ELEC 312
Systems I

Modeling Electrical and Mechanical Systems
(Derived from Notes by Dr. Robert Barsanti)

(Images from Nise, 7^{th} Edition)

Required Reading: Chapter 2, *Control Systems Engineering*

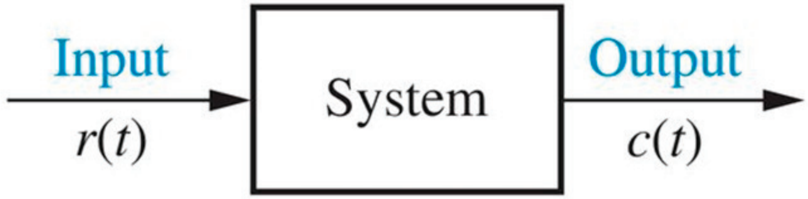
January 21, 2015

Modeling Linear, Time-Invariant (LTI) Systems

Mathematical models for systems are not unique and may assume many different forms. Depending on the particular system and the particular circumstances in which the system is used, one model may be better suited than another.

For example, in optimal control problems, it is advantageous to use state-space representations (see Chapter 3 of *Control Systems Engineering* or ELEC 407 – Systems II).

On the other hand, for the **transient response** or **frequency response** analysis of single-input-single-output (SISO), linear, time-invariant (LTI) systems, the **transfer function** representation may be more convenient.



Transfer Functions

$$\boxed{\text{Transfer Function } = G(s) = \left.\frac{\mathcal{L}\,[\text{Output}]}{\mathcal{L}\,[\text{Input}]}\right|_{\text{IC's}=0} = \frac{C(s)}{R(s)}}$$

For a **linear, time-invariant system (LTI)** described by a LCCDE of form

$$a_n c^{(n)}(t) + a_{n-1}c^{(n-1)}(t) + \ldots + a_0 c(t) = b_m r^{(m)}(t) + b_{m-1}r^{(m-1)}(t) + \ldots + b_0 r(t),$$

where $r(t)$ is a known function, $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ are known real constants, and m and n are the highest derivative orders of $r(t)$ and $c(t)$, respectively.

Taking the Laplace Transform and setting all initial conditions to zero, we have

$$C(s)\left[a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0\right] = R(s)\left[b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0\right].$$

The transfer function is given by

$$\boxed{G(s) = \left.\frac{C(s)}{R(s)}\right|_{\text{IC's}=0} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}}$$

Notes

1. Transfer functions are **only** defined for LTI systems.
2. A transfer function is a **system** property, independent of the magnitude or nature of the input.
3. The transfer function carries any necessary **units** to relate input to the output.
4. The transfer function is a **mathematical model** in that it is an operational method of expressing the differential equation that relates input variables to output variables.
5. If the transfer function of a system is known, the **system response** can be studied for various forms of inputs.

Convolution and the Impulse Response

For an LTI System: $G(s) = \frac{C(s)}{R(s)}\Big|_{\text{IC's}=0}$. Therefore, $C(s) = G(s)R(s)$.

Since multiplication in the complex domain (s-domain) corresponds to convolution in the time domain $\left(G(s)R(s) \overset{\mathcal{L}^{-1}}{\longleftrightarrow} g(t) * r(t)\right)$, then

$$c(t) = \int_0^t r(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)r(t-\tau)d\tau,$$

where the system is causal ($g(t) = r(t) = 0$ for $t < 0$).

Note: The convolution expression above for $y(t)$ ignores the output resulting from any initial conditions that are not actually equal to zero. In other words, this is the **forced response** or **zero-state response** of the system only.

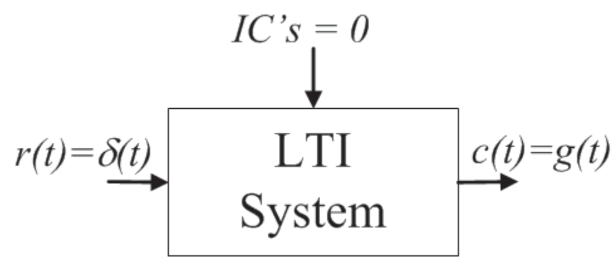
The Impulse Response

Consider the output (response) of a system to a unit impulse ($\delta(t)$) **when the initial conditions are zero**. Since $\mathcal{L}\{\delta(t)\} = 1$, then Laplace transform of the output is

$$C(s) = G(s)R(s) = G(s) \cdot 1 = G(s), \text{ the system transfer function.}$$

Therefore the output $c(t)$ is $\mathcal{L}^{-1}[G(s)]$, or $c(t) = g(t)$. We call this the **system impulse response**.

> The **impulse response** $g(t)$ is the response of an LTI system to a unit impulse input when the initial conditions are zero.

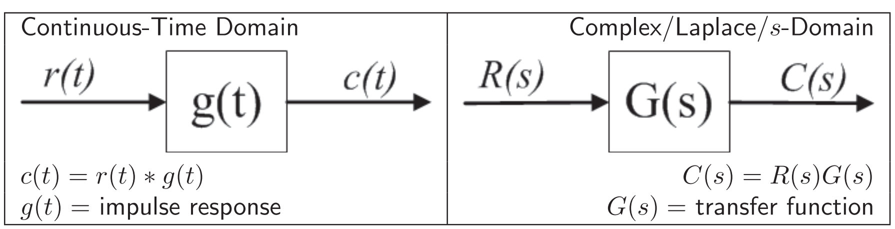


The Impulse Response vs. System Transfer Function

The Laplace transform of the system impulse response gives the transfer function.

Therefore, the transfer function and impulse response of an LTI system **contain the same information**.

Hence, it is possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse and measuring the response.



Three Ways to Find $G(s)$

1. Given $g(t)$, then

$$G(s) = \mathcal{L}\{g(t)\}.$$

2. Given $r(t)$ and $c(t)$, then

$$G(s) = \frac{\mathcal{L}\{c(t)\}}{\mathcal{L}\{r(t)\}}.$$

3. Given the LCCDE

$$a_n c^{(n)}(t) + a_{n-1}c^{(n-1)}(t) + \ldots + a_0 c(t) = b_m r^{(m)}(t) + \ldots + b_0 r(t), \text{ then}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

Example 1:

Given system impulse response $g(t) = 2e^{-2t}$ and input $r(t) = e^{3t}$, determine the system transfer function and the output response $c(t)$.

Example 1 (continued):

Example 2:

Given $c'(t) + 2c(t) = 2r(t)$, determine the system transfer function $G(s)$ and the system impulse response $g(t)$.

Example 2 (continued):

Electrical Network Transfer Function

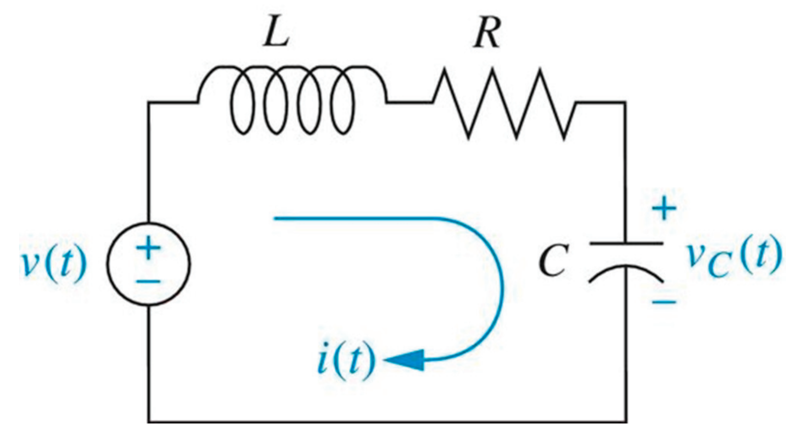
TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Electrical Network Transfer Function

Consider:



Find the transfer function $G(s)$ where $G(s) = \left.\frac{V_C(s)}{V(s)}\right|_{\text{IC's}=0}$.

Electrical Network Transfer Function

1. The KVL differential equation is given by

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t).$$

2. Using $i(t) = \frac{dq(t)}{dt}$ and letting IC's $= 0$, we have

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{q(t)}{C} = v(t).$$

3. Since $q(t) = Cv_C(t)$, we have

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t).$$

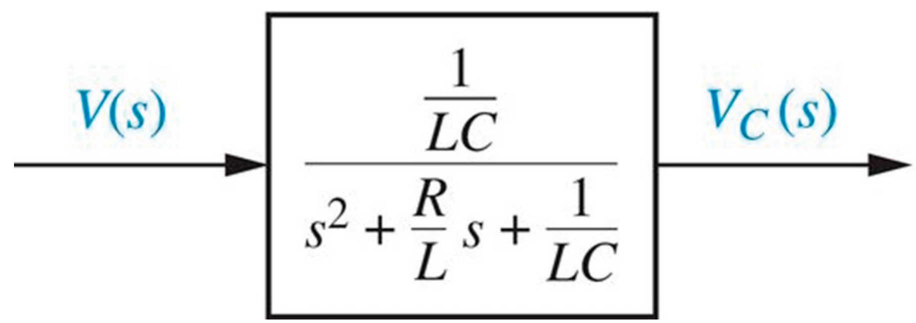
4. Taking Laplace Transforms with IC's $= 0$, we have

$$V_C(s)\left[LCs^2 + RCs + 1\right] = V(s).$$

Electrical Network Transfer Function

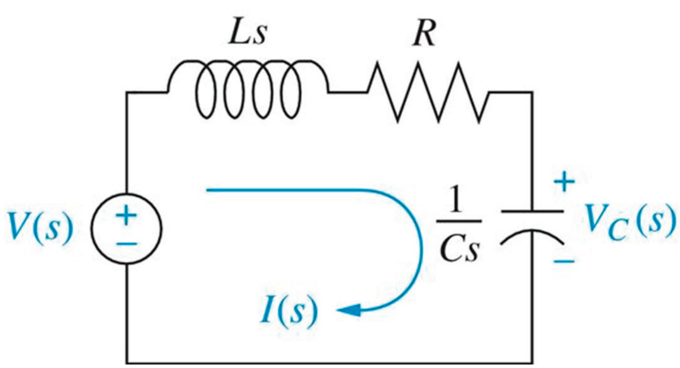
5. Therefore, the transfer function $G(s) = \left.\frac{V_C(s)}{V(s)}\right|_{\text{IC's}=0}$ is given by

$$\boxed{G(s) = \frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.}$$



Electrical Network Transfer Function

An easier way:



1. Use Laplace impedances of circuit elements to write the KVL equation as

$$\left[Ls + R + \frac{1}{Cs}\right] I(s) = V(s).$$

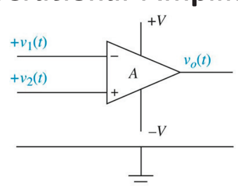
2. The voltage across the capacitor is given by

$$V_C(s) = I(s)\frac{1}{Cs} = \frac{\frac{1}{Cs}V(s)}{Ls + R + \frac{1}{Cs}} = \left[\frac{\frac{1}{Cs}V(s)}{Ls + R + \frac{1}{Cs}}\right]\left(\frac{\frac{s}{L}}{\frac{s}{L}}\right) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}V(s).$$

3. The transfer function is given by

$$\boxed{G(s) = \frac{V_C(s)}{V(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.}$$

Operational Amplifiers



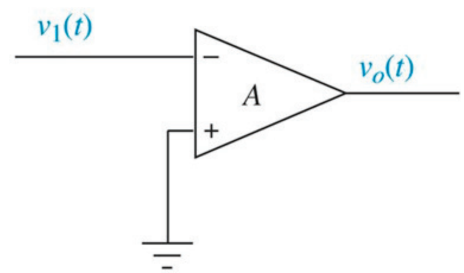
An **operational amplifier** is an electronic amplifier that is used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $v_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

The output $v_o(t)$ is given by

$$v_o(t) = A\left(v_2(t) - v_1(t)\right).$$

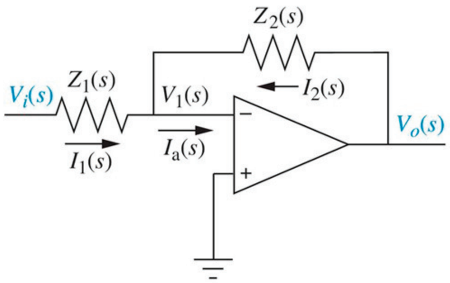
Inverting Operational Amplifier



If $v_2(t)$ is ground, the amplifier is called an **inverting operational amplifier**, and the output $v_o(t)$ is given by

$$v_o(t) = -Av_1(t).$$

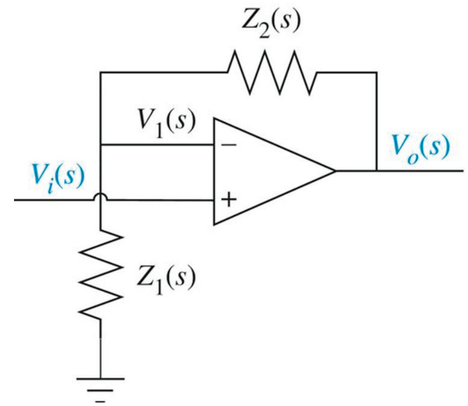
Transfer Function of Inverting Operational Amplifier



The transfer function $G(s)$ for an **inverting operational amplifier** is given by

$$\boxed{G(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}.}$$

Transfer Function of Noninverting Operational Amplifier



The transfer function $G(s)$ for an **noninverting operational amplifier** is given by

$$\boxed{G(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_1(s)+Z_2(s)}{Z_1(s)} = 1 + \frac{Z_2(s)}{Z_1(s)}.}$$

Translational Mechanical System Transfer Function

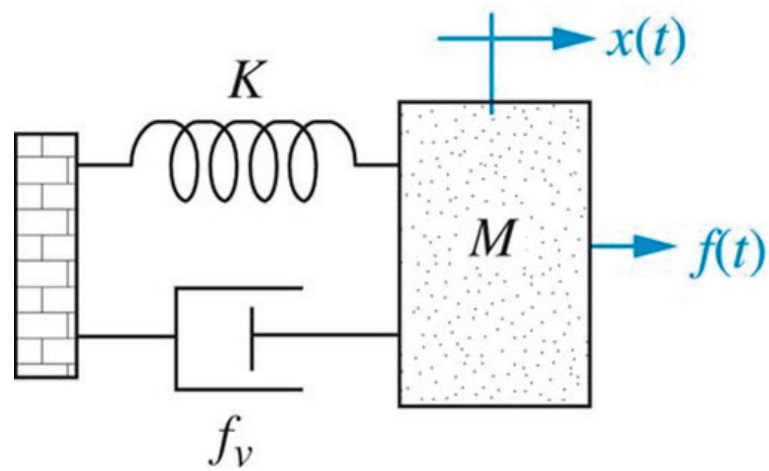
TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring (K)	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper (f_v)	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass (M)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Translational Mechanical System Transfer Function

Consider the system:



Find the transfer function $G(s)$ where $G(s) = \left.\frac{X(s)}{F(s)}\right|_{\text{IC's}=0}$.

Translational Mechanical System Transfer Function

1. The equation of motion is given by:

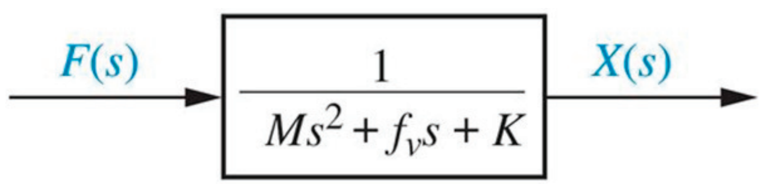
$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

2. Taking the Laplace transform, we have have

$$X(s)\left[Ms^2 + f_v s + K\right] = F(s)$$

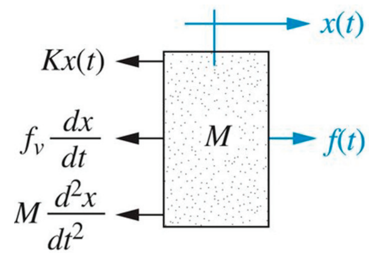
3. Therefore, the transfer function is given by

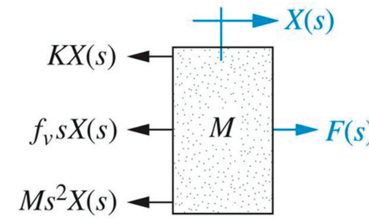
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$



Translational Mechanical System Transfer Function

An easier way is to represent the free body diagram using Laplace impedances:

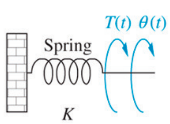
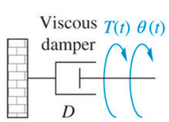
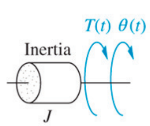




$$F(s) = \left[Ms^2 + f_v s + K\right] X(s)$$
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}.$$

Rotational Mechanical System Transfer Function

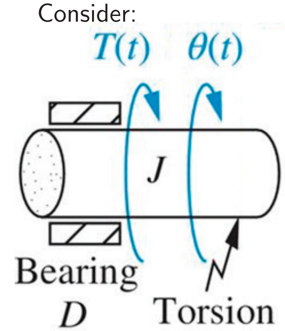
TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad (radians), $\omega(t)$ – rad/s (radians/second), K – N-m/rad (newton-meters/radian), D – N-m-s/rad (newton-meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

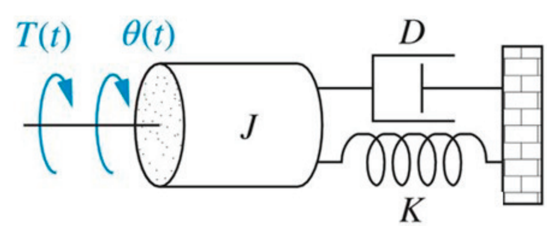
Rotational Mechanical System Transfer Function

Consider:



System

$\Longleftrightarrow$



System Diagram

Find $G(s) = \frac{\Theta(s)}{T(s)}$.

Rotational Mechanical System Transfer Function

1. Equation of Motion:

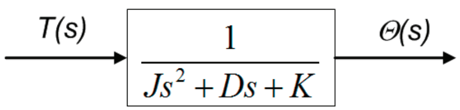
$$T(t) = K\theta(t) + D\frac{d\theta(t)}{dt} + J\frac{d^2\theta(t)}{dt^2}$$

2. Laplace Transform:

$$T(s) = \Theta(s)\left[K + Ds + Js^2\right]$$

3. Transfer function:

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Ds + K}$$



Zero-Input and Zero-State Response of LTI Systems

Consider the LCCDE given by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t). \tag{1}$$

This corresponds to $\{L = 2, C = 1/2, R = 4\}$ or $\{M = J = 1, f_v = D = 2, K = 1\}$ in our previous second-order systems.

Taking Laplace Transforms, we have

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = X(s).$$

Zero-Input and Zero-State Response of LTI Systems

Solving for $Y(s)$, we have

$$Y(s) = \underbrace{\frac{(s+3)y(0) + y'(0)}{s^2 + 3s + 2}}_{Y_N(s)} + \underbrace{\frac{1}{s^2 + 3s + 2}X(s)}_{Y_F(s)}. \tag{2}$$

$Y_N(s)$ = term due to the system's **initial conditions**.
This response results when the input $x(t)$ is zero.
It is called the **ZERO-INPUT (or NATURAL) RESPONSE**.

$Y_F(s)$ = term due to the **system input**.
This response results when all initial conditions are zero.
It is called the **ZERO-STATE (or FORCED) RESPONSE**.

Characteristic Polynomial and Zero-Input (Natural) Response

If we set the input or forcing function to zero, then Equation (1) becomes

$$y''(t) + 3y'(t) + 2y(t) = 0,$$

which is the **homogeneous** form of Equation (1).

Similarly, Equation (2) becomes

$$Y_N(s) = \frac{(s+3)y(0) + y'(0)}{(s+2)(s+1)} = \frac{C_1}{s+2} + \frac{C_2}{s+1}, \tag{3}$$

and the output due to the initial conditions alone, or the **zero-input response** is

$$y_N(t) = C_1e^{-2t} + C_2e^{-t}$$

Characteristic Polynomial and Zero-Input (Natural) Response

...The **zero-input response** is

$$y_N(t) = C_1 e^{-2t} + C_2 e^{-t}$$

In this example, the zero-input response will always be a linear combination of e^{-2t} and e^{-t} regardless of the values of the initial conditions.

So the **form** of the zero-input response is determined by the denominator of Equation (3), which is the **SYSTEM CHARACTERISTIC POLYNOMIAL**.

Setting the system characteristic polynomial equal to zero yields the **SYSTEM CHARACTERISTIC EQUATION**.

Note: This is the same characteristic equation defined when solving differential equations by classical methods.

The **characteristic equation** governs the transient response of the system.

Transfer Function and Zero-State (Forced) Response

Setting all initial conditions to ZERO in Equation (2) gives

$$Y_F(s) = \frac{1}{(s+2)(s+1)} X(s) = G(s)X(s)$$

where, as before, $G(s) = \left.\frac{Y(s)}{X(s)}\right|_{\text{IC's}=0} = \frac{1}{(s+2)(s+1)}$.

Therefore, the **zero-state response** is governed by the transfer function.

Definitions

Given a system transfer function $G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}$,

- $G(s)$ is **PROPER** if the degree of $N(s) \leq$ degree of $D(s)$
- $G(s)$ is **STRICTLY PROPER** if the degree of $N(s) <$ degree of $D(s)$
- p is a **POLE** of $G(s)$ if $|G(p)| = \infty$.
- z is a **ZERO** of $G(s)$ if $|G(z)| = 0$.
- $G(s)$ is **IRREDUCIBLE** if $N(s)$ and $D(s)$ have no common factors, i.e. factors of the form $s - k_i$ where k_i is complex, in general.

If $G(s)$ is **irreducible**, then $D(s)$ is the **characteristic polynomial** of the system, and $D(s) = 0$ is the **characteristic equation** of the system.

Example 3:

Given a causal LTI system described by

$$y''(t) + 3y'(t) - 4y(t) = 2x(t),$$

(a) find the **zero-input response** for $y(0) = 1$ and $y'(0) = 0$ and

Example 3 (continued):

(b) find the **zero-state response** to a unit step input.

Example 4:

Given $G(s) = \frac{1}{s+1}$ and input $x(t) = e^{-2t}u(t)$, find the **zero-state response**.

Example 5:

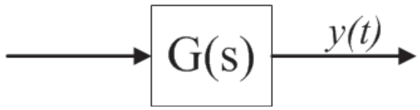
Given $G(s) = \frac{3s-1}{(s+1)(s+2)}$, find the zeros and poles.

Example 6:

Given $G(s) = \frac{s+1}{s^2+3s+2}$, find the zeros and poles.

NOTE: $s = \infty$ is ALWAYS a zero of a strictly proper $G(s)$.

Example 7:



Consider a causal LTI system described by

$$y''(t) + 5y'(t) + 4y(t) = x'(t) + x(t).$$

1. Find the system transfer function $G(s)$.

Example 7 (continued):

2. Plot the poles and zeros.

Example 7 (continued):

3. What is the system impulse response?

Example 7 (continued):

4. What is the system response to input $x(t) = \delta(t)$?

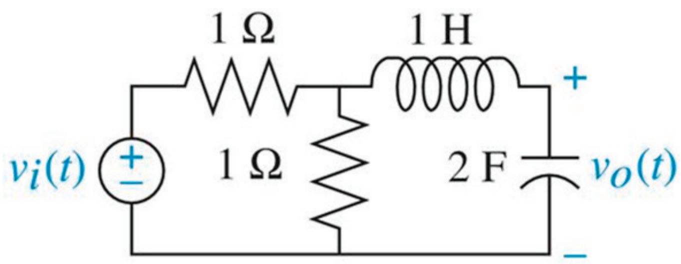
Example 7 (continued):

5. What is the system response to input $x(t) = 4\delta(t-2)$?

Example 7 (continued):

6. What is the system response to input $x(t) = u(t)$?

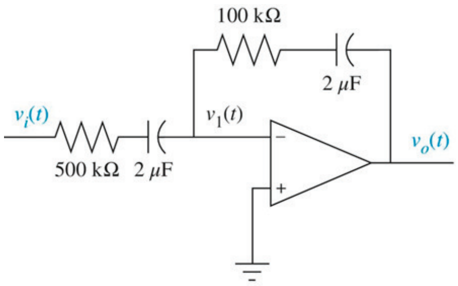
Example 8:



Find the LCCDE and transfer function for the above circuit with input $v_i(t)$ and output $v_o(t)$.

Example 8 (continued):

Example 9:



Find the LCCDE and transfer function for the above circuit with input $v_i(t)$ and output $v_o(t)$.

Example 9 (continued):

Example 9 (continued):

Example 10:

Given $G(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^3+8s^2+9s+2}$, what is the corresponding LCCDE?