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Spring 2015

ELEC 318 – *Electromagnetic Fields*

Lecture 3(c)

**Review of Vector Calculus:
Differential Length/Area/Volume, Integrals**

THE CITADEL, THE MILITARY COLLEGE OF SOUTH CAROLINA
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Diff. Length, Area, Volume: **Cartesian**

differential length, area, volume:

- useful for integration along a path, open/closed surface
- allows us to answer questions like

“What is the strength of \mathbf{E} along this path?”

“What is the density of \mathbf{H} across this surface?”

“How much of \mathbf{q} is contained within this shape?”

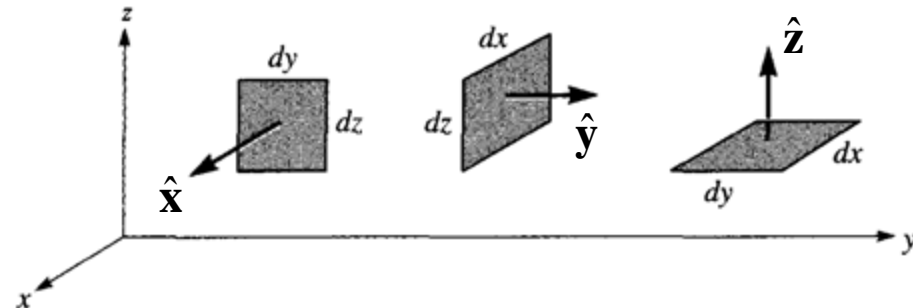
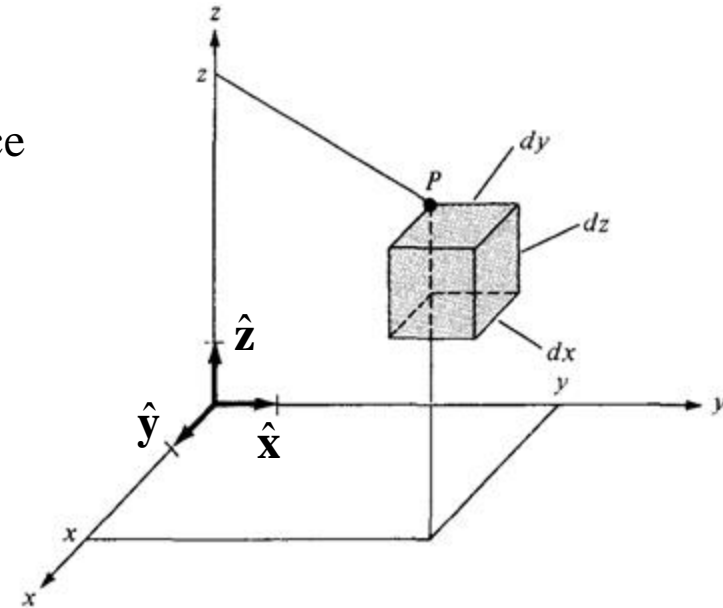
differential length (displacement, distance):

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$

differential area:

$$\begin{aligned} d\mathbf{S} &= dS \hat{\mathbf{n}} \\ &= dx dy \hat{\mathbf{z}} \quad \text{or} \quad dy dz \hat{\mathbf{x}} \quad \text{or} \quad dz dx \hat{\mathbf{y}} \end{aligned}$$

differential volume: $dv = dx dy dz$



Diff. Length, Area, Volume: Cylindrical

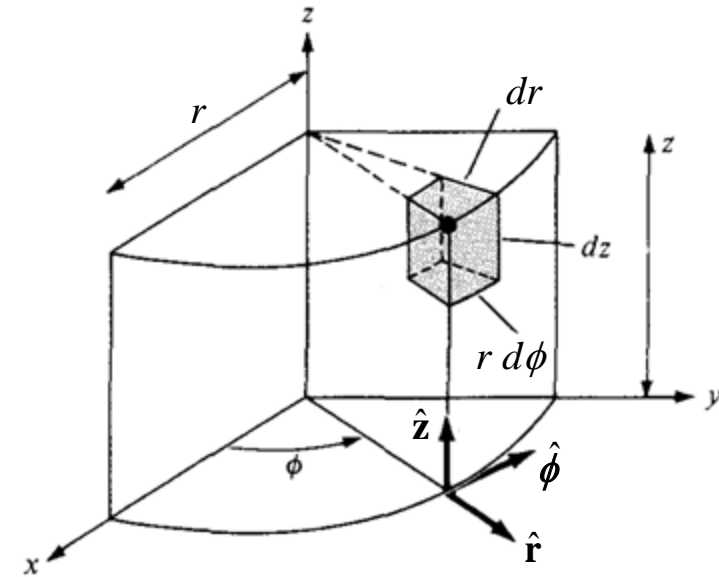
differential length (displacement, distance):

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$$

differential area:

$$\begin{aligned} d\mathbf{S} &= dS \hat{\mathbf{n}} \\ &= r d\phi dz \hat{\mathbf{r}} \quad \text{or} \quad dr dz \hat{\boldsymbol{\phi}} \quad \text{or} \quad r dr d\phi \hat{\mathbf{z}} \end{aligned}$$

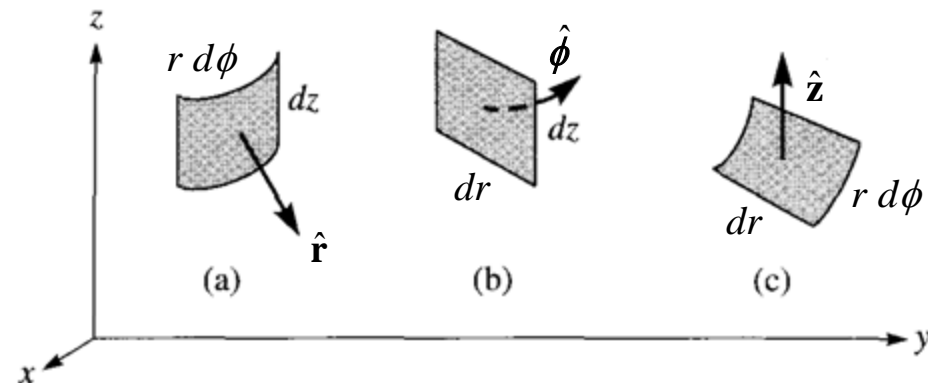
differential volume: $dv = r dr d\phi dz$



Example:

If the charge density in the wedge (above, right), is $\rho_v = 2r - 3z$ (C/cm³), then the amount of charge contained in the tiny curved cube is

$$(2r^2 - 3rz) dr d\phi dz \quad (\text{coulombs})$$



Diff. Length, Area, Volume: Spherical

differential length (displacement, distance):

$$d\mathbf{l} = dR \hat{\mathbf{R}} + R d\theta \hat{\boldsymbol{\theta}} + R \sin \theta d\phi \hat{\boldsymbol{\phi}}$$

differential area: $d\mathbf{S} = dS \hat{\mathbf{n}}$

$$= R^2 \sin \theta d\theta d\phi \hat{\mathbf{R}} \quad \text{or}$$

$$= R \sin \theta dR d\phi \hat{\boldsymbol{\theta}} \quad \text{or}$$

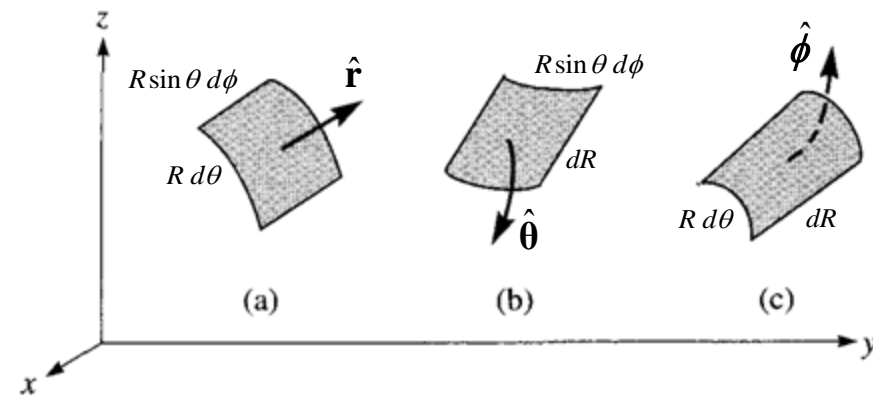
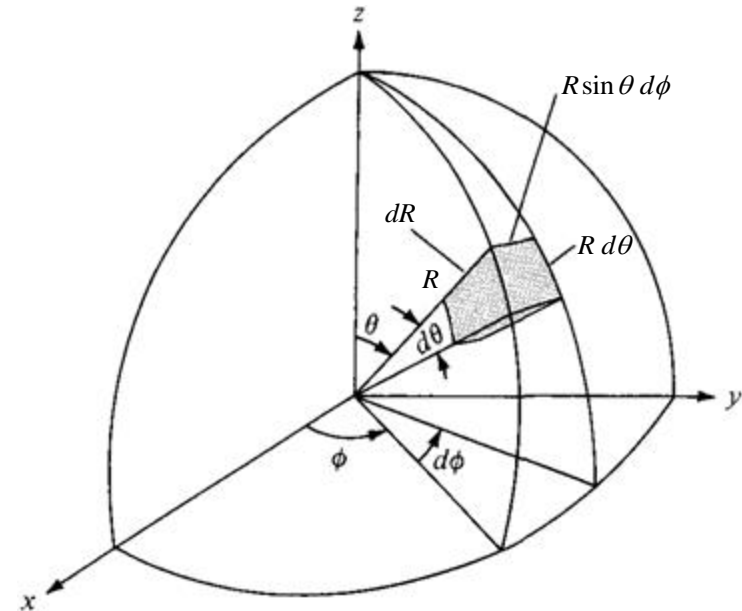
$$= R dR d\theta \hat{\boldsymbol{\phi}}$$

differential volume: $dv = R^2 \sin \theta dR d\theta d\phi$

Example:

If the magnetic flux density crossing the sphere (above, right), is $\mathbf{B} = 4 \sin \phi \hat{\mathbf{R}}$ (Wb/m²), then the amount of flux through the tiny curved square (on the outside of the sphere) is

$$4 R^2 \sin \phi \sin \theta d\theta d\phi \quad (\text{Wb})$$



Line Integrals

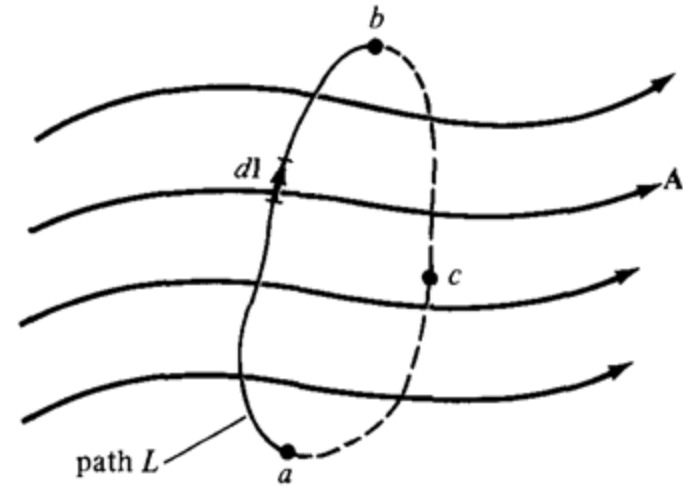
line (single) integral:

- integration of the tangential (parallel) component of a vector (\mathbf{A}) along a path (L):

$$\int_L \mathbf{A} \cdot d\mathbf{l}$$

- allows us to answer the question
“How much of \mathbf{A} is projected along a path?”

If the path (L) is closed (forms a *surface*),
then the integral becomes $\oint_L \mathbf{A} \cdot d\mathbf{l}$



Examples:

$$V_{21} = -\int_{P1}^{P2} \mathbf{E} \cdot d\mathbf{l}$$

$$I_{\text{enc}} = \oint \mathbf{H} \cdot d\mathbf{l}$$

As before, for rectangular coordinates,
for cylindrical coordinates,
and for spherical coordinates,

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$$

$$d\mathbf{l} = dR \hat{\mathbf{R}} + R d\theta \hat{\boldsymbol{\theta}} + R \sin \theta d\phi \hat{\boldsymbol{\phi}}$$

Example: Mixed Coordinates

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1} \frac{y}{x}, \quad z = z$$

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z$$

$$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$$

$$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\phi \hat{\phi} + dz \hat{\mathbf{z}}$$

$$\hat{\mathbf{x}} = \cos \phi \hat{\mathbf{r}} - \sin \phi \hat{\phi} \quad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

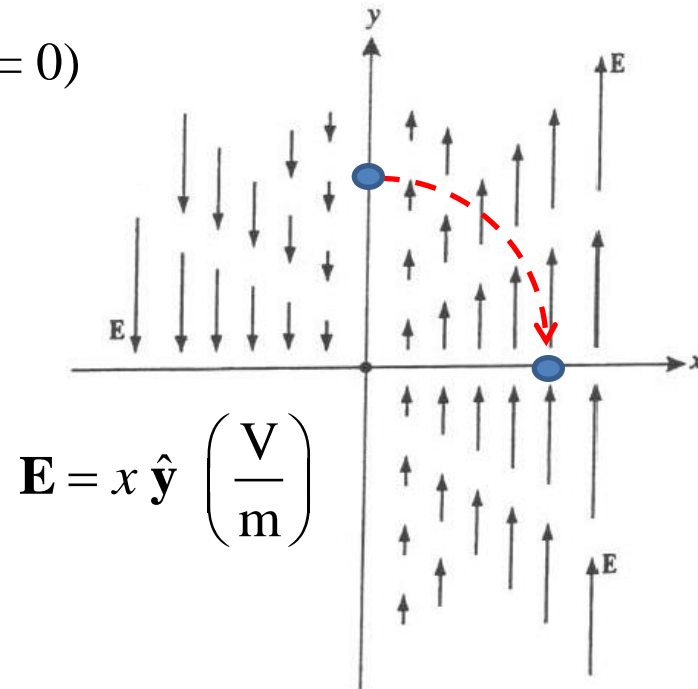
$$\hat{\mathbf{y}} = \sin \phi \hat{\mathbf{r}} + \cos \phi \hat{\phi}$$

$$\hat{\mathbf{r}} = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}} \quad \hat{\mathbf{z}} = \hat{\mathbf{z}}$$

$$\hat{\phi} = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}$$

For the path from P_1 ($x = 0$, $y = 2$ m) to P_2 ($x = 2$ m, $y = 0$) and \mathbf{E} illustrated, compute

$$-\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$



Surface Integrals

surface (double) integral:

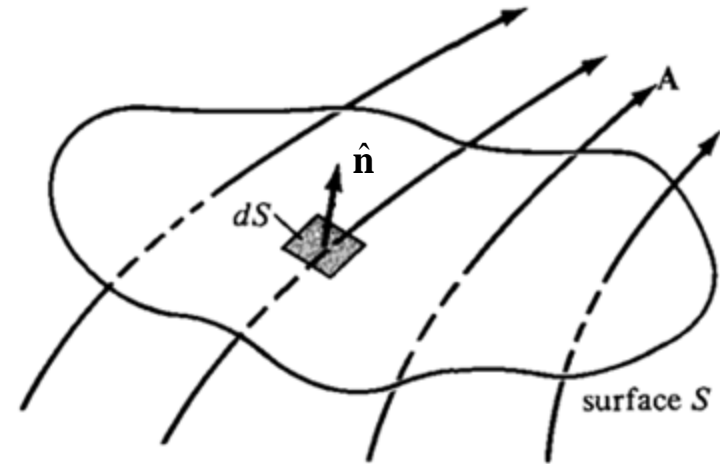
- integration of the normal component of a vector (\mathbf{A}) across a surface (S):

$$\int_S \mathbf{A} \cdot d\mathbf{S}$$

(“flux”)

- allows us to answer the question

“How much of \mathbf{A} crosses a given surface?”



If the surface (S) is closed (forms a *volume*), then the integral becomes

$$\oint_S \mathbf{A} \cdot d\mathbf{S} \quad (\text{“net outward flux”})$$

Examples:

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$Q_{\text{enc}} = \oint \mathbf{D} \cdot d\mathbf{S}$$

As before, for rectangular coordinates, $d\mathbf{S} = dx dy \hat{\mathbf{z}}$ or $dy dz \hat{\mathbf{x}}$ or $dz dx \hat{\mathbf{y}}$

for cylindrical coordinates, $d\mathbf{S} = r d\phi dz \hat{\mathbf{r}}$ or $d\rho dz \hat{\phi}$ or $r dr d\phi \hat{\mathbf{z}}$

for spherical coordinates, $d\mathbf{S} = R^2 \sin \theta d\theta d\phi \hat{\mathbf{R}}$ or $R \sin \theta dR d\phi \hat{\theta}$ or $R dR d\theta \hat{\phi}$

Volume Integrals

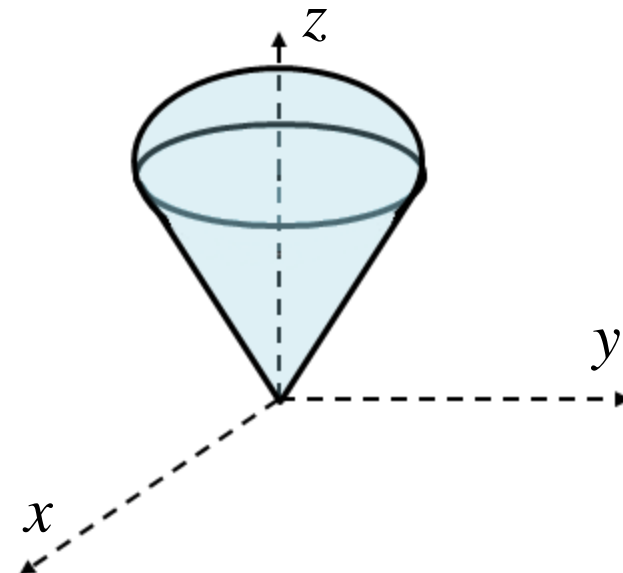
volume (triple) integral:

- integration of a scalar *density* throughout a volume (V):

$$\int_v A_v \cdot dv$$

- allows us to answer the question

“How much of [a particular quantity] is contained within [a given volume] ?”



Examples:

$$Q = \int_v \rho_v dv$$

$$W_E = \frac{1}{2} \epsilon \int_v |\mathbf{E}|^2 dv$$

As before, for rectangular coordinates, $dv = dx dy dz$

for cylindrical coordinates, $dv = r dr d\phi dz$

for spherical coordinates, $dv = R^2 \sin \theta dR d\theta d\phi$