Math 335 Exam 1 Key

1.) *[6 points]* Shift the series below so that the variable is x^n.

a.) $\sum_{k=2}^{\infty} a_k x^{k+1}$

$n = k+1$

$n-1 = k$

$k=2 \rightarrow n=3$

$$\boxed{\sum_{n=3}^{\infty} a_{n-1} x^n}$$

b.) $\sum_{i=4}^{\infty} (i+3) b_{i-1} x^{i-2}$

$i-2 = n$

$i = n+2$

$i+3 = n+5$

$i-1 = n+1$

$i=4 \rightarrow n=2$

$$\boxed{\sum_{n=2}^{\infty} (n+5) b_{n+1} x^n}$$

2.) *[6 points]* Find all singular points of the ODE below and classify the points as regular or irregular. Show work to justify your classification.

$$x^2(x+2)^2 y'' - xy' + y = 0$$

$$y'' - \underbrace{\frac{1}{x(x+2)^2}}_{P(x)} y' + \underbrace{\frac{1}{x^2(x+2)^2}}_{Q(x)} y = 0$$

Singular points $x = 0, -2$

$$\left.\begin{array}{l} xP(x) = \dfrac{1}{(x+2)^2} \\ x^2 Q(x) = \dfrac{1}{(x+2)^2} \end{array}\right] \to \text{Analytic at } x=0 \Rightarrow \boxed{x=0 \text{ is regular}}$$

$$(x+2)P(x) = \frac{1}{x(x+2)} \Rightarrow \text{Not analytic at } x=-2$$

$$\Rightarrow \boxed{x=-2 \text{ irregular}}$$

3.) *[10 points]* Find the first 5 terms (through x^4) of the series solution about x=0 of the ODE

$$3y''-2xy=0$$

Write your coefficients in the blanks below in terms of a_0 and a_1.

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$3y''-2xy=0$$

$$3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} 3n(n-1)a_n x^{n-2}} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$n-2=k+1, \quad n=k+3$

$n-1=k+2, n=2 \to k=-1$

$$\sum_{k=-1}^{\infty} 3(k+3)(k+2)\, a_{k+3}\, x^{k+1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$-6a_2 + \sum_{n=0}^{\infty} \left[3(n+3)(n+2)\, a_{n+3} - 2a_n\right] x^{n+1} = 0$$

$\underline{x^0}$: $-6a_2 = 0 \to a_2 = 0$

$\underline{x^1}$: $18a_3 - 2a_0 = 0 \to a_3 = \frac{1}{9}a_0$

$\underline{x^2}$: $36a_4 - 2a_1 = 0$

$a_4 = \frac{1}{18}a_1$

$y = a_0 + a_1 x +$ ___ 0 ___ $x^2 +$ ___ $\frac{1}{9}a_0$ ___ $x^3 +$ ___ $\frac{1}{18}a_1$ ___ $x^4 + \cdots$

4.) *[8 points]* Use your answer to #3 to find the solution of the Initial Value Problem

$$y'' - 3xy = 0, \qquad y(0) = 2, \qquad y'(0) = 1.$$

Write the series through x^4, as in the last problem.

$$y = a_0 + a_1 x + 0x^2 + \frac{1}{9} a_0 x^3 + \frac{1}{18} a_1 x^4 + \dots$$

$$y(0) = 2 = a_0$$

$$y'(0) = 1 = a_1$$

$$\boxed{y = 2 + x + 0x^2 + \frac{2}{9} a_0 x^3 + \frac{1}{18} x^4 + \dots}$$

5.) *[20 points]* Note x=0 is a regular singular point of the ODE

$$xy'' + 3y' + 10y = 0$$

Using the Method of Frobenius about x=0, find the indicial roots of the ODE and the general recurrence relation in terms of n and r. *(You do not need to find the Frobenius series soultions. The next page is left blank if you need more room for your work.)*

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$xy'' + 3y' + 10y = 0$$

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + 3\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + 10 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + 3\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \underbrace{10 \sum_{n=0}^{\infty} c_n x^{n+r}}_{} = 0$$

Pull out $n=0$ terms

Shift to $k+r-1$

$n+r = k+r-1$

$n = k-1$

$n=0 \Rightarrow k=1$

$$\sum_{k=1}^{\infty} 10 c_{k-1} x^{k+r-1}$$

$$r(r-1) c_0 x^{r-1} + 3r c_0 x^{r-1}$$

$$+ \sum_{n=1}^{\infty} \left[(n+r)(n+r-1) c_n + 3(n+r) c_n + 10 c_{n-1} \right] x^{n+r-1} = 0$$

$$x^{r-1} \Big| \quad [r(r-1) + 3r] c_0 = 0$$

$$r^2 - r + 3r = 0$$

$$r^2 + 2r = 0$$

$$r(r+2) = 0$$

$$\Rightarrow \boxed{r = 0, -2}$$

#5 continued...

$$x^{n+r-1}\Big| \quad (n+r)(n+r-1)c_n + 3(n+r)c_n + 10c_{n-1} = 0$$

$$\left[(n+r)(n+r-1) + 3(n+r)\right]c_n = -10c_{n-1}$$

$$\boxed{c_n = \frac{-10c_{n-1}}{(n+r)(n+r-1)+3(n+r)}}$$