Math Modeling Homework 1

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1 Whaling

For a given population of whales, we define the following values.

- The intrinsic population growth rate, $r = 0.05$.
- The maximum sustainable population, $K = 300,000$.
- The current population size, x.
- The annual growth rate of the population without harvesting, $G = rx(1 - x/K)$.
- The number of boat days allowed per year, E.
- The harvest rate, $H = 0.00001Ex$.
- The total annual population growth rate, $A = G - H$.

The population reaches a steady state when the total annual population growth rate $A = 0$.

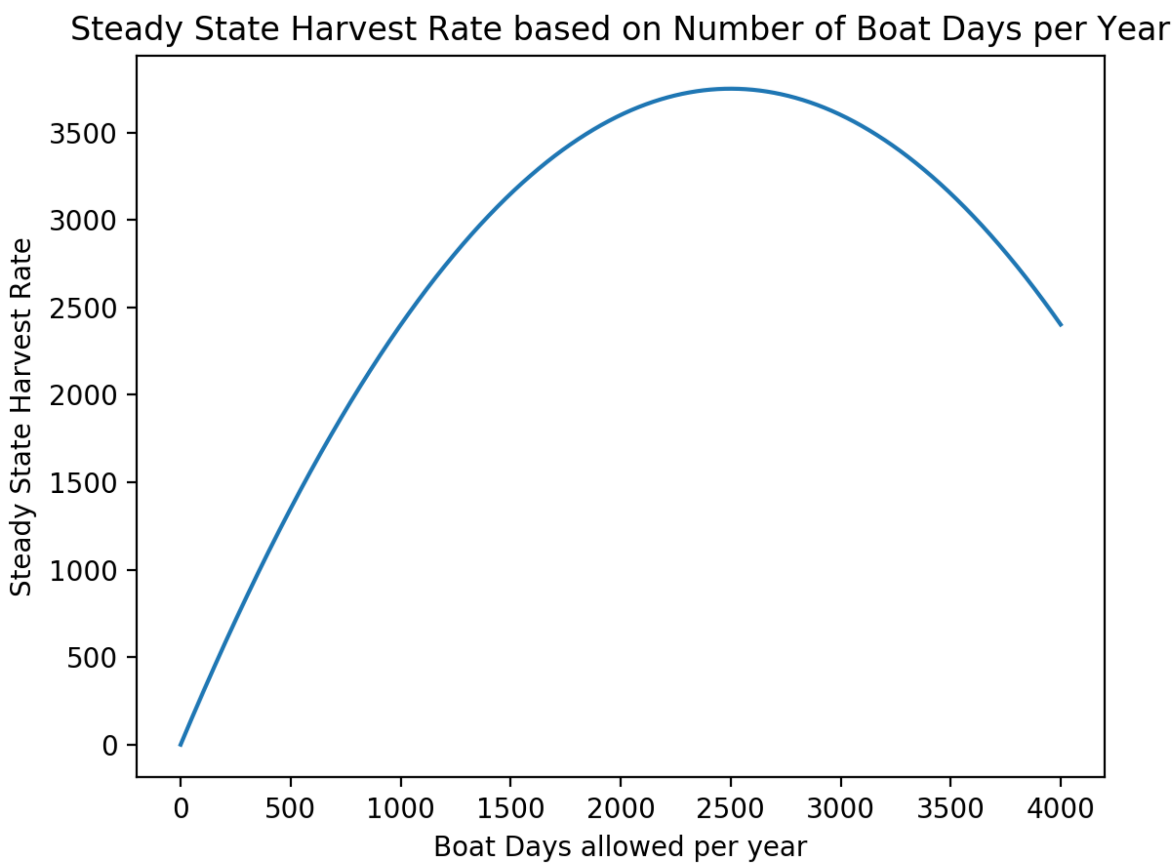
(a) We want to maximize the steady-state harvest rate with respect to the amount of boat days allowed per year. We know that the population reaches a steady state when $A = G - H = 0$. So H reaches a steady state when $H = G$. Let's call the steady state harvest rate H_s. So we want to get H_s as a function of E, and then we can take the derivative and set it equal to zero in order to maximize it. If x_s is the steady state population, then $H_s = 0.00001Ex_s$. If G_s is the steady state growth rate without harvesting, we can use the constraint that $G_s = H_s$ to find x_s.

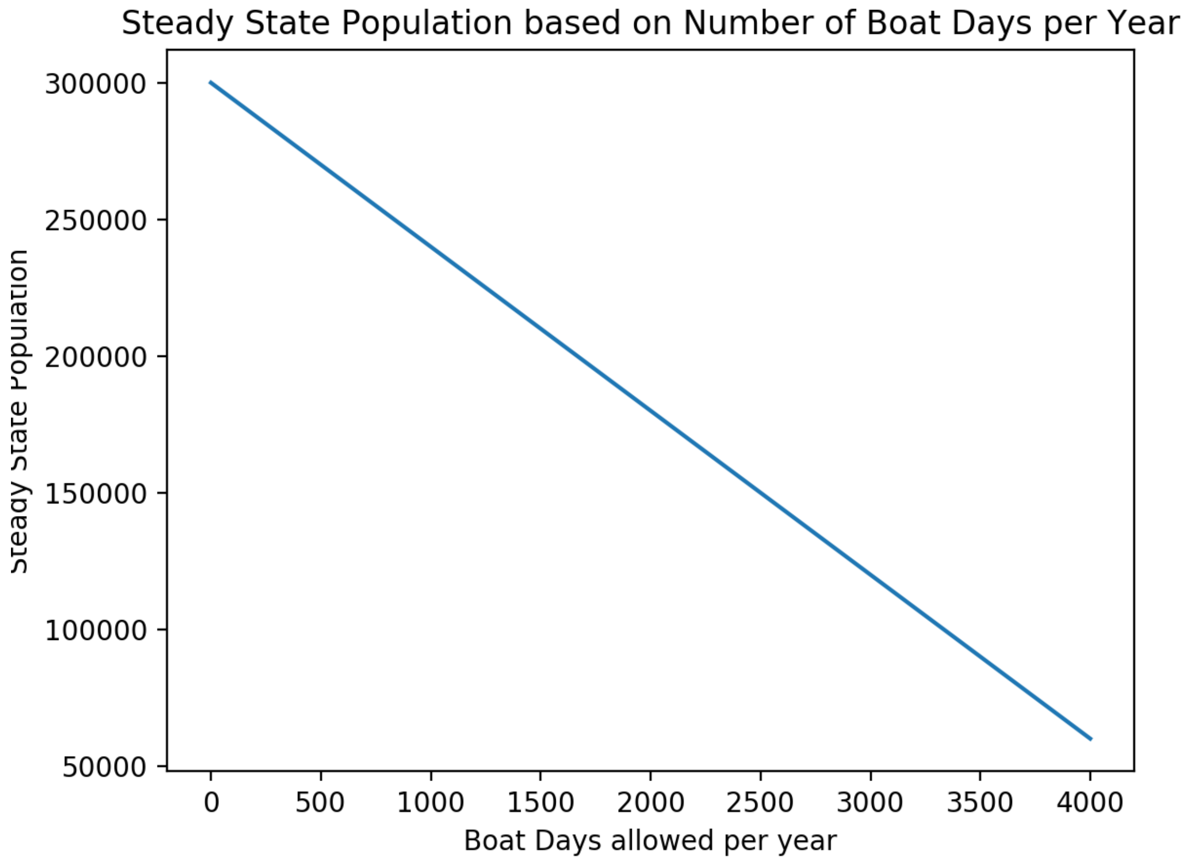
$$\begin{aligned} G_s &= H_s \\ rx_s(1 - x_s/K) &= 0.00001Ex_s \\ r(1 - x_s/K) &= 0.00001E \\ x_s/K &= 1 - \frac{0.00001E}{r} \\ x_s &= K \cdot \left(1 - \frac{0.00001E}{r}\right) \end{aligned}$$

Now that we've found x_s in terms of known constants, we can plug it into the equation for H_s and maximize with respect to E.

$$
\begin{aligned}
H_s &= 0.00001Ex_s \\
H_s &= 0.00001E \cdot K \cdot \left(1 - \frac{0.00001E}{r}\right) \\
H_s &= 0.00001 \cdot K \cdot E - \frac{1}{r} \cdot 0.00001^2 \cdot K \cdot E^2 \\
H_s &= 3E - 0.0006E^2 \\
\frac{dH_s}{dE} &= 3 - 0.0012E = 0 \\
E &= \frac{3}{0.0012} = 2,500
\end{aligned}
$$

So we've found that $E = 2,500$ maximizes the steady state harvest rate. We know that this is a maximum because $\frac{d^2H_s}{dE^2} = -0.0012$, and so it is always negative, which means that H_s is always concave down, so $E = 2,500$ in fact maximizes the steady state harvest rate. Plugging this back in, we see that $H_s = 3(2,500) - 0.0006(2,500)^2$, and so the maximum steady state harvest rate H_s is 11,250 whales per year. We also know that since $x_s = K \cdot \left(1 - \frac{0.00001(2,500)}{r}\right)$, then the steady state population x_s is 150,000. We can plot both the steady state harvest rate and steady state population as functions of E.





(b) We want to measure the sensitivity of the optimal boat days, let's call it E_o, to r. This is defined as $S(E_o, r) = \frac{dE_o}{dr} \cdot \frac{r}{E_o}$. We want to get a function that gives us the E_o in terms of r. From above, we have the following.

$$
\begin{aligned}
H_s &= 0.00001 \cdot K \cdot E - \frac{1}{r} \cdot 0.00001^2 \cdot K \cdot E^2 \\
H_s &= 3E - 0.00003 \cdot \frac{1}{r} E^2 \\
\frac{dH_s}{dE} &= 3 - 0.00006 \cdot \frac{1}{r} E_o = 0 \\
0.00006 \cdot \frac{1}{r} E_o = 3 & \\
E_o &= \frac{3r}{0.00006} = 50,000r \\
\frac{dE_o}{dr} &= 50,000
\end{aligned}
$$

So we have $S(E_o, r) = 50,000 \cdot \frac{r}{E_o}$. When we evaluate at $r = 0.05$, we get that $S(E_o, r = 0.05) = 50,000 \cdot \frac{0.05}{2,500}$, so $S(E_o, r = 0.05) = 1$. This means that a 1% increase in the intrinsic growth rate would result in a 1% increase in the optimal number of boat days allowed. This makes sense because if the intrinsic growth were larger, then there would be more whales to harvest. It also makes sense that a relatively small change in r would only affect E_o a small amount, because they are linearly related and r is very small.

We now want to measure the sensitivity of the steady state population x_s to r. This is defined as $S(x_s, r) = \frac{dx_s}{dr} \cdot \frac{r}{x_s}$. We want to get a function that gives us the x_s in terms of r. From above, we have the following.

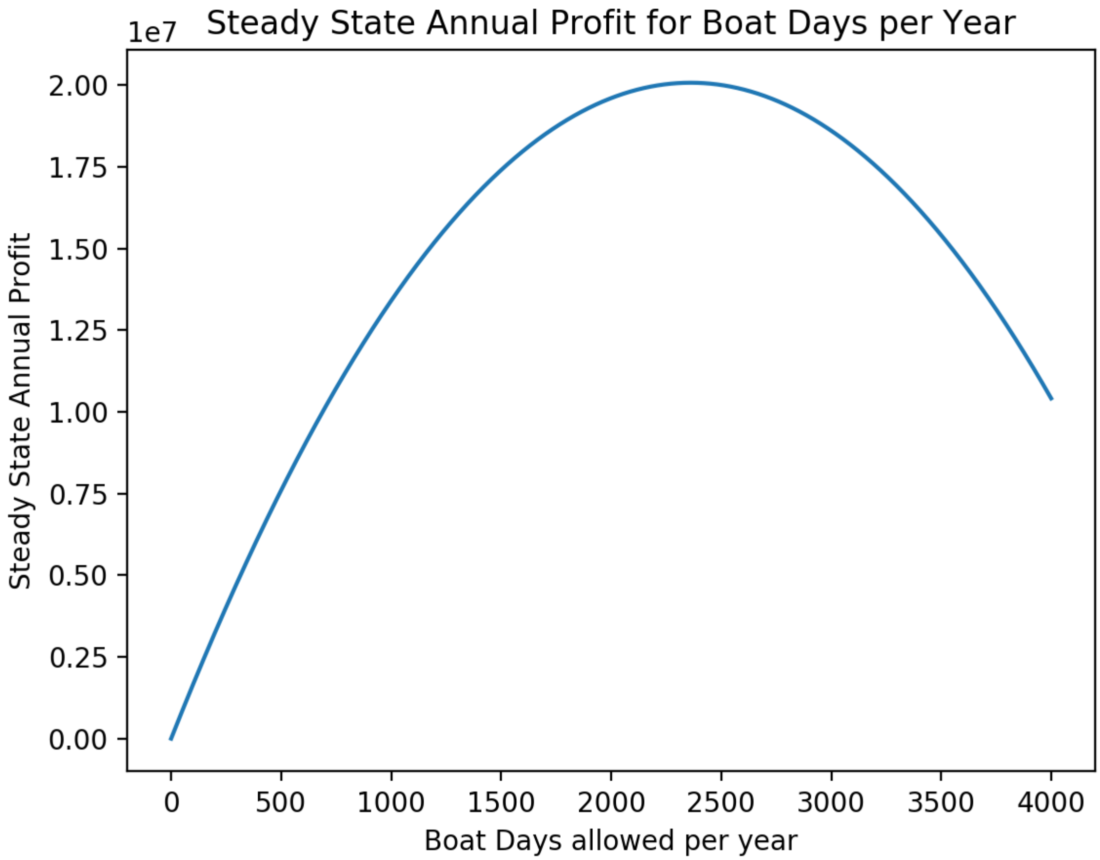
$$
\begin{aligned}
x_s &= K \cdot \left(1 - \frac{0.00001E}{r}\right) \\
x_s &= 300,000 \cdot \left(1 - \frac{0.00001(2,500)}{r}\right) \\
x_s &= 300,000 - \frac{7,500}{r} \\
\frac{dx_s}{dr} &= \frac{7,500}{r^2}
\end{aligned}
$$

So we have $S(x_s, r) = \frac{7,500}{r^2} \cdot \frac{r}{x_s} = \frac{7,500}{rx_s}$. When we evaluate at $r = 0.05$, we get that $S(x_s, r = 0.05) = \frac{7,500}{0.05 \cdot 150,000} = \frac{7,500}{7,500}$, so $S(x_s, r = 0.05) = 1$. This means that a 1% increase in the intrinsic growth rate would result in a 1% increase in the steady state population. This makes sense because if the intrinsic growth were larger, then there would be more whales.

(c) We want to maximize the steady state annual profit, let's call it P. We know that P is defined as follows.

$$
\begin{aligned}
P &= 6,000H_s - 500E \\
P &= 6,000(3E - 0.0006E^2) - 500E \\
P &= 18,000E - 3.6E^2 - 500E \\
P &= 17,500E - 3.6E^2 \\
\frac{dP}{dE} &= 17,500 - 7.2E_o = 0 \\
E_o &= \frac{17,500}{7.2} = 2,430.\bar{5} \approx 2,431
\end{aligned}
$$

So we've found that $E = 2,430.\bar{5}$ maximizes the steady state harvest rate. We know that this is a maximum because $\frac{d^2P}{dE^2} = -7.2$, and so it is always negative, which means that P is always concave down, so $E = 2,430.\bar{5}$ in fact maximizes the steady state harvest rate. But E_o must be a whole number. We know that $E_o = 2,431$ and not $2,430$ because P is a parabola, so rounding the real value found for E gives the maximum possible value of P. Plugging this back in, we see that $P = 17,000(2,431) - 3.6(2,431)^2$, and so the maximum steady state annual profit P is \$20,051,860.40. We also know that since $x_s = K \cdot \left(1 - \frac{0.00001(2,431)}{r}\right)$, then the steady state population $x_s = 154,140$. This is higher than the value found in (a). This is because the optimal number of boat days per year is less, so fewer whales will be harvested every year. We can plot the total profit as a function of E on the next page.



(d) We want to measure the sensitivity of the optimal boat days, E_o, to the cost per boat day, let's call it c. This is defined as $S(E_o, c) = \frac{dE_o}{dc} \cdot \frac{c}{E_o}$. We want to get a function that gives us the E_o in terms of c. From above, we have the following.

$$
\begin{aligned}
P &= 6,000H_s - cE \\
P &= 6,000(3E - 0.0006E^2) - cE \\
P &= 18,000E - 3.6E^2 - cE \\
P &= (18,000 - c)E - 3.6E^2 \\
\frac{dP}{dE} &= 18,000 - c - 7.2E_o = 0 \\
E_o &= \frac{18,000 - c}{7.2} \\
\frac{dE}{dc} &= -\frac{1}{7.2} = -0.13\bar{8}
\end{aligned}
$$

So we have $S(E_o, c) = -0.13\bar{8} \cdot \frac{c}{E_o}$. When we evaluate at $c = 500$, we get that $S(E_o, c = 500) = -0.13\bar{8} \cdot \frac{500}{2{,}431}$, so $S(E_o, c = 500) \approx -0.029$. This means that a 1% increase in the cost per boat day would result in a 0.029% decrease in the optimal number of boat days allowed. This makes sense because if the cost were larger, then more boat days would cause the total profit to go down, so the optimal number of boat days will be less. This result tells us that even a large change in the cost per boat day would result in a small change in the optimal number of boat days, because the change in c is 2 orders of magnitude larger than the resulting change in E_o. However, sensitivity is really only defined for a small change in c, so the previous statement would have to be further analyzed to be confirmed.

2 Blood Types

The four blood types A, B, O, and AB reflect six gene pairs (genotypes), with blood type A corresponding to gene pairs AA and AO, blood type B corresponding to gene pairs BB and BO, blood type O corresponding to gene pair OO, and blood type AB corresponding to gene pair AB. We say that p is the proportion of gene A in the population, q is the proportion of gene B in the population, and r is the proportion of gene O in the population. So $p + q + r = 1$.

(a) The probability of an individual having an A gene is p, a B gene is q, and an O gene is r. Therefore, the probability of an individual having genes AA, BB, or OO is p^2, q^2 and r^2, respectively. This means that the probability of someone having two different genes is $1 - (p^2 + q^2 + r^2)$.

(b) We want to find the maximum percentage of the population with two different genes. We will find this using two different methods. First, we will find a function for the proportion of the population with two different genes, let's call it T, that depends on only two variables and maximize it. We have the following.

$$\begin{aligned}
T &= 1 - (p^2 + q^2 + r^2) \\
1 &= p + q + r \\
r &= 1 - p - q \\
T &= 1 - (p^2 + q^2 + (1 - p - q)^2) \\
T &= 1 - p^2 - q^2 - (1 - 2p - 2q + 2pq + p^2 + q^2) \\
T &= -2p^2 - 2q^2 + 2p + 2q - 2pq \\
T &= 2(p + q - pq - p^2 - q^2)
\end{aligned}$$

So now we have T as a function of p and q. In order to maximize T, we want to take $\frac{\partial T}{\partial p} = \frac{\partial T}{\partial q} = 0$. We have the following.

$$\begin{aligned}
T &= 2(p + q - pq - p^2 - q^2) \\
\frac{\partial T}{\partial p} &= 2 - 2q - 4p \\
\frac{\partial T}{\partial q} &= 2 - 2p - 4q \\
2 - 2q - 4p &= 2 - 2p - 4q \\
2p &= 2q \\
p &= q
\end{aligned}$$

We have found that $\frac{\partial T}{\partial p} = \frac{\partial T}{\partial q} = 0$ when $p = q$. We can check that this is a maximum of T because $\frac{\partial^2 T}{\partial p^2} = \frac{\partial^2 T}{\partial q^2} = -4$, and so since both partial second derivatives are always negative, then $p = q$ maximizes T.

Since we know that $p + q + r = 1$, then $p = q = r = \frac{1}{3}$, and so we have:

$$\begin{aligned}
T_{max} &= 2\Big(\frac{1}{3} + \frac{1}{3} - \Big(\frac{1}{3}\Big)^2 - \Big(\frac{1}{3}\Big)^2 - \Big(\frac{1}{3}\Big)^2\Big)\\
T_{max} &= 2\Big(\frac{2}{3} - \frac{3}{9}\Big)\\
T_{max} &= 2\Big(\frac{2}{3} - \frac{1}{3}\Big)\\
T_{max} &= \frac{2}{3}
\end{aligned}$$

We will now find T_{max} using the method of Lagrange multipliers. We have $T = 1 - p^2 - q^2 - r^2$ and the constraint that $p + q + r = 1$. So we can say the following.

$$\begin{aligned}
\mathcal{L}(p, q, r, \lambda) &= 1 - p^2 - q^2 - r^2 + \lambda(1 - p - q - r)\\
\frac{\partial \mathcal{L}}{\partial p} &= -2p - \lambda\\
\frac{\partial \mathcal{L}}{\partial q} &= -2q - \lambda\\
\frac{\partial \mathcal{L}}{\partial r} &= -2r - \lambda\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 1 - p - q - r
\end{aligned}$$

Setting all the partials equal to zero yields the following.

$$\begin{aligned}
0 &= -2p - \lambda\\
0 &= -2r - \lambda\\
0 &= -2q - \lambda\\
0 &= 1 - p - q - r\\
p + q + r &= 1\\
p = q = r &= -\frac{\lambda}{2}\\
p = q = r &= \frac{1}{3} = -\frac{\lambda}{2}\\
\lambda &= -\frac{4}{3}
\end{aligned}$$

So again we have found that T reaches a maximum when $p = q = r = \frac{1}{3}$, so $T_{max} = \frac{2}{3}$. So both of these methods tell us that the maximum proportion of the population that can have two different genes is $\frac{2}{3}$.

(c) If we say $g(n) = p+q+r$, then when using the method of Lagrange multipliers, we used the constraint $g(n) = c$ for $c = 1$. The multiplier λ represents $\frac{dT}{dc}$, but since c cannot change in this context (because p, q, and r are probabilities that encompass all possibilities, so their sum must equal 1), then λ does not represent something very meaningful. However, say there is a possibility that there were a new gene with an unknown proportion in the population. Then $p+q+r = c < 1$ and now c could be a number of values. Then λ would represent the change in T with respect to c.