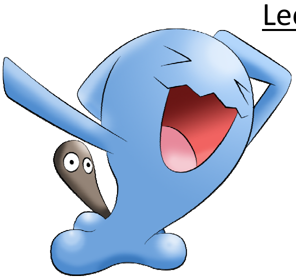


Lecture 11: Surface Integrals

Wobbuffet's Goals for the Day

- Learn how to compute the surface integral of a function
- Discuss applications to surface area and flux

9.13 Surface Integrals

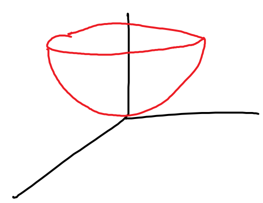
Def A surface S is explicit if it can be written as

$$z = f(x,y).$$

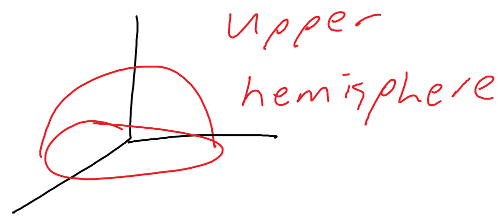
Or solve for another variable $x = f(y,z)$.

Ex Paraboloid "Bowl"

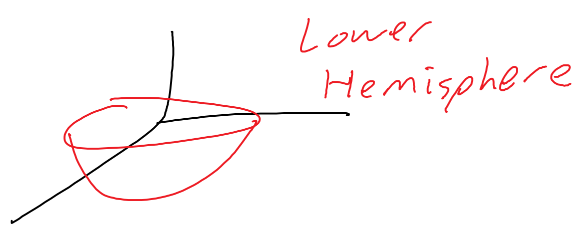
$$z = x^2 + y^2$$



Sphere $x^2+y^2+z^2=1$



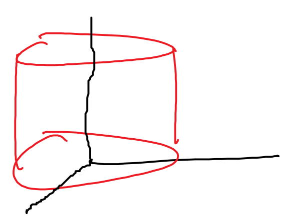
$$z = \sqrt{1-x^2-y^2}$$

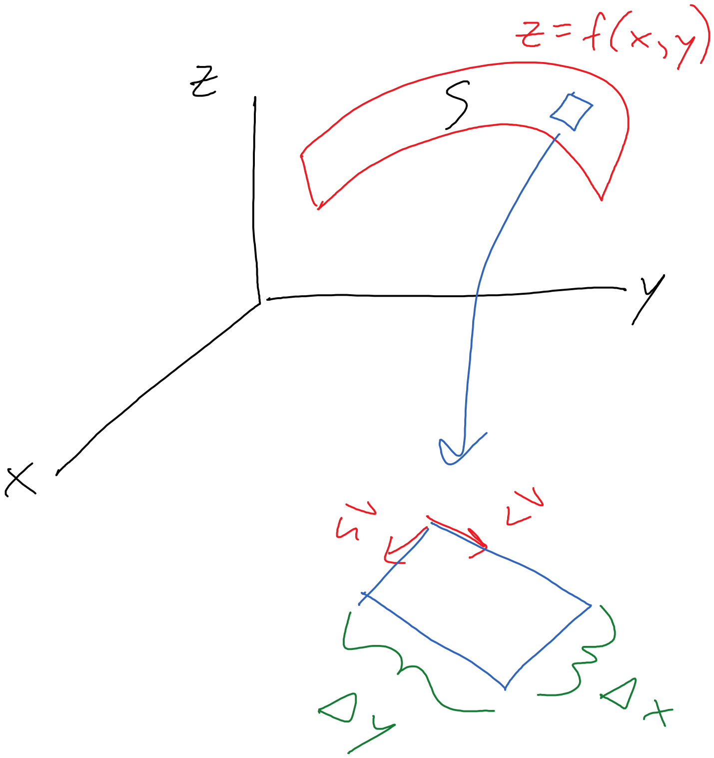


$$z = -\sqrt{1-x^2-y^2}$$

Cylinder $x^2+y^2=1$

Not explicit





Find surface area of S by adding up small rectangular patches.

$$\vec{u} = \langle \Delta x, 0, f_x \Delta x \rangle$$

$$\vec{v} = \langle 0, \Delta y, f_y \Delta y \rangle$$

Fact: The area of a parallelogram bounded by vectors $\vec{u}$ and $\vec{v}$ is given by

$$A = \|\vec{u} \times \vec{v}\|.$$

$$\text{Area Patch} = \|\vec{u} \times \vec{v}\|$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 0 & f_x \Delta x \\ \Delta y & f_y \Delta y \end{vmatrix} - \vec{j} \begin{vmatrix} \Delta x & f_x \Delta x \\ 0 & f_y \Delta y \end{vmatrix} + k \begin{vmatrix} \Delta x & 0 \\ 0 & \Delta y \end{vmatrix}$$

$$= i(0 - f_x \Delta x \Delta y) - j(f_y \Delta x \Delta y - 0) + k(\Delta x \Delta y - 0)$$

$$= \langle -f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y \rangle$$

Area Patch $= \|\vec{u} \times \vec{v}\| = \sqrt{(-f_x \Delta x \Delta y)^2 + (-f_y \Delta x \Delta y)^2 + (\Delta x \Delta y)^2}$

$$= \Delta x \Delta y \sqrt{f_x^2 + f_y^2 + 1}$$

Surface Area of S $\approx$ Sum of Areas of Patches

$$\approx \sum_x \sum_y \Delta x \Delta y \sqrt{f_x^2 + f_y^2 + 1}$$

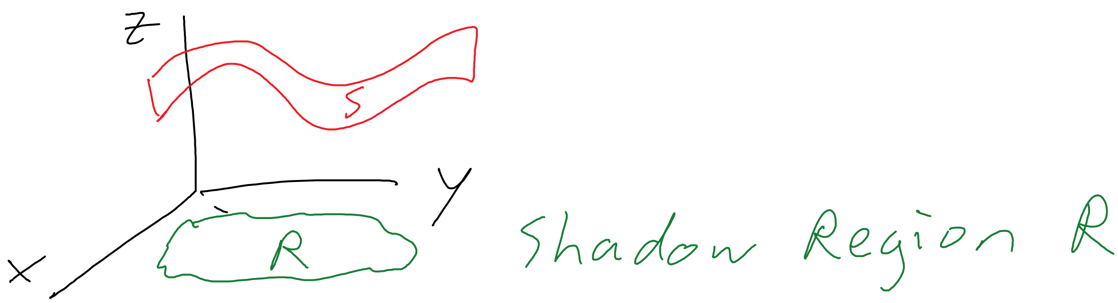
Let $\Delta x, \Delta y \to 0$

Surface Area $= \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$

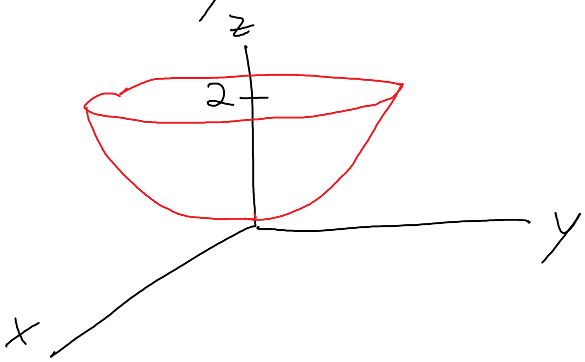
The <u>surface area</u> of a surface S given by $z = f(x,y)$ is

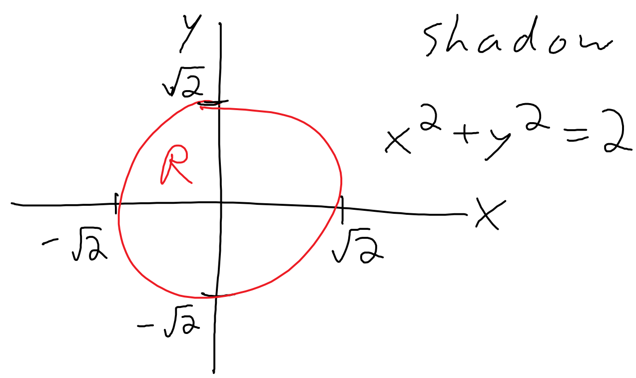
$$\iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

where R is the region spanned in the xy-plane.



<u>Ex</u> Find surface area of the parabaloid $z = x^2 + y^2$ that is below the plane $z=2$.





$$S.A. = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA \quad \leftarrow f(x,y)=x^2+y^2$$

$$= \iint_R \sqrt{1+(2x)^2+(2y)^2}\, dA$$

$$= \iint_R \sqrt{1+4(x^2+y^2)}\, dA$$

Polar!

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\; r \; dr\, d\theta$$

Jacobian

$$= \left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\sqrt{2}} r\sqrt{1+4r^2}\, dr\right]$$

$$= \left[\theta\Big|_0^{2\pi}\right]\left[\frac{1}{12}(1+4r^2)^{3/2}\Big|_0^{\sqrt{2}}\right]$$

$$= [2\pi - 0]\left[\frac{1}{12}(9)^{3/2} - \frac{1}{12}(1)^{3/2}\right]$$

$$= \frac{\pi}{6}[27-1]$$

$$= \boxed{\frac{13\pi}{3}}$$

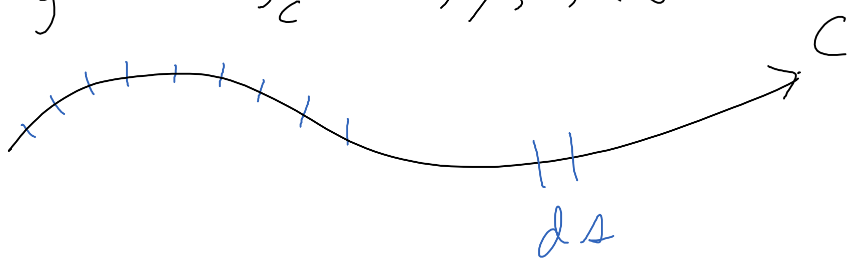
Surface Area: $\iint\limits_S dS = \iint\limits_R \sqrt{1+f_x^2+f_y^2}\, dA$

surface integral   double integral

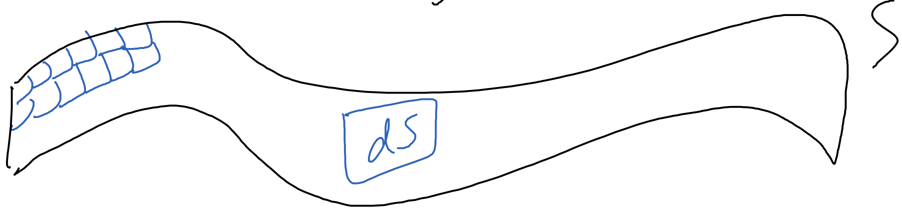
<u>Def</u> The <u>surface integral</u> of $G(x,y,z)$ over a surface S given $z=f(x,y)$ with shadow region R in the xy-plane is

$$\iint\limits_S G(x,y,z)\,dS = \iint\limits_R G(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\, dA$$

Line Integral: $\int_C G(x,y,z)\,ds$



Surface Integral: $\iint\limits_S G(x,y,z)\,dS$



Applications of Surface Integrals $\iint_S G(x,y,z)\,dS$

① $G(x,y,z) = 1$

$\iint_S dS$ = Surface Area of S

② $G(x,y,z)$ = density per unit area

$\iint_S G(x,y,z)\,dS$ = mass of surface S

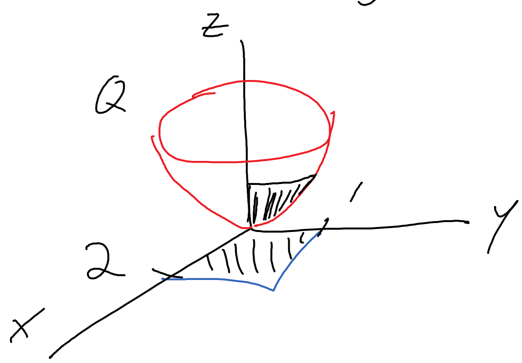
③ $G(x,y,z)$ = charge density per unit area

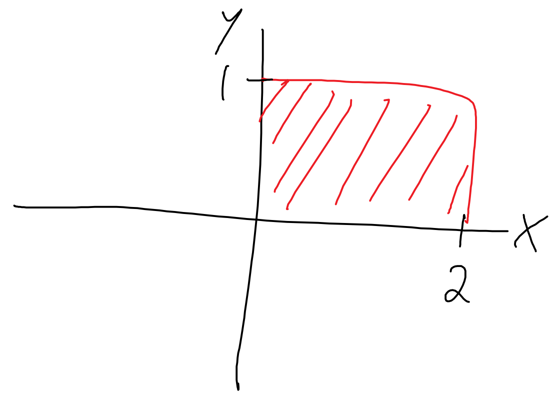
$\iint_S G(x,y,z)\,dS$ = total charge on surface S

④ $G(x,y,z)$ = force exerted at point (x,y,z)

$\iint_S G(x,y,z)\,dS$ = Work done on surface

Ex Set up $\iint_Q yz^2\,dS$ where Q is the parabaloid $z = x^2 + 4y^2$ that is over the rectangle $0 \le x \le 2$, $0 \le y \le 1$.

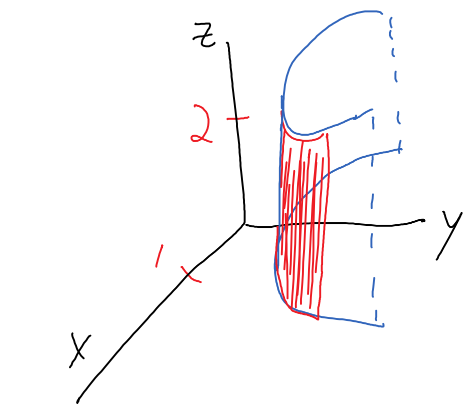


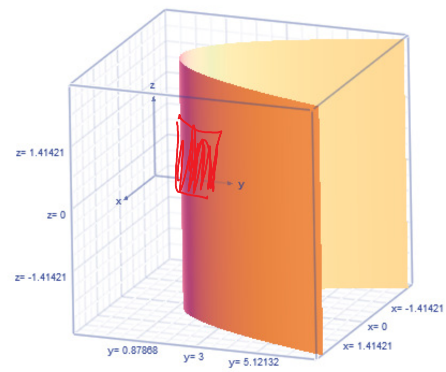


$$\iint_Q yz^2\,dS = \int_0^1\int_0^2 \underbrace{y(x^2+4y^2)^2}_{yz^2}\underbrace{\sqrt{1+(2x)^2+(8y)^2}}_{\substack{\text{Jacobian} \\ \sqrt{1+f_x^2+f_y^2}}}\,dx\,dy$$

Sometimes we have to adjust the dependent variable to something other than z.

Ex Calculate $\iint_Q x z^3 \, dS$ where Q is the surface $y = x^2 + 1$ for $0 \le x \le 1$, $0 \le z \le 2$.





$$y = \underbrace{x^2 + 1}_{f(x,z)} \qquad \sqrt{1 + f_x^2 + f_z^2}$$

$$\iint_Q x z^3 \, dS = \int_0^1 \int_0^2 x z^3 \sqrt{1 + (2x)^2 + (0)^2} \, dz \, dx$$

$$= \int_0^1 \int_0^2 x z^3 \sqrt{1 + 4x^2} \, dz \, dx$$

$$= \left[\int_0^2 z^3 \, dz \right] \left[\int_0^1 x \sqrt{1 + 4x^2} \, dx \right]$$

$\downarrow u = 1 + 4x^2$

$$= \left[\frac{1}{4} z^4 \Big|_0^2 \right] \left[\frac{1}{12} (1 + 4x^2)^{3/2} \Big|_0^1 \right]$$

$$= \left[4 \right] \left[\frac{1}{12} (5)^{3/2} - \frac{1}{12} \right]$$

$$= \boxed{\frac{1}{3} (5)^{3/2} - \frac{1}{3}}$$