Probability and Random Variables Test # 2 Note Sheet

DeMoivre-Laplace Theorem: Many distributions approach a gaussian when certain conditions are met.

$$\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(k-m)^2/(2\sigma^2)}$$

where $n \gg 1$, $m = np$, and $\sigma^2 = npq$

RV Transformations: $f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|\Big|_{x=g^{-1}(y)}$

Generation of Random Numbers:

- Generate a uniform RV outcome x_i
- Calculate the outcome y_i using the mapping

$$y_i = F_Y^{-1}(x_i)$$

Linear Transformations:

$$Y = g(X) = aX + b$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{Y-b}{a}\right)$$

Expectation Operator: $E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

Mean $\equiv E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\, dx = m$

Variance $\equiv E\{X^2\} - m^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \sigma^2$

Standard deviation $\equiv \sqrt{\sigma^2} = \sigma$

$E\{\alpha\} = \alpha$ for α constant

$E\{\alpha g(X)\} = aE\{g(X)\}$

$E\{g_1(X) + g_2(X)\} = E\{g_1(X)\} + E\{g_2(X)\}$

For discrete pdfs: $E\{g(X)\} = \sum_k g(x_k) p_k$

Normalizing: $Y = \frac{X-m}{\sigma}$

Moments of an RV: $M_X(v) = E\{e^{jvX}\}$

$$E\{X^n\} = (-j)^n \left[\frac{d^n}{dv^n} M_X(v)\right]\Bigg|_{v=0}$$

Conditional CDFs & PDFs:

$$F_X(x|B) = P(X \le x|B) = \frac{P(X \le x, B)}{P(B)}$$

$$F_X(x|a < X \le b) = \begin{cases} 0 & x \le a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a < x \le b \\ 1 & x > b \end{cases}$$

$$f_X(x|a < X \le b) = \begin{cases} 0 & x \le a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)} & a < x \le b \\ 0 & x > b \end{cases}$$

$$F_X(x|X \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x \le b \\ 1 & x > b \end{cases}$$

$$f_X(x|X \le b) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} & x \le b \\ 0 & x > b \end{cases}$$

$$F_X(x|X > a) = \begin{cases} 0 & x \le a \\ \dfrac{F_X(x) - F_X(a)}{1 - F_X(a)} & x > a \end{cases}$$

$$f_X(x|X > a) = \begin{cases} 0 & x \le a \\ \dfrac{f_X(x)}{1 - F_X(a)} & x > a \end{cases}$$

Conditional CDF and Total Prob. Theorem: Given the sample space partitioning

$$S = \{A_1, A_2, \ldots, A_n\}$$

then $F_X(x) = \sum_{k=1}^{n} F_X(x|A_k) P(A_k)$

Chebyshev's Inequality:

$$P(|X-m| \ge k\sigma) \le \frac{1}{k^2} \text{ or } P(|X-m| \le k\sigma) \ge 1 - \frac{1}{k^2}$$

Markov's Inequality:

$$P(X \ge \delta) \le \frac{m}{\delta} \text{ or } P(X \le \delta) \ge 1 - \frac{m}{\delta}$$

Monte Carlo Method: Given

$$A = \int_a^b f(x) g(x) dx \text{ where } \int_a^b g(x) dx = 1$$

generate set of outcomes, x_i from RV X with pdf g(x)

$$A \approx \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

Weak Law of Large Numbers: As the number of samples increases, the sample mean is close to the true mean

Strong Law of Large Numbers: As the number of samples increases, the sample mean approaches the true mean

Joint CDFs: Given two RVs X and Y, the joint CDF is

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

Properties:

- $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = F_{XY}(-\infty, -\infty) = 0$
- $F_{XY}(\infty, \infty) = 1$
- $F_{XY}(x, \infty) = F_X(x)$ and $F_{XY}(\infty, y) = F_Y(y)$
- Positive-side continuous
- Monotonically non-decreasing along the positive direction

Joint PDFs: Given a joint CDF

$$f_{XY}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x,y)$$

Properties:

- $f_{XY}(x,y) \ge 0$ for all x,y
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dxdy = 1$
- $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

Joint Conditional Probability:

$$F_{X|Y}(x)(x|B) = \frac{P(X \le x, B)}{P(B)}$$

$$F_{X|Y}(X|a < Y \le b) = \frac{F_{XY}(x,b) - F_{XY}(x,a)}{F_Y(b) - F_Y(a)}$$

$$F_{X|Y}(x|Y \le b) = \frac{F_{XY}(x,b)}{F_Y(b)}$$

$$f_{X|Y}(x|Y \le b) = \frac{1}{F_Y(b)}\left[\frac{\partial}{\partial x} F_{XY}(x,b)\right]$$

$$F_{X|Y}(x|Y = a) = \frac{\frac{\partial}{\partial y} F_{XY}(x,y)\Big|_{y=a}}{f_Y(a)}$$

$$f_{X|Y}(x|a) = \frac{f_{XY}(x,a)}{f_Y(y)}$$