RESEARCH PROPOSAL

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1. Background

I study higher rank Teichmuller theory, a subfield of differential geometry which re-examines the classical theory of riemann surfaces, and aims to generalize as much as possible from $\mathrm{PSL}_2\mathbb{R}$ to other Lie groups. The jump from $\mathrm{PSL}_2\mathbb{R}$ to other Lie groups is quite nontrivial and each aspect seems to generalize in a different way. After over 30 years of discovery, there are still many aspects of classical Teichmuller theory that may have higher rank versions yet to be discovered. The field is very focused on specific concrete problems, but is also relevant to mathematics as a whole because, like its predecessor, higher rank Teichmuller theory is a crossroads for differential geometry, algebraic geometry, analysis, dynamics, and theoretical physics.

My research is on generalizing two fundamental constructions in Techmuller theory: the identification of Teichmuller space, the space of complex structures on a surface, with a component of the $\mathrm{PSL}_2\mathbb{R}$ character variety via the uniformization theorem; and Thurston's compactification of Teichmuller space by projective measured laminations. In both cases, I restrict attention to the quintessential examples of higher Teichmuller spaces: $\mathrm{PSL}_n\mathbb{R}$ Hitchin components.

Definition 1. *The* $\mathrm{PSL}_n\mathbb{R}$ *Hitchin component,* $\mathrm{Hit}^n(S)$*, for an oriented surface* S *of genus at least 2, is the component of* $\mathrm{Rep}(\pi_1(S), \mathrm{PSL}_n\mathbb{R})$ *containing the compositions of the holonomy representations for hyperbolic structures* $\pi_1 S \to \mathrm{PSL}_2\mathbb{R}$ *with the symmetric power* $\mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_n\mathbb{R}$*.*

Hitchin planted the seed of higher Teichmüller theory in 1992 [15] by showing that this component resembles ordinary Teichmüller space in a striking way: $\mathrm{Hit}^n(S)$ is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$. For both higher uniformization, and higher Thurston compactification, I describe here my recent results, work in progress, and broader research goals.

2. Higher rank Thurston Compactification

Let $\mathcal{L}(S)$ denote the set of homotopy classes of closed curves in S, and let $\mathcal{T}(S)$ be Teichmüller space, thought of as the space of hyperbolic structures on S. Taking marked length spectra gives a natural embedding

$$\mathcal{T}(S) \to \mathbb{R}^{\mathcal{L}(S)}$$

and Thurston's compactification is the closure of $\mathcal{T}(S)$ after projecting to $\mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$. He showed that this compactifiaction is a closed ball, and interpreted the sphere at infinity as parametrizing projective measured laminations. He used this compactification to great effect, for instance using a Brower fixed point argument to give a qualitative classification of automorphisms of surfaces [31]. Measured laminations have since become an indispensable tool in the theory of surfaces.

One can compactify $\mathrm{Hit}^n(S)$ in the same way by replacing the geodesic length of a loop γ with the logarithm of the top eigenvalue

$$\log|\lambda_1(\rho(\gamma))|$$

where $\rho \in \mathrm{Hit}^n(S)$ is a Hitchin representation. This is sometimes called the spectral radius compactification; here we opt for the shorter name λ_1-compactification and refer to the boundary as $\partial_{\lambda_1}\mathrm{Hit}^n(S)$. Our mission is to understand what geometric structures $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is parametrizing. We a have a complete understanding of part of $\partial_{\lambda_1}\mathrm{Hit}^3(S)$ [30], and promising results towards a general theory for all n.

Once this new geometric interpretation of $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is more developed, it will provide a new approach to some important open questions, in particular whether $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is a sphere, and whether there is a Fock-Goncharov duality map for closed surfaces.

The first observation leading to Thurston's compactification is that a degenerating hyperbolic surface, re-scaled to have diameter 1, can converge to a metric graph, in which case the length spectrum converges to geometric intersection number with a weighted simple multicurve. In general, the length spectra of a sequence in $\mathcal{T}(S)$ leaving all compact sets subconverges to geometric intersection number with a measured lamination. We can take the preimage of this measured lamination in the universal cover, and construct a dual metric $\mathbb{R}$-tree X with $\pi_1 S$ action. Geometric intersection numbers with the geodesic lamination become translation lengths of $\pi_1 S$ acting on X. The quotient $X/\pi_1 S$ is generally non-Hausdorff, but when X is an ordinary tree, $X/\pi_1 S$ is the aforementioned graph. We aim to construct metric spaces which play the role of this $\mathbb{R}$-tree for boundary points of $\mathrm{Hit}^n(S)$.

2.1. **New behaviour for** $\mathrm{SL}_3\mathbb{R}$**.** There is a natural family of asymmetric Finsler metrics with triangular unit balls which I recently demonstrated arise in $\partial_{\lambda_1}\mathrm{Hit}^3(S)$.

Definition 2. *Let μ be a cubic differential on a Riemann surface C; that is, a holomorphic section of $(T^*C)^{\otimes 3}$. We define F^Δ_μ to be the maximum of twice the real parts of the cube roots of μ.*

$$F^\Delta_\mu(v) := \max_{\{\alpha\in T^*_xC:\alpha^3=\mu_x\}} 2Re(\alpha(v))$$

Here $x \in C$ is a point, and $v \in T_xC$ is a tangent vector.

We call these Δ-Finsler metrics, and call the space of them up to isotopy $\Delta\,\mathrm{Fins}(S)$. My work uses the identification of $\mathrm{Hit}^3(S)$ with the space of convex projective structures on S [6], and the parametrization of convex projective structures by pairs (J, μ) where J is a complex structure, and μ is a holomorphic cubic differential constructed by Labourie [21] and Loftin [23] using the hyperbolic affine spheres constructed by Cheng and Yau [5]. We show that when J_i converges and μ_i diverges, the $\log|\lambda_1|$-spectrum becomes the length spectrum for a Δ-Finsler metric.

Theorem 1 (Theorem A in [30])**.** *Let μ_i be a sequence of cubic differentials on a smooth oriented surface S of genus at least 2, each holomorphic with respect to some complex structure, such that $a_i^3\mu_i$ converges uniformly to μ, for some sequence of positive real numbers a_i tending to 0. Let $\gamma \in \pi_1(S)$. Let $F^\Delta_\mu(\gamma)$ denote the infimal length of loops representing γ in the Finsler metric F^Δ_μ.*

$$\lim_{i\to\infty} a_i \log(\lambda_1(\rho_i(\gamma))) = F^\Delta_\mu(\gamma)$$

This is a strengthening, and altenative formulation of a very similar recent result by Loftin, Tamburelli, and Wolf [22] using somewhat different methods. The result is also similar to Parreau's construction of metric spaces corresponding to certain Fock-Goncharov tropical points [29]. The proof uses a π_1-invariant version of the Funk metric [11] studied by Danciger and Stecker, and 2D affine sphere analysis modeled on [26].

Theorem 1 shows that taking marked length spectrum is a continuous map from the sphere bundle $\Delta\,\mathrm{Fins}(S)/\mathbb{R}_+$ to $\partial_{\lambda_1}\mathrm{Hit}^n(S)$.

Conjecture 1. $\Delta\,\mathrm{Fins}(S)/\mathbb{R}_+$ *is embedded in $\partial_{\lambda_1}\mathrm{Hit}^3(S)$, and comprises a dense open subset. [30]*

The first claim is motivated by the fact that marked length spectrum is a very strong invariant in the setting of negative curvature; for instance negatively curved Riemannian metrics on closed surfaces are determined by their marked length spectra [27]. The second claim is founded on a similar result [28] for a different compactification. I now have an outline of a procedure for reconstructing a Δ-Finsler metric from its length spectrum, and surprisingly the key construction works for the whole λ_1-boundary of $\mathrm{Hit}^n(S)$ for any n, so this has grown into a much larger project.

2.2. **Work in progress for** $\mathrm{PSL}_n\mathbb{R}$**.** The construction is based on *geodesic currents* which were introduced [4] to give a complete, concise treatment of Thurston's compactification. A geodesic current is simply a $\pi_1 S$ invariant Radon measure on $\mathcal{G} = \partial\pi_1 S \times \partial\pi_1 S\backslash\Delta$, the space of distinct pairs of points in the Gromov boundary. Let $\mathcal{C}$ be the space of geodesic currents on S. Bonohon defined a natural embedding of $\mathcal{T}(S)$ into $\mathcal{C}$, and showed that after projecting to $\mathbb{P}(\mathcal{C})$, the boundary consists of symmetric currents with zero self intersection.

Labourie [20] generalized Bonohon's embedding to $\mathrm{Hit}^n(S)$ using limit maps. A Hitchin representation $\rho : \pi_1(S) \to GL(V)$ determines equivariant limit maps $\xi : \partial\pi_1 S \to \mathbb{P}(V)$ and $\xi^* : \partial\pi_1 S \to \mathbb{P}(V^*)$ such that the point $\xi(x)$ is on the hyperplane $\xi^*(y)$ only when $x = y$. Pulling back the natural symplectic structure on $\mathbb{P}(V) \times \mathbb{P}(V^*) \backslash \Delta$ via $\xi \times \xi^*$ gives a measure μ_ρ on $\mathcal{G}$. The space $\mathbb{P}(\mathcal{C})$ of projective geodesic currents is compact, so we get a compactification of $\mathrm{Hit}^n(S)$ for each n which we call $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$.

Most likely, $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S) = \partial_{\lambda_1}\mathrm{Hit}^n(S)$. This conjecture is based on the fact that both the current, and the λ_1 spectrum are complete invariants of the *flow* associated to a representation defined in [20], so they determine each other. I am working on extending the flow construction to the boundary of the Hitchin component, so one can conclude that the λ_1-spectra of a sequence of representations converges if and only if the sequence of currents converges. It would be quite surprising if there was a discrepancy, but any case, $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$ is also a natural compactification.

Labourie not only showed how to embed Hit^n into $\mathcal{C}$, he also characterized the image in terms of vanishing and non-vanishing of certain determinants. At the boundary, these equations tropicalize, imposing combinatorial conditions on currents in $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$.

For $SL_2\mathbb{R}$ the conditions are transparent: symmetry and non-self-intersection, but for higher n they become much more complicated. From my work with $\mathrm{SL}_3\mathbb{R}$, I learned that Lioville currents of Δ-Finsler metrics must be examples of boundary points, giving me a foot in the door to start gleaning geometric meaning from these tropical determinant relations.

This lead to the definition of a metric space X_μ, for $[\mu] \in \partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n$ which is a space of certain partitions $\mu = \mu^+ + \mu^-$ into "clockwise" and "counterclockwise" submeasures. The definition is inspired by the observation that if one puts a negatively curved metric on S, each point $p \in \tilde{S}$ gives a partition of $\mathcal{G}$ by direction of rotation of oriented geodesics around p. The following result shows that X_μ is much nicer than one should expect.

Theorem 2. *For any $[\mu] \in \partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$ such that μ has discrete support, X_μ is a polyhedral complex of dimension less than or equal to $n-1$.*

This follows directly from the tropical determinant condition after setting things up correctly. I am in the process of making this result into a more complete story which I hope to finish in the coming months. The ideal result is the following. Let l and μ be the renormalized limiting $\log|\lambda_1|$-spectrum and geodesic current respectively of a sequence in $\mathrm{Hit}^n(S)$ escaping to infinity.

Project 1. *Show that there is a (generally asymmetric) metric on X_μ such that the translation length of γ is $l(\gamma)$. Show that if $n = 2$, X_μ is an $\mathbb{R}$ tree, and if $n = 3$, X_l is a union of Δ-Finsler surface and $\mathbb{R}$-tree regions.*

So far, I have a natural symmetric metric on X_l whose translation lengths are $l(\gamma) + l(\gamma^{-1})$, and have checked the low rank descriptions in several examples. There are two clear next steps for this line of research.

Project 2. *Construct a version of X_μ with a Weyl cone valued metric corresponding to the compactification of $\mathrm{Hit}^n(S)$ which takes into account all eigenvalues λ_k. Generalize to other split real Lie groups.*

Project 3. *Characterize exactly what metric spaces arise as X_μ. This will entail a combinatorial classification of local models, and a way of constructing a path of representations converging to a space built from these local models.*

There are several exciting potential applications of the X_μ construction. First, one can look for a higher rank version of the identification between measured laminations and holomorphic quadratic differentials for a fixed complex structure on S. Quadratic differentials should be replaced with the Hitchin base. Optimistically, This could lead to a proof that $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is a sphere analogous to Wolf's proof for $\mathrm{SL}_2\mathbb{R}$ [32].

Secondly, it could help us find the Langlands dual regular function to an integral tropical point of $\mathrm{Hit}^n(S)$. The Fock-Goncharov duality conjecture [9], corrected and proven in [13], states that in the setting of character varieties of decorated punctured surfaces, (some) integral tropical points index canonical regular functions on the Langlands dual character variety, but nothing is known for closed surfaces in higher rank. For $n = 3$ we conjectured that integral Δ-Finsler metrics correspond to certain web functions.

Lastly, the construction of X_μ seems adaptable to higher dimensions, and could be used to investigate limits of degenerating families of higher dimensional convex projective manifolds. This could reveal a lot about the topology of convex projective manifolds. This is inspired by the beautiful discoveries of Morgan, Shalen and Culler [7][24] using tropical limit points of $\mathrm{SL}_2\mathbb{C}$ character varieties of 3-manifolds.

3. Higher Uniformization

By the uniformization theorem, a Riemann surface of genus at least 2 admits a unique hyperbolic metric compatible with the complex structure, which endows its fundamental group with a representation into $\mathrm{PSL}_2\mathbb{R}$. In this way, the space of complex structures up to isotopy, $\mathcal{T}(S)$, is identified with a component of $\mathrm{Rep}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$. The fact that $\mathcal{T}(S)$ simultaneously parameterizes complex structures and real representations gives rise to extra structure. For instance, the natural complex structure on $\mathcal{T}(S)$ combines with the natural symplectic structure on $\mathrm{Rep}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ to give the Weil-Peterson Kähler metric.

A $\mathrm{PSL}_n\mathbb{R}$ version will involve new differential geometric structures that can be put on surfaces which have a correspondence with $\mathrm{PSL}_n\mathbb{R}$ representations. A hyperbolic metric on a Riemann surface is a solution to a nonlinear PDE, and we expect the higher rank version to also involve nonlinear PDE. Recently, new objects called *n-complex structures* have been proposed, and we have recently made very promising progress towards identifying these with Hitchin representations.

There was a lot of excitement around the idea of complex analytic higher Teichmuller space in the early 1990s in the conformal field theory community centering around $\mathcal{W}$-algebras, [1][2][17][12]. I found Witten's talk for the 1990 strings conference while prepairing this research statement, and was amazed to find how much of the story he had predicted. $\mathcal{W}_n$-algebras are examples of vertex operator algebras, certain algebraic structures which can be extracted from a 2D conformal field theory. Other vertex operator algebras have well known relationships with the complex analytic moduli spaces $Bun_G(C)$, and $\mathcal{M}_g$, which are both related to real character varieties, via the Narasimhan-Seshadri, and Uniformization theorems respectively. As soon as physicists heard of Hitchin's work with $\mathrm{PSL}_n\mathbb{R}$, they conjectured that he had found "$\mathcal{W}_n$-moduli space". The mathematical understanding of $\mathcal{W}$-algebras has progressed immensely in the last few decades but there is still no concrete mathematical proposal I am aware of for what $\mathcal{W}$-moduli space is.

In 2018 V. Fock and A. Thomas [10] introduced new objects called *n-complex structures* and conjectured that they are parameterized by $\mathrm{PSL}_n\mathbb{R}$ Hitchin components. A higher complex is an infinitesimal thickening of the zero section of the complexified cotangent bundle of S, which one considers up to Hamiltonian diffeomorphisms of T^*S. The moduli space $\mathcal{T}^n(S)$ of n complex structures is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$.

I found, the following reformulation of higher complex structures which is entirely complex analytic, even algebro-geometric.

Theorem 3. *n-complex structures modulo higher diffeomorphisms are in bijection with order $n-2$ symplectic thickenings.*

I define a symplectic thickening to be an infinitesimal thickening of a complex curve C with a Poisson bracket on its structure sheaf, defined modulo a power of the nil-radical. The first examples are truncated cotangent bundles, i.e. a k-th order infinitesimal thickening of the zero-section of C in T^*C. These have non-trivial deformations owing to the failure of the Weinstien neighborhood theorem in complex geometry. The moduli space $\mathcal{S}_g^k$ of order k symplectic thickenings is an iterated affine bundle over the moduli of complex curves $\mathcal{M}_g$. It has a $\mathbb{C}^*$ action by scaling the Poisson bracket whose fixed locus is the truncated cotangent bundles.

Though symplectic thickenings have their historical roots in $\mathcal{W}$-algebras, it not at all clear what the mathematical relationship is, but now that we have an algebraic moduli space, we are in a good position to collaborate with vertex algebra experts and work out if $\mathcal{S}_g^k$ is the $\mathcal{W}$-moduli space in any sense.

For each symplectic thickening T of C I also construct a moduli space F_T of connected length $k+1$ subschemes transverse to C, and a nilpotent Higgs bundle on F_T. F_T is an iterated affine bundle over C, and one obtains a higher complex structure by choosing a smooth section. Isotopies between sections give higher diffeomorphisms. Since F_T is homotopy equivalent to C, solving the Hitchin equation would give a representation of $\pi_1 C$. Unfortunately since F_T is non-compact, current technology doesn't tell us weather there is a unique solution to the Hitchin equation. Though we haven't shown this Higgs bundle has a solution yet, it is worth thinking about what a solution would look like:

Question 1. *If we have a solution to the Hitchin equation on the natural Higgs bundle on F_T, does the kernel of the Higgs field define a local diffeomorphism from F_T to $\mathbb{CP}^n$? If so, is the image related to the limit curve of the representation?*

This would relate to the other interpretation of higher uniformization [18], [14], [8] involving higher dimensional locally homogeneous manifolds. The early $\mathcal{W}$-moduli literature, [2], [12] also predicts a relationship to $\mathbb{CP}^{n-1}$ structures.

Even though symplectic thickenings give a more complete conceptual picture, it might be easier to solve the PDE directly in terms of higher complex structures. This is because the Hitchin equation is overdetermined on manifolds of complex dimension greater than 1 so we can expect uniqueness in solving the relevant equation on a smooth section of F_T. Together with A. Thomas and G. Kydonakis, we have very promising results in this direction.

Theorem 4 (Theorem 5.6 in [19])**.** *There is a canonical map from a neighborhood of $\mathcal{T}(S)$ in $\mathcal{T}^n(S)$ to $\mathrm{Hit}^n(S)$.*

The map can be characterized in 2 sentences: A higher complex structure can be encoded in what we call a Fock bundle (E, g, Φ), where E is a complex vector bundle, g is a complex symmetric pairing, and Φ is a smooth $\mathrm{End}(E)$ valued 1-form whose coefficients are commuting self-adjoint nilpotent matrices at each point. We show that for all compatible hermitian metrics h, there is a unique unitary connection A_h preserving g such that $d_{A_h}\Phi = 0$, and conjecture that there exists a unique h such that

$$F_{A_h} + [\Phi \wedge \Phi^{*_h}] = 0 \tag{1}$$

, i.e. $A_h + \Phi + \Phi^{*_h}$ is a flat connection. We were able to use ellipticity properties of this equation to solve it for small perturbations of Φ. Equation 1 is similar to Hitchin's equation, but the solution method will have to be different because Φ doesn't satisfy a differential equation.

There are several clear next steps in this higher uniformization program:

Project 4. *Show uniqueness and existance for equation 1.*

Project 5. *Investigate how and when one can split a given real connection into the form $A_h + \Phi + \Phi^{*_h}$, thus recovering a higher complex structure.*

Project 6. *Ascertain whether a Kähler structure appears on the locus where Hitchin representations are parametrized by higher complex structures.*

Project 6 is work in progress with A. Thomas and G. Kydonakis. We expect the Kähler structure based on an infinite dimensional Kähler reduction picture, but it is unclear at this point what the signature is.

3.1. **Deformations of theories of class S.** There is a close relationship between character varieties of surfaces and certain supersymmetric field theories. For every compact Lie group G, and Riemann surface C there is a $4D$ $\mathcal{N} = 2$ supersymmetric theory $\mathcal{T}(C, G)$ referred to as a theory of class S. Teichmuller space thus gives a family of theories for each G. In [25] it was proposed that higher complex structures also give deformations of theories of class S. While deformations of complex structure give *marginal* deformations of the theory, meaning deformations which are equally relevant at all length scales, deformations of higher complex structure should give *irrelevant* deformations. Here,"irrelevant" means that these deformations disappear at large scales, but they can completely change the theory at small scales.

This interpretation suggests that there is a lot more to the story of higher complex structures. It has long been understood that the moduli of G Higgs bundles on C is the moduli of Vacua of the $3D$ $\mathcal{N} = 4$ theory obtained from $\mathcal{T}(G, C)$ by reducing on a circle. If higher complex structures truly also correspond to theories, then these theories should also have moduli of vacua. Translating back to mathematics, there should be a notion of a Higgs bundle on a surface with higher complex structure, and the moduli space of these objects should be hyperkähler.

3.2. **Hyperkähler thickenings.** There is a larger more conjectural picture which could explain why symplectic thickenings are appearing in the context of character varieties. Taking n to infinity, one might imagine infinitesimal symplectic thickenings being replaced by genuine holomorphic symplectic thickenings. To my surprise, this already featured in Witten's 1990 talk, in which he conjectured that the space of holomorphic symplectic thickenings is the $\mathcal{W}_\infty$-moduli space. In a sense, finding a Hyperkähler metric on T^*C where C is a Riemann surface is like solving Hitchin's equation for an infinite dimensional Higgs bundle. This idea appeared in [16] where Hitchin studied hyperkähler metrics on disk bundles in T^*C with "fold" singularities on a the boundary, though the idea was already communicated to Witten in 1990.

There is a unique $U(1)$ invariant hyperkähler metric on the unit disk bundle in T^*C which restricts to the hyperbolic metric on C, and it has a fold singularity on the unit cotangent bundle. Buiquard showed [3] that for arbitrary small variations of the holomorphic-symplectic structure, and arbitrary small displacements of the unit cotangent bundle, there is a unique compatible folded hyperkähler metric.

Theorems 3 and 4, suggest, in my view, that hyperkähler thickenings are not only an $n = \infty$ analogue of $\mathrm{Hit}^n(S)$, but also have a direct relationship with $\mathrm{Hit}^n(S)$. Evidence for this suspicion comes from the Labourie's observation that Hitchin representations can be encoded in a deformation of the geodesic flow of a hyperbolic surface, and the fact that such a flow appears naturally on the boundary of a hyperkähler thickening as Reeb flow for one of the symplectic structures.

The biggest missing piece of this puzzle is how a representation of $\pi_1 C$ should relate to the hyperkähler structure. There are various types of bundles with connection on hyperkähler four manifolds which could be the answer. Whatever it is, it would answer the question from physics of "what is a Higgs bundle for a higher complex structure". Certain moduli of sheaves on K3 surfaces are well known to be hyperkähler, so it is reasonable to expect that certain moduli of sheaves on hyperkähler 4-manifolds with boundary are also hyperkähler, given appropriate boundary conditions.

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