ELEC 309
Signals and Systems

Frequency-Domain Analysis of Continuous-Time (Chapter 5, *Schaum's Outline of Signals and Systems*) and Discrete-Time (Chapter 6, *Schaum's Outline of Signals and Systems*) Systems

Dr. Jason S. Skinner

October 28, 2014

System Representation in the Continuous-Time Domain

Previously, we showed that the response $y(t)$ of a continuous-time LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$, or

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

where

$$y(t) = x(t) * h(t).$$

System Representation in the Frequency Domain

Applying the convolution property to the previous convolution equation, we have

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

where

$$Y(\omega) = X(\omega)H(\omega),$$

and $X(\omega)$, $Y(\omega)$, and $H(\omega)$ are the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively.

The System Frequency Response

Solving for $H(\omega)$, we have

$$\mathcal{F}\{h(t)\} = \boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}} = \frac{\mathcal{F}\{y(t)\}}{\mathcal{F}\{x(t)\}}.$$

The Fourier transform $H(\omega)$ of $h(t)$ is called the **frequency response** of the system.

The System Frequency Response: Magnitude Response and Phase Response

Recall that the frequency response is, in general, complex and therefore can be written as

$$H(\omega) = \left|H(\omega)\right| e^{j\theta_H(\omega)}.$$

The term $\left|H(\omega)\right|$ is called the **magnitude response** of the system.

The term $\theta_H(\omega)$ is called the **phase response** of the system.

Frequency Response of Continuous-Time LTI Systems: Time Domain

Recall that

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow h(t),$$

and

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = h(t) * x(t).$$

Frequency Response of Continuous-Time LTI Systems: Frequency Domain

Recall that

$$1 \longrightarrow \boxed{\text{LTI System}} \longrightarrow H(\omega),$$

and

$$X(\omega) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(\omega) = H(\omega)X(\omega).$$

Frequency Response of Continuous-Time LTI Systems

Consider a continuous-time LTI system with an input $x(t) = e^{j\omega_0 t}$, which has Fourier transform

$$X(\omega) = 2\pi\delta(\omega - \omega_0).$$

The Fourier transform of the output of this continuous-time LTI system is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= H(\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(\omega_0)\delta(\omega - \omega_0). \end{aligned}$$

Frequency Response of Continuous-Time LTI Systems

Taking the inverse Fourier transform of $Y(\omega)$, we have

$$y(t) = H(\omega_0)e^{j\omega_0 t}.$$

Therefore, for a single-frequency-component input,

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{\quad H(\omega) \quad} \longrightarrow y(t) = H(\omega_0)e^{j\omega_0 t}.$$

From our previous analyses of signals using Fourier transformations, we observed that we can represent signals as linear expressions of terms of the form $e^{j\omega_0 t}$.

Response of Continuous-Time LTI Systems: Periodic Inputs

Since the Fourier transform is linear, then if the input $x(t)$ is periodic with the Fourier series representation

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

then the corresponding output of the continuous-time LTI system is also periodic with the Fourier series representation

$$y(t) = \sum_{n=-\infty}^{\infty} D_n H\left(n\omega_0\right) e^{jn\omega_0 t}.$$

Response of Continuous-Time LTI Systems: Aperiodic Inputs

If $x(t)$ is aperiodic, then, from the definition of the inverse Fourier transform, we have

$$x(t) = \mathcal{F}^{-1}\left\{X(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega,$$

and the corresponding output of the continuous-time LTI system can be written as

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega = \mathcal{F}^{-1}\left\{Y(\omega)\right\}. \end{aligned}$$

Response of Continuous-Time LTI Systems

The behavior of a continuous-time LTI system in the frequency domain is completely characterized by its frequency response $H(\omega)$.

Let

$$X(\omega) = \left|X(\omega)\right|e^{j\theta_X(\omega)}$$

and

$$Y(\omega) = \left|Y(\omega)\right|e^{j\theta_Y(\omega)}.$$

Response of Continuous-Time LTI Systems: Magnitude Spectrum

The magnitude spectrum of the output can be written as

$$\left|Y(\omega)\right| = \left|H(\omega)\right|\left|X(\omega)\right|,$$

which says that the magnitude spectrum $\left|Y(\omega)\right|$ of the output is the product of the magnitude spectrum $\left|X(\omega)\right|$ of the input and the magnitude response $\left|H(\omega)\right|$ of the system.

The magnitude response $\left|H(\omega)\right|$ is sometimes referred to as the **gain** of the system.

Response of Continuous-Time LTI Systems: Phase Spectrum

The phase spectrum of the output can be written as

$$\theta_Y(\omega) = \theta_H(\omega) + \theta_X(\omega),$$

which says that the phase spectrum $\theta_Y(\omega)$ of the output is the sum of the phase spectrum $\theta_X(\omega)$ of the input and the phase response $\theta_H(\omega)$ of the system.

Response of Continuous-Time LTI Systems: Distortionless Transmission

For **distortionless transmission** through a continuous-time LTI system, we require that the signal retain its overall shape. In other words, the system may only alter the amplitude (by a multiplicative constant) of the input signal or delay it in time. Therefore, if $x(t)$ is the input signal to a continuous-time LTI system, then the input-output equation

$$y(t) = Kx(t - t_d)$$

is required for distortionless transmission, where t_d is the **time delay** and $K > 0$ is a **gain constant**.

Response of Continuous-Time LTI Systems: Distortionless Transmission

Taking the Fourier transform of both sides, we see that

$$Y(\omega) = Ke^{-j\omega t_d}X(\omega).$$

Therefore, for distortionless transmission, the system must have a frequency response given by

$$H(\omega) = \left|H(\omega)\right|e^{j\theta_H(\omega)} = Ke^{-j\omega t_d}.$$

Therefore,

$$\left|H(\omega)\right| = K \text{ and} \tag{1}$$

$$\theta_H(\omega) = -\omega t_d. \tag{2}$$

Response of Continuous-Time LTI Systems: Amplitude Distortion

For distortionless transmission, the amplitude of $H(\omega)$ must be **constant** over all frequencies ω.

When the magnitude spectrum $|H(\omega)|$ of the system is not constant with the frequency band of interest, the frequency components of the input signal are transmitted with a different amount of **gain** or **attenuation**.

This effect is referred to as **amplitude distortion**.

Response of Continuous-Time LTI Systems: Phase Distortion

For distortionless transmission, the phase of $H(\omega)$ must be **linear** with respect to frequency ω.

When the phase spectrum $\theta_H(\omega)$ of the system is not linear over the frequency band of interest, the different frequency components of the input signal encounter different delays in passing through the system, which results in a different waveform at the output of the system.

This effect is referred to as **phase distortion**.

Frequency Response for Continuous-Time LTI Systems Described by LCCDEs

Previously, we considered a continuous-time LTI system for which input $x(t)$ and output $y(t)$ satisfy a general N^{th}-order linear constant-coefficient differential equation (LCCDE) given by

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

with $M \leq N$.

Frequency Response for Continuous-Time LTI Systems Described by LCCDEs

Taking the Fourier transform of both sides, and applying the linearity and time-differentiation properties of the Fourier transform, we have

$$\sum_{n=0}^{N} a_n (j\omega)^n Y(\omega) = \sum_{m=0}^{M} b_m (j\omega)^m X(\omega).$$

or

$$Y(\omega) \sum_{n=0}^{N} a_n (j\omega)^n = X(\omega) \sum_{m=0}^{M} b_m (j\omega)^m.$$

Frequency Response for Continuous-Time LTI Systems Described by LCCDEs

Therefore, a continuous-time LTI system whose input $x(t)$ and output $y(t)$ satisfy a general N^{th}-order LCCDE has a **frequency response** given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}, \tag{3}$$

which is a rational function of ω.

Example:

Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

Find the impulse response $h(t)$ of the system.

Example (continued):

Filtering

Filtering is one of the most basic operations in any signal processing system.

Filtering is the process by which the relative amplitudes of the frequency components in a signal are changed or even suppressed.

For continuous-time LTI system: $Y(\omega) = X(\omega)H(\omega)$.

Therefore, an LTI system acts as a filter on the input signal, and the term **filter** is used to describe any system that exhibits some type of frequency-selective behavior.

Ideal Frequency-Selective Filters

An **ideal frequency-selective filter** exactly passes signal in one set of frequencies and completely suppresses the rest.

The band of frequencies passed by the filter is referred to as the **pass band**.

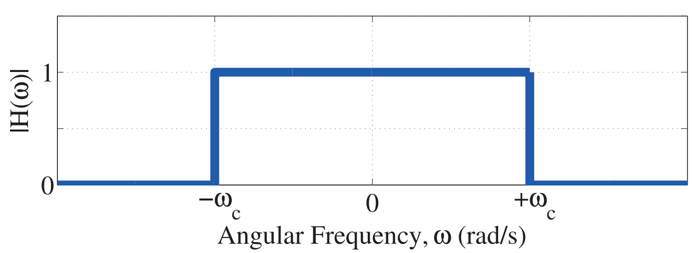
The band of frequencies rejected by the filter is referred to as the **stop band**.

Ideal Frequency-Selective Filters: Ideal Low-Pass Filter Amplitude Response

An ideal low-pass filter (LPF) amplitude response is given by

$$\left|H(\omega)\right| = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c. \end{cases}$$

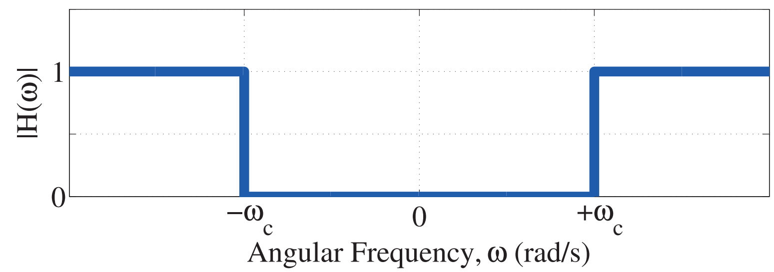
The frequency ω_c is called the **cutoff frequency**.



Ideal Frequency-Selective Filters: Ideal High-Pass Filter Amplitude Response

An ideal high-pass filter (HPF) amplitude response is given by

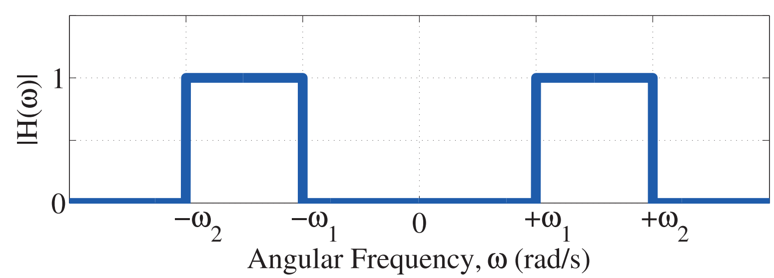
$$\left|H(\omega)\right| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| \geq \omega_c. \end{cases}$$



Ideal Frequency-Selective Filters: Ideal Band-Pass Filter Amplitude Response

An ideal band-pass filter (BPF) amplitude response is given by

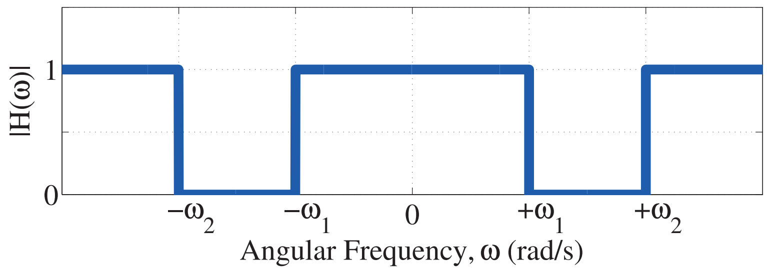
$$\left|H(\omega)\right| = \begin{cases} 1 & \omega_1 \leq |\omega| \leq \omega_2 \\ 0 & \text{otherwise.} \end{cases}$$



Ideal Frequency-Selective Filters: Ideal Band-Stop Filter Amplitude Response

An ideal band-stop filter (BSF) amplitude response is given by

$$\left|H(\omega)\right| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise.} \end{cases}$$



Ideal Frequency-Selective Filters: Phase Response

To avoid phase distortion in the filtering process, a filter should have a linear phase characteristic over the pass band of the filter, or

$$\theta_H(\omega) = -\omega t_d,$$

where t_d is a constant.

All ideal frequency-selective filters are *noncausal* systems.

Example:

Consider the periodic square wave given by

$$x(t) = \begin{cases} 1 & 2\pi k \le t < 2\pi(k+0.5) \\ 0 & 2\pi(k+0.5) \le t < 2\pi(k+1) \end{cases}$$

for all integers k. Suppose this signal passes through an ideal low-pass filter with ideal phase response and amplitude response given by

$$\left|H(\omega)\right| = \begin{cases} 1 & |\omega| \le 6 \\ 0 & |\omega| > 6. \end{cases}$$

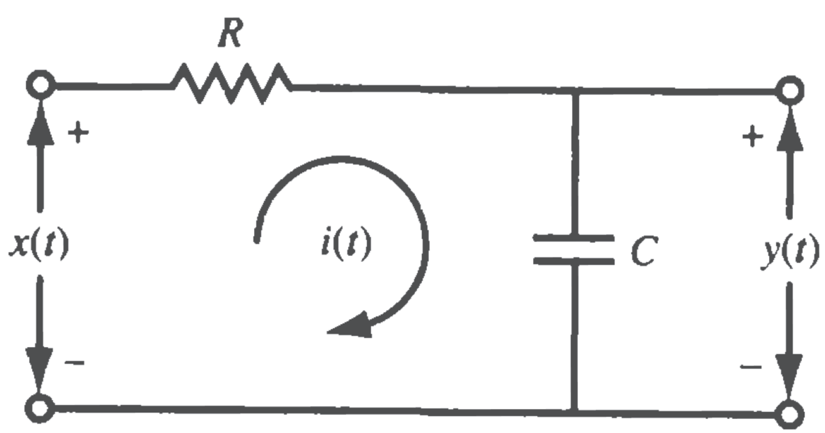
Determine the output $y(t)$ of this filter.

Example (continued):

Example (continued):

Nonideal Frequency-Selective Filters

Consider the RC filter shown below:



Nonideal Frequency-Selective Filters

The input-output equation for this circuit is given by

$$RC\frac{dy(t)}{dt} + y(t) = x(t).$$

Taking the Fourier transform of both sides of the input-output equation, the frequency response $H(\omega)$ of the RC filter is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

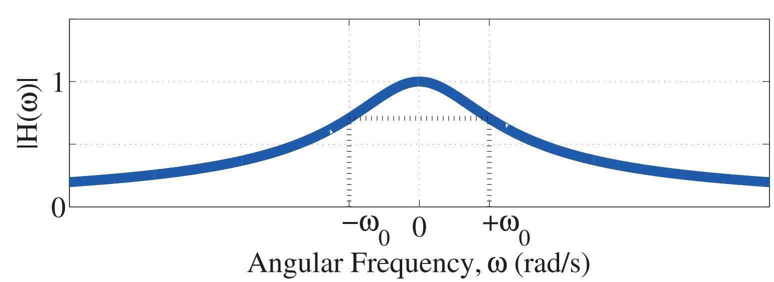
where $\omega_0 = \frac{1}{RC}$ is the **cutoff frequency**.

This is a **first-order low-pass Butterworth filter**.

Nonideal Frequency-Selective Filters

The amplitude response is given by

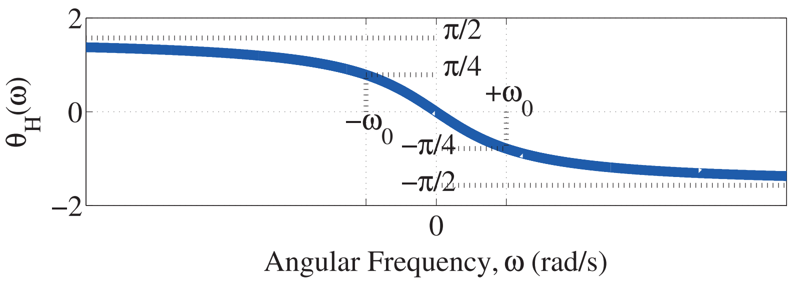
$$\left|H(\omega)\right| = \frac{1}{\left|1 + j\frac{\omega}{\omega_0}\right|} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}.$$



Nonideal Frequency-Selective Filters

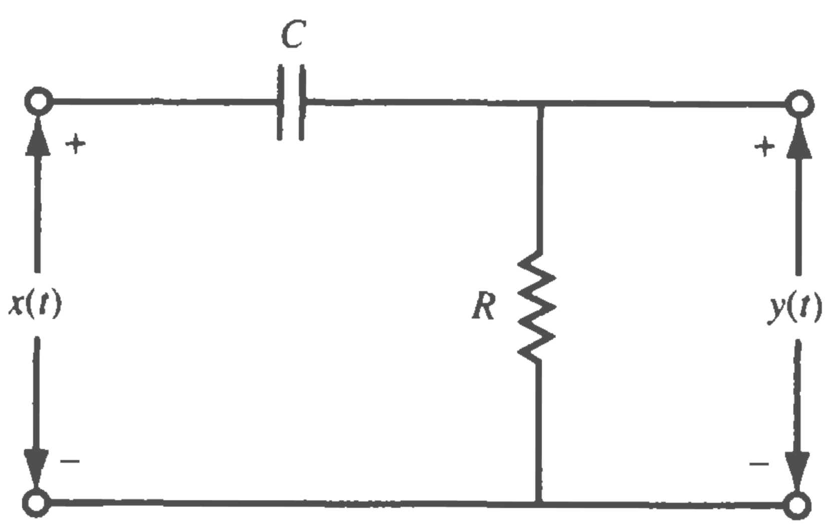
The phase response is given by

$$\theta_H(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right).$$



Example:

Consider the RC filter shown below.



Nonideal Frequency-Selective Filters

The input-output equation for this circuit is given by

$$RC\frac{dy(t)}{dt} + y(t) = RCx'(t).$$

Taking the Fourier transform of both sides of the input-output equation, the frequency response $H(f)$ of the RC filter is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega RC}{1 + j\omega RC} = \frac{1}{1 - j\frac{\omega_0}{\omega}}$$

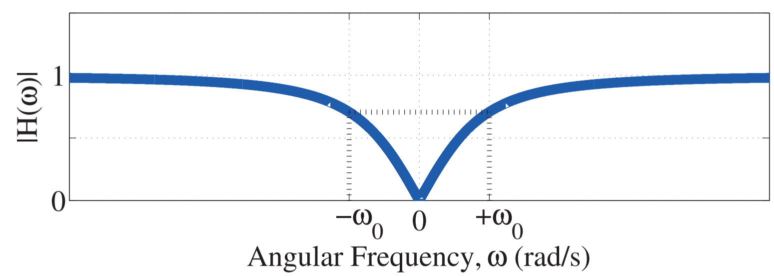
where $\omega_0 = \frac{1}{RC}$ is the **cutoff frequency**.

This is a **first-order high-pass Butterworth filter**.

Nonideal Frequency-Selective Filters

The amplitude response is given by

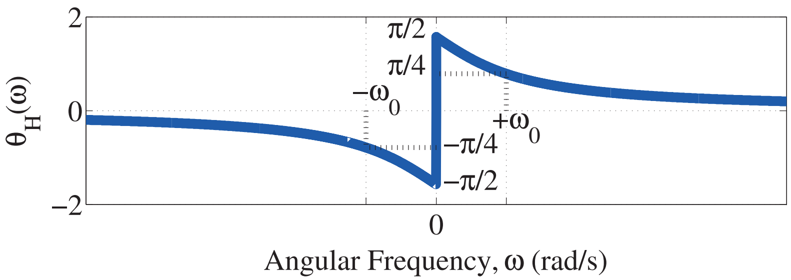
$$\left|H(\omega)\right| = \frac{1}{\left|1 - j\frac{\omega_0}{\omega}\right|} = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}}.$$



Nonideal Frequency-Selective Filters

The phase response is given by

$$\theta_H(\omega) = \tan^{-1}\left(\frac{\omega_0}{\omega}\right).$$



Filter or System Bandwidth

One important concept in system analysis is the **bandwidth** of a filter or system.

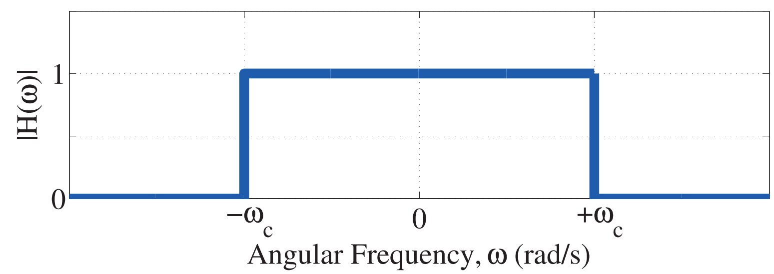
The **bandwidth** of a filter or system is defined as the width of the range of positive frequencies of components that are not filtered out when passing through a filter or system.

There are *many* different definitions of bandwidth.

Absolute Bandwidth of Ideal Filters

The **absolute bandwidth** is defined only for *ideal* low-pass or band-pass filters.

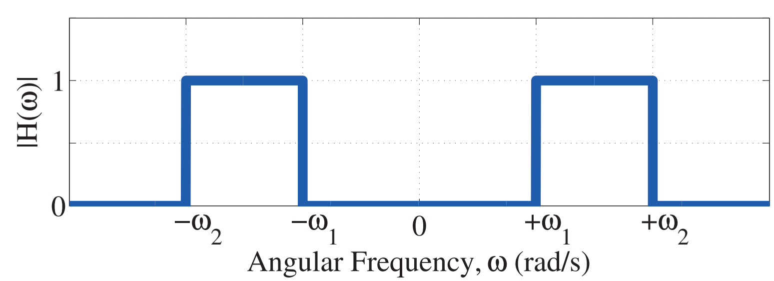
The absolute bandwidth W_B of an ideal low-pass filter is its cutoff frequency. In other words, $W_B = f_c = \frac{\omega_c}{2\pi}$ Hz.



Absolute Bandwidth of Ideal Filters

The absolute bandwidth W_B of an ideal band-pass filter is given by $W_B = f_2 - f_1 = \frac{\omega_2 - \omega_1}{2\pi}$ Hz.

A band-pass filter is called **narrowband** if $W_B \ll f_m$, where $f_m = \frac{f_1+f_2}{2} = \frac{\omega_1+\omega_2}{4\pi}$ is the center frequency of the filter.



Half-Power Bandwidth of Nonideal Filters

For causal or practical filters, a commonly-used definition of filter or system bandwidth is the **half-power (or 3-dB) bandwidth** $W_{\text{3-dB}}$.

The **half-power bandwidth** is defined as the width of the range of positive frequencies for which

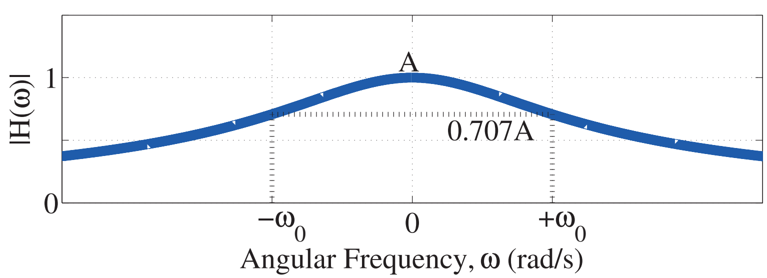
$$\left|H(\omega)\right| \geq \frac{\max\left\{\left|H(\omega)\right|\right\}}{\sqrt{2}}.$$

The half-power bandwidth is also known as the **3-dB bandwidth** because a power attenuation by a factor of 2 is equivalent to a attenuation of 3 dB.

Half-Power Bandwidth of Nonideal Low-Pass Filters

For a causal or practical low-pass filter, we have $\max\left\{H(\omega)\right\} = \left|H(0)\right|$ and its cutoff frequency ω_0 must meet the condition

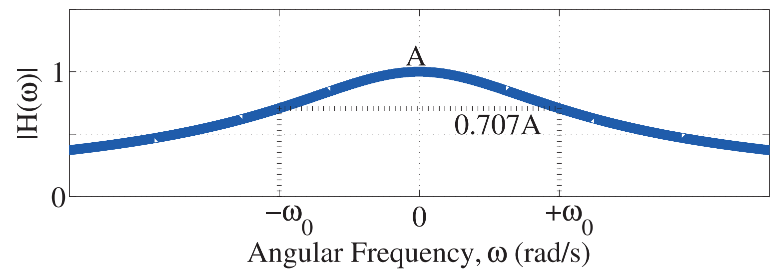
$$\left|H(\omega_0)\right| = \frac{\left|H(0)\right|}{\sqrt{2}}.$$



Half-Power Bandwidth of Nonideal Low-Pass Filters

For a causal or practical low-pass filter, the **half-power (or 3-dB) bandwidth** is given by

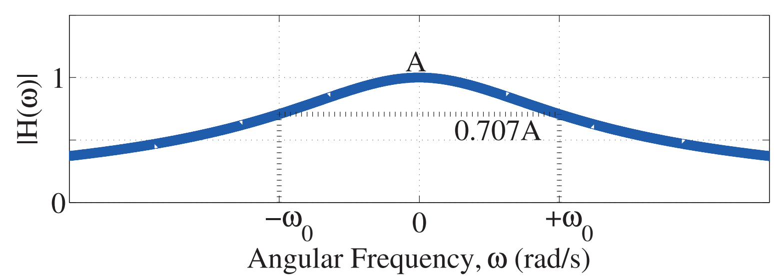
$$W_{\text{3-dB}} = f_0 = \frac{\omega_0}{2\pi} \text{ Hz.}$$



Half-Power Bandwidth of Nonideal Low-Pass Filters

For the low-pass RC filter discussed previously, the **half-power (or 3-dB) bandwidth** is given by

$$W_{\text{3-dB}} = f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi RC} \text{ Hz.}$$

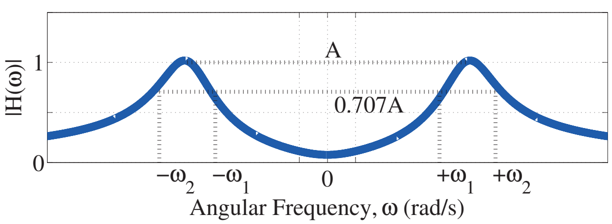


Half-Power Bandwidth of Nonideal Band-Pass Filters

For a causal or practical band-pass filter, its two cutoff frequencies ω_1 and ω_2 must meet the conditions

$$\omega_1 = \min\left\{\text{positive } \omega : \left|H(\omega)\right| \geq \frac{\left|H(0)\right|}{\sqrt{2}}\right\} \text{ and}$$

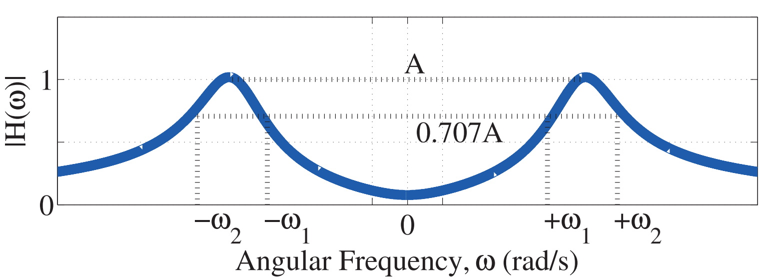
$$\omega_2 = \max\left\{\text{positive } \omega : \left|H(\omega)\right| \geq \frac{\left|H(0)\right|}{\sqrt{2}}\right\}.$$



Half-Power Bandwidth of Nonideal Band-Pass Filters

For a causal or practical band-pass filter, the **half-power (or 3-dB) bandwidth** is given by

$$W_{\text{3-dB}} = f_2 - f_1 = \frac{\omega_2 - \omega_1}{2\pi} \text{ Hz.}$$



Signal Bandwidth

The **bandwidth** of a signal is defined as the width of the range of "significant" positive frequencies contained in a signal.

The **absolute bandwidth** of a signal is defined as the width of the range of positive frequency components contained in a signal.

The **bandwidth** of a signal can also be defined as the width of the range of positive frequencies in which "most" of the energy or power lies.

Half-Power Bandwidth of a Signal

The bandwidth of a signal $x(t)$ can also be defined on a similar basis as a filter bandwidth, such as the half-power (or 3-dB) bandwidth, using the amplitude spectrum $\left|X(\omega)\right|$ of the signal.

If we replace $\left|H(\omega)\right|$ by $\left|X(\omega)\right|$ in our previous amplitude response plots, we will have frequency-domain plots of **low-pass**, **high-pass**, and **band-pass** signals.

Band-Limited Signal

A signal $x(t)$ is said to be **band-limited** if

$$\left|X(\omega)\right| = 0 \text{ if } |\omega| > \omega_M$$

The bandwidth of a band-limited signal is given by ω_M radians per second or $f_M = \frac{\omega_M}{2\pi}$ Hz.

System Representation in the Discrete-Time Domain

Previously, we showed that the response $y[n]$ of a discrete-time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$, or

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

where

$$y[n] = x[n] * h[n].$$

System Representation in the Frequency Domain

Applying the convolution property to the previous convolution equation, we have

$$X(\Omega) \longrightarrow \boxed{H(\Omega)} \longrightarrow Y(\Omega)$$

where

$$Y(\Omega) = X(\Omega)H(\Omega),$$

and $X(\Omega)$, $Y(\Omega)$, and $H(\Omega)$ are the Fourier transforms of $x[n]$, $y[n]$, and $h[n]$, respectively.

The System Frequency Response

Solving for $H(\Omega)$, we have

$$\mathcal{F}\left\{h[n]\right\} = \boxed{H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}} = \frac{\mathcal{F}\left\{y[n]\right\}}{\mathcal{F}\left\{x[n]\right\}}.$$

The Fourier transform $H(\Omega)$ of $h[n]$ is called the **frequency response** of the system.

The System Frequency Response: Magnitude Response and Phase Response

Recall that the frequency response is, in general, complex and therefore can be written as

$$H(\Omega) = \left|H(\Omega)\right| e^{j\theta_H(\Omega)}.$$

The term $\left|H(\Omega)\right|$ is called the **magnitude response** of the system.

The term $\theta_H(\Omega)$ is called the **phase response** of the system.

Frequency Response of Discrete-Time LTI Systems: Time Domain

Recall that

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n],$$

and

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = h[n] * x[n].$$

Frequency Response of Discrete-Time LTI Systems: Frequency Domain

Recall that

$$1 \longrightarrow \boxed{\text{LTI System}} \longrightarrow H(\Omega),$$

and

$$X(\Omega) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(\Omega) = H(\Omega)X(\Omega).$$

Frequency Response of Discrete-Time LTI Systems

Consider a discrete-time LTI system with an input $x[n] = e^{j\Omega_0 n}$, which has Fourier transform

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0).$$

The Fourier transform of the output of this discrete-time LTI system is given by

$$\begin{aligned} Y(\Omega) &= H(\Omega)X(\Omega) \\ &= H(\Omega)2\pi\delta(\Omega - \Omega_0) = 2\pi H(\Omega_0)\delta(\Omega - \Omega_0). \end{aligned}$$

Frequency Response of Discrete-Time LTI Systems

Taking the inverse Fourier transform of $Y(\Omega)$, we have

$$y[n] = H(\Omega_0)e^{j\Omega_0 n}.$$

Therefore, for a single-frequency-component input,

$$x[n] = e^{j\Omega_0 n} \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n] = H(\Omega_0)e^{j\Omega_0 n}.$$

From our previous analyses of signals using Fourier transformations, we observed that we can represent signals as linear expressions of terms of the form $e^{j\Omega_0 n}$.

Response of Discrete-Time LTI Systems: Periodic Inputs

Since the Fourier transform is linear, then if the input $x[n]$ is periodic with the Fourier series representation

$$x[n] = \sum_{k=<N_0>} \mathcal{D}_n e^{jk\Omega_0 n},$$

then the corresponding output of the discrete-time LTI system is also periodic with the Fourier series representation

$$y[n] = \sum_{k=<N_0>} \mathcal{D}_n H\left(k\Omega_0\right) e^{jk\Omega_0 n}.$$

Response of Discrete-Time LTI Systems: Aperiodic Inputs

If $x[n]$ is aperiodic, then, from the definition of the inverse Fourier transform, we have

$$x[n] = \mathcal{F}^{-1}\left\{X(\Omega)\right\} = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega,$$

and the corresponding output of the discrete-time LTI system can be written as

$$\begin{aligned} y[n] &= \frac{1}{2\pi}\int_{2\pi} H(\Omega)X(\Omega)e^{j\Omega n}d\Omega \\ &= \frac{1}{2\pi}\int_{2\pi} Y(\Omega)e^{j\Omega n}d\Omega = \mathcal{F}^{-1}\left\{Y(\Omega)\right\}. \end{aligned}$$

Response of Discrete-Time LTI Systems

The behavior of a discrete-time LTI system in the frequency domain is completely characterized by its frequency response $H(\Omega)$.

Let

$$X(\Omega) = \left|X(\Omega)\right|e^{j\theta_X(\Omega)}$$

and

$$Y(\Omega) = \left|Y(\Omega)\right|e^{j\theta_Y(\Omega)}.$$

Response of Discrete-Time LTI Systems: Magnitude Spectrum

The magnitude spectrum of the output can be written as

$$\left|Y(\Omega)\right| = \left|H(\Omega)\right|\left|X(\Omega)\right|,$$

which says that the magnitude spectrum $\left|Y(\Omega)\right|$ of the output is the product of the magnitude spectrum $\left|X(\Omega)\right|$ of the input and the magnitude response $\left|H(\Omega)\right|$ of the system.

The magnitude response $\left|H(\Omega)\right|$ is sometimes referred to as the **gain** of the system.

Response of Discrete-Time LTI Systems: Phase Spectrum

The phase spectrum of the output can be written as

$$\theta_Y(\Omega) = \theta_H(\Omega) + \theta_X(\Omega),$$

which says that the phase spectrum $\theta_Y(\Omega)$ of the output is the sum of the phase spectrum $\theta_X(\Omega)$ of the input and the phase response $\theta_H(\Omega)$ of the system.

Periodic Nature of the Frequency Response

Note that

$$H\left(\Omega + 2\pi\right) = H(\Omega).$$

Unlike the frequency response of continuous-time systems, the frequency response of discrete-time systems is periodic with period 2π.

Therefore, we need to consider the frequency response of a discrete-time system only over the range $0 \leq \Omega < 2\pi$ or $-\pi \leq \Omega < \pi$.

Frequency Response for Discrete-Time LTI Systems Described by LCCDEs

Previously, we considered a discrete-time LTI system for which input $x[n]$ and output $y[n]$ satisfy a general N^{th}-order linear constant-coefficient difference equation (LCCDE) given by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

with $M \leq N$.

Frequency Response for Discrete-Time LTI Systems Described by LCCDEs

Taking the Fourier transform of both sides, and applying the linearity and time-shifting properties of the Fourier transform, we have

$$\sum_{k=0}^{N} a_k e^{-jk\Omega} Y(\Omega) = \sum_{m=0}^{M} b_m e^{-jm\Omega} X(\Omega).$$

or

$$Y(\Omega) \sum_{k=0}^{N} a_k e^{-jk\Omega} = X(\Omega) \sum_{m=0}^{M} b_m e^{-jm\Omega}.$$

Frequency Response for Discrete-Time LTI Systems Described by LCCDEs

Therefore, a discrete-time LTI system whose input $x[n]$ and output $y[n]$ satisfy a general N^{th}-order LCCDE has a **frequency response** given by

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum\limits_{m=0}^{M} b_m e^{-jm\Omega}}{\sum\limits_{k=0}^{N} a_k e^{-jk\Omega}}. \tag{4}$$

Example:

Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] - 0.5y[n-1] = x[n].$$

Find the impulse response $h[n]$ of the system.

Example (continued):