RESEARCH STATEMENT

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1. Background

I study higher rank Teichmüller theory, a subfield of differential geometry which re-examines the classical theory of Riemann surfaces, and aims to generalize as much as possible from $\mathrm{PSL}_2\mathbb{R}$ to other Lie groups. The jump from $\mathrm{PSL}_2\mathbb{R}$ to other Lie groups is quite nontrivial and each aspect seems to generalize in a different direction. The field is focused on specific concrete problems, and is also relevant to mathematics as a whole because, like ordinary Teichmüller theory, higher rank Teichmüller theory is a crossroads for differential geometry, algebraic geometry, analysis, dynamics, and theoretical physics.

My research aims to provide higher rank versions of two fundamental constructions in Teichmüller theory: (i) Thurston's compactification of Teichmüller space by projective measured laminations, and (ii) the identification of Teichmüller space, the space of complex structures on a surface, with a component of the $\mathrm{PSL}_2\mathbb{R}$ character variety via the uniformization theorem. In both cases, I restrict attention to the quintessential examples of higher Teichmüller spaces: $\mathrm{PSL}_n\mathbb{R}$ Hitchin components.

Definition 1. *The* $\mathrm{PSL}_n\mathbb{R}$ *Hitchin component,* $\mathrm{Hit}^n(S)$*, for an oriented surface* S *of genus at least 2, is the component of* $\mathrm{Rep}(\pi_1 S, \mathrm{PSL}_n\mathbb{R})$ *containing the compositions of the holonomy representations for hyperbolic structures* $\pi_1 S \to \mathrm{PSL}_2\mathbb{R}$ *with the irreducible projective representation* $\mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_n\mathbb{R}$.

Hitchin planted the seed of higher Teichmüller theory in [Hit92] by showing that this component resembles ordinary Teichmüller space in a striking way: $\mathrm{Hit}^n(S)$ is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$. In Section 2, I discuss a Thurston compactification for $\mathrm{Hit}^n(S)$, and progress towards a geometric understanding of this compactification. In Section 3, I discuss progress towards a higher rank uniformization theorem which would identify $\mathrm{Hit}^n(S)$ with a moduli space of complex analytic objects.

2. Higher rank Thurston Compactification

Let $\mathcal{L}(S)$ denote the set of homotopy classes of closed curves in S, and let $\mathcal{T}(S)$ be the space of hyperbolic structures on S up to isotopy. Each hyperbolic structure gives a function $l : \mathcal{L}(S) \to \mathbb{R}$, called the marked length spectrum, which assigns to each homotopy class the length of the geodesic representative. Marked length spectrum gives an embedding

$$\mathcal{T}(S) \to \mathbb{R}^{\mathcal{L}(S)}$$

and Thurston's compactification is the closure of $\mathcal{T}(S)$ after projecting to $\mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$. Thurston showed that this compactifiaction is a closed ball, and interpreted the sphere at infinity as parametrizing projective measured laminations. He used this compactification to great effect—for instance he proved a qualitative classification of automorphisms of surfaces [Thu88] using a fixed point argument. Measured laminations have since become an indispensable tool in the theory of surfaces.

One can compactify $\mathrm{Hit}^n(S)$ in the same way by replacing the geodesic length of a loop γ with the logarithm of the eigenvalue of greatest magnitude

$$\log|\lambda_1(\rho(\gamma))|$$

where $\rho \in \mathrm{Hit}^n(S)$ is a Hitchin representation. This is sometimes called the spectral radius compactification; here we opt for the shorter name λ_1-compactification and refer to the boundary as $\partial_{\lambda_1}\mathrm{Hit}^n(S)$. My goal is to understand what geometric structures $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is parametrizing. **We have an understanding of part of $\partial_{\lambda_1}\mathrm{Hit}^3(S)$ [Rei23], and initial results towards a general theory for all n.**

Once this geometric interpretation of $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is more developed, it will provide a new approach to some important open questions, in particular whether $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is a sphere, and whether there is a Fock-Goncharov duality map for closed surfaces. The latter would give a natural basis of functions on $\mathrm{Hit}^n(S)$ generalizing the basis of trace functions on $\mathcal{T}(S)$ indexed by simple closed multicurves.

One can arrive at Thurston's compactification starting from the observation that a sequence of hyperbolic surfaces, re-scaled to have diameter 1, can converge to a metric graph. In this case, the length spectrum converges to geometric intersection number with a weighted simple multicurve. This describes a dense subset of Thurston's compactification. In general, a divergent sequence in $\mathcal{T}(S)$ converges in the compactification if the renormalized length spectra converge to geometric intersection number with a measured lamination. We can take the preimage of this measured lamination in the universal cover, and construct a dual metric $\mathbb{R}$-tree X with $\pi_1 S$ action. Geometric intersection numbers with the geodesic lamination become translation lengths of $\pi_1 S$ acting on X. The quotient $X/\pi_1 S$ is generally non-Hausdorff, but when X is an ordinary tree, $X/\pi_1 S$ is the aforementioned graph. We aim to construct metric spaces which play the role of this $\mathbb{R}$-tree for boundary points of $\mathrm{Hit}^n(S)$.

2.1. **New behaviour for** $\mathrm{SL}_3\mathbb{R}$**.** There is a natural family of asymmetric Finsler metrics with triangular unit balls which I recently demonstrated arise in $\partial_{\lambda_1}\mathrm{Hit}^3(S)$.

Definition 2. *Let μ be a cubic differential on a Riemann surface C; that is, a holomorphic section of $(T^*C)^{\otimes 3}$. We define the Finsler metric F_μ^Δ by*

$$F_\mu^\Delta(v) := \max_{\{\alpha \in T_x^*C : \alpha^3 = \mu_x\}} 2Re(\alpha(v))$$

where $x \in C$ is a point, and $v \in T_x C$ is a tangent vector.

We call F_μ^Δ a Δ-Finsler metric, and denote the space of Δ-Finsler metrics up to isotopy by $\Delta\,\mathrm{Fins}(S)$. My work uses the identification of $\mathrm{Hit}^3(S)$ with the space of convex projective structures on S [CG93], and the parametrization of convex projective structures by pairs (J, μ), where J is a complex structure and μ is a holomorphic cubic differential, constructed by Labourie [Lab07b] and Loftin [Lof01] using the hyperbolic affine spheres of Cheng and Yau [CY77]. We show that when J_i converges and μ_i diverges, the $\log|\lambda_1|$-spectrum becomes the length spectrum for a Δ-Finsler metric.

Theorem 1 (Theorem A in [Rei23])**.** *Let μ_i be a sequence of cubic differentials on a smooth oriented surface S of genus at least 2, each holomorphic with respect to some complex structure, such that $a_i^3\mu_i$ converges uniformally to μ, for some sequence of positive real numbers a_i tending to 0. Let $\gamma \in \pi_1 S$. Let $F_\mu^\Delta(\gamma)$ denote the infimal length of loops representing γ in the Finsler metric F_μ^Δ.*

$$\lim_{i\to\infty} a_i \log|\lambda_1(\rho_i(\gamma))| = F_\mu^\Delta(\gamma)$$

This is a strengthening, and alternative formulation of a similar recent result by Loftin, Tamburelli, and Wolf [LTW22], proved using somewhat different methods. The result is also similar to Parreau's construction of metric spaces corresponding to certain Fock-Goncharov tropical points [Par15]. The proof uses a π_1-invariant version of the Funk metric [Fun30] studied by Danciger and Stecker, and 2D affine sphere analysis modeled on [Nie22].

A corollary of Theorem 1 is that the projective λ_1-spectrum embedding $\mathrm{Hit}^3(S) \to \mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$ extends continuously to the sphere bundle $\Delta\,\mathrm{Fins}(S)/\mathbb{R}_+$. By considering Benoist limit cones [Ben97], one checks that length spectra of Δ-Finsler metrics, and $\log|\lambda_1|$-spectra of Hitchin representaions are disjoint, so $\Delta\,\mathrm{Fins}(S)/\mathbb{R}_+$ maps to $\partial_{\lambda_1}\mathrm{Hit}^n(S)$.

Conjecture. *$\Delta\,\mathrm{Fins}(S)/\mathbb{R}_+$ is embedded in $\partial_{\lambda_1}\mathrm{Hit}^3(S)$, and comprises a dense open subset. [Rei23]*

The first claim is motivated by the fact that marked length spectrum is a very strong invariant in the setting of negative curvature; for instance negatively curved Riemannian metrics on closed surfaces are determined by their marked length spectra [Ota90]. The second claim is motivated by a similar result [OT21] for a different compactification. I now have an outline of a procedure for reconstructing a Δ-Finsler metric from its length spectrum. Surprisingly, the key construction works for the whole λ_1-boundary of $\mathrm{Hit}^n(S)$ for any $n \geq 2$, so it has inspired a much larger project.

2.2. **Work in progress for** $\mathrm{PSL}_n\mathbb{R}$**.** The construction is based on *geodesic currents* which were introduced in [Bon88] to give a complete, concise treatment of Thurston's compactification. A geodesic current is simply a $\pi_1 S$-invariant Radon measure on $\mathcal{G} = (\partial\pi_1 S \times \partial\pi_1 S)\backslash\Delta$, the space of distinct pairs of points in the Gromov boundary. Let $\mathcal{C}$ be the space of geodesic currents on S. Bonohon defined a natural embedding of $\mathcal{T}(S)$

into $\mathcal{C}$, and showed that after projecting to $\mathbb{P}(\mathcal{C})$, the boundary consists of symmetric currents with zero self intersection, i.e. measured laminations.

Labourie [Lab07a] generalized Bonohon's embedding to $\mathrm{Hit}^n(S)$ using limit maps. A Hitchin representation $\rho : \pi_1 S \to \mathrm{GL}(V)$ determines equivariant limit maps $\xi : \partial\pi_1 S \to \mathbb{P}(V)$ and $\xi^* : \partial\pi_1 S \to \mathbb{P}(V^*)$ such that the point $\xi(x)$ is on the hyperplane $\xi^*(y)$ if and only if $x = y$. Pulling back the natural symplectic structure on $\mathbb{P}(V) \times \mathbb{P}(V^*)\backslash\Delta$ via $\xi \times \xi^*$ gives a measure μ_ρ on $\mathcal{G}$. The space $\mathbb{P}(\mathcal{C})$ of projective geodesic currents is compact, so we get a compactification of $\mathrm{Hit}^n(S)$ for each n which we call $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$.

Most likely, $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S) = \partial_{\lambda_1}\mathrm{Hit}^n(S)$, but as far as I am aware this is not known. This conjecture is based on the fact that both the current and the $\log|\lambda_1|$-spectrum are complete invariants of the flow associated to a representation defined in [Lab07a], so they determine each other. I am working on extending the flow construction to the boundary of the Hitchin component, so one can conclude that the λ_1-spectra of a sequence of representations converges if and only if the sequence of currents converges. It would be quite surprising if there was a discrepancy, but in any case, $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$ is also a natural compactification.

Labourie not only showed how to embed $\mathrm{Hit}^n(S)$ into $\mathcal{C}$, he also characterized the image in terms of vanishing and non-vanishing of certain determinants. At the boundary, these equations tropicalize, imposing combinatorial conditions on currents in $\partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$.

For $n = 2$ the conditions are transparent: symmetry and non-self-intersection, but for higher n they become much more complicated. From my work with $\mathrm{SL}_3\mathbb{R}$, I learned that Liouville currents of Δ-Finsler metrics must be examples of boundary points, giving me a foot in the door to start gleaning geometric meaning from these tropical determinant relations.

This led to the definition of a metric space X_μ, for $[\mu] \in \partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n$ which is a space of partitions $\mu = \mu^+ + \mu^-$ into "clockwise" and "counterclockwise" submeasures. The definition is inspired by the observation that if one puts a negatively curved metric on S, each point $p \in \tilde{S}$ gives a partition of oriented geodesics by direction of rotation of around p. The following result shows that X_μ is much nicer than one might expect.

Theorem 2. *For any $[\mu] \in \partial^{\mathcal{C}}_{\lambda_1}\mathrm{Hit}^n(S)$ such that μ has discrete support, X_μ is a polyhedral complex of dimension less than or equal to $n-1$.*

This follows from the tropical determinant condition. I am in the process of making this result into a more complete story. Let l and μ be the renormalized limit $\log|\lambda_1|$-spectrum and geodesic current respectively of a sequence in $\mathrm{Hit}^n(S)$ escaping to infinity.

Project 1. *Show that there is a (generally asymmetric) metric on X_μ such that the translation length of γ is $l(\gamma)$. Show that if $n = 2$, X_μ is an $\mathbb{R}$-tree, and if $n = 3$, X_l is a union of Δ-Finsler surface and $\mathbb{R}$-tree regions.*

So far, I have defined a symmetric metric on X_l whose translation lengths are $l(\gamma) + l(\gamma^{-1})$, and have checked the low rank descriptions in several examples. There are two clear next steps after project 1.

Project 2. *Construct a version of X_μ with a Weyl cone valued metric corresponding to a point in the compactification of* $\mathrm{Hit}^n(S)$ *which takes into account all eigenvalues λ_k. Generalize to other split real Lie groups.*

Project 3. *Characterize exactly what metric spaces arise as X_μ. This will entail a combinatorial classification of local models, and a way of constructing a path of representations converging to a space built from these local models.*

There are several exciting potential applications of the X_μ construction. First, one can look for a higher rank version of the identification of measured laminations with holomorphic quadratic differentials for a fixed complex structure on S. Quadratic differentials should be replaced with the Hitchin base. Optimistically, this could lead to a proof that $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is a sphere, analogous to Wolf's proof for $n = 2$ [Wol89].

Secondly, it could aid in the search for the Langlands dual regular function to an integral tropical point of $\mathrm{Hit}^n(S)$. The Fock-Goncharov duality conjecture [FG06], corrected and proven in [Gro+18], states that in the setting of character varieties of decorated punctured surfaces, (some) integral tropical points index canonical regular functions on the character variety for the Langlands dual group, but nothing is known for closed surfaces in higher rank. For $n = 3$ I conjectured that integral Δ-Finsler metrics correspond to certain web functions.

Lastly, the construction of X_μ seems adaptable to higher dimensions, and could be used to investigate limits of degenerating families of higher dimensional convex projective manifolds. Morgan, Shalen and Culler made beautiful topological discoveries using tropical limit points of $\mathrm{SL}_2\mathbb{C}$ character varieties of 3-manifolds. Similarly, tropical limit points of the moduli space of convex projective manifolds could reveal topological properties of these manifolds.

3. Higher Uniformization

By the uniformization theorem, a Riemann surface of genus at least 2 admits a unique hyperbolic metric compatible with the complex structure, which endows its fundamental group with a representation into $\mathrm{PSL}_2\mathbb{R}$. In this way, the space of complex structures up to isotopy, $\mathcal{T}(S)$, is identified with a component of $\mathrm{Rep}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$. The fact that $\mathcal{T}(S)$ simultaneously parameterizes complex structures and real representations gives rise to extra structure. For instance, the natural complex structure on $\mathcal{T}(S)$ combines with the natural symplectic structure on $\mathrm{Rep}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ to give the Weil-Peterson Kähler metric.

A $\mathrm{PSL}_n\mathbb{R}$ version will involve new differential geometric structures that can be put on surfaces which have a correspondence with $\mathrm{PSL}_n\mathbb{R}$ representations. A hyperbolic metric on a Riemann surface is a solution to a nonlinear PDE, and we expect the higher rank version to also involve nonlinear PDE. In 2018, new objects called *n-complex structures* were proposed, and we have recently proved results towards identifying these with Hitchin representations [KRT23].

3.1. **Historical Context.**

In the early 1990s, there was excitement around the idea of complex analytic higher Teichmüller space in the conformal field theory community, centering around $\mathcal{W}_n$-algebras [Wit90] [BFK91][Hul93][GJ95]. I found the notes from Witten's talk at the Strings '90 conference while preparing this research statement, and was surprised by how much of the story he had foreseen. $\mathcal{W}_n$-algebras are examples of vertex operator algebras, certain algebraic structures which can be extracted from a 2D conformal field theory. Other vertex operator algebras have well known relationships with the complex analytic moduli spaces $\mathrm{Bun}_G(C)$, and $\mathcal{M}_g$, which are both related to real character varieties, via the Narasimhan-Seshadri, and uniformization theorems respectively. As soon as physicists learned of Hitchin's work with $\mathrm{PSL}_n\mathbb{R}$, they conjectured that $\mathrm{Hit}^n(S)$ is "$\mathcal{W}_n$-moduli space", but as far as I know a precise complex analytic definition of $\mathcal{W}_n$-moduli space never appeared. Though higher complex structures have their historical roots in $\mathcal{W}_n$-algebras, it not yet clear what the mathematical relationship is. In upcoming work, I show that higher complex structures are equivalent to algebraic objects, namely *symplectic thickenings*, which could provide an avenue to reconcile higher complex structures with their $\mathcal{W}_n$-algebra origins.

There was another well-known program to construct a complex structure on $\mathrm{Hit}^n(S)$, based on Labourie's conjecture that for each Hitchin representation there is a unique equivariant minimal surface in the symmetric space [Lab06]. Now that this conjecture has been disproven [SS22], there is added motivation for the approach via higher complex structures, though before our paper [KRT23], the approach had not been formulated precisely.

3.2. **Higher complex structures and symplectic thickenings.**

In 2018, V. Fock and A. Thomas [FT21] introduced new objects called *n-complex structures* and conjectured that they are parameterized by $\mathrm{PSL}_n\mathbb{R}$ Hitchin components. A higher complex structure is an infinitesimal thickening of the zero section of the complexified cotangent bundle of S, which one considers up to Hamiltonian diffeomorphisms of T^*S. The moduli space $\mathcal{T}^n(S)$ of n-complex structures is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$.

I found the following reformulation of higher complex structures which is entirely complex analytic, even algebro-geometric.

Theorem 3 (In preparation)**.** *n-complex structures modulo higher diffeomorphisms are in bijection with order $n-2$ symplectic thickenings.*

A symplectic thickening is an infinitesimal holomorphic symplectic thickening of a complex curve, which I define precisely as follows. For a scheme T, let $\mathfrak{n}_T$ denote its nilradical, and T_{red} denote its reduced locus.

Definition 3. *An order n symplectic thickening is a scheme T over $\mathbb{C}$ such that T_{red} is a smooth curve, $(\mathfrak{n}_T)^{n+1} = 0$, and $(\mathfrak{n}_T)^k/(\mathfrak{n}_T)^{k+1}$ are invertable sheaves for $1 \le k \le n$, together with an antisymmetric bilinear map $\{\cdot,\cdot\} : \mathcal{O}_T \times \mathcal{O}_T \to \mathcal{O}/(\mathfrak{n}_T)^n$ of sheaves on T_{red} which satisfies the Leibniz rule, satisfies the Jaccobi identity, and descends to a symplectic form on the cotangent space to each closed point of T_{red}.*

The first examples are truncated cotangent bundles, i.e. a k-th order infinitesimal thickening of the zero-section of C in T^*C. These are not the only examples, owing to the failure of the Weinstien neighborhood theorem in complex geometry. The moduli space $\mathcal{S}_g^k$ of order k symplectic thickenings is an iterated affine bundle over the moduli of complex curves $\mathcal{M}_g$. It has a $\mathbb{C}^*$ action by scaling the Poisson bracket whose fixed locus parametrizes the truncated cotangent bundles.

For each symplectic thickening T of C, I also construct a moduli space F_T of connected length $k+1$ subschemes transverse to C, and a nilpotent Higgs bundle on F_T. F_T is an iterated affine bundle over C, and one obtains a higher complex structure by choosing a smooth section. Isotopies between sections give higher diffeomorphisms. Since F_T is homotopy equivalent to C, solving the Hitchin equation would give a representation of $\pi_1 C$. Unfortunately since F_T is non-compact, current technology doesn't tell us whether there is a solution to the Hitchin equation. Nonetheless, we could ask what a solution would look like:

Question. *Given a solution to the Hitchin equation on the natural Higgs bundle on F_T, the kernel of the Higgs field defines a map from the universal cover of F_T to $\mathbb{CP}^n$. Is this a local diffeomorphism? Is the image related to the Anosov limit curve in the sense of Labourie [Lab06]?*

This would relate to the other interpretation of higher uniformization [KLP18][GW12][DS21] involving higher dimensional locally homogeneous manifolds. The early $\mathcal{W}_n$-moduli literature, [BFK91][GJ95], also predicts a relationship to $\mathbb{CP}^{n-1}$ structures.

3.3. **Higher complex structures and Hitchin components.** Even though symplectic thickenings give a more complete conceptual picture, it has so far been easier to solve the PDE directly in terms of higher complex structures. This is because the Hitchin equation is overdetermined on manifolds of complex dimension greater than 1, so we can expect uniqueness in solving the relevant equation on a smooth section of F_T. This is the approach we take with A. Thomas and G. Kydonakis.

Theorem 4 (Theorem 5.6 in [KRT23])**.** *There is a canonical map from a neighborhood of $\mathcal{T}(S)$ in $\mathcal{T}^n(S)$ to $\mathrm{Hit}^n(S)$.*

Of course, it is the nature of the map that is interesting, not its mere existence. A higher complex structure can be encoded in what we call a Fock bundle (E, g, Φ), where E is a complex vector bundle, g is a complex symmetric pairing, and Φ is a smooth $\mathrm{End}(E)$ valued 1-form whose coefficients are commuting self-adjoint nilpotent endomorphisms at each point. We show that for all compatible Hermitian metrics h, there is a unique unitary connection A_h preserving g such that $d_{A_h}\Phi = 0$, and conjecture that there exists a unique h such that

$$F_{A_h} + [\Phi \wedge \Phi^{*_h}] = 0 \tag{1}$$

which is equivalent to $A_h + \Phi + \Phi^{*_h}$ being a flat connection. We were able to use ellipticity properties of this equation to solve it for small perturbations of Φ. Equation (1) is similar to Hitchin's equation, but the solution method will have to be different because Φ does not satisfy an elliptic equation.

In [KRT23] we also find an explanation for why higher diffeomorphic higher complex structures should map to isomorphic flat connections. Just as for solutions to the Hitchin equation, a solution to (1) gives not just one connection, but a whole $\mathbb{C}^*$-family: $\nabla(\lambda) = \lambda^{-1}\Phi + A_h + \lambda\Phi^{*_h}$. We show that Hamiltonian deformations of the higher complex structure induced by Φ come from applying λ-dependent gauge transformations to $\nabla(\lambda)$. More specifically, the Hamiltonian deformation coming from a function on T^*S which is a polynomial of vector fields $v_1 \cdots v_k$ is realized by the infinitesimal gauge transformation $\lambda^{-1}\eta + \lambda\eta^{*_h}$ where $\eta = \Phi(v_1)\cdots\Phi(v_k)$.

There are several clear next steps in this higher uniformization program:

Project 4. *Show uniqueness and existence for Equation (1).*

Project 5. *Investigate how and when one can split a given real connection into the form $A_h + \Phi + \Phi^{*_h}$, thus recovering a higher complex structure.*

Project 6. *Ascertain whether a Kähler structure appears on the locus where Hitchin representations are parametrized by higher complex structures.*

Project 6 is work in progress with A. Thomas and G. Kydonakis. We expect the Kähler structure based on an infinite dimensional Kähler reduction picture, but it is unclear at this point what the signature is.

3.4. **Deformations of theories of class S.** There is a parallel story in physics which has deeply influenced my thinking about higher uniformization, and which illuminates the longer term outlook of the project. For every ADE Lie algebra $\mathfrak{g}$, and Riemann surface C, there is a $4D$ $\mathcal{N} = 2$ supersymmetric theory $S[\mathfrak{g}, C]$ referred to as a theory of class S. Teichmüller space thus gives a family of theories for each $\mathfrak{g}$. In [NS21] it was proposed that higher complex structures also give deformations of theories of class S. While deformations of complex structure give *marginal* deformations of the theory, meaning deformations which are equally relevant at all length scales, deformations of higher complex structure should give *irrelevant* deformations. Here, "irrelevant" means that these deformations disappear at large scales, but they can completely change the theory at small scales.

The realization of higher complex structures as symplectic thickenings fits with a picture introduced to me by A. Neitzke, in which $S[\mathfrak{g}, C]$ arises in M-theory compactified on T^*C as the worldvolume theory for an $M5$-brane supported on the zero section. From this picture it is natural that deformations of T^*C should give deformations of $S[\mathfrak{g}, C]$.

This interpretation of higher complex structures suggests that there is a lot more to the story. It has long been understood that the moduli of G Higgs bundles on C is the moduli of vacua of the $3D$ $\mathcal{N} = 4$ theory obtained from $S[\mathfrak{g}, C]$ by reducing on a circle. If higher complex structures truly also correspond to theories, then these theories should also have moduli of vacua. Translating back to mathematics, there should be a notion of a Higgs bundle on a surface with higher complex structure, and the moduli space of these objects should be hyperkähler.

3.5. **Hyperkähler thickenings.** I end with a more speculative picture which could explain why symplectic thickenings are appearing in the context of character varieties, and could help answer the question raised by the interpretation of higher complex structures as deformations of class S theories.

Taking n to infinity, one might imagine infinitesimal symplectic thickenings being replaced by genuine holomorphic symplectic thickenings. Holomorphic symplectic thickenings already featured in Witten's Strings '90 talk, in which he conjectured that they are parametrized by $\mathcal{W}_\infty$-moduli space. In a sense, finding a hyperkähler metric on T^*C where C is a Riemann surface is like solving Hitchin's equation for an infinite dimensional Higgs bundle. This idea appeared in [Hit16], where Hitchin studied hyperkähler metrics on disk bundles in T^*C with "fold" singularities on the boundary, though Hitchin had already communicated the basic idea to Witten in 1990.

There is an explicit $U(1)$-invariant hyperkähler metric on the unit disk bundle in T^*C which restricts to the hyperbolic metric on C, and it has a fold singularity on the unit cotangent bundle. Biquard showed [Biq19] that for arbitrary small variations of the holomorphic-symplectic structure, and arbitrary small displacements of the unit cotangent bundle, there is a unique compatible deformation of folded hyperkähler structure.

Theorems 3 and 4, suggest, in my view, that hyperkähler thickenings are not only an $n = \infty$ analogue of $\mathrm{Hit}^n(S)$, but also have a direct relationship with $\mathrm{Hit}^n(S)$. Evidence for this suspicion comes from Labourie's observation [Lab07a] that Hitchin representations can be encoded in a deformation of the geodesic flow of a hyperbolic surface, and the fact that such a flow appears naturally on the boundary of a hyperkähler thickening as Reeb flow for one of the symplectic structures.

The biggest missing piece of this puzzle is how a representation of $\pi_1 C$ should relate to the hyperkähler structure. There are various types of objects on hyperkähler four manifolds which could be the answer. Whatever it is, it would answer the question from physics of what a Higgs bundle should be on a surface with higher complex structure. Moduli of certain sheaves on K3 surfaces are well known to be hyperkähler, so it is reasonable to expect that moduli of certain sheaves on hyperkähler 4-manifolds with boundary are also hyperkähler, given appropriate boundary conditions.

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