# Unit 1: Calculus

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# Limits

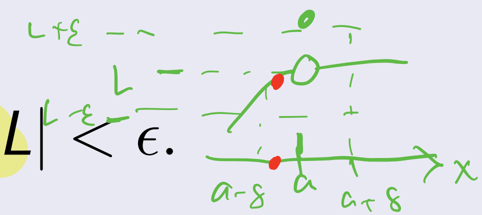
# Definition

## Definition

(a) Define $\lim_{x \to a} f(x) = L$ to mean for all $\epsilon > 0$ there exists a $\delta > 0$ such that

we don't care about f at a

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |f(x) - L| < \epsilon.$$



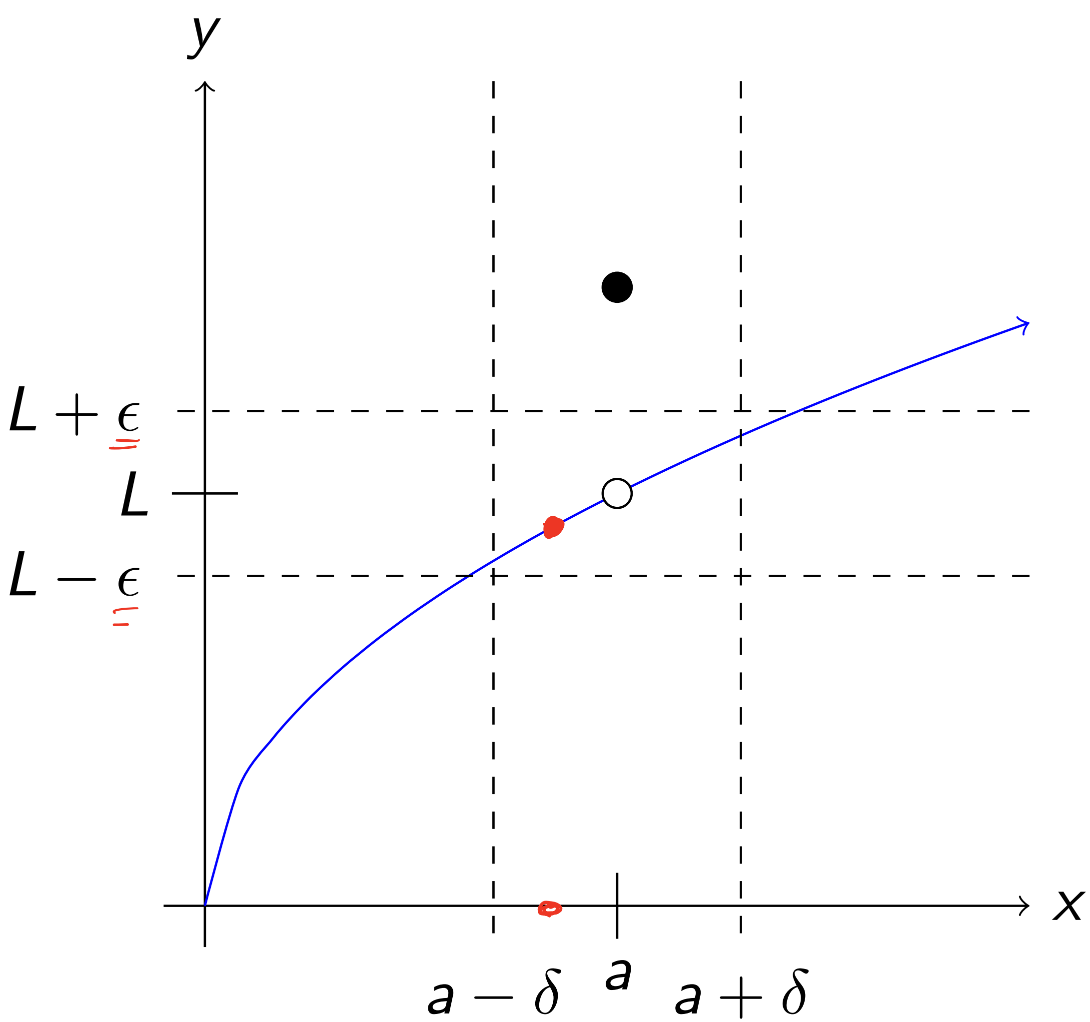

(b) Define $\lim_{x \to \infty} f(x) = L$ to mean for all $\epsilon > 0$ there exists an $N$ such that

$$x \geq N \qquad \text{implies} \qquad |f(x) - L| < \epsilon.$$

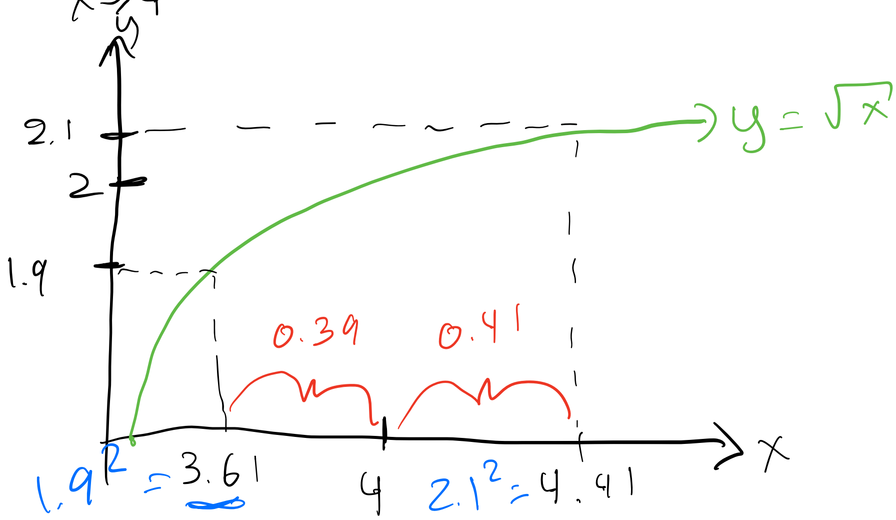
(c) Define $\lim_{x \to -\infty} f(x) = L$ to mean for all $\epsilon > 0$ there exists an $N$ such that

$$x \leq N \qquad \text{implies} \qquad |f(x) - L| < \epsilon.$$

# $\lim_{x \to a} f(x) = L$



$$\lim_{x \to 4} \sqrt{x} = 2$$




$\sqrt{3.61} = 1.9$

$\sqrt{x} = 1.9$

$\Rightarrow x = 1.9^2$

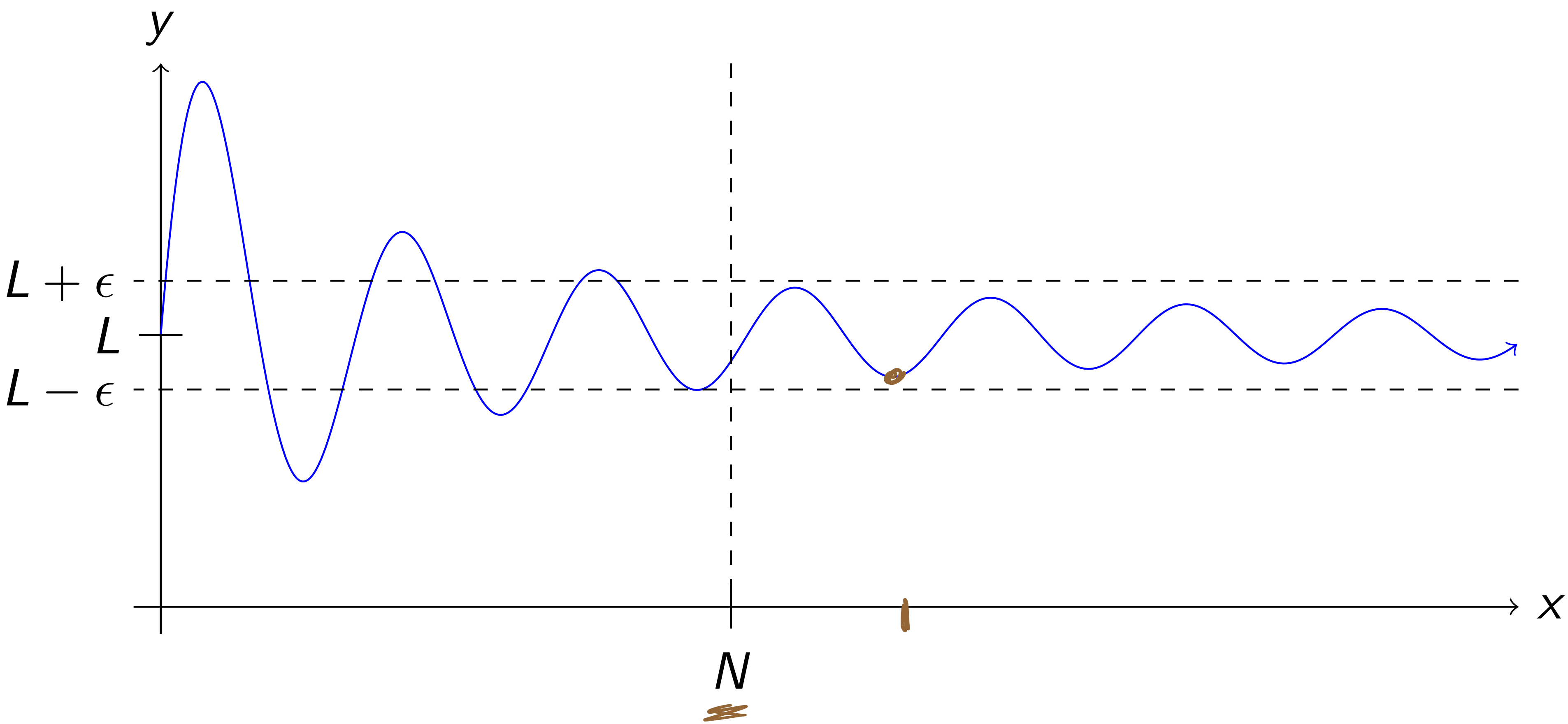
Pick $\varepsilon = 0.1$

so, $\delta = 0.39$

$$0 < |x - 4| < 0.39 \implies |\sqrt{x} - 2| < 0.1$$

A smaller $\delta$ would also work, e.g. in this case $\delta = 0.01$.

## $\lim_{x \to \infty} f(x) = L$



# Using the Definition

## Example

Prove

$$\lim_{x \to \infty} \frac{1}{x^p} = 0$$

for all $p > 0$.

Sol The definition $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$x \geq N \implies \left| \frac{1}{x^p} - 0 \right| < \varepsilon.$$

Pick $\varepsilon > 0$. Then Consider

$$\left| \frac{1}{x^p} - 0 \right| = \left| \frac{1}{x^p} \right| = \frac{1}{|x|^p} < \varepsilon$$

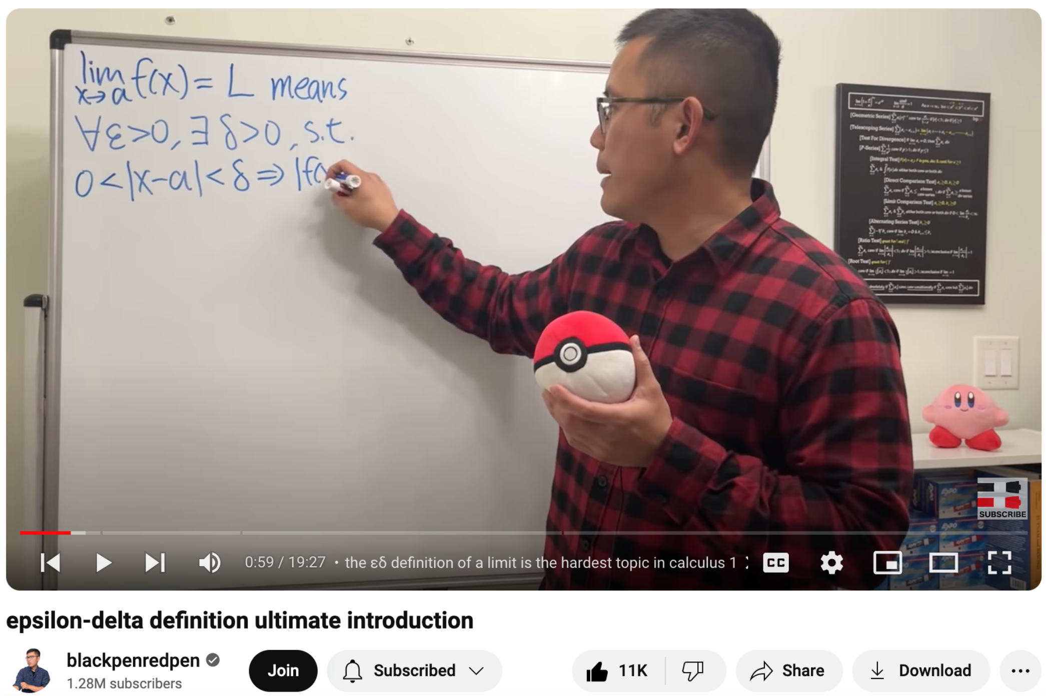
$$\iff \frac{1}{\varepsilon} < |x|^p \iff \sqrt[p]{1/\varepsilon} < |x|$$

Let $N = \left\lceil \sqrt[p]{\frac{1}{\varepsilon}} \right\rceil$. ✓

So, we've come up with a rule, You give me $\varepsilon > 0$ and I can give you an $N$ s.t. $x \geq N \Rightarrow \left| \frac{1}{x^p} \right| < \varepsilon$

# $\epsilon$-$\delta$ Limit Definition on YouTube

Watch BlackPenRedPen explain the $\epsilon$-$\delta$ limit definition (https://www.youtube.com/watch?v=DdtEQk_DHQs). There's another video where he goes over the $x \to \infty$ case (https://youtu.be/9JMFLzHtljA?si=1WPW-fmaf2DBe3Ph).



# Properties of Limits

## Theorem

*Suppose a is in the interval* $[-\infty, \infty]$. *Let*

$$\lim_{x \to a} f(x) = L_1 \qquad \textit{and} \qquad \lim_{x \to a} g(x) = L_2.$$

(a) $\lim_{x \to a} \alpha f(x) + \beta g(x) = \alpha L_1 + \beta L_2$ *for any real constants* $\alpha$ *and* $\beta$

(b) $\lim_{x \to a} f(x) \cdot g(x) = L_1 \cdot L_2$

(c) $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L_1}$ *if* $L_1 \neq 0$.

# Useful Limits

**Theorem**

*Suppose $p > 0$.*

(a) $\lim_{x\to\infty} \frac{1}{x^p} = 0$

(b) $\lim_{x\to\infty} p^{1/x} = 1$

(c) $\lim_{x\to\infty} x^{1/x} = 1$

(d) *If* $a > 0$, $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

(e) *If* $|r| < 1$, *then* $\lim_{x\to\infty} r^x = 0$

# Python Example

## Example

Define

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Graph $f$ in Python to see that $\lim_{x \to 0} f(x)$ does not exist.

# Python Example Cont.

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use latex
plt.rcParams['text.usetex'] = True

# Use Seaborn style
plt.style.use('seaborn')

# Define f
f = lambda x: 0 if x == 0 else np.sin(1/x)

# Let's graph on the interval [-pi, pi]
x_vals = np.arange(-np.pi, np.pi, np.pi/200)

# Calculate the y-values
y_vals = [f(x) for x in x_vals]

# Generate the plot
plt.plot(x_vals, y_vals)

# Label the x-axis
plt.xlabel(r'$x$')

# Label the y-axis
plt.ylabel(r'$y$')

# Give the graph a title
plt.title(r'Graph of $y = f(x)$')

# Save the figure
plt.savefig(path + r'Images/ex1-1.png')

# Display the plot
plt.show()
```

By default in radians

```
def f(x):
  → If x == 0:
    → return 0
  else:
    → return np.sin(1/x)
```

start stop

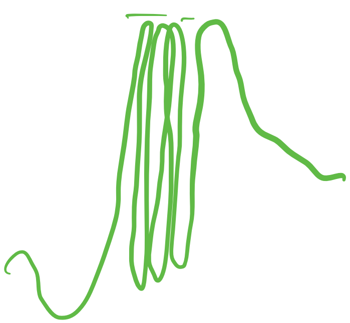
np.arange(_, _, _)

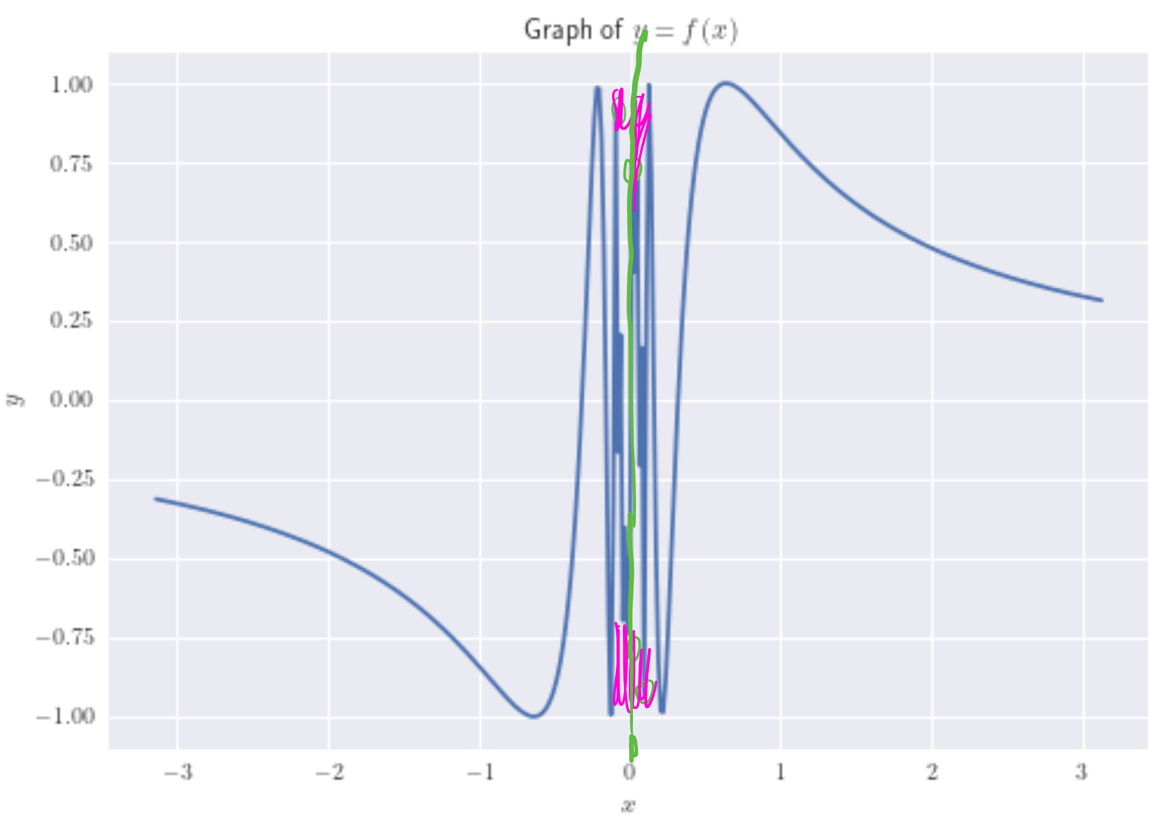
step size

# Python Example Result

The graph isn't perfect, but it's enough to see that $f$ doesn't approach anything in particular as $x$ approaches 0.

$\lim_{x \to 0} f(x) = \text{DNE} \leftarrow$ does not exist





# Continuity

# Continuity

**Definition**

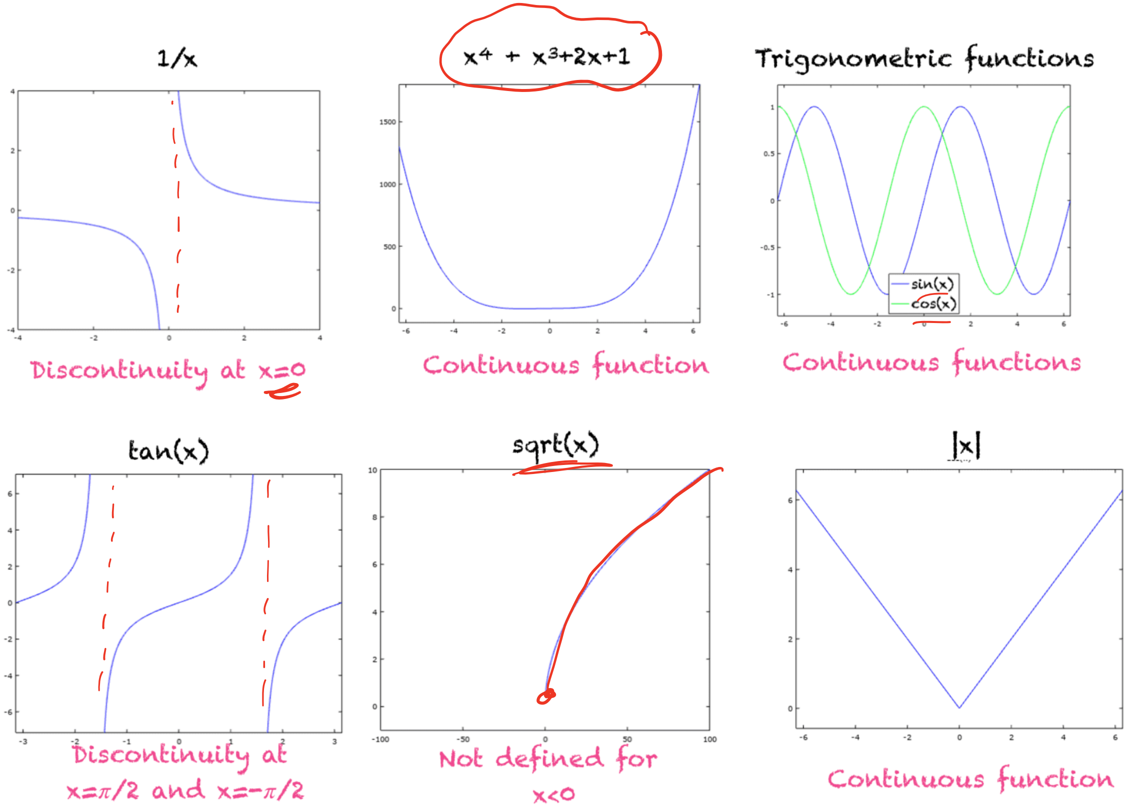
(a) A function $f$ is continuous at $a$ if $\lim_{x \to a} f(x) = f(a)$.

(b) A function $f$ is continuous on the set $A$ if

$$a \in A \quad \text{implies} \quad \lim_{x \to a} f(x) = f(a).$$

# Continuity Idea

Continuous functions have no breaks, i.e. if you were to draw them you would never need to lift your pencil.




$X = \frac{k\pi}{2}$, where k odd

https://machinelearningmastery.com/continuous-functions/

# Useful Theorem

## Theorem

*If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then*

$$\lim_{x \to a} f\Big(g(x)\Big) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

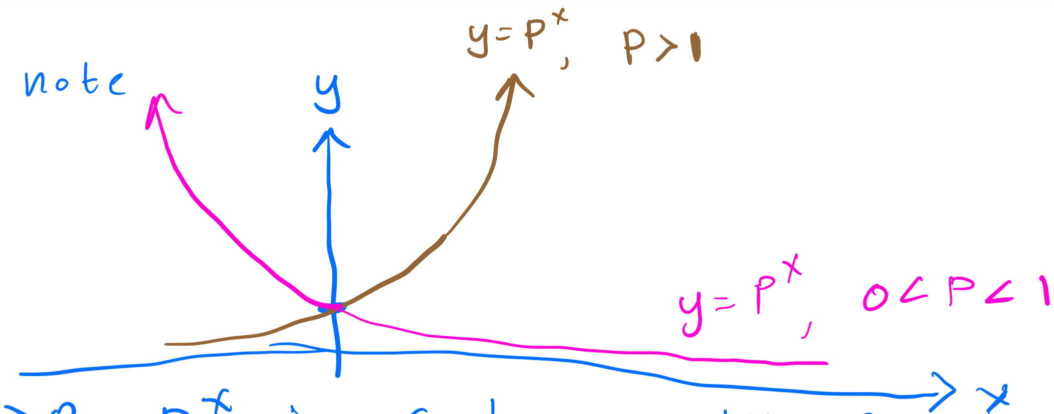
$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

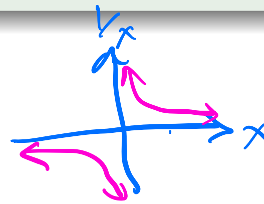
# Example

## Example

Suppose $p > 0$. Compute $\lim_{x \to \infty} p^{1/x}$.

Sol First, note




Note: For $p > 0$, $p^x$ is cont. everywhere. So, ...

$$\lim_{x \to \infty} p^{1/x} = p^{\lim_{x \to \infty} 1/x} = p^0 = 1.$$

# Derivatives

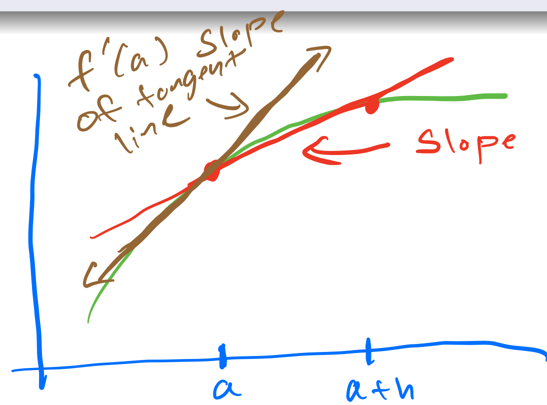
# Derivatives

## Definition

(a) The **derivative of a function *f*** at a number $a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(b) The function $f$ is differentiable on a set $A$ if $f'(a)$ exists for all $a$ in $A$.




$$\frac{f(a+h) - f(a)}{h}$$

# Notation

Leibniz notation is frequently used:

$$\frac{df}{dx} = f'(x) \qquad \text{and} \qquad \left.\frac{df}{dx}\right|_{x=a} = f'(a).$$

The respective second and third derivatives are written

$$\frac{d^2f}{dx^2} = f''(x) \qquad \text{and} \qquad \frac{d^3f}{dx^3} = f'''(x)$$

For the $k$-th derivatives, where $k > 3$, we use the notation

$$\frac{d^kf}{dx^k} = f^{(k)}(x).$$

# Derivatives Example

## Example

Let

$$f(x) = \begin{cases} xe^{-x^2 - x^{-2}}, & x \neq 0 \\ \underline{0}, & x = 0. \end{cases}$$

Compute $f'(0)$.

Sol

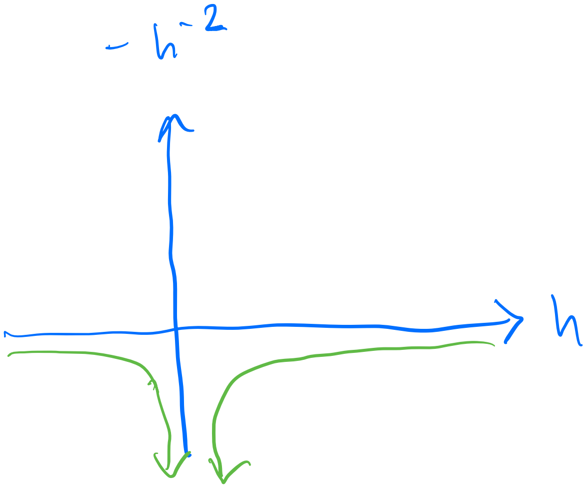
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\cancel{h} e^{-h^2 - h^{-2}} - 0}{\cancel{h}}$$

$$= \lim_{h \to 0} e^{-h^2 - h^{-2}} = e^{-h^2} \cdot e^{-h^{-2}}$$

$$= \lim_{h \to 0} e^{-h^2 \to -0^2} \cdot \lim_{h \to 0} e^{-h^{-2} \to -\infty}$$

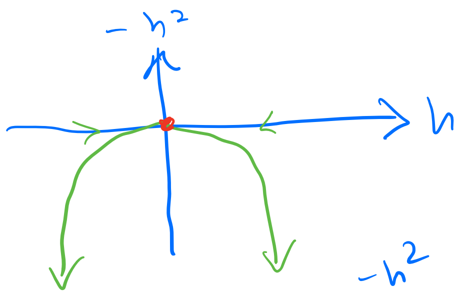
$$= 1 \cdot e^{\lim_{h \to 0} -h^{-2}} \quad -\infty$$

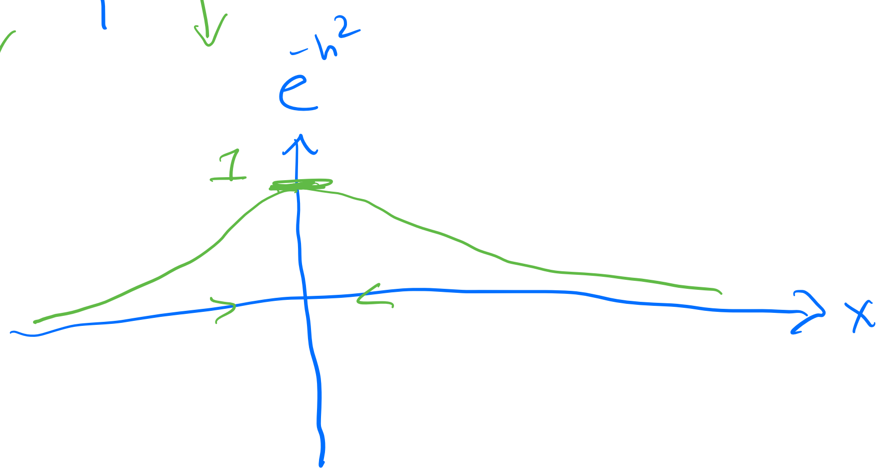
$$= 1 \cdot 0 = \boxed{0}$$




$$\lim_{h\to 0} e^{-h^2} \overset{?}{=} 1$$

$$\lim_{h\to 0} e^{-h^2} = e^{\lim_{h\to 0} -h^2} = e^0 = 1$$



# Numerical Approximation

It is often helpful to numerically approximate $f'$. This can be done by choosing a small value of $h$ and calculating

$$\frac{f(x+h)-f(x)}{h}.$$

The value $h$ can be positive or negative. Often, a better numerical approach is to consider positive and negative values of $h$ at the same time and take the average:

Tends to converge to $f'$ a little faster

$$\frac{1}{2}\cdot\frac{f(x+h)-f(x)}{h}+\frac{1}{2}\cdot\frac{f(x-h)-f(x)}{-h}=\frac{f(x+h)-f(x-h)}{2h}.$$

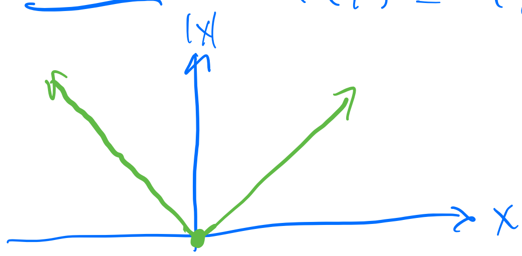
where $h>0$.

Note

$$\frac{f(x+h) - f(x-h)}{2h} \quad h>0$$

tends to go to $f'$ faster when $f'$ exists. But the ratio something odd if $f'$ doesn't exist.

Example $f(x) = |x|$




$f'(0)$ DNE

But

$$\frac{f(0+h) - f(0-h)}{2h}$$

$$= \frac{|h| - |-h|}{2h}$$

$$= \frac{h-h}{2h}$$

$$= 0$$

Since $f'(0)$ doesn't exist, we bad answer

# Python Example

## Example

Use Python to graph $f$ and $f'$ on the interval $[-2, 2]$, where

$$f(x) = \begin{cases} xe^{-x^2 - x^{-2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Use a numerical approximation for $f'$ with $h = 0.001$.

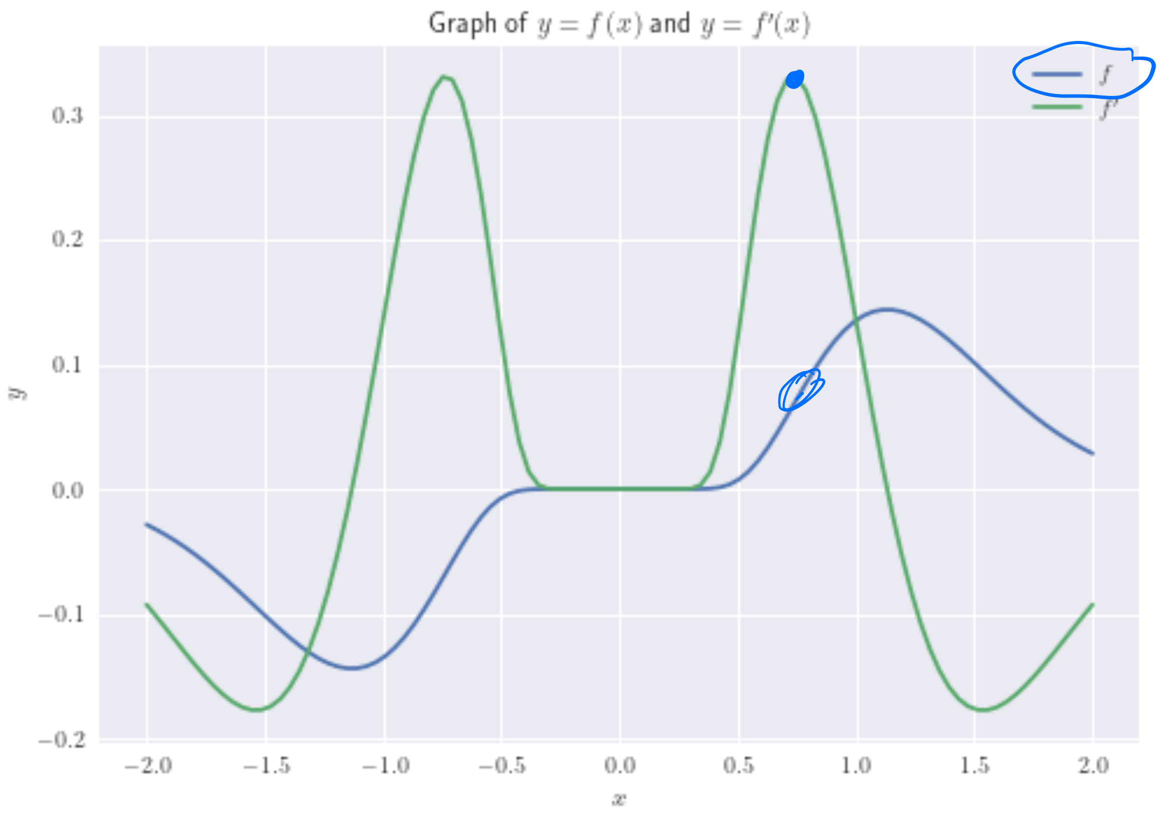
# Python Example Cont.

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use latex
plt.rcParams['text.usetex'] = True

# Use Seaborn style
plt.style.use('seaborn')

# Define f
f = lambda x: x * np.exp(-x**2 - x**-2) if
    x != 0 else 0

# Define h
h = 0.001

# Use numerical approximation
f_prime = lambda x: (f(x + h) - f(x - h))
    /(2 * h)

# Get the x-values
x_vals = np.linspace(-2, 2, 100)

# Get the two sets of y-values
y1_vals = [f(x) for x in x_vals]
y2_vals = [f_prime(x) for x in x_vals]
```

means not equal

```
# Generate the plot for f
plt.plot(x_vals, y1_vals, label = r"$f$")

# Generate the plot for f'
plt.plot(x_vals, y2_vals, label = r"$f'$")

# Label the x-axis
plt.xlabel(r'$x$')

# Label the y-axis
plt.ylabel(r'$y$')

# Give the graph a title
plt.title(r"Graph of $y = f(x)$ and $y = f
    '(x)$")

# Create a legend
plt.legend()

# Save the figure
plt.savefig(path + r'Images/ex1-2.png')

# Display the plot
plt.show()
```

# Python Example Result



# Derivative Properties

## Theorem

*Suppose $\alpha$ and $\beta$ are constants and $f'$ and $g'$ exist.*

(a) $\dfrac{d}{dx}(\alpha f + \beta g) = \alpha f' + \beta g'$.

(b) $\dfrac{d}{dx}(f \cdot g) = g \cdot f' + f \cdot g'$ ← Product rule

(c) $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{f' \cdot g - f \cdot g'}{g^2}$ ← quotient rule

(d) $\dfrac{d}{dx}(f \circ g) = (f' \circ g) \cdot g'$ ← Chain rule

$$\frac{d}{dx}\left(f(u)\right) = f'(u) \cdot \frac{du}{dx}$$

# Useful Derivative Formulas

Suppose $a > 0$.

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \ x \neq 0$
- $\frac{d}{dx}(\log_a|x|) = \frac{1}{x \ln a}, \ x \neq 0$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

# Derivative Example

## Example

$$\frac{d}{dx}\left(e^{1/x}\sin x\right) = \frac{d}{dx}\left(e^{x^{-1}}\sin x\right)$$

We will need

- Product rule
- Chain rule
- Power rule

$$\frac{d}{dx}\left(e^{x^{-1}}\right) = e^{+x^{-1}}\cdot\left(-1\cdot x^{-1-1}\right) = e^{-x}\cdot\left(-x^{-2}\right)$$

**Sol**

$$\frac{d}{dx}\left(e^{1/x}\sin x\right) = \frac{d}{dx}\left(e^{1/x}\right)\cdot\sin x + e^{1/x}\cdot\frac{d}{dx}\left(\overbrace{\sin x}^{\cos x}\right)$$

$$= -\frac{1}{x^2}e^{1/x}\sin x + e^{1/x}\cos x$$

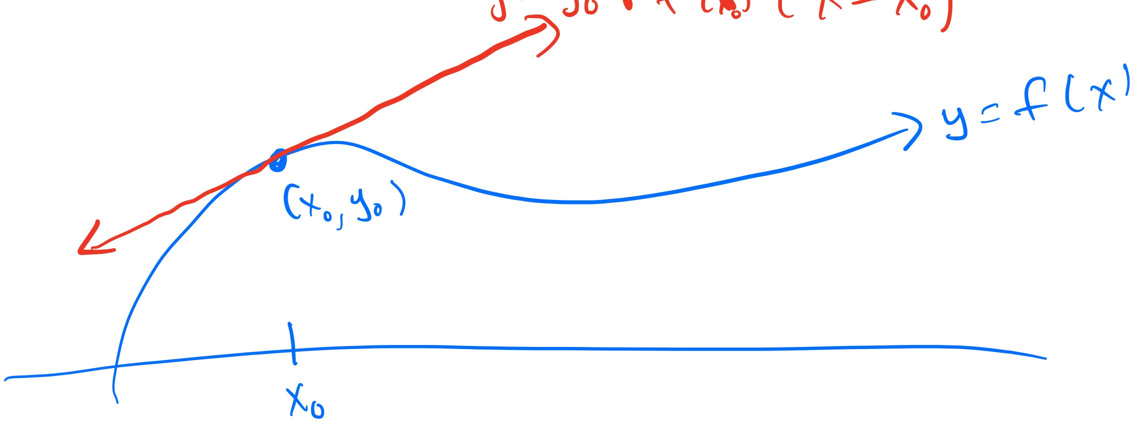
$$\boxed{= \left(\cos x - \frac{\sin x}{x^2}\right)e^{1/x}}$$

# Tangent Lines

The tangent line of the graph of $y = f(x)$ at $(x_0, y_0)$ is

$$\boxed{y = y_0 + f'(x_0)(x - x_0).}$$

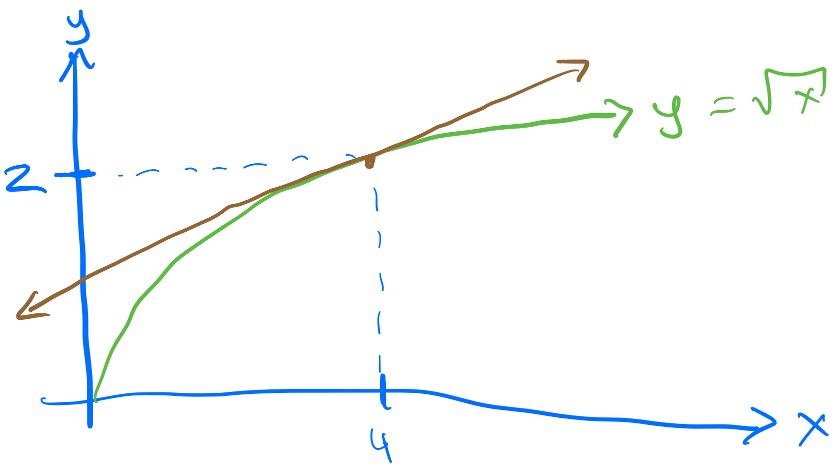
tangent line
$y = y_0 + f'(x_0) \cdot (x - x_0)$



# Tangent Line Example

## Example

Approximate $\sqrt{3.9}$.




- Calculate the tangent line at $x = 4$
- Plug in 3.9

This should give us a value close to $\sqrt{3.9}$

Sol $(x_0, y_0) = (4, 2)$; $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2} x^{1/2 - 1}$

$= \frac{1}{2} x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \frac{1}{4}$$

Hence, our tangent line is

$$y = 2 + \frac{1}{4}(x-4)$$
$$= 2 + \frac{1}{4}x - 1$$
$$= 1 + \frac{1}{4}x.$$

good approx of $\sqrt{x}$ for $x$ close to 4

$$\Rightarrow \quad \sqrt{3.9} \approx 1 + \frac{1}{4}\cdot 3.9 = 1 + 0.975$$
$$= 1.975$$

$$\begin{array}{r} .975 \\ 4\overline{)3.900} \\ -36 \\ \hline 30 \\ -28 \\ \hline 20 \end{array}$$

**Check:** $\sqrt{3.9} \approx 1.974841766\ldots$

# Local and Global Extrema

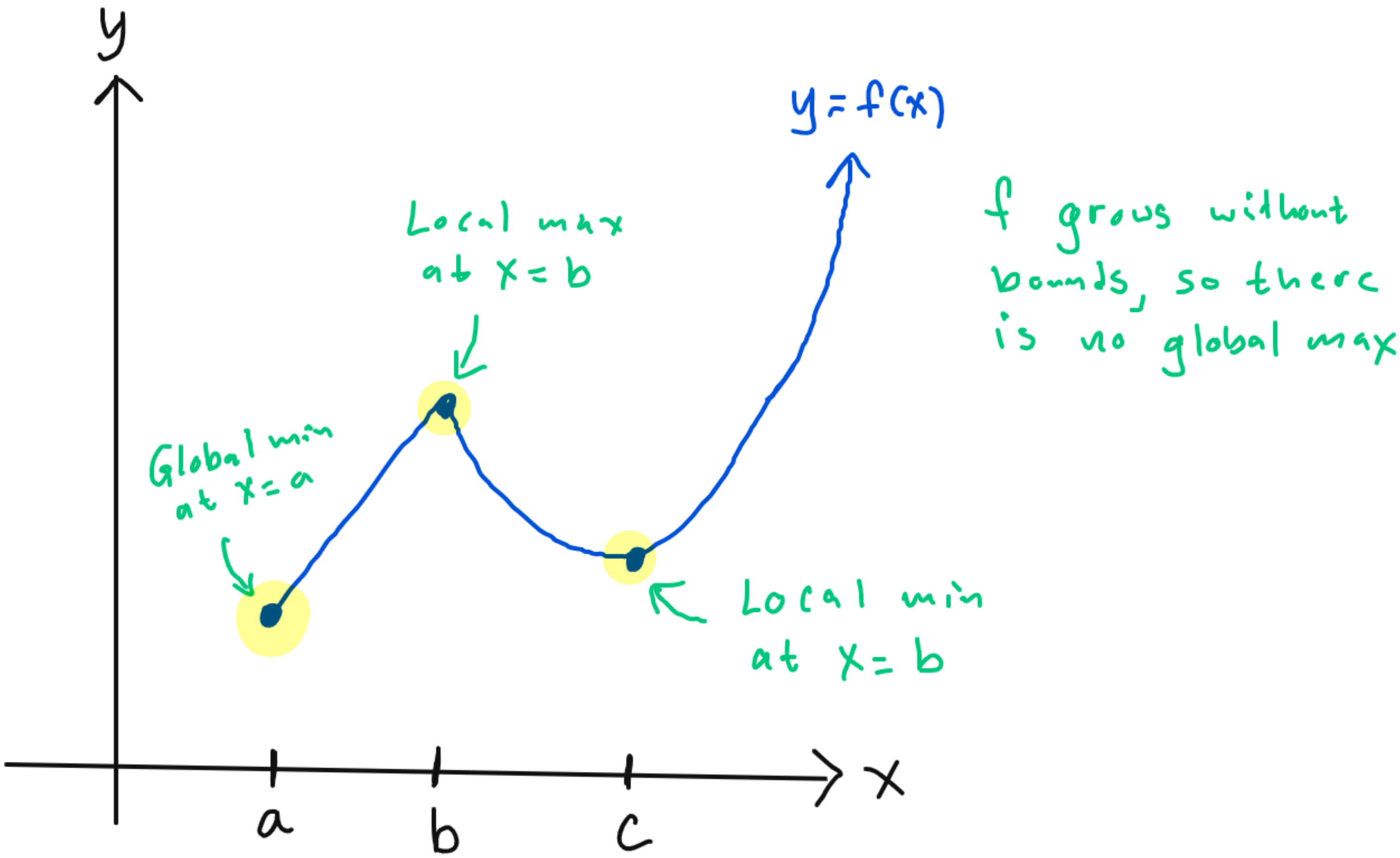
## Definition

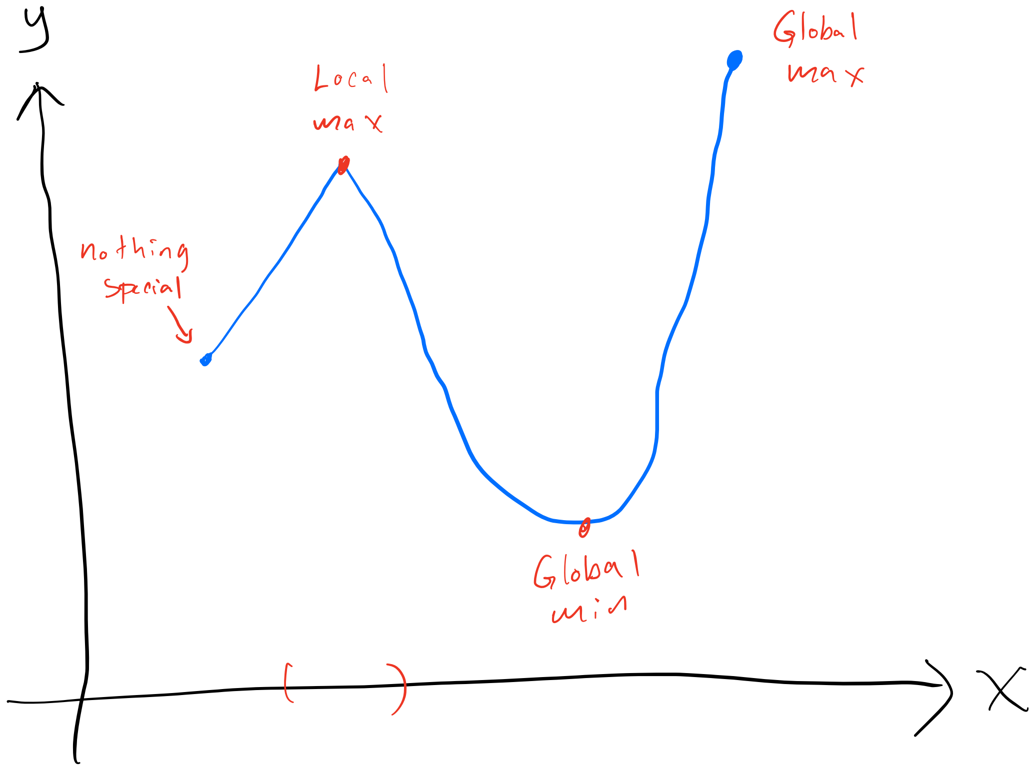
Let $f$ be a function defined on domain $D$.

(a) The **global maximum** of $f$ is at $c$ if $f(c) \geq f(x)$ for all $x$ in $D$. The **global minimum** of $f$ is at $c$ if $f(c) \leq f(x)$ for all $x$ in $D$. The global maximum and global minimum values of $f$ are called the **global extrema** of $f$.

(b) A **local maximum** of $f$ is at $c$ if there is an interval $(a, b)$ such that $f(c) \geq f(x)$ for all $x$ in $(a, b)$ and $a < c < b$. Similarly, a **local minimum** of $f$ is at $c$ there is an interval $(a, b)$ such that $f(c) \leq f(x)$ for all $x$ in $(a, b)$ and $a < c < b$. The local maximum and local minimum values of $f$ are called the **local extrema** of $f$.

# Local and Global Extrema Example




y
Local
max
Global
max
nothing
special
Global
min
x

# Extreme Value Theorem

**Theorem (Extreme Value Theorem)**

*If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains a global maximum value $f(c)$ and a global minimum value $f(d)$ for some numbers $c$ and $d$ in $[a, b]$.*

# Increasing and Decreasing Functions

For $f$ differentiable.

- If $f'(x) > 0$, then $f$ is increasing at $x$.
- If $f'(x) < 0$, then $f$ is decreasing at $x$.

Increasing means: $f(x+h) - f(x) > 0$ for $h > 0$ small.

$$\Rightarrow \quad \frac{f(x+h) - f(x)}{h} > 0$$

$$\Rightarrow \quad f'(x) \geq 0$$

Decreasing means: $f(x+h) - f(x) < 0$ for $h > 0$ small

$$\Rightarrow \quad \frac{f(x+h) - f(x)}{h} < 0$$

$$\Rightarrow \quad f'(x) \leq 0$$

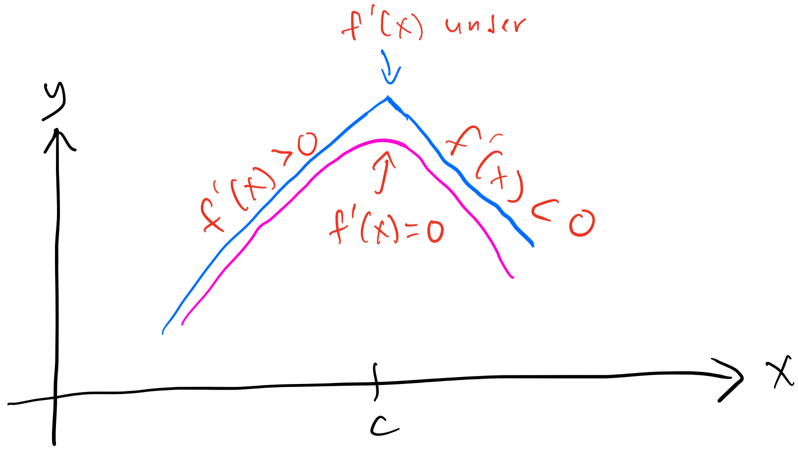
# First Derivative Test

Suppose $x = c$ is a critical number, i.e. $c$ is in the domain of $f$ and $f'(c)$ is 0 or undefined.

(a) If $f'$ changes from positive to negative at $x = c$, then $f$ has a local maximum at $x = c$.

(b) If $f'$ changes from negative to positive at $x = c$, then $f$ has a local minimum at $x = c$.

(c) If $f'$ does not change sign at $x = c$, then $f$ has no local extremum at $x = c$.




Local max @ $x = c$ since $f'$ changed from positive to negative at $x = c$.

Same idea for a local min.

# Optimization Example

## Example

Find the local and global extrema of the function $f(x) = x^3(x-2)^2$. Suppose the domain of $f$ is the closed interval $[-1, 3]$.

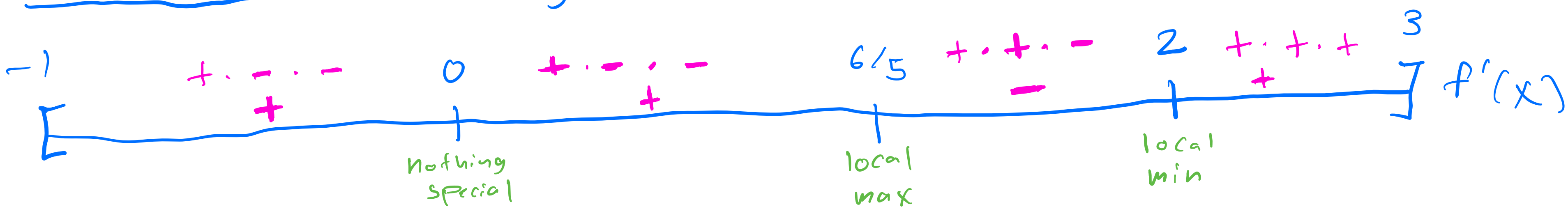
Observation: Global extrema occur at endpoints or local extrema

To find local extrema, I'll use the first deriv test.

Sol $f(x) = x^3(x-2)^2 \Rightarrow f'(x) = 3x^2(x-2)^2 + x^3 \cdot 2(x-2)^1 \cdot 1$

$$= 3x^2(x-2)^2 + 2x^3(x-2)$$
$$= x^2\big(3(x-2) + 2x\big)(x-2)$$
$$= x^2(5x-6)(x-2)$$

Critical numbers: $x = 0, 6/5, 2$



$f(x) = x^3(x-2)^2$

$f(-1) = (-1)^3(-3)^2 = -1 \cdot 9 = -9$

$f(6/5) = (6/5)^3 (\underbrace{6/5 - 2}_{-4/5})^2 = \frac{216}{125} \cdot \frac{16}{25} \approx 1.10592$

$f(2) = 2^3 \cdot (2-2)^2 = 0$

$f(3) = 3^3 \cdot (3-2)^2 = 27$

$$\boxed{\begin{array}{l} \text{Global min of } -9 \text{ @ } x=-1 \\ \text{Global max of } 27 \text{ @ } x=3 \\ \text{Local max of } \sim 1.10592 \text{ @ } x=6/5 \\ \text{Local min of } 0 \text{ @ } x=2 \end{array}}$$

Suppose $f'(c) = 0$.

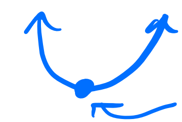
(a) If $f''(c) > 0$, then $f$ has a local minimum at $x = c$.

(b) If $f''(c) < 0$, then $f$ has a local maximum at $x = c$.

(c) If $f''(c) = 0$ or is undefined, then the test fails.

Concave up

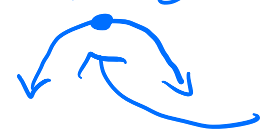
$\Rightarrow f'' > 0$



local min

Concave down

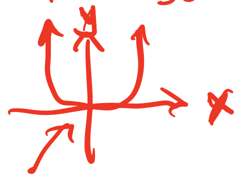
$f'' < 0$



local max

The test fails in some obvious cases.

$f(x) = x^4$



local min

$f'(x) = 4x^3$ ← easy to see crit. at $x = 0$

$\Rightarrow f''(x) = 12x^2$

$\Rightarrow f''(0) = 0$ ← test fails :(

# Local Extrema Example

## Example

Find all local extrema of $g(x) = x^4 - 4x^3$.

Sol $g(x) = x^4 - 4x^3 \Rightarrow g'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

The crit. values are when $g'(x) = 0$. So, when

$$4x^3 - 12x^2 = 0$$
$$\Rightarrow 4x^2(x-3) = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x = 3$$

Diff. again to find whether these are local mins or maxs:

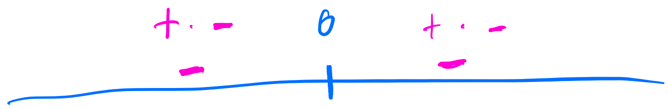
$$g''(x) = 12x^2 - 24x$$

$g''(0) = 0$

The test failed, let's use first deriv. test

$g''(3) = 12 \cdot (3)^2 - 24(3) = 108 - 72 = 36$

Since $g''(3) > 0$, local min at $x = 3$.




Nothing special at $x=0$

Let's plug 3 into $g$, so we can find y-value of local min

$$g(3) = 3^4 - 4(3)^3$$
$$= 81 - 4 \cdot 27$$
$$= 81 - 108$$
$$= -27$$

Local min of $-27$ @ $x = 3$.
No local max.

# L'Hôpital's Rule

## Theorem (L'Hôpital's Rule)

*Suppose $f$ and $g$ are differentiable on the open interval $(a, b)$ and $g'(x) \neq 0$ for all $x$ in $(a, b)$, where $a \leq \cancel{x} < b \leq \infty$. If*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

*and either*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \qquad \textit{or} \qquad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty,$$

*then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

## Example

Prove $\lim_{x \to \infty} x^{1/x} = 1$.

Sol

$\lim_{x\to\infty} x^{1/x} \to \infty^0$ ← not a ratio, no so clear

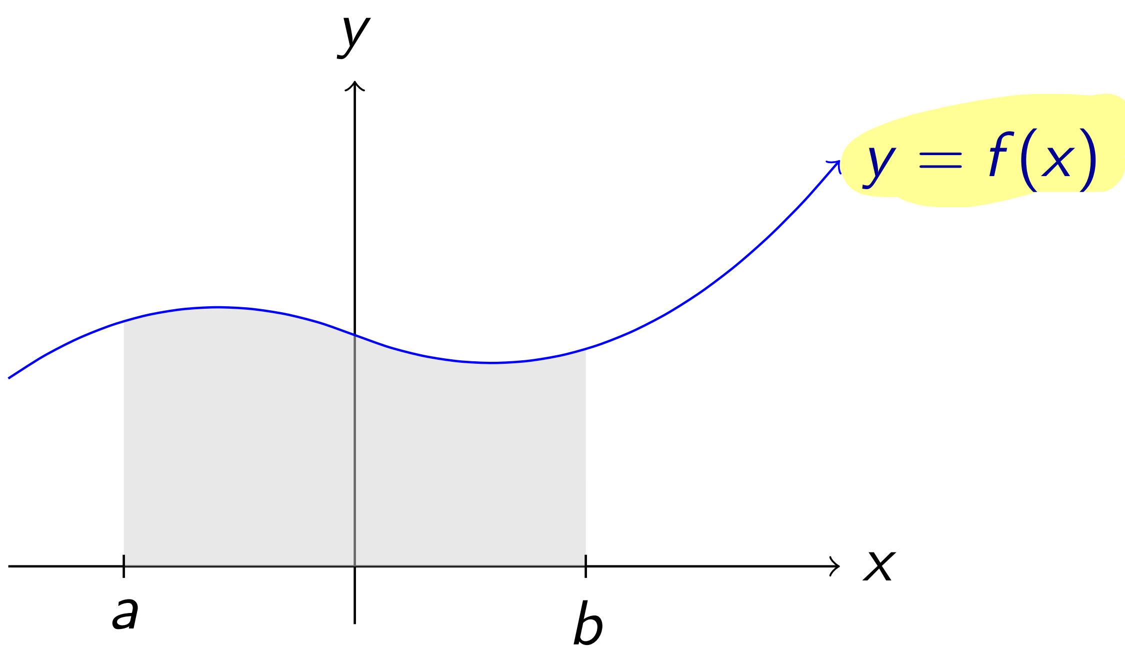
Note: $e^{\ln x} = x$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} (e^{\ln x})^{1/x} = \lim_{x\to\infty} e^{\ln x/x}$$

$$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \stackrel{LH}{=} e^{\lim_{x\to\infty} \frac{1/x}{1}} = e^{\lim_{x\to\infty} 1/x} = e^0 = \boxed{1}$$

# Integration

# Definite Integration




The motivating problem for the definite integral is finding area under the graph $y = f(x)$ for $a \leq x \leq b$.

# Riemann Sum

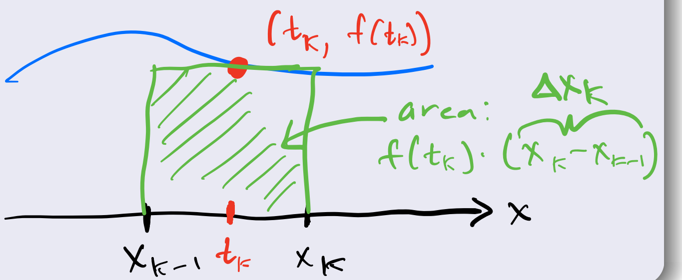
## Definition

Suppose we have a function $f$ defined on the interval $[a, b]$. Consider a **partition pair** $P$ and $T$; $P = (x_0, x_1, \dots x_n)$ and $T = (t_1, t_2, \dots, t_n)$, where

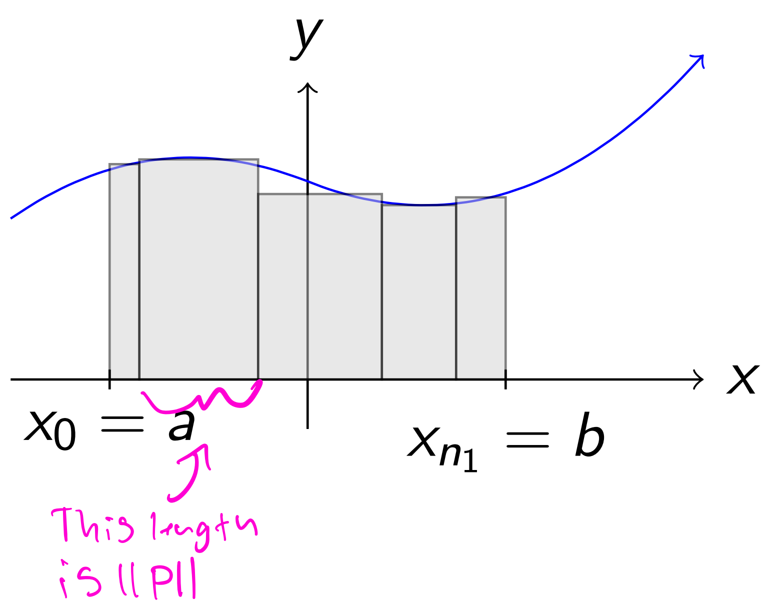
$$a = x_0 \le t_1 \le x_1 \le \dots \le x_{n-1} \le t_n \le x_n = b.$$

The **Riemann sum** corresponding to the partition pair $P$ and $T$ is defined to be

$$\sum_{k=1}^{n} f(t_k)\Delta x_k.$$



# Finer and Finer Partition




The rectangles in the right figure do a better job of approximating the area under the curve. This is because the partition in the second figure is "finer", i.e. uses more $x$-values, than the first.

# Partition Mesh

**Definition**

The **mesh** of partition $P$ is

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots \Delta x_n\}.$$

As $\|P\| \to 0$, the approximation becomes better and better.

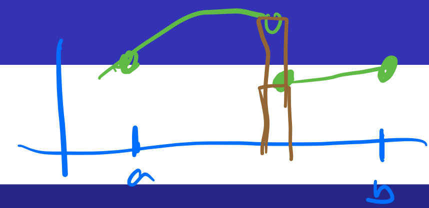
## Definition

The **Riemann integral** of $f$ over the interval $[a, b]$ is

$$\int_a^b f(x)\ dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(t_k) \Delta x_k$$

whenever the limit converges.

# Integral Theorems




## Theorem

*A bounded function on an interval $[a, b]$ is Riemann integrable if it is continuous for all but a finite number of points.*

## Theorem

*If $f$ and $g$ are Riemann-integrable on $[a, b]$ and $\alpha$ and $\beta$ are constants, then the following hold.*

(a) $\displaystyle \int_a^b \alpha f(x) + \beta g(x)\ dx = \alpha \int_a^b f(x)\ dx + \beta \int_a^b g(x)\ dx$

(b) $\displaystyle \int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ *for any $c$ in $[a, b]$*

# Analytic Example

## Example

Prove $\int_1^a \frac{dx}{x} = \ln a$ for $a > 1$. Use partition pairs of the form

$P = (1, a^{1/n}, a^{2/n} \ldots, a)$ and $T = (1, a^{1/n}, a^{2/n} \ldots, a^{(n-1)/n})$.

(n+1)-values  n-values  left-end points

Sol
Let's compute the Riemann sum:

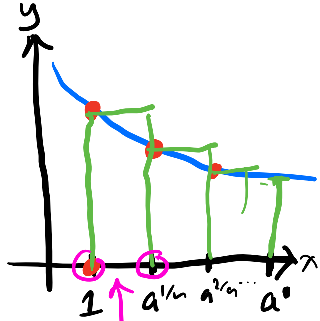
$$\sum_{k=1}^{n} f(t_k)\Delta x_k = \sum_{k=1}^{n} \frac{1}{a^{(k-1)/n}} \underbrace{\left(a^{k/n} - a^{(k-1)/n}\right)}_{\text{width}}$$

height of rectangle

$t_k = a^{(k-1)/n}$

$\Delta x_k = a^{k/n} - a^{(k-1)/n}$

$f(x) = \frac{1}{x}$




$$= \sum_{k=1}^{n} \frac{a^{k/n}}{a^{(k-1)/n}} - 1$$

$$= \sum_{k=1}^{n} a^{1/n} - 1$$

no k here, so just summing $a^{1/n} - 1$
$a^{1/n} - 1$ n-times

$$= n\left(a^{1/n} - 1\right)$$

$$\Rightarrow \int_1^a \frac{dx}{x} = \lim_{n\to\infty} \underbrace{n^{\infty}\left(a^{1/n}-1\right)^{1-1}}_{\infty \cdot 0} = \lim_{n\to\infty} \frac{a^{1/n}-1}{1/n} = \lim_{x\to\infty} \frac{a^{1/x}-1 \to 0}{1/x \to 0}$$

Recall:

$$\frac{d}{du}(a^u) = a^u \cdot u' \ln a$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{a^{1/x} \cdot -1/x^2 \cdot \ln a - 0}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{a^{1/x} \cdot \cancel{-1/x^2} \cdot \ln a}{\cancel{-1/x^2}}$$

$$= \lim_{x\to\infty} \underbrace{a^{1/x}}_{1} \cdot \ln a$$

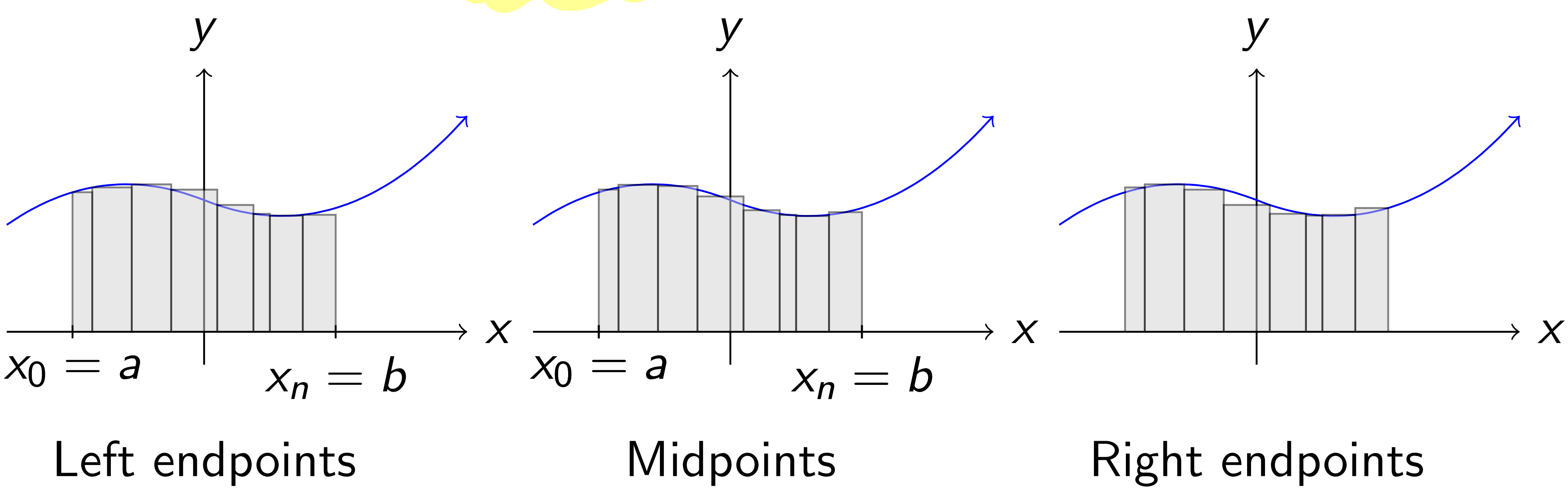
$$= 1 \cdot \ln a$$

$$\boxed{= \ln a}$$

# Selection of $T$

The most common choices for $T$ are

- Left endpoints: $t_k = x_{k-1}$
- Midpoints: $t_k = \frac{x_{k-1}+x_k}{2}$
- Right endpoints: $t_k = x_k$



# Selection of $P$

The most common choice for $P$, by far, is the uniform partition

$$x_k = a + k\Delta x \qquad \text{and} \qquad \Delta x = \frac{b-a}{n}.$$

# Useful formulas

When working analytic problems with a uniform partition, these formulas come up a lot

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

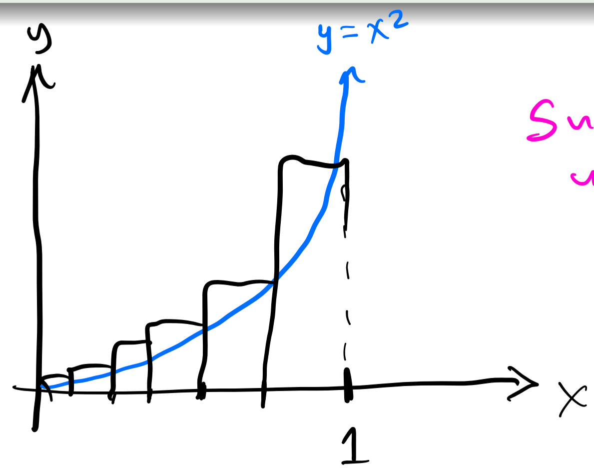
and

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2.$$

# Uniform Partition Example

## Example

Use uniform partitions and right endpoints to find $\int_0^1 x^2 \, dx$.




Supposed to be uniform horizontal length!

Steps: 1. Find $\Delta x_k = \Delta x$

2. Find $x_k = a + k\Delta x$ ← right endpoints

3. Find $f(x_k)\Delta x$

4. Find $\sum_{k=1}^{n} f(x_k)\Delta x$

5. Find $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$

Sol

1. $\Delta x = \frac{1-0}{n} = 1/n$

2. $x_k = 0 + k \cdot \Delta x$ ← right end points

$= k/n$

3. $f(x_k)\Delta x = (k/n)^2 \cdot 1/n = k^2/n^3$

4. $\sum_{k=1}^{n} k^2/n^3 = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$

5. $\lim_{n\to\infty} \frac{1}{n^{\cancel{3}2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \cdot \frac{1/n}{1/n^2}$

$= \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6}$

$= \frac{(1+0)(2+0)}{6}$

$= \boxed{1/3}$

# Python Code

```python
# Define Riemann sum
def riemann_sum(f, P, pts):

    # Sort values
    P = np.sort(P)
```

makes sure values are from smallest to biggest

```python
    # Calculate Delta x
    dx_vals = np.diff(P)
```

Takes diff. of consecutive terms

```python
    # Calculate the number of terms
    # Note: dx_vals has one fewer element than P
    N = len(dx_vals)

    # Define T
    if pts == 'left':
```

$0, 1, 2, \ldots, N-1$ ← last value is not included

```python
        T = [P[i] for i in range(0, N)]
```

range(N)

$P = (x_0, x_1, x_2, \ldots, x_{N-1}, x_N)$

```python
    elif pts == 'right':
        T = [P[i + 1] for i in range(0, N)]
    elif pts == 'mid':
        T = [0.5 * P[i] + 0.5 * P[i + 1] for i in range(0, N)]
```

$T = \left(\frac{x_0+x_1}{2}, \frac{x_1+x_2}{2}, \ldots, \frac{x_{N-1}+x_N}{2}\right)$

```python
    # Get area of rectangles
    rectangles = [f(T[i]) * dx_vals[i] for i in range(0, N)]
```

height

width

```python
    # Return sum
    return np.sum(rectangles)
```

Take sum of rectangle areas to get Riemann sum

# Improper Integral

## Definition

(a) If the integrals exists for every $t \geq a$ and for every $s \leq b$, then

$$\int_a^{\infty} f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

and

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{s\to-\infty} \int_s^b f(x)\ dx.$$

(b) For any $c$ in $\mathbb{R}$, if both $\int_c^{\infty} f(x)\ dx$ and $\int_{-\infty}^{c} f(x)\ dx$ converge, then

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{c} f(x)\ dx + \int_c^{\infty} f(x)\ dx.$$

# Python Example

## Example

Use pandas to create a data frame of Riemann sums with left endpoints, midpoints, and right endpoints, and uniform partitions to approximate $\int_0^\infty e^{-x^2/2}\,dx$. Consider $n = 10, 50, 100, 500$, and $1000$.

# Python Example Cont.

**Solution.** Since $e^{-x^2/2}$ goes to 0 rapidly, it's safe to use $b = 10$.

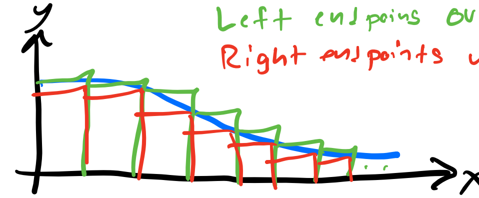
Assume we've already imported numpy as np

$e^{-10^2/2} = e^{-50} \approx 1.93\times10^{-22}$

```
# Import pandas
import pandas as pd

# Define function
f = lambda x: np.e**(-x**2/2)

# Define the n-values
n_vals = [10, 50, 100, 500, 1000]

# Define the data frame
results = pd.DataFrame(index = n_vals, columns = ['left', 'mid', 'right'])

# Loop over values
for n in n_vals:
    # We can use np.linspace for a uniform partition
    partition = np.linspace(0, 10, n + 1)

    # Get left endpoint results
    results.loc[n, 'left'] = riemann_sum(f, partition, 'left')

    # Get midpoint results
    results.loc[n, 'mid'] = riemann_sum(f, partition, 'mid')

    # Get right endpoint results
    results.loc[n, 'right'] = riemann_sum(f, partition, 'right')

# Note: screenshot of output is ex1-3
results
```

note includes endpoints

# Python Example Result




Left endpoints over estimate
Right endpoints under-estimate

The limit as $\|P\| \to 0$ is $\sqrt{\pi/2} \approx 1.253.$

| n | left | mid | right |
|---|---|---|---|
| **10** | 1.753314 | 1.253314 | 0.753314 |
| **50** | 1.353314 | 1.253314 | 1.153314 |
| **100** | 1.303314 | 1.253314 | 1.203314 |
| **500** | 1.263314 | 1.253314 | 1.243314 |
| **1000** | 1.258314 | 1.253314 | 1.248314 |

Little big
Little small

# Indefinite Integral

## Definition

The function $F$ is an **indefinite integral** or **antiderivative** of $f$ if $F'(x) = f(x)$. We write

$$\int f(x)\, dx = F(x)$$

to denote this.

Indefinite integrals are only unique up to a constant. For example, two antiderivatives of $2x$ are $x^2 + 1$ and $x^2 - 4$. To handle all possibilities, we write

$2x+0 = 2x-0$

$$\int 2x\, dx = x^2 + C.$$

# Antiderivative Theorems

**Theorem**

*Let $f$ and $g$ be continuous functions on some domain and let $\alpha$ and $\beta$ be real numbers. Then*

$$\int \alpha f(x) + \beta g(x)\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx.$$

# Useful Antiderivative Formulas

- $\displaystyle\int x^n \; dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$
- $\displaystyle\int \frac{dx}{x} = \ln|x| + C$
- $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$
- $\displaystyle\int e^x \; dx = e^x + C$
- $\displaystyle\int a^x \; dx = \frac{a^x}{\ln a} + C$
- $\displaystyle\int \sin x \; dx = -\cos x + C$
- $\displaystyle\int cosx \; dx = \sin x + C$
- $\displaystyle\int \tan x \; dx = -\ln|cosx| + C$

# Fundamental Theorem of Calculus

**Theorem (Fundamental Theorem of Calculus)**

*Suppose $f$ is continuous on the closed interval $[a, b]$. Then*

(a) $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, *where* $F'(x) = f(x)$.

(b) $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,tx\right) = f(x)$

Just idea of why true. Not proof

(a) $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$

$$\frac{F(x_k) - F(x_{k-1})}{\Delta x_k} \approx f(x_k), \qquad P = (\overset{\overset{a}{\shortparallel}}{x_0}, x_1, x_2, \dots, \overset{\overset{b}{\shortparallel}}{x_n})$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

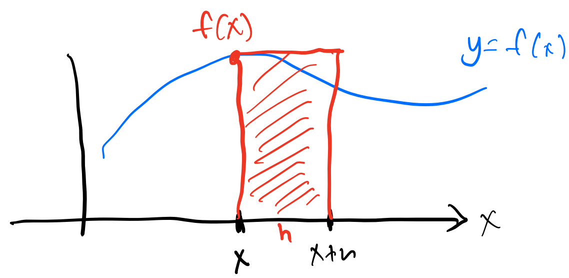
$$\approx \lim_{n\to\infty} \sum_{k=1}^{n} \frac{F(x_k) - F(x_{k-1})}{\Delta \cancel{x_k}} \cdot \Delta \cancel{x_k}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(F(x_k) - F(x_{k-1})\right)$$

$$= \lim_{n\to\infty} \left(\cancel{F(x_1)} - F(a)\right) + \left(F(x_2) - \cancel{F(x_1)}\right) + \dots + \left(F(b) - \cancel{F(x_{n-1})}\right)$$

$$= \lim_{n\to\infty} F(b) - F(a)$$

$$= F(b) - F(a)$$




(b) $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

$$\lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\cdot \cancel{h}}{\cancel{h}}$$

$$= f(x)$$

# Fundamental Theorem of Calculus Example

## Example

$$\int_0^2 \max\{x, 1\} \; dx =$$

Sol Notice:

$$\max\{x, 1\} = \begin{cases} 1, & x \le 1 \\ x, & x > 1. \end{cases}$$

So,

$$\int_0^2 \max\{x,1\}\,dx = \int_0^1 \max\{x,1\}\,dx + \int_1^2 \max\{x,1\}\,dx$$

$$= \int_0^1 1\,dx + \int_1^2 x\,dx$$

$$= \left[x\right]_0^1 + \left[\tfrac{1}{2}x^2\right]_1^2$$

$$= (1-0) + \left(\tfrac{1}{2}\cdot 2^2 - \tfrac{1}{2}\cdot 1^2\right)$$

Note: $\frac{d}{dx}(x+c) = 1$

$\frac{d}{dx}\left(\frac{1}{2}x^2 + c\right) = x$

$$= 1 + \frac{1}{2} \cdot 4 - \frac{1}{2}$$

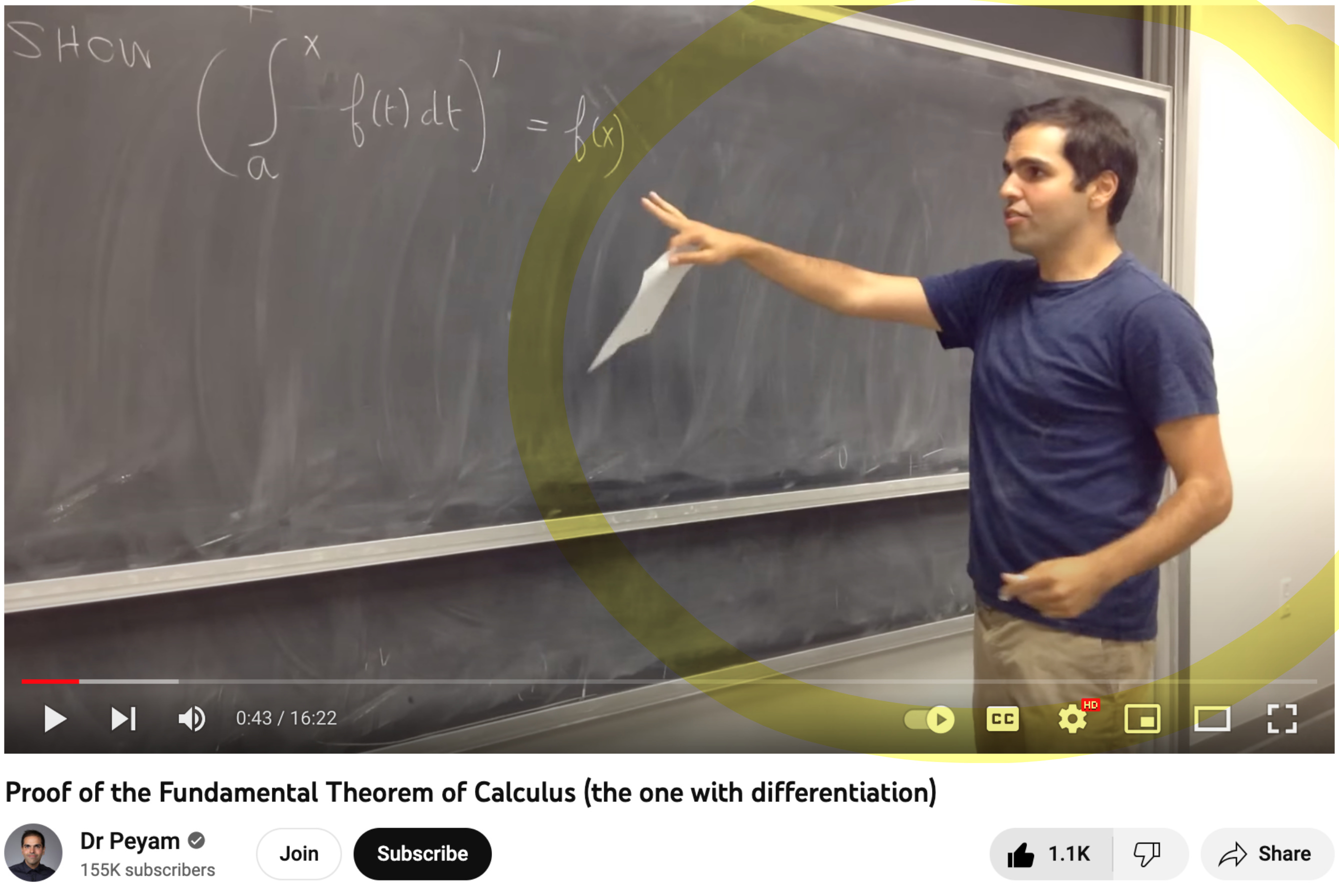
$$= 1 + 2 - \frac{1}{2}$$

$$= \frac{6}{2} - \frac{1}{2}$$

$$= 5/2$$

# Proof of the Fundamental Theorem on YouTube

Watch Peyam prove part (b) of the Fundamental Theorem of Calculus (https://youtu.be/4DrCKhCECHo).



# u-Substitution

## Theorem (u-Substitution)

*Suppose $g'$ is continuous on the closed interval $[a, b]$ and $f$ is continuous on the range of $g$. Then*

$$\int_a^b (f \circ g)(x) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

$u = g(x)$

# Example

## Example

$$\int xe^{-x^2/2}\, dx =$$

Sol

$$\int x e^{-x^2/2} dx = -\int e^{\overbrace{-x^2/2}^{u}} \cdot \overbrace{(-x\, dx)}^{du}$$

Let $u = -x^2/2$

$\Rightarrow du = -x\, dx$

$$= - \int e^u du$$

$$= - e^u + C$$

$$\boxed{= - e^{-x^2/2} + C}$$

# Integration by Parts

## Theorem (Integration by parts)

*Suppose $F$ and $G$ are differentiable functions, $F'(x) = f(x)$, and $G'(x) = g(x)$, where $f$ and $g$ are continuous. Then*

$$\int F(x)g(x)\,dx = F(x)G(x) - \int f(x)G(x)\,dx.$$

$$\int u\,dv = uv - \int v\,du$$

Strategy: Choose $u$ so that it becomes less complex when diff.
E.g.
Polynomials, $\ln x$, inverse trig functions
Choose $dv$ so that it's at least not more complex when you integrate
E.g. $e^x$, trig functions

# Example

## Example

$$\int \ln x \; dx =$$

Clear we won't $u = \ln x$ because $du = \frac{1}{x}\,dx \leftarrow$ way easier to deal with than $\ln x$

we can let $dv = dx$, because $v = x \leftarrow$ $x$ isn't much harder to deal with than 1

Sol

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$dv = dx \qquad v = x$

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int \ln x\,dx = \overbrace{(\ln x)}^{u} \cdot \underbrace{x}_{v} - \int \overbrace{x}^{v} \underbrace{\left(\frac{1}{x}\,dx\right)}_{du}$$

$$= x \ln x - \int 1\,dx$$

$$\boxed{= x \ln x - x + C}$$

# Ordinary Differential Equations

# Ordinary Differential Equations

## Definition

(a) An **ordinary differential equation** (ODE) involves an unknown function of a single variable and some of its derivatives.

(b) The **order** of a differential equation is the order of the highest derivative that appears in the equation.

For example, $xy' = e^{xy}$ is a first order ordinary differential equation, while

$$\frac{d^3x}{dt^3} - 2t\frac{d^2x}{dt^2} + t^2x = \cos t$$

is a third order ordinary differential equation.

# Separable ODEs

## Definition

An ODE of the form

$$\frac{dy}{dx} = F(x, y)$$

is separable if $F(x, y) = f(x)g(y)$.

It's relatively easy to solve separable differential equations

$$\frac{dy}{dx} = f(x)g(y) \quad \text{implies} \quad \frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Hence,

$$\int \frac{1}{g(y)}\frac{dy}{dx}\, dx = \int f(x)\, dx \qquad \text{implies} \qquad \boxed{\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.}$$

# Separable ODE Example

## Example

Solve the differential equation

$$\frac{dy}{dt} = \frac{ty + 3t}{t^2 + 1}$$

subject to the initial condition $y(0) = 2$.

Sol

$$\frac{dy}{dt} = \frac{ty + 3t}{t^2+1} = \frac{(y+3)t}{t^2+1} = (y+3) \cdot \frac{t}{t^2+1}$$

$$\Rightarrow \frac{1}{y+3}\frac{dy}{dt} = \frac{t}{t^2+1} \Rightarrow \int \frac{1}{y+3}\overbrace{\frac{dy}{dt}dt}^{dy} = \int \frac{t}{t^2+1}\,dt$$

$$\Rightarrow \int \frac{dy}{y+3} = \frac{1}{2}\int \frac{\overbrace{2t\,dt}^{du}}{t^2+1}$$

$u = t^2 + 1$

$du = 2t\,dt$

$$\Rightarrow \ln|y+3| = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$\Rightarrow \ln|y+3| = \frac{1}{2}\ln|t^2+1| + C$$

$$\Rightarrow \cancel{e}^{\cancel{\ln}|y+3|} = e^{\frac{1}{2}\ln|t^2+1| + C}$$

$$\Rightarrow |y+3| = e^C \cdot \left(\cancel{e}^{\cancel{\ln}|t^2+1|}\right)^{1/2}$$

$$= e^C\, |t^2+1|^{1/2}$$

$$\Rightarrow y+3 = \underbrace{\pm e^C}_{A}\sqrt{t^2+1}$$

$$\Rightarrow y = A\sqrt{t^2+1} - 3$$

We have initial condition: $y(0) = 2$.

$$\Rightarrow 2 = A\sqrt{0^2+1} - 3$$

$$\Rightarrow 2 = A - 3$$

$$\Rightarrow A = 5$$

Solution

$$\Rightarrow \boxed{y = 5\sqrt{t^2+1} - 3}$$

# Linear ODEs

## Definition

A first order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The idea behind solving these is to find $\mu = f(x)$ such that

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \overbrace{\mu P(x)}^{d\mu/dx}y.$$

Using the product rule, it becomes clear

$$\frac{d\mu}{dx} = \mu P(x) \qquad \text{implies} \qquad \boxed{\mu = \exp\left(\int P(x)\ dx\right).}$$

# Linear ODEs Example

## Example

Solve the differential equation

$$x\frac{dy}{dx} + 3x^3y = 6x^3.$$

Sol Put into standard form:

$$x\frac{dy}{dx} + 3x^3 y = 6x^3 \Rightarrow \frac{dy}{dx} + 3x^2 y = 6x^2$$

$$\Rightarrow \underbrace{\mu\frac{dy}{dx} + 3x^2\mu y}_{\frac{d}{dx}(\mu y)} = 6x^2\mu$$

$$\frac{d\mu}{dx} = 3x^2\mu$$

$$\Rightarrow \int\frac{1}{\mu}\frac{d\mu}{dx}dx = \int 3x^2\,dx$$

$$\Rightarrow \ln|\mu| = x^3 + \cancel{C}^{\,0}$$

$$\Rightarrow \mu = e^{x^3}$$

$$\Rightarrow e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(e^{x^3}y\right) = 6x^2e^{x^3}$$

$$\Rightarrow \quad e^{x^3} y = \int 6x^2 e^{x^3}\, dx$$

$$= 2 \int e^{\overbrace{x^3}^{u}} \cdot \overbrace{(3x^2\, dx)}^{du}$$

$$= 2 \int e^{u}\, du$$

$$= 2e^{u} + C$$

$$= 2e^{x^3} + C$$

$$\Rightarrow \quad \boxed{y = 2 + Ce^{-x^3}}$$

Let $u = x^3$.

$\Rightarrow du = 3x^2 dx$

# Sequences and Series

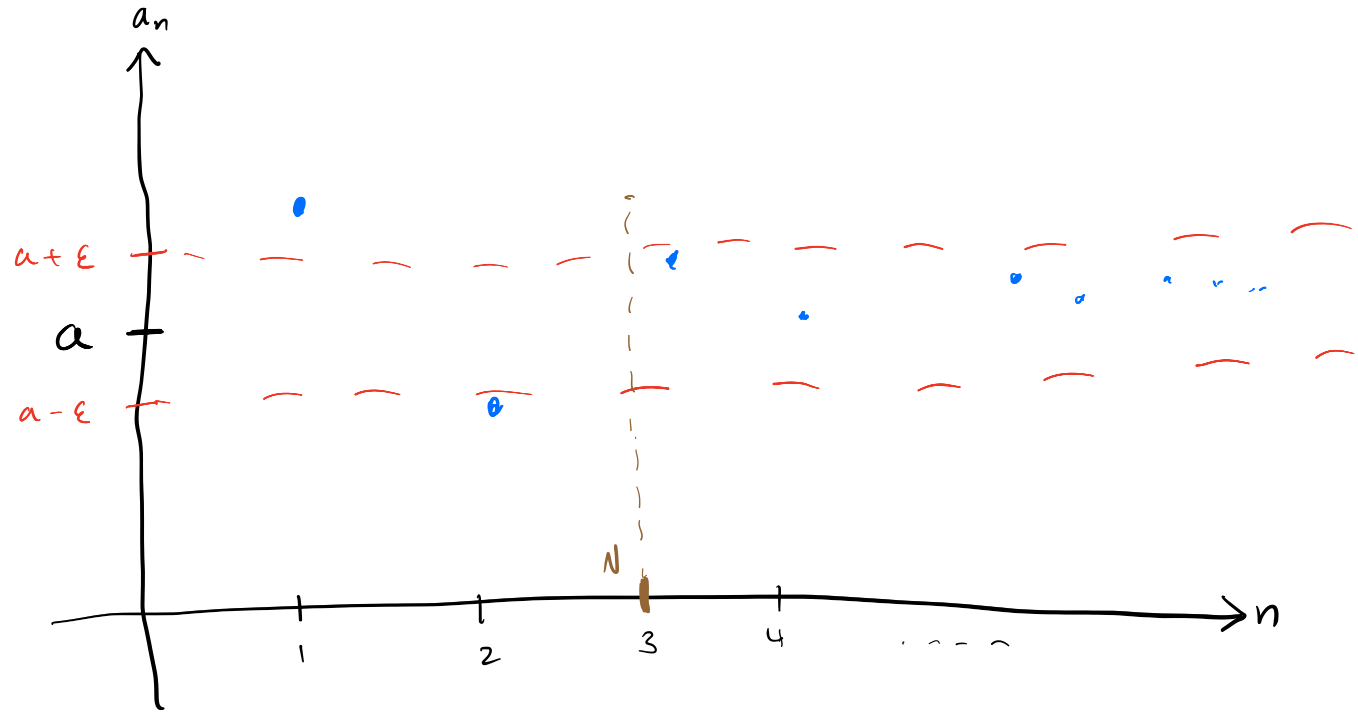
# Sequences

## Definition

A sequence $(a_n)_{n=1}^{\infty}$ is said to **converge**, if there is a value $a$ in $\mathbb{R}$ which has the property that: For all $\epsilon > 0$, there exists an integer $N$ such that $n \geq N$ implies that $|a_n - a| < \epsilon$. We often write

$$a_n \to a \qquad \text{or} \qquad \lim_{n\to\infty} a_n = a$$

when $(a_n)_{n=1}^{\infty}$ converges to $a$. If $(a_n)_{n=1}^{\infty}$ does not converge, then it **diverges**.




Pick $\varepsilon > 0$. If the series converges to $a$, there exists $N$ s.t. $n \geq N$ implies $a_n$ and $a$ are less than $\varepsilon$ distance apart, i.e. $|a_n - a| < \varepsilon$.

# Sequences Example

## Example

Determine which of the following sequences converge/diverge. If the sequence converges, find its limit.

(a) $a_n = \dfrac{1}{n}$ Converge. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = \underline{\underline{0}}$

(b) $b_n = \sqrt{n}$ Diverges

(c) $c_n = (-1)^n$ $-1, 1, -1, 1, \ldots$ Since this doesn't approaching anything it diverges.

(d) $d_n = 1 + \dfrac{(-1)^n}{n}$ $\lim_{n\to\infty} 1 + \underbrace{\frac{(-1)^n}{n}}_{0} = 1 + 0 = \underline{\underline{1}}$ Converges

# Python Sequences Example

**Example**

Use Python to graph the four sequences in the previous example. Graph them on separate subplots, and for the sequences that converge use horizontal lines to show their respective limits.

# Python Sequences Example Solution

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use LaTeX
plt.rcParams['text.usetex'] = True

# Use Seaborn style
plt.style.use('seaborn')

# Define functions
a = lambda n: 1/n
b = lambda n: np.sqrt(n)
c = lambda n: (-1)**n
d = lambda n: 1 + (-1)**n/n

# Define the limits
a_lim, d_lim = 0, 1

# Get the n-values
n_vals = np.arange(1, 21)

# Get the sequence values
# Functions already vectorized
a_vals = a(n_vals)
b_vals = b(n_vals)
c_vals = c(n_vals)
d_vals = d(n_vals)

# Set up subplots
fig, ax = plt.subplots(2, 2, sharey = True,
    figsize = (10, 6))
```

define sequences

start

stop

The default step value is 1

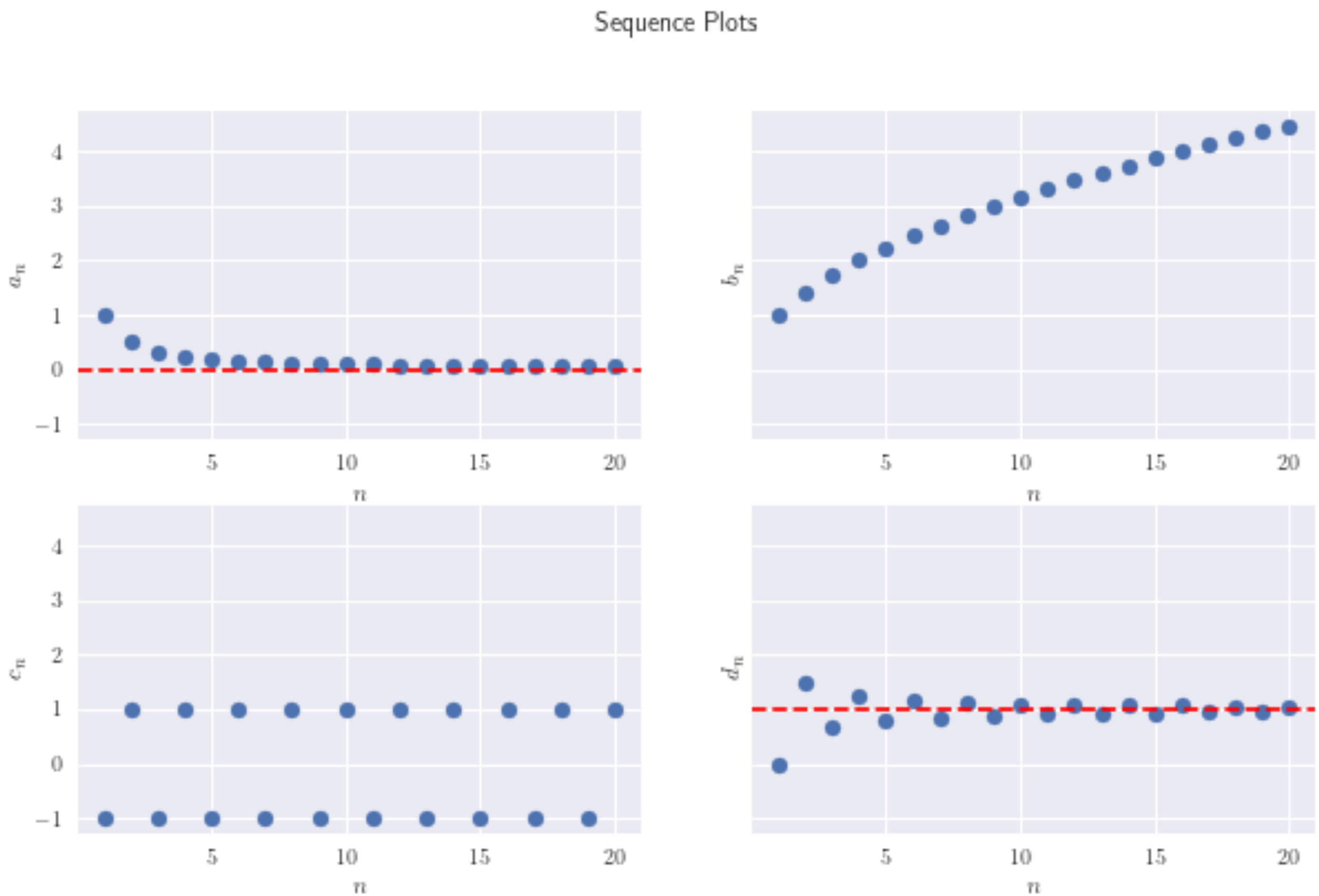
Plot for an

red

Adds title on top of everything

```
# Plot a_n and its limit
ax[0, 0].scatter(n_vals, a_vals)
ax[0, 0].axhline(y = a_lim, color = 'r',
    linestyle = 'dashed')
ax[0, 0].set_xlabel(r'$n$')
ax[0, 0].set_ylabel(r'$a_n$')

# Plot b_n and its limit
ax[0, 1].scatter(n_vals, b_vals, label = r'
    $b_n$')
ax[0, 1].set_xlabel(r'$n$')
ax[0, 1].set_ylabel(r'$b_n$')

# Plot c_n and its limit
ax[1, 0].scatter(n_vals, c_vals)
ax[1, 0].set_xlabel(r'$n$')
ax[1, 0].set_ylabel(r'$c_n$')

# Plot d_n and its limit
ax[1, 1].scatter(n_vals, d_vals)
ax[1, 1].axhline(y = d_lim, color = 'r',
    linestyle = 'dashed')
ax[1, 1].set_xlabel(r'$n$')
ax[1, 1].set_ylabel(r'$d_n$')

plt.suptitle(r'Sequence Plots')

# Save the figure
plt.savefig(path + r'ex1-4.png')

plt.show()
```

# Python Sequences Example Result



## Triangle Inequality

For any real numbers $x$, $y$, and $z$,

$$|x - y| \leq |x - z| + |z - y|.$$

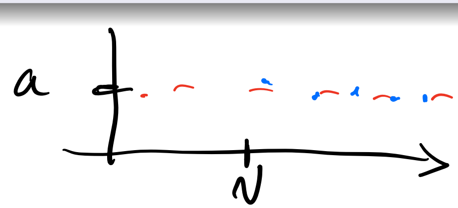
# Sequences

## Theorem

(a) *The sequence $(a_n)_{n=1}^{\infty}$ converges to $a$ in $\mathbb{R}$ if and only if for every $\epsilon > 0$, we have $a_n$ in the interval $(a - \epsilon, a + \epsilon)$ for all but finitely many $n$.* More-or-less definition of convergence

(b) *If $(a_n)_{n=1}^{\infty}$ converges to both $a$ and $b$, then $a = b$.*

(c) *If $(a_n)_{n=1}^{\infty}$ converges, then it is bounded. That is, convergence of $(a_n)_{n=1}^{\infty}$ implies there exists a real number $B$ such that $|a_n| \leq B$ for all $n$.*




All of terms eventually a finite distance from a.

(b) $a_n \to a$ and $a_n \to b \overset{?}{\Rightarrow} a = b$

Pick $\varepsilon > 0$. Then $\exists\, N_1$ and $N_2$ such that

$$|a_n - a| < \varepsilon \quad \text{for} \quad n \geq N_1$$

and

$$|a_n - b| < \varepsilon \quad \text{for} \quad n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then $n \geq N$, implies

$$|a_n - a| < \varepsilon \quad \text{and} \quad |a_n - b| < \varepsilon.$$

From the triangle inequality,

$$\begin{aligned} |a - b| &\leq |a_n - a| + |a_n - b| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon. \end{aligned}$$

This must hold for all $\varepsilon$. Since $a$ and $b$ are not functions of $n$, this means $a = b$.

# Sequence Properties

## Theorem

*Suppose* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *are real numbered sequences and*

$$\lim_{n\to\infty} a_n = a \qquad \textit{and} \qquad \lim_{n\to\infty} b_n = b.$$

*Let* $\alpha$ *and* $\beta$ *be real constants.*

(a) $\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha a + \beta b$

(b) $\lim_{n\to\infty} a_n b_n = ab$

(c) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ *if* $a_n \neq 0$ *and* $a \neq 0$.

# Monotonic Sequences

## Definition

- A real sequence $(a_n)_{n=1}^{\infty}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all $n$.

monotonically increasing

- A real sequence $(a_n)_{n=1}^{\infty}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all $n$.

monotonically decreasing

## Theorem

*Suppose that $(a_n)_{n=1}^{\infty}$ is monotonic. Then it converges if and only if it is bounded.* Really hard proof!

# Series

## Definition

Consider a series $S = \sum_{k=1}^{\infty} a_k$. Its ***n*-th partial sum** is $S_n = \sum_{k=1}^{n} a_k$. The series $S$ **converges** if the sequence $(S_n)_{n=1}^{\infty}$ converges, and it **diverges** otherwise.

$S = \sum_{k=1}^{\infty} a_k \longrightarrow$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + \ldots + a_n$

$\longrightarrow (S_n)_{n=1}^{\infty}$

The series converges if the induced sequence converges.

# Example

## Example

For what values of $r$ does the geometric series $\sum_{k=1}^{\infty} r^{k-1}$ converge?

Sol Consider the partial sums.

$$S_1 = 1$$
$$S_2 = 1 + r$$
$$S_3 = 1 + r + r^2$$
$$\vdots$$
$$S_n = 1 + r + r^2 + \ldots + r^{n-1}$$

So,

$$\begin{array}{rcl} S_n & = & 1 + \cancel{r} + \cancel{r^2} + \ldots + \cancel{r^{n-1}} \\ -(r S_n & = & \cancel{r} + \cancel{r^2} + \ldots + \cancel{r^{n-1}} + r^n) \\ \hline S_n - r S_n & = & 1 - r^n \end{array}$$

$$\Rightarrow (1-r) S_n = 1 - r^n$$

$$\Rightarrow \quad S_n = \frac{1-r^n}{1-r}$$

Hence, the series converges if and only if

$$\Big(\underbrace{\frac{1-r^n}{1-r}}_{S_n}\Big)_{n=1}^{\infty}$$

converges. So, consider

$$\lim_{n\to\infty} \frac{1-\cancel{r^n}^{\,0}}{1-r}.$$

When does this converge?

Converge if $|r|<1$

The series $S = \sum_{k=1}^{\infty} r^{k-1}$ converges if and only if $|r|<1$.

If $|r|<1$,

$$\boxed{S = \sum_{k=1}^{\infty} r^{k-1} = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r}.}$$

# Geometric Series

The geometric series is extremely important in finance. Remember these formulas.

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$$

and

$$\sum_{k=1}^{\infty} ar^{k-1} = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ DNE, & |r| \geq 1 \end{cases}$$

Make sure you know these well!!

# Partial Fraction Decomposition

- $$\frac{ax+b}{(x-c)(x-d)} = \frac{A}{x-c} + \frac{B}{x-d}, \quad c \neq d.$$
- $$\frac{ax+b}{(x-c)^2} = \frac{A}{x-c} + \frac{B}{(x-c)^2}.$$

# Partial Fraction Decomposition Example

## Example

Rewrite $\dfrac{x-2}{x(x-3)}$ using partial fraction decomposition.

Sol

$$\frac{x-2}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3}$$

Holds for all $x$ values

$$\Rightarrow \quad x-2 = A(x-3) + Bx$$

Let $x=0$. $\Rightarrow -2 = -3A + 0$

$\Rightarrow A = 2/3$

Let $x=3$. $\Rightarrow 1 = 0 + 3B$

$\Rightarrow B = 1/3$

So,

$$\boxed{\frac{x-2}{x(x-3)} = \frac{2/3}{x} + \frac{1/3}{x-3}}$$

# Convergent Series

## Example

Show $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges.

Sol We want to consider the partial sums.

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

We want to use partial fraction decomposition to break down $\frac{1}{k(k+1)}$

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} \Rightarrow 1 = A(k+1) + Bk$$

let $k=0$: $1 = A(0+1) + 0 \Rightarrow A = 1$

Let $k=-1$: $1 = A(-1+1) + B \cdot -1 \Rightarrow B = -1$

$$\Rightarrow \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$$

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{}^{k=1} + \overbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}^{k=2} + \cdots + \overbrace{\left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right)}^{k=n-1} + \overbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)}^{k=n}$$

$$= 1 - \frac{1}{n+1}$$

To prove $S = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges, all we need to do is

show the sequence $S_n = 1 - \frac{1}{n+1}$ converges,

$$\lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = 1$$

$$\Rightarrow \boxed{S = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1}$$

we proved it converged! we even found what it converged to!

---

Telescoping Series

$$S_n = \sum_{k=1}^{n}(b_k - b_{k+1}) = (b_1 - \cancel{b_2}) + (\cancel{b_2} - \cancel{b_3}) + \cdots + (\cancel{b_n} - b_{n+1})$$

$$= b_1 - b_{n+1}$$

$b_{n+1} \to 0$ as $n \to \infty$

$$S = \sum_{k=1}^{\infty}(b_k - b_{k+1}) = \lim_{n\to\infty} b_1 - b_{n+1} = b_1$$

# Divergence Test

- If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.
- If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ may or may not converge.

Does converge $\downarrow$ $\sum_{k=1}^{\infty} \frac{1}{k^2}$

Does not converge $\downarrow$ $\sum_{k=1}^{\infty} \frac{1}{k}$, though $\lim_{k\to\infty} \frac{1}{k} = 0$

# Property of Series

## Theorem

*Suppose that* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *converge. For any real constants* $\alpha$ *and* $\beta$

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k.$$

# Dominating Series

**Theorem**

*If a series* $\sum_{k=1}^{\infty} b_k$ *dominates a series* $\sum_{k=1}^{\infty} a_k$ *in the sense that for all sufficiently large* $k$, $|a_k| \leq b_k$, *then convergence of* $\sum_{k=1}^{\infty} b_k$ *implies convergence of* $\sum_{k=1}^{\infty} a_k$.

# Dominating Series Example

## Example

Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$ converges.

Note: $-1 \le \sin k \le 1 \Rightarrow |\sin k| \le 1$

Sol It's clear

$$\left|\frac{\sin k}{2^k}\right| \le \frac{1}{2^k}.$$

Furthermore,

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \qquad \sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}, \ |r|<1$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}$$

$$= \frac{1}{2}\cdot\frac{1}{1-1/2} = \frac{1}{2-1} = 1$$

So, since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, it must be that $\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$ converges.

# Integral Test

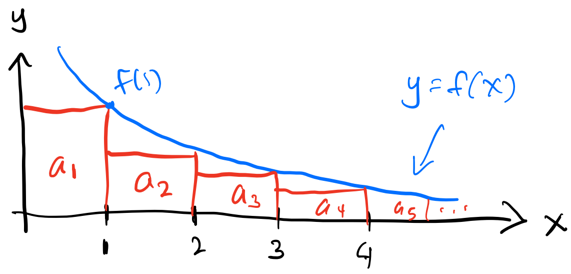
## Theorem (Integral Test)

*Suppose $f$ is continuous and $f(k) = a_k$.*

(a) *If $|a_k| \leq f(x)$ for all sufficiently large $k$ and all $x$ in the interval $(k-1, k]$, then convergence of $\int_1^\infty f(x)\,dx$ implies convergence of $\sum_{k=1}^{\infty} a_k$.*

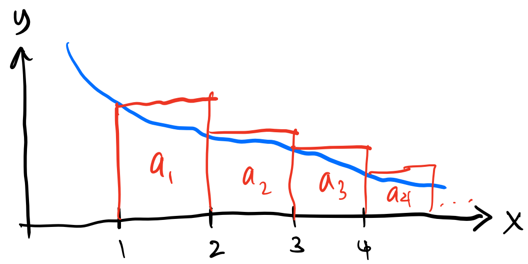
(b) *If $|f(x)| \leq a_k$ for all sufficiently large $k$ and all $x$ in the interval $[k, k+1)$ then diverges of $\int_1^\infty f(x)\,dx$ implies divergence of $\sum_{k=1}^{\infty} a_k$.*

(a)




$$\sum_{k=2}^{\infty} a_k \leq \int_1^{\infty} f(x)\,dx$$

(b)




$$\sum_{k=1}^{\infty} a_k \geq \int_1^{\infty} f(x)\,dx$$

## Example (p-Series)

### Example

Prove that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ ← p-series converges for $p > 1$ and diverges for $p \leq 1$.

Sol

$p > 1$: Integral test. Let $f(x) = \frac{1}{x^p}$. Then $\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty$

$= \frac{1}{1-p} \cdot \left.\frac{1}{x^{p-1}}\right|_1^\infty = \frac{1}{1-p} \lim_{b\to\infty} \left.\frac{1}{x^{p-1}}\right|_1^b = \frac{1}{1-p} \lim_{b\to\infty} \frac{1}{b^{p-1}} - \frac{1}{1^{p-1}} = \frac{1}{1-p} \lim_{b\to\infty} \underbrace{\frac{1}{b^{p-1}}}_{0} - 1$

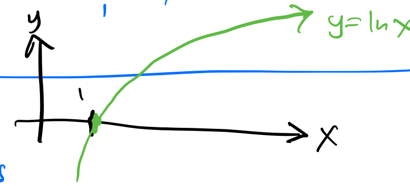
$= \frac{-1}{1-p} < \infty$

Since $\int_1^\infty \frac{dx}{x^p}$ converges, so does $\sum_{k=1}^{\infty} \frac{1}{k^p}$ when $p > 1$.

$p-1>0$

$P=1$: Integral test. Let $f(x) = \frac{1}{x}$. Then $\int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \lim_{b\to\infty} \ln x \Big|_1^b$

$= \lim_{b\to\infty} \ln b - \cancel{\ln 1}^{\,0} = \lim_{b\to\infty} \ln b = \infty.$




Hence, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges because $\int_1^\infty \frac{dx}{x}$ diverges

$P<1$: Just like $p>1$, but $p-1<0$, which makes

$$\lim_{b\to\infty} \frac{1}{1-p} \frac{1}{x^{p-1}} - 1 = \infty.$$

So $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges in this case.

# Alternating Series Test

## Theorem

*Suppose the alternating series* $\sum_{k=1}^{\infty}(-1)^{k+1} b_k$ *is such that* $0 \leq b_{k+1} \leq b_k$ monotonically decreasing *for sufficiently large* $k$. *Then the series converges if* $\lim_{n\to\infty} b_k = 0$.

Much easier for alternating Series to converge.

# Alternating Series Test Example

## Example

Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

Sol Note:

- $\frac{1}{k} \geq 0$ ✓
- $\frac{1}{k+1} \leq \frac{1}{k}$ ✓
- $\lim_{k \to \infty} \frac{1}{k} = 0$ ✓

Hence, by the alternating series test, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

# Absolutely and Conditionally Convergent Series

## Definition

A series $\displaystyle\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges. A series is **conditionally convergent** if $\displaystyle\sum_{k=1}^{\infty} a_k$ converges but $\displaystyle\sum_{k=1}^{\infty} |a_k|$ does not.

For example, the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

is conditionally convergent but not absolutely, because $\displaystyle\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

# Ratio Test

## Theorem

(a) If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

(b) If $\lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent.

(c) If $\lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L = 1$, then the test fails.

Proof uses comparison with a geometric series.

# Ratio Test Example

## Example

What can be said about the convergence of $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{k!}{k^k}$?

Sol. We will use the ratio test. Consider

$$\lim_{k\to\infty}\left|\frac{\cancel{(-1)^{k+1}}\,\frac{(k+1)!}{(k+1)^{k+1}}}{\cancel{(-1)^k}\,\frac{k!}{k^k}}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^k}{k!}\right|$$

$$= \lim_{k\to\infty}\frac{(k+1)\cdot \cancel{k!}}{(k+1)^{k+1}}\cdot\frac{k^k}{\cancel{k!}}$$

$$= \lim_{k\to\infty}\frac{k^k\cdot \cancel{(k+1)}}{(k+1)^{k+\cancel{1}}}$$

$$= \lim_{k\to\infty}\frac{k^k}{(k+1)^k}$$

$$(k+1)! = (k+1)\cdot \overbrace{k\cdot(k-1)\cdots 2\cdot 1}^{k!}$$

$$= (k+1)\cdot k!$$

$$= \lim_{k\to\infty} \left( \frac{k \cdot \frac{1}{k}}{(k+1)\cdot\frac{1}{k}} \right)^k$$

$$= \lim_{k\to\infty} \left( \frac{1}{1+1/k} \right)^k$$

$$= \lim_{k\to\infty} \frac{1}{(1+1/k)^k},$$

$$\boxed{\lim_{k\to\infty} (1+1/k)^k = e \approx 2.718}$$

$$= \frac{1}{e}$$

$$< 1$$

There series is absolutely convergent.

---

$$\lim_{k\to\infty} (1+1/k)^k = \lim_{x\to\infty} (1+1/x)^x$$

$$= \lim_{x\to\infty} \left[ e^{\ln(1+1/x)} \right]^x$$

$$= \lim_{x\to\infty} e^{\overset{\infty}{\cancel{x}} \ln(\overset{0}{\cancel{1+1/x}})}$$

$$= \lim_{x\to\infty} e^{\frac{\ln(1+1/x)}{1/x}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+\cancel{1/x})^{0}}{\cancel{1/x}^{0}}}$$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot(0-\cancel{1}/x^2)}{-\cancel{1}/x^2}}$$

$$= e^{\lim_{x\to\infty} \frac{1}{1+\cancel{1/x}^{0}}}$$

$$= e^{\frac{1}{1+0}}$$

$$= e^1$$

$$= e$$

# Power Series

## Definition

A **power series** centered at $c$ is a series of the form

$$\sum_{k=0}^{\infty} a_k(x-c)^k.$$

If the series converges for $|x-c| < R$ and diverges for $|x-c| > R$, then $R$ is the **radius of convergence**. The **interval of convergence** $I$ is the set of all $x$ values where the series converges.

**Remark:** We assume $0^0 = 1$ within our power series, so the power series always converges at $x = c$.

# Power Series Example

## Example

Find the radius and interval of convergence of the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-3)^k(x+1)^k}{\sqrt{k+1}}$.

- Use ratio test to find the radius of convergence $R$
- Check endpoints to see if they are inclusive or exclusive.

<u>Sol</u> Ratio test:

$$\lim_{k\to\infty} \left| \frac{\frac{(-3)^{k+1}(x+1)^{k+1}}{\sqrt{k+2}}}{\frac{(-3)^k(x+1)^k}{\sqrt{k+1}}} \right| = \lim_{k\to\infty} \left| \frac{(-3)^{\cancel{k+1}}(x+1)^{\cancel{k+1}}}{\sqrt{k+2}} \cdot \frac{\sqrt{k+1}}{\cancel{(-3)^k}\cancel{(x+1)^k}} \right|$$

$$= \lim_{k\to\infty} \left| \underbrace{\sqrt{\frac{k+1}{k+2}}}_{\to 0} \cdot \underbrace{3}_{\geq 0}(x+1) \right|$$

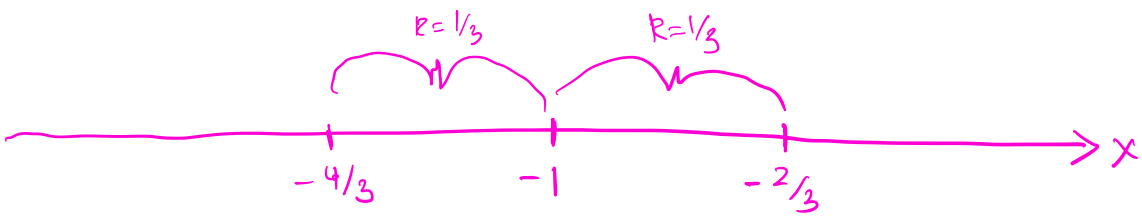
$$= \lim_{k\to\infty} \sqrt{\frac{k+1}{k+2}} \cdot 3 \cdot |x+1|$$

$$= 3\,|x+1|$$

$$< 1$$ ← ratio test says converges when this is true

$$\Rightarrow \quad |x+1| < 1/3$$

The radius of convergence is $R = 1/3$.




Let's check our endpoints!

$x = -4/3$:

$$\sum_{k=1}^{\infty} \frac{(-3)^k \, (-4/3+1)^k}{\sqrt{k+1}} = \sum_{k=1}^{\infty} \frac{(-3)^k \cdot (-1/3)^k}{\sqrt{k+1}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}} \qquad \ell = k+1$$

$$= \sum_{\ell=2}^{\infty} \frac{1}{\sqrt{\ell}}$$ ← p-series with $p = 1/2$

Diverges

$x = -2/3$:

$$\sum_{k=1}^{\infty} \frac{(-3)^k \cdot (-2/3+1)^k}{\sqrt{k+1}} = \sum_{k=1}^{\infty} \frac{(-3)^k \cdot (1/3)^k}{\sqrt{k+1}}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$$ ← Converges due to alt. Series test

Interval of convergence $(-4/3, -2/3] = \{x \mid -4/3 < x \le -2/3\}$.

# Differentiation and Integration of Power Series

## Theorem

*If the power series* $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ *has radius of convergence* $R > 0$*,*

*then both*

(a) $f'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$ *and*

(b) $\int f(x)\,dx = C + \sum_{k=0}^{\infty} a_k \frac{(x-c)^{k+1}}{k+1}$

*have radii of convergence* $R$.

Integration and differentiation don't affect the radius of convergence. They may affect the endpoints of the interval of convergence.

# Differentiation of Power Series Example

## Example

$$\sum_{k=1}^{\infty} \frac{k}{1.10^k} =$$

Note: $\frac{1}{1-x} = \sum_{k=1}^{\infty} x^{k-1} \Rightarrow \frac{x}{1-x} = \sum_{k=1}^{\infty} x^k, \quad |x| < 1$

Sol $\frac{x}{1-x} = \sum_{k=1}^{\infty} x^k \Rightarrow \frac{d}{dx}\left(\frac{x}{1-x}\right) = \sum_{k=1}^{\infty} \frac{d}{dx}(x^k)$

$$\Rightarrow \frac{(1-x)\cdot 1 - x(-1)}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\Rightarrow \frac{1-\cancel{x}+\cancel{x}}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\Rightarrow \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k$$

multiply both sides by $x$ so the powers match the coefs.

This is the formula we want. We just let $x = \frac{1}{1.10}$ (fine because $\left|\frac{1}{1.10}\right| = \frac{1}{1.10} < 1$).

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k \xrightarrow{x = \frac{1}{1.10}} \boxed{\frac{\frac{1}{1.10}}{\left(1-\frac{1}{1.10}\right)^2}} = \sum_{k=1}^{\infty} k \left(\frac{1}{1.10}\right)^k$$

$$= \sum_{k=1}^{\infty} \frac{k}{1.10^k}$$

We'll just do some math to make result look nice

$$\frac{\frac{1}{1.10}}{\left(1-\frac{1}{1.10}\right)^2} = \frac{\frac{1}{1.10}}{\left(\frac{1.10-1}{1.10}\right)^2} = \frac{\frac{1}{1.10}}{\left(\frac{0.10}{1.10}\right)^2} = \frac{1}{1.10} \cdot \left(\frac{1.10}{0.10}\right)^2$$

$$= \frac{1}{\cancel{1.10}} \cdot \frac{1.10^{\cancel{2}}}{0.10^2}$$

$$= \frac{1.10}{0.10^2}$$

$$= \frac{1.10}{0.01}$$

$$\boxed{= 110}$$

# Taylor's Theorem

## Theorem

*Suppose*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \qquad \text{for} \qquad |x-c| < R.$$

*Then*

$$a_k = \frac{f^{(k)}(c)}{k!}.$$

If you check, this makes all the derivatives line up.

# Popular Taylor Series Centered at Zero

- $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $x \in (-1,1)$

Geometric Series

- $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$ for $x \in \mathbb{R}$

Very well known

- $\sin x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{(2k+1)!}$ for $x \in \mathbb{R}$
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k}}{(2k)!}$ for $x \in \mathbb{R}$
- $\arctan x = \sum_{k=0}^{\infty} (-1)^k \dfrac{x^{2k+1}}{2k+1}$ for $x \in [-1,1]$

# Taylor's Theorem Example

## Example

Prove $\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ for $x \in [-1, 1]$.

Recall $\int_0^x \frac{dt}{1+t^2} = \arctan x$

Idea: start with geometric series. Then use that to find a series for $\frac{1}{1+t^2}$. Integrate to find the result.

Sol Geometric Series

$$\frac{1}{1-x} = \sum_{k=1}^{\infty} x^{k-1}, \quad |x| < 1$$

Let $x = -t^2$. Then we have

$$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{k=1}^{\infty} (-t^2)^{k-1} = \sum_{k=1}^{\infty} (-1)^{k-1} t^{2k-2}.$$

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k=1}^{\infty} (-1)^{k-1} t^{2k-2}\, dt$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^x t^{2k-2}\, dt$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, t^{2k-1}}{2k-1} \Big|_0^x$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, x^{2k-1}}{2k-1}$$

Let $\ell = k-1$.

$\Rightarrow k = \ell + 1$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+1-1}\, x^{2(\ell+1)-1}}{2(\ell+1)-1}$$

$$\boxed{\arctan x = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\, x^{2\ell+1}}{2\ell+1}}$$

# Time Value of Money

# Time Value of Money

For a time $t$ cash flow $C_t$ discounted at rate $r$, the *present value* is

$$PV = \frac{C_t}{(1+r)^t}.$$

The time $T$ *future value* is

$$FV_T = PV \cdot (1+r)^T = C_t \cdot (1+r)^{T-t}.$$

# Compound Interest

We assumed that interest is compounded once per unit of time. However, if it is compounded $n$ times per unit of time the formulas become

$$PV = \frac{C_t}{\left(1+\frac{r}{n}\right)^{nt}} \qquad \text{and} \qquad FV_T = PV \cdot \left(1+\frac{r}{n}\right)^{nT}.$$

Since

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

for continuous compounding (i.e. $n=\infty$) the formulas are

$$PV = C_t e^{-rt} \qquad \text{and} \qquad FV_T = PV \cdot e^{rT} = C_t \cdot e^{r(T-t)}$$

# Multiple Cash Flows

Suppose we have a sequence of cash flows $C_0, C_1, C_2, \ldots, C_T$, where the subscript denotes the time of the cash flow, the *net present value* (NPV) of these cash flows discounted at the constant rate $r$ is

$$NPV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^T}.$$

# Time Value of Money Python Example

## Example

| Time | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Cash Flow | -100 | 50 | 20 | 70 | 10 |

Calculate the net present value given a continuously compounded discount rate of 5%.

# Time Value of Money Python Solution

```python
# Import module
import numpy as np
# Record rate
rate = 0.05
# Record time of cash flows
time = np.array([0, 1, 2, 3, 4])
# Record cash flows
cash_flows = np.array([-100, 50, 20, 70, 10])
# Get the NPV
NPV = np.sum(cash_flows * np.exp(-rate * time))
print(f'The NPV of the cash flows is {NPV:.2f}.')
```

round to two decimal places

np.exp exponentiation : $x \mapsto e^x$
vectorized.

$$np.exp(-r \cdot time)$$

$$= np.exp(-r \cdot (0, 1, 2, 3, 4))$$

$$= np.exp((0, -r, -2r, -3r, -4r))$$

$$= (e^0, e^{-r}, e^{-2r}, e^{-3r}, e^{-4r})$$

$$= (1, e^{-r}, e^{-2r}, e^{-3r}, e^{-4r})$$

$$(-100, 50, 20, 70, 10) \cdot (1, e^{-r}, e^{-2r}, e^{-3r}, e^{-4r})$$

$$= (-100, 50e^{-r}, 20e^{-2r}, 70e^{-3r}, 10e^{-4r})$$

$NPV \approx 34.10$

# numpy_finance Module

There is a numpy_finance module that has a net present value function (https://numpy.org/numpy-financial/latest/npv.html). However, we would need the annual rate to use it in the last example, i.e. we would have to use the rate

$$100\% \times (e^{0.05} - 1) \approx 5.127\%.$$

## Example

Jain borrows \$1,000,000 to purchase a house. The loan is for thirty years and her first payment is one month from when she initially borrows the money. If her annualized rate is 12%, what will be her monthly payments? Ignore fees.

Sol Convert the annualized rate to the monthly rate.

$$\frac{12\%}{12} = \underline{\underline{1\%}}$$

Say we have payments of P each month. Then the NPV of these payments is \$1,000,000. So,

$$\frac{P}{1.01} + \frac{P}{1.01^2} + \dots + \frac{P}{1.01^{360}} = 1{,}000{,}000$$

12 payments per year. 30 years so, $12 \cdot 30 = 360$ payments.

LHS

$$\Rightarrow P\left(\frac{1}{1.01} + \frac{1}{1.01^2} + \dots + \frac{1}{1.01^{360}}\right) = P \cdot \frac{\frac{1}{1.01} \cdot \left(1 - \frac{1}{1.01^{360}}\right)}{1 - \frac{1}{1.01}}$$

$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$$

$$P \cdot \frac{1 - \frac{1}{1.01^{360}}}{1.01 - 1} = P \cdot \frac{1 - \frac{1}{1.01^{360}}}{0.01}$$

LHS = RHS:

$$P \cdot \frac{1 - \frac{1}{1.01^{360}}}{0.01} = 1,000,000$$

$$\Rightarrow \qquad P = 1,000,000 \times \frac{0.01}{1 - \frac{1}{1.01^{360}}}$$

$$\approx 10,286.13$$

Jain makes monthly payments of \$10,286.13

# Growing Payments

Suppose 1 is payed at time 1, and payments increase at a rate of $g$ each subsequent period until a final payment of $(1+g)^{n-1}$ is made at time $n$. If cash flows are discounted at rate $r$, then the NPV of the cash flows is

$$\frac{1 - \left(\frac{1+g}{1+r}\right)^{n}}{r-g}.$$

# Growing Payments Example

## Example

Calculate the NPV of the series of end-of-year cash flows. Assume

- \$100 is payed in the first year,
- each subsequent year payments increase by 5%,
- the final payment is made at the end of year ten, and
- the discount rate is 8%.