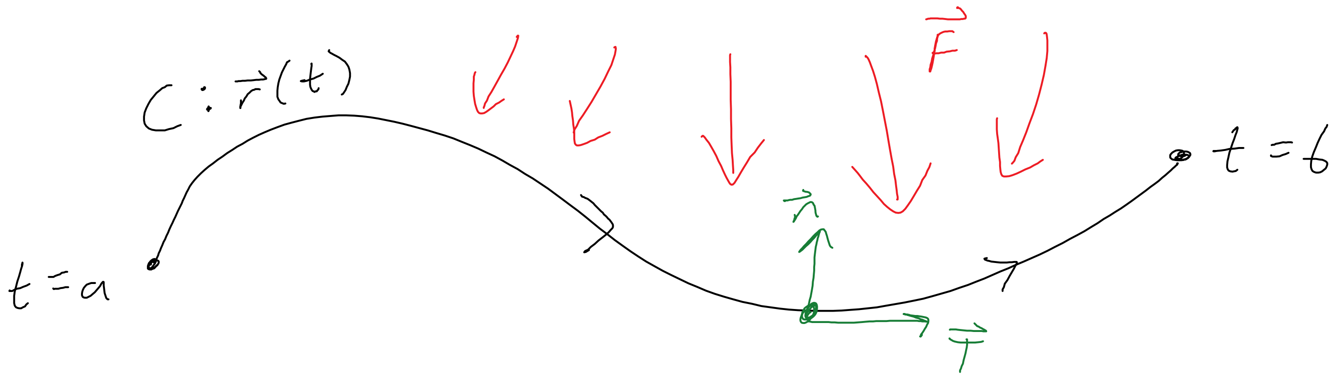
Lecture 7: Line Integrals over Vector Fields

Tepig's Goals for the Day

- Define how to compute the line integral over a vector field
- Discuss applications to flow and flux of a velocity field
- Discuss different notations for line integrals
- Define conservative vector fields and how to find potential functions

9.8 Line Integrals

Recall $\int_C f(x,y,z)\,ds = \int_a^b f(t)\,\|\vec{r}\,'(t)\|\,dt$



$\vec{F}\cdot\vec{T} > 0$ Force is in same direction as motion of object (tailwind)

$\vec{F}\cdot\vec{T} < 0$ Force opposes motion (headwind)

$\int_C \vec{F}\cdot\vec{T}\,ds$ = Total work done on object by $\vec{F}$

Line Integral over Vector Field

Curve C: $\vec{r}(t) \quad a \le t \le b$

Vector field $\vec{F}$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$$

Note Unit tangent vector $\vec{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(t) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt = \int_a^b \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$$

Notation $\int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

Applications

① $\vec{F}$ = force

$\int_C \vec{F}\cdot\vec{T}\,ds$ = work done by $\vec{F}$ on an object moving along curve C

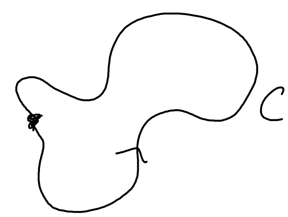
② $\vec{F}$ = velocity field of fluid

$\int_C \vec{F}\cdot\vec{T}\,ds$ = "flow"

$\int_C \vec{F}\cdot\vec{n}\,ds$ = "flux"

If C is a closed curve (starts and ends at same place), then

$\oint_C \vec{F}\cdot\vec{T}\,ds$ = "circulation"

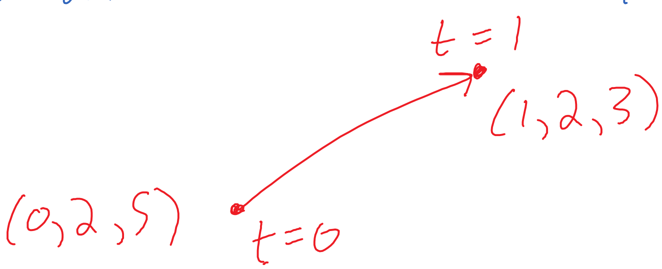


Ex Tepig swims in a straight line from (0,2,5) to (1,2,3). The ocean currents have a velocity of

$$\vec{F} = \langle x^2, yz, z-y \rangle.$$

Calculate the flow.

Parametrize the path.



$$\vec{r}(t) = \langle \underbrace{t}_{x}, \underbrace{2}_{y}, \underbrace{5-2t}_{z} \rangle \qquad 0 \le t \le 1$$

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_a^b \vec{F}(t) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle t^2, (2)(5-2t), 5-2t-2 \rangle \cdot \langle 1, 0, -2 \rangle dt$$

$$= \int_0^1 t^2 + 0 + (-2t+3)(-2)\ dt$$

$$= \int_0^1 t^2 + 4t - 6 \, dt$$

$$= \frac{1}{3}t^3 + 2t^2 - 6t \Big|_0^1$$

$$= \frac{1}{3} + 2 - 6 \quad - 0 - 0 + 0$$

$$= \boxed{-\frac{11}{3}}$$

Flow opposes the motion

<u>Note</u> In previous example

$$\int_0^1 t^2 (1dt) + [2(5-2t)](0dt) + (5-2t-2)(-2dt)$$

$$= \int_C \underline{x^2} \, \underline{dx} + \underline{yz} \, \underline{dy} + \underline{(z-y)} \, \underline{dz}$$

Components of vector field $\vec{F}$

Components of velocity $\vec{r}'(t)$

$\underline{\text{Ex}}$ Calculate $\int_C y\,dx + x^2\,dy$ where C is the curve given by

$$\vec{r}(t) = \langle \underbrace{3t}_{x}, \underbrace{2t^2}_{y} \rangle \qquad 0 \le t \le 2.$$

Think: $\int_C \vec{F}\cdot\vec{T}\,ds$ where $\vec{F} = \langle y, x^2 \rangle$

$$x = 3t \qquad\qquad y = 2t^2$$

$$dx = 3\,dt \qquad\qquad dy = 4t\,dt$$

$$\int_C y\,dx + x^2\,dy = \int_0^2 (2t^2)(3\,dt) + (3t)^2(4t\,dt)$$

$$= \int_0^2 6t^2 + 36t^3\,dt$$

$$= 2t^3 + 9t^4 \Big|_0^2$$

$$= 2(2)^3 + 9(2)^4 - 0 - 0$$

$$= 16 + 144$$

$$= \boxed{160}$$

Recall $\langle a, b\rangle \perp \langle -b, a\rangle$

Note 2D Vector Field $\vec{F} = \langle M, N\rangle$

$$\text{Flow} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy$$

$$\text{Flux} = \int_C \vec{F}\cdot\vec{n}\,ds = \int_C -N\,dx + M\,dy$$

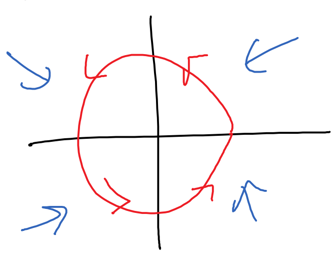
Ex Tepig swims in a circle given by

$$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle \qquad 0 \le t \le 2\pi.$$

The velocity field of the ocean is

$$\vec{F} = \langle x^2, y^3\rangle.$$

Calculate circulation and flux.



$$\text{Circulation} = \oint_C \vec{F}\cdot\vec{T}\,ds$$

$$= \oint_C x^2\,dx + y^3\,dy$$

$$= \int_0^{2\pi} (2\cos t)^2(-2\sin t\,dt)$$

$$+(2\sin t)^3(2\cos t\,dt)$$

$$= \int_0^{2\pi} -8\cos^2 t \sin t \quad + 16\sin^3 t \cos t\,dt$$

$$= \frac{8}{3}\cos^3 t + 4\sin^4 t \Big|_0^{2\pi}$$

$$= \frac{8}{3} + 0 - \frac{8}{3} - 0$$

$$= \boxed{0}$$

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\,ds$$

$$= \oint_C -y^3\,dx + x^2\,dy$$

$$= \int_0^{2\pi} -(2\sin t)^3(-2\sin t\,dt) + (2\cos t)^2(2\cos t\,dt)$$

$$= \int_0^{2\pi} 16\,\underline{\sin^4 t} + 8\,\underline{\cos^3 t}\,dt$$

Look up formulas in integration formulas.

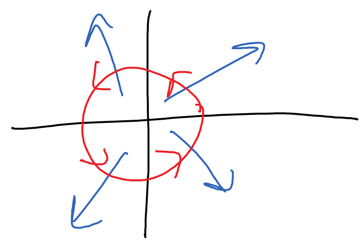
$$= 16\left[\frac{\cancel{\sin^3 t}\cos t}{4} + \frac{3}{4}\left(\frac{t}{2} - \frac{\cancel{\sin t}}{4}\right)\right]$$

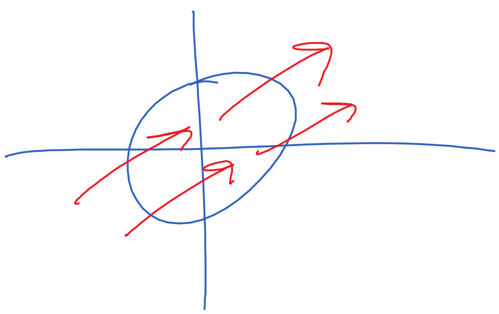
$$+ 8\left[\frac{1}{3}\cos^2 t\,\cancel{\sin t} - \frac{2}{3}\cancel{\sin t}\right]_0^{2\pi}$$

$$= 16\left[\frac{3}{4}\left(\frac{2\pi}{2}\right)\right]$$

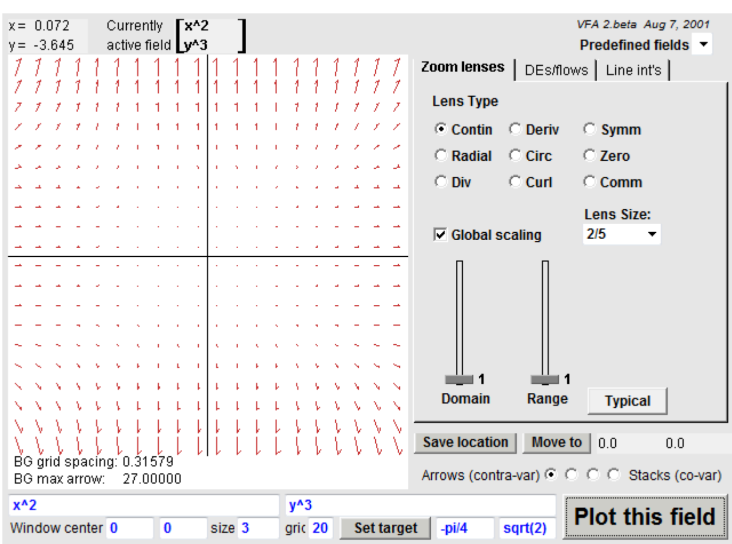
$$= \boxed{12\pi}$$

Fluid flowing outwards



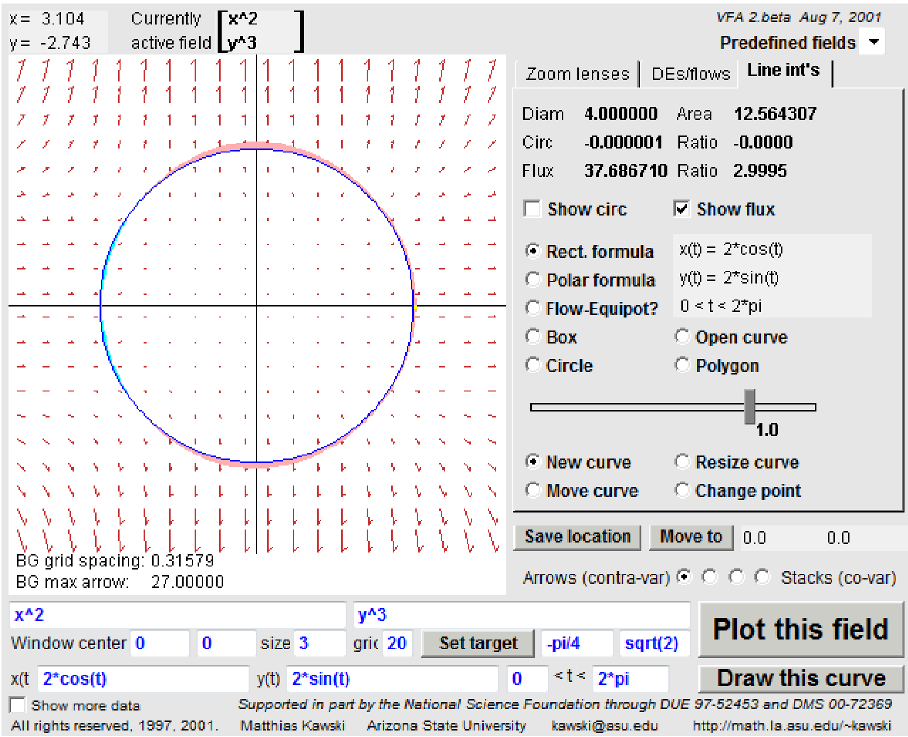


Flux = 0



Use the Vector Field Analyzer linked on our website.

Type in our vector field: x^2, y^3.
Press "Plot this field"



Under the "Line ints" tab, select "Rect formula".

Type in the curve
  x(t) = 2*cos(t)
  y(t) = 2*sin(t)

Set the limits on t to be 0 to 2*pi.

Note it calculates
  Circulation = 0
  Flux = 37.69 (or 12*pi)

Same as our answer!

9.9 Independence of the Path

Def A vector field $\vec{F}$ is conservative if there exists a potential function f such that $\nabla f = \vec{F}$.

Ex Find the vector field with potential function $f(x,y) = x^2 y - y^3$.

$$\vec{F} = \nabla f = \boxed{\langle 2xy,\ x^2 - 3y^2 \rangle}$$

Test for Conservative Vector Field

2D $\vec{F} = \langle M, N \rangle$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $\vec{F}$ is conservative.

3D $\vec{F} = \langle M, N, P \rangle$

If $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

To find a potential function, integrate each component of $\vec{F}$ w.r.t. the corresponding variable and assemble the pieces.

Ex Determine if

$$\vec{F} = \langle \overset{M}{2xy}, \overset{N}{x^2 - 3y^2} \rangle$$

is conservative and if so, find its potential function.

i.) Test for conservative.

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2x$$

Equal $\Rightarrow \vec{F}$ is conservative

ii.) Find potential function.

$$\vec{F} = \langle \underbrace{2xy}_{\frac{\partial f}{\partial x}}, \underbrace{x^2 - 3y^2}_{\frac{\partial f}{\partial y}} \rangle$$

1. $f = \int 2xy \, dx = x^2 y + g(y)$
2. $f = \int x^2 - 3y^2 \, dy = x^2 y - y^3 + h(x)$

Assemble the pieces.

$$f(x,y) = \boxed{x^2 y - y^3}$$

Potential functions may differ up to a constant.
This is one potential function that generates the field F.