Unit 1: Calculus

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Limits

Definition

Definition

(a) Define $\lim_{x \to a} f(x) = L$ to mean for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |f(x) - L| < \epsilon.$$

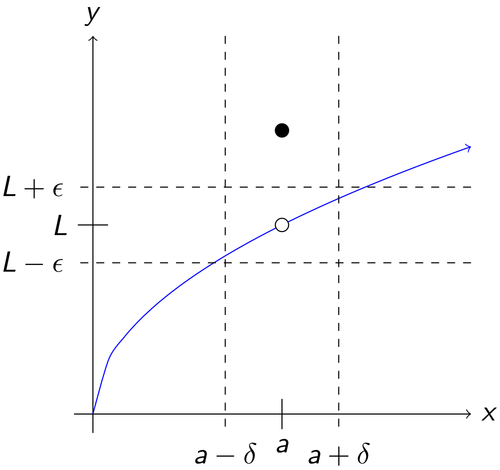
(b) Define $\lim_{x \to \infty} f(x) = L$ to mean for all $\epsilon > 0$ there exists an N such that

$$x \geq N \qquad \text{implies} \qquad |f(x) - L| < \epsilon.$$

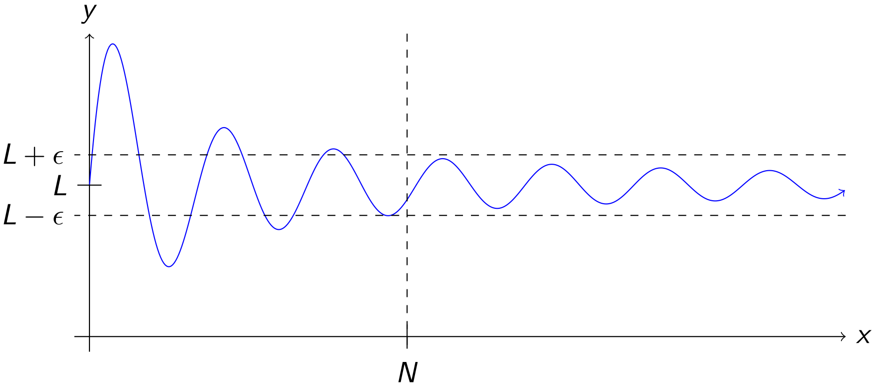
(c) Define $\lim_{x \to -\infty} f(x) = L$ to mean for all $\epsilon > 0$ there exists an N such that

$$x \leq N \qquad \text{implies} \qquad |f(x) - L| < \epsilon.$$

$\lim_{x \to a} f(x) = L$



$\lim_{x \to \infty} f(x) = L$



Using the Definition

Example

Prove

$$\lim_{x \to \infty} \frac{1}{x^p} = 0$$

for all $p > 0$.

Python Example

Example

Consider

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases} \qquad \text{and} \qquad \lim_{x \to 2} f(x) = 4.$$

For $\epsilon = 1, 0.1, 0.01$, and 0.001, find δ-values that satisfy

$$0 < |x - 2| < \delta \qquad \text{implies} \qquad |f(x) - 4| < \epsilon.$$

Python Example

```
import numpy as np

# Create function to find a delta, given epsilon, for lim_{x\to a} f(x) = L
def get_delta(f, a, L, epsilon, delta = 1, n_iter = 20):

    for _ in range(n_iter):

        # Get points within delta distance of a; a + delta not included since 0 < |x - a|
      < delta
        x_vals = np.linspace(a - delta, a + delta, 100, endpoint = False)

        # Remove a and a - delta since 0 < |x - a| < delta
        x_vals = x_vals[~np.isin(x_vals, [a - delta, a])]

        # Get y-values; not assuming f is vectorized
        y_vals = np.array([f(x) for x in x_vals])

        # Check to see if any outside of epsilon distance of L
        if np.max(np.abs(y_vals - L)) > epsilon:

            # If there are, then half delta
            delta *= 0.5

        # If not, return delta
        else:
            return delta

    # If no value after n_iter iterations, return np.nan to indicate no delta exists
    return np.nan
```

Python Example

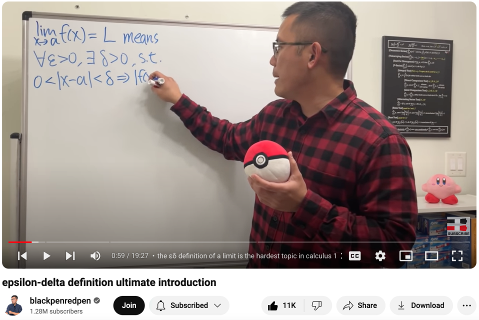
```
# Define f for example
f = lambda x: x**2 if x != 2 else 5
# In this case a is 2
a = 2
# In this case, the limit is 4
L = 4
# This gives the listed epsilon values
epsilon_vals = np.logspace(0, -3, num = 4)
# Find delta for each epsilon
for epsilon in epsilon_vals:
    # Get delta corresponding to epsilon
    delta = get_delta(f, a, L, epsilon)
    print(f'If 0 < |x - {a}| < {delta:.4f}, then |f(x) - {L}| < {epsilon:.4f}')
```

The output is shown below.

```
If 0 < |x - 2| < 0.1250, then |f(x) - 4| < 1.0000
If 0 < |x - 2| < 0.0156, then |f(x) - 4| < 0.1000
If 0 < |x - 2| < 0.0020, then |f(x) - 4| < 0.0100
If 0 < |x - 2| < 0.0002, then |f(x) - 4| < 0.0010
```

ϵ-δ Limit Definition on YouTube

Watch BlackPenRedPen explain the ϵ-δ limit definition (https://www.youtube.com/watch?v=DdtEQk_DHQs). There's another video where he goes over the $x \to \infty$ case (https://youtu.be/9JMFLzHtljA?si=1WPW-fmaf2DBe3Ph).



Properties of Limits

Theorem

Suppose a is in the interval $[-\infty, \infty]$. *Let*

$$\lim_{x \to a} f(x) = L_1 \qquad \textit{and} \qquad \lim_{x \to a} g(x) = L_2.$$

(a) $\lim\limits_{x \to a} \alpha f(x) + \beta g(x) = \alpha L_1 + \beta L_2$ *for any real constants* α *and* β

(b) $\lim\limits_{x \to a} f(x) \cdot g(x) = L_1 \cdot L_2$

(c) $\lim\limits_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{L_1}$ *if* $L_1 \neq 0$.

Useful Limits

Theorem

Suppose $p > 0$.

(a) $\lim\limits_{x \to \infty} \dfrac{1}{x^p} = 0$

(b) $\lim\limits_{x \to \infty} p^{1/x} = 1$

(c) $\lim\limits_{x \to \infty} x^{1/x} = 1$

(d) *If* $a > 0$, $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a$

(e) *If* $|r| < 1$, *then* $\lim\limits_{x \to \infty} r^x = 0$

Python Example

Example

Define

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

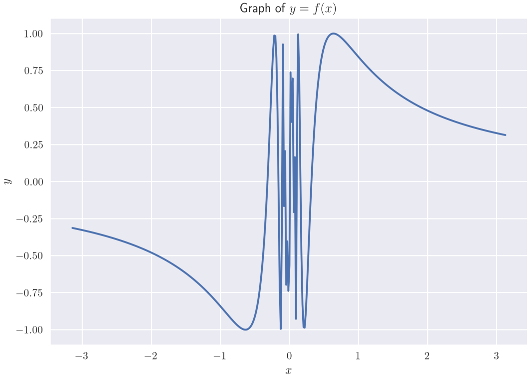
Graph f in Python to see that $\lim_{x \to 0} f(x)$ does not exist.

Python Example

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use LaTex and increase image resolution
plt.rcParams['text.usetex'] = True
plt.rcParams['figure.dpi'] = 300

# Use Seaborn style
plt.style.use('seaborn-v0_8')

# Define f
f = lambda x: 0 if x == 0 else np.sin(1/x)

# Let's graph on the interval [-pi, pi)
x_vals = np.arange(-np.pi, np.pi, np.pi/200)

# Calculate the y-values
y_vals = [f(x) for x in x_vals]

# Generate the plot
plt.plot(x_vals, y_vals)

# Label the x-axis
plt.xlabel('$x$')

# Label the y-axis
plt.ylabel('$y$')

# Give the graph a title
plt.title('Graph of $y = f(x)$')

# Save the figure
plt.savefig(path + r'ex1-1.png')

# Display the plot
plt.show()
```

Python Example Result

The graph isn't perfect, but it's enough to see that f doesn't approach anything in particular as x approaches 0.



Continuity

Continuity

Definition

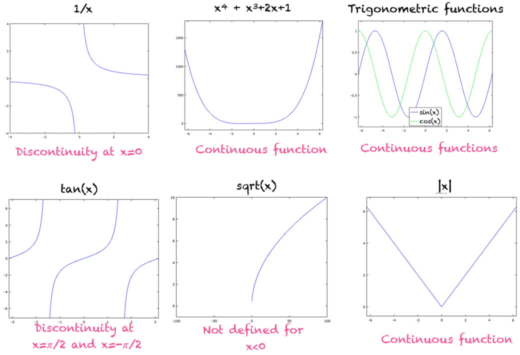
(a) A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the set A if

$$a \in A \qquad \text{implies} \qquad \lim_{x \to a} f(x) = f(a).$$

Continuity Idea

Continuous functions have no breaks, i.e. if you were to draw them you would never need to lift your pencil.



https://machinelearningmastery.com/continuous-functions/

Useful Theorem

Theorem

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f\Big(g(x)\Big) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

Example

Example

Suppose $p > 0$. Compute $\lim_{x \to \infty} p^{1/x}$.

Derivatives

Derivatives

Definition

(a) The **derivative of a function f** at a number a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(b) The function f is differentiable on a set A if $f'(a)$ exists for all a in A.

Notation

Leibniz notation is frequently used:

$$\frac{df}{dx} = f'(x) \qquad \text{and} \qquad \left.\frac{df}{dx}\right|_{x=a} = f'(a).$$

The respective second and third derivatives are written

$$\frac{d^2 f}{dx^2} = f''(x) \qquad \text{and} \qquad \frac{d^3 f}{dx^3} = f'''(x)$$

For the k-th derivatives, where $k > 3$, we use the notation

$$\frac{d^k f}{dx^k} = f^{(k)}(x).$$

Derivatives Example

Example

Let

$$f(x) = \begin{cases} xe^{-x^2 - x^{-2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Compute $f'(0)$.

Numerical Approximation

It is often helpful to numerically approximate f'. This can be done by choosing a small value of h and calculating

$$\frac{f(x+h) - f(x)}{h}.$$

The value h can be positive or negative. Often, a better numerical approach is to consider positive and negative values of h at the same time and take the average:

$$\frac{1}{2} \cdot \frac{f(x+h) - f(x)}{h} + \frac{1}{2} \cdot \frac{f(x-h) - f(x)}{-h} = \frac{f(x+h) - f(x-h)}{2h},$$

where $h > 0$.

Python Example

Example

Use Python to graph f and f' on the interval $[-2, 2]$, where

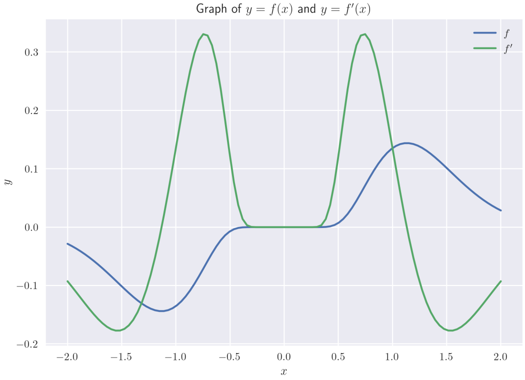
$$f(x) = \begin{cases} xe^{-x^2 - x^{-2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Use a numerical approximation for f' with $h = 0.001$.

Python Example

```python
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use LaTex and increase image resolution
plt.rcParams['text.usetex'] = True
plt.rcParams['figure.dpi'] = 300

# Use Seaborn style
plt.style.use('seaborn-v0_8')

# Define f
f = lambda x: x * np.exp(-x**2 - x**-2) if
     x != 0 else 0

# Define h
h = 0.001

# Use numerical approximation
f_prime = lambda x: (f(x + h) - f(x - h))
     /(2 * h)

# Get the x-values
x_vals = np.linspace(-2, 2, 100)

# Get the two sets of y-values
y1_vals = [f(x) for x in x_vals]
y2_vals = [f_prime(x) for x in x_vals]

# Generate the plot for f
plt.plot(x_vals, y1_vals, label = r"$f$")

# Generate the plot for f'
plt.plot(x_vals, y2_vals, label = r"$f'$")

# Label the x-axis
plt.xlabel('$x$')

# Label the y-axis
plt.ylabel('$y$')

# Give the graph a title
plt.title("Graph of $y = f(x)$ and $y = f'(
     x)$")

# Create a legend
plt.legend()

# Save the figure
plt.savefig(path + r'ex1-2.png')

# Display the plot
plt.show()
```

Python Example Result



Derivative Properties

Theorem

Suppose α and β are constants and f' and g' exist.

(a) $\dfrac{d}{dx}(\alpha f + \beta g) = \alpha f' + \beta g'$.

(b) $\dfrac{d}{dx}(f \cdot g) = g \cdot f' + f \cdot g'$

(c) $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{g \cdot f' - f \cdot g'}{g^2}$

(d) $\dfrac{d}{dx}(f \circ g) = (f' \circ g) \cdot g'$

Useful Derivative Formulas

Suppose $a > 0$.

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \ln a$
- $\frac{d}{dx}(\ln|x|) = \frac{1}{x},\ \ x \neq 0$
- $\frac{d}{dx}(\log_a|x|) = \frac{1}{x \ln a},\ \ x \neq 0$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Derivative Example

Example

$$\frac{d}{dx}\left(e^{1/x}\sin x\right) =$$

Tangent Lines

The tangent line of the graph of $y = f(x)$ at (x_0, y_0) is

$$y = y_0 + f'(x_0)(x - x_0).$$

Tangent Line Example

Example

Approximate $\sqrt{3.9}$.

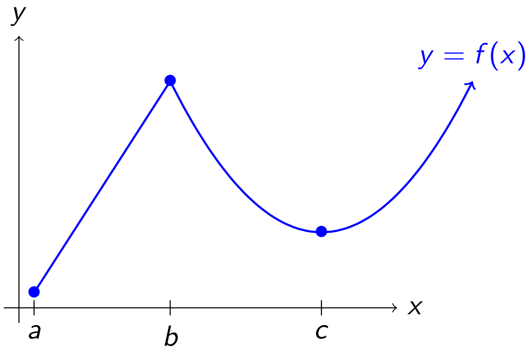
Local and Global Extrema

Definition

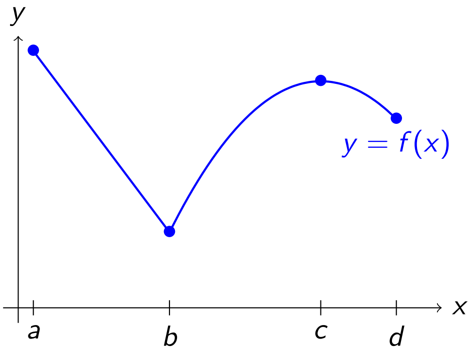
Let f be a function defined on domain D.

(a) The **global maximum** of f is at c if $f(c) \geq f(x)$ for all x in D. The **global minimum** of f is at c if $f(c) \leq f(x)$ for all x in D. The global maximum and global minimum values of f are called the **global extrema** of f.

(b) A **local maximum** of f is at c if there is an interval (a, b) such that $f(c) \geq f(x)$ for all x in (a, b) and $a < c < b$. Similarly, a **local minimum** of f is at c there is an interval (a, b) such that $f(c) \leq f(x)$ for all x in (a, b) and $a < c < b$. The local maximum and local minimum values of f are called the **local extrema** of f.



- Absolute minimum at $x = a$
- Local maximum at $x = b$
- Local minimum at $x = c$
- The function f grows without bounds so there is no global maximum



- Absolute maximum at $x = a$
- Absolute minimum at $x = b$
- Local maximum at $x = c$
- Nothing special at $x = d$

Extreme Value Theorem

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains a global maximum value $f(c)$ and a global minimum value $f(d)$ for some numbers c and d in $[a, b]$.

Increasing and Decreasing Functions

Assume f' exists.

- If $f'(x) > 0$, then f is increasing at x.
- If $f'(x) < 0$, then f is decreasing at x.

First Derivative Test

Suppose $x = c$ is a critical number, i.e. c is in the domain of f and $f'(c)$ is 0 or undefined.

(a) If f' changes from positive to negative at $x = c$, then f has a local maximum at $x = c$.

(b) If f' changes from negative to positive at $x = c$, then f has a local minimum at $x = c$.

(c) If f' does not change sign at $x = c$, then f has no local extremum at $x = c$.

Optimization Example

Example

Find the local and global extrema of the function $f(x) = x^3(x-2)^2$. Suppose the domain of f is the closed interval $[-1, 3]$.

Second Derivative Test

Suppose $f'(c) = 0$.

(a) If $f''(c) > 0$, then f has a local minimum at $x = c$.

(b) If $f''(c) < 0$, then f has a local maximum at $x = c$.

(c) If $f''(c) = 0$ or is undefined, then the test fails.

Local Extrema Example

Example

Find all local extrema of $g(x) = x^4 - 4x^3$.

L'Hôpital's Rule

Theorem (L'Hôpital's Rule)

Suppose the functions f *and* g *are defined on the open interval* (a, b)*,* $a < c < b$*,* f *and* g *are differentiable on* $(a, b) \setminus \{c\}$*, and* $g'(x) \neq 0$ *for all* x *in* $(a, b) \setminus \{c\}$*. If*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$$

and either

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \qquad \text{or} \qquad \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

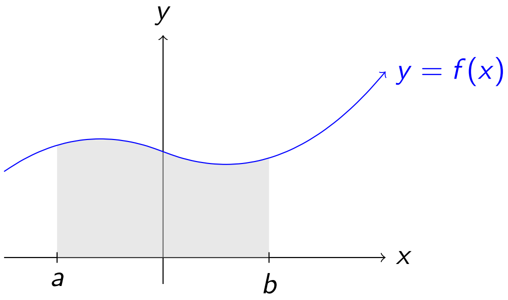
L'Hôpital's Rule

Example

Prove $\lim_{x \to \infty} x^{1/x} = 1$.

Integration

Definite Integration



The motivating problem for the definite integral is finding area under the graph $y = f(x)$ for $a \leq x \leq b$.

Riemann Sum

Definition

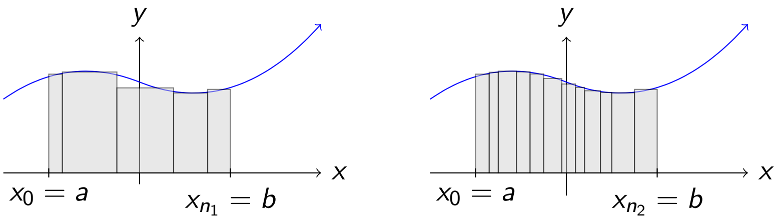
Suppose we have a function f defined on the interval $[a, b]$. Consider a **partition pair** P and T; $P = (x_0, x_1, \ldots x_n)$ and $T = (t_1, t_2, \ldots, t_n)$, where

$$a = x_0 \leq t_1 \leq x_1 \leq \ldots \leq x_{n-1} \leq t_n \leq x_n = b.$$

The **Riemann sum** corresponding to the partition pair P and T is defined to be

$$\mathcal{R}(f, P, T) = \sum_{k=1}^{n} f(t_k)\Delta x_k.$$

Finer and Finer Partition



The rectangles in the right figure do a better job of approximating the area under the curve. This is because the partition in the second figure is "finer", i.e. uses more x-values, than the first.

Partition Mesh

Definition

The **mesh** of partition P is

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}.$$

As $\|P\| \to 0$, the approximation becomes better and better.

Riemann Integral

Definition

The **Riemann integral** of f over the interval $[a, b]$ is

$$\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} \mathcal{R}(f, P, T)$$

whenever the limit converges.

Integral Theorems

Theorem

A bounded function on an interval $[a, b]$ is Riemann integrable if it is continuous for all but a finite number of points.

Theorem

If f and g are Riemann-integrable on $[a, b]$ and α and β are constants, then the following hold.

(a) $\displaystyle\int_a^b \alpha f(x) + \beta g(x)\ dx = \alpha \int_a^b f(x)\ dx + \beta \int_a^b g(x)\ dx$

(b) $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ *for any c in $[a, b]$*

Analytic Example

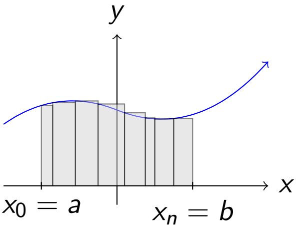
Example

Prove $\displaystyle\int_1^a \frac{dx}{x} = \ln a$ for $a > 1$. Use partition pairs of the form $P = (1, a^{1/n}, a^{2/n} \ldots, a)$ and $T = (1, a^{1/n}, a^{2/n} \ldots, a^{(n-1)/n})$.

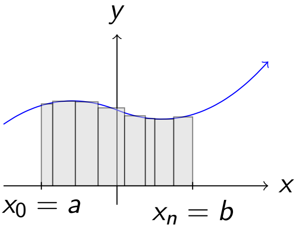
Selection of T

The most common choices for T.

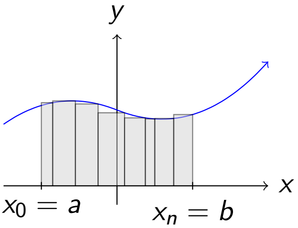
- Left endpoints: $t_k = x_{k-1}$
- Midpoints: $t_k = \frac{x_{k-1}+x_k}{2}$
- Right endpoints: $t_k = x_k$



Left endpoints



Midpoints



Right endpoints

Selection of P

The most common choice for P, by far, is the uniform partition

$$x_k = a + k\Delta x \qquad \text{and} \qquad \Delta x = \frac{b-a}{n}.$$

Useful formulas

When working analytic problems with a uniform partition, these formulas come up a lot:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

and

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2.$$

Uniform Partition Example

Example

Use uniform partitions and right endpoints to find $\int_0^1 x^2\, dx$.

Python Code

```
# Define Riemann sum
def riemann_sum(f, P, pts):

    # Sort values
    P = np.sort(P)

    # Calculate Delta x
    dx_vals = np.diff(P)

    # Define T
    if pts == 'left':

        T = P[:-1]

    elif pts == 'right':

        T = P[1:]

    elif pts == 'mid':

        T = (P[:-1] + P[1:])/2

    else:

        raise Exception('Currently only left, right, and midpoints are supported!')

    # Get area of rectangles; assumes f is vectorized
    rectangle_areas = f(T) * dx_vals

    # Return sum
    return np.sum(rectangle_areas)
```

Improper Integral

Definition

(a) If the integrals exists for every $t \geq a$ and for every $s \leq b$, then

$$\int_a^\infty f(x)\ dx = \lim_{t \to \infty} \int_a^t f(x)\ dx$$

and

$$\int_{-\infty}^b f(x)\ dx = \lim_{s \to -\infty} \int_s^b f(x)\ dx.$$

(b) For any c in $\mathbb{R}$, if both $\int_c^\infty f(x)\ dx$ and $\int_{-\infty}^c f(x)\ dx$ converge, then

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^c f(x)\ dx + \int_c^\infty f(x)\ dx.$$

Python Example

Example

Use `pandas` to create a data frame of Riemann sums with left endpoints, midpoints, and right endpoints, and uniform partitions to approximate $\int_0^{\infty} e^{-x^2/2}\,dx$. Consider $n = 10, 50, 100, 500$, and 1000.

Python Example Cont.

Solution. Since $e^{-x^2/2}$ goes to 0 rapidly, it's safe to use $b = 10$.

```
# Import pandas
import pandas as pd

# Define function
f = lambda x: np.e**(-x**2/2)

# Define the n-values
n_vals = [10, 50, 100, 500, 1000]

# Define the data frame
results = pd.DataFrame(index = n_vals, columns = ['left', 'mid', 'right'])

# Loop over values
for n in n_vals:
    # We can use np.linspace for a uniform partition
    partition = np.linspace(0, 10, n + 1)

    # Get left endpoint results
    results.loc[n, 'left'] = riemann_sum(f, partition, 'left')

    # Get midpoint results
    results.loc[n, 'mid'] = riemann_sum(f, partition, 'mid')

    # Get right endpoint results
    results.loc[n, 'right'] = riemann_sum(f, partition, 'right')

# Note: screenshot of output is ex1-3
results
```

Python Example Result

The limit as $\|P\| \to 0$ is $\sqrt{\pi/2} \approx 1.253$.

	left	mid	right
10	1.753314	1.253314	0.753314
50	1.353314	1.253314	1.153314
100	1.303314	1.253314	1.203314
500	1.263314	1.253314	1.243314
1000	1.258314	1.253314	1.248314

Indefinite Integral

Definition

The function F is an **indefinite integral** or **antiderivative** of f if $F'(x) = f(x)$. We write

$$\int f(x)\, dx = F(x)$$

to denote this.

Indefinite integrals are only unique up to a constant. For example, two antiderivatives of $2x$ are $x^2 + 1$ and $x^2 - 4$. To handle all possibilities, we write

$$\int 2x\, dx = x^2 + C.$$

Antiderivative Theorems

Theorem

Let f and g be continuous functions on some domain and let α and β be real numbers. Then

$$\int \alpha f(x) + \beta g(x)\ dx = \alpha \int f(x)\ dx + \beta \int g(x)\ dx.$$

Useful Antiderivative Formulas

- $\int x^n \; dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$
- $\int \frac{dx}{x} = \ln|x| + C$
- $\int \frac{dx}{1+x^2} = \arctan x + C$
- $\int e^x \; dx = e^x + C$
- $\int a^x \; dx = \frac{a^x}{\ln a} + C$
- $\int \sin x \; dx = -\cos x + C$
- $\int cosx \; dx = \sin x + C$
- $\int \tan x \; dx = -\ln|cosx| + C$

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Suppose f is continuous on the closed interval $[a, b]$. Then

(a) $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, *where* $F'(x) = f(x)$

(b) $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

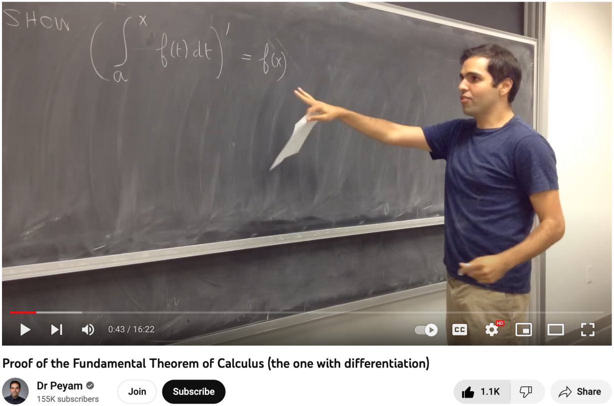
Fundamental Theorem of Calculus Example

Example

$$\int_0^2 \max\{x, 1\}\ dx =$$

Proof of the Fundamental Theorem on YouTube

Watch Peyam prove part (b) of the Fundamental Theorem of Calculus (`https://youtu.be/4DrCKhCECHo`).



u-Substitution

Theorem (u-Substitution)

Suppose g' is continuous on the closed interval $[a, b]$ and f is continuous on the range of g. Then

$$\int_a^b (f \circ g)(x) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Example

Example

$$\int xe^{-x^2/2}\, dx =$$

Integration by Parts

Theorem (Integration by parts)

Suppose F and G are differentiable functions, $F'(x) = f(x)$, and $G'(x) = g(x)$, where f and g are continuous. Then

$$\int F(x)g(x)\ dx = F(x)G(x) - \int f(x)G(x)\ dx.$$

Example

Example

$$\int \ln x \; dx =$$

Ordinary Differential Equations

Ordinary Differential Equations

Definition

(a) An **ordinary differential equation** (ODE) involves an unknown function of a single variable and some of its derivatives.

(b) The **order** of a differential equation is the order of the highest derivative that appears in the equation.

For example, $xy' = e^{xy}$ is a first order ordinary differential equation, while

$$\frac{d^3x}{dt^3} - 2t\frac{d^2x}{dt^2} + t^2x = \cos t$$

is a third order ordinary differential equation.

Separable ODEs

Definition

An ODE of the form
$$\frac{dy}{dx} = F(x,y)$$
is separable if $F(x,y) = f(x)g(y)$.

It's relatively easy to solve separable differential equations

$$\frac{dy}{dx} = f(x)g(y) \qquad \text{implies} \qquad \frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Hence,

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx \qquad \text{implies} \qquad \int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

Separable ODE Example

Example

Solve the differential equation

$$\frac{dy}{dt} = \frac{ty + 3t}{t^2 + 1}$$

subject to the initial condition $y(0) = 2$.

Linear ODEs

Definition

A first order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

The idea behind solving these is to find $\mu = f(x)$ such that

$$\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \mu P(x) y.$$

Using the product rule, it becomes clear

$$\frac{d\mu}{dx} = \mu P(x) \qquad \text{implies} \qquad \mu = \exp\left(\int P(x)\, dx\right).$$

Linear ODEs Example

Example

Solve the differential equation

$$x\frac{dy}{dx} + 3x^3 y = 6x^3.$$

Sequences and Series

Sequences

Definition

A sequence $(a_n)_{n=1}^{\infty}$ is said to **converge**, if there is a value a in $\mathbb{R}$ which has the property that: For all $\epsilon > 0$, there exists an integer N such that $n \geq N$ implies that $|a_n - a| < \epsilon$. We often write

$$a_n \to a \qquad \text{or} \qquad \lim_{n\to\infty} a_n = a$$

when $(a_n)_{n=1}^{\infty}$ converges to a. If $(a_n)_{n=1}^{\infty}$ does not converge, then it **diverges**.

Sequences Example

Example

Determine which of the following sequences converge/diverge. If the sequence converges, find its limit.

(a) $a_n = \frac{1}{n}$

(b) $b_n = \sqrt{n}$

(c) $c_n = (-1)^n$

(d) $d_n = 1 + \frac{(-1)^n}{n}$

Python Sequences Example

Example

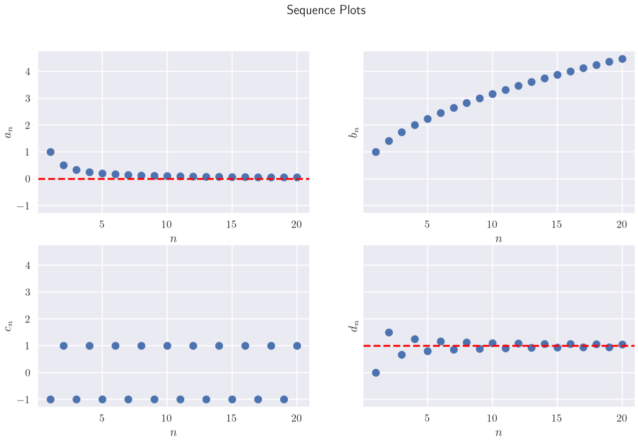
Use Python to graph the four sequences in the previous example. Graph them on separate subplots, and for the sequences that converge use horizontal lines to show their respective limits.

Python Sequences Example Solution

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use LaTeX and increase resolution
plt.rcParams['text.usetex'] = True
plt.rcParams['figure.dpi'] = 300

# Use Seaborn style
plt.style.use('seaborn-v0_8')

# Define functions
a = lambda n: 1/n
b = lambda n: np.sqrt(n)
c = lambda n: (-1)**n
d = lambda n: 1 + (-1)**n/n

# Define the limits
a_lim, d_lim = 0, 1

# Get the n-values
n_vals = np.arange(1, 21)

# Get the sequence values
# Functions already vectorized
a_vals = a(n_vals)
b_vals = b(n_vals)
c_vals = c(n_vals)
d_vals = d(n_vals)

# Set up subplots
fig, ax = plt.subplots(2, 2, sharey = True,
        figsize = (10, 6))
```

```
# Plot a_n and its limit
ax[0, 0].scatter(n_vals, a_vals)
ax[0, 0].axhline(y = a_lim, color = 'r',
        linestyle = 'dashed')
ax[0, 0].set_xlabel('$n$')
ax[0, 0].set_ylabel('$a_n$')

# Plot b_n and its limit
ax[0, 1].scatter(n_vals, b_vals, label = '
        $b_n$')
ax[0, 1].set_xlabel('$n$')
ax[0, 1].set_ylabel('$b_n$')

# Plot c_n and its limit
ax[1, 0].scatter(n_vals, c_vals)
ax[1, 0].set_xlabel('$n$')
ax[1, 0].set_ylabel('$c_n$')

# Plot d_n and its limit
ax[1, 1].scatter(n_vals, d_vals)
ax[1, 1].axhline(y = d_lim, color = 'r',
        linestyle = 'dashed')
ax[1, 1].set_xlabel('$n$')
ax[1, 1].set_ylabel('$d_n$')

plt.suptitle('Sequence Plots')

# Save the figure
plt.savefig(path + r'ex1-4.png')

plt.show()
```

Python Sequences Example Result



Triangle Inequality

For any real numbers x, y, and z,

$$|x - y| \leq |x - z| + |z - y|.$$

Sequences

Theorem

(a) *The sequence* $(a_n)_{n=1}^{\infty}$ *converges to* a *in* $\mathbb{R}$ *if and only if for every* $\epsilon > 0$, *we have* a_n *in the interval* $(a - \epsilon, a + \epsilon)$ *for all but finitely many* n.

(b) *If* $(a_n)_{n=1}^{\infty}$ *converges to both* a *and* b, *then* $a = b$.

(c) *If* $(a_n)_{n=1}^{\infty}$ *converges, then it is bounded. That is, convergence of* $(a_n)_{n=1}^{\infty}$ *implies there exists a real number* B *such that* $|a_n| \leq B$ *for all* n.

Sequence Properties

Theorem

Suppose $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *are real numbered sequences and*

$$\lim_{n\to\infty} a_n = a \qquad \textit{and} \qquad \lim_{n\to\infty} b_n = b.$$

Let α *and* β *be real constants.*

(a) $\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha a + \beta b$

(b) $\lim_{n\to\infty} a_n b_n = ab$

(c) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ *if* $a_n \neq 0$ *and* $a \neq 0$

Monotonic Sequences

Definition

- A real sequence $(a_n)_{n=1}^{\infty}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n.
- A real sequence $(a_n)_{n=1}^{\infty}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n.

Theorem

Suppose that $(a_n)_{n=1}^{\infty}$ is monotonic. Then it converges if and only if it is bounded.

Series

Definition

Consider a series $S = \sum_{k=1}^{\infty} a_k$. Its ***n*-th partial sum** is $S_n = \sum_{k=1}^{n} a_k$. The series S **converges** if the sequence $(S_n)_{n=1}^{\infty}$ converges, and it **diverges** otherwise.

Example

Example

For what values of r does the geometric series $\displaystyle\sum_{k=1}^{\infty} r^{k-1}$ converge?

Geometric Series

The geometric series is extremely important in finance. Remember these formulas.

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$$

and

$$\sum_{k=1}^{\infty} ar^{k-1} = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ DNE, & |r| \geq 1 \end{cases}$$

Partial Fraction Decomposition

- $\dfrac{ax+b}{(x-c)(x-d)} = \dfrac{A}{x-c} + \dfrac{B}{x-d}, \quad c \neq d.$
- $\dfrac{ax+b}{(x-c)^2} = \dfrac{A}{x-c} + \dfrac{B}{(x-c)^2}.$

Partial Fraction Decomposition Example

Example

Rewrite $\dfrac{x-2}{x(x-3)}$ using partial fraction decomposition.

Convergent Series

Example

Show $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges.

Divergence Test

- If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.
- If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ may or may not converge.

Property of Series

Theorem

Suppose that $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *converge. For any real constants* α *and* β

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k.$$

Dominating Series

Theorem

If a series $\sum_{k=1}^{\infty} b_k$ *dominates a series* $\sum_{k=1}^{\infty} a_k$ *in the sense that for all sufficiently large* k, $|a_k| \leq b_k$, *then convergence of* $\sum_{k=1}^{\infty} b_k$ *implies convergence of* $\sum_{k=1}^{\infty} a_k$.

Dominating Series Example

Example

Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{2^k}$ converges.

Integral Test

Theorem (Integral Test)

(a) *If* $|a_k| \leq f(x)$ *for all sufficiently large* k *and all* x *in the interval* $(k-1, k]$, *then convergence of* $\int_1^\infty f(x)\,dx$ *implies convergence of* $\sum_{k=1}^\infty a_k$.

(b) *If* $|f(x)| \leq a_k$ *for all sufficiently large* k *and all* x *in the interval* $[k, k+1)$ *then diverges of* $\int_1^\infty f(x)\,dx$ *implies divergence of* $\sum_{k=1}^\infty a_k$.

p-Series Example

Example

Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Python p-Series Example

Example

Graph

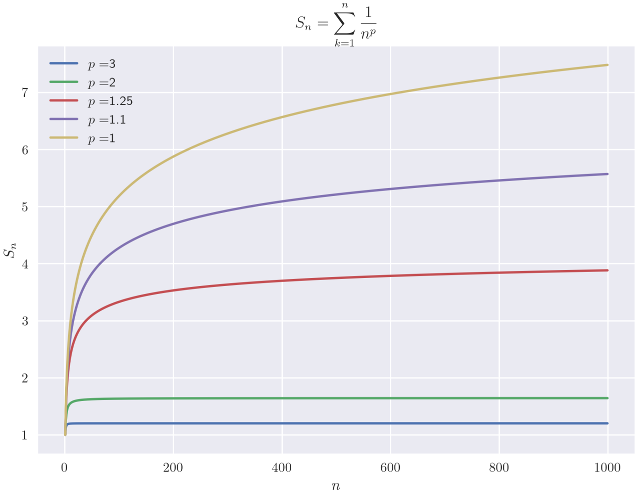
$$S_n = \sum_{k=1}^{n} \frac{1}{k^p}$$

as a function of n for $p = 3, 2, 1.25, 1.1$, and 1.

Python Sequences Example Solution

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use LaTeX and increase resolution
plt.rcParams['text.usetex'] = True
plt.rcParams['figure.dpi'] = 300

# Use Seaborn style
plt.style.use('seaborn-v0_8')

# Define p-values
p_vals = [3, 2, 1.25, 1.1, 1]

# Get n-values
n_vals = np.arange(1, 1_000, dtype = float)

# Loop over p-values
for p in p_vals:

    # Calculate terms of sum
    Sn = n_vals**(-p)

    # Take the cumulative sum
    Sn = np.cumsum(Sn)

    # Plot result
    plt.plot(n_vals, Sn, label = '$p = $' +
      str(p))

# Create x- and y-labels
plt.xlabel('$n$')
plt.ylabel('$S_n$')

# Add a legend
plt.legend()

# Add a title
plt.title(r'$S_n = \displaystyle\sum_{k =
    1}^n \frac{1}{n^p}$')

# Save the figure
plt.savefig(path + r'ex1-5.png')

plt.show()
```

Python p-Series Example Result



Alternating Series Test

Theorem

Suppose the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$ *is such that* $0 \leq b_{k+1} \leq b_k$ *for sufficiently large* k*. Then the series converges if* $\lim_{n\to\infty} b_k = 0$.

Alternating Series Test Example

Example

Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

Absolutely and Conditionally Convergent Series

Definition

A series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ converges. A series is **conditionally convergent** if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ does not.

For example, the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

is conditionally convergent but not absolutely.

Ratio Test

Theorem

(a) *If* $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent.*

(b) *If* $\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right| > 1$*, then the series* $\sum_{k=1}^{\infty} a_k$ *is divergent.*

(c) *If* $\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$*, then the test fails.*

Ratio Test Example

Example

What can be said about the convergence of $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{k!}{k^k}$?

Power Series

Definition

A **power series** centered at c is a series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k.$$

If the series converges for $|x-c|<R$ and diverges for $|x-c|>R$, then R is the **radius of convergence**. The **interval of convergence** I is the set of all x values where the series converges.

Remark: We assume $0^0 = 1$ within our power series, so the power series always converges at $x = c$.

Power Series Example

Example

Find the radius and interval of convergence of the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-3)^k (x+1)^k}{\sqrt{k+1}}$.

Differentiation and Integration of Power Series

Theorem

If the power series $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ *has radius of convergence* $R > 0$, *then both*

(a) $f'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$ *and*

(b) $\int f(x)\, dx = C + \sum_{k=0}^{\infty} a_k \frac{(x-c)^{k+1}}{k+1}$

have radii of convergence R.

Differentiation of Power Series Example

Example

Use power series manipulation to find the value of $\displaystyle\sum_{k=1}^{\infty} \frac{k}{1.10^k}$.

Power Series Manipulation on YouTube

Watch BlackPenRedPen explain how to use power series manipulation to find the power series of $\ln(1 + x)$ (https://youtu.be/qxzVGCuiU6c?si=aNo9hDDWZyLAxDVl).

Taylor Polynomial

Definition

The r-th order **Taylor polynomial** centered at c of a function f is

$$P(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(r)}(c)}{r!}(x-c)^r$$
$$= \sum_{k=0}^{r} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

For the r-th order Taylor polynomial to exist, f must be differentiable at least r times at c.

Taylor's Approximation Theorem

Theorem

Assume that f is differentiable at least r times at c.

(a) *The function $f(x)$ can be expressed as the sum of its Taylor polynomial of degree r centered at c and a remainder term $R(x-c)$:*

$$f(x) = \sum_{k=0}^{r} \frac{f^{(k)}(c)}{k!}(x-c)^k + R(x-c),$$

where

$$\lim_{x \to c} \frac{R(x-c)}{(x-c)^r} = 0.$$

(b) *The r-th order Taylor polynomial centered at c is the only polynomial of degree less than or equal to r with the property described in (a).*

(c) *If f is differentiable $r+1$ times on an open interval containing both c and x, then for any given x in that interval, there exists a number z between x and c such that the remainder term can be expressed in the form:*

$$R(x-c) = \frac{f^{(r+1)}(z)}{(r+1)!}(x-c)^{r+1}.$$

Popular Taylor Series Centered at Zero

- $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$ for $x \in (-1, 1)$
- $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$
- $\sin x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for $x \in \mathbb{R}$
- $\cos x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ for $x \in \mathbb{R}$
- $\arctan x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ for $x \in [-1, 1]$

Taylor's Theorem Example

Example

Prove $\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ for $x \in [-1, 1]$.

Error Bound

Part (c) of Taylor's theorem induces an error bound between f and its r-th order Taylor polynomial centered at c. In particular,

$$|f(x) - P(x)| \leq \frac{\max_{x \in I} \left| f^{(r+1)}(x) \right|}{(r+1)!} |x - c|^{r+1}$$

where $P(x)$ is the Taylor polynomial and I is a closed interval containing x and c.

Error Bound Example

Example

The first order Taylor polynomial $1 - x$ is used to approximate e^{-x} on the closed interval $[0, 1]$. Find the error bound.

Python Error Bound Example

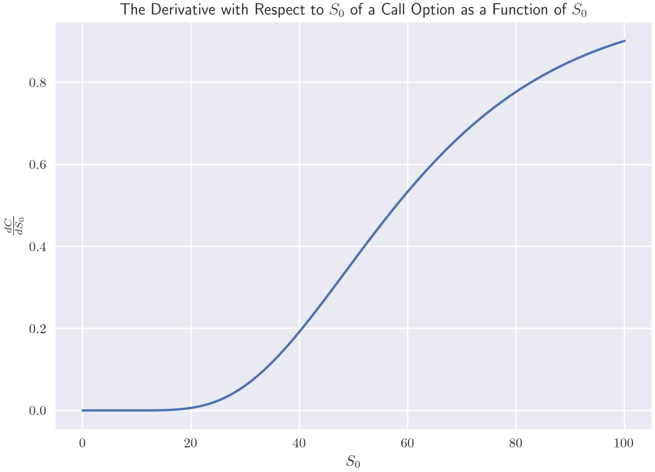
Example

Graph the Taylor polynomial $1 - x$, the original function e^{-x}, and the error bound on the closed interval $[0, 1]$.

Python Error Bound Example

```
# Import modules
import numpy as np
import matplotlib.pyplot as plt

# Use LaTeX and increase resolution
plt.rcParams['text.usetex'] = True
plt.rcParams['figure.dpi'] = 300

# Use Seaborn style
plt.style.use('seaborn-v0_8')

# Define Taylor polynomial
P = lambda x: 1 - x

# Get error function
error_fun = lambda x: 1/2 * np.abs(x)**2

# Get x-values
x_vals = np.linspace(0, 1, 100)

# Get y-values of approximation; P is
    vectorized
y_approx = P(x_vals)

# Get true y-values; np.exp is vectorized
y_true = np.exp(-1 * x_vals)

# Get the error; error_fun is vectorized
y_error = error_fun(x_vals)

# Plot graph of approximation
plt.plot(x_vals, y_approx, label = '$y = 1
    - x$')

# Plot graph of truth
plt.plot(x_vals, y_true, linestyle = '
    dashed', label = '$y = e^{-x}$')

# Plot the error bound
plt.fill_between(x_vals, y_approx - y_error
    , y_approx + y_error, label = 'Error
    Bound', alpha = 0.25)

# Create x- and y-labels
plt.xlabel('$x$')
plt.ylabel('$y$')

# Add legend
plt.legend()

# Add title
plt.title('Approximation of $e^{-x}$ Using
    Taylor Polynomial $1- x$ and
    Corresponding Error Bound')

# Save the figure
plt.savefig(path + r'ex1-6.png')

plt.show()
```

Python Error Bound Result



Time Value of Money

Time Value of Money

For a time t cash flow C_t discounted at rate r, the *present value* is

$$PV = \frac{C_t}{(1+r)^t}.$$

The time T *future value* is

$$FV_T = PV \cdot (1+r)^T = C_t \cdot (1+r)^{T-t}.$$

Compound Interest

We assumed that interest is compounded once per unit of time. However, if it is compounded n times per unit of time the formulas become

$$PV = \frac{C_t}{\left(1+\frac{r}{n}\right)^{nt}} \qquad \text{and} \qquad FV_T = PV \cdot \left(1+\frac{r}{n}\right)^{nT}.$$

Since

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

for continuous compounding (i.e. $n=\infty$) the formulas are

$$PV = C_t e^{-rt} \qquad \text{and} \qquad FV_T = PV \cdot e^{rT} = C_t \cdot e^{r(T-t)}$$

Multiple Cash Flows

Suppose we have a sequence of cash flows $C_0, C_1, C_2, \ldots, C_T$, where the subscript denotes the time of the cash flow, the *net present value* (NPV) of these cash flows discounted at the constant rate r is

$$NPV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^T}.$$

Time Value of Money Python Example

Example

Time	0	1	2	3	4
Cash Flow	-100	50	20	70	10

Calculate the net present value given a continuously compounded discount rate of 5%.

Time Value of Money Python Example

```python
# Import module
import numpy as np

# Record rate
rate = 0.05

# Record time of cash flows
time = np.array([0, 1, 2, 3, 4])

# Record cash flows
cash_flows = np.array([-100, 50, 20, 70, 10])

# Get the NPV
NPV = np.sum(cash_flows * np.exp(-rate * time))

print(f'The NPV of the cash flows is {NPV:.2f}.')
```

$NPV \approx 34.10$

`numpy_finance` Module

There is a `numpy_finance` module that has a net present value function (`https://numpy.org/numpy-financial/latest/npv.html`). However, we would need the annual rate to use it in the last example, i.e. we would have to use the rate

$$100\% \times (e^{0.05} - 1) \approx 5.127\%.$$

Time Value of Money Example

Example

Jain borrows \$1,000,000 to purchase a house. The loan is for thirty years and her first payment is one month from when she initially borrows the money. If her annualized rate is 12%, what will be her monthly payments? Ignore fees.

Growing Payments

Suppose 1 is payed at time 1, and payments increase at a rate of g each subsequent period until a final payment of $(1+g)^{n-1}$ is made at time n. If cash flows are discounted at rate r, then the NPV of the cash flows is

$$\frac{1-\left(\frac{1+g}{1+r}\right)^{n}}{r-g}.$$

Growing Payments Example

Example

Calculate the NPV of the series of end-of-year cash flows. Assume

- \$100 is payed in the first year,
- each subsequent year payments increase by 5%,
- the final payment is made at the end of year ten, and
- the discount rate is 8%.

Options

Call and Put Options

Definition

- A **call option** gives the owner the right—but not obligation—to purchase an underlying security at a predetermined price.
- A **put option** gives the owner the right—but not obligation—to sell an underlying security at a predetermined price.

European and American Options

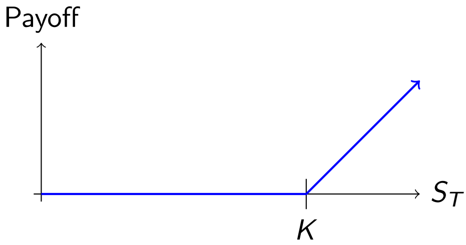
Definition

- A **European option** gives the owner the right to excersie her option *only* at maturity.
- An **American option** gives the owner the right to excercise her option at any time before maturity.

So, American options give their owners more chances to exercise their right to buy or sell. However, that also makes American options more expensive to purchase!

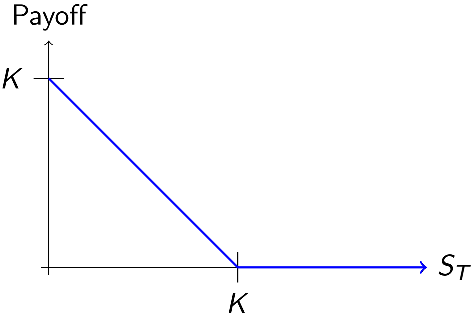
Call Option Payoff Diagram

Suppose we have a European call option, where we may buy stock S_0 at time T for price K. Then the payoff diagram, i.e. the amount we would make at time T given the stock is at price S_T is shown below.



Put Option Payoff Diagram

Suppose we have a European put option, where we may sell stock S_0 at time T for price K. Then the payoff diagram, i.e. the amount we would make at time T given the stock is at price S_T is shown below.



Strike Price

Definition

The **strike price** K is the value at which the owner of a call can buy the underlying or the owner of a put can sell the underlying.

Payoff Formula

For a call option with strike price K on underlying security S_0, the formula for the payoff is

$$\max\{S_T - K, 0\} = (S_T - K)^+.$$

For a put option with strike price K on underlying security S_0, the formula for the payoff is

$$\max\{K - S_T, 0\} = (K - S_T)^+.$$

Put-Call Parity

- Suppose C_t is the price of a call option with strike price K on underlying S_t.
- Suppose P_t is the price of a put option with strike price K on the same underlying S_t.

$$C_t - P_t = S_t - PV(K).$$

This formula assumes that the underlying does *not* pay dividends.

Black-Scholes Call Price

The price of a European call option under the Black-Scholes framework is

$$C_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

We're assuming that the underlying asset does *not* pay dividends!

Black-Scholes Call and Put Price in Python

```
# Import module
import numpy as np
from scipy.stats import norm

# Function to give the value of a European call option
def bs_call(S, K, T, r, sigma):

    # Calculate d_1 and d_2
    d_1 = (np.log(S/K) + (r + sigma**2/2) * T)/(sigma * np.sqrt(T))
    d_2 = d_1 - sigma * np.sqrt(T)

    # Calculate C
    C = S * norm.cdf(d_1) - K * np.exp(-r * T) * norm.cdf(d_2)

    return C
# Function to give the value of a European put option
def bs_put(S, K, T, r, sigma):

    # Get the price of a call
    C = bs_call(S, K, T, r, sigma)

    # Use put-call parity
    P = C - S + K * np.exp(-r * T)

    return P
```

Black-Scholes Python Example

Example

Suppose that for a call option with underlying security S_0, we have $K = 70$, $T = 2$, $r = 5\%$, and $\sigma = 30\%$. Plot the derivative of the price of the call as a function of S_0.

Black-Scholes Python Example

```python
import matplotlib.pyplot as plt

# Use LaTeX and increase resolution
plt.rcParams['text.usetex'] = True
plt.rcParams['figure.dpi'] = 300

# Define h
h = 0.001

# Define Black-Scholes vars
K, T, r, sigma = 70, 2, 0.05, 0.30

# Calculate C'
C_prime = lambda S: (bs_call(S + h, K, T, r, sigma) - bs_call(S - h, K, T, r, sigma))/(2
    * h)

# Define s-values
s_vals = np.linspace(1e-2, 100, 100)

# Calculate C_prime values; function vectorized
c_prime_vals = C_prime(s_vals)

# Plot relationship
plt.plot(s_vals, c_prime_vals)

# Give plot x- and y-labels
plt.xlabel('$S_0$')
plt.ylabel(r"$\frac{dC}{dS_0}$")

# Add a title
plt.title("The Derivative with Respect to $S_0$ of a Call Option as a Function of $S_0$")

# Save the figure
plt.savefig(path + r'ex1-7.png')

plt.show()
```

Black-Scholes Python Result