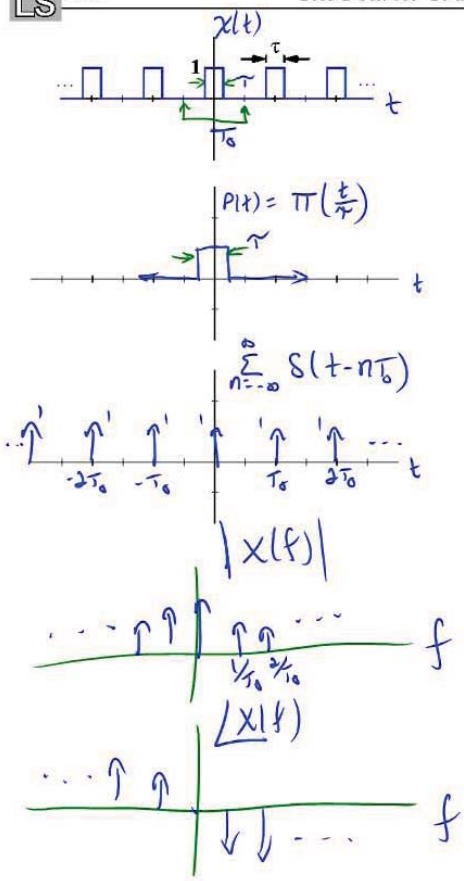
## The Fourier Transform of a Periodic Signal






$$X(f) = \Im\left\{\sum_{n=-\infty}^{\infty}\Pi\left(\frac{t-nT_0}{\tau}\right)\right\}$$

$$x(t) = p(t) * \sum_{n=-\infty}^{\infty}\delta(t-nT_0)$$

$$= \sum_{n=-\infty}^{\infty}\underbrace{p(t)*\delta(t-nT_0)}_{p(t-nT_0)}$$

$$X(f) = \Im\{x(t)\} = \Im\left\{p(t) * \sum_{n=-\infty}^{\infty}\delta(t-nT_0)\right\}$$

$$= \underbrace{\Im\{p(t)\}}_{\underbrace{P(f)}_{\tau\,\mathrm{sinc}(f\tau)}}\ \underbrace{\Im\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT_0)\right\}}_{\frac{1}{T_0}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T_0}\right)}$$

$$X(f) = \tau\,\mathrm{sinc}(f\tau)\cdot\frac{1}{T_0}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T_0}\right)$$

$$= \sum_{k=-\infty}^{\infty}\left[\frac{\tau}{T_0}\,\mathrm{sinc}(f\tau)\right]\delta\Big(\underbrace{f-\frac{k}{T_0}}\Big) = \sum_{k=-\infty}^{\infty}\frac{\tau}{T_0}\,\mathrm{sinc}\left(\frac{k}{T_0}\tau\right)\delta\left(f-\frac{k}{T_0}\right)$$

$$f-\frac{k}{T_0}=0 \Rightarrow f=\frac{k}{T_0}$$

ASIDE

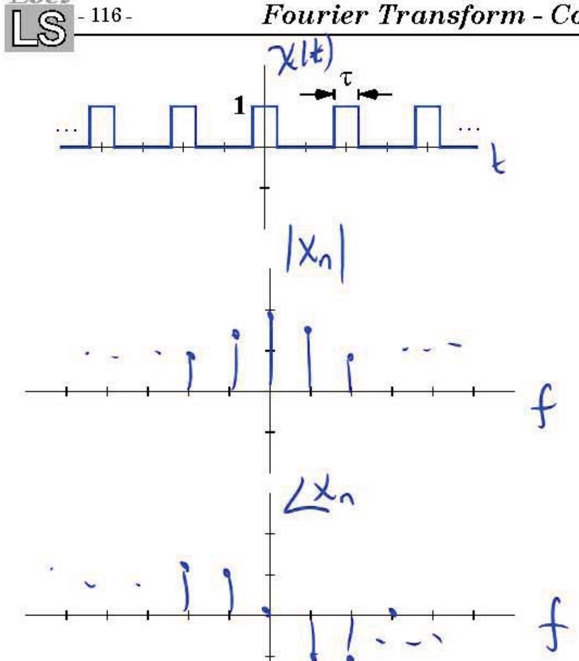
$$a(t)\,\delta(t-t_0) = a(t_0)\,\delta(t-t_0)$$

$$a(t) * \delta(t-t_0) = a(t-t_0)$$



## Fourier Transform - Complex Exp Fourier Series Relationship






$$x(t) = \sum_{n=-\infty}^{\infty}X_n e^{jn\omega_0 t}$$

$$X_n = \frac{1}{T_0}\int_{T_0}x(t)e^{-jn\omega_0 t}\,dt$$

$X_n$ = WEIGHT OF δ FUNC IN $X(f)$ AT $f = \frac{n}{T_0}$

## The Fourier Transform Can Solve Differential Eqs



❖ **Find the signal $h(t)$ defined by the DE**

ASSUME ALL INITIAL ARE ZERO.

$$RC\frac{d}{dt}h(t) + h(t) = \delta(t)$$

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

$\mathcal{F}$

$$\mathcal{F}\left\{RC\frac{d}{dt}h(t) + h(t)\right\} = \mathcal{F}\{\delta(t)\}$$

$$RC\,\underbrace{\mathcal{F}\left\{\frac{d}{dt}h(t)\right\}}_{(j2\pi f)\underbrace{\mathcal{F}\{h(t)\}}_{H(f)}} + \underbrace{\mathcal{F}\{h(t)\}}_{H(f)} = \underbrace{\mathcal{F}\{\delta(t)\}}_{1}$$

LINEARITY THEOREM

$$RC(j2\pi f)H(f) + H(f) = 1$$

$$H(f) = \frac{1}{1 + j2\pi fRC} = \frac{1}{RC}\left[\frac{1}{\frac{1}{RC} + j2\pi f}\right]$$

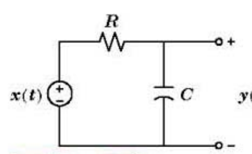


## System Analysis Using the Fourier Transform



❖ **Find the *impulse response* for the circuit shown.**

$$i(t) = C\frac{d}{dt}v(t)$$




METHOD #1: FIND SYSTEM EQ, SOLVE TO FIND $h(t)$

$$RC\frac{d}{dt}y(t) + y(t) = x(t)$$

$x(t) \to \boxed{\text{DE}} \to y(t)$

FIND $h(t)$ BY SOLVING

$$RC\frac{d}{dt}h(t) + h(t) = \delta(t)$$

$\delta(t) \to \boxed{\text{DE}} \to h(t)$

ASSUME I.C. ARE ZERO

SEE PREVIOUS SLIDE

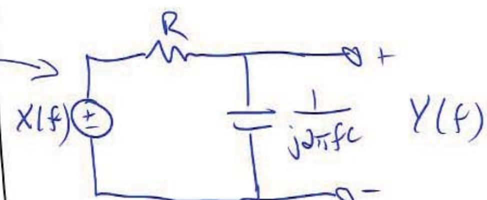
$$h(t) = \frac{1}{RC}e^{-\frac{1}{RC}t}u(t)$$

$\Rightarrow$ NOW CIRCUIT OUTPUT IS

$$y(t) = x(t) * h(t)$$

METHOD #2: CONVERT CIRCUIT INTO THE FREQ. DOMAIN COMPONENT AND SOLVE.

$\mathcal{F}$




VOLTAGE DIVISION

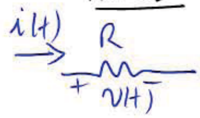
$$Y(f) = X(f)\left[\frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}}\right]\left(\frac{j2\pi fC}{j2\pi fC}\right)$$

$$\frac{Y(f)}{X(f)} = \frac{1}{1 + j2\pi fRC} = H(f)$$

$$Y(f) = X(f)H(f)$$

$$\frac{Y(f)}{X(f)} = H(f)$$

ASIDE

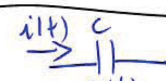



$$v(t) = i(t)R$$

$$\mathcal{F}\{v(t)\} = \mathcal{F}\{i(t)R\}$$

$$V(f) = I(f)R$$

$$R = \frac{V(f)}{I(f)}$$

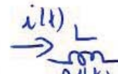



$$i(t) = C\frac{dv(t)}{dt}$$

$$\mathcal{F}\{i(t)\} = \mathcal{F}\left\{C\frac{dv(t)}{dt}\right\}$$

$$I(f) = C(j2\pi f)V(f)$$

$$\frac{V(f)}{I(f)} = \frac{1}{j2\pi fC} = Z(f)$$



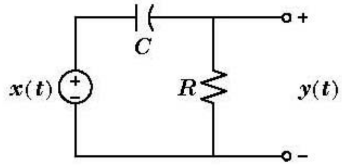

$$v(t) = L\frac{di(t)}{dt} \Rightarrow V(f) = L(j2\pi f)I(f)$$

$$\frac{V(f)}{I(f)} = j2\pi fL = Z(f)$$

❖ **Find the *impulse response* for the circuit shown.**

TRY THIS ON YOUR OWN

$$i(t) = C\frac{d}{dt}v(t)$$

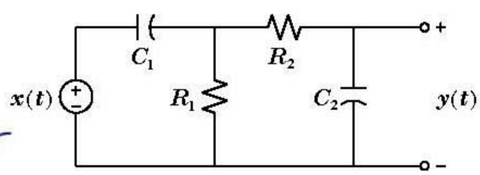


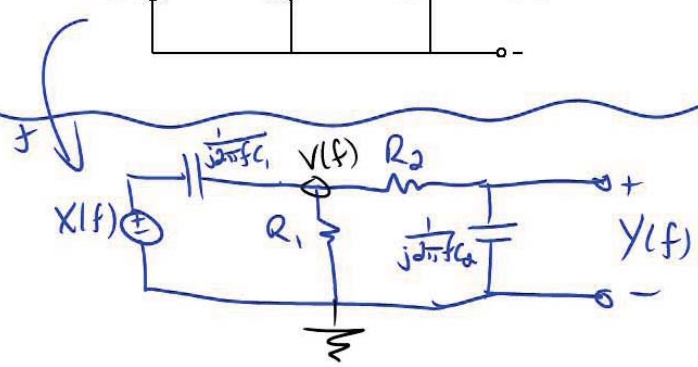




❖ **Find the *impulse response* for the circuit shown.**

$$i(t) = C\frac{d}{dt}v(t)$$




FIND V(f) USING NODAL ANALYSIS, THEN USE VOLTAGE DIVISION TO RELATE V(f) TO Y(f).

## The Transfer Function of a System



- An LTI system with impulse $h(t)$ has *Transfer Function*

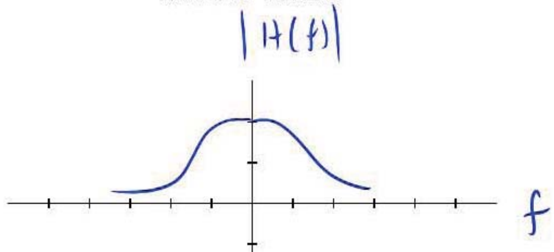
$$H(f) = \Im\{h(t)\}$$

REAL SIGNAL

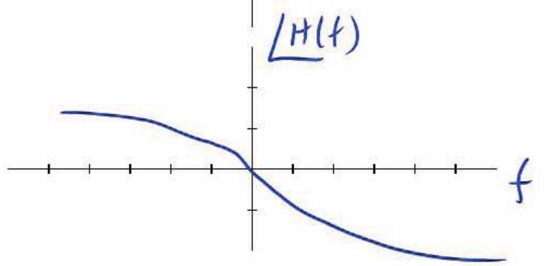
GENERALLY BE COMPLEX

Note: $H(f)$ is generally a *complex* function

**KNOW THIS!**




EVEN




ODD



## An Important Problem



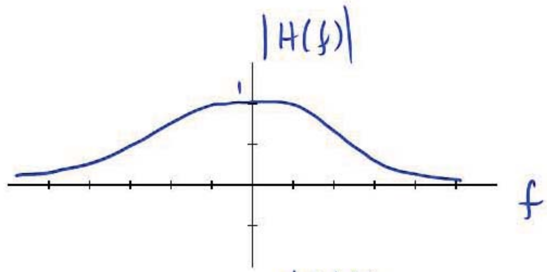
- Find $y(t)$ for an LTI system with Transfer Function $H(f)$ and with input

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

L.T.I.

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t) = x(t) * h(t)$$

$$y(t) = A\,|H(f_0)|\cos\left[2\pi f_0 t + \theta + \angle H(f_0)\right]$$




LOWPASS FILTER