Symmetry Properties of the Fourier Transform

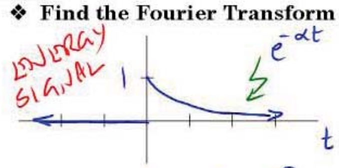
- **If $x(t)$ is *real*, then $X(f)$ has *even* magnitude and *odd* phase**
- **If $x(t)$ is *real* and *even*, then $X(f)$ has *even* magnitude and is *purely real***
- **If $x(t)$ is *real* and *odd*, then $X(f)$ has *even* mag. and is *purely imaginary***

$0 + j(\ \ )$

Another Fourier Transform Calculation

- **Find the Fourier Transform of $x(t) = e^{-\alpha t}u(t)$ for $\alpha > 0$.**

ENERGY SIGNAL



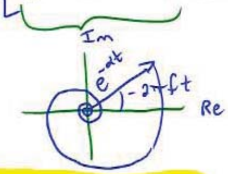
$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$

$= \int_{-\infty}^{\infty} e^{-\alpha t} u(t)\, e^{-j2\pi f t}\, dt$

$= \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt$

$= \frac{1}{-(\alpha + j2\pi f)} e^{-(\alpha + j2\pi f)t}\Big|_0^{\infty}$

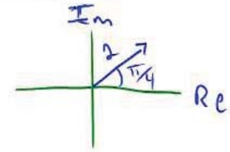
$= \frac{-1}{\alpha + j2\pi f}\left[e^{-\alpha(\infty)} e^{-j2\pi f(\infty)} - 1\right]$



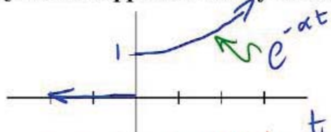
$$\boxed{X(f) = \frac{1}{\alpha + j2\pi f}}$$

ASIDE

$2e^{j\pi/4}$

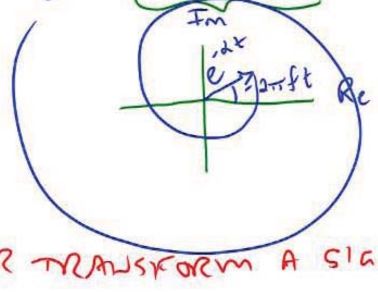


Q: **What happens if we try to redo this calculation with $\alpha < 0$?**



NEITHER ENERGY NOR POWER

$X(f) = \cdots = \frac{-1}{\alpha + j2\pi f}\left[e^{-\alpha(\infty)} e^{-j2\pi f(\infty)} - 1\right] = \text{NO SOLUTION}$



NOTE: WE CANNOT FOURIER TRANSFORM A SIGNAL THAT IS NOT ENERGY NOR POWER

Fourier Transform Table of Pairs

See the website!

A list of Fourier Transforms are given on the website

KNOW THESE!

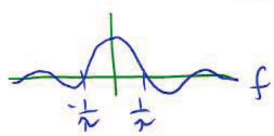
BE ABLE TO USE THESE!

Fourier Transform Pairs

ASIDE

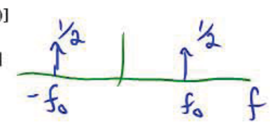
$\frac{1}{j} = -j$

Time-domain	Fourier-domain
$\delta(t)$	1
$u(t)$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\Pi(t/\tau)$	$\tau\,\mathrm{sinc}(\tau f)$
$\Lambda(t/\tau)$	$\tau\,\mathrm{sinc}^2(\tau f)$
$e^{-\alpha t}u(t)$	$\frac{1}{\alpha + j2\pi f}$, provided $\mathrm{Re}\{\alpha\} > 0$
$te^{-\alpha t}u(t)$	$\frac{1}{(\alpha + j2\pi f)^2}$, provided $\mathrm{Re}\{\alpha\} > 0$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(\alpha + j2\pi f)^n}$, provided $\mathrm{Re}\{\alpha\} > 0$
$e^{-\alpha t^2}$	$\sqrt{\frac{\pi}{\alpha}}e^{-(\pi f)^2/\alpha}$
$\sin(2\pi f_0 t)$	$(1/2j)[\delta(f-f_0) - \delta(f+f_0)]$
$\cos(2\pi f_0 t)$	$(1/2)[\delta(f-f_0) + \delta(f+f_0)]$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$

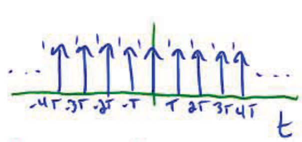


$\tau f = k$

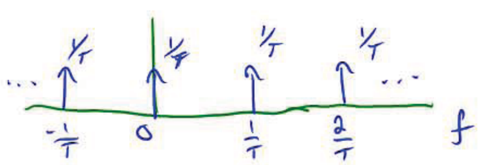
$f = \frac{k}{\tau}, k = \pm 1, \pm 2, \ldots$



SMALL SCALE STRUCTURE IN TIME-DOMAIN GIVES RISE TO BROAD FREQ-DOMAIN STRUCTURE AND VICE VERSA.



"TRAIN"



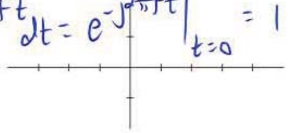
"TRAIN"

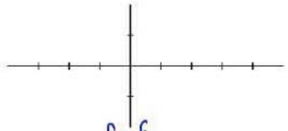
Fourier Transform Pairs

❖ $\Im\{\underbrace{\delta(t)}\} = \mathbf{1}$

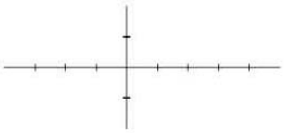
$$\int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = e^{-j2\pi f t}\Big|_{t=0} = 1$$

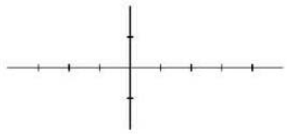
BY SIFTING PROPERTY

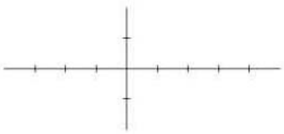


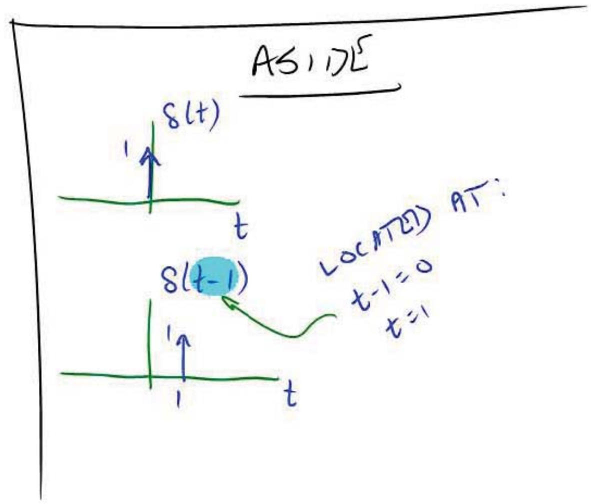


❖ $\Im\left\{e^{j2\pi f_0 t}\right\} = \delta(f - f_0)$ $\Rightarrow$ $\mathcal{F}^{-1}\{\delta(f-f_0)\} = \int_{-\infty}^{\infty} \delta(\overbrace{f-f_0}^{\substack{f=f_0 \\ f-f_0=0}}) e^{+j2\pi f t} df = e^{j2\pi f t}\Big|_{f=f_0} = e^{j2\pi f_0 t}$



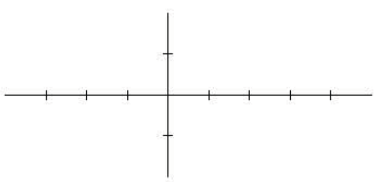


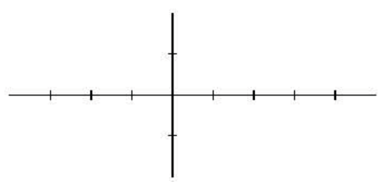




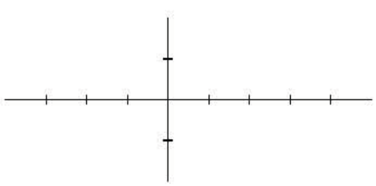
More Transform Pairs

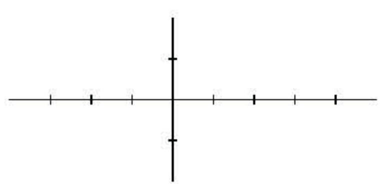
❖ $\Im\{\Lambda(t)\} = \mathbf{sinc}^2(\boldsymbol{f})$





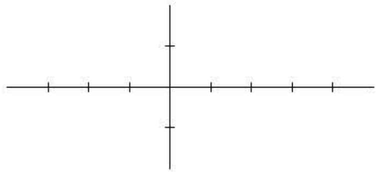
❖ $\Im\{e^{-\alpha t}u(t)\} = \dfrac{1}{\alpha + j2\pi f}$ **provided** $\mathbf{Re}\{\alpha\} > 0$

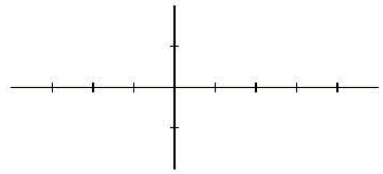




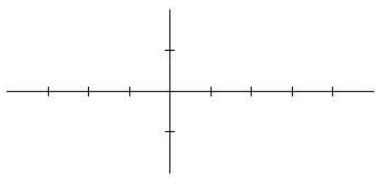
More Transform Pairs

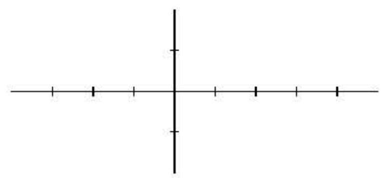
- $\Im\{\sin(2\pi f_0 t)\} = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$





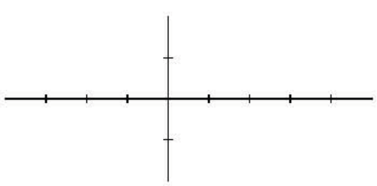
- $\Im\{\cos(2\pi f_0 t)\} = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$

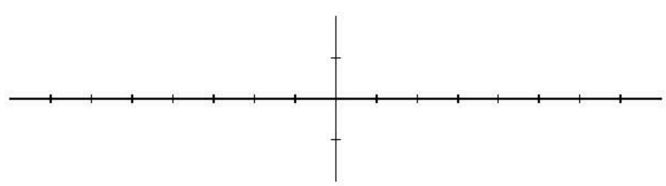




More Transform Pairs

- $\Im\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\} = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$





There is a duality between time and frequency!

Rayleigh's Energy Theorem

- **The Fourier Transform can only be computed for:**

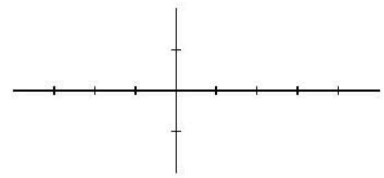
  Energy signals and Periodic signals

- ***Rayleigh's Energy Theorem*: If $x(t)$ is an *energy signal*, then**

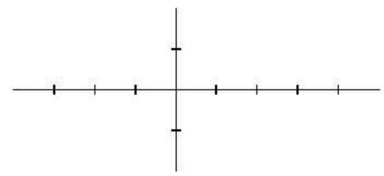
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$X(f) = \mathcal{F}\{x(t)\}$

- **Energy Calculations**
  - The energy in the time interval $[t_1, t_2]$ is: $E_{t_1, t_2} = \int_{t_1}^{t_2} |x(t)|^2 dt$



  - The energy in the frequency range $[f_1, f_2]$ is: $E_{f_1, f_2} = \int_{f_1}^{f_2} |X(f)|^2 df$



Fourier Transform Theorems

$x(t) \to \boxed{\mathcal{F}} \to X(f)$

	Time-domain	Fourier-domain
Linearity:	$K_a a(t) + K_b b(t)$	$K_a A(f) + K_b B(f)$
Time Shift:	$x(t - t_0)$	$X(f) e^{-j2\pi f t_0}$
Frequency Shift:	$x(t) e^{j2\pi f_0 t}$	$X(f - f_0)$
Convolution:	$a(t) * b(t)$	$A(f) B(f)$
Multiplication:	$a(t) b(t)$	$A(f) * B(f)$
Differentiation:	$\frac{d}{dt} x(t)$	$(j2\pi f) X(f)$
Integration (zero mean $x(t)$):	$\int_{-\infty}^{t} x(\lambda) d\lambda$	$\left(\frac{1}{j2\pi f}\right) X(f)$
Scale Change:	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$

$A(f) = \mathcal{F}\{a(t)\}$
$B(f) = \mathcal{F}\{b(t)\}$
$X(f) = \mathcal{F}\{x(t)\}$

Duality: If $\Im\{x(t)\} = X(f)$, then $\Im\{X(t)\} = x(-f)$