The Fourier Series of a signal is unique

- **The Complex Exponential Fourier Series of $x(t)$ results in only one possible set of coefficients X_n**

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t}\, dt$$

- **What is the Complex Exponential Fourier Series of**

$$x(t) = [\sin 4\pi t][\cos 6\pi t]$$

$$\left[\frac{e^{j4\pi t} - e^{-j4\pi t}}{2j}\right]\left[\frac{e^{j6\pi t} + e^{-j6\pi t}}{2}\right]$$

$$= \frac{1}{4j} e^{j4\pi t} e^{j6\pi t} + \frac{1}{4j} e^{j4\pi t} e^{-j6\pi t} + \frac{1}{4j}\left(-e^{-j4\pi t}\right)\left(e^{j6\pi t}\right) + \frac{1}{4j}\left(-e^{-j4\pi t}\right)\left(e^{-j6\pi t}\right)$$

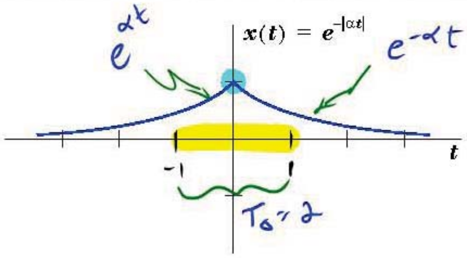
$$= \frac{1}{4j} e^{j10\pi t} + \frac{1}{4j} e^{-j2\pi t} + \left(\frac{-1}{4j}\right) e^{j2\pi t} + \left(\frac{-1}{4j}\right) e^{-j10\pi t}$$

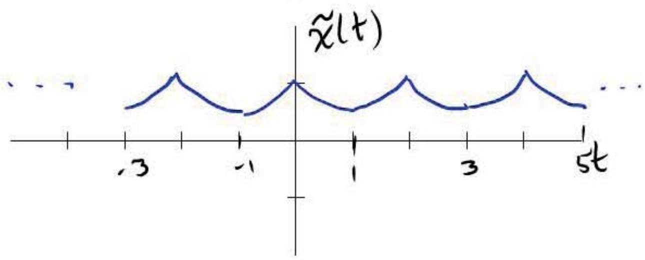
$$\omega_0 = 2\pi$$

$$X_n = \begin{cases} \frac{1}{4j}, & n = 5 \\ \frac{1}{4j}, & n = -1 \\ -\frac{1}{4j}, & n = 1 \\ \frac{-1}{4j}, & n = -5 \\ 0, & \text{ELSE} \end{cases}$$

- **What is the Trigonometric Fourier Series of the same signal?**

- **Find the Complex Exponential Fourier Series of $x(t)$ over $[-1, 1]$**





$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T_0} = \pi = \omega_0$$

$$X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} x(t)\, e^{-jn\pi t}\, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{0} \left[e^{\alpha t}\right] e^{-jn\pi t}\, dt + \int_{0}^{1} \left[e^{-\alpha t}\right] e^{-jn\pi t}\, dt\right]$$

⋮ CALCULUS

$X_n =$ ⋮

Parseval's Theorem

❖ **If X_n are the Complex Exponential Fourier Series Coefficients of $x(t)$, then**

$$\underbrace{\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt}_{} = \sum_{n=-\infty}^{\infty} |X_n|^2$$

KNOW THIS!

AVE POWER CALCULATION

P_{t_0, t_0+T_0} OF $x(t)$

SEE SLIDE 91

$$x(t) = \sin 4\pi t \cos 6\pi t$$

$$P_{AVE} = \left|\frac{1}{4j}\right|^2 + \left|\frac{1}{4j}\right|^2 + \left|\frac{-1}{4j}\right|^2 + \left|\frac{-1}{4j}\right|^2$$

$$= \left(\frac{1}{16}\right)4 = \frac{1}{4} = P_{AVE}$$

Complex Exponential Fourier Series Example

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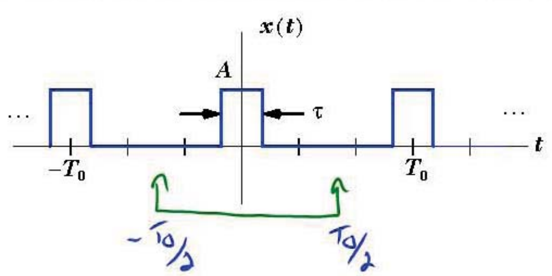
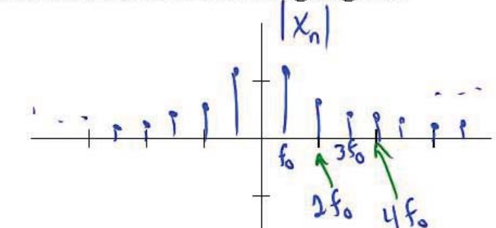
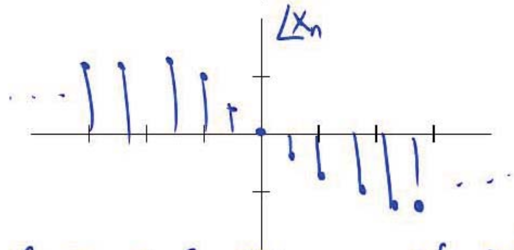
❖ **Find the Complex Exponential Fourier Series of the following signal**





MAG PLOT

$f_0 = \frac{1}{T_0}$



PHASE PLOT

$$\omega_0 = \frac{2\pi}{T_0}$$

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt$$

AS T_0 GETS LARGE, f_0 GETS SMALL $\quad n f_0 \to f$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn2\pi f_0 t} = \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn2\pi f_0 t} dt\right]}_{X_n} e^{jn2\pi f_0 t}$$

The Fourier Transform Integrals

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \qquad x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

$$X(f) = \Im\{x(t)\} \qquad x(t) = \Im^{-1}\{X(f)\}$$

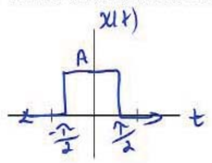
$= \mathcal{F}\{x(t)\}$ $\qquad = \mathcal{F}^{-1}\{X(f)\}$

TIME-DOMAIN → FREQUENCY DOMAIN

$x(t) \xrightarrow{\mathcal{F}\{\}} X(f)$

Example Fourier Transform Calculation

❖ **Find the Fourier Transform of** $x(t) = A\Pi\left(\frac{t}{\tau}\right)$



$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

$= \int_{-\infty}^{\infty} A\Pi\left(\frac{t}{\tau}\right)e^{-j2\pi ft}dt$

$= \int_{-\tau/2}^{\tau/2} Ae^{-j2\pi ft}dt$

$= A \cdot \frac{e^{-j2\pi ft}}{-j2\pi f}\Bigg|_{-\tau/2}^{\tau/2}$

$= \frac{-A}{j2\pi f}\left[e^{-j2\pi f\tau/2} - e^{+j2\pi f(\tau/2)}\right]$

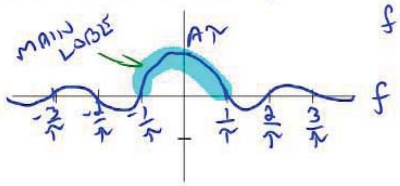
$= \frac{A}{\pi f}\underbrace{\left[\frac{e^{-j\pi f\tau} - e^{j\pi f\tau}}{-2j}\right]}_{\sin(\pi f\tau)} = A\tau\underbrace{\left[\frac{\sin(\pi f\tau)}{\pi f\tau}\right]}_{\mathrm{sinc}(f\tau)}$

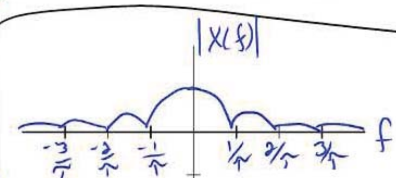
ZERO CROSSINGS OCCUR WHERE

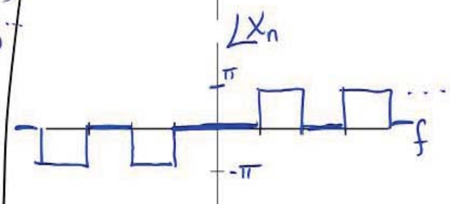
$f\tau = K$ (NON-ZERO INTEGER)

$f = \frac{K}{\tau}, \; K = \pm 1, \pm 2, \cdots$

$X(f) = A\tau\,\mathrm{sinc}(f\tau)$



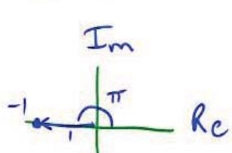




$X(f) = A\tau\,\mathrm{sinc}(f\tau)$

$= |A\tau\,\mathrm{sinc}(f\tau)|e^{j\Theta(f)}$

$\Theta(f) = ?$



$-1 = e^{j\pi}$

$= e^{-j\pi}$

$= e^{j3\pi}$

$= e^{-j3\pi}$