Math 335 HW 12 NAME: KEY

Due Wednesday 11/20 5:15pm

Practice Problems *(Do not turn in.)*
Sec 12.3 #11, 15, 19
Sec 13.1 #1, 3, 11, 13

Print out this page and write all answers directly on this worksheet. Show all work. Your answers must be clear and legible. All pages must be stapled.

1.) *[5 points]* Find the Fourier Sine Series on $(0, \pi)$ for the function

$$f(x) = \begin{cases} 2 & if\ x \le 1 \\ 3 & if\ x > 1 \end{cases}$$

$L = \pi$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{\pi}\left[\int_0^1 2\sin(nx)\,dx + \int_1^\pi 3\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{2}{n}\cos(nx)\Big|_0^1 - \frac{3}{n}\cos(nx)\Big|_1^\pi\right]$$

$$= \frac{2}{\pi}\Big[-\frac{2}{n}\cos(n) + \frac{2}{n}\cancelto{1}{\cos(0)}$$

$$- \frac{3}{n}\underbrace{\cos(n\pi)}_{(-1)^n} + \frac{3}{n}\cos(n)\Big]$$

$$= \frac{2}{\pi}\left[\frac{2}{n} + \frac{1}{n}\cos(n) - \frac{3}{n}(-1)^n\right]$$

$$= \frac{4}{n\pi} + \frac{2}{n\pi}\cos(n) - \frac{6}{n\pi}(-1)^n$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}\left[\frac{4}{n\pi} + \frac{2}{n\pi}\cos(n) - \frac{6}{n\pi}(-1)^n\right]\sin(nx)}$$

2.) *[5 points]* Find the Fourier Cosine Series on $(0,\pi)$ for the function

$$f(x) = \begin{cases} 2 & if\ x \le 1 \\ 3 & if\ x > 1 \end{cases} \qquad L = \pi$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^1 2\,dx + \int_1^\pi 3\,dx\right]$$

$$= \frac{2}{\pi}\left[2x\Big|_0^1 + 3x\Big|_1^\pi\right]$$

$$= \frac{2}{\pi}\left[2 - 0 + 3\pi - 3\right] = -\frac{2}{\pi} + 6$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^1 2\cos(nx)\,dx + \int_1^\pi 3\cos(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\sin(nx)\Big|_0^1 + \frac{3}{n}\sin(nx)\Big|_1^\pi\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\sin(n) + \frac{2}{n}\underbrace{\sin(0)}_{0} + \frac{3}{n}\underbrace{\sin(n\pi)}_{0} - \frac{3}{n}\sin(n)\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\sin(n)\right] = -\frac{2}{n\pi}\sin(n)$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \boxed{-\frac{1}{\pi} + 3 + \sum_{n=1}^{\infty} -\frac{2}{n\pi}\sin(n)\cos(nx)}$$

3.) *[10 points]* *(Sec 13.1 #11)* Use separation of variables to find product solutions $u(x,t)$ to

$$16u_{xx} = u_{tt}$$

a.) First assume the solution is separable as $u(x,t) = v(x)w(t)$. Separate the x and t functions and then set them equal to a separation constant $-\lambda$.

$$16(vw)_{xx} = (vw)_{tt}$$

$$16v_{xx}w = vw_{tt}$$

$$\boxed{\frac{16v_{xx}}{v} = \frac{w_{tt}}{w} = -\lambda}$$

b.) Find the solution $u_1(x,t) = v_1(x)w_1(t)$ assuming $\lambda = 0$

$$\frac{16v_{xx}}{v} = 0$$

$$v_{xx} = 0$$

$$v_x = C_1$$

$$v = C_1x + C_2$$

$$\frac{w_{tt}}{w} = 0$$

$$w_{tt} = 0$$

$$w_t = C_3$$

$$w = C_3t + C_4$$

$$u = vw = (C_1x + C_2)(C_3t + C_4)$$

$$= \boxed{A_1xt + A_2t + A_3x + A_4}$$

#3 continued...

c.) Find the solution $u_2(x,t) = v_2(x)w_2(t)$ assuming $\lambda = \alpha^2$ (So $-\lambda = -\alpha^2$).

$$\frac{16v_{xx}}{v} = -\alpha^2$$

$$16v_{xx} = -\alpha^2 v$$

$$16v_{xx} + \alpha^2 v = 0$$

$$16r^2 + \alpha^2 = 0$$

$$r^2 = -\frac{\alpha^2}{16}$$

$$r = \pm \frac{\alpha}{4} i$$

$$v = C_1 \cos \frac{\alpha}{4} x + C_2 \sin \frac{\alpha}{4} x$$

$$\frac{w_{tt}}{w} = -\alpha^2$$

$$w_{tt} = -\alpha^2 w$$

$$w_{tt} + \alpha^2 w = 0$$

$$r^2 + \alpha^2 = 0$$

$$r^2 = -\alpha^2$$

$$r = \pm \alpha i$$

$$w = C_3 \cos \alpha t + C_4 \sin \alpha t$$

$$u = vw = \boxed{\left(C_1 \cos \frac{\alpha}{4} x + C_2 \sin \frac{\alpha}{4} x\right)\left(C_3 \cos \alpha t + C_4 \sin \alpha t\right)}$$

d.) Find the solution $u_3(x,t) = v_3(x)w_3(t)$ assuming $\lambda = -\alpha^2$ (So $-\lambda = \alpha^2$).

$$\frac{16v_{xx}}{v} = \alpha^2$$

$$16v_{xx} = \alpha^2 v$$

$$16v_{xx} - \alpha^2 v = 0$$

$$16r^2 - \alpha^2 = 0$$

$$r^2 = \frac{\alpha^2}{16}$$

$$r = \pm \frac{\alpha}{4}$$

$$v = C_1 e^{\alpha x/4} + C_2 e^{-\alpha x/4}$$

$$\frac{w_{tt}}{w} = \alpha^2$$

$$w_{tt} = \alpha^2 w$$

$$w_{tt} - \alpha^2 w = 0$$

$$r^2 - \alpha^2 = 0$$

$$r^2 = \alpha^2$$

$$r = \pm \alpha$$

$$w = C_3 e^{\alpha t} + C_4 e^{-\alpha t}$$

$$u = vw = \boxed{\left(C_1 e^{\alpha x/4} + C_2 e^{-\alpha x/4}\right)\left(C_3 e^{\alpha t} + C_4 e^{-\alpha t}\right)}$$