\cdot\frac{-j\omega+1}{-j\omega+1}$$
$$= \frac{-j\omega^3+3\omega^2+j2\omega}{\omega^2+1} = \frac{3\omega^2}{\omega^2+1} + j\frac{2\omega-\omega^3}{\omega^2+1}$$
$$\Rightarrow 2\omega-\omega^3 = \omega(2-\omega^2) = \omega\left(\sqrt{2}-\omega\right)\left(\sqrt{2}+\omega\right) = 0$$

Therefore, the imaginary-axis crossings are at $\boxed{s=0}$ **and at** $\boxed{s=\pm j\sqrt{2} \text{ or } s=\pm j1.41421}$.

(l) (3 points) Determine the gain K at all imaginary-axis crossings.

The imaginary-axis crossing at $s=0$ corresponds to $K=0$, and the imaginary-axis crossings at $s=\pm j\sqrt{2}$ correspond to

$$K = \left.\frac{-1}{G(s)H(s)}\right|_{s=\pm j\sqrt{2}} = \left.\frac{3\omega^2}{\omega^2+1}\right|_{\omega=\pm\sqrt{2}} = \boxed{2}.$$

(m) (3 points) Give the range of positive gain K to ensure system stability.

Therefore, in order for this system to be stable, we require $\boxed{K>2.}$

Routh Test Approach:
The closed-loop transfer function is given by

$$G_0(s) = \frac{\frac{K(s+1)}{s^2-2s}}{1+\frac{K(s+1)}{s^2-2s}} = \frac{Ks+K}{s^2+(K-2)s+K}.$$

The Routh table is given by

	s^2	1	K
$k_1 = \frac{1}{K}$	s^1	$K-2$	
	s^0	K	

Looking at the s^1 row, we need $K>0$. Looking at the s^0 row, we need $K-2>0$. Therefore, a lower bound on K for stability is $K>2$.
Therefore, in order for this system to be stable, we require $\boxed{K>2.}$
By inspection, the imaginary-axis crossing corresponding to $K=0$ is $s=0$. The imaginary-axis crossings corresponding to $K=2$ can be determined by finding the closed-loop poles of $G_0(s)$ when $K=2$, which are $s=\pm j\sqrt{2}=\pm j1.41421$. Therefore, the imaginary-axis crossings are at $\boxed{s=0}$ **and at** $s=\boxed{\pm j\sqrt{2}=\pm j1.41421}$.

2. Consider a unity-feedback control system with the feedforward transfer function given by

$$G(s) = \frac{K(s+1)}{s(s-2)}.$$

where $K = 6$. Note that when $K = 6$, the closed-loop poles are $s = -2 \pm j\sqrt{2}$.

(a) (3 points) Determine the damping ratio.

$$\boxed{\boldsymbol{\zeta = \frac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}} = \frac{2}{\sqrt{2^2 + \left(\sqrt{2}\right)^2}} = \frac{2}{\sqrt{6}} = \boxed{0.81650}}}$$

(b) (3 points) Determine the natural frequency.

$$\boxed{\boldsymbol{\omega_n = \sqrt{\sigma^2 + \omega_d^2} = \sqrt{4+2} = \boxed{\sqrt{6} \text{ radians per second} = 2.44949 \text{ radians per second}}}}$$

(c) (2 points) Estimate the rise time for the unit-step response.

$$\boxed{\boldsymbol{t_r = \frac{0.8 + 2.5\zeta}{\omega_n} = \boxed{1.15993 \text{ seconds}}}}$$

(d) (2 points) Estimate the settling time for the unit-step response.

$$\boxed{\boldsymbol{t_s = \frac{4}{\zeta\omega_n} = \boxed{2 \text{ seconds}}}}$$

(e) (2 points) Determine the peak time for the unit-step response.

$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} = \boxed{2.22144 \text{ seconds}}}$$

(f) (2 points) Determine the percent overshoot for the unit-step response.

$$\boxed{M_p = \exp\left\{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right\} \times 100\% = \boxed{1.17620\%}}$$

(g) (1 point) (**Circle the correct answer.**) The system is

A. Type 0.

B. Type 1.

C. Type 2.

D. None of the above.

(h) (3 points) Determine the appropriate static error constant.

$$\boxed{K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{K(s+1)}{(s-2)} = \frac{6}{-2} = \boxed{-3}}$$

(i) (3 points) Determine the appropriate steady-state error.

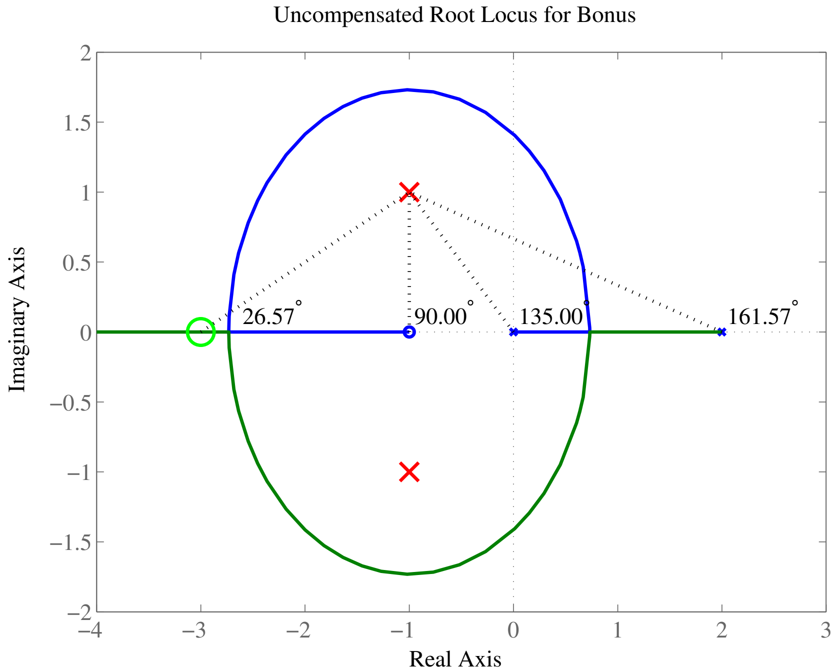
$$\boxed{e_v = \frac{1}{K_v} = \boxed{-\frac{1}{3} \text{ or } -0.33333}}$$

Bonus Question (5 Extra Credit Points):

Consider a unity-feedback control system with the feedforward transfer function given by

$$G(s) = \frac{K(s+1)}{s(s-2)}.$$

Design a PD compensator so that the dominant closed-loop poles are at $s = -1 \pm j$.



The PD compensator has the form

$$G_c(s) = K(s - z).$$

The angle for the single additional zero at $s = z$ is given by

$$\theta_z + 90^\circ - \left(180 - \tan^{-1}\left(\frac{1}{1}\right) + 180 - \tan^{-1}\left(\frac{1}{3}\right)\right) = \pm 180^\circ$$

$$\theta_z + 90^\circ - (135^\circ + 161.56505^\circ) = \pm 180^\circ$$

$$\theta_z - 206.56505^\circ = -180^\circ \Rightarrow \qquad \boxed{\theta_z = 26.56505^\circ.}$$

For $\theta_z = 26.56505^\circ$, the single additional zero must be at $s = -1 - \frac{1}{\tan(26.56505)} = \boxed{-3.}$
The new feedforward transfer function is given by

$$G(s)G_c(s) = \frac{K(s-z)(s+1)}{s(s-2)} = \frac{K(s+3)(s+1)}{s(s-2)}.$$

We can determine the gain K by

$$K = \left.\frac{-1}{\frac{(s+3)(s+1)}{s(s-2)}}\right|_{s=-1\pm j} = \left.\frac{-s(s-2)}{(s+3)(s+1)}\right|_{s=-1\pm j} = \boxed{2.}$$

Therefore, the PD compensator has the form $\boxed{G_c(s) = 2(s+3) = 2s + 6.}$