Lecture 20: Fourier Sine and Cosine Series

Machop's Goals for the Day

- Practice computing Fourier Series
- Learn how to compute Fourier Series on a half-range interval
- Discuss complex-valued Fourier Series

12.3 Fourier Sine + Cosine Series

Fourier Series of $f(x)$ on $(-L, L)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

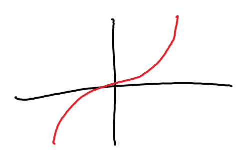
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Odd-Even Theorem

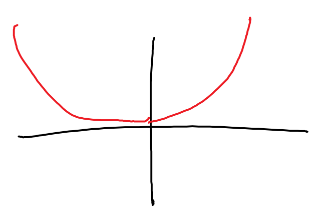
$f(x)$ is odd if $f(-x) = -f(x)$

$f(x)$ odd $\Rightarrow a_n = 0$



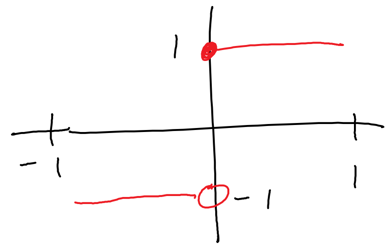
$f(x)$ is even if $f(-x) = f(x)$

$f(x)$ even $\Rightarrow b_n = 0$



Ex Find Fourier series on $(-1,1)$ of the square wave

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$$



$f(x)$ is odd $\Rightarrow a_n = 0$

$L = 1$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \int_{-1}^{1} f(x) \sin(n\pi x)\, dx$$

$$= \int_{-1}^{0} (-1) \sin(n\pi x)\, dx + \int_{0}^{1} (1) \sin(n\pi x)\, dx$$

$$= \frac{1}{n\pi} \cos(n\pi x) \Big|_{-1}^{0} + -\frac{1}{n\pi} \cos(n\pi x) \Big|_{0}^{1}$$

$$= \frac{1}{n\pi} \underbrace{\cos(0)}_{1} - \frac{1}{n\pi} \underbrace{\cos(-n\pi)}_{(-1)^n} - \frac{1}{n\pi} \underbrace{\cos(n\pi)}_{(-1)^n} + \frac{1}{n\pi} \underbrace{\cos(0)}_{1}$$

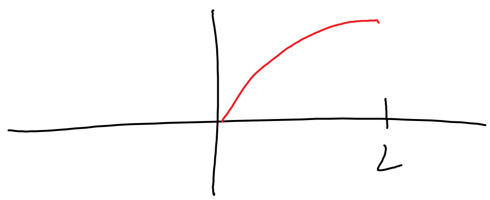
$$= \frac{2}{n\pi} \left[1 - (-1)^n \right]$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 - (-1)^n \right] \sin(n\pi x)}$$

Half-Range Series

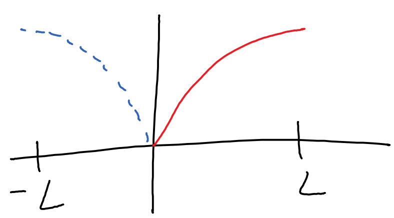
Fourier Series on $(0, L)$

$f(x)$ on $(0, L)$



If we only know the values on $(0, L)$, then we cannot say if $f(x)$ is even or odd.

Even extension



Pretend $f(x)$ is even.

$\Rightarrow b_n = 0$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

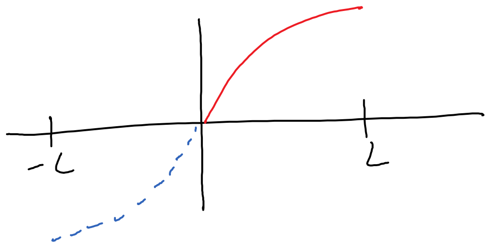
$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \frac{2}{L} \int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Odd extension

Pretend $f(x)$ is odd,

$\Rightarrow a_n = 0$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Half-Range Series on $(0, L)$

① Fourier Cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

② Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Ex</u> Find Fourier Sine series of $f(x)=x$ on $(0,\pi)$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \leftarrow L=\pi$$

$$= \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx$$

Integration by Parts

$u = x \qquad v = -\frac{1}{n}\cos(nx)$

$du = dx \qquad dv = \sin(nx)dx$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) - \int_0^\pi -\frac{1}{n}\cos(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^\pi$$

$$= \frac{2}{\pi}\Big[-\frac{\pi}{n}\underbrace{\cos(n\pi)}_{(-1)^n} + \frac{1}{n^2}\underbrace{\sin(n\pi)}_{0}$$

$$+\underbrace{\frac{0}{n}\cos(0)}_{0} - \frac{1}{n^2}\underbrace{\sin(0)}_{0}\Big]$$

$$= -\frac{2}{n}(-1)^n$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin(nx)}$$

Ex Find Fourier Cosine Series of $f(x) = x$ on $(0, \pi)$,

Left as an exercise (HW 11 #3)

12.4 Complex Fourier Series

Euler's Formula: $e^{ix} = \cos x + i \sin x$

Useful Formulas

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Fourier Series on $(-L, L)$

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} + b_n \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right]$$

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[e^{in\pi x/L} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) + e^{-in\pi x/L} \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) \right]$$

$\hookrightarrow \frac{1}{i} = -i$

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[e^{in\pi x/L}\left(\frac{a_n - i b_n}{2}\right) + e^{-in\pi x/L}\left(\frac{a_n + i b_n}{2}\right)\right]$$

Let $c_0 = \frac{1}{2}a_0$, $\quad c_n = \frac{1}{2}(a_n - i b_n)$.

$$c_0 + \sum_{n=1}^{\infty}\left[e^{in\pi x/L}\, c_n + e^{-in\pi x/L}\, c_n^*\right]$$

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

<u>Def</u> The complex Fourier series of $f(x)$ on $(-L, L)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{in\pi x/L}\, dx$$

This shows that it is possible to view a Fourier Series as a collection of complex exponentials, rather than real-valued sine and cosine waves.
The series are equivalent. The sine/cosine version involves more writing, but less headaches tracking the complex numbers.
In this course, we focus exclusively on the real-valued sine/cosine version. You will see both the real and complex-valued versions of the Fourier Series in a signal processing course.