Solution to April 2025 Jane Street Puzzle

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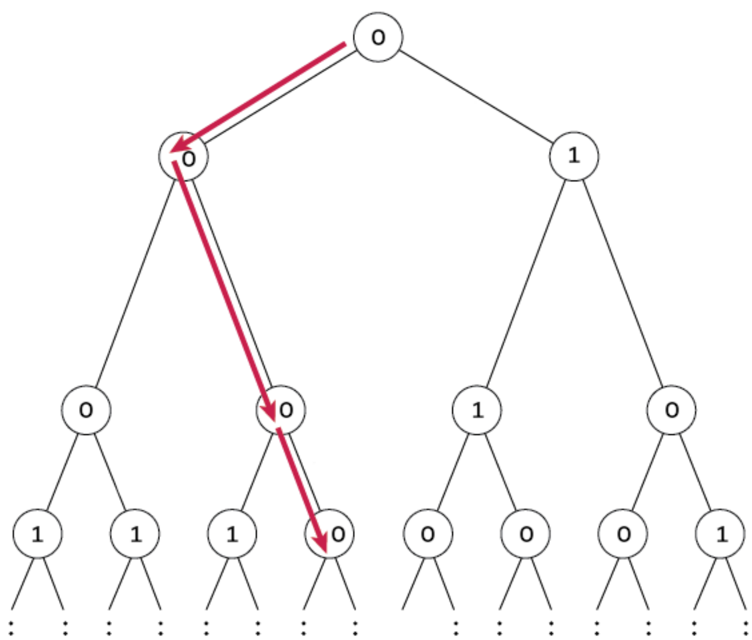
Problem Statement

For a fixed p, independently label the nodes of an infinite complete binary tree 0 with probability p, and 1 otherwise. For what p is there exactly a $\frac{1}{2}$ probability that there exists an infinite path down the tree that sums to at most 1 (that is, all nodes visited, with the possible exception of one, will be labeled 0). Find this value of p accurate to 10 decimal places.



Solution

Main System of Equations

The key insight to our solution is the recrusive nature of the problem. We can think about the problem as an infinite series of events, where each event occurs at a given node, and determines which of the two children to descend to for the next step of the path.

1. If we have not visited any nodes with value 1, then we write the probabillity of outcome A "that there exists an infinite path down the tree that sums to at most 1" as $P(A)$.
2. If we have already visited a node with value 1, then we write the probability of outcome A_0 "that there exists an infinite path down the tree that sums to exactly 0" as $P(A_0)$.

Due to the recursive nature of the problem, we can express $P(A)$in terms of $P(A_0)$ (and rewrite $1-(1-x)(1-x)$ as $2x-x^2$):

$$\begin{aligned} P(A) &= p \cdot (1-(1-P(A))(1-P(A))) + (1-p) \cdot (1-(1-P(A_0))(1-P(A_0))) \\ &= p \cdot \left(2P(A) - P(A)^2\right) + (1-p) \cdot \left(2P(A_0) - P(A_0)^2\right) \end{aligned}$$

This equation expresses the probability of a particular level in the tree having an infinite path down the tree with sum of at most 1, by breaking it up into the sum of probabilities of two possible outcomes:

1. The next choice in the path (next level in the tree) is labeled 0, which has probability p (first term of expression)
2. The next choice in the path (next level in the tree) is labeled 1, which has probability $1-p$ (second term of expression)

Side note on where first term comes from: if the next choice in the path is labeled 0, then the probability of event A in the sub-trees of the child nodes can also be expressed in terms of $P(A)$. Mathematically, we want the union of the probability of event A in the left child subtree with the probability of event A in the right child subtree, which are ultimately the same as the probability of event A in the parent tree:

$$\begin{aligned} P_{left}(A) \bigcup P_{right}(A) &= ! \left(!P_{left}(A) \bigcap !P_{right}(A) \right) \\ &= 1 - \left((1 - P_{left}(A)) \bigcap (1 - P_{right}(A)) \right) \\ &= 1 - ((1-P(A))(1-P(A))) \\ &= 2P(A) - P(A)^2 \end{aligned}$$

Similarly, we can use the recursive nature of the problem to express $P(A_0)$ in terms of $P(A_0)$. The difference between $P(A)$ in terms of $P(A_0)$ and $P(A_0)$ in terms of $P(A_0)$ is, once we have a path that has already visited a node with value 1, the recursive relation must eliminate any subtree options that contain a 1. Rather than having two possible outcomes, there is now only one possible outcome:

1. The next choice in the path (next level in the tree) is labeled 0, which has probability p

Writing that recursive expression gives:

$$\begin{aligned} P(A_0) &= p \cdot (1 - (1 - P(A_0))\,(1 - P(A_0))) \\ &= p \cdot \left(2P(A_0) - P(A_0)^2\right) \end{aligned}$$

Last but not least, we have a target value for the event A, which is exactly $\frac{1}{2}$, so that gives a system of 3 equations and 3 unknowns ($P(A)$, $P(A_0)$, p):

$$P(A) = p \cdot \left(2P(A) - P(A)^2\right) + (1-p) \cdot \left(2P(A_0) - P(A_0)^2\right) \tag{1}$$

$$P(A_0) = p \cdot \left(2P(A_0) - P(A_0)^2\right) \tag{2}$$

$$P(A) = \frac{1}{2} \tag{3}$$

Cubic Equation

Solving (2) for $P(A_0)$ gives:

$$P(A_0) = \frac{2p-1}{p}$$

Substituting into (1), and using (3), we get:

$$p \cdot \left(2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2\right) + (1-p)\left(2\left(\frac{2p-1}{p}\right) - \left(\frac{2p-1}{p}\right)^2\right) = \frac{1}{2}$$

Rearranging and simplifying, we get a cubic relation in terms of p:

$$3p^3 - 10p^2 + 12p - 4 = 0$$

Solving numerically, we get:

$$\boxed{p = 0.5306035754}$$