# Probability and Random Variables Test # 3 Note Sheet

**Important Statistics**:
- Correlation: $R_{XY} = E\{XY\}$
- Covariance: $C_{XY} = E\{[X - m_X][Y - m_Y]\}$
  $= R_{XY} - m_X m_Y$
- Correlation Coeff.: $\rho_{XY} = \dfrac{C_{XY}}{\sigma_X \sigma_Y}$

**Sum of 2 RVs**: RV Z = X + Y

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)$$

**The sum of n RVs**: RV $Z = X_1 + X_2 + \cdots + X_n$

$$f_Z(z) = f_{X_1}(z) * f_{X_2}(z) * \cdots * f_{X_n}(z)$$

**The Central Limit Theorem**: If $X_i$ for $i = 1, 2, \ldots, n$ are RVs with mean m and variance $\sigma^2$and and

$$Z = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i - m}{\sigma}\right)$$

RV Z tends toward a Gaussian as $n \to \infty$ and $E\{Z\} = 0$

**Estimators**: Let $x_i$, $i = 1, 2, \ldots, n$ be n samples of RV X with pdf $f_X(x)$

$$\overline{m} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\overline{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2 \qquad \overline{\sigma^2} = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^2\right] - m^2$$

$$\overline{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{m})^2$$

$$\overline{C_{XY}} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{m_x})(y_i - \overline{m_y})$$

$$\overline{\rho_{XY}} = \frac{\overline{C_{XY}}}{\overline{\sigma_x \sigma_y}}$$

**Estimator Characteristics**:

Suppose we have n samples from RV X and the statistic $\theta$ is estimated by $\bar{\theta}$. Let $f_Z(z)$ be the distribution of the estimator $\bar{\theta}$

If $E\{Z\} = \theta$, then the estimator is *unbiased*.

If $\sigma_z^2 \to 0$ as $n \to \infty$, then the estimator is consistent.

If all other estimates, $\overline{\theta_i}$, with RV $Z_i$, are such that $\sigma_z^2 \le \sigma_{z_i}^2$ for all i, then the estimator is minimum-variance and efficient.

**The Confidence Interval**:

For $100(1-\alpha)\%$ CI of m with known $\sigma^2$is

$$\left[\overline{m} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \overline{m} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right]$$

For $100(1-\alpha)\%$ CI of m with unknown $\sigma^2$is

$$\left[\overline{m} + t_{n-1,1-\alpha/2}\left(\frac{\overline{\sigma}}{\sqrt{n}}\right), \overline{m} + t_{n-1,\alpha/2}\left(\frac{\overline{\sigma}}{\sqrt{n}}\right)\right]$$

For $100(1-\alpha)\%$ CI of $\sigma^2$as $n \to \infty$ is

$$\left[\overline{\sigma^2} - z_{a/2}\left(\sqrt{\frac{2}{n}}\right)\overline{\sigma^2}, \overline{\sigma^2} - z_{a/2}\left(\sqrt{\frac{2}{n}}\right)\overline{\sigma^2}\right]$$

For few n ($n < 50$)

$$\left[\frac{(n-1)\overline{\sigma^2}}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)\overline{\sigma^2}}{\chi^2_{n-1,\alpha/2}}\right]$$

**Linear Regression**: Given RVs X & Y, $\rho_{XY}$ is near $\pm 1$.
Find $Y = aX + b$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} B & \overline{m_x} \\ \overline{m_x} & 1 \end{bmatrix}^{-1} \begin{bmatrix} A \\ \overline{m_y} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Ergodicity**: Ergotic means that time average and ensemble average are the same

Time Average Eq.:

$$< g(X(t)) >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(X(t))dt$$

Ergodic-in-the-mean:

$$E\{X(t)\} = m =< X(t) >$$

Ergodic-in-the-variance:

$$E\left\{X^2(t)\right\} - m^2(t) = \sigma^2 =< X^2(t) > - [< X(t) >]^2$$

**Autocorrelation Function**:

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

**Wide-Sense Stationary (WSS) RPs**:
- Stationary-in-the-mean
- Stationary-in-the-variance
- $R_X$ is only a function of time difference

**$\mathbf{R_X}$ Properties**: If the RP is WSS,
- $R_X(0) \ge |R_X(\tau)|$
- $R_X(\tau) = R_X(-\tau)$
- DC Power = $\lim_{\tau\to\infty} R_X(\tau) = [E\{X(t)\}]^2$
- Total Power = $R_X(0)$
- AC Power = Total Power - DC Power = $\sigma^2$

**White Gaussian Noise (WGN)**:
- WSS
- Amplitude pdf is Gaussian with variance given by the AC power
- $R_X(\tau) = \dfrac{N_0}{2}\delta(\tau)$

**Spectral Density Function (SDF)**:

$$S_X(f) = \mathfrak{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau$$

**SDF Passed through a transfer fnc**:

$$S_Y(f) = S_X(f)|H(f)|^2$$

**Stationarity and Ergodicity Example**:

$$f_X(x) = A\cos(\omega_0 t + \theta) \text{ where } f_\Theta(\theta) = \frac{1}{2\pi}\Pi\left(\frac{\theta}{2\pi}\right)$$

is stationary and ergodic in the mean and variance.