

# SLAM-related Notes

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# Notes on SLAM Tutorial Part I

## (Durrant-Whyte and Bailey 2006)

### (paper)

- Problem:
  - Whether a mobile robot placed in an unknown location of an unknown environment can incrementally *build a consistent map* of this environment while simultaneously *determining its location* within this map

## 1. History

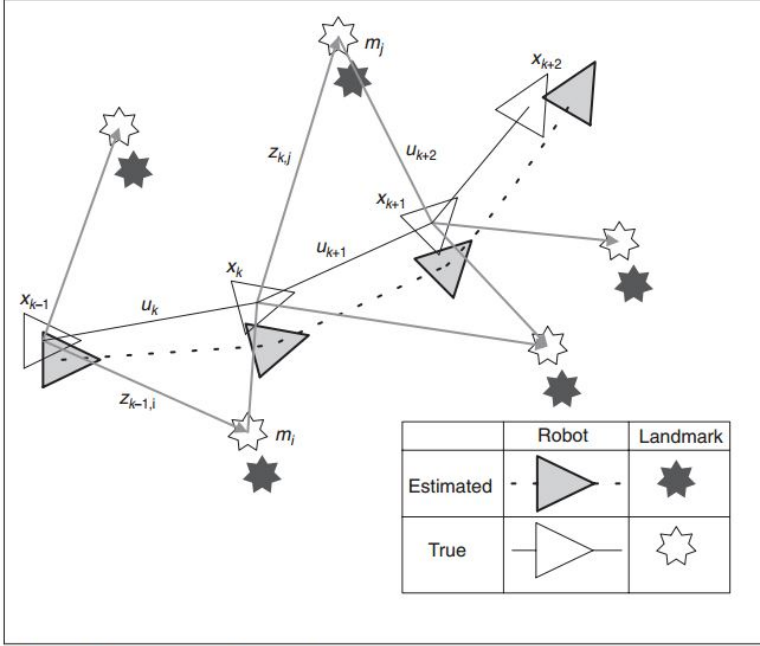
- As a mobile robot moves through an unknown environment taking relative observations of landmarks, the estimates of these landmarks are all necessarily correlated with each other because of the common error in estimated vehicle location
  - Implication: consistent full solution of SLAM problem would require *a joint state* composed of the *vehicle pose* and *every landmark position*, to be updated following *each landmark observation*
    - Pose: combination of position & orientation
- Once SLAM is formulated as a single *estimation problem*, the estimated map errors are *actually convergent*
- The more these correlations between landmarks grow, the better the solution

## 2. Formulation & Structure of the SLAM Problem

- SLAM: a process by which a mobile robot can *build a map of an environment* and at the same time *use this map to deduce its location*
- Both the *trajectory of the platform* and the *location of all landmarks* can be estimated online without any *a priori knowledge of location*

## 2.1 Preliminaries

- SLAM problem illustration:



**Figure 1.** The essential SLAM problem. A simultaneous estimate of both robot and landmark locations is required. The true locations are never known or measured directly. Observations are made between true robot and landmark locations.

- $\mathbf{x}_k$ : the state vector indicating vehicle location & orientation at time  $k$
- $\mathbf{u}_k$ : the control vector, applied at time  $k - 1$  to drive the vehicle to state vector  $\mathbf{x}_k$  at time  $k$
- $\mathbf{m}_i$ : a vector indicating the location of  $i$ th landmark whose true location is assumed to be time invariant
- $\mathbf{z}_{ik}$ : an observation of  $i$ th landmark taken from the vehicle at time  $k$ 
  - May be abbreviated as  $\mathbf{z}_k$  indicating multiple observations at time  $k$  or if the specific landmark is not relevant to the discussion
- $\mathbf{X}_{0:k} = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k\} = \{\mathbf{X}_{0:k-1}, \mathbf{x}_k\}$ : the history of previous vehicle locations
- $\mathbf{U}_{0:k} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\} = \{\mathbf{U}_{0:k-1}, \mathbf{u}_k\}$ : the history of control inputs
- $\mathbf{m} = \{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n\}$ : the set of all landmarks
- $\mathbf{Z}_{0:k} = \{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k\} = \{\mathbf{Z}_{0:k-1}, \mathbf{z}_k\}$ : the set of all landmark observations

## 2.2 Probabilistic SLAM

- SLAM problem requires the computation of  $P(\mathbf{x}_k, \mathbf{m} | \mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)$  at all times  $k$ 
  - Joint posterior density of the *vehicle state* and the *landmark locations* (at time  $k$ )
  - Recursive computation: compute joint posterior density at time  $k$  by  $P(\mathbf{x}_{k-1}, \mathbf{m} | \mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k-1}, \mathbf{x}_0)$  following control  $\mathbf{u}_k$  and observation  $\mathbf{z}_k$
  - Requirement: the definition of a **state transition model** and an **observation model** describing the control input and observation respectively
    - **Observation model:**  $P(\mathbf{z}_k | \mathbf{x}_k, \mathbf{m})$ 
      - The probability of making an observation at time  $k$  given the *vehicle location* and *landmark locations*
      - The observations are **conditionally independent** of current vehicle state ( $\mathbf{x}_k$ ) and the map ( $\mathbf{m}$ ) once the vehicle location and map are defined

- **Motion model:**  $P(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{u}_k)$ 
  - Probability distribution on state transitions
  - Assume state transition is a **Markov process**:
    - The next state  $\mathbf{x}_k$  depends only on the previous state  $\mathbf{x}_{k-1}$  and the applied control  $\mathbf{u}_k$
    - State transition is independent of both the observations and the map
- SLAM Algorithm: two-step recursive (*sequential*) prediction (*time-update*) correction (*measurement-update*) form
  - Step 1: Time-update

$$P(\mathbf{x}_k, \mathbf{m}|\mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k}, \mathbf{x}_0) = \int P(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{u}_k)P(\mathbf{x}_{k-1}, \mathbf{m}|\mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k-1}, \mathbf{x}_0)d\mathbf{x}_{k-1}$$

- Notes:
  - Only vehicle location  $\mathbf{x}_k$  is updated? Map data  $\mathbf{m}$  should be of time  $k-1$
  - Law of total probability:  $P(A) = \sum_n P(A|B_n)P(B_n)$ 
    - Event A: get prediction of current vehicle location & measurement of *previous?* landmark locations
    - Event B: get measurement of previous vehicle location & landmark locations
- Step 2: Measurement-update

$$P(\mathbf{x}_k, \mathbf{m}|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0) = \frac{P(\mathbf{z}_k|\mathbf{x}_k, \mathbf{m})P(\mathbf{x}_k, \mathbf{m}|\mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k}, \mathbf{x}_0)}{P(\mathbf{z}_k|\mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k})}$$

- Notes:
  - Bayes' theorem:  $P(A|B, C) = \frac{P(B|A, C)P(A|C)}{P(B|C)}$ 
    - Event A: get current vehicle location & landmark locations
      - $\mathbf{x}_k, \mathbf{m}$
    - Event B: get current observation
      - $\mathbf{z}_k$
    - Event C: get historical observations & control inputs
      - $\mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k-1}, \mathbf{x}_0$
  - Current observation is only related to current vehicle state and map
    - $P(B|A, C) = P(\mathbf{z}_k|(\mathbf{x}_k, \mathbf{m}), (\mathbf{Z}_{0:k-1}, \mathbf{U}_{0:k-1}, \mathbf{x}_0)) = P(\mathbf{z}_k|\mathbf{x}_k, \mathbf{m}) = P(B|A)$

## 2.3 Structure of Probabilistic SLAM

- Observations are **dependent** on both vehicle & landmark locations from the *observation model*  $P(\mathbf{z}_k|\mathbf{x}_k, \mathbf{m})$ , therefore:

$$P(\mathbf{x}_k, \mathbf{m}|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0) \neq P(\mathbf{x}_k|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)P(\mathbf{m}|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)$$

- Errors in landmark locations are highly correlated
  - Relative location between any 2 landmarks  $\mathbf{m}_i, \mathbf{m}_j$ , may be **known with high accuracy**, even when the absolute location of a landmark  $\mathbf{m}_i$  is quite uncertain
  - Meaning in probabilistic form: joint probability density for the pair of landmarks

$P(\mathbf{m}_i, \mathbf{m}_j)$  is highly peaked even when the marginal densities  $P(\mathbf{m}_i)$  may be quite dispersed

- Correlations between landmark estimates increase monotonically as more and more observations are made
  - Knowledge of relative landmark locations always improves and never diverges, regardless of robot motion
    - $P(\mathbf{m})$  becomes monotonically peaked as more observations are made
  - All known relative landmark locations are updated at each time segment even though some of the landmarks are not seen by the vehicle at that time

### 3. Solutions to the SLAM Problem

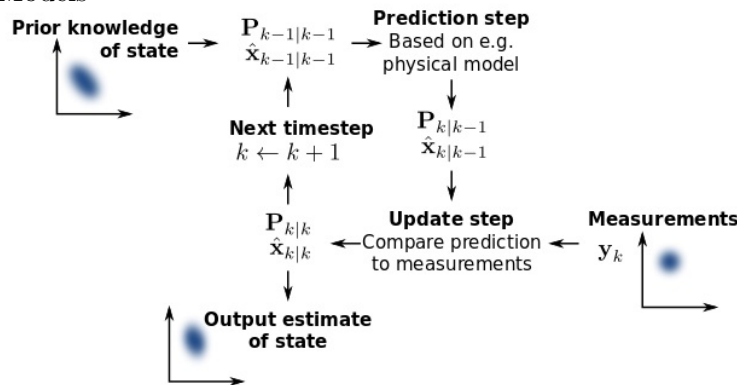
- Solutions involve *finding an appropriate representation* for both the **observation model** and the **motion model**
  - Representation 1: a state-space model with additive Gaussian noise
    - Use **extended Kalman filter** (EKF) to solve the SLAM problem (**EKF-SLAM**)
  - Representation 2: describe vehicle motion model  $P(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_k)$  as a set of samples of a more general non-Gaussian probability distribution
    - Use **Rao-Blackwellized particle filter** or **FastSLAM** algorithm to solve the SLAM problem

#### 3.1 Prerequisites: Kalman Filter & EKF

- Tutorials on Kalman filter & extended Kalman filter:
  - Kalman Filter: [#1](#) | [#2](#) & [extra](#) | [#3](#) (Faragher 2012) | [#4](#) (Wiki)
  - Extend Kalman Filter: [#1](#) | [#2](#) | [#3](#) (Wiki)

##### 3.1.1 Kalman Filter

- Models



- **State model** ( $P(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_k)$ ):

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_k$$

- $\mathbf{x}_k$ : the state vector containing the parameters for the system at time  $k$



- $\mathbf{F}_k$ : state-transition model to transfer the state vector  $\mathbf{x}_{k-1}$  (such as the position & velocity of a robot) at time  $k-1$  to the next state  $\mathbf{x}_k$  at time  $k$
- $\mathbf{B}_k$ : control-input model - control matrix applying on the control vector  $\mathbf{u}_k$  (external force to lead the robot to the next state) at time  $k$
- $\mathbf{w}_k$ : process noise at time  $k$  which assumes to have a multivariate Gaussian distribution with zero mean and a covariance of  $\mathbf{Q}_k$ 
  - $\mathbf{w}_k \sim \mathcal{N}(0, \mathbf{Q}_k)$
- **Observation model** ( $P(\mathbf{z}_k|\mathbf{x}_k)$ ):

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k$$

- $\mathbf{z}_k$ : observation/measurement at time  $k$  made by true state  $\mathbf{x}_k$  (not an estimation)
- $\mathbf{H}_k$ : observation model (transformation matrix) which maps the state space/domain into the observation/measurement space/domain
- $\mathbf{v}_k$ : observation noise which assumes to have a multivariate Gaussian distribution of zero mean and a covariance of  $\mathbf{R}_k$ 
  - $\mathbf{v}_k \sim \mathcal{N}(0, \mathbf{R}_k)$
- 2 steps: prediction & update
  - Step 1: Kalman filter produces estimates of the *current state variables*, along with their uncertainties

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

- General notation  $\mathbf{X}_{a|b}$ :  $\mathbf{X}$  at time  $a$  given observations up to and including at time  $b \leq a$
- State prediction:
  - $\hat{\mathbf{x}}_{k|k-1}$ : the **estimate** of state vector  $\mathbf{x}$  at time  $k$  given observations up to and including at time  $k-1$
- Covariance prediction:
  - $\mathbf{P}_{k|k-1}$ : covariance matrix at time  $k$  (describes the correlations between any two parameters inside the state vector  $\mathbf{x}$ )
  - $\mathbf{Q}_k$ : covariance matrix of the process noise (additive Gaussian noise) at time  $k$
  - Derivation (notice the uncorrelation between state estimation errors and process noise):

$$\begin{aligned}
 \mathbf{P}_{k|k-1} &= \mathbb{E}[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})^T] \\
 &= \mathbb{E}[(\mathbf{F}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}) + \mathbf{w}_k)(\mathbf{F}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}) + \mathbf{w}_k)^T] \\
 &= \mathbb{E}[(\mathbf{F}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})^T \mathbf{F}_k^T)] + \mathbb{E}[\mathbf{F}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})\mathbf{w}_k^T] \\
 &\quad + \mathbb{E}[\mathbf{w}_k(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})^T \mathbf{F}_k^T] + \mathbb{E}[\mathbf{w}_k \mathbf{w}_k^T] \\
 &= \mathbf{F}_k \mathbb{E}[(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1})^T] \mathbf{F}_k^T + 0 + 0 + \mathbb{E}[\mathbf{w}_k \mathbf{w}_k^T] \\
 &= \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k
 \end{aligned}$$

- Step 2: estimates are updated using a *weighted average* once the outcome of the *next*

measurement is observed

$$\begin{aligned}\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k \\ \mathbf{P}_{k|k} &= (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}\end{aligned}$$

where

$$\begin{aligned}\mathbf{K}_k &= \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1} \\ \tilde{\mathbf{y}}_k &= \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1} \\ \mathbf{S}_k &= \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k\end{aligned}$$

- $\tilde{\mathbf{y}}_k$  &  $\mathbf{S}_k$ : innovation / measurement residual and its covariance at time  $k$
- Both *estimated observation* ( $\mathbf{z}_{k,\text{estimated}} = \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$ ) and *actual observation* ( $\mathbf{z}_k$ ) have uncertainties (a range of possible observation data (assuming Gaussian distributions))
- $\mathbf{z}_{k,\text{estimated}} \sim \mathcal{N}(\mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}, \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T) = \mathcal{N}(\mathbf{H}_k \boldsymbol{\mu}_0, \mathbf{H}_k \boldsymbol{\Sigma}_0 \mathbf{H}_k^T)$
- $\mathbf{z}_{k,\text{actual}} \sim \mathcal{N}(\mathbf{z}_k, \mathbf{R}_k) = \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$
- Choose the **overlapped** region of the 2 ranges as the best guess of the real observation:
  - Also a Gaussian distribution (multiplying the 2 densities):  $\mathbf{z}_{k,\text{fused}} \sim \mathcal{N}(\boldsymbol{\mu}_{\text{fused}}, \boldsymbol{\Sigma}_{\text{fused}})$
  - $\boldsymbol{\mu}_{\text{fused}} = \boldsymbol{\mu}_0 + \mathbf{K}_k(\boldsymbol{\mu}_1 - \mathbf{H}_k \boldsymbol{\mu}_0)$
  - $\boldsymbol{\Sigma}_{\text{fused}} = \boldsymbol{\Sigma}_0 - \mathbf{K}_k \mathbf{H}_k \boldsymbol{\Sigma}_0$
  - $\mathbf{K}_k = \boldsymbol{\Sigma}_0 \mathbf{H}_k^T (\mathbf{H}_k \boldsymbol{\Sigma}_0 \mathbf{H}_k^T + \boldsymbol{\Sigma}_1)^{-1}$  ( $\mathbf{K}_k$ : **Kalman gain** at time  $k$ )

### 3.1.2 Extended Kalman Filter (EKF)

Notes: mathematical notations are almost the same with those in KF

- KF limitations:
  - Linear function/model/system
  - Gaussian distribution for observed variables
- EKF: Nonlinear version of KF
  - Use **estimated** mean & covariance of observed variables for filter response computation
  - Mean & covariance is estimated using linear approximations through Taylor Series (multivariate Taylor Series)
    - Linear because only the first order (partial) derivatives are computed (using the Jacobian matrix)
- Models: use **differentiable** functions ( $f$  &  $h$ ) instead of linear functions (i.e., matrices  $\mathbf{F}_k$ ,  $\mathbf{B}_k$ , and  $\mathbf{H}_k$ )
  - **State model**:

$$\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k) + \mathbf{w}_k$$

- **Observation model**:

$$\mathbf{z}_k = h(\mathbf{x}_k) + \mathbf{v}_k$$

- 2 steps: prediction & update
  - Step 1: prediction

$$\hat{\mathbf{x}}_{k|k-1} = f(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

- $\mathbf{F}_k$ : Jacobian matrix whose elements are the linear approximations (partial derivatives) of state transition function  $f$  defined as follows (assume state vector has a dimension of  $n \times 1$ )

$$\mathbf{F}_{k,n \times n} = \frac{\partial f_{n \times 1}}{\partial \mathbf{x}_{n \times 1}}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)$$

- Step 2: update

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

where

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - h(\hat{\mathbf{x}}_{k|k-1})$$

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$

- $\mathbf{H}_k$ : Jacobian matrix whose elements are the linear approximations (partial derivatives) of measurement-to-state mapping function  $h$  defined as follows (assume state vector has a dimension of  $n \times 1$  and observation vector has a dimension of  $m \times 1$ )

$$\mathbf{H}_{k,m \times n} = \frac{\partial h_{m \times 1}}{\partial \mathbf{x}_{n \times 1}}(\hat{\mathbf{x}}_{k|k-1})$$

### 3.2 EKF-SLAM

- Models

- Motion model:

$$P(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_k) \iff \mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k) + \mathbf{w}_k$$

- $f(\cdot)$ : models the vehicle kinematics
- $\mathbf{w}_k$ : *additive, uncorrelated* Gaussian motion disturbances, and  $\mathbf{w}_k \sim \mathcal{N}(0, \mathbf{Q}_k)$

- Observation model:

$$P(\mathbf{z}_k | \mathbf{x}_k, \mathbf{m}) \iff \mathbf{z}_k = h(\mathbf{x}_k, \mathbf{m}) + \mathbf{v}_k$$

- $h(\cdot)$ : models the geometry of the observation
- $\mathbf{v}_k$ : *additive, uncorrelated* Gaussian observation errors, and  $\mathbf{v}_k \sim \mathcal{N}(0, \mathbf{R}_k)$

- Applying EKF to SLAM problem by modeling joint posterior distribution  $P(\mathbf{x}_k, \mathbf{m} | \mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)$  as  $\mathbf{z}_{k,\text{fused}}$

$$\mathbf{z}_{k,\text{fused}} \sim \mathcal{N}(\boldsymbol{\mu}_{\text{fused}}, \boldsymbol{\Sigma}_{\text{fused}})$$

$$\boldsymbol{\mu}_{\text{fused}} = \begin{bmatrix} \hat{\mathbf{x}}_{k|k} \\ \hat{\mathbf{m}}_k \end{bmatrix} = \mathbb{E} \left[ \begin{bmatrix} \mathbf{x}_k \\ \mathbf{m} \end{bmatrix} \middle| \mathbf{Z}_{0:k} \right]$$

$$\boldsymbol{\Sigma}_{\text{fused}} = \begin{bmatrix} \mathbf{P}_{xx} & \mathbf{P}_{xm} \\ \mathbf{P}_{xm}^T & \mathbf{P}_{mm} \end{bmatrix}_{k|k} = \mathbb{E} \left[ \begin{pmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \\ \mathbf{m} - \hat{\mathbf{m}}_k \end{pmatrix} \begin{pmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} \\ \mathbf{m} - \hat{\mathbf{m}}_k \end{pmatrix}^T \right] = \mathbb{E} \left[ \begin{pmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \mathbf{m} - \hat{\mathbf{m}}_k \end{pmatrix} \begin{pmatrix} \mathbf{x}_k - \hat{\mathbf{x}}_k \\ \mathbf{m} - \hat{\mathbf{m}}_k \end{pmatrix}^T \middle| \mathbf{Z}_{0:k} \right]$$

- EKF applying: **time-update** (prediction step) & **observation-update** (update step)

- Time-update is only for state vectors (except when landmarks can move)
- Observation-update:  $\hat{\mathbf{x}}_{k|k} \rightarrow \begin{bmatrix} \hat{\mathbf{x}}_{k|k} \\ \hat{\mathbf{m}}_k \end{bmatrix}$
- $\mathbf{F}_k = \nabla f = \frac{\partial f}{\partial \mathbf{x}}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_k)$
- $\mathbf{H}_k = \nabla h = \frac{\partial h}{\partial \mathbf{x}}(\hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{m}}_{k-1})$
- $\mathbf{W}_k$ : Kalman gain ( $\mathbf{K}_k$ )
- Map convergence: map convergence matrix  $\mathbf{P}_{mm,k}$  and all landmark pair submatrices are monotonically convergent toward zero
- Computation grows **quadratically** with **the number of landmarks** because all landmarks and joint covariance matrix will be updated every time an observation is made
- Standard formulation of EKF-SLAM solution is fragile to incorrect association of observations to landmarks (**loop-closure** problem)
- Linearization of nonlinear motion & observation models (**EKF**) leads to inconsistency in solutions

### 3.3 Rao-Blackwellized Filter (FastSLAM)

- Basis: recursive Monte Carlo sampling / particle filtering
- Applying Rao-Blackwellization (R-B) on sample-space (state-space)
  - Partition joint state according to the product rule:  $P(\mathbf{x}_1, \mathbf{x}_2) = P(\mathbf{x}_2|\mathbf{x}_1)P(\mathbf{x}_1)$ 
    - Only  $P(\mathbf{x}_1)$  needs be sampled ( $\mathbf{x}_1^{(i)} \sim P(\mathbf{x}_1)$ ) if  $P(\mathbf{x}_2|\mathbf{x}_1)$  can be represented analytically
  - Joint distribution is represented by the set  $\{\mathbf{x}_1^{(i)}, P(\mathbf{x}_2|\mathbf{x}_1^{(i)})\}$
  - Statistics can be obtained by sampling over the joint space
    - E.g., the marginal:

$$P(\mathbf{x}_2) \approx \frac{1}{N} \sum_i^N P(\mathbf{x}_2|\mathbf{x}_1^{(i)})$$

- Joint SLAM state may be factored into a **vehicle component** and a **conditional map component**:

$$\begin{aligned} P(\mathbf{X}_{0:k}, \mathbf{m}|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0) &= P(\mathbf{X}_{0:k}|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)P(\mathbf{m}|\mathbf{X}_{0:k}, \mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0) \\ &= P(\mathbf{X}_{0:k}|\mathbf{Z}_{0:k}, \mathbf{U}_{0:k}, \mathbf{x}_0)P(\mathbf{m}|\mathbf{X}_{0:k}, \mathbf{Z}_{0:k}) \end{aligned}$$

- Note:  $P(A, B|C) = P(A|C)P(B|A, C)$
- Map landmarks **become independent** if the distribution is conditioned on the trajectory  $\mathbf{X}_{0:k}$  rather than a single pose  $\mathbf{x}_k$

$$P(\mathbf{m}|\mathbf{X}_{0:k}, \mathbf{Z}_{0:k}) = \prod_{j=1}^n P(\mathbf{m}_j|\mathbf{X}_{0:k}, \mathbf{Z}_{0:k})$$

- Conditional independence of landmarks:
  - As landmark status is dependent with history of observations & robot poses, if all the history info are known, any landmark status in the landmark set can be extracted using the same history info
    - Thus there's no dependence between any two of the landmarks given the history info (observation history is manifestly known)
  - $P(\mathbf{m}_i|\mathbf{m}_j, \mathbf{X}_{0:k}, \mathbf{Z}_{0:k}) = P(\mathbf{m}_i|\mathbf{X}_{0:k}, \mathbf{Z}_{0:k})$

- The map is represented as a set of **independent Gaussians** in FastSLAM with linear complexity rather than a joint map of covariance with quadratic complexity ( $n^2$  elements in the covariance matrix at any given time  $k$ )
- Rao-Blackwellized joint distribution ( $N$ -sample set) at time  $k$ :  $\{w_k^{(i)}, \mathbf{X}_{0:k}^{(i)}, P(\mathbf{m}|\mathbf{X}_{0:k}^{(i)}, \mathbf{Z}_{0:k})\}_i^N$ 
  - $w_k^{(i)}$ : weight for trajectory particle/sample  $\mathbf{X}_{0:k}^{(i)}$
  - Rao-Blackwellized conditional map component for each particle in the above set  

$$P(\mathbf{m}|\mathbf{X}_{0:k}^{(i)}, \mathbf{Z}_{0:k}) = \prod_{j=1}^n P(\mathbf{m}_j|\mathbf{X}_{0:k}^{(i)}, \mathbf{Z}_{0:k})$$
- Particle filter theory is derived from a recursive form of sampling known as **sequential importance sampling** (SIS)
  - Sampling recursively via the product rule rather than directly sampling the joint state

$$P(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_T | \mathbf{Z}_{0:T}) = P(\mathbf{x}_0 | \mathbf{Z}_{0:T}) P(\mathbf{x}_1 | \mathbf{x}_0, \mathbf{Z}_{0:T}) \dots P(\mathbf{x}_T | \mathbf{x}_{0:T-1}, \mathbf{Z}_{0:T})$$

- General form of R-B particle filter for SLAM
  - Assume the joint state is represented as  $\{w_{k-1}^{(i)}, \mathbf{X}_{0:k-1}^{(i)}, P(\mathbf{m}|\mathbf{X}_{0:k-1}^{(i)}, \mathbf{Z}_{0:k-1})\}_i^N$  at time  $k-1$
  - Step 1: draw a sample from a proposal distribution conditioned on specific particle history

$$\mathbf{x}_k^{(i)} \sim \pi(\mathbf{x}_k | \mathbf{X}_{0:k-1}^{(i)}, \mathbf{Z}_{0:k}, \mathbf{u}_k)$$

- Implicitly update the particle history:  $\mathbf{X}_{0:k}^{(i)} \triangleq \{\mathbf{X}_{0:k-1}^{(i)}, \mathbf{x}_k^{(i)}\}$
- Step 2: compute sample weight

$$w_k^{(i)} = w_{k-1}^{(i)} \frac{P(\mathbf{z}_k | \mathbf{X}_{0:k}^{(i)}, \mathbf{Z}_{0:k-1}) P(\mathbf{x}_k^{(i)} | \mathbf{x}_{k-1}^{(i)}, \mathbf{u}_k)}{\pi(\mathbf{x}_k^{(i)} | \mathbf{X}_{0:k}^{(i)}, \mathbf{Z}_{0:k})}$$

- Numerator terms: **observation model** (map info being marginalized) & **motion model**
  - Map marginalization for *observation model*:

$$P(\mathbf{z}_k | \mathbf{X}_{0:k}, \mathbf{Z}_{0:k-1}) = \int P(\mathbf{z}_k | \mathbf{x}_k, \mathbf{m}) P(\mathbf{m} | \mathbf{X}_{0:k-1}, \mathbf{Z}_{0:k-1}) d\mathbf{m}$$

- Step 3: perform resampling if necessary
  - Select particles with replacement from the set  $\{\mathbf{X}_{0:k}^{(i)}\}_i^N$  and set uniform weight  $w_k^{(i)} = 1/N$
- Step 4: update observed landmarks using EKF as a simple mapping function with known vehicle pose

# Notes on SLAM Tutorial Part II

## (Bailey and Durrant-Whyte 2006)

### (paper)

#### 1. Focus

- Tutorial focus: recursive Bayesian formulation of the SLAM problem where probability distributions/estimates of absolute/relative landmark locations & vehicle poses are obtained
- 3 Key areas in SLAM research in this tutorial:
  - a. Computational complexity
    - Approaches to reduce the computational complexity
      - Linear-time state augmentation
      - Sparsification in information form
      - Partitioned updates
      - Submapping methods
  - b. Data association
    - Methods to correctly associate landmark observations with landmarks in the map
      - Batch-validation methods
      - Appearance-based methods
      - Multihypothesis techniques
  - c. Environment representation
    - Related to appearance-based models of landmarks and maps
      - Delayed mapping
      - Use of nongeometric landmarks
      - Trajectory estimation methods
- Use mathematical notations defined in SLAM Part I tutorial

#### 2. Computational Complexity

- Exploit the structure to limit computational complexity of SLAM algorithm
  - Process/Motion model  $P(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_k)$ : only affect vehicle pose states
  - Observation model  $P(\mathbf{z}_k | \mathbf{x}_k, \mathbf{m})$ : only make reference to single vehicle-landmark pair
- Category of techniques:

- **Optimal:** *reduce* required computation and get estimates & covariances that are *equal to* the full-form SLAM algorithm
- **Conservative:** computationally more efficient but *less accurate*
  - Termed *inconsistent* & regarded as *invalid solutions* to the SLAM (or any estimation) problem
- Methods:
  - State augmentation:
    - Re-formulating time-update equations
    - Optimal
  - Partitioned updates:
    - Re-formulating observation-update equations
    - Optimal
  - Sparsification in information form
    - Re-formulating standard state-space SLAM representation into information form
    - Usually conservative
  - Submapping methods
    - Provide estimates in global frame
    - Conservative

## 2.1 State Augmentation

- SLAM state vector  $\mathbf{x}_k = [\mathbf{x}_{vk}^T, \mathbf{m}^T]^T$  (where  $\mathbf{x}_{vk}$  denotes the vehicle state vector at time  $k$ )
- Covariance computation in a naive implementation of EKF-SLAM:

$$\mathbf{P}_{k|k-1} = \nabla \mathbf{f}_{\mathbf{x}} \mathbf{P}_{k-1|k-1} \nabla \mathbf{f}_{\mathbf{x}}^T + \nabla \mathbf{f}_{\mathbf{u}} \mathbf{U}_k \nabla \mathbf{f}_{\mathbf{u}}^T$$

where  $\nabla \mathbf{f}_{\mathbf{x}} = \partial \mathbf{f} / \partial \mathbf{x}_{k-1}$ ,  $\nabla \mathbf{f}_{\mathbf{u}} = \partial \mathbf{f} / \partial \mathbf{u}_k$ , and  $\mathbf{U}_k$  is the covariance characterising uncertainty on control vector

- Exploit the fact that motion model only affects vehicle pose states:

$$\mathbf{P}_{k|k-1} = \begin{bmatrix} \mathbf{P}_{vv} & \mathbf{P}_{vm} \\ \mathbf{P}_{vm}^T & \mathbf{P}_{mm} \end{bmatrix} = \begin{bmatrix} \nabla \mathbf{f}_{v_{\mathbf{x}}} \mathbf{P}_{vv} \nabla \mathbf{f}_{v_{\mathbf{x}}}^T + \nabla \mathbf{f}_{v_{\mathbf{u}}} \mathbf{U}_k \nabla \mathbf{f}_{v_{\mathbf{u}}}^T & \nabla \mathbf{f}_{v_{\mathbf{x}}} \mathbf{P}_{vm} \\ \mathbf{P}_{vm}^T \nabla \mathbf{f}_{v_{\mathbf{x}}}^T & \mathbf{P}_{mm} \end{bmatrix}$$

where  $\nabla \mathbf{f}_{v_{\mathbf{x}}} = \partial \mathbf{f}_{v_{\mathbf{x}}} / \partial \mathbf{x}_{v(k-1)}$  and  $\nabla \mathbf{f}_{v_{\mathbf{u}}} = \partial \mathbf{f}_{v_{\mathbf{u}}} / \partial \mathbf{u}_k$

- Adding new landmarks to the SLAM state vector at a new state  $k$ :

$$\mathbf{x}_k^+ = \begin{bmatrix} \mathbf{x}_{vk} \\ \mathbf{m} \\ \mathbf{m}_{new} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{vk} \\ \mathbf{m} \\ g(\mathbf{x}_{vk}, \mathbf{z}_k) \end{bmatrix}$$

where the new map landmark(s) are initialized by a function of vehicle pose  $\mathbf{x}_{vk}$  and the current observation  $\mathbf{z}_k$

- Apply state augmentation whenever new states are a function of a subset of existing states:

$$\mathbf{x}^+ = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ f(\mathbf{x}_2, \mathbf{q}) \end{bmatrix}$$

$$\mathbf{P}^+ = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{12} \nabla \mathbf{f}_{\mathbf{x}_2}^T \\ \mathbf{P}_{12}^T & \mathbf{P}_{22} & \mathbf{P}_{22} \nabla \mathbf{f}_{\mathbf{x}_2}^T \\ \nabla \mathbf{f}_{\mathbf{x}_2} \mathbf{P}_{12}^T & \nabla \mathbf{f}_{\mathbf{x}_2} \mathbf{P}_{22} & \nabla \mathbf{f}_{\mathbf{x}_2} \mathbf{P}_{22} \nabla \mathbf{f}_{\mathbf{x}_2}^T + \nabla \mathbf{f}_{\mathbf{q}} \mathbf{Q} \nabla \mathbf{f}_{\mathbf{q}}^T \end{bmatrix}$$

where the dimensions are:

$$\dim \mathbf{x}^+ = \begin{bmatrix} a \times 1 \\ b \times 1 \\ (b+c) \times 1 \end{bmatrix}$$

$$\dim \mathbf{P}^+ = \begin{bmatrix} a \times a & a \times b & a \times (b+c) \\ b \times a & b \times b & b \times (b+c) \\ (b+c) \times a & (b+c) \times b & (b+c) \times (b+c) \end{bmatrix}$$

## 2.2 Partitioned Updates

- Instead of updating *all vehicle & map states* every time when a new measurement is made, only update a *small local region* of states and update the global map only at a *much slower frequency*
- 2 basic types:
  - Type 1: operating on a local region of the global map and maintains globally referenced coordinates
    - Key methods:
      - Compressed EKF (CEKF)
      - Postponement algorithm
  - Type 2: generating a short-term submap with its own local coordinate frame
    - Key methods:
      - Constrained local/relative submap filter (CLSF/CRSF)
      - Local map sequencing algorithm
    - Simpler & more numerically stable & less affected by linearization errors
- CLSF/CRSF:
  - Maintain 2 independent SLAM estimates at all times:

$$\mathbf{x}_G = \begin{bmatrix} \mathbf{x}_F^G \\ \mathbf{m}_G \end{bmatrix}, \quad \mathbf{x}_R = \begin{bmatrix} \mathbf{x}_v^R \\ \mathbf{m}_R \end{bmatrix}$$

- $\mathbf{x}_G$ : map composed of global reference pose of a submap coordinate frame  $\mathbf{x}_F^G$  and a set of global referenced landmarks  $\mathbf{m}_G$
- $\mathbf{x}_R$ : local submap with a locally referenced vehicle pose  $\mathbf{x}_v^R$  and locally referenced landmarks  $\mathbf{m}_R$
- Conventional SLAM updates are performed entirely within the local submap



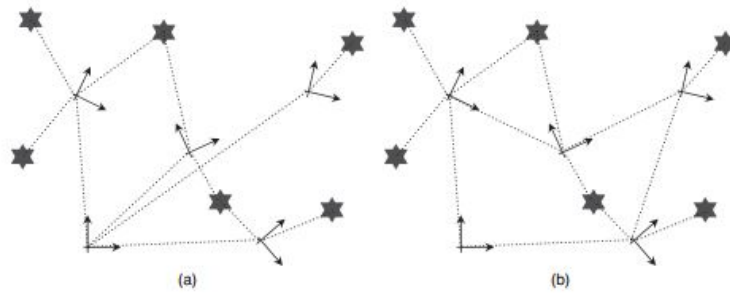
- Only those landmarks held in the local submap will be updated
- Obtain global vehicle pose estimate at any time by **vector summation** of *locally referenced pose* and the *global estimate* of the submap coordinate frame
- Optimal global estimate is obtained periodically by **registering the submap with the global map** and **applying constraint updates on any common features of both maps**
- Advantages:
  - At any one instance, the number of landmarks that must be updated is limited to those described in the local submap coordinate frame only
    - Observation-rate update is independent of the total map size
    - Full update can be performed in a background task at a much lower rate as long as the observation-rate update can be performed
  - Uncertainty is lower in a locally referenced frame
    - Linearization error is reduced
  - Association robustness can be improved by using *batch-validation gating* in submap registration

## 2.3 Sparsification

- Represent Gaussian probability density in canonical/information form rather than using the 1st & 2nd moment  $\hat{\mathbf{x}}_k$  &  $\mathbf{P}_k$ 
  - Information matrix  $\mathbf{Y}_k = \mathbf{P}_k^{-1}$
  - Information vector  $\hat{\mathbf{y}}_k = \mathbf{Y}_k \hat{\mathbf{x}}_k$
- Many off-diagonal components of the *normalized* information matrix are very close to 0
- An exactly sparse solution can be obtained by augmenting the state vector with new vehicle pose estimate at every time step and retaining all past robot poses:
  - $\mathbf{x}_k = [\mathbf{x}_{vk}^T, \mathbf{x}_{vk-1}^T, \dots, \mathbf{x}_{v1}^T, \mathbf{m}^T]^T$
  - Nonzero off-diagonal components: only for poses & landmarks that are directly related by measurement data
  - Control sparsity by utilizing **marginalization** to remove past states
- Caveat: necessary to recover mean & covariance of the state at every time step
  - Mean estimate is required to perform linearization of process & observation models
    - Efficient recovery by conjugate gradient method
  - Both mean & covariance are required to compute validation gate for data association
    - Robust batch gating methods require the full recovery of covariance matrix, which has a **cubic** complexity in the number of landmarks

## 2.4 Submapping Methods

- Another category of methods to deal with the **quadratically scaling computation complexity** with the number of landmarks during measurement updates
- 2 categories: globally (a) & locally referenced submaps (b)



- Global submaps
  - Estimate global locations of submap coordinate frames relative to a common base frame
  - Cannot alleviate linearization issues arising from large pose uncertainties
- Local/Relative submaps
  - Location of any given submap is recorded only by its neighboring submaps
  - Submaps are connected in a graphical network
  - Obtain global estimate by vector summation along a path in the network
  - Can avoid linearization issues because global-level data fusion is eschewed

### 3. Data Association

- *New measurements* are associated with *existing map landmarks* before fusing data into the map
  - The associations ***cannot be revised*** after fusion
- Any single incorrect data association can lead to failure of the SLAM algorithm

#### 3.1 Batch Validation

- Individual gating
  - Consider each measurement-to-landmark association individually
  - Test whether an observed landmark is close to a predicted location
  - Extremely unreliable if the vehicle pose is very uncertain
- Batch gating
  - Multiple associations are considered simultaneously
  - Exploit geometric relationship between landmarks

#### 3.2 Appearance Signatures

- Extract landmark appearance features as its signature
- Use ***similarity metrics*** such as an image similarity metric to predict a possible association

#### 3.3 Multihypothesis Data Association

- Essential for robust target tracking in cluttered environment
- Generate separate track estimate for each association hypothesis, and create an ever-branching tree of tracks over time
- Prune low-likelihood tracks from the hypothesis tree

## 4. Environment Representation

- Assumption in early SLAM works: the world can be reasonably modeled as *a set of simple discrete landmarks* described by *geometric primitives* such as points, lines or circles
  - Assumption does not hold for more complex & unstructured environments

### 4.1 Partial Observability and Delayed Mapping

- Limitations on sensing modality
  - Sonar: produce accurate range measurement but unusable bearing estimate
  - Vision (camera): provide bearing info without accurate range indication
- Single measurement (range-only / bearing-only sensors) is *insufficient* to constrain a landmark location
  - Need be observed from multiple vantage points (e.g., the intersection of regions mapped by multiple measurements)
  - Single measurement can only generate a non-Gaussian distribution over landmark location
    - Example for a Gaussian landmark estimate: central position & radius
- Delayed mapping: obtaining Gaussian landmark estimate from range-only / bearing-only sensors by *delaying initialization & accumulating raw measurement data*
  - Record vehicle pose for each deferred measurement
    - Augment SLAM states with recorded vehicle pose estimates
$$(\mathbf{x}_k = [\mathbf{x}_{v_k}^T, \mathbf{x}_{v_{k-1}}^T, \dots, \mathbf{x}_{v_{k-n}}^T, \mathbf{m}^T]^T)$$
    - Store corresponding measurements in an auxiliary list  $(\{\mathbf{z}_k, \dots, \mathbf{z}_{k-n}\})$
  - Initialize a landmark by a batch update once sufficient information is accumulated
  - Remove recorded poses that do not have any associated measurements
- Delayed fusion
  - General concept for increasing robustness by *accumulating information* and *permitting delayed decision making*
  - Advantages:
    - Reduce linearization errors by performing a batch update (such as *bundle adjustment* (BA)) on accumulated data set
    - Facilitate batch validation gating & aid reliable data association

### 4.2 Nongeometric Landmarks

- EKF-SLAM is usually applied to geometric landmarks
- Nongeometric landmarks: landmarks with arbitrary shape
  - Described by a shape model with an embedded coordinate frame whose origin defines the landmark origin
- EKF-SLAM with nongeometric landmarks: separate shape parameter estimation from landmark locations

### 4.3 3-D SLAM

- 3 essential forms

- a. 2-D SLAM with additional map building ability in 3rd dimension
- b. Direct extension of 2-D SLAM
- c. SLAM with joint state containing history of past vehicle poses
  - Obtaining 3-D environment scan at each pose

#### 4.4 Trajectory-Oriented SLAM

- Estimate *vehicle trajectory* instead of pose + landmark
- Type 1:
  - Joint state  $\mathbf{x}_k$ :  $[\mathbf{x}_{vk}^T, \mathbf{x}_{v_{k-1}}^T, \dots, \mathbf{x}_{v_1}^T]$
  - Particularly suited to environments where
    - Discrete landmarks are not easily discerned
    - Direct alignment of sensed data is easier / more reliable
  - Map is not part of the state but forms an auxiliary data set
    - Map is formed by aligning associated scan of sensed data from each pose estimate
- Type 2 (Consistent Pose Estimation (CPE)):
  - Connect pose in a graphical network instead of forming a joint state vector
- Type 3:
  - Sparse-information-form SLAM with sparse estimation of joint state vector of type 1
- Caveats:
  - Unbounded state-space & quantity of stored measurement data
  - Necessary to coalesce data to bound storage costs

#### 4.5 Embedded Auxiliary Information

- Soil salinity, humidity, temperature, terrain characteristics, etc...
- May be used to assist mapping, aid data association or for tasks unrelated to mapping (path planning / data gathering)
- More difficult to be incorporated within traditional SLAM framework

#### 4.6 Dynamic Environments

- Dynamic landmarks:
  - Moving objects such as people
  - Temporary structures that appear static but later moved such as chairs / parked cars
- Property:
  - Landmarks can be removed from the map without loss of consistency
    - Large number of landmarks can be removed without change of convergence rate
- Ideas:
  - Removing landmarks that are obsolete (due to environment changes)
  - Implement auxiliary identification routine to remove dynamic info from a data scan before sending it to SLAM algorithm
  - Tracking both stationary & moving targets
    - Reduce cost by implementing stationary SLAM update followed by separate tracking of moving objects

# Notes on SLAM Survey (Cadena et al. 2016) ([site](#)|[paper](#))

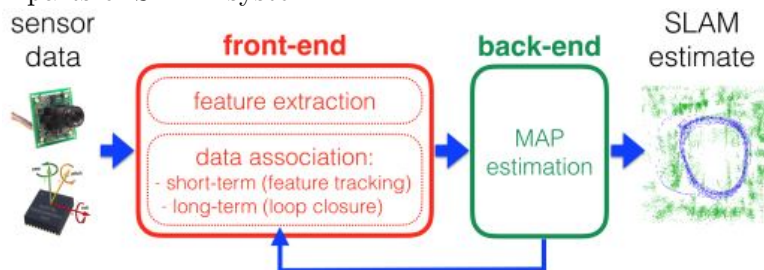
## 1. Introduction

- SLAM development:
  - Classical age (1986 - 2004): covered in SLAM tutorial [I](#) (Durrant-Whyte and Bailey 2006) & [II](#) (Bailey and Durrant-Whyte 2006)
  - Algorithmic-analysis age (2004 - 2015): covered in (Dissanayake et al. 2011 ([link](#)))
  - Robust-perception age (2015 - )
    - Robust performance
    - High-level understanding
    - Resource awareness
    - Task-driven inference
- 2 questions:
  - a. Do autonomous robots really need SLAM?
    - Visual-inertial Navigation (VIN) *is* SLAM
      - A *reduced* SLAM system without loop closure / place recognition module
    - Advantage of SLAM: getting real topology of the environment by enabling loop closure
      - Offers a natural defense against wrong data association & perceptual aliasing, which will deceive place recognition
    - Key to robust operation
    - Many applications implicitly/explicitly require a globally consistent map
  - b. Is SLAM solved?
    - Need to specify the robot/environment/performance combination
      - Robot
        - Type of motion
        - Available sensors
        - Available computational resources
      - Environment
        - Dimension: planar or three-dimensional
        - Presence of natural/artificial landmarks
        - Amount of dynamic elements
        - Amount of symmetry

- Risk of perceptual aliasing
- May depend on the sensor-environment pair
- Performance requirements
  - Desired accuracy of the robot state estimation
  - Accuracy & type of representation of the environment (landmark-based or dense)
  - Success rate
  - Estimation latency
  - Maximum operation time
  - Maximum size of mapped area

## 2. Anatomy of a Modern SLAM System

- 2 parts of SLAM system



- Front-end: abstract sensor data into estimation models
  - Data association module
    - Short-term data block: associate corresponding features in consecutive sensor measurements
    - Long-term data block (loop closure): associate new measurements to older landmarks
  - Pre-processing: sensor dependent
- Back-end: perform inference on abstracted data from front-end
  - Feed feedback info to the front-end to support loop closure detection & validation

### 2.1 SLAM Back-end

#### 2.1.1 Fisher Information Matrix

- Abbreviated as information matrix

- Fisher information  $\mathcal{J}(\theta)$  over parameter  $\theta$ :

$$\begin{aligned}
\mu &= \mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(X;\theta)|\theta\right] \\
&= \int \frac{\partial}{\partial\theta}(\log f(X;\theta))f(X;\theta)dX \\
&= \int \frac{\frac{\partial}{\partial\theta}f(X;\theta)}{f(X;\theta)}f(X;\theta)dX \\
&= \frac{\partial}{\partial\theta} \int f(X;\theta)dX \\
&= \frac{\partial}{\partial\theta}1 \\
&= 0 \\
\mathcal{J}(\theta) &= \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X;\theta) - \mu\right)^2|\theta\right] \\
&= \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X;\theta)\right)^2|\theta\right]
\end{aligned}$$

- $\mu$ : 1st moment / expected value of the “score” (i.e., the partial derivative with respect to  $\theta$  of the log-likelihood function  $f(X;\theta)$ )
- Fisher information: 2nd central moment / variance of log-likelihood function  $f(X;\theta)$
- If  $\log f(X;\theta)$  is twice differentiable w.r.t.  $\theta$ , then under certain regularity conditions

$$\mathcal{J}(\theta) = \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(X;\theta)\right)^2|\theta\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)|\theta\right]$$

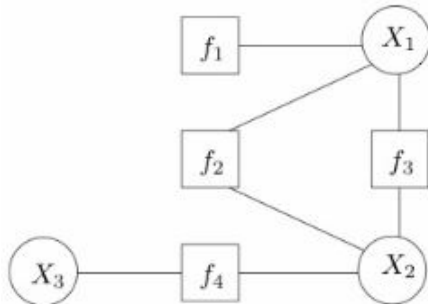
- Fisher information matrix: Fisher information of a  $N \times N$  vector of parameter  $\theta = [\theta_1, \theta_2, \dots, \theta_N]^T$ 
  - Element  $(i, j)$  of matrix  $\mathcal{J}$

$$[\mathcal{J}(\theta)]_{i,j} = \mathbb{E}\left[\left(\frac{\partial}{\partial\theta_i}\log f(X;\theta)\right)\left(\frac{\partial}{\partial\theta_j}\log f(X;\theta)\right)|\theta\right]$$

- $\dim \mathcal{J} = N \times N$
- [Example](#)

### 2.1.2 Maximum-a-posteriori Estimation

- Maximum-a-posteriori (MAP) estimation as SLAM formulation
  - Formulate SLAM as MAP estimation problem
  - Use *factor graphs* to reason about the interdependence among variables
    - Example:  $g(X_1, X_2, X_3) = f_1(X_1)f_2(X_1, X_2)f_3(X_1, X_2)f_4(X_2, X_3)$



- Math notations:

- $X$ : robot trajectory (discrete set of poses) & landmark positions in the environment
- $Z = \{z_k : k = 1, \dots, m\}$ : set of measurements
- $z_k = h_k(X_k) + \epsilon_k$ , where  $X_k \subseteq X$ ,  $h_k(\cdot)$  being the observation/measurement model (known *nonlinear* function)
- MAP estimation: estimate  $X$  by computing variables  $X^*$  that attains the maximum of the posterior  $P(X|Z)$  (the *belief* over  $X$  given the measurements)

$$X^* \triangleq \arg \max_X P(X|Z) = \arg \max_X \frac{P(Z|X)P(X)}{\int_X P(Z|X)P(X)dX} = \arg \max_X P(Z|X)P(X)$$

- The denominator of the posterior distribution (marginal likelihood) plays no role in the optimization because it is always positive & does not depend on  $X$
- Assume the measurements are independent (i.e., the noise terms are not correlated with the measurements)

$$X^* = \arg \max_X P(X) \prod_{k=1}^m P(z_k|X) = \arg \max_X P(X) \prod_{k=1}^m P(z_k|X_k) \quad (3-1)$$

- Assume  $\epsilon_k$  is a zero-mean Gaussian noise with information matrix  $\Omega_k$ , and the prior is normally distributed

$$P(z_k|X_k) \propto \exp\left(-\frac{1}{2}\|h_k(X_k) - z_k\|_{\Omega_k}^2\right) \quad (3-2)$$

$$P(X) \propto \exp\left(-\frac{1}{2}\|h_0(X) - z_0\|_{\Omega_0}^2\right) \quad (3-3)$$

- $P(z_k|X_k)$ : measurement likelihood
- $P(X)$ : the prior
- $\|e\|_{\Omega}^2 = e^T \Omega e$
- According to (3-2) and (3-3), (3-1) becomes the cost function below (nonlinear (generalized) least squares problem):

$$\begin{aligned} X^* &= \arg \min_X (-\log(P(X) \prod_{k=1}^m P(z_k|X_k))) \\ &= \arg \min_X \sum_{k=0}^m \frac{1}{2} \|h(X_k) - z_k\|_{\Omega_k}^2 \end{aligned} \quad (3-4)$$

- Derived from the assumption of normally distributed noise
- Different assumptions for the noise distribution lead to different cost functions
- Similarity & differences between (3-4) & bundle adjustment (BA):
  - Similarity: both stem from MAP formulation
  - Differences:
    - Factors in (3-4) are not constrained to model projective geometry in BA but include a broad variety of sensor models
    - (3-4) is designed to be solved **incrementally** because new measurements are made available at each time step as the robot moves
- Minimization problem of (3-4) is solved via successive **linearizations** using Gauss-Newton (GN) or Levenberg-Marquardt or other modern methods



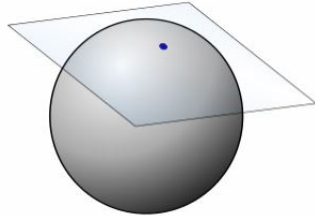
- GN approximate (3-4) as

$$\delta_X^* = \arg \min_{\delta_X} \frac{1}{2} \sum_{k=0}^m \|A_k \delta_X - b_k\|_{\Omega_k}^2 = \arg \min_{\delta_X} \frac{1}{2} \|A \delta_X - b\|_{\Omega}^2 \quad (3-5)$$

- $\delta_X$ : a small “correction” w.r.t. the linearization point  $\hat{X}$
- $A_k = \frac{\partial h_k(X)}{\partial X}$ : the Jacobian of the measurement function  $h_k(\cdot)$  w.r.t.  $X$
- $b_k \triangleq z_k - h_k(\hat{X})$ : the *residual error* at  $\hat{X}$
- $A$  &  $b$  are obtained by stacking  $A_k$  &  $b_k$ 
  - $\dim X = \dim \delta_X \triangleq n \times 1$
  - $\dim A_k = n \times n$
  - $\dim b_k = \dim \delta_X = n \times 1$
  - Stacking:
    - $\dim A = ((m+1) \times n) \times n$
    - $\dim b = ((m+1) \times n) \times 1$
- $\Omega_k$ : measurement information matrix
  - $\dim \Omega_k = n \times n$
- $\Omega$ : block diagonal matrix obtained by stacking  $\Omega_k$  as its diagonal blocks
  - $\dim \Omega = ((m+1) \times n) \times ((m+1) \times n)$
- Optimal correction  $\delta_X^*$  to minimize (3-5)

$$\delta_X^* = (A^T \Omega A)^{-1} A^T \Omega b \quad (3-6)$$

- $A^T \Omega A$ : the *Hessian*
- Update of linearization point: at each iteration,
  - $\hat{X} \leftarrow \hat{X} + \delta_X^*$  if  $X$  belongs to a vector space
  - $\hat{X} \leftarrow \hat{X} \oplus \delta_X^*$  if  $X$  includes variables belonging to a smooth manifold
    - $\delta_X^*$  belongs to the tangent space of  $X$
  - Operator  $\oplus$ : retraction operator that maps the vector  $\delta_X^*$  in the tangent space of manifold at  $\hat{X}$  to an element of the manifold
    - The mapping from one tangent space to another one
    - Example: move the point on a sphere by a small amount, and the result point will have a different tangent space (2D plane) with the original point
    - Tangent space illustration where the sphere can be seen as a 2D manifold



### 2.1.3 Summary

- Key insights behind SLAM solvers
  - Jacobian matrix  $A$  in (3-6) is *sparse* and its sparsity structure is dictated by *the topology of underlying factor graph*

- Enable the use of fast linear solvers to compute  $\delta_X^*$
- Allow the designing of *incremental/online* solvers (update the estimate of  $X$  as new observations are acquired)
- Current SLAM libraries:
  - GTSAM, g2o, Ceres, iSAM, and SLAM++
- Modern SLAM formulation (variant names):
  - Maximum-a-posteriori estimation, factor graph optimization, graph-SLAM, full smoothing, or smoothing and mapping (SAM)
- Nonlinear filtering (covered in [Section 1](#) & [Section 2](#)) vs. MAP estimation
  - MAP estimation is more accurate & efficient than Nonlinear filtering
  - Performance mismatch between Nonlinear filtering & MAP estimation gets smaller for nonlinear filtering methods if:
    - Linearization point for EKF becomes more accurate
    - Sliding-window filters are used
    - Potential sources of inconsistency of EKF are taken care of
- Why referred to as SLAM **back-end**?
  - MAP estimation is performed on pre-processed sensor data

## 2.2 SLAM Front-end

- Motivation
  - Hard to directly write the sensor measurements as an analytic state function required in MAP estimation
  - Unable to design a sufficiently general yet tractable environment representation
- Sensor-dependent SLAM front-end
  - Set up a module called **front-end** before the SLAM back-end
  - Features
    - Extract relevant features from sensor data
    - Associate each measurement to a specific landmark (*data association*)
    - Provide an initial guess for the variables in the nonlinear optimization ([3-4](#))

## 3. Long-term Autonomy I: Robustness

- SLAM system failure
  - Algorithmic
    - Limitation of current SLAM algorithms
  - Hardware-related
    - Sensor/actuator degradation
  - Software-related
- Algorithmic robustness challenges
  - Data association failures
    - Feature-based visual SLAM: *perceptual aliasing* → wrong measurement-state matches (false positives) → wrong estimates from the back-end
    - Unmodeled dynamics in the environment → deceive short/long-term data association

- Typical SLAM system: system based on *static world assumption*
  - Examples:
    - Drastic illumination changes  $\rightarrow$  failure in system relying on the repeatability of visual features
    - Disappearances of old structures  $\rightarrow$  failure in system relying on methods leveraging the geometry of the environment
  - Harsh environment with limited visibility & constantly changing conditions & impossibility of using conventional sensors
  - Solutions for data association failures
    - Make front-end reliable for data association establishment
      - Short-term data association: ensure sensors are of high sampling rate (high frame-rate)
        - Viewpoint of the sensor does not change significantly from time  $t$  to  $t + 1$
    - Long-term data association
      - Involving loop closure *detection* and *validation*
      - Loop closure detection
        - Detect features in environment and then try to match them with all previously detected features
        - Use searching algorithms with low computational complexity: ***Bag-of-words***
          - Quantizing feature space for more efficient search
        - Improve bag-of-words based techniques to handle severe illumination variations
      - Loop closure validation
        - Ascertain the quality of loop closures using additional geometric steps
        - Vision-based application: *RANSAC* is used for geometric verification & outlier rejection
        - Laser-based application: check residual error of laser scan matching
  - Make back-end resilient against spurious measurements
    - Spurious measurements (wrong loop closures): caused by perceptual aliasing
    - Methods:
      - During optimization: reason on the loop closure constraints validity by looking at the residual error induced by the constraints
      - Before optimization: identify incorrect loop closures that are not supported by the odometry
  - Make SLAM system better deal with dynamic environments
    - Challenge 1: SLAM system has to detect, discard/track changes
      - Include dynamic elements as part of the model
    - Challenge 2: SLAM system has to model permanent/semi-permanent changes & understand how and when to update the map accordingly
      - Maintain multiple (time-dependent) maps of the same location
- Open problems
  - a. Failsafe SLAM and recovery
    - Current SLAM solvers (based on iterative optimization of nonconvex costs) are vulnerable to outliers

- Goal: *fail-safe & failure-aware*
  - Be aware of imminent failures & provide recovery mechanisms that can re-establish proper operation
- b. Robustness to HW failure
  - Goal: detect degraded sensor operation & adjust sensor noise statistics (covariances & biases) accordingly
- c. Metric relocalization: estimate relative pose w.r.t. previously built map
  - Hard/impossible for feature-based methods (rely on visual sensors) to detect loop closures among different times (day/night/diff. season/...)
  - Localizing from significantly different viewpoints with heterogeneous sensing
- d. Time varying and deformable maps
  - Limitations on mainstream SLAM methods: rigid & static world assumption
  - Ideal SLAM method: be able to deal with environment dynamics
- e. Automatic parameter tuning
  - SLAM systems require extensive parameter tuning for a specific scenario
    - Especially for data association module
    - Parameters:
      - Thresholds to control feature matching
      - RANSAC parameters
      - Criteria to decide when to add new factors to the graph or when to trigger a loop closing algorithm to search for matches
  - Goal: automatic parameter tuning for arbitrary scenarios

## 4. Long-term Autonomy II: Scalability

- SLAM methods should be designed to be scalable to the specific scenario
  - Different scenarios require different computational time & memory footprint
  - Computational & memory complexity should remain bounded
- Approaches to control/reduce growth of SLAM problem size
  - **Sparsification** methods: trade-off information loss for memory & computational efficiency
    - Node & edge sparsification
      - **Reducing** number of nodes *added* to the graph, or **pruning** less “informative” nodes & factors to address scalability
  - **Out-of-core** & **multi-robot** methods: split computation among many robots/processors
    - Out-of-core (parallel) SLAM
      - Split computation & memory load of factor graph optimization among multiple processors
      - Key idea: global factor graph optimization → localized subgraph optimization + global refinement
      - Called *submapping* algorithms
  - Distributed multi robot SLAM
    - Divide the scenario in smaller areas by deploying multiple robots
      - Map each area by a different robot

- Two main variants
  - *Centralized* method: robots build submaps & transfer local info to a central station that performs inference
  - *Decentralized* method: no central data fusion & agents leverage local communication to reach consensus on a common estimate
- Open problems
  - a. Map maintenance
    - Localization against a compressed known map
    - Memory-efficient dense reconstruction
  - b. Robust distributed mapping
    - Outliers are ignored in multi-robot SLAM
    - Challenges
      - Robots may not share a common reference frame which makes it harder to detect & reject wrong loop closures
      - Robots have to detect outliers from very partial/local information
  - c. Learning, forgetting, remembering
    - Question for long-term mapping: determine the followings
      - How often to update the map information
      - How to decide when the information becomes outdated & can be discarded
      - When is it fine to forget
      - What can be forgotten & what is essential to maintain
  - d. Resource-constrained platforms
    - How to adapt existing SLAM algorithms for robot platforms with severe computational constraints
    - How to guarantee reliable operation for multi robot teams given tight bandwidth constraints & communication dropout

## 5. Representation I: Metric Reasoning

- ***Metric representation*** (metric map): symbolic structure that encodes the geometry of the environment
- Geometric modeling in SLAM in 2D case
  - *Landmark-based maps*: model the environment as a sparse set of landmarks
  - *Occupancy grid maps*: discretize the environment in cells and assign a probability of occupation to each cell
- Geometric modeling in SLAM in 3D case
  - a. Landmark-based sparse representations
    - Represent scene as a set of sparse 3D landmarks corresponding to discriminative features (lines/corners/...) in the environment
    - Referred to as landmark-based / feature-based representations
    - Application:
      - Localization & mapping in mobile robotics
      - *Structure from motion* in computer vision

- Assumption: landmarks are distinguishable
  - Sensor data measure geometric aspect of a landmark & provide a descriptor for data association between measurement & landmark
- Features: point (mainly), lines, segments, or arcs
- b. Low-level raw dense representations
  - Provide high-resolution models for 3D geometry
  - Application
    - Obstacles avoidance & path planning in robotics
    - Visualization & rendering
  - Raw representations
    - Large unstructured set of points: *point clouds*
      - In conjunction with stereo & RGB-D cameras, and 3D laser scanners
    - Polygons: *polygon soup*
    - *Surfels maps*: encode the geometry as a set of disks
  - Application in monocular SLAM: in conjunction with the use of *direct methods*
    - Estimate robot trajectory & 3D model directly from image pixel intensity values
  - Pros & cons
    - Pro: visually pleasant
    - Cons:
      - Cumbersome (require storing of large amount of data)
      - Low-level geometry description which neglects obstacle topology
- c. Boundary and spatial-partitioning dense representations
  - Attempt to explicitly represent surfaces/*boundaries* and volumes
  - Boundary representations (b-reps): define 3D objects in terms of their surface boundary
    - Plane-based models
    - *Curve-based representations*: tensor product of NURBS or B-splines
    - *Surface mesh models*: connected set of polygons
    - *Implicit surface representations*
      - Specify the surface of a solid as the zero crossing of a function defined on  $\mathbb{R}^3$
      - Function examples
        - Radial-basis functions
        - Signed-distance function
          - *Truncated signed-distance function (TSDF)*
  - Spatial-partitioning representations
    - Spatial-occupancy enumeration: decompose 3D space into identical cubes (voxels) arranged in a regular 3D grid
    - Octree: used for 3D mapping
      - Occupancy grid maps: probabilistic variants of space-partitioning representations
    - Polygonal Map octree
    - Binary Space-partitioning tree
    - 2.5D elevation maps
- d. High-level object-based representations

- Features of object & solid shape representations
  - Model objects as 3D objects rather than 1D (points) & 2D (surfaces)
  - Associate physical notions (volume, mass, ...) to each object
- Solid representations in CAD & computer graphics (no applications in SLAM yet) ([intro](#))
  - *Parameterized Primitive Instancing*
    - Define a families of objects (cylinder, sphere, ...)
    - Define a set of parameters (radius, height, ...) to each member of the family
  - *Sweep representations* (see the intro above)
    - Define a solid as the sweep of a 2D/3D object along a trajectory through space
  - *Constructive solid geometry* (see the intro above)
- Feature-based models in CAD
- Dictionary-based representations
  - Define a solid as a combination of atoms in a dictionary
  - Dictionary source:
    - Learned from data
    - Existing repositories of object models
- Affordance-based models
- Generative & procedural models
- Scene graphs
- Sparse (feature-based) vs. dense representations/algorithms in visual SLAM
  - Disadvantages of sparse representations
    - Dependence of feature type
    - Reliance on numerous detection & matching thresholds
    - Necessity for robust estimation methods to deal with wrong correspondences
    - Most feature detectors are optimized for speed rather than precision
  - Dense/direct methods
    - Exploit all image information
    - Outperform feature-based methods in scenes with poor texture, defocus & motion blur
    - Require high computing power for real-time performance
    - Currently unable to jointly estimate dense structure & motion
  - Alternatives to feature-based methods
    - *Semi-dense* methods: only exploit pixels with strong gradients
    - *Semi-direct* methods: leverage both sparse features & direct methods
      - Most efficient alternative
      - Allow joint estimation of structure & motion
- Open problems
  - a. High-level, expressive representations in SLAM
    - Point clouds or TSDF are mainly used to model 3D geometry
      - Drawbacks
        - Wasteful for environment modeling
        - No high-level understanding of 3D geometry
    - Advantages of higher-level representations

- Compact representations → compressed map in large-scale mapping
- Higher-level object description facilitates data association, place recognition & semantic understanding
- Enable interactions with existing modern building construction & management standards
- Current representations in SLAM are only low-level point clouds, mesh models, surfels models, and TSDFs
- b. Optimal representations
  - Lack of criteria to choose an appropriate representation for specific environment
- c. Automatic & adaptive representations
  - Choice of representation is entrusted to SLAM system designer
    - Drawbacks
      - Representation design is time-consuming & require an expert
      - No flexibility: representation cannot be changed once the system is designed

## 6. Representation II: Semantic Reasoning

- Features of semantic mapping
  - Associate semantic concepts to geometric entities in robot's surroundings
  - Enhance robot's autonomy & robustness
  - Facilitate more complex tasks (e.g. avoid muddy-road while driving)
  - Move from path-planning to task-planning
  - Enable advanced human-robot interaction
  - Typical semantic parsing: formulated as a classification problem
    - Simple mapping between sensory data & semantic concepts
- Semantic vs. topological SLAM
  - Semantic: interested in classifying the places according to semantic labels
  - Topological: do place recognition only for topological mapping (disregard the actual function of a place (kitchen/corridor/...))
- Design of a semantic representation
  - Level/detail of semantic concepts: determined by specific tasks
  - Organization of semantic concepts
    - Flat/hierarchical organization
    - Sharing/not sharing properties
- Semantic mapping in SLAM
  - SLAM helps semantics
    - Segmenting the map built from classical SLAM systems into semantic concepts
  - Semantics helps SLAM
    - Improve map estimation by recognizing semantic concepts in the map
  - Joint SLAM & semantics inference
    - Perform monocular SLAM & map segmentation within a joint formulation
- Open problems
  - a. Semantic SLAM lacks cohesive formulation



- b. Semantic mapping is much more than a categorization problem
  - How to represent interrelationships among semantic concepts
- c. Ignorance, awareness & adaption
  - Given prior knowledge, a robot should be able to reason about new concepts & their semantic representations
    - Discover new objects
    - Learn new properties
    - Adapt representations to slow/abrupt changes in the environment
- d. Semantic-based reasoning
  - Robots are currently unable to effectively localize & continuous map using semantic concepts in the environment
    - Unable to update & refine map by detecting new semantic concepts

## 7. New Theoretical Tools for SLAM

- Focus
  - Factor graph optimization approaches
  - *Algorithmic properties*
- MAP estimation  $\rightarrow$  MLE if no prior is given
  - SLAM inherits all properties of maximum likelihood estimators
  - MLE properties
    - Consistent
    - Asymptotically Gaussian & asymptotically efficient
    - Invariant to transformations in the Euclidean space
  - Estimator will no longer be invariant if priors are given
- SLAM: *nonconvex* problem
  - Iterative optimization can only guarantee local convergence
  - Local minimum  $\rightarrow$  completely wrong estimation & unsuitable for navigation
  - State-of-the-art iterative solvers fail to converge to global minimum for relatively small noise levels
- Progress on theoretical analysis of SLAM problem
  - Rotation estimation can be solved uniquely in closed form in 2D
  - Convergence improvement approaches
    - (Convex) dual ***semidefinite programming (SDP)***: under certain conditions often encountered in practice (strong duality), ML estimation is unique & pose graph optimization can be solved globally via this approach
    - Convex relaxation: avoid local minima convergence
    - Compute a suitable initialization for iterative nonlinear optimization
      - Effective idea: solve for rotations first, and then use the resulting estimate to bootstrap nonlinear iteration
  - Verification techniques to judge whether the quality of given SLAM solution is optimal or not
    - Based on Lagrangian duality in SLAM

- Current progress: perform verification by solving a sparse linear system and provide a correct answer as long as strong duality holds
- Open problems
  - a. Generality, guarantees, verification
    - Can guaranteed solution & verification techniques in context of pose graph optimization be generalized to arbitrary factor graphs?
  - b. Weak or strong duality?
    - SLAM can be solved globally when strong duality holds
    - Given a set of sensors & a factor graph, does the strong duality hold?
  - c. Resilience to outliers
    - Design of global techniques & verification techniques that are resilient to outliers remains open

## 8. Active SLAM

- Active SLAM
  - Control robot's motion to minimize the uncertainty of its map representation & localization
  - Other names:
    - Simultaneous Planning Localization and Mapping (SPLAM)
    - Autonomous/Active/Integrated Exploration
- General frameworks for decision making in active SLAM
  - Goal: solve for exploration-exploitation decisions
  - Frameworks
    - Theory of Optimal Experimental Design (TOED)
      - Allow selecting future robot motion based on predicted map uncertainty
    - Information theoretic approaches
      - Decision making is guided by *information gain*
    - Control theoretic approaches
      - Use Model Predictive Control
    - Partially Observably Markov Decision Process (POMDP)
      - Computationally intractable
    - Bayesian Optimization
    - Efficient Gaussian beliefs propagation
- Framework based on selecting future robot motion among a finite set of alternatives
  - Step 1 - *selecting vantage points*: robot identifies possible locations (vantage locations) to explore/exploit
    - Only a small subset of locations are selected as evaluation grows exponentially with the search place
    - Related approaches only guarantee local optimal policies
  - Step 2 - *computing the utility of an action*: robot computes the utility of visiting each vantage point and select the action with highest utility
    - Computationally intractable for precise information gain of each action
    - Compute approximations: utility defined as linear combination of metrics that quantify

- robot & map uncertainties
  - Drawback: scale of numerical uncertainties not comparable
  - TOED can also be applied
- Step 3 - *executing actions or terminate exploration*: robot carries out selected action and decides if necessary to continue/terminate the task
  - Open challenge for the decision whether exploration task is done
- Open problems
  - a. Fast and accurate predictions of future states
    - Necessary because each robot action should reduce the uncertainty in map & improve localization accuracy in active SLAM
    - Loop closure is important: no efficient methods exist
    - Predicting effects of future actions is computationally expensive
  - b. Enough is enough: when to stop active SLAM?
    - Active SLAM is computationally expensive, need to stop and focus resources on other tasks
  - c. Performance guarantees
    - Need mathematical guarantees for active SLAM & near-optimal policies
      - Exact solution is computationally intractable
      - Desirable for approximation algorithms with clear performance bounds

## 9. New Sensors for SLAM

- Progress in SLAM are partly triggered by novel sensors
  - 2D laser range finders: enable very robust SLAM algorithms
  - 3D lidars: main thrust behind applications such as autonomous cars
  - Vision sensors: used in augmented reality & vision-based navigation
- Limitation of current vision sensors
  - High latency & temporal discretization
    - Latency: time to capture image & time to process it
    - Temporal discretization: time to wait until the next frame arrives
  - Prevent the advent of high-agility autonomous robot
- ***Event-based cameras***
  - May substitute current vision sensors
  - Typical cameras:
    - Dynamic Vision Sensor (DVS)
    - Asynchronous Time-based Image Sensor (ATIS)
  - Event cameras vs. standard frame-based cameras
    - Standard frame-based cameras: send entire images at fixed frame rates
    - Event cameras: only send local pixel-level changes caused by movement in a scene at the time it they occur
      - Event is triggered if relative intensity change exceeds a threshold
  - *Microsecond time* resolution for event cameras
  - 5 key advantages over standard frame-based cameras

- a. Temporal latency: 1 microsecond
- b. Measurement rate: up to 1MHz
- c. Dynamic range: up to 140 dB (vs. 60-70 dB of standard cameras)
- d. Power consumption: 20 mW (vs. 1.5W of standard cameras)
- e. Bandwidth for data transfer: kilobytes/s scale (vs. Megabytes/s)
- Disadvantages
  - Traditional frame-based computer-vision algorithms are not applicable
    - Sensor output composes of a sequence of asynchronous events instead of image frame
  - Require a *paradigm shift* from traditional computer vision approaches
- Progress on event camera applications
  - Algorithmic adaptations:
    - Feature detection & tracking
    - LED tracking
    - Epipolar geometry
    - Image reconstruction
    - Motion estimation
    - Optical flow
    - 3D reconstruction
    - Iterative closest points
  - Event-based visual odometry, localization & SLAM
    - Design goal: each incoming event can asynchronously change estimated system state
    - Adaptions of continuous-time trajectory-estimation methods
      - Pose trajectory can be *approximated* by a smooth curve in rigid-body motion space using basis functions (cubic splines) & *optimized* according to the observed events
- Open problems
  - a. Modeling, sensitivity and resolution
    - Events in event-based cameras are highly susceptible to noise
    - Lack of full characterization of sensor noise & sensor non idealities
    - Event pipelines may be failed for scenes with smooth edges as events are triggered by brightness changes
    - Low spatial resolution (128x128 pixels)
  - b. Event-based or standard camera for SLAM?
    - No criteria for choosing algorithm-sensor pair

# Notes on Multiple View Geometry Book (Hartley and Zisserman 2004) (site)

## Part 0 - The Background: Projective Geometry, Transformations and Estimation

### 2. Projective Geometry and Transformations of 2D

#### 2.2 The 2D Projective Plane

##### 2.2.1 Points and Lines

- Homogeneous representation
  - Line in a plane:  $\mathbf{l} : ax + by + c = 0 \rightarrow \text{vector } (a, b, c)$ 
    - $(ka)x + (kb)y + (kc) = 0$  are the same line for  $k \neq 0$
  - Point:  $\mathbf{x} = (x, y) \rightarrow (x, y, 1)$ 
    - $(x_1, x_2, x_3) \rightarrow (x_1/x_3, x_2/x_3)$  in  $\mathbb{R}^2$  for  $x_3 \neq 0$
- Point on a line:  $\mathbf{x} \cdot \mathbf{l} = 0$  or  $\mathbf{x}^T \mathbf{l} = 0$
- Degrees of freedom (dof): both point & line are 2
- (Point) Intersection of lines:  $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$ 
  - $\mathbf{l} \cdot (\mathbf{l} \times \mathbf{l}') = \mathbf{l}' \cdot (\mathbf{l} \times \mathbf{l}') = 0$
  - $\mathbf{l}^T \mathbf{x} = \mathbf{l}'^T \mathbf{x} = 0$
- Line passing through 2 points  $\mathbf{x}$  &  $\mathbf{x}'$ :  $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$ 
  - $\mathbf{x} \cdot \mathbf{l} = \mathbf{x}' \cdot \mathbf{l} = 0$

##### 2.2.2 Ideal Points and the Line at Infinity

- Intersection of parallel lines  $\mathbf{l} = (a, b, c)$  &  $\mathbf{l}' = (a, b, c')$ 
  - $\mathbf{l} \times \mathbf{l}' = (c' - c)(b, -a, 0)$ 
    - Intersection: point  $(b, -a, 0)$
- Ideal point & the line at infinity
  - Homogeneous vectors  $\mathbf{x} = (x_1, x_2, x_3)$ 
    - Vectors with  $x_3 \neq 0$ : finite points in  $\mathbb{R}^2$
    - Vectors with any  $x_3$ : set of all homogeneous 3-vectors, i.e., projective space  $\mathbb{P}^2$

- Ideal point:  $(b, -a, 0)$ 
  - Set of ideal points:  $(x_1, x_2, 0)$  specified by the ratio  $x_1 : x_2$
- Line at infinity:  $\mathbf{l}_\infty = (0, 0, 1)$
- Inhomogeneous notation  $(b, -a)$ : represent line's direction
- Projective plane model

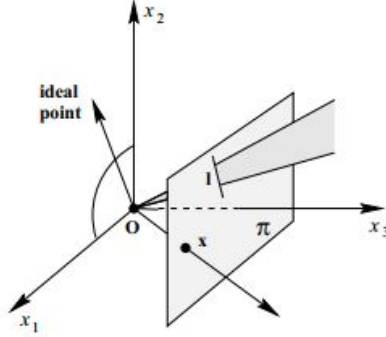


Fig. 2.1. **A model of the projective plane.** Points and lines of  $\mathbb{P}^2$  are represented by rays and planes, respectively, through the origin in  $\mathbb{R}^3$ . Lines lying in the  $x_1x_2$ -plane represent ideal points, and the  $x_1x_2$ -plane represents  $\mathbf{l}_\infty$ .

- $\mathbb{P}^2 \rightarrow \mathbb{R}^3$
- $k(x_1, x_2, x_3)$ 
  - Point in  $\mathbb{P}^2$ : ray through the origin
  - Line in  $\mathbb{P}^2$ : plane passing through the origin
- 2 points form 1 line  $\rightarrow$  2 rays are on exactly one plane
- 2 lines intersect at 1 point  $\rightarrow$  2 planes intersect in one ray
- Duality
  - Symmetry between line & point in homogeneous representation
    - $\mathbf{l}^T \mathbf{x} = 0 \leftrightarrow \mathbf{x}^T \mathbf{l} = 0$
  - **Duality principle:** To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the roles of points and lines in the original theorem
    - Concepts of incidence must be appropriately translated
    - Example: line through 2 points  $\leftrightarrow$  point through (point of intersection of) 2 lines

## 2.3 Projective Transformations

- Projectivity: an invertible mapping  $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  such that 3 points  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ , and  $\mathbf{x}_3$  lie on the same line if and only if  $h(\mathbf{x}_1)$ ,  $h(\mathbf{x}_2)$ , and  $h(\mathbf{x}_3)$  do
  - Synonym: *collineation*, *projective transformation*, *homography*
    - Collineation: collinear points remain collinear after the mapping
- A mapping  $h : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  is a **projectivity** if and only if there exists a non-singular matrix  $\mathbf{H}$  such that for any point in  $\mathbb{P}^2$  represented by a vector  $\mathbf{x}$  it is true that  $h(\mathbf{x}) = \mathbf{H}\mathbf{x}$ 
  - $\mathbf{x}' = \mathbf{H}\mathbf{x}$  where  $\mathbf{x}$  and  $\mathbf{x}'$  are homogeneous 3-vectors
    - $$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
  - $\mathbf{H}$ : *homogeneous* matrix (only ratio of matrix elements is significant)
    - DOF: 8 because there are 8 ratios among 9 elements of  $\mathbf{H}$

### 2.3.1 Transformations of Lines and Conics

- Transformation of lines under mapping  $\mathbf{H}$

$$\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$$

- $\mathbf{x}$  lies on line  $\mathbf{l}$  and  $\mathbf{x}' = \mathbf{H}\mathbf{x}$  lies on line  $\mathbf{l}'$ 
  - $\mathbf{x}'^T \mathbf{l}' = \mathbf{x}^T \mathbf{l} = 0$
  - $\mathbf{x}^T \mathbf{H}^T \mathbf{l}' = \mathbf{x}^T \mathbf{l}$
  - $\mathbf{l}' = \mathbf{H}^{-T} \mathbf{l}$

### 2.4 A Hierarchy of Transformations

- Projective transformations form a group called *projective linear group*
- $GL(n)$ : (real) general linear group on  $n$  dimensions
  - Group of invertible  $n \times n$  matrices with real elements
- $PL(n)$ : projective linear group
  - Quotient group of  $GL(n)$  (matrices related by a scalar multiplier)
- Important subgroups of  $PL(3)$ :
  - **Affine group**: matrices whose last row is  $(0, 0, 1)$
  - **Euclidean group**: subgroup of affine group
    - Upper left hand  $2 \times 2$  matrix has determinant 1
- Subsequent 4 classes of transformations are from most specialized *class I* to most generalized *class IV* transformations

#### 2.4.1 Class I: Isometries

- Isometries are transformations of plane  $\mathbb{R}^2$
- Euclidean distance is preserved
- Representation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \epsilon \cos \theta & -\sin \theta & t_x \\ \epsilon \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- $\epsilon = \pm 1$
- When  $\epsilon = 1$ :
  - Isometry is orientation-preserving
  - Isometry is an Euclidean transformation (composition of translation & rotation)
- When  $\epsilon = -1$ : orientation is reversed
- Euclidean transformations model the motion of a *rigid object*

$$\mathbf{x}' = \mathbf{H}_E \mathbf{x} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{x}$$

- $\mathbf{R}$ :  $2 \times 2$  rotation matrix (orthogonal matrix:  $\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}$ )
- $\mathbf{t}$ : translation 2-vector
- $\mathbf{0}$ : null 2-vector

- Special cases
  - Pure translation ( $\mathbf{R} = \mathbf{I}$ )
  - Pure rotation ( $\mathbf{t} = \mathbf{0}$ )
- Also known as ***displacement***
- DOF for planar Euclidean transformation: 3 (1 + 2)
  - 1 for rotation & 2 for translation
  - Transformation can be computed from 2 point correspondences
- Invariants
  - Length: distance between 2 points
  - Angle: angle between 2 lines
  - Area
- Orientation-preserving transformations form a group while orientation-reversing ones do not

### 2.4.2 Class II: Similarity Transformations

- Similarity transformation (*similarity*): isometry composed with isotropic scaling
- Representation in case of Euclidean composed with scaling (no reflection) (matrix & block matrix form)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s \cos \theta & -s \theta & t_x \\ s \sin \theta & s \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mathbf{x}' = \mathbf{H}_s \mathbf{x} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{x}$$

- $s$ : isotropic scaling
- Also known as *equi-form* transformation as it preserves “shape” (form)
- DOF for planar similarity transformation: 4 (1 + (1 + 2))
  - 1 for scaling and 3 for isometry
- Invariants
  - Angle: parallel lines are mapped to parallel lines
  - *Ratio* of 2 lengths (before & after the transformation)
  - *Ratio* of 2 areas
- *Metric structure*: the structure is defined up to a similarity

### 2.4.3 Class III: Affine Transformations

- Affine transformation (*affinity*): non-singular linear transformation followed by a translation
- Representation (matrix & block matrix form)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mathbf{x}' = \mathbf{H}_A \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{x}$$

- $\mathbf{A}$ :  $2 \times 2$  non-singular matrix
- DOF for planar affine transformation: 6
  - 4 for transformation  $\mathbf{A}$  & 2 for translation
  - Transformation can be computed from 3 point correspondences



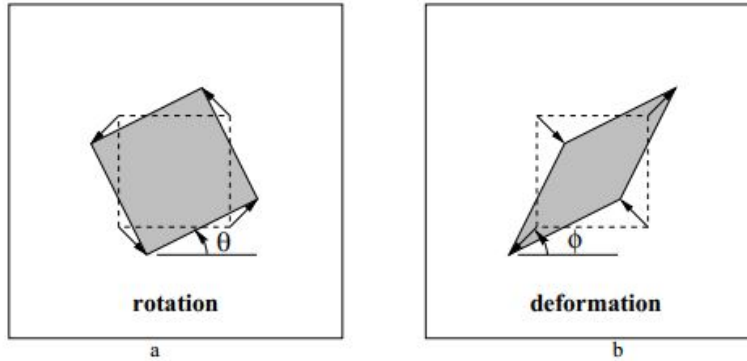
- Decomposition of  $\mathbf{A}$  (using SVD decomposition):

$$\begin{aligned}
\mathbf{A} &= \mathbf{U}\mathbf{D}\mathbf{V}^T \\
&= \mathbf{U}\mathbf{V}^T\mathbf{V}\mathbf{D}\mathbf{V}^T \\
&= (\mathbf{U}\mathbf{V}^T)(\mathbf{V}\mathbf{D}\mathbf{V}^T) \\
&= (\mathbf{R}(\theta))(\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)) \\
&= (\text{rotation})(\text{deformation})
\end{aligned}$$

- $\mathbf{R}(\theta)$  &  $\mathbf{R}(\phi)$ : rotations by  $\theta$  &  $\phi$
- $\mathbf{D}$ : a diagonal matrix representing non-isotropic scaling

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- $\mathbf{U}$  &  $\mathbf{V}$  are orthogonal matrices
- Affine matrix  $\mathbf{A}$  is concatenation of the followings:
  - Deformation (non-isotropic scaling)
    - A rotation by  $\phi$
    - A non-isotropic scaling by  $\lambda_1$  &  $\lambda_2$  respectively in (rotated)  $x$  &  $y$  directions
    - A rotation back by  $\phi$
  - Rotation
    - A rotation by  $\theta$
- Non-isotropic scaling (deformation):
  - Contribute 2 extra DOF to affinity over similarity
    - Angle  $\phi$  specifying the scaling direction
    - Ratio of scaling parameters  $\lambda_1 : \lambda_2$
- Illustration of rotation & deformation



**Fig. 2.7. Distortions arising from a planar affine transformation.** (a) Rotation by  $\mathbf{R}(\theta)$ . (b) A deformation  $\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)$ . Note, the scaling directions in the deformation are orthogonal.

- Invariants
  - Parallel lines
  - Ratio of lengths of parallel line segments
  - Ratio of areas
- $\det\mathbf{A} = \lambda_1\lambda_2$ 
  - Sign of the determinant (scalings) determines whether the affinity is orientation-preserving

(positive sign) or orientation-reversing (negative sign)

#### 2.4.4 Class IV: Projective Transformations

- Projective transformation is a general non-singular linear transformation of *homogeneous* coordinates
- Generalization of affinity, which is the composition of a general non-singular linear transformation of *inhomogeneous* coordinates and a translation
- Block form representation

$$\mathbf{x}' = \mathbf{H}_P \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix} \mathbf{x}$$

- $\mathbf{v} = (v_1, v_2)^T$
- DOF: 8
  - 9 elements with only their ratio significant  $\rightarrow$  8 parameters
  - Projective transformation between 2 planes can be computed from 4 point correspondences (no 3 collinear points on either plane)
- Invariants
  - Cross ratio (ratio of ratios) of 4 collinear points

#### 2.4.7 The Number of Invariants

- “Number”: counting argument for the number of *functionally independent* invariants
- number of invariants  $\geq \text{DOF}_{\text{configuration}} - \text{DOF}_{\text{transformation}}$
- Summary



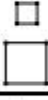

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, <b>order of contact</b> : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $l_\infty$ .
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, <b>I, J</b> (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Table 2.1. **Geometric properties invariant to commonly occurring planar transformations.** The matrix  $\mathbf{A} = [a_{ij}]$  is an invertible  $2 \times 2$  matrix,  $\mathbf{R} = [r_{ij}]$  is a 2D rotation matrix, and  $(t_x, t_y)$  a 2D translation. The distortion column shows typical effects of the transformations on a square. Transformations higher in the table can produce all the actions of the ones below. These range from Euclidean, where only translations and rotations occur, to projective where the square can be transformed to any arbitrary quadrilateral (provided no three points are collinear).

## 2.7 Recovery of Affine and Metric Properties from Images

### 2.7.1 The Line at Infinity

- $l_\infty$  remains a fixed line under projective transformation  $\mathbf{H}$  if and only if  $\mathbf{H}$  is an affinity
- Fixed lines are fixed as a set, not fixed pointwise
  - A point on the line is mapped to another point on this line

## 2.9 Fixed Points and Lines

- Fixed point/line of a transformation: correspond to an *eigenvector*
  - $\mathbf{H}\mathbf{e} = \lambda\mathbf{e}$ , where  $\mathbf{e}$  and  $\lambda\mathbf{e}$  represents the same point (eigenvector of  $\mathbf{H}$ )
  - $\mathbf{H}^{-T}\mathbf{e} = \lambda\mathbf{e}$ , where  $\mathbf{e}$  and  $\lambda\mathbf{e}$  represents the same line (eigenvector of  $\mathbf{H}^T$ )
- If 2 of the eigenvalues are identical, the corresponding fixed line will be fixed pointwise

## 3. Projective Geometry and Transformations of 3D

### 3.1 Points and Projective Transformations

- Point  $\mathbf{X}$  in 3-space is represented in homogeneous coordinates as a 4-vector
  - $\mathbf{X} = (X_1, X_2, X_3, X_4)$  with  $X_4 \neq 0$  represents the point  $(X, Y, Z)$  of  $\mathbb{R}^3$  with inhomogeneous coordinates
    - $X = X_1/X_4, Y = X_2/X_4, Z = X_3/X_4$
- Projective transformation acting on  $\mathbb{P}^3$ : non-singular  $4 \times 4$  matrix  $\mathbf{H}$ :  $\mathbf{X}' = \mathbf{H}\mathbf{X}$ 
  - DOF: 15 (16 elements and less one for overall scaling)

### 3.2 Representing and Transforming Planes, Lines and Quadrics

- Duality in  $\mathbb{P}^3$ : points & *planes* are dual
  - Similar to point-line duality in  $\mathbb{P}^2$
  - Lines are self-dual in  $\mathbb{P}^3$

#### 3.2.1 Planes

- A plane in 3-space may be written as  $\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$ 
  - Homogeneous representation:  $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ 
    - DOF: 3
- $\boldsymbol{\pi}^T \mathbf{X} = 0 \rightarrow$  point  $\mathbf{X}$  in on plane  $\boldsymbol{\pi}$
- $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$ :
  - First 3 components: plane normal of Euclidean geometry
  - Plane equation in inhomogeneous notation:  $\mathbf{n} \cdot \tilde{\mathbf{X}} + d = 0$ 
    - $\mathbf{n} = (\pi_1, \pi_2, \pi_3)$
    - $\tilde{\mathbf{X}} = (X, Y, Z)$  with  $X_4 = 1$
    - $d = \pi_4$
    - $d/\|\mathbf{n}\|$ : distance of the plane from the origin

- 3 points define a plane

$$\begin{bmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \mathbf{X}_3^T \end{bmatrix} \boldsymbol{\pi} = \mathbf{A} \boldsymbol{\pi} = \mathbf{0}$$

- The  $3 \times 4$  matrix has a rank of 3 if the 3 points are not collinear (linearly independent)
- Plane  $\boldsymbol{\pi}$  is in the (right) nullspace (1-dimensional)
  - $\mathbf{A} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ 
    - $\mathbf{X}_i \in C(\mathbf{A}^T)$
    - $\boldsymbol{\pi} \in N(\mathbf{A})$
    - $\dim C(\mathbf{A}^T) + \dim N(\mathbf{A}) = 3 + 1 = 4$
  - 1-dimensional: the plane is defined uniquely (up to scale)
- 3 planes define a point

$$\begin{bmatrix} \boldsymbol{\pi}_1^T \\ \boldsymbol{\pi}_2^T \\ \boldsymbol{\pi}_3^T \end{bmatrix} \mathbf{X} = \mathbf{A} \mathbf{X} = \mathbf{0}$$

- Dual to the case of 3 points define a plane
- The point: defined in the (right) nullspace (1-dimensional)
- Projective transformation (plane)
  - $\mathbf{X}' = \mathbf{H} \mathbf{X} \rightarrow \boldsymbol{\pi}' = \mathbf{H}^{-T} \boldsymbol{\pi}$

### 3.2.2 Lines

- DOF of lines in  $\mathbb{P}^3$ : 4
  - Can be defined by two points on 2 orthogonal planes, 2 DOF for each point
  - Hard to be represented normally because it needs a homogeneous 5-vector
- Line representations
  - Nullspace and span representation
    - Line: pencil (1-parameter family) of collinear points
      - Defined by any of these 2 points
        - 3 DOF (homogeneous 4-vector) for each point
    - Line represented as the span of row space of  $2 \times 4$  matrix  $\mathbf{W}$  composed of 2 transposed 4-vectors (point)  $\mathbf{A}^T$  &  $\mathbf{B}^T$

$$\mathbf{W} = \begin{bmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{bmatrix}$$

- Span of  $\mathbf{W}^T$ : pencil of points  $\lambda \mathbf{A} + \mu \mathbf{B}$  on the line
- Span of 2-dimensional  $N(\mathbf{W})$  (nullspace) of  $\mathbf{W}$ : pencil of planes with the line as axis
  - Suppose  $\mathbf{P}$  &  $\mathbf{Q}$  are a basis of the nullspace
  - $\mathbf{A}^T \mathbf{P} = \mathbf{B}^T \mathbf{P} = \mathbf{0}$ :  $\mathbf{P}$  is a plane containing the points  $\mathbf{A}$  &  $\mathbf{B}$ 
    - Similarly for  $\mathbf{Q}$ : a distinct plane
  - As  $\mathbf{A}$  and  $\mathbf{B}$  lie on both (linearly independent) plane  $\mathbf{P}$  &  $\mathbf{Q}$ , the line defined by  $\mathbf{W}$  is the plane intersection
  - Any plane of the pencil with the line as axis can be represented by the span

$$\lambda' \mathbf{P} + \mu' \mathbf{Q}$$

- Dual representation: line represented as the span (of row space) of  $2 \times 4$  matrix  $\mathbf{W}^*$  composed of plane  $\mathbf{P}^T$  &  $\mathbf{Q}^T$ :

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{P}^T \\ \mathbf{Q}^T \end{bmatrix}$$

- 1) Span of  $\mathbf{W}^{*T}$ : pencil of planes  $\lambda' \mathbf{P} + \mu' \mathbf{Q}$  with the line as axis
- 2) Span of 2-dimensional  $N(\mathbf{W}^*)$ : pencil of points on the line
- Relation of the 2 representations:  $\mathbf{W}^* \mathbf{W}^T = \mathbf{W} \mathbf{W}^{*T} = \mathbf{0}_{2 \times 2}$ 
  - $\mathbf{0}_{2 \times 2}$ :  $2 \times 2$  null matrix

b. Pluecker matrices

- Line represented by a  $4 \times 4$  skew-symmetric homogeneous matrix
  - Line  $\mathbf{L}$  joining 2 points  $\mathbf{A}$  &  $\mathbf{B}$ 
    - Elements:  $l_{ij} = A_i B_j - B_i A_j$
    - Vector notation:  $\mathbf{L} = \mathbf{A} \mathbf{B}^T - \mathbf{B} \mathbf{A}^T$
  - Properties of  $\mathbf{L}$ 
    - 1) Rank of  $\mathbf{L}$ : 2
      - $\mathbf{A}^T \mathbf{P} = \mathbf{B}^T \mathbf{P} = \mathbf{A}^T \mathbf{Q} = \mathbf{B}^T \mathbf{Q} = \mathbf{0}$ 
        - $\mathbf{P}$  &  $\mathbf{Q}$ : distinct planes containing  $\mathbf{A}$  &  $\mathbf{B}$
      - $\mathbf{L} \mathbf{P} = \mathbf{L} \mathbf{Q} = \mathbf{0} \rightarrow \mathbf{L} \begin{bmatrix} \mathbf{P} & \mathbf{Q} \end{bmatrix} = \mathbf{0} \rightarrow \mathbf{L} \mathbf{W}^{*T} = \mathbf{0}_{4 \times 2}$ 
        - $\mathbf{P} \in N(\mathbf{L})$  &  $\mathbf{Q} \in N(\mathbf{L})$
        - $\dim N(\mathbf{L}) = 1 + 1 = 2$
    - 2) The representation has the 4 DOF requirement:
      - 6 independent non-zero elements in the  $4 \times 4$  skew-symmetric matrix (upper/lower triangle) with 5 of their ratios significant in homogeneous representation
      - $\det \mathbf{L} = 0$  & elements satisfy a (quadratic) constraint  $\rightarrow$  net DOF number: 4
    - 3)  $\mathbf{L} = \mathbf{A} \mathbf{B}^T - \mathbf{B} \mathbf{A}^T$ : generalization to 4-space of the vector product formula  $\mathbf{l} = \mathbf{x} \times \mathbf{y}$  of  $\mathbb{P}^2$  for a line  $\mathbf{l}$  defined by 2 points  $\mathbf{x}$  &  $\mathbf{y}$  represented by 3-vectors
    - 4)  $\mathbf{L}$  is independent of the points used to define it
      - Choose another point  $\mathbf{C}$  on the line containing  $\mathbf{A}$  &  $\mathbf{B}$  and  $\mathbf{C} = \mathbf{A} + \mu \mathbf{B}$

$$\begin{aligned} \hat{\mathbf{L}} &= \mathbf{A} \mathbf{C}^T - \mathbf{C} \mathbf{A}^T \\ &= \mathbf{A} (\mathbf{A}^T + \mu \mathbf{B}^T) - (\mathbf{A} + \mu \mathbf{B}) \mathbf{A}^T \\ &= \mathbf{A} \mathbf{B}^T - \mathbf{B} \mathbf{A}^T \\ &= \mathbf{L} \end{aligned}$$

- 5) Under point transformation  $\mathbf{X}' = \mathbf{H} \mathbf{X}$ , the matrix transforms as  $\mathbf{L}' = \mathbf{H} \mathbf{L} \mathbf{H}^T$ , a valency-2 tensor (?)
- Dual Pluecker representation  $\mathbf{L}^*$  as the intersection of 2 planes  $\mathbf{P}$  &  $\mathbf{Q}$ 
  - $\mathbf{L}^* = \mathbf{P} \mathbf{Q}^T - \mathbf{Q} \mathbf{P}^T$
  - Has similar properties of  $\mathbf{L}$
  - $\mathbf{X}' = \mathbf{H} \mathbf{X} \rightarrow \mathbf{L}^{*'} = \mathbf{H}^{-T} \mathbf{L}^* \mathbf{H}^{-1}$

- Join & incidence properties
  - 1) Plane defined by the join of point  $\mathbf{X}$  & line  $\mathbf{L}$ :
    - $\pi = \mathbf{L}^* \mathbf{X}$
    - $\mathbf{L}^* \mathbf{X} = 0$  if and only if  $\mathbf{X}$  is on  $\mathbf{L}$
  - 2) Point defined by the intersection of line  $\mathbf{L}$  & plane  $\pi$ 
    - $\mathbf{X} = \mathbf{L} \pi$
    - $\mathbf{L} \pi = 0$  if and only if  $\mathbf{L}$  is on  $\pi$
- c. Pluecker line coordinates
  - 6 non-zero elements of the  $4 \times 4$  skew-symmetric Pluecker matrix  $\mathbf{L}$ 
    - $\mathcal{L} = \{l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34}\}$
    - Use  $l_{42}$  instead of  $l_{24}$  to eliminate negatives in some related formulae
  - Homogeneous 6-vector in  $\mathbb{P}^5$
  - $\det \mathbf{L} = 0 \rightarrow l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0$

### 3.4 The Hierarchy of Transformations

- Specializations with additional properties of 3-space transformations over 2-space counterparts (check 2.4.7 for general properties)





Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, $\pi_\infty$ , (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$		The absolute conic, $\Omega_\infty$ , (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$		Volume.

Table 3.2. **Geometric properties invariant to commonly occurring transformations of 3-space.** The matrix  $\mathbf{A}$  is an invertible  $3 \times 3$  matrix,  $\mathbf{R}$  is a 3D rotation matrix,  $\mathbf{t} = (t_X, t_Y, t_Z)^T$  a 3D translation,  $\mathbf{v}$  a general 3-vector,  $v$  a scalar, and  $\mathbf{0} = (0, 0, 0)^T$  a null 3-vector. The distortion column shows typical effects of the transformations on a cube. Transformations higher in the table can produce all the actions of the ones below. These range from Euclidean, where only translations and rotations occur, to projective where five points can be transformed to any other five points (provided no three points are collinear, or four coplanar).

- DOF (class I  $\rightarrow$  IV) in  $\mathbb{P}^3$ 
  - Euclidean: 6 (3 + 3)

- 3 for rotation
- 3 for translation
- Similarity: 7 (1 + 6)
  - 1 for isotropic scaling
  - 6 for Euclidean
- Affinity: 12 (9 + 3)
  - 9 for transformation  $\mathbf{A}$
  - 3 for translation  $\mathbf{t}$
- Projectivity: 15
  - 16 elements with only their ratio significant  $\rightarrow$  15 parameters

### 3.5 The Plane at Infinity

- Plane at infinity:  $\pi_\infty = (0, 0, 0, 1)$  in affine 3-space
- 2 lines are parallel if and only if their line intersection is on  $\pi_\infty$
- A line is parallel to another line/plane if the point of intersection is on  $\pi_\infty$
- $\pi_\infty$  is a fixed plane under projective transformation  $\mathbf{H}$  if and only if  $\mathbf{H}$  is an affinity
  - $\pi_\infty$  is in general only fixed under an affinity as a set; it is not fixed pointwise
  - There may be additional planes fixed under a particular affinity, but only  $\pi_\infty$  is fixed under any affinity

## 4. Estimation - 2D Projective Transformations

- Some concrete estimation problems
  - a. 2D homography
    - Given a set of points  $\mathbf{x}_i$  & corresponding  $\mathbf{x}'_i$  in  $\mathbb{P}^2$ , compute projective transformation that maps each  $\mathbf{x}_i$  to  $\mathbf{x}'_i$
  - b. 3D to 2D camera projection
    - Given a set of points  $\mathbf{X}_i$  in 3D space and a set of corresponding points  $\mathbf{x}_i$  in an image, find the 3D to 2D projective mapping that maps  $\mathbf{X}_i$  to  $\mathbf{x}_i$  carried out by a projective camera
  - c. Fundamental matrix computation
    - Given a set of points  $\mathbf{x}_i$  in one image, and corresponding set of points  $\mathbf{x}'_i$  in another image, compute the fundamental matrix  $\mathbf{F}$  consistent with these correspondences
    - Fundamental matrix  $\mathbf{F}$ : a singular  $3 \times 3$  matrix satisfying  $\mathbf{x}_i^T \mathbf{F} \mathbf{x}'_i = 0$  for all  $i$
  - d. Trifocal tensor computation
    - Given a set of point correspondences  $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i \leftrightarrow \mathbf{x}''_i$  across 3 images, compute the trifocal tensor
    - Trifocal tensor  $\mathcal{T}_i^{jk}$ : a tensor relating points or lines in 3 views
- Focus on 2D homography problem in this chapter

#### 4.1 The Direct Linear Transformation (DLT) Algorithm

$$\cdot \mathbf{x}'_i = \mathbf{H}\mathbf{x}_i \rightarrow \mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \mathbf{0}$$

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \begin{pmatrix} y'_i \mathbf{h}^{3T} \mathbf{x}_i - w'_i \mathbf{h}^{2T} \mathbf{x}_i \\ w'_i \mathbf{h}^{1T} \mathbf{x}_i - x'_i \mathbf{h}^{3T} \mathbf{x}_i \\ x'_i \mathbf{h}^{2T} \mathbf{x}_i - y'_i \mathbf{h}^{1T} \mathbf{x}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0}^T \mathbf{h}^1 - w'_i \mathbf{x}_i^T \mathbf{h}^2 + y'_i \mathbf{x}_i^T \mathbf{h}^3 \\ w'_i \mathbf{x}_i^T \mathbf{h}^1 + \mathbf{0}^T \mathbf{h}^2 - x'_i \mathbf{x}_i^T \mathbf{h}^3 \\ -y'_i \mathbf{x}_i^T \mathbf{h}^1 + x'_i \mathbf{x}_i^T \mathbf{h}^2 + \mathbf{0}^T \mathbf{h}^3 \end{pmatrix}$$

$$\mathbf{A}_i \mathbf{h} = \begin{bmatrix} \mathbf{0}^T & -w'_i \mathbf{x}_i^T & y'_i \mathbf{x}_i^T \\ w'_i \mathbf{x}_i^T & \mathbf{0}^T & -x'_i \mathbf{x}_i^T \\ -y'_i \mathbf{x}_i^T & x'_i \mathbf{x}_i^T & \mathbf{0}^T \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = \mathbf{0} \quad (4-1)$$

$$\cdot \mathbf{x}'_i = (x'_i, y'_i, w'_i)$$

· Notations for (4-1)

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} = \begin{pmatrix} \mathbf{h}^{1T} \\ \mathbf{h}^{2T} \\ \mathbf{h}^{3T} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} \quad (4-2)$$

·  $\mathbf{h}$ : a 9-vector

·  $\mathbf{h}^{jT}$ : a vector denoting the  $j$ -th row of  $\mathbf{H}$

· Basic DLT for 2D homography problem (without normalization)

##### Objective

Given  $n \geq 4$  2D to 2D point correspondences  $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$ , determine the 2D homography matrix  $\mathbf{H}$  such that  $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$ .

##### Algorithm

- (i) For each correspondence  $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$  compute the matrix  $\mathbf{A}_i$  from (4.1). Only the first two rows need be used in general.
- (ii) Assemble the  $n$   $2 \times 9$  matrices  $\mathbf{A}_i$  into a single  $2n \times 9$  matrix  $\mathbf{A}$ .
- (iii) Obtain the SVD of  $\mathbf{A}$  (section A4.4(p585)). The unit singular vector corresponding to the smallest singular value is the solution  $\mathbf{h}$ . Specifically, if  $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$  with  $\mathbf{D}$  diagonal with positive diagonal entries, arranged in descending order down the diagonal, then  $\mathbf{h}$  is the last column of  $\mathbf{V}$ .
- (iv) The matrix  $\mathbf{H}$  is determined from  $\mathbf{h}$  as in (4.2).

· Over-determined solution

· No exact solution  $\rightarrow$  find an approximate solution:

· A vector  $\mathbf{h}$  that minimizes a cost function

· Minimize  $\|\mathbf{A}\mathbf{h}\|$  with constraint  $\|\mathbf{h}\| = 1$

## Part 1 - Camera Geometry and Single View Geometry

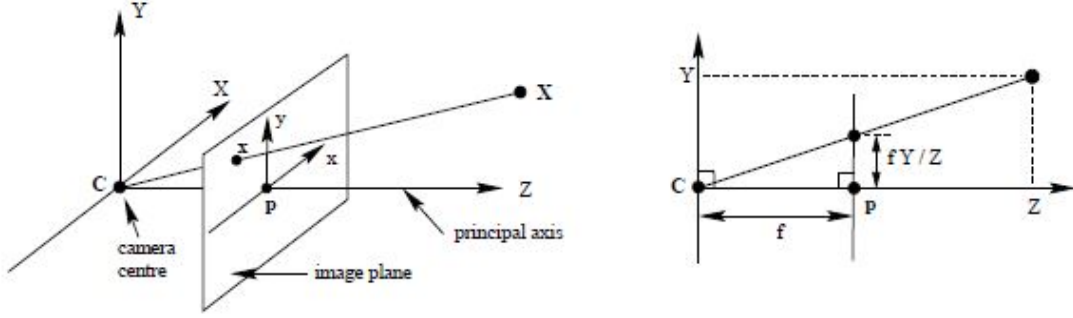
### 6. Camera Models

· Camera: a mapping between the 3D world (object space) and a 2D image



## 6.1 Finite Cameras

- The basic pinhole model:



**Fig. 6.1. Pinhole camera geometry.**  $C$  is the camera centre and  $p$  the principal point. The camera centre is here placed at the coordinate origin. Note the image plane is placed in front of the camera centre.

- $\mathbf{X}$ : 3D point  $\mathbf{X} = (X, Y, Z)$
- $\mathbf{x}$ : 2D point on image/focal plane  $Z = f$
- $\mathbf{p}$ : principle point (where the principle axis meets the image plane)
- $(X, Y, Z)$  mapped to  $(fX/Z, fY/Z, f)$  on image plane
  - An Euclidean 3-space  $\mathbb{R}^3$  to Euclidean 2-space  $\mathbb{R}^2$  mapping

$$(X, Y, Z) \mapsto (fX/Z, fY/Z) \quad (6-1)$$

- Central projection using homogeneous coordinates
  - Written (6-1) as

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ & f & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \quad (6-2)$$

- (6-2) can be written compactly as

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

- Camera matrix  $\mathbf{P} = \text{diag}(f, f, 1) \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$
- Principle point offset
  - For principle point at  $(p_x, p_y)$  in the image plane, the  $3D \rightarrow 2D$  mapping becomes  $(X, Y, Z) \mapsto (fX/Z + p_x, fY/Z + p_y)$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & p_x & 0 \\ & f & p_y \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \quad (6-3)$$

where

$$\mathbf{K} = \begin{bmatrix} f & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

is called *camera calibration matrix*

- (6-3) has the concise form

$$\mathbf{x} = \mathbf{K} \left[ \mathbf{I} \mid \mathbf{0} \right] \mathbf{X}_{\text{cam}} \quad (6-5)$$

- $\mathbf{X}_{\text{cam}} = (X, Y, Z, 1)$  is in *camera coordinate frame*
- Camera rotation and translation
  - Let  $\tilde{\mathbf{X}}$  &  $\tilde{\mathbf{X}}_{\text{cam}}$  be inhomogeneous 3-vectors representing coordinates of a point in *world coordinate frame* and *camera coordinate frame*, respectively
    - $\tilde{\mathbf{X}}_{\text{cam}} = \mathbf{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$ 
      - $\tilde{\mathbf{C}}$  represents the coordinate of the camera center in the world coordinate frame
      - $\mathbf{R}$ :  $3 \times 3$  rotation matrix representing the orientation of the camera coordinate frame
  - Homogeneous version:

$$\mathbf{X}_{\text{cam}} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ 0 & 1 \end{bmatrix} \mathbf{X} \quad (6-6)$$

- Putting (6-5) and (6-6) together:

$$\mathbf{x} = \mathbf{K}\mathbf{R} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right] \mathbf{X}$$

- $\mathbf{X}$ : coordinate in the world coordinate frame
- $\mathbf{P} = \mathbf{K}\mathbf{R} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right]$ : general pinhole camera
- DOF for  $\mathbf{P}$ : 9
  - 3 for  $\mathbf{K}$  ( $f, p_x, p_y$ ), called *internal* camera parameters, or *internal orientation* of the camera
  - 3 for  $\mathbf{R}$ , and 3 for  $\tilde{\mathbf{C}}$ , which are called *external* parameters, or the *exterior orientation*
- The world to image transformation can be represented as  $\tilde{\mathbf{X}}_{\text{cam}} = \mathbf{R}\tilde{\mathbf{X}} + \mathbf{t}$ , and then the camera matrix can be

$$\mathbf{P} = \mathbf{K} \left[ \mathbf{R} \mid \mathbf{t} \right] \quad (6-8)$$

where  $\mathbf{t} = -\mathbf{R}\tilde{\mathbf{C}}$

- CCD Cameras
  - Possibly having non-square pixels
  - General form of calibration matrix

$$\mathbf{K} = \text{diag}(m_x, m_y, 1) \begin{bmatrix} f & p_x \\ f & p_y \\ & 1 \end{bmatrix} = \begin{bmatrix} \alpha_x & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix}$$

where

- $m_x$  &  $m_y$ : number of pixels in per unit distance in image coordinates in the  $x$  and  $y$  directions
- $\alpha_x = fm_x$  &  $\alpha_y = fm_y$  represent camera focal length in terms of pixel dimensions in  $x$  and  $y$  direction respectively
- $\tilde{\mathbf{x}}_0 = (x_0, y_0) = (m_x p_x, m_y p_y)$ : principal point

- DOF: 10
- Finite projective camera
  - Add a *skew* parameter  $s$  to the calibration matrix

$$\mathbf{K} = \begin{bmatrix} \alpha_x & s & x_0 \\ & \alpha_y & y_0 \\ & & 1 \end{bmatrix} \quad (6-10)$$

- Finite projective camera: a camera (6-8) with calibration matrix  $\mathbf{K}$  of the form (6-10)
- DOF: 11, same with a  $3 \times 4$  matrix defined up to arbitrary scale
  - The set of camera matrices of *finite projective cameras* is **identical** with the set of *homogeneous  $3 \times 4$  matrices* for which the left hand  $3 \times 3$  submatrix is **non-singular**
- Transformation from world coordinate  $\mathbf{X} = (X, Y, Z, 1)$  to pixel coordinate  $w\mathbf{x} = (wx, wy, w)$  (ref):

$$w\mathbf{x} = \mathbf{K}\mathbf{R} \left[ \mathbf{I} \mid -\tilde{\mathbf{C}} \right] \mathbf{X}$$

- General projective camera
  - Arbitrary homogeneous  $3 \times 4$  matrices with rank 3
  - DOF: 11
  - Left-hand  $3 \times 3$  submatrix needs not be non-singular

## Part 4 - N-View Geometry

### 18. N-View Computational Methods

- Topic: methods for estimating a projective/affine reconstruction from a set of images
  - Number of views is large
- Contents
  - Bundle adjustment (BA) for projective reconstruction (general case)
  - Specialization of BA to affine cameras  $\rightarrow$  factorization algorithm
  - Generalization of the factorization algorithm to non-rigid scenes
  - Specialization of BA for scenes containing planes
  - Methods for obtaining point correspondences throughout image sequence and projective reconstruction from these correspondences

#### 18.1 Projective Reconstruction - Bundle Adjustment

- Formulation
  - A set of 3D points  $\mathbf{X}_j$  is viewed by a set of cameras with matrices  $\mathbf{P}^i$
  - $\mathbf{x}_j^i$ : coordinate of  $j$ -th point as seen by  $i$ -th camera
  - Reconstruction problem: given the set of image coordinates  $\mathbf{x}_j^i$ , find the set of camera matrices  $\mathbf{P}^i$  and the points  $\mathbf{X}_j$ , such that  $\mathbf{P}^i \mathbf{X}_j = \mathbf{x}_j^i$ 
    - Projective reconstruction if no constraints given on  $\mathbf{P}^i$  or  $\mathbf{X}_j$ 
      - $\mathbf{X}_j$  may differ by an arbitrary 3D projective transformation from true reconstruction
- Bundle adjustment

- $\mathbf{x}_j^i = \mathbf{P}^i \mathbf{X}_j$  will not be satisfied exactly if image measurements are noisy
- Maximum Likelihood (ML) solution: assume measurement noise is Gaussian
  - Estimate  $\hat{\mathbf{P}}^i$  and  $\hat{\mathbf{X}}_j$  such that  $\hat{\mathbf{x}}_j^i = \hat{\mathbf{P}}^i \hat{\mathbf{X}}_j$
  - Minimize image distance between reprojected point  $\hat{\mathbf{x}}_j^i$  and detected/measured point  $\mathbf{x}_j^i$  for every view where 3D point appears

$$\min_{\hat{\mathbf{P}}^i, \hat{\mathbf{X}}_j} \sum_{i,j} d(\hat{\mathbf{P}}^i \hat{\mathbf{X}}_j, \mathbf{x}_j^i)^2 \quad (18-1)$$

- $d(\mathbf{x}, \mathbf{y})$ : geometric image distance between homogeneous point  $\mathbf{x}$  and  $\mathbf{y}$
- Bundle adjustment: the above estimation involving minimizing the reprojection error
  - Adjusting the bundle of rays between each camera center and the set of 3D points
- Should generally be used as a final step of any reconstruction algorithm
- Pros & cons
  - + Tolerance of missing data while providing true ML estimate
  - + Allow individual covariances (or more general PDFs) assignment to each measurement
  - + May be extended to include the following estimates: priors, constraints on camera parameters, or point positions
  - - Require good initialization
  - - Can become extremely large optimization problem as the number of parameters involved increases

## 18.2 Affine Reconstruction - the Factorization Algorithm

- Algorithm

**Objective**

Given  $n \geq 4$  image point correspondences over  $m$  views  $\mathbf{x}_j^i$ ,  $j = 1, \dots, n$ ;  $i = 1, \dots, m$ , determine affine camera matrices  $\{\mathbf{M}^i, \mathbf{t}^i\}$  and 3D points  $\{\mathbf{X}_j\}$  such that the reprojection error

$$\sum_{ij} \|\mathbf{x}_j^i - (\mathbf{M}^i \mathbf{X}_j + \mathbf{t}^i)\|^2$$

is minimized over  $\{\mathbf{M}^i, \mathbf{t}^i, \mathbf{X}_j\}$ , with  $\mathbf{M}^i$  a  $2 \times 3$  matrix,  $\mathbf{X}_j$  a 3-vector, and  $\mathbf{x}_j^i = (x_j^i, y_j^i)^T$  and  $\mathbf{t}^i$  are 2-vectors.

**Algorithm**

- (i) **Computation of translations.** Each translation  $\mathbf{t}^i$  is computed as the centroid of points in image  $i$ , namely
$$\mathbf{t}^i = \langle \mathbf{x}^i \rangle = \frac{1}{n} \sum_j \mathbf{x}_j^i.$$
- (ii) **Centre the data.** Centre the points in each image by expressing their coordinates with respect to the centroid:
$$\mathbf{x}_j^i \leftarrow \mathbf{x}_j^i - \langle \mathbf{x}^i \rangle.$$

Henceforth work with these centred coordinates.
- (iii) **Construct the  $2m \times n$  measurement matrix  $\mathbf{W}$**  from the centred data, as defined in (18.5), and compute its SVD  $\mathbf{W} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ .
- (iv) Then the matrices  $\mathbf{M}^i$  are obtained from the first three columns of  $\mathbf{U}$  multiplied by the singular values:
$$\begin{bmatrix} \mathbf{M}^1 \\ \mathbf{M}^2 \\ \vdots \\ \mathbf{M}^m \end{bmatrix} = \begin{bmatrix} \sigma_1 \mathbf{u}_1 & \sigma_2 \mathbf{u}_2 & \sigma_3 \mathbf{u}_3 \end{bmatrix}.$$

The vectors  $\mathbf{t}^i$  are as computed in step (i) and the 3D structure is read from the first three columns of  $\mathbf{V}$

$$\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \dots & \mathbf{X}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}^T.$$

- $\hat{\mathbf{W}}$ : rank 3 matrix closest to  $\mathbf{W}$  in Frobenius norm
  - $\hat{\mathbf{W}} = \mathbf{U}_{2m \times 3} \mathbf{D}_{3 \times 3} \mathbf{V}_{3 \times n}^T = \hat{\mathbf{M}} \hat{\mathbf{X}}$
- Choice of  $\hat{\mathbf{M}}$  &  $\hat{\mathbf{X}}$  is not unique
  - E.g.,  $\hat{\mathbf{M}} = \mathbf{U}_{2m \times 3} \mathbf{D}_{3 \times 3}$  &  $\hat{\mathbf{X}} = \mathbf{V}_{3 \times n}^T$ ; or  $\hat{\mathbf{M}} = \mathbf{U}_{2m \times 3}$  &  $\hat{\mathbf{X}} = \mathbf{D}_{3 \times 3} \mathbf{V}_{3 \times n}^T$ , etc.
- Affine ambiguity:  $\hat{\mathbf{W}} = \hat{\mathbf{M}} \hat{\mathbf{X}} = \hat{\mathbf{M}} \mathbf{A} \mathbf{A}^{-1} \hat{\mathbf{X}} = (\hat{\mathbf{M}} \mathbf{A})(\mathbf{A}^{-1} \hat{\mathbf{X}})$
- Affinity in  $\mathbb{P}^3$ :  $\begin{bmatrix} \mathbf{A}_{3 \times 3} & \mathbf{t}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1_{1 \times 1} \end{bmatrix}_{4 \times 4}$
- Camera matrices  $\mathbf{M}^i$  & 3D points  $\mathbf{X}_j$  obtained from  $\hat{\mathbf{M}}$  &  $\hat{\mathbf{X}}$  (estimates) are determined up to multiplication by a common matrix  $\mathbf{A}$ 
  - The MLE reconstruction is affine

## Part 5 - Appendices

### Appendix 4: Matrix Properties and Decompositions

#### A4.2 Symmetric and Skew-symmetric Matrices

- Cross products

- $\mathbf{a} = (a_1, a_2, a_3) \rightarrow [\mathbf{a}]_{\times}$

$$[\mathbf{a}]_{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- $\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b} = (\mathbf{a}^T [\mathbf{b}]_{\times})^T$

### Appendix 6: Iterative Estimation Methods

#### A6.1 Newton Iteration

- Formulation

- Given a hypothesized function relation  $\mathbf{X} = \mathbf{f}(\mathbf{P})$ 
    - $\mathbf{X}$ : a *measurement vector* in Euclidean space  $\mathbb{R}^N$
    - $\mathbf{P}$ : a *parameter vector* in Euclidean space  $\mathbb{R}^M$
  - Measured value  $\mathbf{X}$  approximating true value  $\bar{\mathbf{X}}$  is provided
  - Find the vector  $\hat{\mathbf{P}}$  satisfying  $\mathbf{X} = \mathbf{f}(\hat{\mathbf{P}}) - \boldsymbol{\epsilon}$  for which  $\|\boldsymbol{\epsilon}\|$  is minimized
    - Linear function  $\mathbf{f}(\cdot)$ : least-squares problem where  $\mathbf{f}(\mathbf{P}) = \mathbf{A}\mathbf{P}$

- Solution

- Start with an initial estimated value  $\mathbf{P}_0$ 
    - Proceed to refine the estimate under the assumption that *function  $\mathbf{f}$  is locally linear*
  - Let  $\boldsymbol{\epsilon} = \mathbf{f}(\mathbf{P}_0) - \mathbf{X}$
  - Assume  $\mathbf{f}$  is approximated at  $\mathbf{P}_0$  by  $\mathbf{f}(\mathbf{P}_0 + \boldsymbol{\Delta}) = \mathbf{f}(\mathbf{P}_0) + \mathbf{J}\boldsymbol{\Delta}$ 
    - $\mathbf{J} = \partial \mathbf{f} / \partial \mathbf{P}$  (Jacobian matrix)
  - Seek a point  $\mathbf{P}_1 = \mathbf{P}_0 + \boldsymbol{\Delta}$  which minimizes  $\mathbf{f}(\mathbf{P}_1) - \mathbf{X} = \mathbf{f}(\mathbf{P}_0) + \mathbf{J}\boldsymbol{\Delta} - \mathbf{X} = \boldsymbol{\epsilon}_0 + \mathbf{J}\boldsymbol{\Delta}$ 
    - Minimizing  $\|\boldsymbol{\epsilon}_0 + \mathbf{J}\boldsymbol{\Delta}\|$  over  $\boldsymbol{\Delta} \rightarrow$  linear minimization problem
  - $\boldsymbol{\Delta}$  is obtained by
    - a. Solving *normal* equations

$$\mathbf{J}^T \mathbf{J} \boldsymbol{\Delta} = -\mathbf{J}^T \boldsymbol{\epsilon}_0 \quad (\text{A6.1})$$

- b. Using the pseudo-inverse  $\boldsymbol{\Delta} = -\mathbf{J}^+ \boldsymbol{\epsilon}_0$
  - Solution vector  $\hat{\mathbf{P}}$  is obtained by
    - Starting with an initial estimate  $\mathbf{P}_0$  and computing successive approximations by the formula  $\mathbf{P}_{i+1} = \mathbf{P}_i + \mathbf{J}\boldsymbol{\Delta}_i$ 
      - $\boldsymbol{\Delta}_i$ : solution to the least-squares problem  $\mathbf{J}\boldsymbol{\Delta}_i = -\boldsymbol{\epsilon}_i$ 
        - $\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{P}}|_{\mathbf{P}_i}$
        - $\boldsymbol{\epsilon}_i = \mathbf{f}(\mathbf{P}_i) - \mathbf{X}$
  - Disadvantage
    - The iteration procedure may converge to a local minimum or not converge at all
    - The algorithm depends very strongly on the initial estimate  $\mathbf{P}_0$

- Weighted iteration

- Measurement  $\mathbf{X}$  satisfies a Gaussian distribution with covariance matrix  $\Sigma_{\mathbf{X}}$ 
  - $\Sigma_{\mathbf{X}}$ : generally be an arbitrary symmetric & positive definite matrix
- Minimize Mahalanobis distance  $\|\mathbf{f}(\mathbf{P}) - \mathbf{X}\|_{\Sigma_{\mathbf{X}}}$  instead of  $\|\mathbf{f}(\mathbf{P}) - \mathbf{X}\|$
- Normal equation becomes  $\mathbf{J}^T \Sigma_{\mathbf{X}}^{-1} \mathbf{J} \Delta_i = -\mathbf{J}^T \Sigma_{\mathbf{X}}^{-1} \epsilon_i$

- Newton's method & the Hessian

- Finding minima of function  $g(\mathbf{P})$  of many variables
  - Expand  $g(\mathbf{P})$  about  $\mathbf{P}_0$  in a Taylor series
    - $g(\mathbf{P}_0 + \Delta) = g + g_{\mathbf{P}} \Delta + \Delta^T g_{\mathbf{PP}} \Delta / 2 + \dots$
  - Set derivative w.r.t.  $\Delta$  to 0

$$g_{\mathbf{PP}} \Delta = -g_{\mathbf{P}} \quad (\text{A6.2})$$

- $g_{\mathbf{PP}}$ : the **Hessian** where the  $(i, j)$ -th entry of which is  $\partial^2 g / \partial p_i \partial p_j$
- **Newton iteration**: starting at an initial value  $\mathbf{P}_0$ , and iteratively computing parameter increment  $\Delta$  using (A6.2) until convergence occurs
- Cost/error function

$$g(\mathbf{P}) = \|\epsilon(\mathbf{P})\|^2 / 2 = \epsilon(\mathbf{P})^T \epsilon(\mathbf{P}) / 2$$

- Gradient vector:  $g_{\mathbf{P}} = \epsilon_{\mathbf{P}}^T \epsilon$ 
  - $\epsilon_{\mathbf{P}}$ :  $\frac{\partial \epsilon(\mathbf{P})}{\partial \mathbf{P}}$
  - $\epsilon$ :  $\epsilon(\mathbf{P})$
- The Hessian

$$g_{\mathbf{PP}} = \epsilon_{\mathbf{P}}^T \epsilon_{\mathbf{P}} + \epsilon_{\mathbf{PP}}^T \epsilon \quad (\text{A6.3})$$

- Assume  $\mathbf{f}(\mathbf{P})$  is linear, 2nd term of (A6.3) becomes 0
  - $g_{\mathbf{PP}} = \epsilon_{\mathbf{P}}^T \epsilon_{\mathbf{P}} = \mathbf{J}^T \mathbf{J}$
  - (A6.2) becomes:  $\mathbf{J}^T \mathbf{J} \Delta = -\mathbf{J}^T \epsilon$ 
    - Same as (A6.1)
- **Gaussian-Newton method**: the iterative procedure where  $\mathbf{J}^T \mathbf{J}$  is used as an approximation of the Hessian (  $g_{\mathbf{PP}}$  )
- Gradient descent
  - $-g_{\mathbf{P}} = -\epsilon_{\mathbf{P}}^T \epsilon$ : negative/down-hill gradient vector defined the direction of most rapid decrease of cost function
  - Parameter increment  $\Delta$ : computed from the equation  $\lambda \Delta = -g_{\mathbf{P}}$ 
    - $\lambda$ : controls the length of the step
  - The Hessian is approximated (somewhat arbitrarily) by matrix  $\lambda \mathbf{I}$ :
$$g_{\mathbf{PP}} \Delta = -g_{\mathbf{P}} \quad \rightarrow \quad \lambda \mathbf{I} \Delta = -g_{\mathbf{P}} \quad \rightarrow \quad \lambda \Delta = -g_{\mathbf{P}}$$
  - Not a good minimization strategy due to slow convergence by zig-zagging
- Summary on minimization of cost function  $g(\mathbf{P}) = \|\epsilon(\mathbf{P})\|^2 / 2$ 
  - a. **Newton**  $\rightarrow$  update equation:

$$g_{\mathbf{PP}} \Delta = -g_{\mathbf{P}}$$

- $g_{\mathbf{PP}} = \epsilon_{\mathbf{P}}^T \epsilon_{\mathbf{P}} + \epsilon_{\mathbf{PP}}^T \epsilon$
- $g_{\mathbf{P}} = \epsilon_{\mathbf{P}}^T \epsilon$

b. *Gauss-Newton*  $\rightarrow$  update equation:

$$\epsilon_{\mathbf{P}}^T \epsilon_{\mathbf{P}} \Delta = -\epsilon_{\mathbf{P}}^T \epsilon$$

c. *Gradient Descent*  $\rightarrow$  update equation:

$$\lambda \Delta = -\epsilon_{\mathbf{P}}^T \epsilon = -g_{\mathbf{P}}$$

## A6.2 Levenberg-Marquardt iteration

- Normal equations  $\mathbf{J}^T \mathbf{J} \Delta = -\mathbf{J}^T \epsilon$  are replaced by *augmented normal equations*  $(\mathbf{J}^T \mathbf{J} + \lambda \mathbf{I}) \Delta = -\mathbf{J}^T \epsilon$ 
  - Typical value for  $\lambda$  is  $10^{-3}$
- Levenberg-Marquardt (LM) iteration
  - Repeatedly solving augmented normal equations for different  $\lambda$  until an acceptable  $\Delta$  is found
    - If  $\Delta$  solved leads to reduction in the error
      - Accept the increment
    - If  $\Delta$  solved leads to increased error
      - Multiply  $\lambda$  by same factor and solve new augmented normal equation



# Notes on State Estimation for Robotics

## Book (Barfoot 2017) ([link](#))

### Part I - Estimation Machinery

#### 2. Primer on Probability Theory

##### 2.1 Probability Density Functions

##### 2.1.2 Bayes' Rule and Inference

- Factorization of joint probability

$$p(\mathbf{x}, \mathbf{y}) = p(\mathbf{x}|\mathbf{y})p(\mathbf{y}) = p(\mathbf{y}|\mathbf{x})p(\mathbf{x})$$

- $\mathbf{x} = (x_1, \dots, x_N)$ ,  $\mathbf{y} = (y_1, \dots, y_M)$
- Rearranging gives *Bayes' rule*:

$$\begin{aligned} p(\mathbf{x}|\mathbf{y}) &= \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} \\ &= \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{\int p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x}} \end{aligned}$$

- $p(\mathbf{x}|\mathbf{y})$ : (*inferred*) *posterior*/likelihood of the state given the measurements
  - *Posterior* density
- $p(\mathbf{x})$ : *prior* probability density function (PDF) over the state
  - *Prior* density
- $p(\mathbf{y}|\mathbf{x})$ : sensor model

##### 2.1.6 Normalized Product

- Normalized product  $p(\mathbf{x})$  of 2 PDFs  $p_1(\mathbf{x})$  &  $p_2(\mathbf{x})$  for  $\mathbf{x}$

$$p(\mathbf{x}) = \eta p_1(\mathbf{x})p_2(\mathbf{x})$$

- $\eta$ : normalization constant

$$\eta = \left( \int p_1(\mathbf{x})p_2(\mathbf{x})d\mathbf{x} \right)^{-1}$$

## Part II - Three Dimensional Machinery

### 7. Matrix Lie Groups

- Lie group: a group that is also a **differentiable manifold**
  - Property: group operations are smooth

#### 7.1 Geometry

- Two specific Lie groups:
  - (1)  $SO(3)$ : special orthogonal group, which can represent rotations
  - (2)  $SE(3)$ : special Euclidean group, which can represent poses

##### 7.1.1 Special Orthogonal and Special Euclidean Groups

- Special orthogonal group: set of valid rotation matrices

$$SO(3) = \{ \mathbf{C} \in \mathbb{R}^{3 \times 3} \mid \mathbf{C}\mathbf{C}^T = \mathbf{I}, \det \mathbf{C} = 1 \}$$

- Special Euclidean group: set of valid transformation matrices

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4} \mid \mathbf{C} \in SO(3), \mathbf{r} \in \mathbb{R}^3 \right\}$$

- Lie group properties for  $SO(3)$  and  $SE(3)$ :

property	$SO(3)$	$SE(3)$
closure	$\mathbf{C}_1, \mathbf{C}_2 \in SO(3)$ $\Rightarrow \mathbf{C}_1 \mathbf{C}_2 \in SO(3)$	$\mathbf{T}_1, \mathbf{T}_2 \in SE(3)$ $\Rightarrow \mathbf{T}_1 \mathbf{T}_2 \in SE(3)$
associativity	$\mathbf{C}_1 (\mathbf{C}_2 \mathbf{C}_3) = (\mathbf{C}_1 \mathbf{C}_2) \mathbf{C}_3$ $= \mathbf{C}_1 \mathbf{C}_2 \mathbf{C}_3$	$\mathbf{T}_1 (\mathbf{T}_2 \mathbf{T}_3) = (\mathbf{T}_1 \mathbf{T}_2) \mathbf{T}_3$ $= \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_3$
identity	$\mathbf{C}, \mathbf{1} \in SO(3)$ $\Rightarrow \mathbf{C} \mathbf{1} = \mathbf{1} \mathbf{C} = \mathbf{C}$	$\mathbf{T}, \mathbf{1} \in SE(3)$ $\Rightarrow \mathbf{T} \mathbf{1} = \mathbf{1} \mathbf{T} = \mathbf{T}$
invertibility	$\mathbf{C} \in SO(3)$ $\Rightarrow \mathbf{C}^{-1} \in SO(3)$	$\mathbf{T} \in SE(3)$ $\Rightarrow \mathbf{T}^{-1} \in SE(3)$

##### 7.1.2 Lie Algebras

- Lie algebra
  - Associated with every matrix Lie group
  - Consists of a vector space  $\mathfrak{g}$ , over some field  $\mathbb{F}$ , together with a binary operation  $[\cdot, \cdot]$  called *Lie bracket*
    - Lie bracket: non-associative, alternating bilinear map  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}; (x, y) \mapsto [x, y]$
    - Visualization of Lie bracket: [link](#)
  - Properties of Lie bracket for all  $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{g}$  and  $a, b \in \mathbb{F}$ 
    - a) Closure:  $[\mathbf{X}, \mathbf{Y}] \in \mathfrak{g}$
    - b) Bilinearity:
      - $[a\mathbf{X} + b\mathbf{Y}, \mathbf{Z}] = a[\mathbf{X}, \mathbf{Z}] + b[\mathbf{Y}, \mathbf{Z}]$

- $[\mathbf{Z}, a\mathbf{X} + b\mathbf{Y}] = a[\mathbf{Z}, \mathbf{X}] + b[\mathbf{Z}, \mathbf{Y}]$
- c) Alternating:  $[\mathbf{X}, \mathbf{X}] = \mathbf{0}$
- d) Jacobi identity:  $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = \mathbf{0}$
- Lie algebra associated with  $SO(3)$  (rotations)
  - Definition:
    - Vector space:  $\mathfrak{so}(3) = \{\Phi = \phi^\wedge \in \mathbb{R}^{3 \times 3} \mid \phi \in \mathbb{R}^3\}$
    - where  $\phi^\wedge$  is a **skew-symmetric matrix** generated from vector  $\phi$ :

$$\phi^\wedge = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}, \phi \in \mathbb{R}^3$$

- Field:  $\mathbb{R}$
- Lie bracket:  $[\Phi_1, \Phi_2] = \Phi_1 \Phi_2 - \Phi_2 \Phi_1$
- $\Phi = \phi^\wedge, \phi = \Phi^\vee$
- Lie algebra associated with  $SE(3)$  (poses)
  - Definition:
    - Vector space:  $\mathfrak{se}(3) = \{\Xi = \xi^\wedge \in \mathbb{R}^{4 \times 4} \mid \xi \in \mathbb{R}^6\}$
    - where

$$\xi^\wedge = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \rho, \phi \in \mathbb{R}^3$$

- $\rho$ : translation component of pose
- $\phi$ : rotation component of pose
- Field:  $\mathbb{R}$
- Lie bracket:  $[\Xi_1, \Xi_2] = \Xi_1 \Xi_2 - \Xi_2 \Xi_1$
- $\Xi = \xi^\wedge, \xi = \Xi^\vee$
- Lie algebra: **tangent space** of the corresponding Lie group (normally, at the *identity*  $\mathbf{I}$  of a Lie group)

- Why *identity*: for Lie group  $G$ , the tangent spaces at all points of  $G$  are *isomorphic*. So we pick the identity as  $\mathbf{I} \in G$  is guaranteed for every Lie group  $G$

- Intuitive explanation:

Think of tangent space of a surface at a point as the space of all “directions” one can move from that point (with all velocities) *while staying on that surface*.

The paths through the point is not on the surface, but a very small portion of the path will be very likely on it.

Start from the identity (in this example,  $\mathbf{I} \in SO(n)$ ):

$$(\mathbf{I} + \alpha \mathbf{t})^T (\mathbf{I} + \alpha \mathbf{t}) = \mathbf{I} + \alpha(\mathbf{t} + \mathbf{t}^T) + O(\alpha^2)$$

where  $\mathbf{t} \in \mathbb{R}^{3 \times 3}$  and  $\alpha \rightarrow 0$ .

If we ignore the  $\alpha^2$  term,  $\mathbf{I} + \alpha \mathbf{t}$  will be in the Lie group if  $\mathbf{t} + \mathbf{t}^T = \mathbf{0}$  for small  $\alpha$ . That is, the skew-symmetric matrices  $\mathbf{t}$  form the Lie algebra  $\mathfrak{so}(n)$ .

- More generally, the Lie algebra gives the set of possible ways (directions) in which you can change the group elements while staying in the group

### 7.1.3 Exponential Map

- Matrix exponential

$$\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n$$

- Matrix logarithm

$$\ln(\mathbf{A}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\mathbf{A} - \mathbf{I})^n$$

- Exponential map:  $\mathfrak{so}(3) \leftrightarrow SO(3)$

- $\mathfrak{so}(3) \rightarrow SO(3)$ :

$$\mathbf{C} = \exp(\phi^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi^\wedge)^n$$

where  $\mathbf{C} \in SO(3)$ , and  $\phi \in \mathbb{R}^3$  ( $\phi^\wedge \in \mathfrak{so}(3)$ )

- Let  $\phi = \phi \mathbf{a}$  where  $\phi = |\phi|$  is the rotation angle, and  $\mathbf{a}$  is a unit-vector representing rotation axis (angle-axis representation)

$$\mathbf{C} = \exp(\phi^\wedge) = \exp(\phi \mathbf{a}^\wedge) = \cos \phi \mathbf{I} + \sin \phi \mathbf{a}^\wedge + (1 - \cos \phi) \mathbf{a} \mathbf{a}^T$$

(Rodrigues' formula)

- $SO(3) \rightarrow \mathfrak{so}(3)$  (not uniquely):

$$\phi = \ln(\mathbf{C})^\vee = (\phi^\wedge)^\vee$$

- Exponential map:  $\mathfrak{se}(3) \leftrightarrow SE(3)$

- $\mathfrak{se}(3) \rightarrow SE(3)$ :

$$\begin{aligned} \mathbf{T} &= \exp(\xi^\wedge) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (\xi^\wedge)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\xi} \end{bmatrix}^\wedge \right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \phi^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix}^n \\ &= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} (\phi^\wedge)^n & \left( \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n \right) \boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in SE(3) \end{aligned}$$

where  $\xi \in \mathbb{R}^6$ ,  $\mathbf{r} = \mathbf{J} \boldsymbol{\rho} \in \mathbb{R}^3$ , and  $\mathbf{J} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n$  ( $\xi^\wedge \in \mathfrak{se}(3)$ )

- $SE(3) \rightarrow \mathfrak{se}(3)$  (not uniquely):

$$\xi = \ln(\mathbf{T})^\vee = (\xi^\wedge)^\vee$$

- From  $\mathbf{T} \in SE(3)$  to  $\xi \in \mathbb{R}^6$ :

- a) Compute  $\phi \in \mathbb{R}^3$  from  $\mathbf{C} \in SO(3)$
- b) Compute  $\mathbf{J} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n$
- c) Compute  $\rho = \mathbf{J}^{-1} \mathbf{r}$
- d) Composite  $\xi$  using computed  $\phi$  and  $\rho$
- Jacobian  $\mathbf{J}$ :  $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n$
- Close-form expressions for  $\mathbf{J}$  and  $\mathbf{J}^{-1}$ :

$$\mathbf{J} = \frac{\sin \phi}{\phi} \mathbf{I} + (1 - \frac{\sin \phi}{\phi}) \mathbf{a} \mathbf{a}^T + \frac{1 - \cos \phi}{\phi} \mathbf{a}^\wedge$$

$$\mathbf{J}^{-1} = \frac{\phi}{2} \cot \frac{\phi}{2} \mathbf{I} + (1 - \frac{\phi}{2} \cot \frac{\phi}{2}) \mathbf{a} \mathbf{a}^T - \frac{\phi}{2} \mathbf{a}^\wedge$$

where  $\phi = |\phi|$  is the rotation angle and  $\mathbf{a} = \phi/\phi$  is the unit-length rotation axis

- $\mathbf{J} \mathbf{J}^T$  and its inverse:

$$\mathbf{J} \mathbf{J}^T = \gamma \mathbf{I} + (1 - \gamma) \mathbf{a} \mathbf{a}^T$$

$$(\mathbf{J} \mathbf{J}^T)^{-1} = \frac{1}{\gamma} \mathbf{I} + (1 - \frac{1}{\gamma}) \mathbf{a} \mathbf{a}^T$$

where  $\gamma = 2 \frac{1 - \cos \phi}{\phi^2}$

- $\mathbf{J} \mathbf{J}^T$  is positive-definite
- $\mathbf{J}$  represented by rotation matrix  $\mathbf{C}$ :

$$\mathbf{J} = \int_0^1 \mathbf{C}^\alpha d\alpha$$

- Rotation matrix  $\mathbf{C}$  represented by Jacobian  $\mathbf{J}$ :

$$\mathbf{C} = \mathbf{I} + \phi^\wedge \mathbf{J}$$

- Impossible to solve  $\mathbf{J}$  because  $\phi$  is invertible in the foregoing expression
- Direct series expression for  $\mathbf{T} \in SE(3)$

$$\mathbf{T} = \mathbf{I} + \xi^\wedge + (\frac{1 - \cos \phi}{\phi^2}) (\xi^\wedge)^2 + (\frac{\phi - \sin \phi}{\phi^3}) (\xi^\wedge)^3$$

- Based on the identity:  $(\xi^\wedge)^4 + \phi^2 (\xi^\wedge)^2 \equiv \mathbf{0}$

#### 7.1.4 Adjoints

- Adjoint of an element of  $SE(3)$ :

$$\mathcal{T}_{6 \times 6} = \text{Ad}(\mathbf{T}_{4 \times 4}) = \text{Ad} \left( \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \right) = \begin{bmatrix} \mathbf{C} & \mathbf{r}^\wedge \mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

where

$$\mathbf{r}^\wedge = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

- Adjoint of all elements of  $SE(3)$ :

$$\text{Ad}(SE(3)) = \{\mathcal{T} = \text{Ad}(\mathbf{T}) \mid \mathbf{T} \in SE(3)\}$$

- $\text{Ad}(SE(3))$  is a Lie group:
  - $\mathcal{T}_1 \mathcal{T}_2 \in \text{Ad}(SE(3))$
  - $\mathcal{T}^{-1} \in \text{Ad}(SE(3))$
- Adjoint of an element  $\Xi = \xi^\wedge \in \mathfrak{se}(3)$ :

$$\text{ad}(\Xi) = \text{ad}(\xi^\wedge) = \xi^\wedge$$

where

$$\xi^\wedge = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge = \begin{bmatrix} \phi^\wedge & \rho^\wedge \\ \mathbf{0} & \phi^\wedge \end{bmatrix} \in \mathbb{R}^{6 \times 6}, \quad \rho, \phi \in \mathbb{R}^3$$

- The Lie algebra associated with  $\text{Ad}(SE(3))$ :
  - Vector space:  $\text{ad}(\mathfrak{se}(3)) = \{\Psi = \text{ad}(\Xi) \in \mathbb{R}^{6 \times 6} \mid \Xi \in \mathfrak{se}(3)\}$
  - Field:  $\mathbb{R}$
  - Lie bracket:  $[\Psi_1, \Psi_2] = \Psi_1 \Psi_2 - \Psi_2 \Psi_1$
- Exponential map:  $\text{ad}(\mathfrak{se}(3)) \leftrightarrow \text{Ad}(SE(3))$ 
  - $\text{ad}(\mathfrak{se}(3)) \rightarrow \text{Ad}(SE(3))$ :

$$\mathcal{T} = \exp(\text{ad}(\xi^\wedge)) = \sum_{n=0}^{\infty} \frac{1}{n!} (\xi^\wedge)^n$$

where  $\mathcal{T} \in \text{Ad}(SE(3))$  and  $\xi \in \mathbb{R}^6$  ( $\xi^\wedge \in \text{ad}(\mathfrak{se}(3))$ )

- $\text{Ad}(SE(3)) \rightarrow \text{ad}(\mathfrak{se}(3))$  (not uniquely):

$$\xi = \ln(\mathcal{T})^\vee = (\xi^\wedge)^\vee$$

- Relationship between various Lie groups and algebras associated with poses

	Lie algebra		Lie group
$4 \times 4$	$\xi^\wedge \in \mathfrak{se}(3)$	$\xrightarrow{\exp}$	$\mathbf{T} \in SE(3)$
	$\downarrow \text{ad}$		$\downarrow \text{Ad}$
$6 \times 6$	$\xi^\wedge \in \text{ad}(\mathfrak{se}(3))$	$\xrightarrow{\exp}$	$\mathcal{T} \in \text{Ad}(SE(3))$

- Direct series expression for  $\mathcal{T} \in \text{Ad}(SE(3))$

$$\begin{aligned} \mathcal{T} = \mathbf{I} &+ \left( \frac{3 \sin \phi - \phi \cos \phi}{2\phi} \right) \xi^\wedge + \left( \frac{4 - \phi \sin \phi - 4 \cos \phi}{2\phi^2} \right) (\xi^\wedge)^2 \\ &+ \left( \frac{\sin \phi - \phi \cos \phi}{2\phi^3} \right) (\xi^\wedge)^3 + \left( \frac{2 - \phi \sin \phi - 2 \cos \phi}{2\phi^4} \right) (\xi^\wedge)^4 \end{aligned}$$

- Based on the identity:  $(\xi^\wedge)^5 + 2\phi^2(\xi^\wedge)^3 + \phi^4 \xi^\wedge \equiv \mathbf{0}$

### 7.1.5 Baker-Campbell-Hausdorff

- BCH formula: solution to  $Z$  in equation  $e^X e^Y = e^Z$  for possibly noncommutative  $X$  and  $Y$  in

the Lie algebra of a Lie group

- General combinatorial formula by Eugene Dynkin (1947):

$$\ln(\exp(\mathbf{A})\exp(\mathbf{B})) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0, \\ 1 \leq i \leq n}} \frac{\left(\sum_{i=1}^n (r_i + s_i)\right)^{-1}}{\prod_{i=1}^n r_i! s_i!} [\mathbf{A}^{r_1} \mathbf{B}^{s_1} \mathbf{A}^{r_2} \mathbf{B}^{s_2} \dots \mathbf{A}^{r_n} \mathbf{B}^{s_n}]$$

where

$$[\mathbf{A}^{r_1} \mathbf{B}^{s_1} \mathbf{A}^{r_2} \mathbf{B}^{s_2} \dots \mathbf{A}^{r_n} \mathbf{B}^{s_n}] = \underbrace{[\mathbf{A}, \dots [\mathbf{A}, \underbrace{[\mathbf{B}, \dots [\mathbf{B}, \dots [\mathbf{A}, \dots [\mathbf{A}, \underbrace{[\mathbf{B}, \dots [\mathbf{B}, \mathbf{B}], \dots]]}_{s_n} \dots] \dots] \dots]}_{r_1} \dots] \dots]$$

- Lie product formula:

$$\exp(\mathbf{A} + \mathbf{B}) = \lim_{\alpha \rightarrow \infty} (\exp(\mathbf{A}/\alpha) \exp(\mathbf{B}/\alpha))^\alpha$$

- BCH in  $SO(3)$  case

- General case

$$\ln(\mathbf{C}_1 \mathbf{C}_2)^\vee = \ln(\exp(\phi_1^\wedge) \exp(\phi_2^\wedge))^\vee = \phi_1 + \phi_2 + \frac{1}{2} \phi_1^\wedge \phi_2 + \frac{1}{12} \phi_1^\wedge \phi_1^\wedge \phi_2 + \frac{1}{12} \phi_2^\wedge \phi_2^\wedge \phi_1 + \dots$$

- Cases where either  $\phi_1$  or  $\phi_2$  is small

$$\ln(\mathbf{C}_1 \mathbf{C}_2)^\vee = \ln(\exp(\phi_1^\wedge) \exp(\phi_2^\wedge))^\vee \approx \begin{cases} \mathbf{J}_l(\phi_2)^{-1} \phi_1 + \phi_2 & \text{if } \phi_1 \text{ is small} \\ \phi_1 + \mathbf{J}_r(\phi_1)^{-1} \phi_2 & \text{if } \phi_2 \text{ is small} \end{cases}$$

where

$$\mathbf{J}_r(\phi)^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} (-\phi^\wedge)^n = \frac{\phi}{2} \cot \frac{\phi}{2} \mathbf{I} + (1 - \frac{\phi}{2} \cot \frac{\phi}{2}) \mathbf{a} \mathbf{a}^T + \frac{\phi}{2} \mathbf{a}^\wedge$$

$$\mathbf{J}_l(\phi)^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} (\phi^\wedge)^n = \frac{\phi}{2} \cot \frac{\phi}{2} \mathbf{I} + (1 - \frac{\phi}{2} \cot \frac{\phi}{2}) \mathbf{a} \mathbf{a}^T - \frac{\phi}{2} \mathbf{a}^\wedge$$

are *right* and *left Jacobians* of  $SO(3)$  respectively. The  $B_n$  are the *Bernoulli numbers*.

$\phi = \phi \mathbf{a}$  where  $\phi = |\phi|$  and  $\mathbf{a} = \phi / \phi$ .  $\mathbf{C} = \exp(\phi^\wedge)$ .

- More about left & right Jacobians of  $SO(3)$

- Expressions for the Jacobians

$$\mathbf{J}_r(\phi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-\phi^\wedge)^n = \int_0^1 \mathbf{C}^{-\alpha} d\alpha = \frac{\sin \phi}{\phi} \mathbf{I} + (1 - \frac{\sin \phi}{\phi}) \mathbf{a} \mathbf{a}^T - \frac{1 - \cos \phi}{\phi} \mathbf{a}^\wedge$$

$$\mathbf{J}_l(\phi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n = \int_0^1 \mathbf{C}^\alpha d\alpha = \frac{\sin \phi}{\phi} \mathbf{I} + (1 - \frac{\sin \phi}{\phi}) \mathbf{a} \mathbf{a}^T + \frac{1 - \cos \phi}{\phi} \mathbf{a}^\wedge$$

- Relationship between the 2 Jacobians

$$\mathbf{J}_l(\phi) = \mathbf{C} \mathbf{J}_r(\phi)$$

$$\mathbf{J}_l(-\phi) = \mathbf{J}_r(\phi)$$

- BCH in  $SE(3)$  and  $\text{Ad}(SE(3))$  cases

- General case

$$\ln(\mathbf{T}_1 \mathbf{T}_2)^\vee = \ln(\exp(\boldsymbol{\xi}_1^\wedge) \exp(\boldsymbol{\xi}_2^\wedge))^\vee = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \frac{1}{2} \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_2 + \frac{1}{12} \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_2 + \frac{1}{12} \boldsymbol{\xi}_2^\wedge \boldsymbol{\xi}_2^\wedge \boldsymbol{\xi}_1 + \dots$$

$$\ln(\mathcal{T}_1 \mathcal{T}_2)^\vee = \ln(\exp(\boldsymbol{\xi}_1^\wedge) \exp(\boldsymbol{\xi}_2^\wedge))^\vee = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \frac{1}{2} \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_2 + \frac{1}{12} \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_2 + \frac{1}{12} \boldsymbol{\xi}_2^\wedge \boldsymbol{\xi}_2^\wedge \boldsymbol{\xi}_1 + \dots$$

- Cases where either  $\boldsymbol{\xi}_1$  or  $\boldsymbol{\xi}_2$  is small

$$\ln(\mathbf{T}_1 \mathbf{T}_2)^\vee = \ln(\exp(\boldsymbol{\xi}_1^\wedge) \exp(\boldsymbol{\xi}_2^\wedge))^\vee \approx \begin{cases} \mathcal{J}_l(\boldsymbol{\xi}_2)^{-1} \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_1 \text{ is small} \\ \boldsymbol{\xi}_1 + \mathcal{J}_r(\boldsymbol{\xi}_1)^{-1} \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_2 \text{ is small} \end{cases}$$

$$\ln(\mathcal{T}_1 \mathcal{T}_2)^\vee = \ln(\exp(\boldsymbol{\xi}_1^\wedge) \exp(\boldsymbol{\xi}_2^\wedge))^\vee \approx \begin{cases} \mathcal{J}_l(\boldsymbol{\xi}_2)^{-1} \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_1 \text{ is small} \\ \boldsymbol{\xi}_1 + \mathcal{J}_r(\boldsymbol{\xi}_1)^{-1} \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_2 \text{ is small} \end{cases}$$

where

$$\mathcal{J}_r(\phi)^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} (-\boldsymbol{\xi}^\wedge)^n$$

$$\mathcal{J}_l(\phi)^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} (\boldsymbol{\xi}^\wedge)^n$$

are *right* and *left Jacobians* of  $SE(3)$  respectively.

- More about left & right Jacobians of  $SE(3)$

- Expressions for the Jacobians

$$\mathcal{J}_r(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-\boldsymbol{\xi}^\wedge)^n = \int_0^1 \mathcal{T}^{-\alpha} d\alpha$$

$$\mathcal{J}_l(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\boldsymbol{\xi}^\wedge)^n = \int_0^1 \mathcal{T}^{\alpha} d\alpha$$

- Relationship between the 2 Jacobians

$$\mathcal{J}_l(\boldsymbol{\xi}) = \mathcal{T} \mathcal{J}_r(\boldsymbol{\xi})$$

$$\mathcal{J}_l(-\boldsymbol{\xi}) = \mathcal{J}_r(\boldsymbol{\xi})$$

- Rewrite BCH approximations using *left Jacobian* ( $\mathbf{J} = \mathbf{J}_l$  and  $\mathcal{J} = \mathcal{J}_l$ )

- $SO(3)$  case:

$$\ln(\mathbf{C}_1 \mathbf{C}_2)^\vee = \ln(\exp(\boldsymbol{\phi}_1^\wedge) \exp(\boldsymbol{\phi}_2^\wedge))^\vee \approx \begin{cases} \mathbf{J}(\boldsymbol{\phi}_2)^{-1} \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_1 \text{ is small} \\ \boldsymbol{\phi}_1 + \mathbf{J}(-\boldsymbol{\phi}_1)^{-1} \boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_2 \text{ is small} \end{cases}$$

- $SE(3)$  case:

$$\ln(\mathbf{T}_1 \mathbf{T}_2)^\vee = \ln(\exp(\boldsymbol{\xi}_1^\wedge) \exp(\boldsymbol{\xi}_2^\wedge))^\vee \approx \begin{cases} \mathcal{J}(\boldsymbol{\xi}_2)^{-1} \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_1 \text{ is small} \\ \boldsymbol{\xi}_1 + \mathcal{J}(-\boldsymbol{\xi}_1)^{-1} \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_2 \text{ is small} \end{cases}$$



$$\ln(\mathcal{T}_1 \mathcal{T}_2)^\vee = \ln(\exp(\xi_1^\wedge) \exp(\xi_2^\wedge))^\vee \approx \begin{cases} \mathcal{J}(\xi_2)^{-1} \xi_1 + \xi_2 & \text{if } \xi_1 \text{ is small} \\ \xi_1 + \mathcal{J}(-\xi_1)^{-1} \xi_2 & \text{if } \xi_2 \text{ is small} \end{cases}$$

### 7.1.6 Distance, Volume, Integration

- Difference of 2 rotations: right difference  $\phi_{12}$  and left difference  $\phi_{21}$

$$\phi_{12} = \ln(\mathbf{C}_1^T \mathbf{C}_2)^\vee$$

$$\phi_{21} = \ln(\mathbf{C}_2 \mathbf{C}_1^T)^\vee$$

where  $\mathbf{C}_1, \mathbf{C}_2 \in SO(3)$

- Inner product for  $\mathfrak{so}(3)$ :

$$\langle \phi_1^\wedge, \phi_2^\wedge \rangle = \frac{1}{2} \text{tr}(\phi_1^\wedge \phi_2^{\wedge T}) = \phi_1^T \phi_2$$

- The metric *distance* between 2 rotations: (i) the square root of the inner product of difference with itself, or (ii) the Euclidean norm of the difference, or (iii) the magnitude of the angle of the rotation difference

$$\phi_{12} = \sqrt{\langle \phi_{12}, \phi_{12} \rangle} = \sqrt{\phi_{12}^T \phi_{12}} = |\phi_{12}|$$

$$\phi_{21} = \sqrt{\langle \phi_{21}, \phi_{21} \rangle} = \sqrt{\phi_{21}^T \phi_{21}} = |\phi_{21}|$$

- Integration on rotations
  - Let  $\mathbf{C} = \exp(\phi^\wedge) \in SO(3)$
  - Let  $\mathbf{C}' = \exp((\phi + \delta\phi)^\wedge)$  by perturbing  $\mathbf{C}$  a bit
  - Right & left differences (relative to  $\mathbf{C}$ ):

$$\ln(\delta \mathbf{C}_r)^\vee = \ln(\mathbf{C}^T \mathbf{C}')^\vee \approx \ln(\mathbf{C}^T \mathbf{C} \exp((\mathbf{J}_r \delta\phi)^\wedge))^\vee = \mathbf{J}_r \delta\phi$$

$$\ln(\delta \mathbf{C}_l)^\vee = \ln(\mathbf{C}' \mathbf{C}^T)^\vee \approx \ln(\exp((\mathbf{J}_l \delta\phi)^\wedge) \mathbf{C} \mathbf{C}^T)^\vee = \mathbf{J}_l \delta\phi$$

where  $\mathbf{J}_r$  and  $\mathbf{J}_l$  are evaluated at  $\phi$

- BCH approximation (assume  $\mathbf{J}_r(\phi) \delta\phi$  is small enough):

$$\begin{aligned} \mathbf{C}' &= \exp((\phi + \delta\phi)^\wedge) \\ &= \exp((\phi + \mathbf{J}_r(\phi)^{-1}(\mathbf{J}_r(\phi) \delta\phi))^\wedge) \\ &\approx \exp(\phi^\wedge) \exp((\mathbf{J}_r(\phi) \delta\phi)^\wedge) \\ &= \mathbf{C} \exp((\mathbf{J}_r(\phi) \delta\phi)^\wedge) \end{aligned}$$

- Infinitesimal volume element

$$\begin{aligned} d\mathbf{C} &= |\det \mathbf{J}| d\phi \\ &= d\mathbf{C}_r = |\det \mathbf{J}_r| d\phi \\ &= d\mathbf{C}_l = |\det \mathbf{J}_l| d\phi \end{aligned}$$

$$\cdot |\det(\mathbf{J})| = 2 \frac{1 - \cos \phi}{\phi^2}$$

- Integrating functions of rotations

$$\int_{SO(3)} f(\mathbf{C}) d\mathbf{C} \rightarrow \int_{|\phi| < \pi} f(\phi) |\det \mathbf{J}| d\phi$$

- Similar for poses ( $SE(3)$  and  $\text{Ad}(SE(3))$ )
  - Inner product defined for  $\mathfrak{se}(3)$  ( $4 \times 4$ ) and  $\text{ad}(\mathfrak{se}(3))$  ( $6 \times 6$ )

$$\langle \xi_1^\wedge, \xi_2^\wedge \rangle = -\text{tr} \left( \xi_1^\wedge \begin{bmatrix} \mathbf{I}/2 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{I} \end{bmatrix} \xi_2^{\wedge^T} \right) = \xi_1^T \xi_2$$

$$\langle \xi_1^\wedge, \xi_2^\wedge \rangle = -\text{tr} \left( \xi_1^\wedge \begin{bmatrix} \mathbf{I}/4 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{I}/2 \end{bmatrix} \xi_2^{\wedge^T} \right) = \xi_1^T \xi_2$$

### 7.1.7 Interpolation

- From typical linear interpolation scheme to interpolation of 2 rotations

$$x = (1 - \alpha)x_1 + \alpha x_2 = \alpha(x_2 - x_1) + x_1$$

$$\Downarrow$$

$$\mathbf{C} = (\mathbf{C}_2 \mathbf{C}_1^T)^\alpha \mathbf{C}_1$$

where  $\alpha \in [0, 1]$ ,  $\mathbf{C}, \mathbf{C}_1, \mathbf{C}_2 \in SO(3)$

- Let  $\mathbf{C} = \exp(\phi^\wedge)$ ,  $\mathbf{C}_1 = \exp(\phi_1^\wedge)$ ,  $\mathbf{C}_2 = \exp(\phi_2^\wedge) \in SO(3)$ , with  $\phi, \phi_1, \phi_2 \in \mathbb{R}^3$

$$\begin{aligned} \phi &= \ln(\mathbf{C})^\vee = \ln((\mathbf{C}_2 \mathbf{C}_1^T)^\alpha \mathbf{C}_1)^\vee \\ &= \ln(\exp(\alpha \phi_{21}^\wedge) \exp(\phi_1^\wedge))^\vee \approx \alpha \mathbf{J}(\phi_1)^{-1} \phi_{21} + \phi_1 \end{aligned}$$

- Perturbed rotations ( $SO(3)$ )

- Let  $\mathbf{C}', \mathbf{C}'_1, \mathbf{C}'_2 \in SO(3)$  be the perturbed rotation matrices with left differences given as

$$\delta \phi = \ln(\mathbf{C}' \mathbf{C}^T)^\vee, \quad \delta \phi_1 = \ln(\mathbf{C}'_1 \mathbf{C}_1^T)^\vee, \quad \delta \phi_2 = \ln(\mathbf{C}'_2 \mathbf{C}_2^T)^\vee$$

- Interpolation scheme must hold for the perturbed rotation matrices

$$\mathbf{C}' = (\mathbf{C}'_2 \mathbf{C}'_1^T)^\alpha \mathbf{C}'_1, \quad \alpha \in [0, 1]$$

- Special case:  $\mathbf{C} = \mathbf{C}_2^\alpha$ ,  $\phi = \alpha \phi_2$  when  $\mathbf{C}_1 = \mathbf{I}$
- Perturbation quantity:

$$\delta \phi = (\mathbf{I} - \mathbf{A}(\alpha, \phi_{21})) \delta \phi_1 + \mathbf{A}(\alpha, \phi_{21}) \delta \phi_2$$

where

$$\mathbf{A}(\alpha, \phi_{21}) = \alpha \mathbf{J}(\alpha \phi) \mathbf{J}(\phi)^{-1}$$

$$\phi_{21} = \ln(\mathbf{C}_2 \mathbf{C}_1^T)^\vee$$

- $\mathbf{A}(\alpha, \phi) \approx \alpha \mathbf{I}$  when  $\phi$  is small
- $\mathbf{A}(\alpha, \phi)$  has a series expression which can be used to compute it (only use the first several terms if  $\phi$  is small)
- Similar for perturbed poses ( $SE(3)$ )

### 7.1.8 Homogeneous Points

- Represent points in  $\mathbb{R}^3$  via  $4 \times 1$  *homogeneous coordinates*:

$$\mathbf{p} = \begin{bmatrix} sx \\ sy \\ sz \\ s \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}$$

where  $\boldsymbol{\varepsilon} \in \mathbb{R}^3$ ,  $\eta$  is a scalar

- Define 2 operators for manipulating homogeneous points:

$$\begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^{\odot} = \begin{bmatrix} \eta \mathbf{I}_{3 \times 3} & -\boldsymbol{\varepsilon}_{3 \times 3}^{\wedge} \\ \mathbf{0}_{1 \times 3}^T & \mathbf{0}_{1 \times 3}^T \end{bmatrix}_{4 \times 6}, \quad \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^{\odot} = \begin{bmatrix} \mathbf{0}_{3 \times 1} & \boldsymbol{\varepsilon}_{3 \times 1} \\ -\boldsymbol{\varepsilon}_{3 \times 3}^{\wedge} & \mathbf{0}_{3 \times 1} \end{bmatrix}_{6 \times 4}$$

- Useful identities:
  - $\boldsymbol{\xi}^{\wedge} \mathbf{p} \equiv \mathbf{p}^{\odot} \boldsymbol{\xi}$ ,  $\mathbf{p}^T \boldsymbol{\xi}^{\wedge} \equiv \boldsymbol{\xi}^T \mathbf{p}^{\odot}$  where  $\boldsymbol{\xi} \in \mathbb{R}^6$  and  $\mathbf{p} \in \mathbb{R}^4$
  - $(\mathbf{T}\mathbf{p})^{\odot} \equiv \mathbf{T}\mathbf{p}^{\odot} \boldsymbol{\mathcal{T}}^{-1}$

### 7.1.9 Calculus and Optimization

- Rotations
  - Directly taking Jacobian of a rotated point  $\mathbf{C}\mathbf{v}$  w.r.t. the Lie algebra vector  $\phi$  representing the rotation  $\mathbf{C}$ :

$$\frac{\partial(\mathbf{C}\mathbf{v})}{\partial \phi} = -(\mathbf{C}\mathbf{v})^{\wedge} \mathbf{J}$$

where  $\mathbf{C} = \exp(\phi^{\wedge}) \in SO(3)$ ,  $\mathbf{v} \in \mathbb{R}^3$  is some arbitrary three-dimensional point, and  $\mathbf{J} = \mathbf{J}_l(\phi)$

- For scalar function  $u(\mathbf{x})$  with  $\mathbf{x} = \mathbf{C}\mathbf{v}$ , applying chain rule of differentiation

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \phi} = -\frac{\partial u}{\partial \mathbf{x}} (\mathbf{C}\mathbf{v})^{\wedge} \mathbf{J}$$

The result is the transpose of the gradient  $u$  w.r.t.  $\phi$ .

- $\dim(\partial u_{1 \times 1} / \partial \phi_{3 \times 1}) = 1 \times 3$
- Gradient descent on function  $u(\cdot)$ : update in the direction of negative gradient, evaluated

at linearization point,  $\mathbf{C}_{op} = \exp(\phi_{op}^\wedge)$

$$\begin{aligned}
\phi &= \phi_{op} - \alpha \left( \frac{\partial u}{\partial \phi} \right)^T \\
&= \phi_{op} - \alpha \left( - \frac{\partial u}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\mathbf{C}_{op} \mathbf{v}} (\mathbf{C}_{op} \mathbf{v})^\wedge \mathbf{J} \right)^T \\
&= \phi_{op} + \alpha \mathbf{J}^T (-\mathbf{C}_{op} \mathbf{v})^\wedge \frac{\partial u}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\mathbf{C}_{op} \mathbf{v}}^T \\
&= \phi_{op} - \underbrace{\alpha \mathbf{J}^T (\mathbf{C}_{op} \mathbf{v})^\wedge \frac{\partial u}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\mathbf{C}_{op} \mathbf{v}}^T}_{\delta}
\end{aligned}$$

where  $\alpha > 0$  is the step size

- Cleaner way to carry out optimization: find an update step for  $\mathbf{C}$  in the form of small rotation on the left rather than directly on the Lie algebra rotation vector  $\phi$

$$\begin{aligned}
\mathbf{C} &= \exp(\phi^\wedge) = \exp(\phi_{op} - \alpha \mathbf{J}^T \delta)^\wedge \\
&\approx \exp(-\alpha (\mathbf{J} \mathbf{J}^T \delta)^\wedge) \exp(\mathbf{C}_{op}) \\
&= \exp(\psi^\wedge) \exp(\mathbf{C}_{op})
\end{aligned}$$

- Let  $\psi = -\alpha \mathbf{J} \mathbf{J}^T \delta$
- To avoid computing Jacobian, drop  $\mathbf{J} \mathbf{J}^T > 0$  and use  $\psi = -\alpha \delta$  instead (slightly different direction for gradient descent)
- Compute Jacobian w.r.t  $\psi$ , where the perturbation is applied on the left

$$\frac{\partial(\mathbf{C} \mathbf{v})}{\partial \phi} \approx \frac{\partial(\exp(\psi^\wedge) \mathbf{C} \mathbf{v})}{\partial \psi} = -(\mathbf{C} \mathbf{v})^\wedge$$

- Think optimization in terms of perturbation:
  - Perturbation scheme

$$\begin{aligned}
\mathbf{C} \mathbf{v} &= \exp(\psi^\wedge) \exp(\phi_{op}^\wedge) \mathbf{v} \\
&= f(\phi_{op} \oplus \psi) \\
&\approx f(\phi_{op}) + \frac{\partial f(\phi_{op})}{\partial \psi} \psi \\
&= \exp(\phi_{op}^\wedge) \mathbf{v} - (\exp(\phi_{op}^\wedge) \mathbf{v})^\wedge \psi \\
&= \mathbf{C}_{op} \mathbf{v} - (\mathbf{C}_{op} \mathbf{v})^\wedge \psi
\end{aligned}$$

where  $\oplus$  denotes the perturbation operation, and 1st-order Taylor series is used for the approximation

- Insert the scheme into the function to be optimized

$$\begin{aligned}
u(\mathbf{C}\mathbf{v}) &= u(\exp(\boldsymbol{\psi}^\wedge)\mathbf{C}_{op}\mathbf{v}) \\
&\approx u(\mathbf{C}_{op}\mathbf{v} - (\mathbf{C}_{op}\mathbf{v})^\wedge\boldsymbol{\psi}) \\
&\approx u(\mathbf{C}_{op}\mathbf{v}) - \underbrace{\frac{\partial u}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{C}_{op}\mathbf{v}}}_{\boldsymbol{\delta}^T} (\mathbf{C}_{op}\mathbf{v})^\wedge\boldsymbol{\psi} \\
&= u(\mathbf{C}_{op}\mathbf{v}) + \boldsymbol{\delta}^T\boldsymbol{\psi}
\end{aligned}$$

- An example perturbation vector for gradient descent:

$$\boldsymbol{\psi} = -\alpha\mathbf{D}\boldsymbol{\delta}$$

where  $\alpha > 0$  a small step size and  $\mathbf{D}$  is any positive-definite matrix

- Update step:

$$\mathbf{C}_{op} \leftarrow \exp(-\alpha\mathbf{D}\boldsymbol{\delta}^\wedge)\mathbf{C}_{op}$$

where  $\mathbf{C}_{op} \in SO(3)$  is guaranteed at each iteration

- Perturbation applied to Gauss-Newton optimization method
  - An example general nonlinear, quadratic cost function of rotation

$$J(\mathbf{C}) = \frac{1}{2} \sum_m (u_m(\mathbf{C}\mathbf{v}_m))^2$$

where  $u_m(\cdot)$  are scalar nonlinear functions and  $\mathbf{v}_m \in \mathbb{R}^3$  are three-dimensional points

- Perturb on an initial guess for the optimal rotation  $\mathbf{C}_{op} \in SO(3)$ :

$$\mathbf{C} = \exp(\boldsymbol{\psi}^\wedge)\mathbf{C}_{op}$$

where  $\boldsymbol{\psi}$  is the perturbation

- Apply perturbation scheme inside each  $u_m(\cdot)$  to get linearized version of  $u_m(\cdot)$  in terms of perturbation  $\boldsymbol{\psi}$

$$u_m(\mathbf{C}\mathbf{v}_m) \approx \underbrace{u_m(\mathbf{C}_{op}\mathbf{v}_m)}_{\beta_m} \underbrace{\frac{\partial u_m}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\mathbf{C}_{op}\mathbf{v}_m}}_{\boldsymbol{\delta}_m^T} (\mathbf{C}_{op}\mathbf{v}_m)^\wedge\boldsymbol{\psi}$$

and then

$$J(\mathbf{C}) \approx \frac{1}{2} \sum_m (\boldsymbol{\delta}_m^T\boldsymbol{\psi} + \beta_m)^2$$

- Take the derivative of  $J$  w.r.t  $\boldsymbol{\psi}$

$$\frac{\partial J}{\partial \boldsymbol{\psi}^T} = \sum_m \boldsymbol{\delta}_m (\boldsymbol{\delta}_m^T\boldsymbol{\psi} + \beta_m)$$

- Set derivative to 0 to find the optimal perturbation  $\psi^*$  that minimizes  $J$

$$\left( \sum_m \delta_m \delta_m^T \right) \psi^* = - \sum_m \beta_m \delta_m$$

- Apply the optimal perturbation for each iteration

$$\mathbf{C}_{op} \leftarrow \exp(\psi^{*\wedge}) \mathbf{C}_{op}$$

where  $\mathbf{C}_{op} \in SO(3)$  is guaranteed for each iteration

- Poses

- The Jacobian of a transformed point w.r.t the Lie algebra vector  $\xi$  representing the transformation:

$$\frac{\partial(\mathbf{T}\mathbf{p})}{\partial \xi} = (\mathbf{T}\mathbf{p})^\odot \mathcal{J}$$

- The Jacobian w.r.t. the perturbation  $\epsilon$  where it is applied on the left of the transformation (i.e. the (left) Lie derivative)

$$\frac{\partial(\mathbf{T}\mathbf{p})}{\partial \epsilon} = (\mathbf{T}\mathbf{p})^\odot \quad \text{if} \quad \mathbf{T} \leftarrow \exp(\epsilon^\wedge) \mathbf{T}$$

where the need to compute  $\mathcal{J}$  matrix is removed

- Gauss-Newton optimization is similar to rotations by using the above perturbation scheme

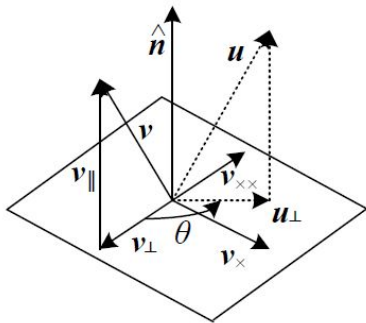
# Miscellaneous

## 1. Basics

### 1.1 3D Rotation

#### 1.1.1 Rotation Representations

- a.  $\mathbf{R}$  itself
  - $3 \times 3$  matrix
- b. Axis/angle
  - Represent rotation by rotation axis  $\hat{\mathbf{n}}$  and angle  $\theta$
  - Derivation of  $\mathbf{R}(\hat{\mathbf{n}}, \theta)$ :



- A rotation from  $\mathbf{v}$  to  $\mathbf{u}$  based on rotation axis  $\hat{\mathbf{n}}$  and rotation angle  $\theta$
  - $[\hat{\mathbf{n}}]_{\times}$ : matrix form of cross product operator with vector  $\hat{\mathbf{n}} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$
  - $[\hat{\mathbf{n}}]_{\times} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix}$
  - $\mathbf{u} = \mathbf{u}_{\perp} + \mathbf{v}_{\parallel} = \mathbf{R}(\hat{\mathbf{n}}, \theta)\mathbf{v}$
  - $\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2$ 
    - Known as *Rodrigues' formula*
- c. Quaternions
    - 4-vector:  $\mathbf{q} = (q_x, q_y, q_z, q_w)$  or  $\mathbf{q} = (x, y, z, w)$
    - Unit quaternions: live on the sphere  $\|\mathbf{q}\| = 1$
    - *Antipodal* (opposite sign) quaternions  $\mathbf{q}$  and  $-\mathbf{q}$  represent the same rotation
    - Representation is *continuous*: rotation matrices vary continuously
    - Quaternions can be derived from the axis/angle representation:
      - $\mathbf{q} = (\mathbf{v}, w) = (\sin \frac{\theta}{2} \hat{\mathbf{n}}, \cos \frac{\theta}{2})$  where  $\hat{\mathbf{n}}$  and  $\theta$  are rotation axis and angle
    - Convert Rodrigues' formula using double-angle formulas:

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin \theta [\hat{\mathbf{n}}]_{\times} + (1 - \cos \theta) [\hat{\mathbf{n}}]_{\times}^2 = \mathbf{I} + 2w[\mathbf{v}]_{\times} + 2[\mathbf{v}]_{\times}^2$$

$$\cdot \mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - zw) & 2(xz + yw) \\ 2(xy + zw) & 1 - 2(x^2 + z^2) & 2(yz - xw) \\ 2(xz - yw) & 2(yz + xw) & 1 - 2(x^2 + y^2) \end{bmatrix}$$

$$\text{where } \mathbf{v} = (x, y, z) \text{ and } [\mathbf{v}]_{\times} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

- Quaternion multiply operator:

$$\mathbf{q}_2 = \mathbf{q}_0 \mathbf{q}_1 = (\mathbf{v}_0 \times \mathbf{v}_1 + w_0 \mathbf{v}_1 + w_1 \mathbf{v}_0, w_0 w_1 - \mathbf{v}_0 \cdot \mathbf{v}_1)$$

$$\cdot \mathbf{R}(\mathbf{q}_2) = \mathbf{R}(\mathbf{q}_0) \mathbf{R}(\mathbf{q}_1)$$

- Multiply operator is *not commutative*

- Quaternion division: just flip the sign of  $\mathbf{v}$  or  $w$  in the multiply operator

$$\mathbf{q}_2 = \mathbf{q}_0 / \mathbf{q}_1 = \mathbf{q}_0 \mathbf{q}_1^{-1} = (\mathbf{v}_0 \times \mathbf{v}_1 + w_0 \mathbf{v}_1 - w_1 \mathbf{v}_0, -w_0 w_1 - \mathbf{v}_0 \cdot \mathbf{v}_1)$$

d. Euler angles

- 3 angles to describe the orientation of a rigid body w.r.t. a fixed coordinate system

- $\alpha, \beta, \gamma$ ; or  $\phi, \theta, \psi$

- 2 groups of angles with 12 possible sequences of rotation axes

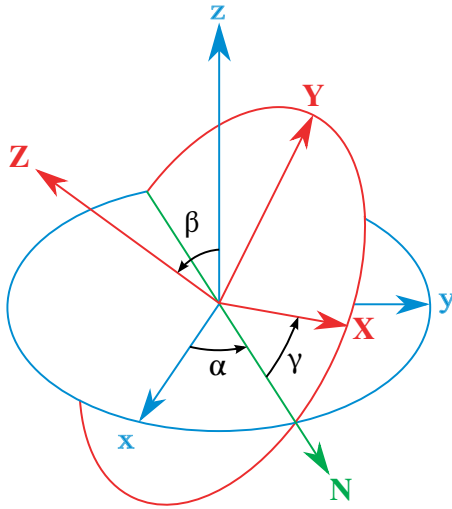
(1) **Proper/Classic Euler angles:**  $z$ - $x$ - $z$ ,  $x$ - $y$ - $x$ ,  $y$ - $z$ - $y$ ,  $z$ - $y$ - $z$ ,  $x$ - $z$ - $x$ ,  $y$ - $x$ - $y$

(2) **Tait-Bryan angles:**  $x$ - $y$ - $z$ ,  $y$ - $z$ - $x$ ,  $z$ - $x$ - $y$ ,  $x$ - $z$ - $y$ ,  $z$ - $y$ - $x$ ,  $y$ - $x$ - $z$

- Also called: **Cardan angles; nautical angles; heading, elevation, and bank; yaw, pitch, and roll**

- Proper Euler angles

- Geometrical definition (one possibility):



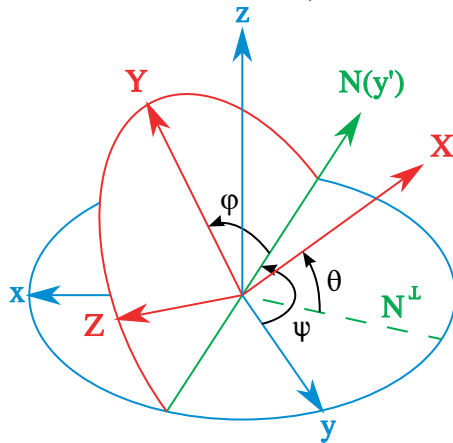
- $x, y, z$ : axes of the original frame
- $X, Y, Z$ : axes of the rotated frame
- $N = z \times Z$  (i.e., the intersection of planes  $xy$  and  $XY$ )
- $\alpha, \beta, \gamma$ : Euler angles

- Definition by intrinsic rotations

- Intrinsic rotations:  $XYZ$  system rotates after each elementary rotation;  $xyz$  system is fixed
- Rotated frame  $XYZ$  from initial orientation to final orientation:
  - Initial:  $x$ - $y$ - $z$  or  $x_0$ - $y_0$ - $z_0$



- After 1st rotation:  $x'-y'-z'$  or  $x_1-y_1-z_1$
- After 2nd rotation:  $x''-y''-z''$  or  $x_2-y_2-z_2$
- Final (after 3rd rotation):  $X-Y-Z$  or  $x_3-y_3-z_3$
- Definition of Euler angles:
  - $\alpha$  (or  $\phi$ ): a rotation around the  $z$  axis
  - $\beta$  (or  $\theta$ ): a rotation around the  $x'$  axis
  - $\gamma$  (or  $\psi$ ): a rotation around the  $z''$  axis
- Definition by extrinsic rotations
  - Extrinsic rotations: elemental rotations occur about the axes of fixed coordinate system  $xyz$
  - Target orientation can be achieved by rotating  $\gamma$ ,  $\beta$ , and  $\alpha$  around  $z$ ,  $x$ ,  $z$  axes respectively
- Tait-Bryan angles
  - Difference with proper Euler angles: Tait-Bryan angles represent rotations about 3 distinct axes (e.g.  $x-y-z$  or  $x-y'-z''$ ), while proper Euler angles use same axis for both 1st & 3rd elemental rotations (e.g.  $z-x-z$  or  $z-x'-z''$ )
  - Geometrical definition (one possibility):



- Different line of nodes compared with proper Euler angles
  - Intersection of planes  $xy$  and  $YZ$  for Tait-Bryan angles vs.  $xy$  and  $XY$  for proper Euler angles
- Rotation sequence:  $z-y'-x''$ 
  - Intrinsic rotations;  $N$  coincides with  $y'$
  - Known as **yaw, pitch, and roll**
- Angle rotation sequence:  $\psi, \theta, \phi$

### 1.1.2 Conversions Between Different Rotation Representations

- References:
  - [wiki page](#)
- Rotation matrix decomposition ([wiki](#)):
  - Any orientation can be achieved by composing 3 elemental rotations
  - Example:  $\mathbf{R} = \mathbf{X}(\alpha)\mathbf{Y}(\beta)\mathbf{Z}(\gamma)$ 
    - 2 possible representations
    - (1) A composition of **extrinsic** rotations about axes  $z-y-x$

(2) A composition of **intrinsic** rotations about axes  $x$ - $y'$ - $z''$

· Rotation matrix from all 12 kinds of Euler angles:

For the sake of simplicity, the following table uses the following nomenclature:

- 1, 2, 3 represent the angles  $\alpha$ ,  $\beta$  and  $\gamma$ , i.e. the angles corresponding to the first, second and third elemental rotations respectively.
- $X$ ,  $Y$ ,  $Z$  are the matrices representing the elemental rotations about the axes  $x$ ,  $y$ ,  $z$  of the fixed frame (e.g.,  $X_1$  represents a rotation about  $x$  by an angle  $\alpha$ ).
- $s$  and  $c$  represent sine and cosine (e.g.,  $s_1$  represents the sine of  $\alpha$ ).
- Each matrix is denoted by the formula used to calculate it. If  $R = Z_1 X_2 Z_3$ , we name it  $Z_1 X_2 Z_3$ .

Proper Euler angles	Tait-Bryan angles
$X_1 Z_2 X_3 = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$X_1 Z_2 Y_3 = \begin{bmatrix} c_2 c_3 & -s_2 & c_2 s_3 \\ s_1 s_3 + c_1 c_3 s_2 & c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 \\ c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 \end{bmatrix}$
$X_1 Y_2 X_3 = \begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$X_1 Y_2 Z_3 = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 \\ s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$Y_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 \\ c_2 s_3 & c_2 c_3 & -s_2 \\ c_1 s_2 s_3 - c_3 s_1 & c_1 c_3 s_2 + s_1 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1 Z_2 Y_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$Y_1 Z_2 X_3 = \begin{bmatrix} c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 \\ s_2 & c_2 c_3 & -c_2 s_3 \\ -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 \end{bmatrix}$
$Z_1 Y_2 Z_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$	$Z_1 Y_2 X_3 = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 s_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix}$
$Z_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$	$Z_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 \\ c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 \\ -c_2 s_3 & s_2 & c_2 c_3 \end{bmatrix}$

To change the formulas for [passive rotations](#) (or find reverse active rotation), transpose the matrices (then each matrix transforms the initial coordinates of a vector remaining fixed to the coordinates of the same vector measured in the rotated reference system; same rotation axis, same angles, but now the coordinate system rotates, rather than the vector).

· Rotation matrix to yaw, pitch, roll in Eigen ([ref](#)):

· `R.eulerAngles(2, 1, 0).reverse;`

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