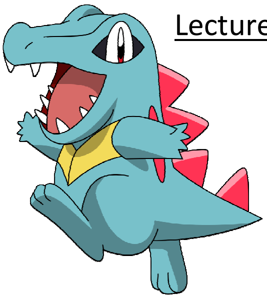


Lecture 9: Green's Theorem

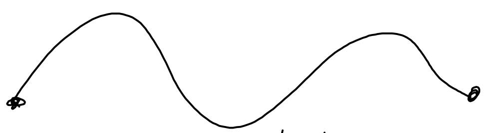
Totodile's Goals for the Day

- Introduce Green's Theorem for 2D closed curves
- Practice finding circulation and flux using Green's Theorem
- Review triple integrals

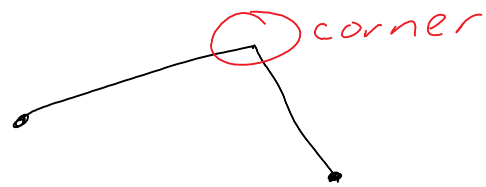
9.12 Green's Theorem

Def Properties of Curves

Smooth: A smooth curve is continuous and all partial derivatives exist and are continuous.

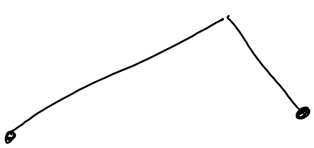


smooth

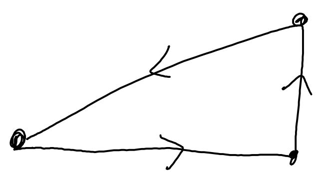


not smooth

Piecewise Smooth: A curve is piecewise smooth if it is continuous and smooth everywhere except at finitely many points.

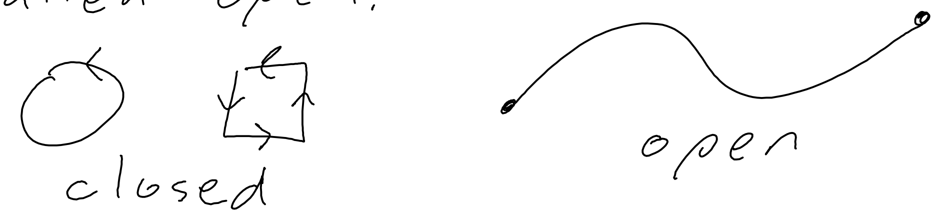


Piecewise Smooth

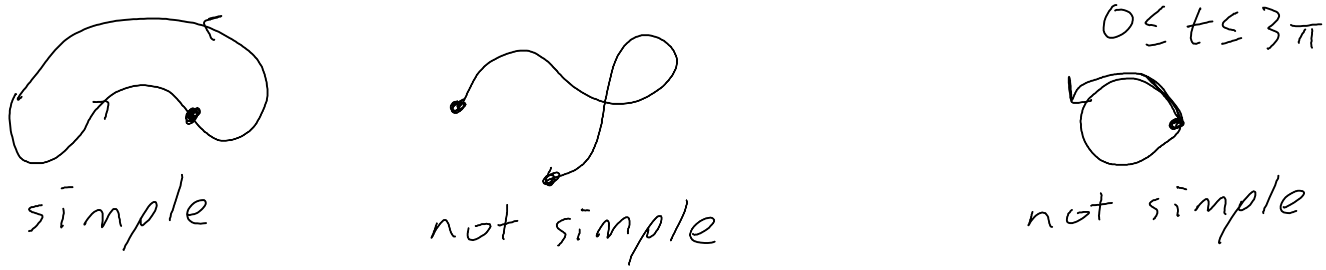


Piecewise Smooth

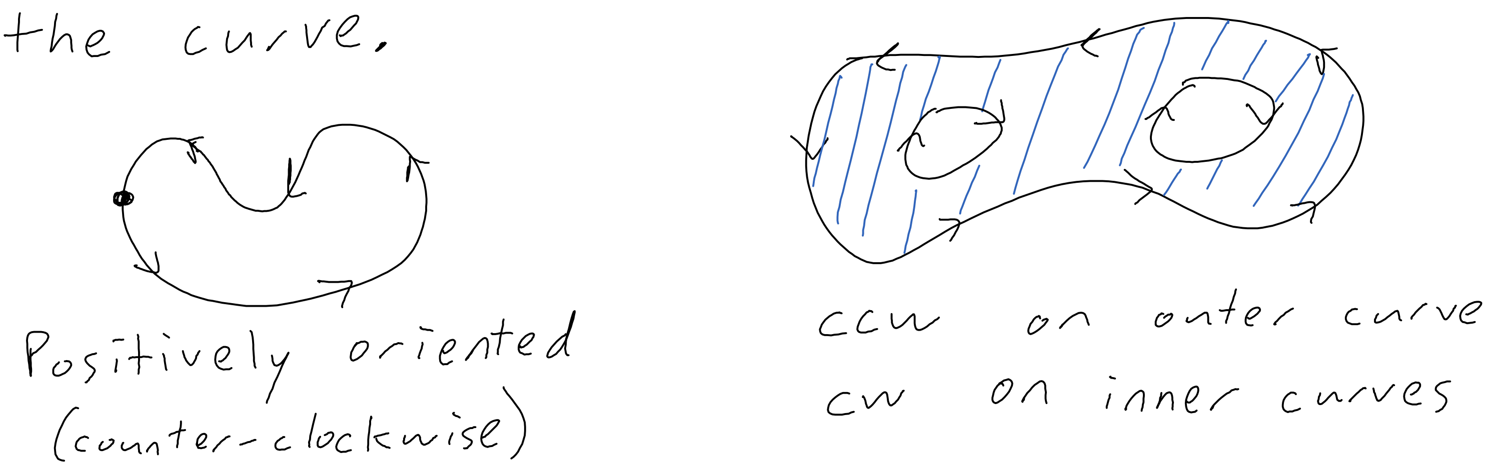
<u>Closed</u>: A closed curve starts and ends at the same position. Otherwise, the curve is called open.



<u>Simple</u>: A simple curve does not intersect itself (except possibly at start/end point).

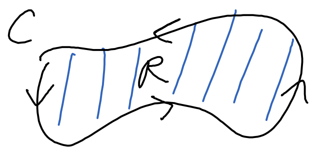


<u>Positively Oriented</u>: A closed curve is positively oriented if it keeps the "outside" on the right-hand side as you walk around the curve.



Idea: Green's Theorem

2D closed simple curve C encloses region R

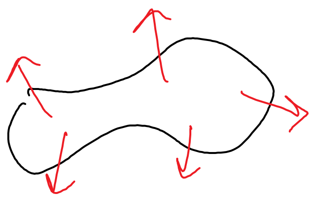


$$\oint_C \vec{F} \cdot \vec{n} \, ds = \text{Outward Flux}$$

= net rate of fluid flowing out of C with velocity $\vec{F}$

= total expansion of $\vec{F}$ in the region R

$$= \iint_R \nabla \cdot \vec{F} \, dA$$



Theorem Green's Theorem

Let C be a simple, closed, piecewise smooth, positively oriented curve.

Let R be the region enclosed by C.

Let $\vec{F} = \langle M, N \rangle$ be a vector field with continuous partial derivatives.

Then

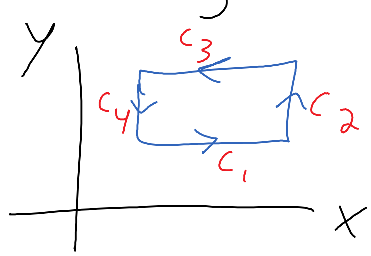
① Outward Flux

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C -N\,dx + M\,dy = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

② Circulation

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Note We use Green's Theorem to calculate line integrals for closed 2D curve.

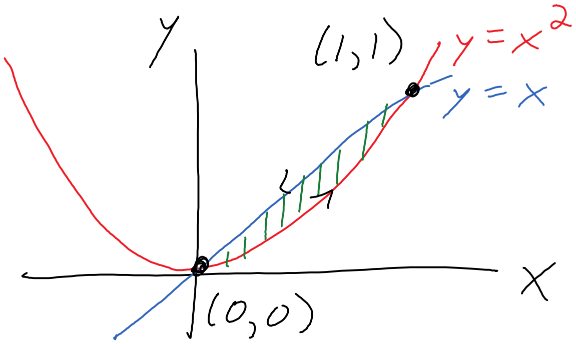


$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_{C_1} \vec{F} \cdot \vec{T}\, ds + \int_{C_2} \vec{F} \cdot \vec{T}\, ds + \int_{C_3} \vec{F} \cdot \vec{T}\, ds + \int_{C_4} \vec{F} \cdot \vec{T}\, ds$$

<u>Ex</u> Let C be the curve that bounds the region between $y=x$ and $y=x^2$.

Compute the counterclockwise circulation and outward flux of

$$\vec{F} = \langle xy, y^2 \rangle.$$



$$\vec{F} = \langle \underbrace{xy}_{M}, \underbrace{y^2}_{N} \rangle$$

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$= \int_0^1 \int_{x^2}^{x} (y + 2y\quad)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} 3y\;dy\right] dx$$

$$= \int_0^1 \left[\frac{3}{2}y^2 \Big|_{y=x^2}^{y=x}\right] dx$$

$$= \int_0^1 \left[\frac{3}{2} x^2 - \frac{3}{2} x^4 \right] dx$$

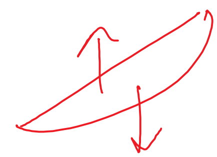
$$= \frac{1}{2} x^3 - \frac{3}{10} x^5 \Big|_{x=0}^{x=1}$$

$$= \frac{1}{2} - \frac{3}{10}$$

$$= \boxed{\frac{1}{5}}$$

Outward Flux > 0

Fluid flows out



$M = xy$

$N = y^2$

$$\text{Circulation} = \oint_C \vec{F} \cdot \vec{T}\, ds$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \int_0^1 \left[\int_{x^2}^{x} (0 - x) \, dy \right] dx$$

$$= \int_0^1 \left[-xy \Big|_{y=x^2}^{y=x} \right] dx$$

$$= \int_0^1 -x^2 + x^3 \, dx$$

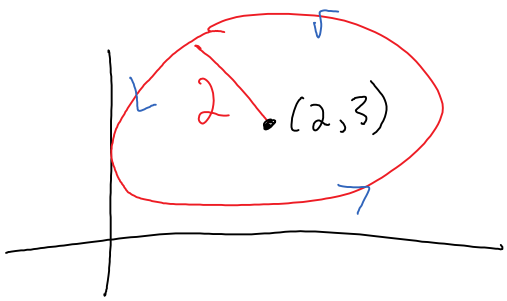
$$= -\frac{1}{3} x^3 + \frac{1}{4} x^4 \Big|_{x=0}^{x=1}$$

$$= -\frac{1}{3} + \frac{1}{4} + 0 - 0$$

$$= \boxed{-\frac{1}{12}}$$

Flow opposes motion

Ex Calculate $\oint_C (6y+x)\,dx + (y+2x)\,dy$ where C is the ccw path around the circle $(x-2)^2+(y-3)^2=4$.



$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$M = 6y+x \qquad N = y+2x$

$$\iint_R (2-6)\,dA = -4 \iint_R dA$$

$$= -4(\text{Area of } R)$$

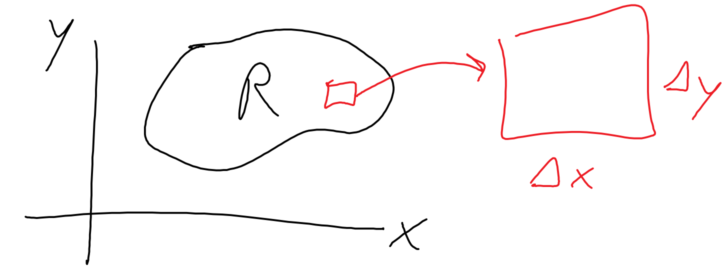
$$= -4(\pi 2^2)$$

$$= \boxed{-16\pi}$$

We just computed a line integral without doing any integration!

9.15 Triple Integrals

A double integral defines an area.



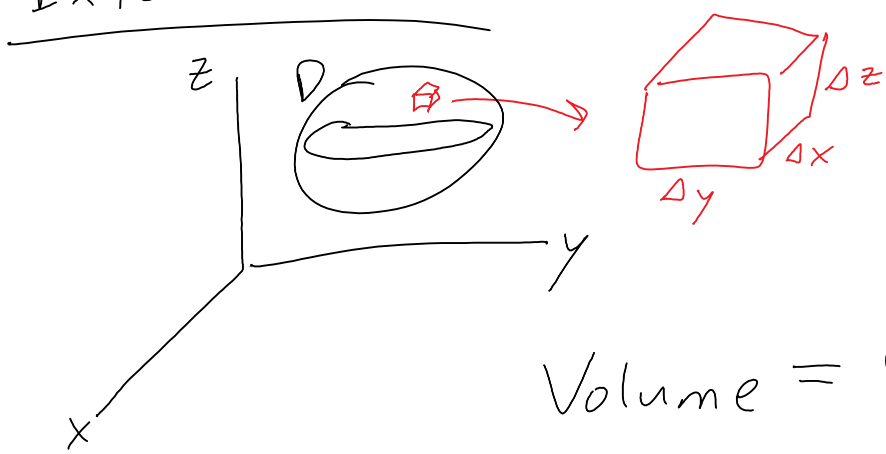
$$\text{Area} \approx \sum_x \sum_y \Delta x \Delta y$$

$$\Big\downarrow \; \Delta x \to 0,\ \Delta y \to 0$$

$$\text{Area} = \iint_R \underbrace{dA}_{\text{Area Element}}$$

Area Element
$dxdy$ OR $dydx$

Extend to 3D



$$\text{Volume} \approx \sum_x \sum_y \sum_z \Delta x \Delta y \Delta z$$

$$\Big\downarrow \; \Delta x \to 0,\ \Delta y \to 0,\ \Delta z \to 0$$

$$\text{Volume} = \iiint_D \underbrace{dV}_{\text{Volume Element}}$$

Volume Element
$dxdydz$

Ex Find volume of the region defined by
$0 \le x \le 2$, $1 \le y \le 3$, $-1 \le z \le 1$.

$$V = \iiint dV$$

$$= \int_{-1}^{1}\int_{1}^{3}\left[\int_{0}^{2} dx\right]dy\,dz$$

$$= \int_{-1}^{1}\int_{1}^{3}\left[\, x \Big|_{0}^{2}\right] dy\,dz$$

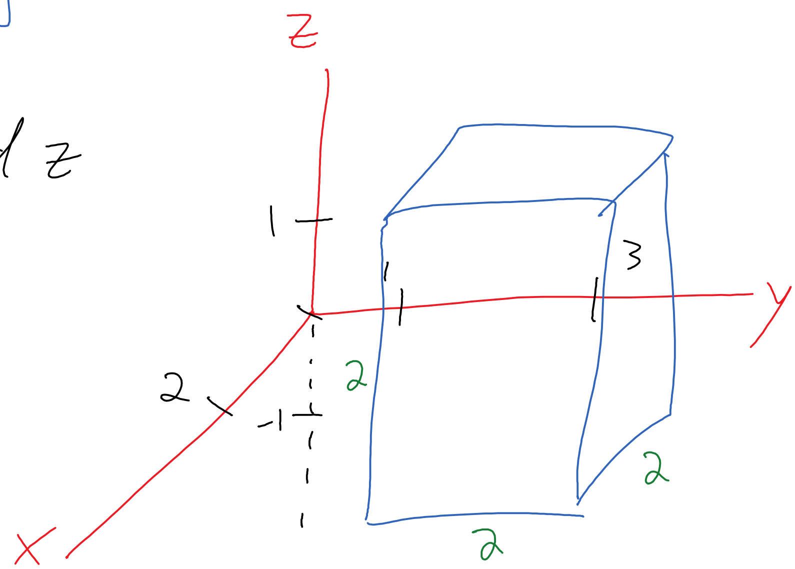
$$= \int_{-1}^{1}\left[\int_{1}^{3} 2\,dy\right]dz$$

$$= \int_{-1}^{1}\left[2y \Big|_{y=1}^{y=3}\right] dz$$

$$= \int_{-1}^{1} 4\,dz$$

$$= 4z \Big|_{-1}^{1}$$

$$= \boxed{8}$$



If the limits are all constants, then we are integrating over a box.

If the limits are not constants, choose the order of integration carefully.

$$V = \int_{a}^{b}\int_{f_1(x)}^{f_2(x)}\int_{g_1(x,y)}^{g_2(x,y)} dz\,dy\,dx$$