

11

Discrete-Time Fourier Transform

The discrete-time Fourier transform has essentially the same properties as the continuous-time Fourier transform, and these properties play parallel roles in continuous time and discrete time. As with the continuous-time Fourier transform, the discrete-time Fourier transform is a complex-valued function whether or not the sequence is real-valued. Furthermore, as we stressed in Lecture 10, the discrete-time Fourier transform is always a periodic function of Ω . If $x(n)$ is real, then the Fourier transform is conjugate symmetric, which implies that the real part and the magnitude are both even functions and the imaginary part and phase are both odd functions. Thus for real-valued signals the Fourier transform need only be specified for positive frequencies because of the conjugate symmetry. Whether or not a sequence is real, specification of the Fourier transform over a frequency range of 2π specifies it entirely. For a real-valued sequence, specification over the frequency range from, for example, 0 to π is sufficient because of conjugate symmetry.

The time-shifting property together with the linearity property plays a key role in using the Fourier transform to determine the response of systems characterized by linear constant-coefficient difference equations. As with continuous time, the convolution property and the modulation property are of particular significance. As a consequence of the convolution property, which states that the Fourier transform of the convolution of two sequences is the product of their Fourier transforms, a linear, time-invariant system is represented in the frequency domain by its frequency response. This representation corresponds to the scale factors applied at each frequency to the Fourier transform of the input. Once again, the convolution property can be thought of as a direct consequence of the fact that the Fourier transform decomposes a signal into a linear combination of complex exponentials each of which is an eigenfunction of a linear, time-invariant system. The frequency response then corresponds to the eigenvalues. The concept of filtering for discrete-time signals is a direct consequence of the convolution property.

The modulation property in discrete time is also very similar to that in continuous time, the principal analytical difference being that in discrete time the Fourier transform of a product of sequences is the periodic convolution

rather than the aperiodic convolution of the individual Fourier transforms. The modulation property for discrete-time signals and systems is also very useful in the context of communications. While many communications systems have historically been continuous-time systems, an increasing number of communications systems rely on discrete-time modulation techniques. Often in digital transmission systems, for example, it is necessary to convert from one type of modulation system to another, a process referred to as transmodulation, and efficient implementation relies on the modulation property for discrete-time signals. As we discuss in this lecture, another important application of the modulation property is the use of modulation to effect a high-pass filter with a lowpass filter or vice versa.

This lecture concludes our discussion of the basic mathematics of Fourier series and Fourier transforms; we turn our attention in the next several lectures to the concepts of filtering, modulation, and sampling. We conclude this lecture with a summary of the basic Fourier representations that we have developed in the past five lectures, including identifying the various dualities. The continuous-time Fourier series is the representation of a periodic continuous function by an aperiodic discrete sequence, specifically the sequence of Fourier series coefficients. Thus, for continuous-time periodic signals there is an inherent asymmetry and lack of duality between the two domains. In contrast, the continuous-time Fourier transform has a strong duality between the time and frequency domains and in fact the Fourier transform of the Fourier transform gets us back to the original signal, time-reversed. In discrete time the situation is the opposite. The Fourier series represents a periodic time-domain sequence by a periodic sequence of Fourier series coefficients. On the other hand, the discrete-time Fourier transform is a representation of a discrete-time aperiodic sequence by a continuous periodic function, its Fourier transform. Also, as we discuss, a strong duality exists between the continuous-time Fourier series and the discrete-time Fourier transform.

Suggested Reading

Section 5.5, Properties of the Discrete-Time Fourier Transform, pages 321–327

Section 5.6, The Convolution Property, pages 327–333

Section 5.7, The Modulation Property, pages 333–335

Section 5.8, Tables of Fourier Properties and of Basic Fourier Transform and Fourier Series Pairs, pages 335–336

Section 5.9, Duality, pages 336–343

Section 5.10, The Polar Representation of Discrete-Time Fourier Transforms, pages 343–345

Section 5.11.1, Calculations of Frequency and Impulse Responses for LTI Systems Characterized by Difference Equations, pages 345–347

DISCRETE-TIME FOURIER TRANSFORM

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

synthesis

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

analysis

$$x[n] \xleftrightarrow{\mathcal{F}} X(\Omega)$$

$$\begin{aligned} X(\Omega) &= \operatorname{Re} \{X(\Omega)\} + j \operatorname{Im} \{X(\Omega)\} \\ &= |X(\Omega)| e^{j\angle X(\Omega)} \end{aligned}$$

TRANSPARENCY

11.1

The discrete-time Fourier transform. [As corrected here, $x[n]$, not $x(t)$, has Fourier transform $X(\Omega)$.]

PROPERTIES OF THE FOURIER TRANSFORM

$$x[n] \xleftrightarrow{\mathcal{F}} X(\Omega)$$

Periodic:

$$X(\Omega) = X(\Omega + 2\pi m)$$

Symmetry:

$$x[n] \text{ real} \Rightarrow X(-\Omega) = X^*(\Omega)$$

$$\left. \begin{array}{l} \operatorname{Re} \{X(\Omega)\} \\ |X(\Omega)| \end{array} \right\} \text{ even}$$

$$\left. \begin{array}{l} \operatorname{Im} \{X(\Omega)\} \\ \angle X(\Omega) \end{array} \right\} \text{ odd}$$

TRANSPARENCY

11.2

Periodicity and symmetry properties of the discrete-time Fourier transform.

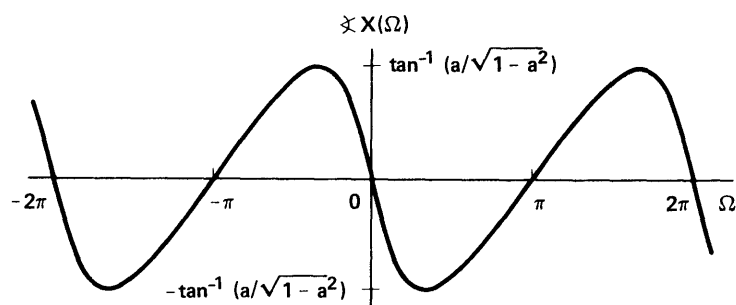
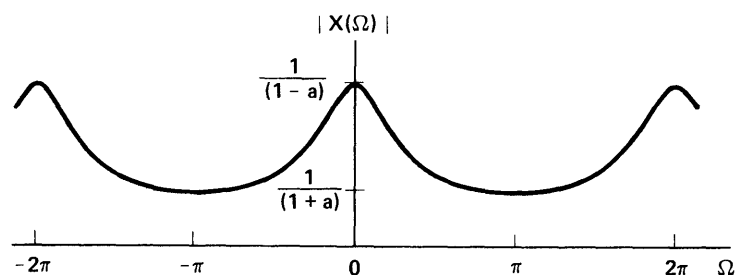
TRANSPARENCY

11.3

Example illustrating the periodicity and symmetry properties.

$$x[n] = a^n u[n] \quad 0 < a < 1$$

$$X(\Omega) = \frac{1}{1 - a e^{-j\Omega}}$$



Time shifting:

$$x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\Omega n_0} X(\Omega)$$

Frequency shifting:

$$e^{j\Omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(\Omega - \Omega_0)$$

Linearity:

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{F}} aX_1(\Omega) + bX_2(\Omega)$$

Parseval's relation:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega$$

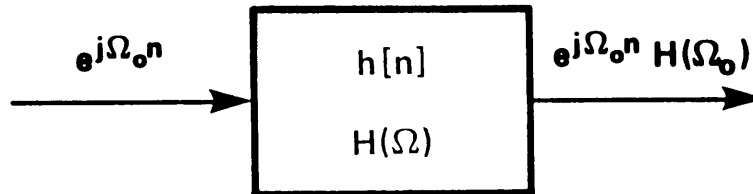
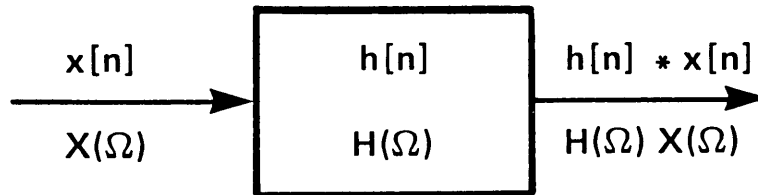
TRANSPARENCY

11.4

Additional properties of the discrete-time Fourier transform.

CONVOLUTION PROPERTY

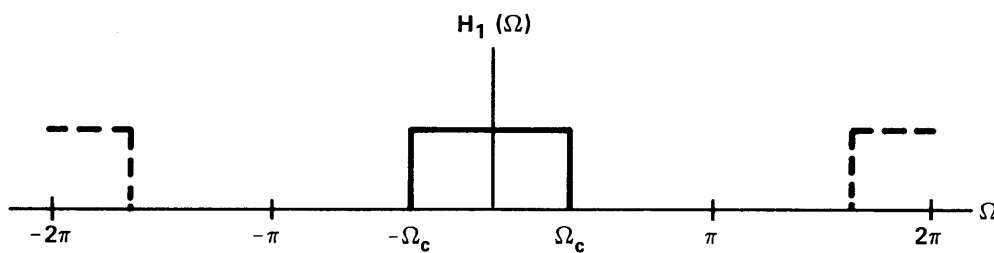
$$h[n] * x[n] \xleftrightarrow{\mathcal{F}} H(\Omega) X(\Omega)$$



TRANSPARENCY

11.5

The convolution property.



TRANSPARENCY

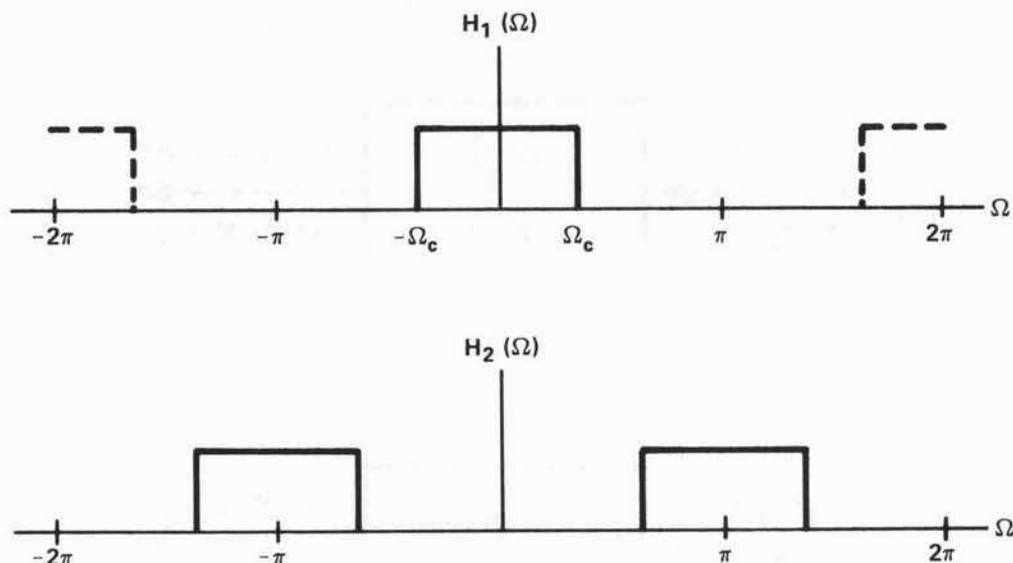
11.6

Frequency response of a discrete-time ideal lowpass filter.

TRANSPARENCY

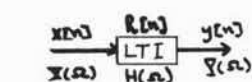
11.7

Frequency response of an ideal lowpass filter and an ideal highpass filter.



MARKERBOARD

11.1 (a)



Causal LTI \iff initial Rest

$$y[n] - ay[n-1] = x[n]$$

Shifting
linearity

$$Y(\Omega) - ae^{-j\Omega} Y(\Omega) = X(\Omega)$$

$$Y(\Omega) = \frac{1}{1 - ae^{-j\Omega}} X(\Omega)$$

$$h[n] \xrightarrow{\mathcal{F}} H(\Omega)$$

$$a^n u[n] \xrightarrow{\mathcal{F}} \frac{1}{1 - ae^{-j\Omega}}$$

MODULATION Property

Example

$$x_1[n] = (-1)^n = e^{j\pi n}$$

periodic



$$x_1[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{\pi}{N})n}$$

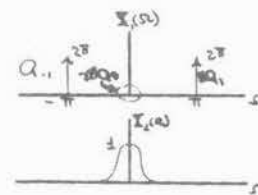
$$N=2$$

$$x_1[n] = a_0 + a_1 e^{j(\frac{\pi}{2})n}$$

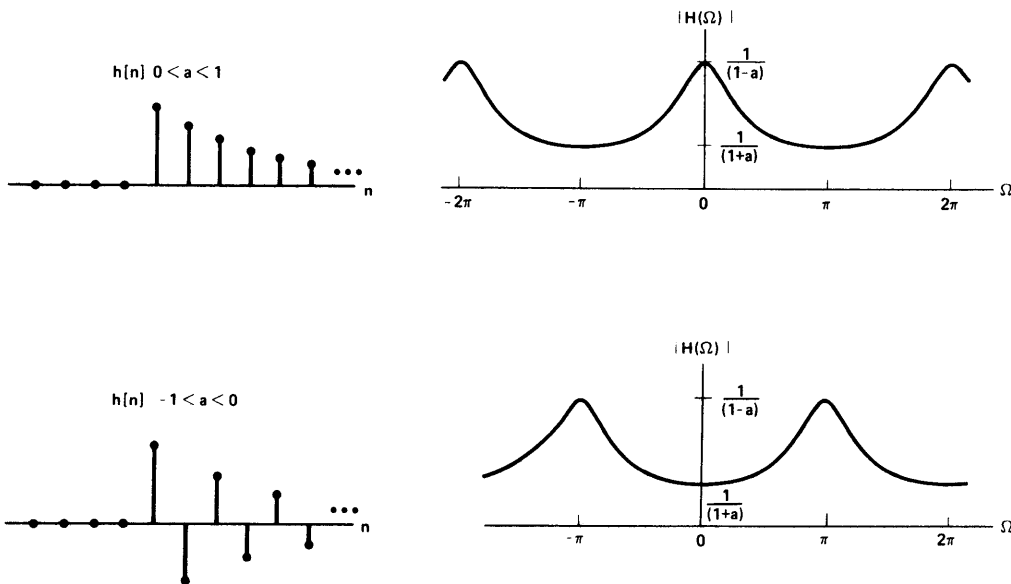
$$a_0 = 0$$

$$a_1 = 1$$

$$X(\Omega) = \sum_k 2\pi a_k \delta(\Omega - k\Omega_0)$$



$$\mathcal{F}\{(-1)^n x_2[n]\}$$

**TRANSPARENCY 11.8**

Impulse response and frequency response for a first-order system approximating a lowpass filter and a first-order system approximating a highpass filter.

MODULATION PROPERTY

Discrete-time:

$$x_1[n] x_2[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{X_1(\theta) X_2(\Omega - \theta) d\theta}_{\text{periodic convolution}}$$

Continuous-time:

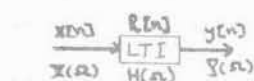
$$x_1(t) x_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{X_1(\rho) X_2(\omega - \rho) d\rho}_{\text{aperiodic convolution}}$$

TRANSPARENCY 11.9

A comparison of the Fourier transform modulation property for continuous time and for discrete time.

MARKERBOARD

11.1 (b)



Causal LTI \iff initial Rest

$$y[n] - ay[n-1] = x[n]$$

\downarrow \downarrow \downarrow
 linear shift linear

$$Y(\omega) - ae^{-j\omega} Y(\omega) = X(\omega)$$

$$Y(\omega) = \frac{1}{1 - ae^{-j\omega}} X(\omega)$$

$$R(\omega) \xleftrightarrow{\mathcal{F}} H(\omega)$$

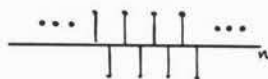
$$a^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - ae^{-j\omega}}$$

MODULATION Property-

Example

$$x_1[n] = (-1)^n = e^{j\pi n}$$

periodic



$$x_1[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{\pi}{2})n}$$

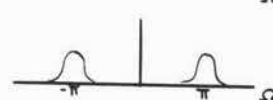
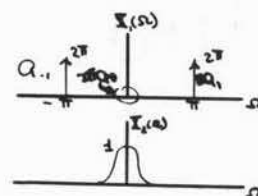
$$N=2$$

$$x_1[n] = a_0 + a_1 e^{j(\frac{\pi}{2})n}$$

$$a_0 = 0$$

$$a_1 = 1$$

$$X(\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0)$$

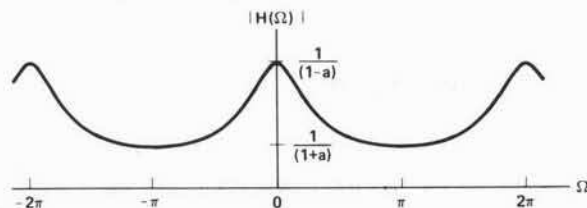
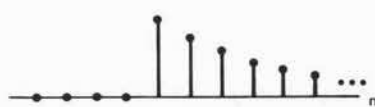


$$\mathcal{F}\{(-1)^n x_2[n]\}$$

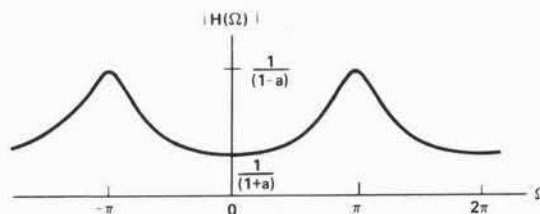
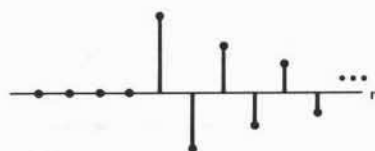
TRANSPARENCY 11.10

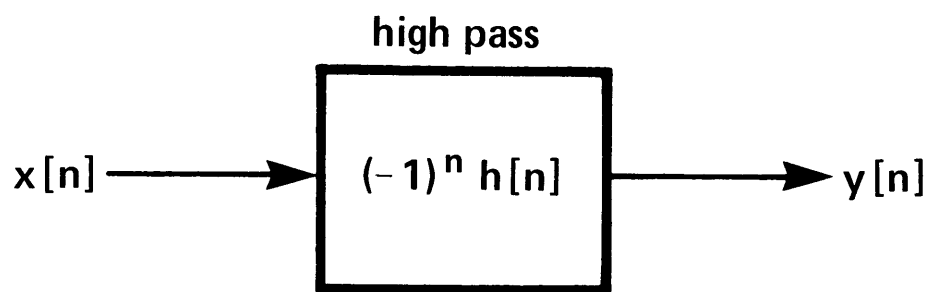
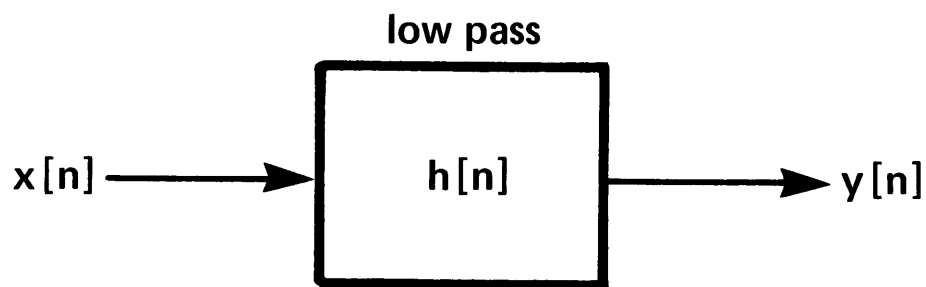
Impulse response and frequency response for a first-order system approximating a lowpass filter and a first-order system approximating a highpass filter.
[Transparency 11.8 repeated]

$$h[n] \quad 0 < a < 1$$

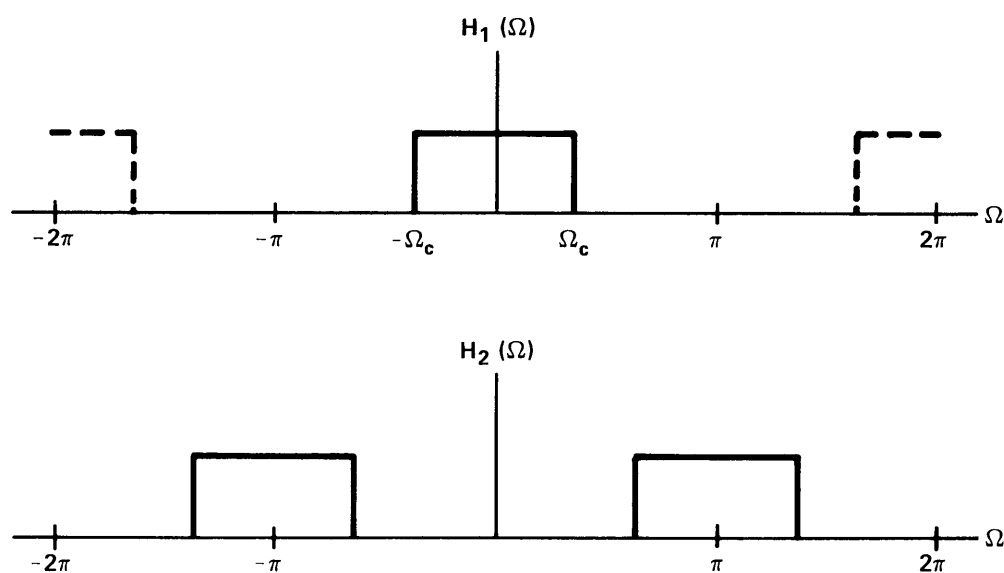


$$h[n] \quad -1 < a < 0$$



**TRANSPARENCY****11.11**

Use of modulation of a system impulse response to convert the system from a lowpass to a highpass filter.

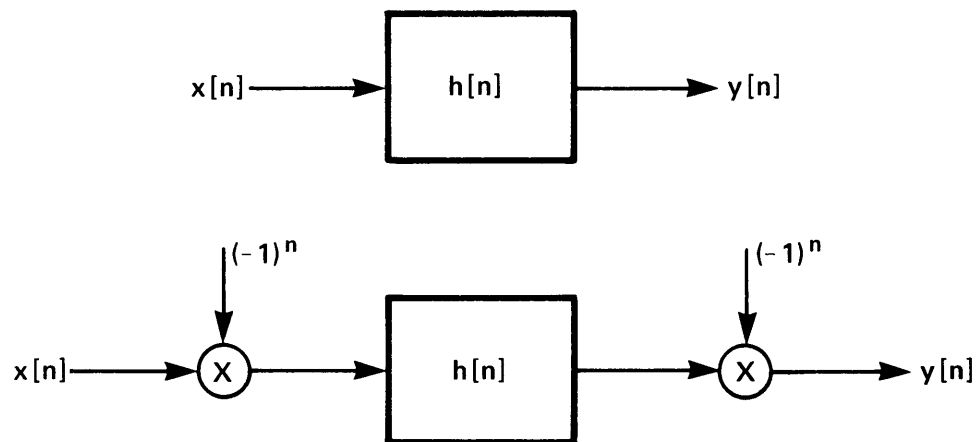
**TRANSPARENCY****11.12**

Effect on the frequency response of modulating the impulse response by an alternating sign change.

TRANSPARENCY

11.13

The use of modulation to implement highpass filtering with a lowpass filter.



MARKERBOARD

11.2

Series (C-T)

$$X(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_s t}$$

(No)

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_s t} dt$$

(Yes)

TRANSFORM (C-T)

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(\omega) e^{j\omega t} d\omega$$

(Yes)

$$\tilde{X}(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Series (D-T)

$$X[n] = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\frac{2\pi}{N})n}$$

(Yes)

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{+\infty} X[n] e^{-jk(\frac{2\pi}{N})n}$$

TRANSFORM (D-T)

$$X[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \tilde{X}(\Omega) e^{j\Omega n} d\Omega$$

(No)

$$\tilde{X}(\Omega) = \sum_{n=-\infty}^{+\infty} X[n] e^{-j\Omega n}$$

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