

**ELEC 309**  
**Signals and Systems**

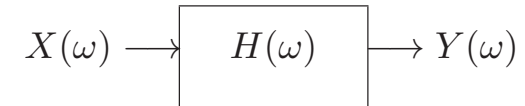
**Frequency-Domain Analysis of**  
**Continuous-Time (Chapter 5, Schaum's Outline of**  
**Signals and Systems)**  
**and Discrete-Time (Chapter 6, Schaum's Outline**  
**of Signals and Systems)**  
**Systems**

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**System Representation in the Frequency Domain**

Applying the convolution property to the previous convolution equation, we have



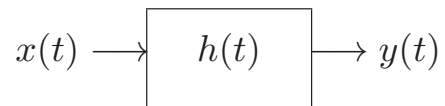
where

$$Y(\omega) = X(\omega)H(\omega),$$

and  $X(\omega)$ ,  $Y(\omega)$ , and  $H(\omega)$  are the Fourier transforms of  $x(t)$ ,  $y(t)$ , and  $h(t)$ , respectively.

**System Representation in the Continuous-Time Domain**

Previously, we showed that the response  $y(t)$  of a **continuous-time** LTI system is the convolution of the input  $x(t)$  with the impulse response  $h(t)$ , or



where

$$y(t) = x(t) * h(t).$$

**The System Frequency Response**

Solving for  $H(\omega)$ , we have

$$\mathcal{F}\{h(t)\} = H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\mathcal{F}\{y(t)\}}{\mathcal{F}\{x(t)\}}.$$

The Fourier transform  $H(\omega)$  of  $h(t)$  is called the **frequency response** of the system.

## The System Frequency Response: Magnitude Response and Phase Response

Recall that the frequency response is, in general, complex and therefore can be written as

$$H(\omega) = |H(\omega)|e^{j\theta_H(\omega)}.$$

The term  $|H(\omega)|$  is called the **magnitude response** of the system.

The term  $\theta_H(\omega)$  is called the **phase response** of the system.

## Frequency Response of **Continuous-Time** LTI Systems: Frequency Domain

Recall that

$$1 \longrightarrow \boxed{\text{LTI System}} \longrightarrow H(\omega),$$

and

$$X(\omega) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(\omega) = H(\omega)X(\omega).$$

## Frequency Response of **Continuous-Time** LTI Systems: Time Domain

Recall that

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow h(t),$$

and

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = h(t) * x(t).$$

## Frequency Response of **Continuous-Time** LTI Systems

Consider a **continuous-time** LTI system with an input  $x(t) = e^{j\omega_0 t}$ , which has Fourier transform

$$X(\omega) = 2\pi\delta(\omega - \omega_0).$$

The Fourier transform of the output of this **continuous-time** LTI system is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= H(\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(\omega_0)\delta(\omega - \omega_0). \end{aligned}$$

## Frequency Response of **Continuous-Time** LTI Systems

Taking the inverse Fourier transform of  $Y(\omega)$ , we have

$$y(t) = H(\omega_0)e^{j\omega_0 t}.$$

Therefore, for a single-frequency-component input,

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) = H(\omega_0)e^{j\omega_0 t}.$$

From our previous analyses of signals using Fourier transformations, we observed that we can represent signals as linear expressions of terms of the form  $e^{j\omega_0 t}$ .

## Response of **Continuous-Time** LTI Systems: Aperiodic Inputs

If  $x(t)$  is aperiodic, then, from the definition of the inverse Fourier transform, we have

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega,$$

and the corresponding output of the **continuous-time** LTI system can be written as

$$\begin{aligned} y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)e^{j\omega t} d\omega = \mathcal{F}^{-1}\{Y(\omega)\}. \end{aligned}$$

## Response of **Continuous-Time** LTI Systems: Periodic Inputs

Since the Fourier transform is linear, then if the input  $x(t)$  is periodic with the Fourier series representation

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

then the corresponding output of the **continuous-time** LTI system is also periodic with the Fourier series representation

$$y(t) = \sum_{n=-\infty}^{\infty} D_n H(n\omega_0) e^{jn\omega_0 t}.$$

## Response of **Continuous-Time** LTI Systems

The behavior of a **continuous-time** LTI system in the frequency domain is completely characterized by its frequency response  $H(\omega)$ .

Let

$$X(\omega) = |X(\omega)|e^{j\theta_X(\omega)}$$

and

$$Y(\omega) = |Y(\omega)|e^{j\theta_Y(\omega)}.$$

### Response of **Continuous-Time** LTI Systems: Magnitude Spectrum

The magnitude spectrum of the output can be written as

$$|Y(\omega)| = |H(\omega)| |X(\omega)|,$$

which says that the magnitude spectrum  $|Y(\omega)|$  of the output is the product of the magnitude spectrum  $|X(\omega)|$  of the input and the magnitude response  $|H(\omega)|$  of the system.

The magnitude response  $|H(\omega)|$  is sometimes referred to as the **gain** of the system.

### Response of **Continuous-Time** LTI Systems: Distortionless Transmission

For **distortionless transmission** through a **continuous-time** LTI system, we require that the signal retain its overall shape. In other words, the system may only alter the amplitude (by a multiplicative constant) of the input signal or delay it in time. Therefore, if  $x(t)$  is the input signal to a **continuous-time** LTI system, then the input-output equation

$$y(t) = Kx(t - t_d)$$

is required for distortionless transmission, where  $t_d$  is the **time delay** and  $K > 0$  is a **gain constant**.

### Response of **Continuous-Time** LTI Systems: Phase Spectrum

The phase spectrum of the output can be written as

$$\theta_Y(\omega) = \theta_H(\omega) + \theta_X(\omega),$$

which says that the phase spectrum  $\theta_Y(\omega)$  of the output is the sum of the phase spectrum  $\theta_X(\omega)$  of the input and the phase response  $\theta_H(\omega)$  of the system.

### Response of **Continuous-Time** LTI Systems: Distortionless Transmission

Taking the Fourier transform of both sides, we see that

$$Y(\omega) = Ke^{-j\omega t_d} X(\omega).$$

Therefore, for distortionless transmission, the system must have a frequency response given by

$$H(\omega) = |H(\omega)|e^{j\theta_H(\omega)} = Ke^{-j\omega t_d}.$$

Therefore,

$$|H(\omega)| = K \text{ and} \tag{1}$$

$$\theta_H(\omega) = -\omega t_d. \tag{2}$$

### Response of **Continuous-Time** LTI Systems: Amplitude Distortion

For distortionless transmission, the amplitude of  $H(\omega)$  must be **constant** over all frequencies  $\omega$ .

When the magnitude spectrum  $|H(\omega)|$  of the system is not constant with the frequency band of interest, the frequency components of the input signal are transmitted with a different amount of **gain** or **attenuation**.

This effect is referred to as **amplitude distortion**.

### Frequency Response for **Continuous-Time** LTI Systems Described by LCCDEs

Previously, we considered a continuous-time LTI system for which input  $x(t)$  and output  $y(t)$  satisfy a general  $N^{\text{th}}$ -order linear constant-coefficient differential equation (LCCDE) given by

$$\sum_{n=0}^N a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^M b_m \frac{d^m x(t)}{dt^m}$$

with  $M \leq N$ .

### Response of **Continuous-Time** LTI Systems: Phase Distortion

For distortionless transmission, the phase of  $H(\omega)$  must be **linear** with respect to frequency  $\omega$ .

When the phase spectrum  $\theta_H(\omega)$  of the system is not linear over the frequency band of interest, the different frequency components of the input signal encounter different delays in passing through the system, which results in a different waveform at the output of the system.

This effect is referred to as **phase distortion**.

### Frequency Response for **Continuous-Time** LTI Systems Described by LCCDEs

Taking the Fourier transform of both sides, and applying the linearity and time-differentiation properties of the Fourier transform, we have

$$\sum_{n=0}^N a_n (j\omega)^n Y(\omega) = \sum_{m=0}^M b_m (j\omega)^m X(\omega).$$

or

$$Y(\omega) \sum_{n=0}^N a_n (j\omega)^n = X(\omega) \sum_{m=0}^M b_m (j\omega)^m.$$

## Frequency Response for **Continuous-Time** LTI Systems Described by LCCDEs

Therefore, a continuous-time LTI system whose input  $x(t)$  and output  $y(t)$  satisfy a general  $N^{\text{th}}$ -order LCCDE has a **frequency response** given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^M b_m(j\omega)^m}{\sum_{n=0}^N a_n(j\omega)^n}, \quad (3)$$

which is a rational function of  $\omega$ .

## Example (continued):

### Example:

Consider a **continuous-time** system whose input  $x(t)$  and output  $y(t)$  are related by

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

Find the impulse response  $h(t)$  of the system.

### Filtering

Filtering is one of the most basic operations in any signal processing system.

**Filtering** is the process by which the relative amplitudes of the frequency components in a signal are changed or even suppressed.

For **continuous-time** LTI system:  $Y(\omega) = X(\omega)H(\omega)$ .

Therefore, an LTI system acts as a filter on the input signal, and the term **filter** is used to describe any system that exhibits some type of frequency-selective behavior.

### Ideal Frequency-Selective Filters

An **ideal frequency-selective filter** exactly passes signal in one set of frequencies and completely suppresses the rest.

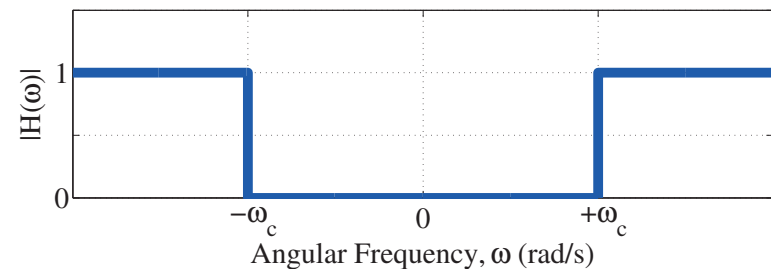
The band of frequencies passed by the filter is referred to as the **pass band**.

The band of frequencies rejected by the filter is referred to as the **stop band**.

### Ideal Frequency-Selective Filters: Ideal High-Pass Filter Amplitude Response

An ideal high-pass filter (HPF) amplitude response is given by

$$|H(\omega)| = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & |\omega| \geq \omega_c \end{cases}$$

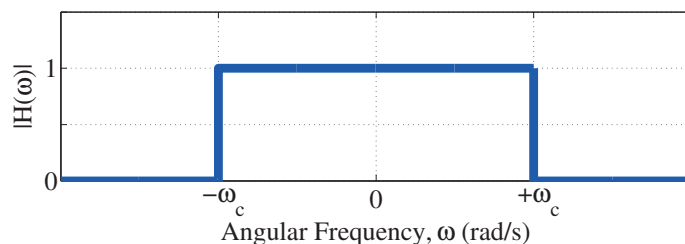


### Ideal Frequency-Selective Filters: Ideal Low-Pass Filter Amplitude Response

An ideal low-pass filter (LPF) amplitude response is given by

$$|H(\omega)| = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

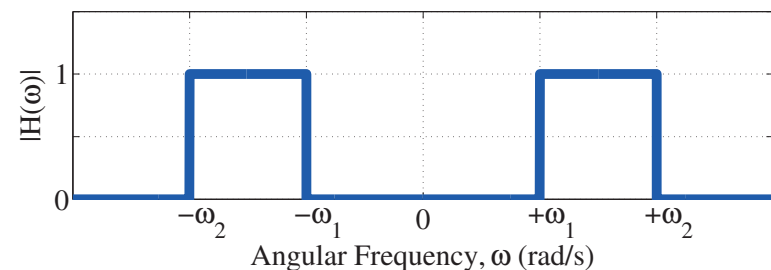
The frequency  $\omega_c$  is called the **cutoff frequency**.



### Ideal Frequency-Selective Filters: Ideal Band-Pass Filter Amplitude Response

An ideal band-pass filter (BPF) amplitude response is given by

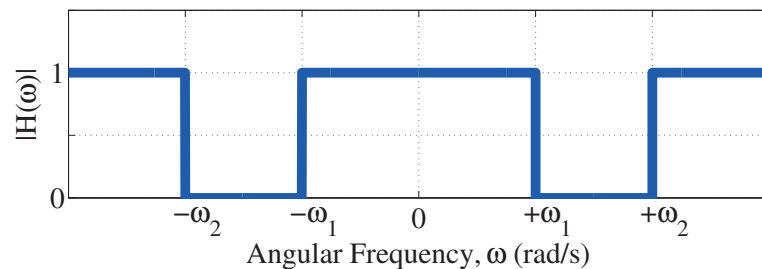
$$|H(\omega)| = \begin{cases} 1 & \omega_1 \leq |\omega| \leq \omega_2 \\ 0 & \text{otherwise.} \end{cases}$$



### Ideal Frequency-Selective Filters: Ideal Band-Stop Filter Amplitude Response

An ideal band-stop filter (BSF) amplitude response is given by

$$|H(\omega)| = \begin{cases} 0 & \omega_1 < |\omega| < \omega_2 \\ 1 & \text{otherwise.} \end{cases}$$



### Example:

Consider the periodic square wave given by

$$x(t) = \begin{cases} 1 & 2\pi k \leq t < 2\pi(k + 0.5) \\ 0 & 2\pi(k + 0.5) \leq t < 2\pi(k + 1) \end{cases}$$

for all integers  $k$ . Suppose this signal passes through an ideal low-pass filter with ideal phase response and amplitude response given by

$$|H(\omega)| = \begin{cases} 1 & |\omega| \leq 6 \\ 0 & |\omega| > 6. \end{cases}$$

Determine the output  $y(t)$  of this filter.

### Ideal Frequency-Selective Filters: Phase Response

To avoid phase distortion in the filtering process, a filter should have a linear phase characteristic over the pass band of the filter, or

$$\theta_H(\omega) = -\omega t_d,$$

where  $t_d$  is a constant.

All ideal frequency-selective filters are **\*noncausal\*** systems.

### Example (continued):



**Example (continued):****Nonideal Frequency-Selective Filters**

The input-output equation for this circuit is given by

$$RC \frac{dy(t)}{dt} + y(t) = x(t).$$

Taking the Fourier transform of both sides of the input-output equation, the frequency response  $H(\omega)$  of the  $RC$  filter is given by

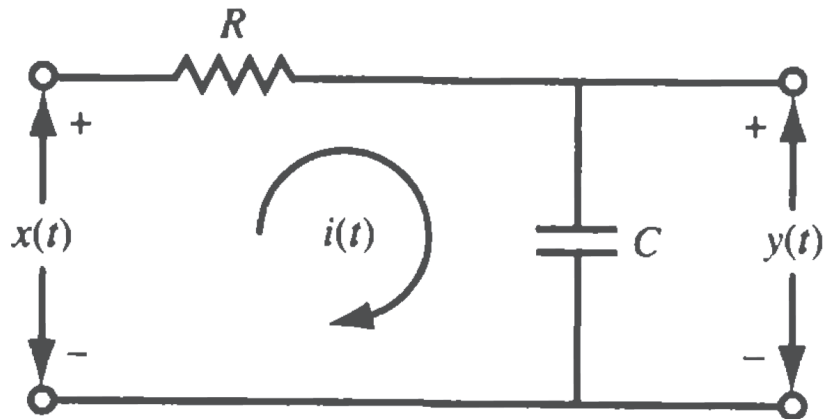
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

where  $\omega_0 = \frac{1}{RC}$  is the **cutoff frequency**.

This is a **first-order low-pass Butterworth filter**.

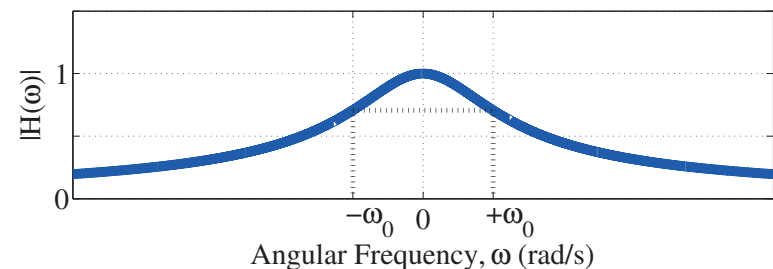
**Nonideal Frequency-Selective Filters**

Consider the  $RC$  filter shown below:

**Nonideal Frequency-Selective Filters**

The amplitude response is given by

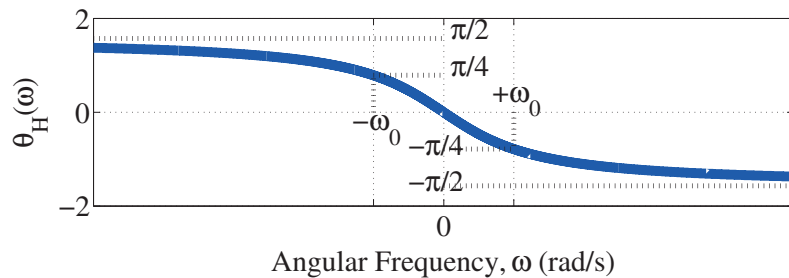
$$|H(\omega)| = \frac{1}{|1 + j\frac{\omega}{\omega_0}|} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}.$$



### Nonideal Frequency-Selective Filters

The phase response is given by

$$\theta_H(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right).$$



### Nonideal Frequency-Selective Filters

The input-output equation for this circuit is given by

$$RC \frac{dy(t)}{dt} + y(t) = RCx'(t).$$

Taking the Fourier transform of both sides of the input-output equation, the frequency response  $H(f)$  of the  $RC$  filter is given by

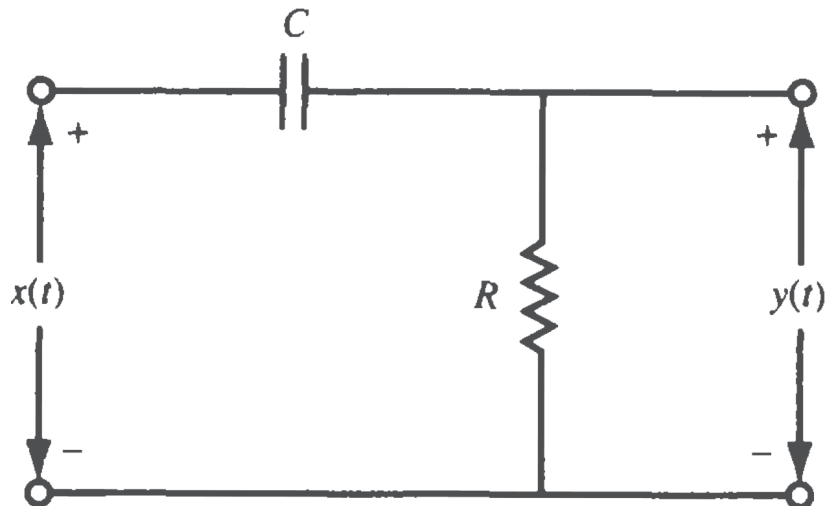
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega RC}{1 + j\omega RC} = \frac{1}{1 - j\frac{\omega_0}{\omega}}$$

where  $\omega_0 = \frac{1}{RC}$  is the **cutoff frequency**.

This is a **first-order high-pass Butterworth filter**.

### Example:

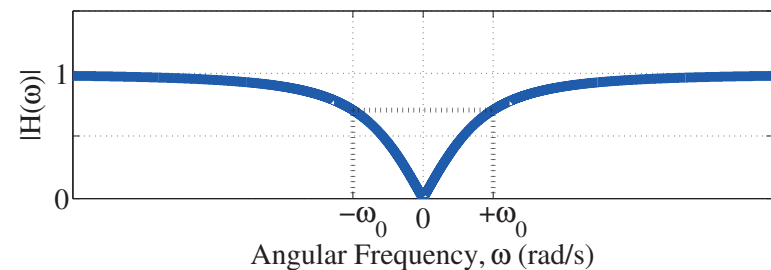
Consider the  $RC$  filter shown below.



### Nonideal Frequency-Selective Filters

The amplitude response is given by

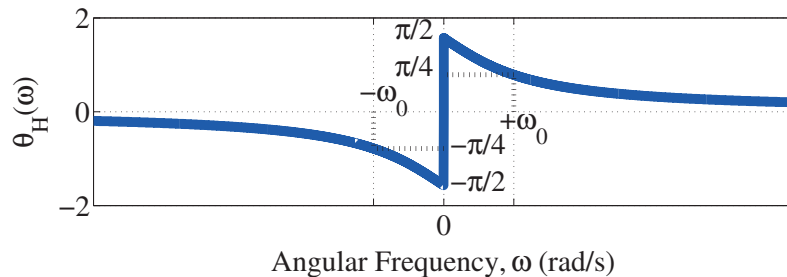
$$|H(\omega)| = \frac{1}{|1 - j\frac{\omega_0}{\omega}|} = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2}}.$$



### Nonideal Frequency-Selective Filters

The phase response is given by

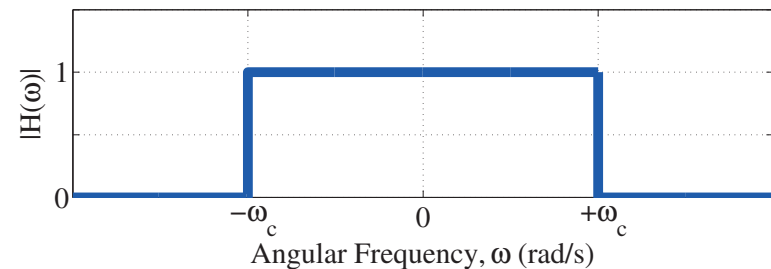
$$\theta_H(\omega) = \tan^{-1} \left( \frac{\omega_0}{\omega} \right).$$



### Absolute Bandwidth of Ideal Filters

The **absolute bandwidth** is defined only for \*ideal\* low-pass or band-pass filters.

The absolute bandwidth  $W_B$  of an ideal low-pass filter is its cutoff frequency. In other words,  $W_B = f_c = \frac{\omega_c}{2\pi}$  Hz.



### Filter or System Bandwidth

One important concept in system analysis is the **bandwidth** of a filter or system.

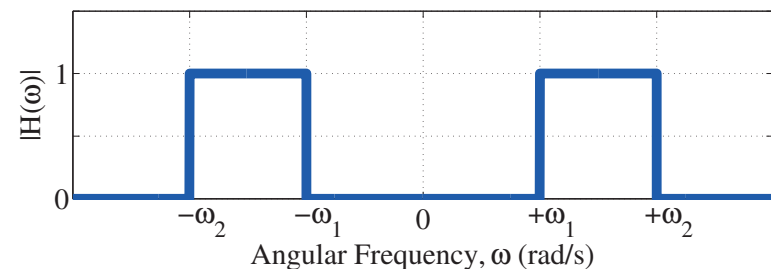
The **bandwidth** of a filter or system is defined as the width of the range of positive frequencies of components that are not filtered out when passing through a filter or system.

There are \*many\* different definitions of bandwidth.

### Absolute Bandwidth of Ideal Filters

The absolute bandwidth  $W_B$  of an ideal band-pass filter is given by  $W_B = f_2 - f_1 = \frac{\omega_2 - \omega_1}{2\pi}$  Hz.

A band-pass filter is called **narrowband** if  $W_B \ll f_m$ , where  $f_m = \frac{f_1 + f_2}{2} = \frac{\omega_1 + \omega_2}{4\pi}$  is the center frequency of the filter.



### Half-Power Bandwidth of Nonideal Filters

For causal or practical filters, a commonly-used definition of filter or system bandwidth is the **half-power (or 3-dB) bandwidth**  $W_{3\text{-dB}}$ .

The **half-power bandwidth** is defined as the width of the range of positive frequencies for which

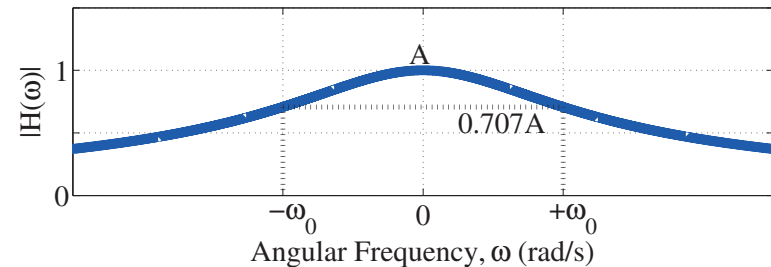
$$|H(\omega)| \geq \frac{\max\{|H(\omega)|\}}{\sqrt{2}}.$$

The half-power bandwidth is also known as the **3-dB bandwidth** because a power attenuation by a factor of 2 is equivalent to an attenuation of 3 dB.

### Half-Power Bandwidth of Nonideal Low-Pass Filters

For a causal or practical low-pass filter, the **half-power (or 3-dB) bandwidth** is given by

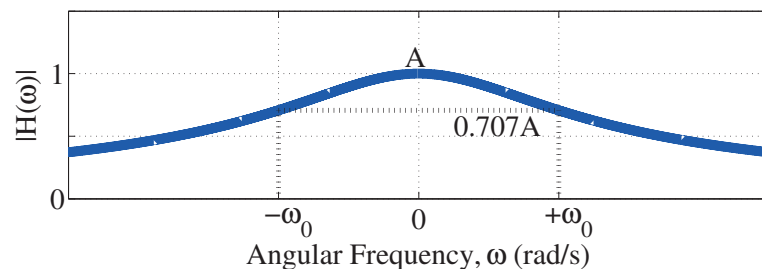
$$W_{3\text{-dB}} = f_0 = \frac{\omega_0}{2\pi} \text{ Hz.}$$



### Half-Power Bandwidth of Nonideal Low-Pass Filters

For a causal or practical low-pass filter, we have  $\max\{|H(\omega)|\} = |H(0)|$  and its cutoff frequency  $\omega_0$  must meet the condition

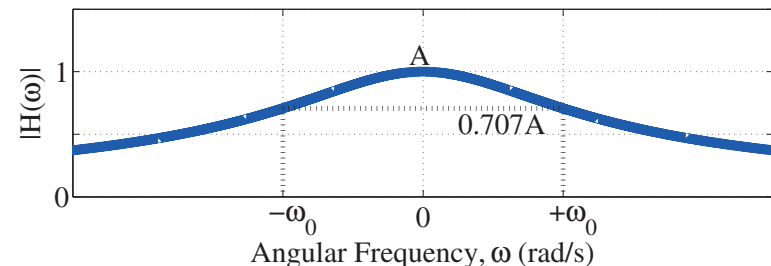
$$|H(\omega_0)| = \frac{|H(0)|}{\sqrt{2}}.$$



### Half-Power Bandwidth of Nonideal Low-Pass Filters

For the low-pass  $RC$  filter discussed previously, the **half-power (or 3-dB) bandwidth** is given by

$$W_{3\text{-dB}} = f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi RC} \text{ Hz.}$$

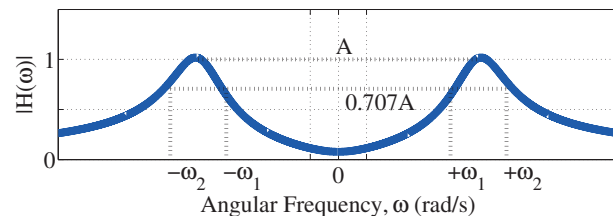


### Half-Power Bandwidth of Nonideal Band-Pass Filters

For a causal or practical band-pass filter, its two cutoff frequencies  $\omega_1$  and  $\omega_2$  must meet the conditions

$$\omega_1 = \min \left\{ \text{positive } \omega : |H(\omega)| \geq \frac{|H(0)|}{\sqrt{2}} \right\} \text{ and}$$

$$\omega_2 = \max \left\{ \text{positive } \omega : |H(\omega)| \geq \frac{|H(0)|}{\sqrt{2}} \right\}.$$



### Signal Bandwidth

The **bandwidth** of a signal is defined as the width of the range of "significant" positive frequencies contained in a signal.

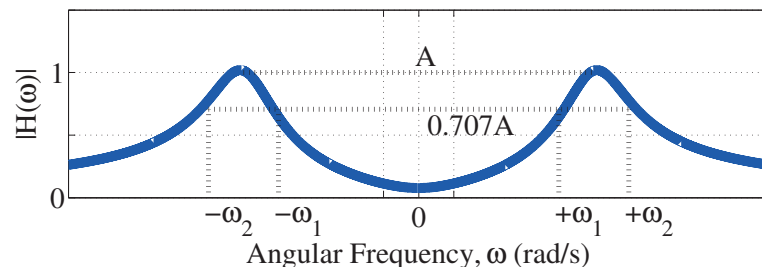
The **absolute bandwidth** of a signal is defined as the width of the range of positive frequency components contained in a signal.

The **bandwidth** of a signal can also be defined as the width of the range of positive frequencies in which "most" of the energy or power lies.

### Half-Power Bandwidth of Nonideal Band-Pass Filters

For a causal or practical band-pass filter, the **half-power (or 3-dB) bandwidth** is given by

$$W_{3\text{-dB}} = f_2 - f_1 = \frac{\omega_2 - \omega_1}{2\pi} \text{ Hz.}$$



### Half-Power Bandwidth of a Signal

The bandwidth of a signal  $x(t)$  can also be defined on a similar basis as a filter bandwidth, such as the half-power (or 3-dB) bandwidth, using the amplitude spectrum  $|X(\omega)|$  of the signal.

If we replace  $|H(\omega)|$  by  $|X(\omega)|$  in our previous amplitude response plots, we will have frequency-domain plots of **low-pass**, **high-pass**, and **band-pass** signals.

## Band-Limited Signal

A signal  $x(t)$  is said to be **band-limited** if

$$|X(\omega)| = 0 \text{ if } |\omega| > \omega_M$$

The bandwidth of a band-limited signal is given by  $\omega_M$  radians per second or  $f_M = \frac{\omega_M}{2\pi}$  Hz.

## System Representation in the Frequency Domain

Applying the convolution property to the previous convolution equation, we have

$$X(\Omega) \longrightarrow \boxed{H(\Omega)} \longrightarrow Y(\Omega)$$

where

$$Y(\Omega) = X(\Omega)H(\Omega),$$

and  $X(\Omega)$ ,  $Y(\Omega)$ , and  $H(\Omega)$  are the Fourier transforms of  $x[n]$ ,  $y[n]$ , and  $h[n]$ , respectively.

## System Representation in the Discrete-Time Domain

Previously, we showed that the response  $y[n]$  of a **discrete-time** LTI system is the convolution of the input  $x[n]$  with the impulse response  $h[n]$ , or

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

where

$$y[n] = x[n] * h[n].$$

## The System Frequency Response

Solving for  $H(\Omega)$ , we have

$$\mathcal{F}\{h[n]\} = \boxed{H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}} = \frac{\mathcal{F}\{y[n]\}}{\mathcal{F}\{x[n]\}}.$$

The Fourier transform  $H(\Omega)$  of  $h[n]$  is called the **frequency response** of the system.

### The System Frequency Response: Magnitude Response and Phase Response

Recall that the frequency response is, in general, complex and therefore can be written as

$$H(\Omega) = |H(\Omega)|e^{j\theta_H(\Omega)}.$$

The term  $|H(\Omega)|$  is called the **magnitude response** of the system.

The term  $\theta_H(\Omega)$  is called the **phase response** of the system.

### Frequency Response of **Discrete-Time** LTI Systems: Frequency Domain

Recall that

$$1 \longrightarrow \boxed{\text{LTI System}} \longrightarrow H(\Omega),$$

and

$$X(\Omega) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(\Omega) = H(\Omega)X(\Omega).$$

### Frequency Response of **Discrete-Time** LTI Systems: Time Domain

Recall that

$$\delta[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow h[n],$$

and

$$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = h[n] * x[n].$$

### Frequency Response of **Discrete-Time** LTI Systems

Consider a **discrete-time** LTI system with an input  $x[n] = e^{j\Omega_0 n}$ , which has Fourier transform

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0).$$

The Fourier transform of the output of this **discrete-time** LTI system is given by

$$\begin{aligned} Y(\Omega) &= H(\Omega)X(\Omega) \\ &= H(\Omega)2\pi\delta(\Omega - \Omega_0) = 2\pi H(\Omega_0)\delta(\Omega - \Omega_0). \end{aligned}$$

### Frequency Response of Discrete-Time LTI Systems

Taking the inverse Fourier transform of  $Y(\Omega)$ , we have

$$y[n] = H(\Omega_0)e^{j\Omega_0 n}.$$

Therefore, for a single-frequency-component input,

$$x[n] = e^{j\Omega_0 n} \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n] = H(\Omega_0)e^{j\Omega_0 n}.$$

From our previous analyses of signals using Fourier transformations, we observed that we can represent signals as linear expressions of terms of the form  $e^{j\Omega_0 n}$ .

### Response of Discrete-Time LTI Systems: Aperiodic Inputs

If  $x[n]$  is aperiodic, then, from the definition of the inverse Fourier transform, we have

$$x[n] = \mathcal{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{j\Omega n} d\Omega,$$

and the corresponding output of the discrete-time LTI system can be written as

$$\begin{aligned} y[n] &= \frac{1}{2\pi} \int_{2\pi} H(\Omega)X(\Omega)e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{2\pi} Y(\Omega)e^{j\Omega n} d\Omega = \mathcal{F}^{-1}\{Y(\Omega)\}. \end{aligned}$$

### Response of Discrete-Time LTI Systems: Periodic Inputs

Since the Fourier transform is linear, then if the input  $x[n]$  is periodic with the Fourier series representation

$$x[n] = \sum_{k=\langle N_0 \rangle} \mathcal{D}_n e^{jk\Omega_0 n},$$

then the corresponding output of the discrete-time LTI system is also periodic with the Fourier series representation

$$y[n] = \sum_{k=\langle N_0 \rangle} \mathcal{D}_n H(k\Omega_0) e^{jk\Omega_0 n}.$$

### Response of Discrete-Time LTI Systems

The behavior of a discrete-time LTI system in the frequency domain is completely characterized by its frequency response  $H(\Omega)$ .

Let

$$X(\Omega) = |X(\Omega)|e^{j\theta_X(\Omega)}$$

and

$$Y(\Omega) = |Y(\Omega)|e^{j\theta_Y(\Omega)}.$$



### Response of Discrete-Time LTI Systems: Magnitude Spectrum

The magnitude spectrum of the output can be written as

$$|Y(\Omega)| = |H(\Omega)||X(\Omega)|,$$

which says that the magnitude spectrum  $|Y(\Omega)|$  of the output is the product of the magnitude spectrum  $|X(\Omega)|$  of the input and the magnitude response  $|H(\Omega)|$  of the system.

The magnitude response  $|H(\Omega)|$  is sometimes referred to as the **gain** of the system.

### Periodic Nature of the Frequency Response

Note that

$$H(\Omega + 2\pi) = H(\Omega).$$

Unlike the frequency response of **continuous-time** systems, the frequency response of **discrete-time** systems is periodic with period  $2\pi$ .

Therefore, we need to consider the frequency response of a **discrete-time** system only over the range  $0 \leq \Omega < 2\pi$  or  $-\pi \leq \Omega < \pi$ .

### Response of Discrete-Time LTI Systems: Phase Spectrum

The phase spectrum of the output can be written as

$$\theta_Y(\Omega) = \theta_H(\Omega) + \theta_X(\Omega),$$

which says that the phase spectrum  $\theta_Y(\Omega)$  of the output is the sum of the phase spectrum  $\theta_X(\Omega)$  of the input and the phase response  $\theta_H(\Omega)$  of the system.

### Frequency Response for Discrete-Time LTI Systems Described by LCCDEs

Previously, we considered a **discrete-time** LTI system for which input  $x[n]$  and output  $y[n]$  satisfy a general  $N^{\text{th}}$ -order linear constant-coefficient difference equation (LCCDE) given by

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

with  $M \leq N$ .

## Frequency Response for Discrete-Time LTI Systems Described by LCCDEs

Taking the Fourier transform of both sides, and applying the linearity and time-shifting properties of the Fourier transform, we have

$$\sum_{k=0}^N a_k e^{-jk\Omega} Y(\Omega) = \sum_{m=0}^M b_m e^{-jm\Omega} X(\Omega).$$

or

$$Y(\Omega) \sum_{k=0}^N a_k e^{-jk\Omega} = X(\Omega) \sum_{m=0}^M b_m e^{-jm\Omega}.$$

## Example:

Consider a discrete-time system whose input  $x[n]$  and output  $y[n]$  are related by

$$y[n] - 0.5y[n-1] = x[n].$$

Find the impulse response  $h[n]$  of the system.

## Frequency Response for Discrete-Time LTI Systems Described by LCCDEs

Therefore, a discrete-time LTI system whose input  $x[n]$  and output  $y[n]$  satisfy a general  $N^{\text{th}}$ -order LCCDE has a **frequency response** given by

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{m=0}^M b_m e^{-jm\Omega}}{\sum_{k=0}^N a_k e^{-jk\Omega}}. \quad (4)$$

## Example (continued):