Lecture 24: The Wave Equation

**Bulbasaur's Goals for the Day**

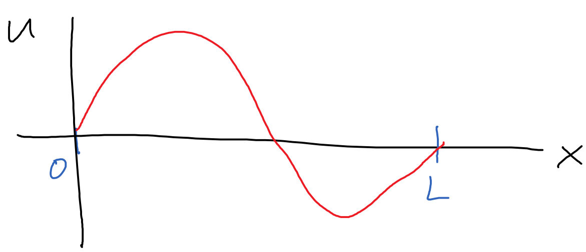
- Derive Bernoulli's Solution of the Wave Equation
- Visualize solutions to the Wave Equation in Matlab
- Practice using formulas to solve PDE word problems

## 13.4 The Wave Equation

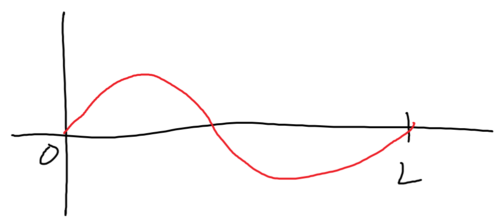
Vibrating string of length $L$

Tension constant $a = \sqrt{\frac{T}{\rho}}$ ← Tension ← Linear density

Let $u(x,t)$ = vertical displacement at time $t$ and position $x$




$t = 0$




$t = 1$

Wave Equation: $u_{tt} = a^2 u_{xx}$

BCs: Clamp the ends at x-axis.

$u(0,t) = 0$ , $u(L,t) = 0$ (Dirichlet BCs)

ICs: Initial shape of string

$u(x,0) = f(x)$ $\quad 0 < x < L$

Initial velocity of string

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \qquad 0 < x < L$$

If released from rest ("plucking"), $g(x) = 0$.

Assume the solution is separable.

$$u(x, t) = v(x)\, w(t)$$

Plug this into the PDE.

$$u_{tt} = a^2 u_{xx}$$

$$(vw)_{tt} = a^2 (vw)_{xx}$$

$$v\, w_{tt} = a^2 v_{xx}\, w$$

$$\frac{w_{tt}}{a^2 w} = \frac{v_{xx}}{v} = -\lambda$$

Separation Constant

3 cases: $\lambda$ is zero, negative, or positive.

① $\underline{\lambda = 0}$

$$\frac{w_{tt}}{a^2 w} = 0 \qquad \frac{v_{xx}}{v} = 0$$

$$w_{tt} = 0 \qquad v_{xx} = 0$$

$$w = C_1 t + C_2 \qquad v = C_3 x + C_4$$

$$u = vw = (C_1 t + C_2)(C_3 x + C_4)$$

$$= D_1 t x + D_2 x + D_3 t + D_4$$

$\underline{BC}$ $u(0,t) = 0 = D_3 t + D_4 = 0 \Rightarrow D_3 = D_4 = 0$

$u(L,t) = 0 = D_1 t L + D_2 L = 0 \Rightarrow D_1 = D_2 = 0$

$\Rightarrow u = 0$ Trivial!

② $\lambda = -\alpha^2$

$$\frac{w_{tt}}{a^2 w} = \alpha^2$$

$$w_{tt} = \alpha^2 a^2 w$$

$$w_{tt} - \alpha^2 a^2 w = 0$$

$$\Downarrow$$

$$r^2 - \alpha^2 a^2 = 0$$

$$r^2 = \alpha^2 a^2$$

$$r = \pm \alpha a$$

$$w = C_1 e^{\alpha a t} + C_2 e^{-\alpha a t}$$

$$\frac{v_{xx}}{v} = \alpha^2$$

$$v_{xx} = \alpha^2 v$$

$$v_{xx} - \alpha^2 v = 0$$

$$\Downarrow$$

$$r^2 - \alpha^2 = 0$$

$$r^2 = \alpha^2$$

$$r = \pm \alpha$$

$$v = C_3 e^{\alpha x} + C_4 e^{-\alpha x}$$

$$u = vw = \left(C_1 e^{\alpha a t} + C_2 e^{-\alpha a t}\right)\left(C_3 e^{\alpha x} + C_4 e^{-\alpha x}\right)$$

$$= D_1 e^{\alpha a t + \alpha x} + D_2 e^{\alpha a t - \alpha x} + D_3 e^{-\alpha a t + \alpha x} + D_4 e^{-\alpha a t - \alpha x}$$

BCs $\quad u(0,t) = 0 = D_1 e^{\alpha a t} + D_2 e^{\alpha a t} + D_3 e^{-\alpha a t} + D_4 e^{-\alpha a t}$

$$= (D_1 + D_2) e^{\alpha a t} + (D_3 + D_4) e^{-\alpha a t}$$

$$\Rightarrow D_1 + D_2 = 0 \qquad \text{and} \qquad D_3 + D_4 = 0$$

$$D_1 = -D_2 \qquad\qquad D_3 = -D_4$$

$$u(L,t) = 0 = D_1 e^{\alpha a t + \alpha L} + D_2 e^{\alpha a t - \alpha L} + D_3 e^{-\alpha a t + \alpha L} + D_4 e^{-\alpha a t - \alpha L}$$

$$0 = D_2\left(-e^{\alpha a t + \alpha L} + e^{\alpha a t - \alpha L}\right) + D_4\left(-e^{\alpha a t + \alpha L} + e^{-\alpha a t - \alpha L}\right)$$

$D_1 = -D_2$ $\qquad$ $D_3 = -D_4$

$$D_2 = D_4 = 0 \quad \Rightarrow \quad D_1 = D_3 = 0$$

$\Rightarrow u = 0$ $\qquad$ Trivial!

(3) $\lambda = \alpha^2$

$$\frac{w_{tt}}{a^2 w} = -\alpha^2$$

$$w_{tt} = -\alpha^2 a^2 w$$

$$w_{tt} + \alpha^2 a^2 w = 0$$

$$\Downarrow$$

$$r^2 + \alpha^2 a^2 = 0$$

$$\frac{v_{xx}}{v} = -\alpha^2$$

Same except without the a

$$r^2 = -\alpha^2 a^2$$

$$r = \pm \alpha a i$$

$$w = C_1 \cos(\alpha a t) + C_2 \sin(\alpha a t) \quad \Big| \quad v = C_3 \cos(\alpha x) + C_4 \sin(\alpha x)$$

$$u = wv = \left[C_1 \cos(\alpha a t) + C_2 \sin(\alpha a t)\right]\left[C_3 \cos(\alpha x) + C_4 \sin(\alpha x)\right]$$

<u>BCs</u> $u(0,t) = 0 = \left[C_1 \cos(\alpha a t) + C_2 \sin(\alpha a t)\right]\left[C_3\right]$

$$\Rightarrow C_3 = 0$$

$$u = \left[C_1 \cos(\alpha a t) + C_2 \sin(\alpha a t)\right]\left[C_4 \sin(\alpha x)\right]$$

<u>BCs</u> $u(L,t) = 0 = \left[C_1 \cos(\alpha a t) + C_2 \sin(\alpha a t)\right]\left[C_4 \sin(\alpha L)\right]$

$$C_4 \sin(\alpha L) = 0$$

If $C_4 = 0$, then we get another trivial solution.

$$\sin(\alpha L) = 0$$

$$\alpha L = n\pi \qquad n \geq 1 \text{ integer}$$

$$\alpha = \frac{n\pi}{L}$$

For each integer $n \geq 1$

$$u_n = \left[C_1 \cos(\alpha a t) + C_2 \sin(\alpha a t)\right]\left[C_4 \sin(\alpha x)\right]$$

$$= A_n \cos(\alpha a t)\sin(\alpha x) + B_n \sin(\alpha a t)\sin(\alpha x)$$

$\alpha = \frac{n\pi}{L} \rightarrow$

$$= A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

By the superposition principle, the sum of solutions to a linear DE is also a solution.

$$u = \sum_{n=1}^{\infty} u_n$$

$$u = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

$\underline{IC}$ $u(x,0) = f(x)$ Initial shape at time $t=0$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \quad \text{on} \quad (0,L)$$

Fourier Sine Series!

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$\underline{IC}$ $\frac{\partial u}{\partial t}(x,0) = g(x)$ Initial velocity at time $t=0$

$$u = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-A_n \frac{n\pi a}{L} \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n \frac{n\pi a}{L}\cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L}\sin\left(\frac{n\pi x}{L}\right) = g(x)$$

Fourier Sine Series with an extra constant $\frac{n\pi a}{L}$

$$B_n = \frac{L}{n\pi a}\,\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

## Bernoulli's Solution of the Wave Equation

$$u_{tt} = a^2 u_{xx}$$

$$u(0,t) = 0, \qquad u(L,t) = 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$