

# CONVOLUTION - COMPLETE EXAMPLE<sup>\*</sup>

Anders Gjendemsjø

Richard Baraniuk

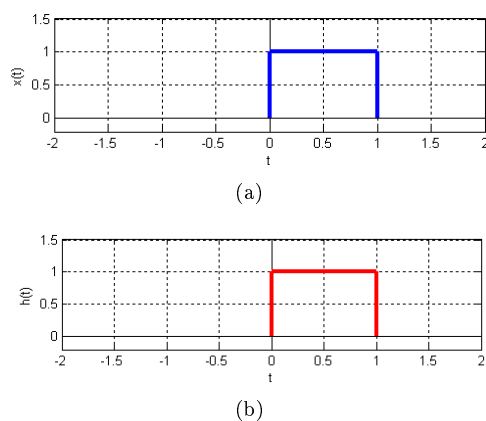
This work is produced by OpenStax-CNX and licensed under the  
Creative Commons Attribution License 1.0<sup>†</sup>

## Abstract

Shows a full example of convolution including math and figures.

## 1 Basic Example

Let us look at a basic continuous-time convolution example to help express some of the important ideas. We will convolve together two square pulses,  $x(t)$  and  $h(t)$ , as shown in Figure 1



**Figure 1:** Two basic signals that we will convolve together.

---

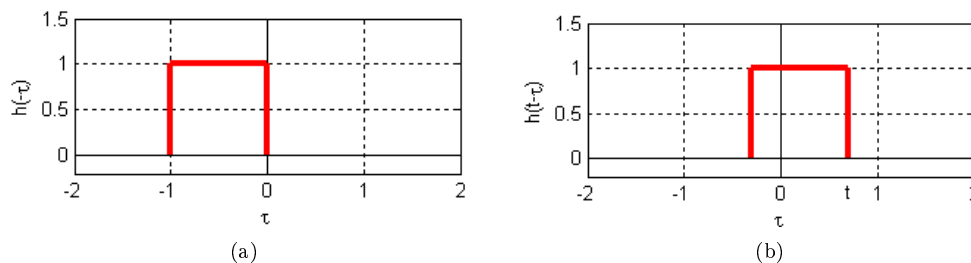
---

<sup>\*</sup>Version 1.7: Jun 26, 2007 4:58 am -0500

<sup>†</sup><http://creativecommons.org/licenses/by/1.0>

### 1.1 Reflect and Shift

Now we will take one of the functions and reflect it around the y-axis. Then we must shift the function, such that the origin, the point of the function that was originally on the origin, is labeled as point  $t$ . This step is shown in Figure 2,  $h(t - \tau)$ .



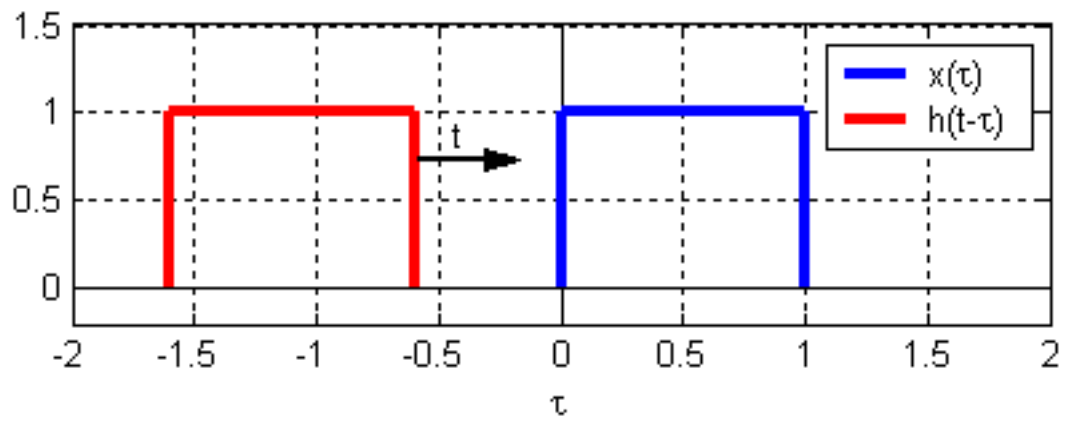
**Figure 2:**  $h(-\tau)$  and  $h(t - \tau)$ . (a) Reflected square pulse. (b) Reflected and **shifted** square pulse.

Note that in Figure 2  $\tau$  is the 1st axis variable while  $t$  is a constant (in this figure). Since convolution is commutative it will never matter which function is reflected and shifted; however, as the functions become more complicated reflecting and shifting the "right one" will often make the problem much easier.

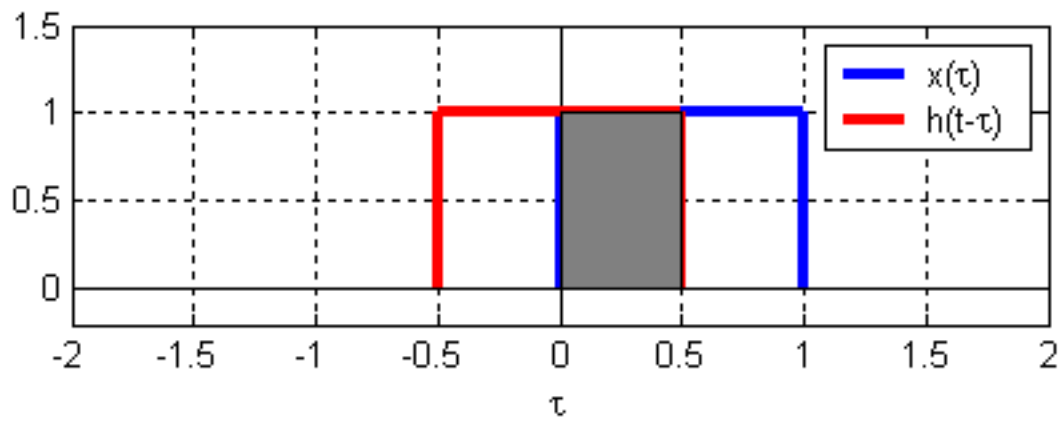
### 1.2 Regions of Integration

We start out with the convolution integral,  $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$ . The value of the function  $y$  at time  $t$  is given by the amount of overlap (to be precise the integral of the overlapping region) between  $h(t - \tau)$  and  $x(\tau)$ .

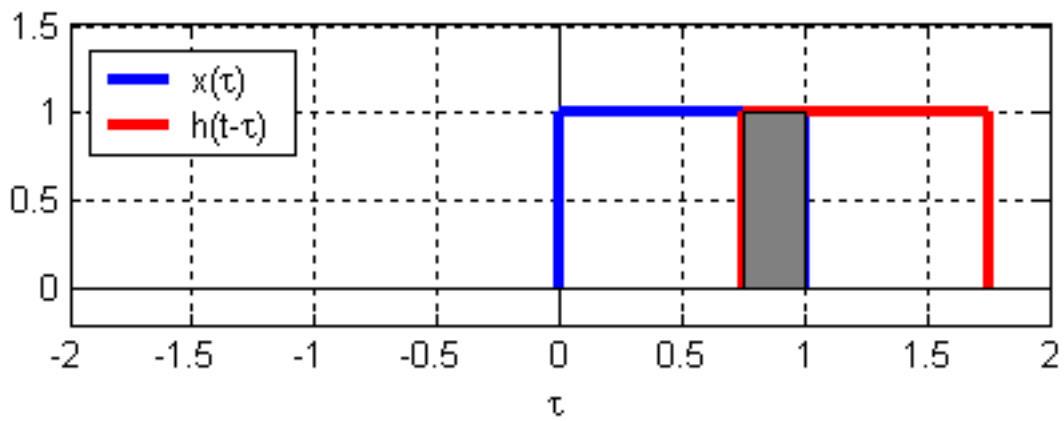
Next, we want to look at the functions and divide the span of the functions into different limits of integration. These different regions can be understood by thinking about how we slide  $h(t - \tau)$  over  $x(\tau)$ , see Figure 3.



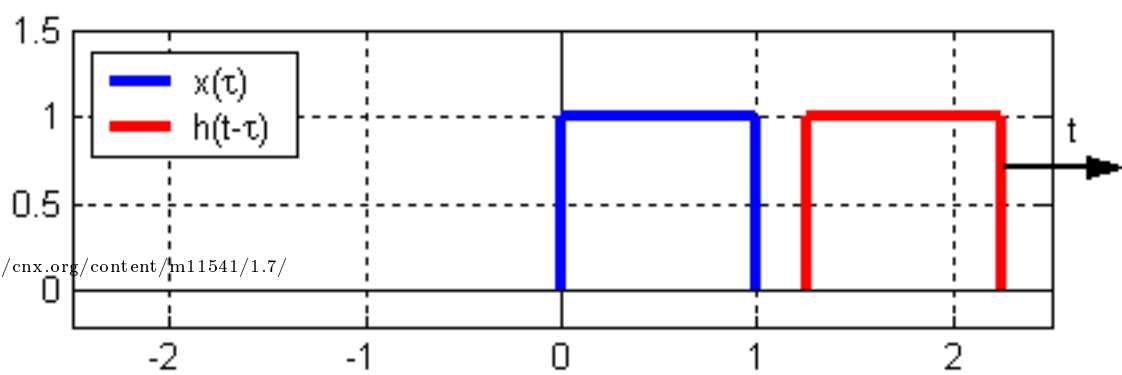
(a)



(b)



(c)



In this case we will have the following four regions. Compare these limits of integration to the four illustrations of  $h(t - \tau)$  and  $x(\tau)$  in Figure 3.

#### Four Limits of Integration

1.  $t < 0$
2.  $0 \leq t < 1$
3.  $1 \leq t < 2$
4.  $t \geq 2$

### 1.3 Using the Convolution Integral

Finally we are ready for a little math. Using the convolution integral, let us integrate the product of  $x(\tau)h(t - \tau)$ . For our first and fourth region this will be trivial as it will always be 0. The second region,  $0 \leq t < 1$ , will require the following math:

$$\begin{aligned} y(t) &= \int_0^t 1 d\tau \\ &= t \end{aligned} \tag{1}$$

The third region,  $1 \leq t < 2$ , is solved in much the same manner. Take note of the changes in our integration though. As we move  $h(t - \tau)$  across our other function, the left-hand edge of the function,  $t - 1$ , becomes our lowlimit for the integral. This is shown through our convolution integral as

$$\begin{aligned} y(t) &= \int_{t-1}^1 1 d\tau \\ &= 1 - (t - 1) \\ &= 2 - t \end{aligned} \tag{2}$$

The above formulas show the method for calculating convolution; however, do not let the simplicity of this example confuse you when you work on other problems. The method will be the same, you will just have to deal with more math in more complicated integrals.

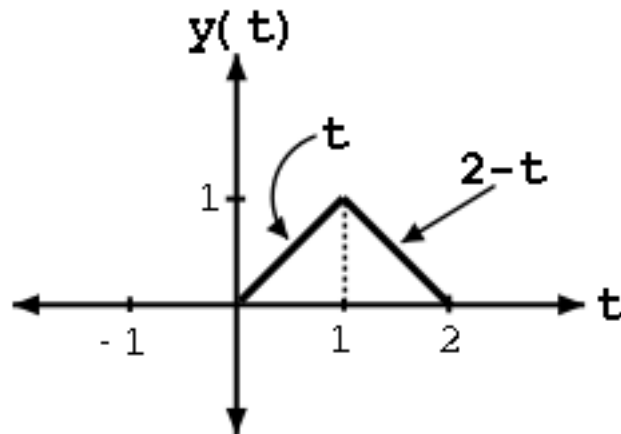
Note that the value of  $y(t)$  at all time is given by the integral of the overlapping functions. In this example  $y$  for a given  $t$  equals the gray area in the plots in Figure 3.

### 1.4 Convolution Results

Thus, we have the following results for our four regions:

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t < 1 \\ 2 - t & \text{if } 1 \leq t < 2 \\ 0 & \text{if } t \geq 2 \end{cases} \tag{3}$$

Now that we have found the resulting function for each of the four regions, we can combine them together and graph the convolution of  $x(t) * h(t)$ .



**Figure 4:** Shows the system's output in response to the input,  $x(t)$ .

### 1.5 Common sense approach

By looking at Figure 3 we can obtain the system output,  $y(t)$ , by "common" sense. For  $t < 0$  there is no overlap, so  $y(t)$  is 0. As  $t$  goes from 0 to 1 the overlap will **linearly** increase with a maximum for  $t = 1$ , the maximum corresponds to the peak value in the triangular pulse. As  $t$  goes from 1 to 2 the overlap will **linearly** decrease. For  $t > 2$  there will be no overlap and hence no output.

We see readily from the "common" sense approach that the output function  $y(t)$  is the same as obtained above with calculations. When convolving to square pulses the result will **always** be a triangular pulse. Its origin, peak value and stretch will, of course, vary.

## 2

- Introduction<sup>1</sup>
- Convolution - Discrete time<sup>2</sup>
- Convolution - Analog<sup>3</sup>
- Properties of convolution<sup>4</sup>

<sup>1</sup>"Introduction to Convolution" <<http://cnx.org/content/m11542/latest/>>

<sup>2</sup>"Convolution - Discrete time" <<http://cnx.org/content/m11539/latest/>>

<sup>3</sup>"Convolution - Analog" <<http://cnx.org/content/m11540/latest/>>

<sup>4</sup>"Properties of Continuous Time Convolution" <<http://cnx.org/content/m10088/latest/>>