Lecture 13: Differential Equations Review

Haunter's Goals for the Day

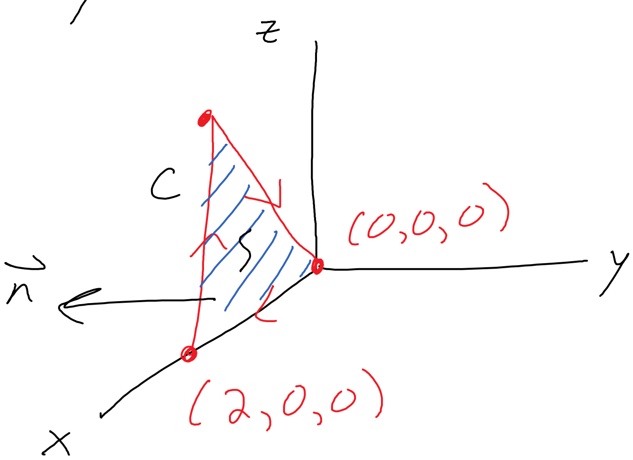
- Calculate flux around a closed path using Stokes' Theorem
- Introduce terminology for Differential Equations (DEs)
- Review solving 1st order DEs using:
  - Separation of Variables
  - Integrating Factor Method

9.13 Stokes' Theorem

Ex Haunter flies in a triangular path from (0,0,0) to (2,0,0) to (0,-2,2) and back to the origin. A windstorm exerts a velocity field of

$$\vec{F} = \langle y, z, x \rangle.$$

Calculate the work done on Haunter by the storm.



$$\text{Work} = \oint_C \vec{F}\cdot\vec{T}\,ds$$

To compute directly, we would add up 3 line integrals, one for each leg of the parametrization.

Stokes' Theorem: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$

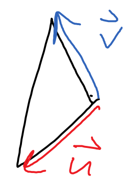
Find equation of the plane containing 3 points.

Hint: $Ax+By+Cz=D$ where $\langle A,B,C\rangle$ are normal to plane.

Form 2 vectors in plane.

$$\vec{u}=(2,0,0)-(0,0,0)=\langle 2,0,0\rangle$$

$$\vec{v}=(0,-2,2)-(0,0,0)=\langle 0,-2,2\rangle$$



The cross product $\vec{u}\times\vec{v}$ is $\perp$ to the plane.

$$\vec{u}\times\vec{v}=\begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 0 & -2 & 2 \end{vmatrix}=\langle 0,-4,-4\rangle$$

$$0x-4y-4z=D \Rightarrow -4y-4z=D$$

To find value of D, plug in any point in plane.

$$(0,0,0) \Rightarrow -4(0)-4(0)=0=D$$

Plane: $-4y-4z=0$

$$y+z=0$$

Compute $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$

① Normal vector $\vec{n}$

$$\underbrace{y + z}_{G(x,y,z)} = 0 \qquad \nabla G = \langle 0, 1, 1 \rangle$$

Choose $\langle 0, -1, -1 \rangle$

Unit vector: $\vec{n} = \dfrac{\langle 0, -1, -1 \rangle}{\sqrt{2}}$

② Integrand $(\nabla \times \vec{F}) \cdot \vec{n}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y} x - \frac{\partial}{\partial z} z, -\left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial z} y\right), \frac{\partial}{\partial x} z - \frac{\partial}{\partial y} y \right\rangle$$

$$= \langle -1, -1, -1 \rangle$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = \langle -1, -1, -1 \rangle \cdot \frac{\langle 0, -1, -1 \rangle}{\sqrt{2}}$$

$$= 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

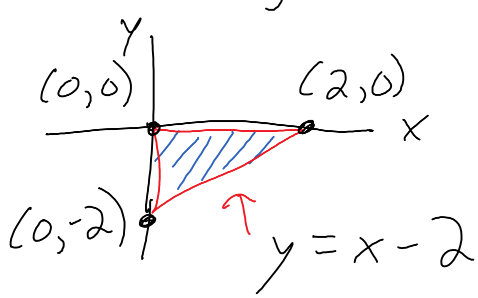
③ Jacobian

Surface $y+z=0 \Rightarrow z=\underbrace{-y}_{f(x,y)}$

$$\sqrt{1+f_x^2+f_y^2}=\sqrt{1+0^2+(-1)^2}=\sqrt{2}$$

④ Shadow Region

Look at just (x,y) components of our 3 points.



Shadow is a triangle

$$x-2 \le y \le 0$$
$$0 \le x \le 2$$

⑤ Integrate

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \iint_R (\nabla \times \vec{F}) \cdot \vec{n} \sqrt{1+f_x^2+f_y^2}\, dA$$

$$= \int_0^2 \int_{x-2}^0 (\sqrt{2})(\sqrt{2})\, dy\, dx$$

$$= 2 \int_0^2 \int_{x-2}^0 dy\, dx$$

$$= 2 \int_0^2 y \Big|_{x-2}^0 dx$$

$$= 2\int_0^2 -x + 2 \, dx$$

$$= 2\left[-\frac{1}{2}x^2 + 2x\right]_0^2$$

$$= 2\left[-\frac{1}{2}(4) + 2(2)\right]$$

$$= 2(2)$$

$$= \boxed{4}$$

Positive work means force is in same direction as Haunter's motion.

We use Stokes' Theorem to calculate work/flux line integrals along complicated paths.

In this example, we could have computed the total work by summing 3 line integrals. It should give the same final answer of 4.

Instead of computing 3 line integrals, we chose to use Stokes' Theorem to compute one surface integral. Surface integrals are always nasty to compute, so it was a marginal improvement.

Stokes' Theorem problems are quite frankly too long to include on an in-class midterm exam.

But you have 3 hours for the final exam, so I could put a Stokes' Theorem problem on the final.
Mwaa-ha-ha!

Chapter 5: Power Series Method

Def A *differential equation* (DE) is an equation which contains a derivative.
The *order* of a DE is the number of the highest derivative in the equation.

$$y''' + 5x^2 y' - x^4 = 2 \qquad \text{Order} = 3$$

Notation for Derivatives

① Newton	$f'(x), f''(x), \ldots, f^{(n)}(x)$	$y'' + 2y' = x$
② Leibnitz	$\frac{df}{dx}, \frac{d^2f}{dx^2}, \ldots, \frac{d^nf}{dx^n}$	$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x$
③ Subscript	$f_x, f_{xx}, \ldots, f_{xxxxxx}$	$y_{xx} + 2y_x = x$

Def An ordinary DE (ODE) has all derivatives with respect to one independent variable.
A partial DE (PDE) has derivatives in more than one independent variables.

Ex Heat Equation $u(x,t)$

$$u_t = K u_{xx}$$

$\hookrightarrow$ conductivity

2nd-order PDE

Def Algebraic terms occur linearly if all terms are separate and to the first power.

x & y both linear: $x+y$, $2x-3y$

not linear in both x & y: x^2+y^3, $e^x+\cos y$, xy

Linear in y: x^2+y, x^3y

Def A DE is linear if the dependent variable and all its derivatives occur linearly.

Ex Classify each DE.

a.) $y'' + y'\cos x = 3$

2nd-order ODE Linear

b.) $f_{xx} + f_x f_y = x + y$

2nd-order PDE Nonlinear (because of $f_x f_y$)

c.) $\dfrac{\partial^2 f}{\partial y^2} = \tan x + xy^2$

2nd-order ODE Linear

Solving 1st-Order ODEs

1. Separation of Variables
2. Integrating Factor Method

Separation of Variables

We say a 1st-order ODE is separable if we can write it as

$$\frac{dy}{dx} = P(x)\,Q(y)$$

Move x+y terms to separate sides.

$$\frac{1}{Q(y)}\,dy = P(x)\,dx$$

Integrate both sides

$$\int \frac{1}{Q(y)}\,dy = \int P(x)\,dx$$

Ex Solve $y' \sec x = x + xy^2$.

$$\frac{dy}{dx}\sec x = x(1+y^2)$$

$$\int \frac{1}{1+y^2}\,dy = \int x\cos x\,dx$$

$$\arctan(y) = x\sin x + \cos x + C$$

Implicit Solution

$$\boxed{y = \tan(x\sin x + \cos x + C)}$$

Explicit Solution

<u>Ex</u> Solve $y' = 2y$.

$$\frac{dy}{dx} = 2y$$

$$\int \frac{1}{y}\,dy = \int 2\,dx$$

$$\ln y = 2x + C$$

$$y = e^{2x+C} = e^{2x}e^{C} = \boxed{Ke^{2x}}$$

Integrating Factor Method

To solve a linear 1st-order DE

$$y' + P(x)\,y = Q(x)$$

Multiply both sides by integrating factor $e^{\int P(x)dx}$.

$$e^{\int P(x)dx}\left[y' + P(x)y\right] = e^{\int P(x)dx}Q(x)$$

The left side becomes the derivative of the integrating factor times y.

$$\frac{d}{dx}\left[e^{\int P(x)dx}\,y\right] = e^{\int P(x)dx}Q(x)$$

Integrate both sides.

Ex Solve $x^2 y' + 3xy = x^5 - 2$.

Rewrite so coefficient of y' is one.

$$y' + \underbrace{\frac{3}{x}}_{P(x)} y = x^3 - \frac{2}{x^2}$$

I.F. $e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$$x^3\left[y' + \frac{3}{x}y\right] = x^3\left[x^3 - \frac{2}{x^2}\right]$$

$$\int \frac{d}{dx}\left[x^3 y\right] = \int x^6 - 2x \quad dx$$

$$x^3 y = \frac{1}{7}x^7 - x^2 + C$$

$$\boxed{y = \frac{1}{7}x^4 - \frac{1}{x} + \frac{C}{x^3}}$$

You should be good at solving simple 1st-order ODEs.

In some problems in this course, we will have to solve several 1st-order DEs just as part of one problem.