Math 335
Chapter 13 Summary: Boundary Value Problems

Some PDE Boundary Value Problems (BVPs) can be solved by using methods we learned for ODEs.

If the PDE is linear with independent variables x and y, we can assume the solution u(x,y) is separable:

$$u(x, y) = v(x)w(y) .$$

Separating the variables and setting each side equal to some constant -λ should give two ODE problems. We then find the choice of λ that leads to non-trivial solutions.

We call the sequence of numbers λ_n the *eigenvalues* and the corresponding solutions $u_n = v_n w_n$ the *eigenfunctions*.

The general solution of the PDE is then:

$$u(x,y) = \sum_n C_n v_n(x) w_n(y)$$

Ensuring that this solution satisfies the given boundary conditions often involves a Fourier series.

The Heat Equation *(Sec 13.3)*

Suppose we have a metal rod of length L with thermal diffusivity constant k.
The temperature u(x,t) of a rod at location x and time t follows the PDE:

$$u_t = k\, u_{xx}$$

If we assume the endpoints x=0 and x=L have temperature zero and the initial temperature profiles is f(x), then we obtain the boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x)$$

Fourier's Solution of the Heat Equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$$

The Wave Equation *(Sec 13.4)*

Suppose we have a vibrating string of length L and tension constant a.
The vertical displacement u(x,t) of the string at location x and time t follows the PDE:

$$u_{tt} = a^2 u_{xx}$$

If we assume the endpoints x=0 and x=L are clamped at the x-axis and the string has initial shape f(x) and initial velocity g(x), then we obtain the boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0, \qquad u(x,0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

Bernoulli's Solution of the Wave Equation is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi a t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Laplace's Equation *(Sec 13.5)*

A function u(x,y) is said to be *harmonic* if it satisfies Laplace's Equation

$$u_{xx} + u_{yy} = 0$$

The *Dirichlet Problem* looks for a solution u(x,y) to Laplace's Equation on some domain with prescribed boundary conditions.

For example, suppose we have a rectangular domain $0 \le x \le a, \;\; 0 \le y \le b$.
If we assume the values on three sides of the rectangle are fixed at zero and the value along the top of the box is some function f(x), then we obtain the boundary conditions:

$$u(0,y) = 0, \quad u(a,y) = 0, \quad u(x,0) = 0, \quad u(x,b) = f(x)$$

Dirichlet's Solution of Laplace's Equation is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{A_n \sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}, \qquad A_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$