❖ ***Linearity* Theorem:** $\Im\{K_a a(t) + K_b b(t)\} = K_a A(f) + K_b B(f)$

$$\int_{-\infty}^{\infty} [K_a a(t) + K_b b(t)] e^{-j2\pi f t} dt$$

$$\int_{-\infty}^{\infty} K_a a(t) e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} K_b b(t) e^{-j2\pi f t} dt$$

$$K_a \underbrace{\int_{-\infty}^{\infty} a(t) e^{-j2\pi f t} dt}_{A(f)} + K_b \underbrace{\int_{-\infty}^{\infty} b(t) e^{-j2\pi f t} dt}_{B(f)}$$

$$= K_a A(f) + K_b B(f)$$

❖ ***Time Delay* Theorem:** $\Im\{x(t-t_0)\} = X(f)e^{-j2\pi f t_0}$

$$\int_{-\infty}^{\infty} x(t-t_0) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} x(t_1) e^{-j2\pi f (t_1+t_0)} dt_1 = e^{-j2\pi f t_0} \underbrace{\int_{-\infty}^{\infty} x(t_1) e^{-j2\pi f t_1} dt_1}_{X(f)}$$

$t_1 = t - t_0$ As $t \to \infty$, $t_1 \to \infty$

$dt_1 = dt$ As $t \to -\infty$, $t_1 \to -\infty$

$t = t_1 + t_0$

❖ ***Frequency Translation* Theorem:** $\Im^{-1}\{X(f-f_0)\} = x(t)e^{j2\pi f_0 t}$

❖ *Convolution* **Theorem:** $\Im\{a(t) \otimes b(t)\} = A(f)B(f)$

❖ *Multiplication* **Theorem:** $\Im\{a(t)b(t)\} = A(f) \otimes B(f)$

❖ *Differentiation* **Theorem:** $\Im\left\{\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right\} = (j2\pi f)X(f)$

❖ *Integration* **Theorem:** $\Im\left\{\int_{-\infty}^{t} x(\lambda)d\lambda\right\} = \left(\frac{1}{j2\pi f}\right)X(f) + \frac{1}{2}X(0)\delta(f)$

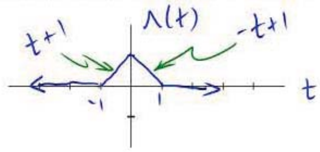
More Theorems

- ***Duality* Theorem: If $\Im\{x(t)\} = X(f)$, then $\Im\{X(t)\} = x(-f)$**

- ***Scale Change* Theorem: $\Im\{x(at)\} = \frac{1}{|a|}X\left(\frac{f}{a}\right)$, where a is a constant**

Example

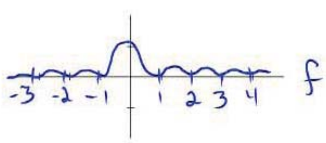
- **Find $\Im\{\Lambda(t)\}$ using only $\Im\{\Pi(t/\tau)\} = \mathrm{Sinc}(f\tau)$ and the theorems**



$$\mathcal{F}\{\Lambda(t)\} = \int_{-\infty}^{-1} 0\,dt + \underbrace{\int_{-1}^{0}(t+1)e^{-j2\pi f t}\,dt + \int_{0}^{1}(-t+1)e^{-j2\pi f t}\,dt}_{\text{NOT EASY TO SOLVE}} + \int_{1}^{\infty} 0\,dt$$

0 0

BUT...



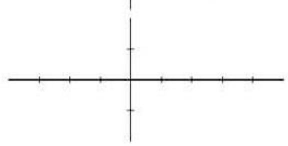
$$\Lambda(t) = \Pi(t) * \Pi(t)$$

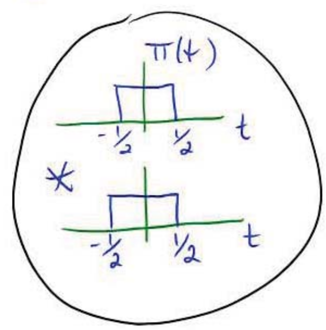
$$\mathcal{F}\{\Lambda(t)\} = \mathcal{F}\{\Pi(t) * \Pi(t)\}$$

$$= \mathcal{F}\{\Pi(t)\}\,\mathcal{F}\{\Pi(t)\}$$

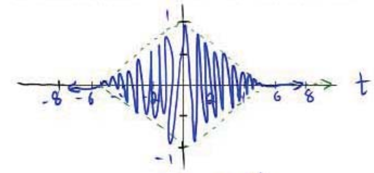
$$= \mathrm{SINC}(f)\cdot\mathrm{SINC}(f)$$

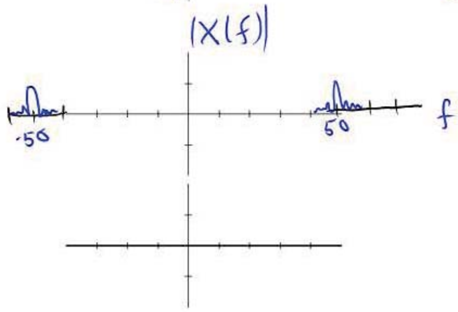
$$\mathcal{F}\{\Lambda(t)\} = \mathrm{SINC}^2(f)$$





❖ **Find** $X(f) = \Im\{\Lambda(t/5)\cos(100\pi t + \pi/4)\}$.





ASIDE

$a(t)\,\delta(t-t_0) = a(t_0)\,\delta(t-t_0)$

$a(t) * \delta(t-t_0) = a(t-t_0)$

$$\mathcal{F}\{\Lambda(t/5)\cos(100\pi t + \pi/4)\}$$

$$= \underbrace{\mathcal{F}\{\Lambda(t/5)\}}_{5\,\text{sinc}^2(5f)} * \mathcal{F}\{\cos(\underbrace{100\pi t + \pi/4}_{[100\pi(t + \frac{1}{100\pi}\cdot\frac{\pi}{4})]})\}$$

$\frac{1}{400}$

$$\underbrace{\mathcal{F}\{\cos(100\pi t)\}}_{\frac{1}{2}[\delta(f-50)+\delta(f+50)]}\, e^{-j2\pi f\left(\frac{-1}{400}\right)}$$

$$\frac{1}{2}\,\delta(f-50)\,e^{-j2\pi(50)\left(\frac{-1}{400}\right)} + \frac{1}{2}\,\delta(f+50)\,e^{-j2\pi(-50)\left(\frac{-1}{400}\right)}$$

$$= 5\,\text{sinc}^2(5f) * \left[\frac{1}{2}e^{j\frac{\pi}{4}}\,\delta(f-50) + \frac{1}{2}e^{-j\frac{\pi}{4}}\,\delta(f+50)\right]$$

$$= \frac{5}{4}e^{j\frac{\pi}{4}}\,\text{sinc}^2[\underbrace{5(f-50)}] + \frac{5}{2}e^{-j\pi/4}\,\text{sinc}^2[\underbrace{5(f+50)}]$$

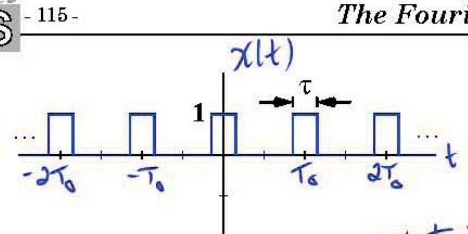
$5(f-50) = k$

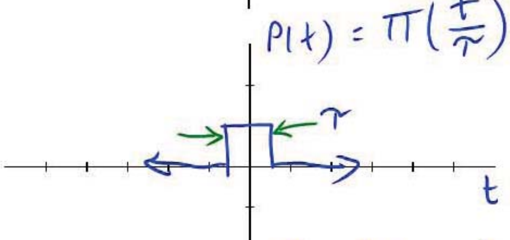
$f = 50 + \frac{k}{5}$

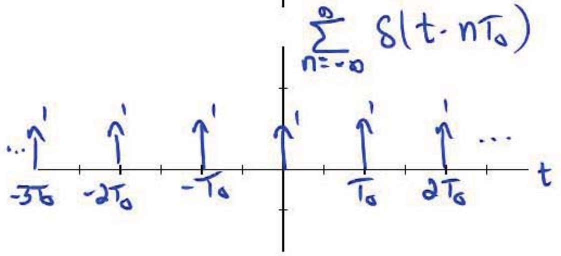
$5(f+50) = k$

$f = -50 + \frac{k}{50}$

$k = \pm 1, \pm 2, \cdots$







$$X(f) = \Im\left\{\sum_{n=-\infty}^{\infty}\Pi\left(\frac{t-nT_0}{\tau}\right)\right\}$$

$$x(t) = p(t) * \sum_{n=-\infty}^{\infty}\delta(t-nT_0)$$

$$= \sum_{n=-\infty}^{\infty}\underbrace{p(t)*\delta(t-nT_0)}_{p(t-nT_0)}$$

$\therefore$

$$X(f) = \mathcal{F}\left\{p(t) * \sum_{n=-\infty}^{\infty}\delta(t-nT_0)\right\}$$

$$= P(f)\left[\frac{1}{T_0}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)\right]$$

$$= \frac{1}{T_0}\sum_{k=-\infty}^{\infty}P(f)\,\delta\left(f-\frac{n}{T_0}\right)$$

$$X(f) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty}P\left(\frac{n}{T_0}\right)\delta\left(f-\frac{n}{T_0}\right)$$