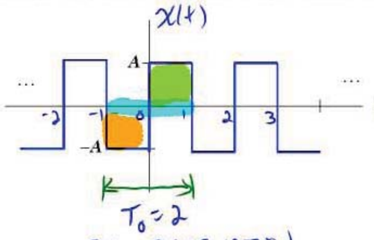
Trigonometric Fourier Series Numeric Example

- **Find the Trigonometric Fourier Series expansion of the following signal**



BY OBSERVATION

$\therefore$ CHOOSE $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi = \omega_0$

NOW CALCULATE COEFFICIENT

$a_0 = \frac{1}{2}\int_{-1}^{1} x(t)\,dt = 0$ BY OBSERVATION

BECAUSE ● AREA IS THE NEGATIVE OF ● AREA

$$a_n = \frac{2}{2}\int_{-1}^{1} x(t)\cos(n\pi t)\,dt$$

$$= \int_{-1}^{0}(-A)\cos(n\pi t)\,dt + \int_{0}^{1}(+A)\cos(n\pi t)\,dt$$

$$= -A\cdot\frac{\sin(n\pi t)}{n\pi}\Big|_{-1}^{0} + A\,\frac{\sin n\pi t}{n\pi}\Big|_{0}^{1}$$

$$= \frac{A}{n\pi}\left\{-[\sin(0) - \sin(-n\pi)] + [\sin n\pi - \sin 0]\right\} = 0$$

$\therefore\ a_n = 0$, ALL n

$$b_n = \frac{2}{2}\int_{-1}^{1} x(t)\sin(n\pi t)\,dt$$

$$= \int_{-1}^{0}(-A)\sin(n\pi t)\,dt + \int_{0}^{1}(+A)\sin(n\pi t)\,dt$$

$$= \frac{A}{n\pi}\left\{-\left[-\underbrace{\cos(0)}_{1} + \underbrace{\cos(-n\pi)}_{(-1)^n}\right] + \left[-\underbrace{\cos(n\pi)}_{(-1)^n} + \underbrace{\cos(0)}_{1}\right]\right\}$$

$$= \frac{A}{n\pi}\left\{1 - (-1)^n - (-1)^n + 1\right\}$$

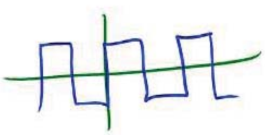
$$b_n = \frac{A}{n\pi}\left[2 - 2(-1)^n\right] = \frac{2A}{n\pi}\underbrace{\left[1-(-1)^n\right]}_{\begin{cases}0, & n \text{ EVEN}\\ 2, & n \text{ ODD}\end{cases}}$$

$$b_n = \begin{cases}\frac{4A}{n\pi}, & n \text{ ODD}\\ 0, & n \text{ EVEN}\end{cases}$$

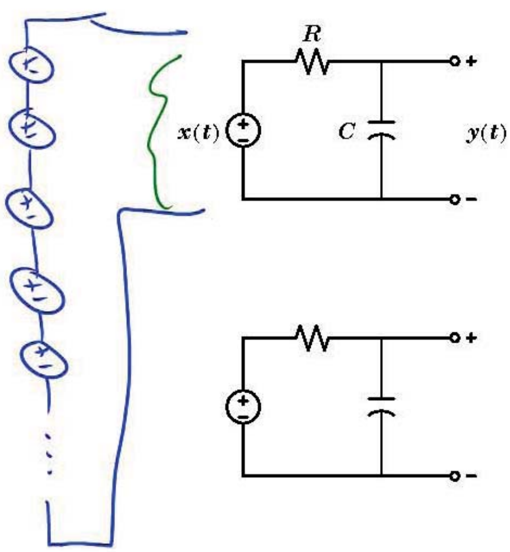
$$\tilde{x}(t) = \sum_{\substack{n=1\\ n \text{ ODD}}}^{\infty}\frac{4A}{n\pi}\sin(n\pi t)$$

Application to Circuits

$$\tilde{x}(t) = \frac{4A}{\pi}\left\{\sin[4\pi(1)t] + \frac{1}{3}\sin[4\pi(3)t] + \frac{1}{5}\sin[4\pi(5)t] + \frac{1}{7}\sin[4\pi(7)t] + \ldots\right\}$$



SUPERPOSITION



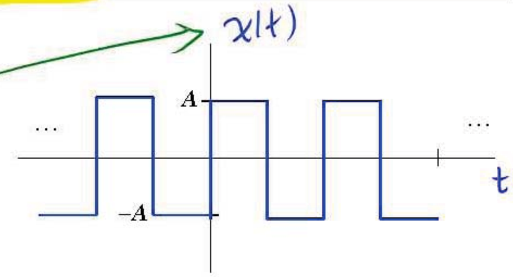
KNOW HOW TO DO SINUSOIDAL STEADY ANALYSIS PROBLEMS

- It is possible to represent a periodic signal using an infinite series of complex exponentials.

$a + a^* = \text{REAL}$

$a \cdot a^* = \text{REAL}$

PURELY REAL



The form of the series is as follows:

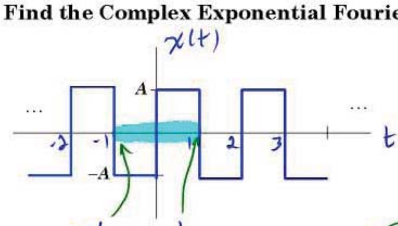
$$x(t) = \sum_{n=-\infty}^{\infty} X_n \underbrace{e^{jn\omega_0 t}}$$

where

$$X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

where the integrals are over the interval $[t_0, t_0 + T_0]$ where t_0 is arbitrary.

- Find the Complex Exponential Fourier Series of the following signal

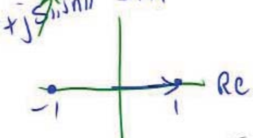


$\omega_0 = 2\pi f_0$ RAD/SEC, CYCLES/SEC Hz

$e^{j\theta} = \cos\theta + j\sin\theta$

$\theta = n\pi$

$e^{jn\pi} = \cos n\pi + j\overset{0}{\sin n\pi}$



$t_0 = -1$, $\boxed{T_0 = 2}$

$[t_0, t_0 + T_0] = [-1, 1]$

$$X_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jn\omega_0 t}\, dt$$

FIND ω_0 THE SAME WAY WE DID FOR TRIGONOMETRIC FOURIER SERIES $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \boxed{\pi = \omega_0}$

CASE I $n \neq 0$

$$= \frac{1}{2}\int_{-1}^{1} x(t) e^{-jn\pi t}\, dt = \frac{1}{2}\int_{-1}^{0}(-A) e^{-jn\pi t}\, dt + \frac{1}{2}\int_{0}^{1}(A) e^{-jn\pi t}\, dt$$

$$= \frac{A}{-j2\pi n}\underbrace{\left[-\left(1 - \underbrace{e^{-jn\pi(-1)}}_{(-1)^n}\right) + \left(\underbrace{e^{-jn\pi}}_{(-1)^n} - 1\right)\right]}_{\underbrace{[2(-1)^n - 2]}_{-4\ n\ \text{ODD};\ 0\ n\ \text{EVEN}}} = \boxed{\begin{cases} \frac{2A}{j\pi n}, & n \text{ ODD} \\ 0, & n \text{ EVEN} \end{cases}}$$

CASE II: $n = 0$

$$X_0 = \frac{1}{2}\int_{-1}^{1} x(t)\, dt = \boxed{0 = X_0}$$

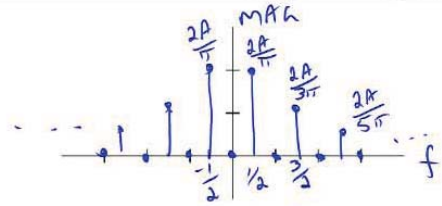
$$\boxed{\tilde{x}(t) = \sum_{\substack{n=-\infty \\ n \text{ ODD}}}^{\infty} \left(\frac{2A}{j\pi n}\right) e^{jn\pi t}}$$

$$3\angle\tfrac{\pi}{3} + \underbrace{2\angle\tfrac{-\pi}{4}}_{2e^{-j\pi/4}}$$

$$3e^{j\pi/3}$$

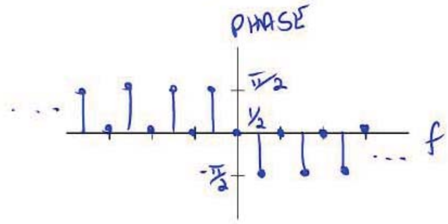
$$3\cos\left(\tfrac{\pi}{3}\right) + j3\sin\tfrac{\pi}{3}$$

Coefficient Magnitude & Phase Plots



$$X_n = \begin{cases} \frac{2A}{j\pi n}, & n \text{ ODD} \\ 0, & n \text{ EVEN} \end{cases}$$

RAD/SEC OR $\frac{n\pi}{2\pi} = \frac{n}{2}$ (HZ)



$$\tilde{x}(t) = \sum_{\substack{n=-\infty \\ n \text{ ODD}}}^{\infty} \underbrace{\frac{2A}{j\pi n} e^{jn\pi t}}_{\text{ROTATING PHASOR}}$$

$n = 0$, 0

$n = 1$, $\frac{2A}{j\pi} = \frac{2A}{\pi} e^{-j\pi/2}$ @ $f = \frac{1}{2}$ HZ

$f = 1$ HZ

$n = 2$, 0

$n = 3$, $\frac{2A}{j\pi 3} = \frac{2A}{3\pi} e^{-j\pi/2}$ @ $f = \frac{3}{2}$ HZ

$n = 4$, 0

$n = 5$, $\frac{2A}{j\pi 5} = \frac{2A}{5\pi} e^{-j\pi/2}$ @ $f = \frac{5}{2}$ HZ

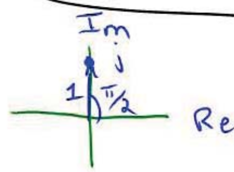
$n = -1$, $\underbrace{\frac{2A}{j\pi(-1)}} = \frac{2A}{\pi} e^{j\pi/2}$ @ $f = -\frac{1}{2}$ HZ

$\frac{2A}{(-j)\pi}\left(\frac{j}{j}\right)$

$-j^2 = 1$

$\frac{2A}{\pi}(j)$

$0 + j = e^{j\pi/2}$



$e^{j\theta} = \cos\theta + j\sin\theta$

Symmetry Properties

❖ **If $x(t)$ is *real*, then the X_n coefficients have *even* magnitude and *odd* phase**

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

$-1 = e^{j\pi} = e^{-j\pi} = e^{j3\pi} = e^{-j3\pi} \ldots$

❖ **If $x(t)$ is *real* and *even*, then X_n has *even* magnitude and is *purely real***

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)dt - j\left[\frac{1}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt\right]$$

❖ **If $x(t)$ is *real* and *odd*, then X_n has *even* mag. and is *purely imaginary***

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\underbrace{e^{-jn\omega_0 t}}_{\cos(-n\omega_0 t) + j\sin(-n\omega_0 t)}dt = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)dt - j\left[\frac{1}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt\right]$$

$\underbrace{\text{ODD} \cdot \text{EVEN}}_{\text{ODD}}$

$\underbrace{\text{ODD} \cdot \text{ODD}}_{\text{EVEN}}$

Trigonometric <=> Complex Exp Fourier Series Relationship

- **Trigonometric Fourier Series:**

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

- **Complex Exponential Fourier Series:**

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$a_0 = X_0$

$a_n = X_n + X_{-n}$

$b_n = j[X_n - X_{-n}]$

Uniqueness of the Fourier Series

The Fourier Series of a signal is unique

- **The Complex Exponential Fourier Series of $x(t)$ results in only one possible set of coefficients X_n**

- **What is the Complex Exponential Fourier Series of**

$$x(t) = [\sin 4\pi t][\cos 6\pi t]$$

- **What is the Trigonometric Fourier Series of the same signal?**