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Boundaries of Hitchin Components

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Boundaries of Hitchin Components

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Abstract

Boundaries of Hitchin Components

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We study the spectral radius compactification of Hitchin components of $\mathrm{PSL}_n \mathbb{R}$ character varieties of surfaces. For $n = 2$ the Hitchin component is simply Teichmuller space, and the compactification is Thurston's compactification. For $n > 2$, a modular interpretation of the boundary points is still lacking. Chapter 2 studies the case $n = 3$ using convex projective geometry and analysis of affine spheres to estimate top eigenvalues along paths to infinity parametrized by rays of cubic differentials. Exponential growth rates are shown to be lengths with respect to singular, flat, Finsler metrics with triangular unit balls. In Chapter 3, for each boundary point of the compactification we construct a metric space with $\pi_1(S)$ action. For $n = 2$ this metric space is an $\mathbb{R}$-tree, for rational boundary points it is a polyhedral complex of dimension at most $n - 1$, and for $n = 3$, for limit points of cubic differential rays, it is the universal cover of S with the symmetrization of the Finsler metric from Chapter 2.

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Chapter 1: Introduction

This thesis is a compilation of two papers, and this introduction will introduce a few background concepts, but we start with a back road to boundaries of Hitchin components.

A vector bundle with flat connection on a manifold M has no local invariants–it looks exactly the same near every point. Consider however a family of flat connections $\nabla(\lambda)$ such that

$$\lim_{\lambda \to \infty} \frac{\nabla(\lambda)}{\lambda} = \Phi$$

where Φ is an endomorphism valued 1-form. As λ approaches ∞, local structure appears, like water crystalizing as it cools. Wherever Φ is diagonalizable, its eigenvalues are a tuple of closed 1-forms. More globally, one obtains some kind of Lagrangian in the cotangent bundle of M.

Suppose that $\nabla(\lambda)$ are connections preserving a volume element so that the eigenvalues of Φ sum to zero, and that the rank of the vector bundle is one more than the dimension of M. Wherever the eigenvalues are linearly independant, they can be integrated to give distinguished local coordinates on M. As the eigenvalues of Φ collide and permute, some intricate geometric structure is induced on M. The leading asymptotics of the holonomy of $\nabla(\lambda)$ should be computable from this structure.

One needs to be much more specific to start proving theorems. One well studied instance is Schrödinger operators [KT05]

$$-\frac{d^2}{dz^2} + \lambda^2 Q(z)$$

where $Q(z)$ is a holomorphic function. This operator corresponds to a connection on a rank 2 vector bundle, and the asymptotics of solutions in the limit $\lambda \to \infty$ are governed by the (1/2)-translation structure given by the quadratic differential $Q(z)dz^2$. In Chapter 2 we study a different situation where this story plays out out nicely. A Riemann surface S with cubic differential α determines, via solving Hitchin's equation for a certain Higgs bundle, a flat connection $\nabla(\alpha)$ on a real rank 3 vector bundle. Taking a limit

$$\Phi := \lim_{R\to\infty} \frac{\nabla(R^3\alpha)}{R}$$

we find [Maz+16] that the eigenvalues of Φ are the twice the real parts of the cube roots of α. These three 1-forms locally give an identification with $\mathbb{R}^2$, though they cyclically permute around zeros of α. The maximum of these three one forms is a Finsler metric which governs the leading asymptotics of holonomy (Theorem 3.50).

Instead of looking for natural families of connections in higher dimensions to study, one could turn to a systematic way of studying divergent families of flat connections: compactifications of character varieties. The spectral radius compactification studied in Chapter 3 just keeps track of asymptotics of holonomy by definition, but the geometry of a $n-1$ dimensional manifold with n 1-forms which sum to zero emerges again on its own.

1.1 Thurston's compactification of Teichmüller space

Perhaps the most well studied compactification of a space of flat connections is Thurston's compactification of Teichmüller space [Thu88] [FLP12]. Let S be a closed oriented surface of genus at least 2, and let $\Gamma := \pi_1(S)$ be its fundamental group. A discrete and faithful representation $\Gamma \to \mathrm{PSL}_2\mathbb{R}$ gives rise to an oriented hyperbolic surface $\Gamma\backslash\mathbb{H}^2$ marked by S. In this way, a connected component of the $\mathrm{PSL}_2\mathbb{R}$ character variety is identified with Teichmüller space $\mathcal{T}(S)$ which is diffeomorphic to $\mathbb{R}^{6g-6}$.

Free homotopy classes of closed curves in S are in bijection with the set $[\Gamma]$ of conjugacy classes in Γ. Each hyperbolic structure gives a function $l : [\Gamma] \to \mathbb{R}$, called the marked length spectrum, which assigns to each homotopy class the length of the geodesic representative. Marked length spectrum gives an embedding

$$\mathcal{T}(S) \to \mathbb{R}^{[\Gamma]}$$

and Thurston's compactification is the closure of $\mathcal{T}(S)$ after projecting to $\mathbb{P}(\mathbb{R}^{[\Gamma]})$. The boundary points can be interpreted as parametrizing geometric structures, namely measured foliations, measured laminations, or metric $\mathbb{R}$-trees with Γ-action, and the compactification makes $\mathcal{T}(S)$ into a closed ball. The same idea can be used to define compactifications of other components of character varieties, but geometric interpretations of boundary points, and topological understandings of the boundaries are only known in a few cases.

1.2 Analogous compactifications of Hitchin components

For $n \geq 2$, composition with the unique irreducible representation $\mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_n\mathbb{R}$, gives an embedding of $\mathcal{T}(S)$ into the $\mathrm{PSL}_n\mathbb{R}$ character variety of S. The *Hitchin component* $\mathrm{Hit}^n(S)$ is the component of the $\mathrm{PSL}_n\mathbb{R}$ character variety containing this copy of $\mathcal{T}(S)$. Quite surprisingly, $\mathrm{Hit}^n(S)$ is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$ [Hit92]. Representations in the Hitchin component also share some properties with holonomy representations of hyperbolic structures. They are discrete and faithful, and have all real eigenvalues [Lab06b].

If $\rho : \pi_1 S \to \mathrm{PSL}_2\,\mathbb{R}$ comes from a hyperbolic structure, then the hyperbolic length of $[\gamma]$ is the logarithm of the absolute value of the larger eigenvalue of $\rho(\gamma)$. By analogy with the hyperbolic case, the marked length spectrum $l_\rho \in \mathbb{R}^{[\Gamma]}$ of a representation $\rho \in \mathrm{Hit}^n(S)$ is defined [BCL20] to be

$$l_\rho([a]) = \log|\lambda_1(\rho(a))|.$$

The map $\mathrm{Hit}^n(S) \to \mathbb{P}(\mathbb{R}^{[\Gamma]})$ given by $\rho \mapsto [l_\rho]$ is an embedding with compact closure. The closure is called the spectral radius compactification because $|\lambda_1(\rho(a))|$ is the spectral radius of $\rho(a)$. Note that $\log|\lambda_1(\rho(a))|$ is approximately the same as $\log|\operatorname{tr}(\rho(a))|$ and these give the same compactification.

An alternative to the spectral radius compactification would be to record asymptotics of all n eigenvalues. Equivalently, one could measure $\log |\operatorname{tr}(\nu(\rho(a)))|$ a set of finite dimensional irreducible representations ν of $\mathrm{SL}_n \mathbb{R}$ whose highest weights generate the cone of dominant weights over $\mathbb{R}$, for instance the antisymmetric powers. This clearly generalizes to other groups. The resulting compactification is called the Weyl chamber length compactification [Par12]. The Weyl chamber length compactification seems in some ways more natural than the spectral radius compactification, though it is the same for $n = 2$ and $n = 3$ and as far as I know it is unknown if it is the same for $n > 3$. Parreau showed that

1.3 Asymptotics of nonabelian Hodge theory

Hitchin showed that $\mathrm{Hit}^n(S)$ is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$ using the nonabelian Hodge correspondence, a tool which has proved extremely powerful for answering questions about the topology and geometry of the moduli space of flat connections. The machine is set in motion by choosing a complex structure on S. For every irreducible representation $\pi_1 S \to SL_n \mathbb{C}$ there is a unique harmonic map to the associated symmetric space $SL_n \mathbb{C} / SU_n$. In bundle language, for every complex vector bundle with irreducible flat connection (E, ∇) with trivial determinant, there is a unique unit volume hermitian metric on E giving a decomposition

$$\nabla = A + \Phi$$

where A is a unitary connection and Φ is a hermitian matrix valued one form, such that $\phi := \Phi^{1,0}$ is a holomorphic matrix valued 1-form. The $(0,1)$ part of A gives E a holomorphic structure. The pair (E, ϕ) where E is considered as a holomorphic vector bundle is called a Higgs bundle. In the reverse direction, if (E, ϕ) is a Higgs bundle with $det(E)$ trivial which is "stable" than there is a unique hermitian metric h on E such that $D_h + \phi + \phi^{*_h}$ is a flat connection, where here D_h is the Chern connection. Equivalently, there is a unique solution h to the Hitchin equation

$$F_{D_h} + [\phi \wedge \phi^{*_h}] = 0.$$

Together, these theorems give a bijection between irreducible flat connections and stable Higgs bundles.

Hitchin showed that $\mathrm{Hit}^n(S)$ is diffeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$ by defining an explicit family of Higgs bundles parametrized by $(\alpha_2, ..., \alpha_n) \in H^0(S, K^2) \oplus ... \oplus H^0(S, K^n)$, where K is the cannonical bundle, and showing that under the nonabelian Hodge correspondence, this family of Higgs bundles fills out a component of the real character variety.

A Higgs bundle (E, ϕ) with $\phi \neq 0$ determines a ray of Higgs bundles $\{(E, R\phi) : R > 0\}$, which gives rise to a path of representations ρ_R. Conjecturally, [GMN13] for every group element $a \in \Gamma$,

$$\log|\operatorname{tr}(\rho_R(a))| \sim l(a)R$$

for some growth rate $l(a)$ which depends in a calculable, but complicated way on a and (E, ϕ), at least when ϕ is sufficiently generic. In other words, ρ_R should converge in the spectral radius compactification, and we should be able to calculate the limit point without solving a partial differential equation. In Chapter 2 we show that this calculation is actually the simplest possible thing when (E, ϕ) is the SL_3 Higgs bundle corresponding to $\alpha_2 = 0$ and $\alpha_3 \neq 0$.

1.4 Relation between the two papers

This thesis consists of two papers. The first one shows that for cubic differential paths in $\mathrm{Hit}^3(S)$, the asymptotic length spectrum is the length spectrum of the triangular Finsler metric defined by the cubic differential. This raised the question of whether these triangular Finsler metrics are determined by their marked length spectrum. To answer this question, we use a tool that is often very powerful for proving marked length spectrum rigidity: geodesic currents.

In the second paper, just as Bonohon did for Teichmuller space [Bon88], we show that the compactification can equivalently be taken in geodesic currents, then give a construction which takes in a geodesic current and produces a metric space with $\pi_1(S)$ action. Generically this construction produces an infinite dimensional space, but for currents associated with triangular Finsler metrics it gives back the universal cover of the triangular Finsler surface. Together, it is shown that part of the boundary of $\mathrm{Hit}^3(S)$ is parametrizing these Finsler metrics. It was a surprise that for currents in boundaries of $\mathrm{Hit}^n(S)$ for other values of n the construction also produces a nice space.

Chapter 2: Limits of Convex Projective Surfaces and Finsler Metrics

Abstract

We show that for certain sequences escaping to infinity in the $\mathrm{SL}_3\,\mathbb{R}$ Hitchin component, growth rates of trace functions are described by natural Finsler metrics. More specifically, as the Labourie-Loftin cubic differential gets big, logarithms of trace functions are approximated by lengths in a Finsler metric which has triangular unit balls and is defined directly in terms of the cubic differential. This is equivalent to a conjecture of Loftin from 2006 [Lof07a] which has recently been proven by Loftin, Tambourelli, and Wolf [LTW22], though phrasing the result in terms of Finsler metrics is new and leads to stronger results with simpler proofs. From our perspective, the result is a corollary of a more local theorem which may have other applications. The key ingredient of the proof is another asymmetric Finsler metric, defined on any convex projective surface, recently defined by Danciger and Stecker, in which lengths of loops are logarithms of eigenvalues. We imitate work of Nie [Nie22] to show that, as the cubic differential gets big, Danciger and Stecker's metric converges to our Finsler metric with triangular unit balls. While [LTW22] addresses cubic differential rays, our methods also address sequences of representations which are asymptotic to cubic differential rays, giving us more insight into natural compactifications of the moduli space of convex projective surfaces.

2.1 Introduction

A projective structure on a closed manifold M is an atlas of charts valued in $\mathbb{RP}^n$ with transition functions in $\mathrm{PGL}_{n+1}\mathbb{R}$. Such a structure gives rise to a developing map $\tilde{M} \to \mathbb{RP}^n$, and a holonomy representation $\pi_1(M) \to \mathrm{PGL}_{n+1}\mathbb{R}$. A projective structure is called convex if the developing map is a homeomorphism onto a properly convex domain in $\mathbb{RP}^n$. Classification of convex projective manifolds in general is a largely open subject, but the case of surfaces is understood: the space of convex projective structures on an oriented closed surface of genus g, which we will denote $\mathrm{Conv}(S)$, is a ball of dimension $16g-16$ [CG93]. Taking holonomy representations identifies $\mathrm{Conv}(S)$ with the Hitchin component of the space of representations $\mathrm{Rep}(\pi_1(S), \mathrm{SL}_3\mathbb{R})$. This is analogous to the fact that the space $\mathrm{Teich}(S)$ of hyperbolic structures on S, is a ball of dimension $6g-6$, and is in bijection with a component of $\mathrm{Rep}(\pi_1(S), \mathrm{PSL}_2\mathbb{R})$. In fact the symmetric square $\mathrm{PSL}_2\mathbb{R} \to \mathrm{SL}_3\mathbb{R}$ embeds $\mathrm{Teich}(S)$ into $\mathrm{Conv}(S)$ as a submanifold, which we call the Fuchsian locus.

This paper is largely motivated by the aspiration to extend Thurston's compactification of $\mathrm{Teich}(S)$ by measured foliations to a compactification of $\mathrm{Conv}(S)$. Features of Thurston's compactification which are desiderata for the convex projective case include the following.

- Thurston's compactification is homeomorphic to a closed ball.
- A boundary point records ratios of growth rates of lengths of closed curves.
- The boundary points parametrize some kind of geometric structures on S.

In this paper we show that certain sequences in $\mathrm{Conv}(S)$ converge to the following class of Finsler metrics in a natural way.

Definition 2.1. Let μ be a cubic differential on a Riemann surface C; that is, a holomorphic section of $(T^*C)^{\otimes 3}$. We define F^{Δ}_{μ} to be the maximum of twice the real parts of the cube roots of μ.

$$F_\mu^\Delta(v) := \max_{\{\alpha \in T_x^* C : \alpha^3 = \mu_x\}} 2Re(\alpha(v))$$

Here $x \in C$ is a point, and $v \in T_x C$ is a tangent vector.

We conjecture that these Finsler metrics, considered up to scaling, comprise an open dense subset of a compactification which extends Thurston's compactification of Teichmüller space, and satisfies the above three desiterata. The sequences we consider are "orthogonal" to Teich(S) in the following sense.

In the early 2000's Labourie [Lab06a] and Loftin [Lof01a] found a beautiful, but non-explicit parametrization of the space of convex projective structures on a closed surface S by pairs (J, μ) where J is a complex structure, and μ is a holomorphic cubic differential. The Fuchsian locus corresponds to $\mu = 0$. We will consider sequences (J_i, μ_i) where J_i converges, and μ_i diverges.

A natural measurement one can make on a convex projective structure is the logarithm of the top eigenvalue of the action of a group element $\gamma \in \pi_1(S)$.

$$\log(\lambda_1(\rho(\gamma)))$$

Here, $\rho : \pi_1(S) \to \mathrm{SL}_3 \mathbb{R}$ is the holonomy representation. If ρ is Fuchsian, then this is the hyperbolic length of the geodesic representing the conjugacy class of γ, so we think of it as a notion of geodesic length for closed curves in convex projective manifolds. We will call $\log(\lambda_1(\rho(\gamma)))$ the asymmetric length, because it is a function of oriented loops, and to distinguish it from the more commonly used Hilbert length: $\log(\lambda_1(\rho(\gamma))/\lambda_3(\rho(\gamma)))$. Hilbert length is the symmetrization of the asymmetric length.

Theorem 2.1. *Let μ_i be a sequence of cubic differentials on a smooth oriented surface S of genus at least 2, each holomorphic with respect to a complex structure J_i, such that $a_i^3 \mu_i$ converges uniformally to μ, for some sequence of positive real numbers a_i*

tending to 0*. (It follows that* J_i *converge to* J*.) Let* $\gamma \in \pi_1(S)$*. Let* $F^\Delta_\mu(\gamma)$ *denote the infimal length of loops representing* γ *in the Finsler metric* F^Δ_μ*.*

$$\lim_{i\to\infty} a_i \log(\lambda_1(\rho_i(\gamma))) = F^\Delta_\mu(\gamma)$$

Part of the appeal of theorem 3.50 is that the left hand side is complicated, necessitating both solution of a PDE and an ODE to compute directly from μ_i, while the right hand side only involves integrals of cube roots of μ. The cubic differential μ is equivalent to a 1/3-translation structure on S in which F^Δ_μ geodesics can always be straightened to be concatenations of straight line segments which have angle at least π on either side at zeros. To compute $F^\Delta_\mu(\gamma)$, it suffices to find such a geodesic representative and add up the F^Δ_μ lengths of the constituent segments. Theorem 3.50 was proved in [LTW22] for the case of cubic differential rays $\mu_i = \mu/a_i^3$. We are not sure if one can directly deduce Theorem 3.50 from [LTW22], but in any case, theorem 3.50 is a useful improvement.

Our main new result is a more local version of this theorem. Danciger and Stecker have recently discovered an asymmetric Finsler metric F^{DS}, defined on any closed convex projective manifold, whose length function is the asymmetric length function.

$$F^{DS}(\gamma) = \log(\lambda_1(\rho(\gamma)))$$

We call it the domain shape metric because its unit balls are all projectively equivalent to the developing image of the convex projective structure. Our main theorem says that when μ is large, the domain shape metric for the projective structure on S specified by (J, μ), looks roughly like a much simpler Finsler metric F^Δ_μ.

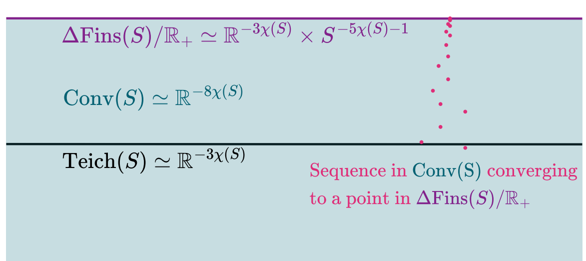
Theorem 2.2. *If* S, μ_i*,* a_i *and* μ *are as in theorem 3.50, then* $a_i F^{DS}_{\mu_i}$ *converges uniformally to* F^Δ_μ *on any compact set in the complement of the zeros of* μ*.*

Pointwise convergence would follow from [Nie22] which describes the Gromov-Hausdorff limit of a sequence of pointed convex domains coming from a sequence of pointed Riemann surfaces with cubic differentials tending to infinity, but we will need uniformity to deduce theorem 3.50 from theorem 2.2. To get uniform convergence away from zeros, we will retrace the steps of [Nie22], with a slightly different setup, and make sure things work uniformly. In section 5 we show that uniform convergence away from zeros is sufficient for deducing convergence of length functions.

2.1.1 What theorem 3.50 says about compactification

Let $\mathcal{L}(S)$ denote the set of homotopy classes of closed loops in S. An attractive way to define a compactification of $\mathrm{Conv}(S)$, which goes by various names, including Morgan-Shalen compactification, tropical compactification, and spectral radius compactification, is to embed it into $\mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$ via taking projectivized marked asymmetric length spectrum, then take the closure. The map $\mathrm{Conv}(S) \to \mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$ is an embedding, but the topology of the closure is still unknown. Part of the difficulty is that we still don't have a good understanding of what geometric structures on S the tropical boundary points might be parametrizing. Theorem 3.50 takes a step towards answering this last question.

The Labourie-Loftin parametrization identifies $\mathrm{Conv}(S)$ with the vector bundle of cubic differentials $Q(S)$ over Teichmuller space which we can compactify in the fiber directions by adding in a point for each $\mathbb{R}_+$ orbit. Let $\bar{Q}(S)$ denote this radial partial compactification. Theorem 3.50 implies that the projectivized marked length spectrum map $\mathrm{Conv}(S) \to \mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$ extends continuously to $\bar{Q}(S)$, and the boundary map is given by the projectivized length spectrum of F^{Δ}_{μ}.



Let $\Delta\,\mathrm{Fins}(S)$ denote the set of triangular Finsler metrics $\{F^{\Delta}_{\mu} : \mu \in Q(S)\backslash\underline{0}\}$. We conjecture that $\Delta\,\mathrm{Fins}(S)/\mathbb{R}_+$, injects into $\mathbb{P}(\mathbb{R}^{\mathcal{L}(S)})$ and comprises an open dense subset of the boundary of $\mathrm{Conv}(S)$.

2.1.2 Relation to other work, and future directions

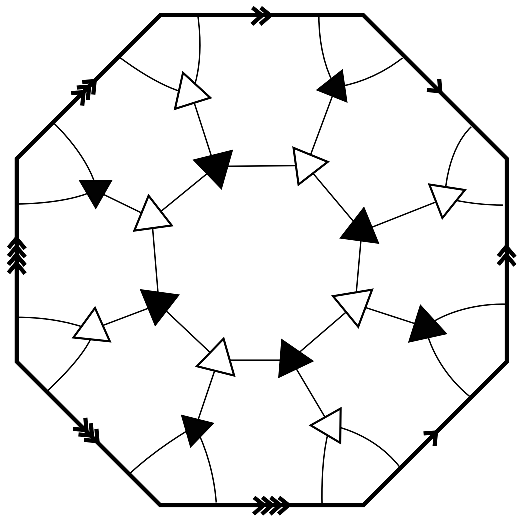
This work fits into a few different bigger stories. Firstly, Gaiotto Moore and Neitzke [GMN13] have a conjecture for the asymptotics of $tr(\rho(\gamma))$ along rays of Higgs bundles $\{(E, R\phi) : R \in \mathbb{R}_+\}$ under the nonabelian hodge correspondence. Our paper, and [LTW22], can be seen as proving the leading order part of the conjecture for the subspace of the SL_3 Hitchin section where the quadratic differential is zero. The methods of [LTW22] seem like they might work in higher rank, while our methods seem very specific to $\mathrm{SL}_3\,\mathbb{R}$. On the other hand, our method is less similar to ideas of [GMN13] in that Stokes lines don't make an appearence, so it might bring a new perspective to the conjectures.

Our Theorem 2.2 is similar in flavor to work of Oyang-Tambourelli [OT21a] where it is shown that when μ is big, the Blaschke metric is close to the singular flat Riemannian metric defined by μ. They are also able to fully understand the closure of the space of projectivised length spectra of Blaschke metrics as mixed structures on S, which consist of a singular flat metric on part of the surface, and a measured lamination on the rest. We hope that the tropical compactification of $\mathrm{Conv}(S)$ has a similar description, but with singular flat triangular Finsler metrics instead of singular flat Riemannian metrics. In [OT21a], it is shown that singular flat metrics comprise an open dense subset of mixed structures, motivating our analogous conjecture about singular, flat, triangular Finsler metrics.

In the case of punctured surfaces, Fock and Goncharov [FG06] have explained how to compactify Hitchin components by the projectivization of the tropical points of the character variety. In the SL_2 case, these tropical points parametrize measured laminations, and the integral tropical points correspond to integral laminations. This has led many people to investigate what kind of objects the Fock-Goncharov tropical points are parametrizing in higher rank.

In [Par15], Parreau shows that a certain cone of Fock-Goncharov tropical points is parametrizing strucutres on the surface which are part 1/3 translation structure, and part tree. She shows that asymptotics of Jordan projections of group elements are encoded in a Weyl cone valued metric. For the 1/3 translation structure part, this is exactly equivalent to the way we encode asymptotics of top eigenvalues with the Finsler metric F_μ. The present paper can be seen as accomplishing, for closed surfaces, something quite analogous to Parreau's work on punctured surfaces. In neither case are all tropical points covered, but the the limitations are not quite the same. Parreau has to restrict to a certain cone of tropical points which depends on the chosen triangulation, but this cone includes some cases which exhibit tree behavior. We, on the other hand, study exactly the cases which do not exhibit tree behavior.

Douglous and Sun [Dou21] have developed a different perspective on the Fock-Goncharov integral SL_3 tropical points, showing that they parametrize certain bipartite trivalent graphs introduced by Sikora-Westbury [SW07] called non-elliptic webs. Webs and 1/3 translation structures are related by a simple geometric construction (which has been contemplated by J. Farre. myself, and possibly other people.) Let S be a closed surface. Suppose we have a filling non-elliptic web: a trivalent bipartite graph W embedded in S whose complementary regions are all disks with at least six sides. We can replace each black vertex of W with the equilateral triangle $\mathrm{Conv}(0, 1, 1/2 + i\sqrt{3}/2) \subset \mathbb{C}$ with cubic differential dz^3, and each white vertex with $\mathrm{Conv}(0, 1, 1/2 - i\sqrt{3}/2) \subset \mathbb{C}$, then glue these triangles according to the edges of the web. This construction produces a surface with cubic differential, which is identified with S up to isotopy. In the other direction, a cubic differential on S with integral periods is the same as a singular 1/3 translation structure with holonomy valued in $\mathbb{Z}/3 \ltimes \mathbb{Z}^2 \subset \mathbb{Z}/3 \ltimes \mathbb{R}^2$. One can take the preimage of a standard hexagonal web on $\mathbb{R}^2$ via the developing map to get a filling non-elliptic web on S. Below is an example of a web on a genus 2 surface specifying a 1/3 translation structure glued from 16 equilateral triangles.



Using this correspondence, we get a map from filling non-elliptic webs to integral tropical points of $Rep(\pi_1 S, \mathrm{SL}_3\,\mathbb{R})$. It would great to extend the results of this paper to punctured surfaces so that they could be compared with work of Parreau, Douglas-Sun, and Fock-Goncharov.

A final, much more wide open, future direction is to use ideas of this paper to understand degenerations of higher dimensional convex projective manifolds. As we will see in the next section, the domain shape metric is defined in any dimension. It is conceivable that one could make sense of limits of domain shape metrics in higher dimensions, and relate these to tropical points of the relevant character varieties. There are various instances throughout geometry where studying more degenerate, combinatorial versions of a class of objects is the key to understanding many important things. Maybe convex projective geometry will be another example of this trend.

2.1.3 Structure of the paper

In Section 2 we define domain shape metrics and prove their important properties. In section 3 we review the Labourie-Loftin correspondence, mostly to recall formulas and set notation. In section 4, the technical heart of the paper, we prove theorem 2.2. In section 5 we prove that length functions of Finsler metrics are sufficiently continuous to deduce theorem 3.50 from theorem 2.2. In section 6 we explain how to apply theorem 3.50 to triangle reflection groups, and present some numerical computations. Some readers may prefer to look at section 6 first.

2.1.4 Acknowledgements

This paper would not exist without many discussions with J. Danciger and F. Stecker. In particular, they told me about domain shape metrics, and F. Stecker did various computer experiments giving evidence for Theorem 3.50. This research was supported in part by NSF grant DMS-1937215, and NSF grant DMS-1945493.

2.2 Domain shape Finsler metrics

In this section, we construct a Finsler metric F^{DS} on any convex projective manifold (of any dimension) whose geodesics are projective lines, and whose length spectrum is the assymmetric length spectrum. The construction depends on a lift of the domain in $\mathbb{RP}^n$ to a convex hypersurface in $\mathbb{R}^{n+1}$. After this section we will specialize to the case where we use the affine sphere as our choice of lift. This metric, and its main properties, were shown to me by Danciger and Stecker, but have not yet appeared in the literature.

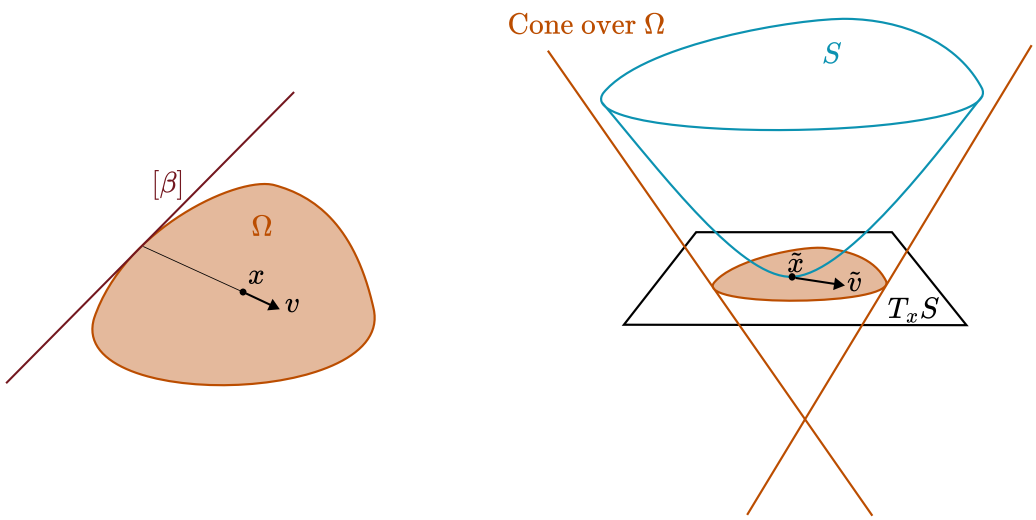
Definition 2.2. Let v be a tangent vector to a point x in a properly convex domain $\Omega \subset \mathbb{R}\mathbb{P}^n$. Let β be a linear functional defining a supporting hyperplane at the point where the ray starting at x and tangent to $-v$ intersects $\partial\Omega$. Let $S \subset \mathbb{R}^{n+1}$ be a convex, differentiable lift, which is asymptotic to the cone over Ω in the sense that

that the line going through two points in S is never in $\bar{\Omega}$. Let $\tilde{x}$ and $\tilde{v}$ be the lifts of x and v to S.

$$F^{DS}(x,v) := \frac{\beta(\tilde{v})}{\beta(\tilde{x})}$$

We call F^{DS} the domain shape metric of Ω for the hypersurface S.

The name is justified by the fact that each unit ball of F^{DS} is projectively equivalent to Ω. More specifically, the unit ball at $x \in \Omega$ is the antipodal image of the intersection of the tangent space of S at $\tilde{x}$ with the cone over Ω.

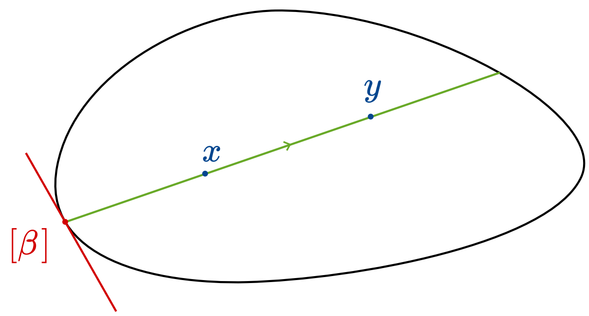


Remark 2.1. Domain shape metrics are a simple generalization of Funk metrics [Fun30], which have been studied since Funk introduced them in 1929, because their geodesics are straight lines. An interpretation of Hilbert's 4th problem is to classify Finsler metrics on euclidean space whose geodesics are straight lines, so there has been interest in such metics for a long time.

It turns out that F^{DS} integrates to the asymmetric path metric d^{DS} which we now define.

Definition 2.3. Let Ω be a convex domain in $\mathbb{RP}^n$, and let S be a lift of Ω to $\mathbb{R}^{n+1}$ which is convex, such the line going through two points of S is never in $\bar{\Omega}$. We define a metric on Ω as follows. Let $x, y \in \Omega$ be distinct points, and let $\tilde{x}, \tilde{y} \in S$ be their lifts. Let p be the point where the projective line starting at y and passing through x first hits the boundary of Ω. Let $[\beta]$ be a supporting hyperplane to Ω at p, defined by a linear functional β.

$$d^{DS}(x, y) := \log \frac{\beta(\tilde{y})}{\beta(\tilde{x})}$$



Remark 2.2. In the case when Ω has a sharp corner at p, there are multiple choices of $[\beta]$, but they give the same value.

Remark 2.3. Nicolas Tholozan pointed out to me an infinite dimensional case where d^{DS} has been studied. Let C be the cone of parametrizations of the geodisic foliation of the unit tangent bundle of a hyperbolic surface (up to weak conjugacy) and let S be the hypersurface of positive, entropy 1 parametrizations. One can restrict the resulting domain shape metric on $\mathbb{P}(C)$ to spaces of Anosov representations, and get asymmetric Finsler metrics studied in [Tho] and [Car+22].

Before we relate d^{DS} to F^{DS}, we check that d^{DS} satisfies the two axioms of an assymmetric metric: reflexivity, and the triangle inequality. Reflexivity is left to the reader. The triangle inequality will follow from the following lemma.

Lemma 2.3. *Let x, y, Ω, S, β, and p be as above. If $[\beta']$ is any hyperplane which doesn't intersect Ω, and doesn't pass through p, then:*

$$\frac{\beta'(\tilde{y})}{\beta'(\tilde{x})} < \frac{\beta(\tilde{y})}{\beta(\tilde{x})}$$

Proof. Let p' be the point where the line through x and y hits the hyperplane $[\beta']$. We can find an affine hyperplane A going through $\tilde{x}$ and $\tilde{y}$ whose intersection with the cone over Ω is as follows.

By assumption, $[\tilde{x} - \tilde{y}]$ cannot be in $\bar{\Omega}$. This means that the dual hyperplane $[\tilde{x} - \tilde{y}]^*$ intersects the dual domain Ω^*. Let α be a linear functional representing a point in the intersection. By construction, $\alpha(\tilde{x}) = \alpha(\tilde{y})$, and we can scale α so that $\alpha(\tilde{x}) = \alpha(\tilde{y}) = 1$. Let A be the hyperplane defined by $\alpha = 1$.

This affine hyperplane A is identified with an affine chart of projective space containing Ω. Arbitrarily choose a euclidean metric d_A on A compatible with the affine structure. Since $\beta|_A$ and $\beta'|_A$ are affine linear functions vanishing $[\beta]$ and $[\beta']$, they are proportional to the affine linear functions which simply measure signed euclidean distance to $[\beta]$ and $[\beta']$. Our desired inequality is thus equated with an inequality involving euclidean distances which is visually clear.

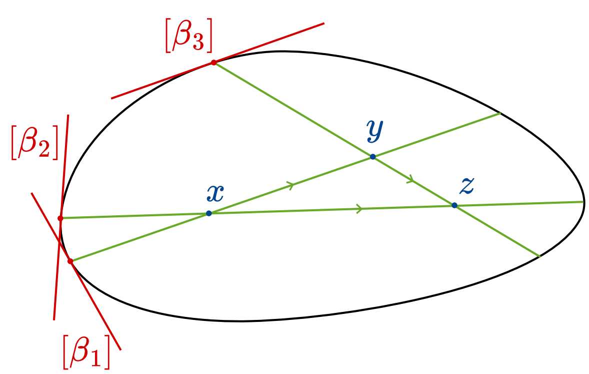
$$\frac{\beta'(\tilde{y})}{\beta'(\tilde{x})} = \frac{d_A(y, p')}{d_A(x, p')} < \frac{d_A(y, p)}{d_A(x, p)} = \frac{\beta(\tilde{y})}{\beta(\tilde{x})}$$

□

Lemma 2.4. *1. d^{DS} satisfies the triangle inequality, $d^{DS}(x, y) + d^{DS}(y, z) \geq d^{DS}(x, z)$.*

2. If x, y, z are collinear, then we have equality.

3. If the domain is strictly convex, and x, y, z are not collinear, then the inequality is strict.



Proof. Let β_1, β_2, and β_3 be linear functionals defining hyperplanes tangent to the points on $\partial\,\Omega$ intersecting the rays $\overrightarrow{yx}$, $\overrightarrow{zy}$, and $\overrightarrow{zx}$ respectively.

$$\frac{\beta_3(z)}{\beta_3(x)} = \frac{\beta_3(z)}{\beta_3(y)}\frac{\beta_3(y)}{\beta_3(x)} \leq \frac{\beta_2(z)}{\beta_2(y)}\frac{\beta_1(y)}{\beta_1(x)}$$

Taking logarithms gives the triangle inequality. If x, y, and z are colinear, then we can choose $\beta_1 = \beta_2 = \beta_3$, so we have equality. If x, y, and z are not colinear, and Ω is strictly convex, then β_1, β_2, and β_3 are supporting hyperplanes at unique, and distinct points, so the previous lemma gives us strict inequality. □

Part 2 of the preceding lemma implies that projective lines are geodesics of d^{DS}, and that d^{DS} is a path metric. Part 3 implies that if Ω is strictly convex, projective lines are the only geodesics. Finally, we check that d^{DS} and F^{DS} agree.

Lemma 2.5. *d^{DS} differentiates to F^{DS}.*

Proof. Let v be a tangent vector at $x \in \Omega$. Let $\gamma : (-\epsilon, \epsilon) \to \Omega$ be a curve with $\gamma'(0) = v$ which we choose for convenience to be a projective line segment. Let β be a linear functional as in the definition of $F^{DS}(x, v)$.

$$\frac{d}{dt}d^{DS}(\tilde{\gamma}(0), \tilde{\gamma}(t))|_{t=0} = \frac{d}{dt}\log(\frac{\beta(\tilde{\gamma}(t))}{\beta(\tilde{\gamma}(0))})|_{t=0} = \frac{\beta(\tilde{\gamma}'(0))}{\beta(\tilde{\gamma}(0))} = F^{DS}(x, v)$$

□

2.2.1 Domain shape metrics for dual hypersurfaces

In this section we show that, if we choose compatible lifts, domain shape metrics for projectively dual domains are pointwise dual, with respect to a natural Riemannian metric. In the case when the lift is an affine sphere, this metric is known as the Blaschke metric. This will be used in section 2.4.6 to turn upper bounds for F^{DS} into lower bounds.

Let V be a real vector space of dimension $n + 1$. Let $\Omega \subset \mathbb{P}(V)$ be a properly convex domain. Let $S \subset V$ be a proper convex lift. Let $\Omega^* \subset \mathbb{P}(V^*)$ be the dual convex domain, and let $S^* \subset V^*$ be the dual convex lift: the set of linear functionals α such that $\inf \alpha(S) = 1$. We say that $x \in S$ and $\alpha \in S^*$ are dual points if $\alpha(x) = 1$. If S and S^* are strictly convex, then each point in S has exactly one dual point in S^*. Assume that S is smooth, with everywhere positive hessian, so that the duality mapping between S and S^* is a diffeomorphism. In this setting, a natural Riemannian metric appears.

Lemma 2.6. *Let $x : \mathbb{R}^n \to S$ and $\alpha : \mathbb{R}^n \to S^*$ be smooth, dual parametrizations. The matrix*

$$g_{ij} = -\langle \partial_i \alpha, \partial_j x \rangle = \langle \alpha, \partial_i \partial_j x \rangle = \langle \partial_i \partial_j \alpha, x \rangle$$

is a Riemannian metric which doesn't depend on the parametrization.

Proof. The first expression $-\langle \partial_i \alpha, \partial_j x \rangle$ is independant of parametrization because we can phrase it in parametrization independant language: if $\beta \in T_\alpha S^*$ and $v \in T_x S$, these both give tangent vectors to S, (or S^*,) and we define $g(\beta, v) = -\langle \beta, v \rangle$. The second expression $\langle \alpha, \partial_i \partial_j x \rangle$ is clearly symmetric and positive definite because partial derivatives commute, and S is convex with positive Hessian. It remains to show that these expressions are indeed equal. That x and α are dual means that $\langle \alpha, \partial_i x \rangle = \langle \partial_i \alpha, x \rangle = 0$ for all i. Applying ∂_j to these equations gives the result. $\square$

The Blaschke metric is $\langle \partial_i \partial_j \alpha, v \rangle$ where v is a certain canonically defined normal vector called the 'affine normal'. An affine sphere (centered at 0) is precisely a hypersurface satisfying $v = x$, so for affine spheres g is the Blaschke metric.

Definition 2.4. Let Ω be a convex subset of a vector space V containing the origin, and let g be a metric on V. The dual of Ω with respect to g is $\Omega^{*_g} := \{v \in V : g(v, \Omega) < 1\} \subset V$

If M is a manifold with Riemannian metric g, and Finsler metric F, let F^{*_g} denote the Finsler metric whose unit balls are pointwise dual to unit balls of F with respect to g.

Lemma 2.7. *If S is a smooth, properly embedded lift of a properly convex domain Ω with positive definite Hessian, and we identify S with S^* by the duality diffeomorphism, then $F_{S^*}^{DS} = (F_S^{DS})^{*_g}$, where g is the metric defined above.*

Proof. Let C denote the cone $\mathbb{R}_+ S$. Let B denote the unit ball of F_S^{DS} at $x \in S$.

$$B = \{v \in T_x S : x - v \in C\}$$

The unit ball of $F_{S^*}^{DS}$ at α is

$$B^* = \{\beta \in T_\alpha S^* : \langle \alpha - \beta, C\rangle > 0\}$$

C is the $\mathbb{R}_+$ span of $x - B$, so equivalently

$$B^* = \{\beta \in T_\alpha S^* : \langle \alpha - \beta, x - v\rangle > 0 \quad \forall v \in B\}$$

Recall $\langle \alpha, x\rangle = 1$, α vanishes on $T_x S$ and x vanishes on $T_\alpha S^*$, so we have $\langle \alpha - \beta, x - v\rangle = 1 + \langle \beta, v\rangle$. Finally we have

$$B^* = \{\beta \in T_\alpha S^* : -\langle \beta, v\rangle < 1 \quad \forall v \in B\}$$

If we identify $T_\alpha S^*$ and $T_x S$ by the differential of the duality map, then $-\langle \beta, v\rangle$ is g evaluated on β and v, so the unit balls of F_S^{DS} and $F_{S^*}^{DS}$ are dual with respect to the g. $\square$

2.3 Review of the Labourie-Loftin correspondence

The Labourie-Loftin correspondence [Lof01a], [Lab06a] provides a bijection between pairs (J, μ), where J is a complex structure, and μ is a holomorphic cubic differential, and convex projective structures, on a surface of genus at least 1. After quotienting both the space of pairs (J, μ), and the space of convex projective structures by $\mathrm{Diff}_0(S)$, the Labourie-Loftin correspondence becomes a diffeomorphism from the bundle of cubic differentials over Teichmuller space, to the Hitchin component. Here, we review how one gets a convex projective structure on S from a pair (J, μ).

Let S be a Riemann surface with cubic differential μ. It turns out that there is a unique hermitian metric g on S satisfying Wang's equation.

$$\kappa_g = |\mu|_g^2 - 1$$

From the data (S, J, μ, g) we will construct a convex projective structure. This means a developing-holonomy pair: a representation $\pi_1(S) \to \mathrm{SL}_3 \mathbb{R}$, and an equivariant embedding $\tilde{S} \to \mathbb{P}(\mathbb{R}^3)$ whose image is a convex set. We will actually construct an equivariant map $\tilde{S} \to \mathbb{R}^3$, whose image is a strictly convex hypersurface, (in fact an affine sphere) which we can compose with the projection to $\mathbb{P}(\mathbb{R}^3)$ to get a developing map.

The construction of the developing holonomy pair necessitates a choice of base point, and it is useful to pay attention to this choice. For each point $x \in S$, we will construct an affine sphere in the three dimensional vector space $T_x S \oplus \underline{\mathbb{R}}$ with $\pi_1(x, S)$ action, but different choices of x will give isomorphic results. There is an explicit formula for a real flat connection ∇ on $TS \oplus \underline{\mathbb{R}}$ in terms of μ and g. We write this formula in terms of the complexification which has a natural line decomposition $TS \oplus \overline{TS} \oplus \underline{\mathbb{C}}$:

$$\nabla = \begin{bmatrix} D^g_{TS} & g^{-1}\bar{\mu} & 1 \\ g^{-1}\mu & D^g_{\overline{TS}} & 1 \\ g & g & 0 \end{bmatrix}$$

Here, D^g_{TS}, and $D^g_{\overline{TS}}$ are the Chern connections, which both coincide with the Levi-Civita connection for g. The off-diagonal entries are maps of various line bundles.

One checks that ∇ is real, and that its flatness is equivalent to Wang's equation. We sketch this second computation here.

Lemma 2.8. ∇ *is flat if and only if* $\kappa_g = |\mu|_g^2 - 1$.

Proof. This statement can be checked locally. Choose a local holomorphic coordinate z on S. Write $g = e^{\phi} dz d\bar{z}$, and $\mu = \mu_0 dz^3$. We get a frame $\partial_z, \partial_{\bar{z}}, \underline{1}$ of $(TS \oplus \mathbb{R}) \otimes \mathbb{C}$. In this frame, we can write ∇ as the de Rahm differential plus a matrix valued 1-form.

$$\nabla = d + A_1 dz + A_2 d\bar{z}$$

$$A_1 = \begin{bmatrix} \partial\phi & 0 & 1 \\ e^{-\phi}\mu_0 & 0 & 0 \\ 0 & e^{\phi} & 0 \end{bmatrix} dz \qquad A_2 = \begin{bmatrix} 0 & e^{-\phi}\bar{\mu}_0 & 0 \\ 0 & \bar{\partial}\phi & 1 \\ e^{\phi} & 0 & 0 \end{bmatrix} d\bar{z} \tag{2.1}$$

The curvature of ∇ is

$$F_\nabla = (-\partial_{\bar{z}} A_1 + \partial_z A_2 + \frac{1}{2}[A_1, A_2])dz \wedge d\bar{z}$$

This comes out to be

$$F_\nabla = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\partial_z \partial_{\bar{z}} \phi + \frac{1}{2}(e^{-2\phi}|\mu_0|^2 - e^{\phi}))dz \wedge d\bar{z}$$

Recall that $\partial_z \partial_{\bar{z}} = \frac{1}{4}\Delta$ where $\Delta = \partial_x^2 + \partial_y^2$ is the laplacian. Vanishing of F_∇ becomes a PDE for ϕ:

$$\frac{1}{2}\Delta\phi = -e^{-2\phi}|\mu_0|^2 + e^{\phi}$$

Recalling the formula $\kappa_g = -\frac{1}{2}e^{-\phi}\Delta\phi$ for the gauss curvature, we get the coordinate independant form of the equation.

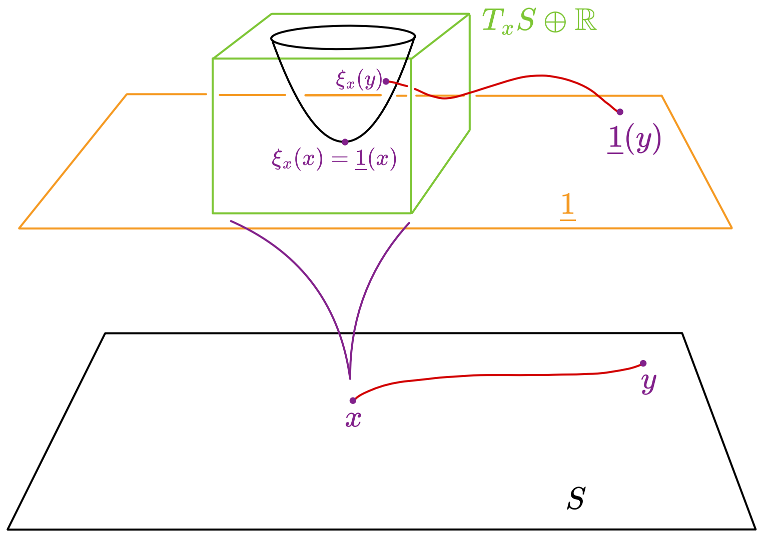
$$\kappa_g = |\mu|_g^2 - 1$$

□

The formula for ∇ may seem a bit mysterious, so we mention two perspectives from which one could derive it. First, for a general strictly convex surface S in three dimensional affine space $\mathbb{R}^3$ (endowed with a translation invariant volume form), there is a canonically defined normal vector field called the affine normal, and a metric called the Blaschke metric. The affine normal lets us identify $T\mathbb{R}^3|_S$ with $TS \oplus \mathbb{R}$, and the complex structure induced by the Blaschke metric gives a line decomposition $TS \otimes \mathbb{C} = TS \oplus \overline{TS}$. If we write the trivial connection on $T\mathbb{R}^3|_S$ with respect to this line decomposition, it takes a form quite similar to (2.1), but with non-trivial tensors in the last column. The fact that the connection takes the form (2.1) is equivalent to S being a hyperbolic affine sphere. We refer to [Lof08] for a survey on affine spheres.

From a completely different point of view, Wang's equation is really a special case of Hitchin's equation: if we use g to identify $\overline{TS}$ with TS^*, we get a harmonic bundle solving the Hitchin equation for the rank 3 Higgs bundle in the Hitchin section corresponding to the cubic differential μ, and the zero quadratic differential.

$TS \oplus \underline{\mathbb{R}}$ has a natural section, namely $\underline{1} \in \Gamma(\underline{\mathbb{R}})$. Let $\tilde{S}_x$ denote the universal cover of S based at $x \in S$, constructed explicitly as the collection of pairs $(y, [\gamma])$, where $y \in S$ and $[\gamma]$ is a homotopy class of path from x to y. For each point $x \in S$, let $\xi_x : \tilde{S}_x \to T_xS \oplus \mathbb{R}$ denote the map which takes a point $(y, [\gamma])$ to the parallel transport of $\underline{1}(y)$ back to x, along γ using the connection ∇. We call ξ_x the affine sphere developing map based at x. One can deduce, from the form of ∇, that the image of ξ_x is indeed an affine sphere: the affine normal at x is a fixed scalar multiple of ξ_x.



The fundamental group $\pi_1(S, x)$ acts on $\tilde{S}_x$ via deck transformations, and on $T_xS \oplus \mathbb{R}$ via the holonomy of ∇. The map ξ_x is equivariant for these two actions by construction. If we choose a different point $x' \in S$, then parallel transport along any path from x to x' will be a volume preserving linear map $T_xS \oplus \mathbb{R} \to T_{x'}S \oplus \mathbb{R}$ identifying the affine spheres $\mathrm{Im}(\xi_x)$ and $\mathrm{Im}(\xi_{x'})$, which intertwines the $\pi_1(x, S)$ and $\pi_1(x', S)$ actions.

2.4 F^{DS} is close to F^{Δ} far from zeros

To make precise what we mean by "far from zeros" we need a metric. A Riemann surface S with cubic differential μ, has a singular, flat Riemannian metric h defined by the equation $|\mu|_h = 1$. We will call this singular flat metric h_μ.

Theorem 2.9. *There exists a function $\epsilon : \mathbb{R}_+ \to (0, \infty]$ with $\lim_{r\to\infty} \epsilon(r) = 0$ such that for any closed Riemann surface S with cubic differential μ,*

$$|\log \frac{F^{DS}_\mu(x, v)}{F^{\Delta}_\mu(x, v)}| < \epsilon(r(x))$$

for any non-zero tangent vector v at any point x, where $r(x)$ denotes the distance from x to the closest zero of μ, with respect to the metric h.

This theorem is simply a more uniform version of the part of Nie's result [Nie22] dealing with triangles, and we will prove it following his method.

2.4.1 Proof of theorem 2.2 assuming theorem 2.9

We will now prove Theorem 2.2 from the introduction as an easy consequence of Theorem 2.9.

Proof. Fix $R > 0$. Let $S_R \subset S$ be the set of points which are at least distance R away from all zeros of μ, in the metric h_μ. It will suffice to show that $F^{DS}_{\mu_i}/a^i$ converges uniformly to F^{Δ}_μ on S_R. Uniform convergence of $a_i^3\mu_i$ to μ implies that for all $\epsilon > 0$, there exists N such that zeros of μ_i for $i > N$ are all in an ϵ neighborhood of the zeros of μ with respect to h_μ. Consequently, the limit of the h_μ distance between the zeros of μ_i and S_R is R. The h_{μ_i} distance between zeros of μ_i and S_R must then go to infinity, because $a_i h_{\mu_i}$ converges uniformly to h_μ. This means that the ratio between $F^{DS}_{\mu_i}$ and $F^{\Delta}_{\mu_i}$ limits to 1 uniformly on S_R. The ratio of $a_i F^{\Delta}_{\mu_i}$ to F^{Δ}_{μ} also goes to 1 uniformally on S_R. It follows that the ratio of $a^i F^{DS}_{\mu_i}$ to F^{Δ}_{μ} goes uniformally to 1 on S_R. □

2.4.2 Blaschke metric estimate

Note that the singular flat metric h_μ, defined by $|\mu|_{h_\mu} = 1$, is a solution to Wang's equation on the complement of the zeros of μ. The first step in proving theorem 2.9 is to show that, far from zeros, the global solution to Wang's equation is close, in C^1 norm, to this singular flat solution.

Lemma 2.10. *There exists a function $\epsilon_{C^1} : \mathbb{R}_+ \to (0, \infty]$ limiting to zero as the input goes to infinity, such that if S is a Riemann surface with holomorphic cubic differential μ such that h_μ is complete, g is the complete solution to Wang's equation, and $g = e^\phi h_\mu$, then*

$$|\phi(x)| + |d\phi_x|_{h_\mu} < \epsilon_{C^1}(r(x))$$

for all $x \in S$, where $r(x)$ denotes the h_μ distance from x to the closest zero of μ.

Wang's equation implies that ϕ satisfies the following PDE.

$$\frac{1}{2}\Delta_{h_\mu}\phi = e^{\phi} - e^{-2\phi}$$

The intuition is that this equation very much wants to force ϕ close to zero, and ellipticity can promote C^0 bounds to C^k bounds for whatever k we want. For us C^1 will be sufficient.

Proof. In [Nie23] it is shown that there are uniform constants C, r_0 such that $\phi(x)$ is bounded between 0 and $\epsilon_{C^0}(r(x))$, where:

$$\epsilon_{C^0}(r) = \begin{cases} C\sqrt{r}e^{-\sqrt{6}r} & \text{if } r \geq r_0 \\ \infty & \text{otherwise} \end{cases}$$

This means in particular that on the disk $B_{r(x)/2}(x)$, ϕ is bounded between 0 and $\epsilon_{C^0}(r(x)/2)$. We now use the following interior gradient estimate.

Lemma 2.11. *If u is a twice differentiable function on the closed disk $\overline{B}_x(R) \subset \mathbb{R}^2$, then we have the following estimate for its gradient at x.*

$$|\nabla u(x)| \leq R|\Delta u|_{C^0(B_x(R))} + \frac{2}{\pi R}|u|_{C^0(\partial B_x(R))}$$

This is proved by writing u as a sum of a function which vanishes on the boundary of the disk and has the same laplacian as u, and a harmonic function which has the same boundary value as u. The gradient of the former can be estimated using its representation in terms of the Green's function for the disk, and the gradient of the latter can be bounded using the maximum principal, and mean value property for harmonic functions. Much more general estimates of this flavor are proved in PDE texts such as [GT01].

Apply this to ϕ on $B_x(r(x)/2)$, recalling that $\Delta\phi = F(\phi)$ where $F(y) = 2(e^y - e^{-2y})$.

$$|\nabla\phi(x)| \leq \frac{r(x)}{2}F(\epsilon_{C^0}(r(x)/2)) + \frac{2n}{\pi r(x)}\epsilon_{C^0}(r(x)/2)$$

Note that this gradient estimate goes to zero as $r(x)$ goes to infinity. We simply add the pointwise estimate and the gradient estimate to get the desired C^1 estimate.

$$\epsilon_{C^1}(r) := \epsilon_{C^0}(r) + \frac{r}{2}F(\epsilon_{C^0}(r/2)) + \frac{2n}{\pi r}\epsilon_{C^0}(r/2)$$

□

2.4.3 F^{DS} and F^{Δ} coincide for Ţiţeica surfaces

In this section, we review the fact that the affine sphere corresponding to the constant cubic differential dz^3 on the complex plane $\mathbb{C}$ is a Ţiţeica surface, and we show that the DS metric is exactly the Δ metric in this case.

Lemma 2.12. $F^{DS}_{dz^3} = F^{\Delta}_{dz^3}$

Proof. The standard hermitian metric $h = dzd\bar{z}$ is the complete solution to Wang's equation. By specializing formula (2.1) to the case $\phi = 0$, we get the flat connection

$$\nabla_0 = d + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} dz + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} d\bar{z} \tag{2.2}$$

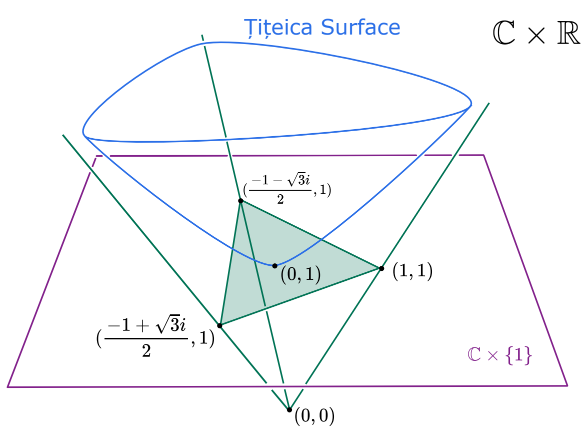
on $T\mathbb{C} \oplus \underline{\mathbb{R}}$, expressed in terms of the frame $(\partial_z, \partial_{\bar{z}}, \underline{1})$ of the complexification $(T\mathbb{C} \oplus \mathbb{R}) \otimes \mathbb{C}$. One directly checks that for each 3rd root of unity ζ,

$$s_\zeta = \begin{bmatrix} \zeta e^{-2Re(\zeta z)} \\ \bar{\zeta} e^{-2Re(\zeta z)} \\ e^{-2Re(\zeta z)} \end{bmatrix}$$

is a flat section. We then express $\underline{1}$ in this flat frame.

$$\underline{1} = \frac{1}{3} \sum_{\zeta \in \sqrt[3]{1}} e^{2Re(\zeta z)} s_\zeta \tag{2.3}$$

Identifying all fibers of $T\mathbb{C} \oplus \mathbb{R}$ via ∇_0, the flat frame $\{s_\zeta\}$ collapses to the basis $\{(\zeta, 1) : \zeta \in \sqrt[3]{1}\} \subset \mathbb{C} \oplus \mathbb{R}$, and $\frac{1}{3} e^{2Re(\zeta z)}$ are the coefficients of the affine sphere developing map in this basis.



This particular affine sphere is called a Ţiţeica surface. All we need to notice about it is that its projection to $\mathbb{C} \times \{1\}$ is indeed the antipodal image of the unit ball for $F^{\Delta}_{dz^3}$ at $z = 0$. Both $F^{DS}_{dz^3}$ and $F^{\Delta}_{dz^3}$ are translation invariant, so agreement at $z = 0$ implies that they are the same everywhere. □

2.4.4 Upper bounds on F^{DS} and F^{Δ}

We won't be able to compare F^{DS} and F^{Δ} directly. Instead, we will define a families of upper and lower bounds for F^{DS} and F^{Δ}, and compare these bounds. In this section we define upper bounds $F^{DS,d} > F^{DS}$, and $F^{\Delta,d} > F^{\Delta}$, and show they are close. In the next section we will use duality arguments to get lower bounds.

The tangent bundle TS embeds into the $\mathbb{RP}^2$ bundle $\mathbb{P}(TS \oplus \mathbb{R})$ as a bundle of affine charts via $v \mapsto [v : 1]$. The projectivization of the affine sphere $Im(\xi_x)$, is a convex subset of the affine chart $T_xS \subset \mathbb{P}(T_xS \oplus \mathbb{R})$, and in fact is the antipodal image of the unit ball of F^{DS} at x. This suggests a family of upper bounds defined by truncating affine spheres.

Definition 2.5. For $d > 0$, let $F^{DS,d}$ denote the Finsler metric whose unit ball at $x \in S$ is $-\operatorname{Conv}(\pi(\xi_x(\tilde{B}_{\tilde{x}}(d)))$, where $\tilde{B}_{\tilde{x}}(d)$ denotes the h_μ-ball of radius d, centered at $\tilde{x}$ in the universal cover $\tilde{S}_x$, and π is the projection from $(T_xS \oplus \mathbb{R})\backslash T_xS$ to the affine chart $T_xS \subset \mathbb{P}(T_xS \oplus \mathbb{R})$.

Now we do a similar thing for F^{Δ}. We let μ_0 and h_0 denote the constant cubic differential and hermitian metric on T_xS which agree with μ_x, and h_x. Let $\xi_{0,p} : T_xS \to T_xS \oplus \mathbb{R}$ denote the affine sphere developing map determined by μ_0 and h_0.

By lemma 2.12, the DS, and Δ metrics for the Ţiţeica surface $\xi_{0,x}(T_pS)$ coincide. At the point x, this means that the unit ball of F^{Δ} at x is $-\pi(\xi_{0,x}(T_pS))$.

Definition 2.6. Let $F^{\Delta,d}$ denote the Finsler metric whose unit ball at p is $-\operatorname{Conv}(\pi(\xi_{0,p}(T_p^{\leq d}S)))$ where $T_p^{\leq d}S$ denotes the ball of radius d in T_pS.

Note that by construction, $F^{\Delta,d}$ converges to F^{Δ}. We can thus find a function $\epsilon_{\Delta,d}$, which goes to zero as d goes to infinity, such that $\log(F_\mu^{\Delta,d}/F_\mu^{\Delta}) < \epsilon_{\Delta,d}$ for all d, for any Riemann surface with cubic differential, on the complement of the zeros.

2.4.5 Closeness of truncated affine spheres

The next step is to show that, far from zeros, $F^{DS,d}$ is close to $F^{\Delta,d}$. This will follow from showing the two affine sphere developing maps ξ_p and $\xi_{p,0}$ are close on the ball of radius d. This will give us a bound on the Hausdorff distance between unit balls of $F^{DS,d}$ and $F^{\Delta,d}$. We will then have to prove a simple lemma relating ratios between norms, and Hausdorff distances between their unit balls.

Lemma 2.13. *There is a function $\epsilon_{d,Haus} : (0,\infty) \to (0,\infty]$ limiting to zero as the argument goes to infinity, such that for a Riemann surface S with cubic differential μ, the Hausdorff distance, with respect to h_μ, between the unit balls of F^{DS}_μ and F^{Δ}_μ at any point p, is bounded above by $\epsilon_{d,Haus}(r(p))$.*

Proof. We define $\epsilon_{d,Haus}(r)$ to be ∞ for $r \leq d$, so we can assume that p is distance at least d from zeros. Let $B(p,d)$ denote the ball centered at p of radius d with respect to h. We have two solutions to Wang's equation on $B(p,d)$: g, and h. These give rise to two affine sphere developing maps $\xi_p, \xi_{0,p} : B(p,d) \to T_pS \oplus \mathbb{R}$ which are constructed by parallel transporting the section $\underline{1}$ back to the fiber over p, via two different connections ∇_0, and ∇. As we did for ∇_0 in equation 2.2, we can write an explicit formula for ∇ by choosing a coordinate z on $B(p,d)$ which takes μ to dz^3, and using the frame $\partial_z, \bar{\partial}_z, \underline{1}$ of the complexification of $TS \oplus \mathbb{R}$. In this frame, we have the following formula for ∇ in terms of $\phi = \log(g/h)$.

$$\nabla = d + \begin{bmatrix} 0 & 0 & 1 \\ e^{-\phi} & \partial\phi & 0 \\ 0 & e^{\phi} & 0 \end{bmatrix} dz + \begin{bmatrix} \bar{\partial}\phi & e^{-\phi} & 0 \\ 0 & 0 & 1 \\ e^{\phi} & 0 & 0 \end{bmatrix} d\bar{z}$$

Let A_0 and A be the matrix valued 1 forms representing ∇_0 and ∇ in this frame: $\nabla_0 = d + A_0$, $\nabla = d + A$. To conclude closeness of ξ_p and $\xi_{0,p}$ from closeness of ∇ and ∇_0, we need the following standard consequence of Gronwall's inequality.

Lemma 2.14. *Suppose $f' = Af$ and $g' = Bg$ where $A, B \in C^0([0,t], \mathbb{R}^{n\times n})$ are matrix valued functions, and $f, g \in C^1([0,t], \mathbb{R}^n)$ are vector valued functions with*

$f(0) = g(0)$. *Then* $|g - f|$ *has the following bound.*

$$|g(t) - f(t)| \leq |B - A|_{C^0} |f|_{C^0} t e^{t|B|_{C^0}}$$

Let $q \in B(p, d)$. Applying Gronwalls inequality to the restriction of ∇ and ∇_0 to the straight line segment from q to p gives the following.

$$|\xi_p(q) - \xi_{0,p}(q)| \leq |A - A_0|_{C^0} |\xi_{0,p}|_{C^0} d e^{d|A|_{C^0}}$$

$|A - A_0|$ is bounded by $C\epsilon_{C^1}(r)$ for some fixed constant C. By equation 2.3, $|\xi_{0,p}|$ is bounded by $\frac{1}{3}e^{4d}$. $|A|$ is bounded by $|A_0| + C\epsilon_{C^1}(r)$. We get

$$|\xi_p(q) - \xi_{0,p}(q)| \leq C\epsilon_{C^1}(r) \frac{1}{3} e^{4d} d e^{d(|A_0| + C\epsilon_{C^1}(r))}$$

Define $\epsilon_d(r)$ to be the right hand side of this inequality, and note that it indeed goes to zero as r goes to infinity for any fixed d which is large enough that ϵ_{C^1} is finite. This bound persists after projecting to T_pS, and negating to get the unit balls of $F^{\Delta,d}$, and $F^{DS,d}$. This is because, by convexity, ξ_0 and ξ are valued in $\{(v, y) \subset TS \oplus \mathbb{R} \mid y \geq 0\}$, a region in which the radial projection π to T_pS is contracting. This bound on $|\pi(\xi_p(q)) - \pi(\xi_{0,p}(q))|$ gives the same bound on the Hausdorff distance between unit balls of $F^{DS,d}(p)$ and $F^{\Delta,d}(p)$. $\square$

Definition 2.7. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be convex sets containing the origin. We can describe Ω_1 and Ω_2 as the unit balls for norms f_1 and f_2 on $\mathbb{R}^n$. Let the radial distance, $d_R(\Omega_1, \Omega_2)$ denote the supremum of $|\log(f_1/f_2)|$.

Note that f_1/f_2 is the scaling factor which takes the boundary of Ω_1 onto the boundary of Ω_2.

Lemma 2.15. *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be convex sets containing the origin. Let* $d_H(\Omega_1, \Omega_2)$ *denote the Hausdorff distance. We have*

$$r d_R(\Omega_1, \Omega_2) \leq d_H(\Omega_1, \Omega_2)$$

where r *is the radius of a ball centered at* 0 *contained in both* Ω_1 *and* Ω_2.

Proof. For $x \in \partial\Omega_1$, let $\alpha(x) := f_1(x)/f_2(x)$ be the scaling factor necessary such that $\alpha x \in \partial\Omega_2$. Let $x \in \partial\Omega_1$ be a point realizing the supremum of $|\log(\alpha(x))|$ and, without loss of generality, assume $\alpha(x) \geq 1$. Suppose we have a supporting hyperplane H_x for Ω_1 at x. We argue that $H_y = \alpha(x)H_x$ must be a supporting hyperplane for Ω_2 at y. Indeed, suppose that there is $y' \in \Omega_2$ on the other side of H_y from 0. Let x' be the point where the line from 0 to z intersects $\partial\Omega^1$, one sees that $\alpha(x') > \alpha(x)$, a contradiction. It follows that there is no y' on the other side of H_y, so H_y is a supporting hyperplane for Ω_2. The Hausdorff distance is at least the euclidean distance from y to Ω_1 which is at least the distance between H_y and H_x, which is at least $(\alpha - 1)r$.

$$d_H(\Omega_1, \Omega_2) \geq (\alpha - 1)r \geq \log(\alpha)r \geq r d_R(\Omega_1, \Omega_2)$$

$\square$

If we assume d is above some threshold d_0, we then the unit balls of F_μ^Δ will contain the unit balls of $r_0^{-1}h_\mu$ for some r_0. Setting $\epsilon_d(r) = \infty$ for $d \leq d_0$, and $\epsilon_d(r) = r_0^{-1}\epsilon_{d,Haus}(r)$ for $d > d_0$ we get our desired control over the ratio between $F^{\Delta,d}$ and $F^{DS,d}$.

Lemma 2.16. *There is a function $\epsilon_d : (0, \infty) \to (0, \infty]$ limiting to zero as the argument goes to infinity, such that for a Riemann surface S with cubic differential μ,*

$$|\log \frac{F_\mu^{DS,d}}{F_\mu^{\Delta,d}}| \leq \epsilon_d(r(p))$$

2.4.6 Lower bounds on F^Δ and F^{DS}

Next, we will construct a family of lower bounds, using projective duality ideas.

Lemma 2.17. *On a Riemann surface with cubic differential μ, inducing flat metric h, we have*

$$(F_\mu^\Delta)^{*2h} = F_{-\mu}^\Delta$$

on the complement of zeros.

Proof. Around any point, there is a local coordinate z such that $\mu = dz^3$, so it suffices to treat the case of of $\mu = dz^3$ on the complex plane. This is an easy computation. Alternatively, it is a special case of the next lemma. $\square$

When we have a Riemann surface with cubic differential μ, and g satisfying $\kappa_g = |\mu|_g^2 - 1$, the Blaschke metric for the corresponding affine sphere is $2g$. The dual affine sphere is given by replacing μ with $-\mu$ in the formula for the connection. It follows from lemma 2.7 that negating the cubic differential corresponds to dualizing the domain shape metric with respect to $2g$.

Lemma 2.18. *If S is a Riemann surface with cubic differential μ, and g is a complete solution to Wang's equation, then $(F_\mu^{DS})^{*_{2g}} = F_{-\mu}^{DS}$ where g is the complete solution to Wang's equation.*

Equivalently, we have $F_\mu^{DS} = (F_{-\mu}^{DS})^{*_{2g}}$. Since taking duals reverses containment of convex sets, $F_{-\mu}^{DS} < F_{-\mu}^{DS,d}$ implies $F_\mu^{DS} > (F_{-\mu}^{DS,d})^{*_{2g}}$. This is our desired lower bound. We will need to show that it is close to F_μ^Δ just like our upper bound. In the last subsection we showed that $F^{DS,d}$ and $F^{\Delta,d}$ are close. The following lemma implies that their duals, with respect to $2g$, are the same amount close.

Lemma 2.19. *Taking dual convex sets is an isometry for d_R.*

$$d_R(\Omega_1, \Omega_2) = d_R(\Omega_1^*, \Omega_2^*)$$

Proof. For $x \in \partial\,\Omega_1$, let $\alpha(x) := f_1(x)/f_2(x)$ be the scaling factor necessary such that $\alpha x \in \partial\,\Omega_2$. Let $x \in \partial\,\Omega_1$ realize the supremum of α. By definition, $d_R(\Omega_1, \Omega_2) = |\log(\alpha)|$. Without loss of generality, assume $\alpha \geq 1$. As in the proof of Lemma 2.15, if H_x is a supporting hyperplane for Ω_1 at x, then H_y is a supporting hyperplane for Ω_2.

Note that $\partial\Omega_1^*$ is identified with the set of supporting hyperplanes of Ω_1. If H is a supporting hyperplane for Ω_1, then let $\beta(H)$ be the positive number such that $\beta(H)H$ is a supporting hyperplane for Ω_2. $d_R(\Omega_1^*, \Omega_2^*)$ is the supremum of $|\log(\beta)|$. We have shown that $d_R(\Omega_1, \Omega_2) \leq d_R(\Omega_1^*, \Omega_2^*)$. The reverse inequality follows from the fact that taking dual convex sets is an involution. $\square$

2.4.7 End of proof of theorem C

We have an upper bound on F_μ^{DS} which is close to F_μ^Δ

$$F_\mu^{DS} \leq F_\mu^{DS,d} \underset{\epsilon_d(r)}{\approx} F_\mu^{\Delta,d} \underset{\epsilon_{\Delta,d}}{\approx} F_\mu^\Delta$$

and a lower bound on F_μ^{DS} which is also close to F_μ^Δ

$$F_\mu^{DS} = (F_{-\mu}^{DS})^{*2g} \geq (F_{-\mu}^{DS,d})^{*2g} \underset{\epsilon_d(r)}{\approx} (F_{-\mu}^{\Delta,d})^{*2g} \underset{\epsilon_{C^0}(r)}{\approx} (F_{-\mu}^{\Delta,d})^{*2h} \underset{\epsilon_{\Delta,d}}{\approx} (F_{-\mu}^{\Delta})^{*2h} = F_\mu^\Delta$$

. Each '$\approx$' symbol means there is a bound, named in the subscript, on the absolute value of the log of the ratio. Combining upper and lower bounds gives bound on the distance between F_μ^{DS} and F_μ^Δ:

$$|\log \frac{F^{DS}}{F^\Delta}| \leq \epsilon_{\Delta,d} + \epsilon_d(r) + \epsilon_{C^0}(r)$$

We have that $\epsilon_{\Delta,d}$ limits to zero as d goes to infinity, and for each fixed d, $\epsilon_d(r)$ goes to zero as r goes to infinity. We just need to choose a function $d(r)$ which limits to infinity, but slowly enough such that $\epsilon_{d(r)}(r)$ goes to zero. Setting $\epsilon(r) = \epsilon_{\Delta,d(r)} + \epsilon_{d(r)}(r) + \epsilon_{C^0}(r)$ finishes the proof.

2.5 Continuity of length functions of Finsler metrics

Depending on context, there are minor variations on what one means by a Finsler metric. So far, we have discussed specific Finsler metrics, so we haven't needed to specify a class of Finsler metrics to work with, but now we need to prove a general fact about Finsler geometry, so we will specify exactly what we mean by Finsler metric.

Definition 2.8. A Finsler metric on a differentiable manifold M is a continuous function $F : TM \to \mathbb{R}_{\geq 0}$ satisfying

- $F_x(\lambda v) = \lambda F_x(v)$ for all $\lambda \in \mathbb{R}_{\geq 0}$
- $F_x(v + v') \leq F_x(v) + F_x(v')$
- $F_x(v) = 0$ iff $v = 0$

Importantly, we don't assume $F_x(v) = F_x(-v)$, and we don't assume that F is differentiable. If F satisfies the first two conditions, but not the third, we call it a degenerate Finsler metric, and we call the subset of M where F is degenerate the degeneracy locus. Let $\mathcal{F}(M)$ denote the set of Finsler metrics on M, and let $\mathcal{F}^{\mathrm{fd}}(M)$ denote the set of Finsler metrics with finite degeneracy locus. These definitions are useful to us because if S is a convex projective surface, F^{DS} is always in $\mathcal{F}(M)$, and F^{Δ} is always in $\mathcal{F}^{\mathrm{fd}}(S)$.

If L is a free homotopy class of loop in M, and $F \in \mathcal{F}(M)$, then we denote by $F(L)$ the infimum of the lengths of paths in L with respect to F.

$$F(L) := \inf_{\gamma \in L} \int_{\gamma} F$$

We give $\mathcal{F}(M)$ the topology in which F_i converge to F if F_i/F converge uniformly to 1 as functions on $TM - \underline{0}$. It is easy to see that $F(\gamma)$ is a continuous function of F with respect to this topology. We need to be a little more thoughtful in choosing a topology for $\mathcal{F}^{\mathrm{fd}}(M)$

Definition 2.9. We endow $\mathcal{F}^{\mathrm{fd}}(M)$ with the topology in which F_i converges to a degenerate Finsler metric F with degeneracy locus X if for every open neighborhood U of X, F_i/F converges uniformly to 1 in $M \backslash U$.

Theorem 2.20. *If $F_i \in \mathcal{F}^{fd}(M)$ is a sequence converging to a Finsler metric F with finite degeneracy locus X, and Y is a free homotopy class of loop in M, then*

$$\lim_{i\to\infty} F_i(Y) = F(Y)$$

Proof. First we show

$$\lim_{i\to\infty} F_i(Y) \leq F(Y)$$

Let $\epsilon > 0$. We can choose $\gamma \in Y$ which doesn't hit X such that $F(\gamma) \leq F(Y) + \epsilon$. This follows from continuity of F, because only very small deformations of a path are necessary to avoid X, and these will change the path's length by a small amount. Because γ avoids the degeneracy locus, we get the following:

$$\lim_{i\to\infty} F_i(Y) \leq \lim_{i\to\infty} F_i(\gamma) = F(\gamma) \leq F(Y) + \epsilon$$

Since this holds for all ϵ, it follows that $\lim F_i(\gamma) \leq F(Y)$.

Now we show the reverse inequality.

$$F(Y) \leq \lim_{i\to\infty} F_i(Y)$$

For every $i \in \mathbb{N}$, let $\gamma_i \in Y$ be a path which nearly realizes minimal length in the metric F_i.

$$F_i(\gamma_i) \leq F_i(Y) + \epsilon$$

Choose $\delta > 0$ such that closed δ-balls around points in X, for the metric F are disjoint. Let $d_{\min}$ be shortest distance between two balls. If a path in the universal cover $\gamma : [0,1] \to \tilde{M}$ hits N different balls, then $F(\gamma) \geq N d_{\min}$. Let $K > 1$. For sufficiently large i, we have $K^{-1} < F_i/F < K$ on the complement of the balls. We get a bound on the number of balls a path can visit in terms of its F_i length:

$$N \leq K F_i(\gamma)/d_{\min}$$

For sufficiently large i, $F_i(Y) \leq F(Y) + \epsilon$, so $F_i(\gamma) \leq F(Y) + 2\epsilon$. Let $\tilde{\gamma}_i : [0,1] \to \tilde{M}$ be a lift of γ_i to the universal cover. For simplicity, choose $\tilde{\gamma}$ so that that $\tilde{\gamma}(0)$ is not in the δ neighborhood of X. Putting things together, we get a bound on the number of balls $\tilde{\gamma}_i$ can visit which holds for all sufficiently large i.

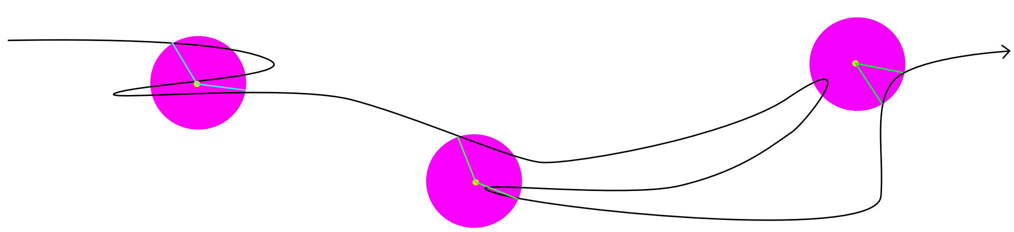
$$N_i \leq K(F(Y) + 2\epsilon)/d_{\min}$$

We now define a path γ_i' whose F length we can estimate. Let B_1 be the first ball that γ_i touches, and let $t_0, t_1 \in [0,1]$ be the first and last points which are sent by $\tilde{\gamma}_i$ to $\overline{B_1}$. Replace $\tilde{\gamma}_i|_{[t_0,t_1]}$ with a path that goes straight to p, and straight out. Now apply the same procedure to the rest of the path $\tilde{\gamma}_i|_{[t_1,1]}$. Proceed inductively, and call the final result $\tilde{\gamma}_i'$. We have an upper bound on the F length of $\tilde{\gamma}_i'$, thus an upper bound on $F(Y)$.

$$F(Y) \leq F(\gamma') \leq KF_i(\gamma_i) + 2N_i\delta \leq K(F_i(Y) + \epsilon) + 2N_i\delta$$

This bound holds for sufficiently large i for any choices of K and δ, and N_i eventually has a uniform bound, so we have $F(Y) \leq \lim_{i\to\infty} F_i(Y) + \epsilon$. This holds for any ϵ, so we get the desired inequality.

$\square$



2.6 Application to a triangle reflection group

The broad aspiration of this project is to understand what happens when we have a sequence of Hitchin representations of a surface group into $\mathrm{SL}_3\,\mathbb{R}$ going to infinity. Replacing the surface group with a triangle reflection group is a great way to probe this question, because the Hitchin component is diffeomorphic to $\mathbb{R}$, so there are only two ways to go to infinity. Before [LTW22] was published, and before we knew of Loftin's work [Lof07a], it was triangle group computations which convinced us that theorem 3.50 should hold.

Consider a triangle group.

$$\Gamma_{pqr} := \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle$$

For $1/p + 1/q + 1/r < 1$, there is a unique conjugacy class of homomorphism $\rho_0 : \Gamma_{pqr} \to SO(2,1)$ giving a proper discontinuous action on the hyperbolic plane $\mathbb{H}^2$, and the quotient $S = \Gamma \backslash \mathbb{H}^2$ is an orbifold. Let $\mathrm{Conv}(\Gamma)$ denote the component of $\mathrm{Rep}(\Gamma, \mathrm{SL}_3 \mathbb{R})$ containing ρ_0.

We can easily make an explicit algebraic parametrization of $\mathrm{Conv}(\Gamma_{pqr})$. Let v_1, v_2, v_3 and $\alpha_1, \alpha_2, \alpha_3$ be bases of $\mathbb{R}^3$ and $(\mathbb{R}^3)^*$ with the following matrix of pairings.

$$\alpha_i(v_j) = \begin{bmatrix} 2 & -2\cos(\frac{\pi}{p})t & -2\cos(\frac{\pi}{r}) \\ -2\cos(\frac{\pi}{p})t^{-1} & 2 & -2\cos(\frac{\pi}{q}) \\ -2\cos(\frac{\pi}{r}) & -2\cos(\frac{\pi}{q}) & 2 \end{bmatrix}$$

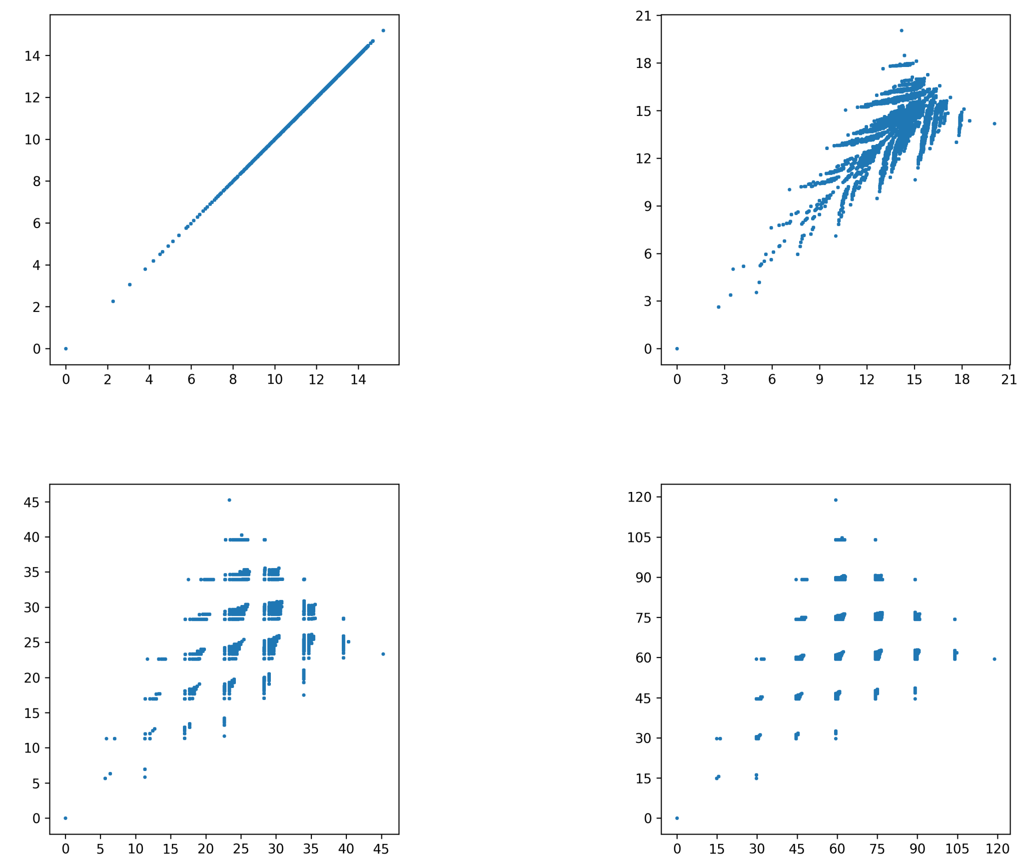
We define $\rho_t : \Gamma_{pqr} \to \mathrm{SL}_3 \mathbb{R}$ to send the generators a, b, c to the three reflections $I - v_i \otimes \alpha_i$. The parameter t is the square root of the triple ratio of the three reflections. It has the following expression:

$$t^2 = \frac{\alpha_1(v_2)\alpha_2(v_3)\alpha_3(v_1)}{\alpha_1(v_3)\alpha_2(v_1)\alpha_3(v_2)}$$

which makes it clear that it is an invariant of the three reflections. By [LLS21], we know that t gives a global parametrization of $\mathrm{Conv}(\Gamma_{pqr})$ by $\mathbb{R}_+$.

A naive way to try to understand the representation ρ_t is to plot the eigenvalues of $\rho_t(g)$ for a bunch of $g \in \Gamma$. Let $\phi : \mathrm{SL}_3 \mathbb{R} \to \mathbb{R}^2$ be the function $(\log|\lambda_1|, -\log|\lambda_3|)$ where λ_1 is the top eigenvalue, and λ_3 is the bottom eigenvalue. ϕ is called the Jordan projection for $\mathrm{SL}_3 \mathbb{R}$ and lands in the cone spanned by $(1,2)$ and $(2,1)$. In general, the Jordan projection of an element of a semi-simple lie group is the Weyl-chamber valued translation length of the the element acting on the associated symmetric space, so Jordan projections are a generalization of hyperbolic translation length.

As an example, we plot here $\phi(g)$ for orientation preserving elements $g \in \Gamma_{444}$ of length at most 24, for triple ratio t^2 set to $1, 10, 1000$, and 10^{12}.



We see that the Jordan projections increasingly tend to lie on an integral lattice. Theorem 3.50 gives a way to compute which lattice point each group element converges to.

From an algebraic perspective, it not so surprising that the Jordan projections collect onto a lattice. Note that when $\lambda_1(g)$ is big, $\log|\lambda_1(g)|$ is approximately $\log|tr(g)|$, and $-\log|\lambda_3(g)|$ is approximately $\log|tr(g^{-1})|$. Trace functions $tr(\rho_t(\gamma))$ for $\gamma \in \Gamma_{pqr}$ are Laurent polynomials in t. This means that when t is big,

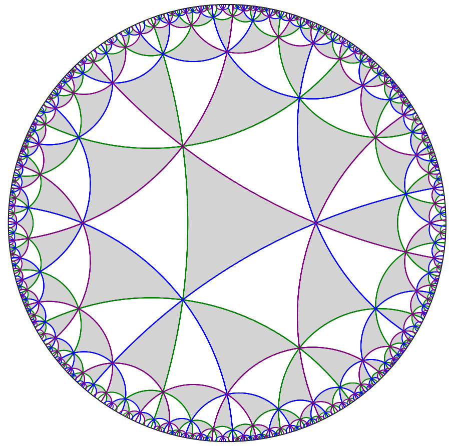
$$\phi(g) \approx \log(t)(d_1, d_2)$$

where d_1 and d_2 are the highest powers of t in $tr(\rho_t(\gamma))$ and $tr(\rho_t(\gamma^{-1}))$ respectively. From this perspective, theorem 3.50 tells us the ratios between the highest power of t in $tr(\rho_t(\gamma))$.

The Labourie-Loftin parametrization can be modified to apply to our reflection orbifold S, by defining a holomorphic cubic differential on S to be a holomorphic cubic differential $\tilde{\mu}$ on a universal cover which is preserved by orientation preserving deck transformations, and complex conjugated by orientation reversing deck transformations. The correspondence for quotient orbifolds (such as our triangle reflection orbifold) follows from the correspondence on a smooth, compact covering space, and the general fact that when solutions to PDE's are unique, they have to be invariant under the symmetry group of the input data. The correspondence between Higgs bundles and Hitchin components for orbifolds was worked out in generality in [ALS20]. Note that this definition of cubic differential for orbifolds forces the fixed loci of reflections to be real trajectories of $\tilde{\mu}$.

As predicted by the Labourie-Loftin correspondence, the space of cubic differentials on S is 1 dimensional. We now construct a nonzero element of this space. Consider a euclidian equilateral triangle $T \subset \mathbb{C}$, whose sides are unit length, and such that the restriction of dz^3 to each side is real. Let $\bar{T}$ denote T with the conjugate cubic differential $d\bar{z}^3$. We can glue copies of T and $\bar{T}$ appropriately to get a Riemann surface with cubic differential which has an action of Γ_{pqr}. Call this cubic differential μ.

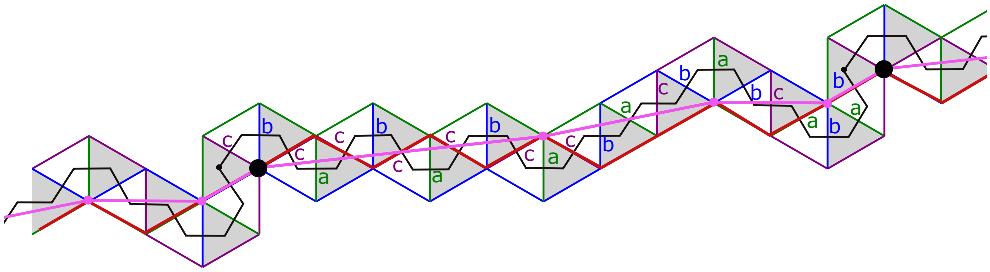
If we color copies of T grey, and color copies of $\bar{T}$ white, and conformally map the resulting surface to the disk, we get the standard picture of the pqr triangulation of the hyperbolic plane. Below is the picture for $p = q = r = 4$.



We now illustrate how to apply theorem 3.50 with an example. The word

$$w = cbcacbcacbcacbacbabcabab$$

was chosen arbitrarily from the words forming the cluster around the lattice point $(6, 5)$ in the $t = 10^{12}$ Jordan projection picture. We can compute the F_μ^Δ translation lengths of w, and w^{-1} as follows. First we construct a curve in the universal cover $\tilde{S}$ which is preserved by w by concatinating straight line segemnts connecting centers of triangles, which cross reflection loci prescribed by the letters of w. Then pull this curve curve tight in the metric h_μ. Then push it onto the reflection locus, while making sure not to change its F_μ^Δ length. F_μ^Δ assigns length 2 to edges going counter clockwise around grey triangles, and length 1 to edges going clockwise around grey triangles (and vice versa for white triangles) so we can just add up these numbers to compute translation length once we have a geodesic representative in the reflection locus.



Here, the thin black line represents the easy to construct w invariant curve, the pink line represents the geodesic for the singular flat metric h_μ, which is also a geodesic for F_μ^Δ, and the red line represents a geodesic for F_μ^Δ which lies in the reflection locus. We see that the translation length of w is 18, and the translation length w^{-1} is 15. Theorem 3.50 thus predicts

$$\lim_{t\to\infty} \frac{\log(\lambda_1(\rho_t(w)))}{\log(\lambda_1(\rho_t(w^{-1})))} = \frac{6}{5}$$

which is what we observed in the Jordan projection picture. We can verify this limit rigorously by using computer algebra software to directly compute that the highest powers of t in $tr(\rho_t(w))$ and $tr(\rho_t(w^{-1}))$ are 6, and 5 respectively. The reader can try these computations for other elements of triangle reflection groups.

Chapter 3: Spectral Radius Compactification of Hitchin Components

Abstract

The spectral radius compactification is the compactification of the Hitchin component of the $\mathrm{SL}_n\,\mathbb{R}$ character variety of a surface which records leading asymptotics of top eigenvalues. For $n = 2$, it is Thurston's compactification of Teichmuller space. The spectral radius compactification is equivalently the closure of the Hitchin component embedded in projective geodesic currents by Labourie's cross ratio construction. We call boundary currents tropical rank n currents. From a tropical rank n current μ one can construct a space X_μ with $\pi_1(S)$ action which records growth rates of top eigenvalues. In the case $n = 2$, X_μ is an $\mathbb{R}$-tree. For $n = 3$, for endpoints of cubic differential rays, X_μ is just the universal cover of S equipped with a triangular Finsler metric. For endpoints of algebraic paths in $\mathrm{Hit}^n(S)$, X_μ is always a polyhedral complex of dimension at most $n - 1$.

3.1 Introduction

Let S be a closed oriented surface of genus at least 2, and let $\Gamma := \pi_1(S)$ be its fundamental group. A discrete and faithful representation $\Gamma \to \mathrm{PSL}_2\mathbb{R}$ gives rise to an oriented hyperbolic surface $\Gamma\backslash\mathbb{H}^2$ marked by S. In this way, a connected component of the $\mathrm{PSL}_2\mathbb{R}$ character variety is identified with Teichmüller space $\mathcal{T}(S)$. Thurston defined a compactification which makes $\mathcal{T}(S)$ into a closed ball. Boundary points of the Thurston compactification parametrize geometric objects which can be thought of as parametrizing measured laminations on S, or $\mathbb{R}$-trees with Γ action. This raises the question of whether there are analogous compactifications of other spaces of representations, and whether boundary points of these compactifications are parametrizing interesting geometric objects.

We will focus on Hitchin components. For $n \geq 2$, composition with the unique irreducible representation $\mathrm{PSL}_2\mathbb{R} \to \mathrm{PSL}_n\mathbb{R}$, gives an embedding of $\mathcal{T}(S)$ into the $\mathrm{PSL}_n\mathbb{R}$ character variety of S. The *Hitchin component* $\mathrm{Hit}^n(S)$ is the component of the $\mathrm{PSL}_n\mathbb{R}$ character variety containing this copy of $\mathcal{T}(S)$.

Just as classical Teichmuller space is diffeomorphic to $\mathbb{R}^{6g-6}$, $\mathrm{Hit}^n(S)$ is diffeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. Hitchin components are the archetypal examples of higher Teichmüller spaces: components of character varieties of surfaces consisting entirely of discrete and faithful representations. We will study a generalization of Thurston's compactification to $\mathrm{Hit}^n(S)$ called the spectral radius compactification.

3.1.1 Spectral radius compactification

The spectral radius of a matrix M, is $|\lambda_1(M)|$ where $\lambda_1(M)$ is the greatest magnitude eigenvalue of M. Sending a representation of Γ to its complete list of spectral radii gives a map from $\mathrm{Hit}^n(S)$ to functions on the set $[\Gamma]$ of conjugacy classes of Γ.

$$\mathrm{Hit}^n(S) \to \mathbb{R}^{[\Gamma]}$$

$$\rho \mapsto (l_\rho : a \mapsto \log |\lambda_1(\rho(a))|)$$

We call the class function l_ρ the marked length spectrum of ρ, because for $n = 2$, $\log |\lambda_1(\rho(a))|)$ is the hyperbolic geodesic length of $[a] \in [\Gamma]$. Let $[l_\rho]$ be the projectivized marked length spectrum: the image of l_ρ in $\mathbb{P}(\mathbb{R}^{[\Gamma]})$. The map $\rho \mapsto [l_\rho]$ is an embedding of $\mathrm{Hit}^n(S)$ into $\mathbb{P}(\mathbb{R}^{[\Gamma]})$ and has compact closure. This closure is the spectral radius compactification. We will denote its boundary by $\partial_{\lambda_1}\mathrm{Hit}^n(S)$. For $n = 2$, $\partial_{\lambda_1}\mathrm{Hit}^2(S)$ is the Thurston boundary of Teichmuller space. More generally the Thurston compactification of $\mathcal{T}(S)$ sits inside the spectral radius compactification of $\mathrm{Hit}^n(S)$ for all n. Our goal is to start to unravel the geometry and combinatorics encoded in these asymptotic λ_1-spectra.

3.1.2 Geodesic currents at infinity

There is an embedding of $\mathrm{Hit}^n(S)$ into a different infinite dimensional space, namely the space of geodesic currents, which will turn out to give the same compactification of $\mathrm{Hit}^n(S)$, but with more geometric understanding of boundary points. Bonohon introduced geodesic currents for exactly this purpose in the case $n = 2$. Bonohon worked with unoriented currents whereas we will use oriented currents throughout.

There are two standard definitions of Geodesic currents, one which is local on S, and one which doesn't require choosing a metric. We start with the local definition.

Definition 3.1. Let g be a hyperbolic metric on S. A **geodesic current** on (S, g) is a locally finite Borel measure on the unit tangent bundle T^1S which is invariant under geodesic flow.

A simple example of a geodesic current is the length measure of a closed orbit. Closed orbits are in bijection with closed geodesics on S, which are in bijection with free homotopy classes of loop, which are in bijection with conjugacy classes $[a] \in [\Gamma]$. The corresponding current is denoted $\delta_{[a]}$. These currents span a dense subspace, so we can think of geodesic currents as a completion of the space of $\mathbb{R}_{>0}$ weighted oriented multi curves. Another important example is the Lioville current for the metric g. This is simply the contact volume form on the unit cotangent bundle.

To get a notion of geodesic current which doesn't depend on a choice of metric, we fix a universal cover $\tilde{S}$ and let $\Gamma = \pi_1(S)$ be its group of deck transformations. The unit tangent bundle of the universal cover $T_1\tilde{S}$ is a principal Γ bundle over T_1S, and a principal $\mathbb{R}$ bundle over the space $\mathcal{G}$ of geodesics in $(\tilde{S}, \tilde{h}_0)$.

$$\begin{tikzcd} & T^1\tilde{S} \arrow[dl] \arrow[dr] & \\ T^1S & & \mathcal{G} \end{tikzcd}$$

For a principal bundle with a chosen Haar measures on the structure group, a measure μ on the base corresponds uniquely to a measure on the total space which in any local trivialization is a product of μ with Haar measure. Flow invariant Radon measures on T^1S are thus in bijection with Γ invariant Radon measures on $\mathcal{G}$. Geodesics in $(\tilde{S}, \tilde{h}_0)$ are determined by their end points in the visual boundary of $\tilde{S}$. By the Milnor-Svark lemma, $\tilde{S}$ is quasi-isometric the Cayley graph of Γ for any finite generating set. Recall that the Gromov boundary $\partial\Gamma$ of Γ is the visual boundary of a Cayley graph of Γ. Visual boundary is a quasi-isometry invariant, so the visual boundary of $\tilde{S}$, is identified with $\partial\Gamma$. We thus have an identification $\mathcal{G} = \partial\Gamma \times \partial\Gamma \backslash \Delta$ where $\partial\Gamma$ denotes the Gromov boundary. This gives the metric-independant definition of geodesic currents:

Definition 3.2. A geodesic current on S is a locally finite, Γ-invariant Borel measure on $\partial\Gamma \times \partial\Gamma \backslash \Delta$.

In this formulation, $\delta_{[a]}$ is a sum of delta measures, one for each point in the Γ orbit of (a^-, a^+) where a^- and a^+ are the repelling and attracting fixed points of a acting on $\partial\Gamma$. A hyperbolic metric identifies $\partial\Gamma$ with $\mathbb{RP}^1$, and the corresponding Lioville measure of the box $[x_1, x_2] \times [y_1, y_2] \subset \mathcal{G}$ is the logarithm of the cross ratio of x_1, y_1, x_2, y_2. This formula for the Lioville formula generalizes directly to $\mathrm{Hit}^n(S)$.

The embedding $\mathrm{Hit}^n(S) \to \mathcal{C}(S)$ is defined using limit maps. A Hitchin representation $\rho : \Gamma \to \mathbb{P}(V)$, with $V \simeq \mathbb{R}^n$ is in particular projective Anosov, meaning that there are continuous equivariant limit maps

$$\xi : \partial\Gamma \to \mathbb{P}(V)$$

$$\xi^* : \partial\Gamma \to \mathbb{P}(V^*)$$

such that $\xi(a^+)$ is the eigenline of top eigenvalue of $\rho(a)$, and $\xi^*(a^+)$ is the eigenline of top eigenvalue of $\rho(a^{-1})^*$, and $\xi^*(x)$ contains $\xi(y)$ if and only if $x = y$. These limit maps define a measure $\mu_\rho \in \mathcal{C}(S)$ by

$$\mu_\rho([x_1, x_2] \times [y_1, y_2]) = \log |\frac{\langle \zeta^*(x_1), \zeta(y_1)\rangle \langle \zeta^*(x_2), \zeta(y_2)\rangle}{\langle \zeta^*(x_1), \zeta(y_2)\rangle \langle \zeta^*(x_2), \zeta(y_1)\rangle}|$$

where $x_1, x_2, y_1, y_2 \in \partial\Gamma$ are cyclically ordered. For $n = 2$, ξ and ξ^* are both the usual identification of $\partial\Gamma$ with the boundary of the hyperbolic plane, and μ_ρ is the Lioville measure for the hyperbolic metric.

Theorem 3.1. *The closure of* $\mathrm{Hit}^n(S)$ *in* $\mathbb{P}(\mathbb{R}^{[\Gamma]})$*, embedded via taking* λ_1 *spectrum, and the closure of* $\mathrm{Hit}^n(S)$ *in* $\mathbb{P}(\mathcal{C}(S))$ *embedded via* $\rho \mapsto \mu_\rho$ *are the same compactification.*

In the case $n = 2$, this is shown using Bonohon's intersection pairing on currents, which is a non degenerate pairing $i : \mathcal{C}(S)^{\mathbb{Z}_2} \times \mathcal{C}(S)^{\mathbb{Z}_2} \to \mathbb{R}$ extending geometric intersection number of unoriented closed curves. Conveniently, the intersection number of a closed curve with a Lioville current is the length of the curve. In other words, the marked length spectrum map $\mathrm{Teich}(S) \to \mathbb{R}^{[\Gamma]}$ factors through the embedding $\mathcal{C}(S)^{\mathbb{Z}/2} \to \mathbb{R}^{[\Gamma]}$ sending a current μ to the intersection function $i(\mu, -)$. Since the closure of $\mathrm{Teich}(S)$ is already compact in $\mathbb{P}(\mathcal{C}(S))$, one gets the same closure in $\mathbb{R}^{[\Gamma]}$.

This strategy doesn't work beyond $n = 2$ because μ_ρ are no longer symmetric currents, and the intersection pairing doesn't distinguish a current from its reverse. Instead, we think of currents and length spectra as coming from the same object, namely equivariant $\mathbb{R}$ bundles on $\mathcal{G}$ with connection. These are similar to reparametrizations of geodesic flow [Bri+15], but allow for more singular behavior. The

The Thurston boundary has two types of points: rational and irrational. The rational points parametrize homotopy classes of simple closed multicurves on S weighted by rational numbers. The and are the points which can be reached along algebraic paths. More generally, we call a geodesic current rational or integral if it is a finite sum of closed curves weighted by rational numbers or integers. For $n > 2$, it is still true that algebraic paths converge to rational boundary points.

Theorem 3.2. *If $\{\rho(t) : t \in \mathbb{R}_+\}$ is an algebraic family of representations in $\mathrm{Hit}^n(S)$ than it converges to a rational point. In fact if $\mu(t)$ is the family of currents corresponding to $\rho(t)$, then $\mu(t)/\log(t)$ converges to an integral current.*

As n increases, multicurves arising in $\partial_{\lambda_1} \mathrm{Hit}^n(S)$ have less and less stringent restrictions on self-intersection. Understanding these restrictions is a major motivation for this paper, though a complete characterization of what multicurves arise in $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ for $n > 2$ is still unknown. We do know that a tropical rank n current has no n-intersection, meaning that the support of μ cannot contain all of (x_i, y_i) where $x_1, ..., x_n, y_1, ..., y_n$ is any $2n$-tuple of cyclically ordered points on $\partial\Gamma$.

3.1.3 Geometry at infinity

There is a third perspective on the Thurston compactification: every point in the Thurston boundary corresponds to an $\mathbb{R}$-tree with Γ action. Rational points correspond to genuine trees while irrational points are more exotic. The translation lenths of Γ acting on this tree will correspond to renormalized limits of hyperbolic lengths. Generalizing this third perspective is the main goal of this paper.

Theorem 3.3. *From a tropical rank n geodesic current μ we construct a metric space X_μ with Γ action, such that the translation length of $\gamma \in \Gamma$ is $l(\gamma) + l(\gamma^{-1})$, where $l \in \mathbb{R}^{[\Gamma]}$ is the length spectrum corresponding to μ.*

This has already been done either by taking an asymptotic cone of the symmetric space, or by finding a representation over a field with valuation to produce a building with Γ action, but the space X_l is much smaller than what has been constructed by these methods, and doesn't require making any extra choices.

When $n > 2$, the top eigenvalue of an $n \times n$ matrix is usually not the same as the top eigenvalue of its inverse, meaning that length spectra in $\mathrm{Hit}^n(S)$, and its λ_1-boundary will usually not satisfy $l(\gamma) = l(\gamma^{-1})$. This asymmetry has been been a notorious stumbling block for many constructions in higher Teichmuller theory. We find a slightly technical way to encode the actual asymptotic λ_1 spectrum geometrically.

Theorem 3.4. *From a tropical rank n geodesic current μ we construct a "reletive metric" on X_μ, such that the translation length of $\gamma \in \Gamma$ is $l(\gamma)$, where $l \in \mathbb{R}^{[\Gamma]}$ is the length spectrum corresponding to μ.*

It is still largely a mystery what X_μ can look like in general, but we know a few things.

Theorem 3.5. *If μ is a discrete tropical rank n geodesic current, then X_μ is a polyhedral complex of dimension at most $n-1$.*

From the definition of X_μ this is far from obvious. The definition of X_μ actually makes sense for a broad class of geodesic currents, including those coming from negatively curved metrics and from Anosov representations, but usually X_μ will be infinite dimensional. We expect that for any tropical rank n current, X_μ has dimension at most $n-1$, but we only know how to formulate and prove this for discrete currents.

For $n = 2$ we reprove the well known fact that if μ represents a point in $\partial_{\lambda_1} \mathrm{Hit}^2(S)$, (which is the Thurston boundary) then it is symmetric, and has no self intersection i.e. it is a geodesic lamination, and X_μ is an $\mathbb{R}$-tree.

For $n = 3$, there are certain boundary points for which X_α has a very nice description, namely endpoints of cubic differential rays. If we equip S with a complex structure and a non-zero holomorphic cubic differential α we can define a certain Higgs bundle (E, ϕ_α). This Higgs bundle lies in the Hitchin section, thus by solving

Hitchin's equation we get a representation in $\mathrm{Hit}^3(S)$. Paths in $\mathrm{Hit}^3(S)$ coming from a ray of cubic differentials $\{R\alpha : R > 0\}$ are called cubic differential rays.

Theorem 3.6. *Let α be a non-zero cubic differential on S. For $R > 0$, let $\mu(R)$ be the geodesic current coming from the Hitchin representation corresponding to the Higgs bundle $(E, \phi_{R\alpha})$. Then as R goes to infinity, $\mu(R)/R$ converges to the current $T(\alpha)$ of descending real tragectories of α, and the corresponding space $X_{T(\alpha)}$ is naturally in bijection with $\tilde{S}$.*

3.1.4 Tropical geometry perspective

The idea of using tropical geometry to compactify character varieties has been extensively studied, for example [**Morgan-Shalen**], [**Fock-Goncharov**], [**Parreau**], [Ale13]. The spectral radius compactification fits into this framework. Specifically, $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is the projectivization of the logaritmic limit set of $\mathrm{Hit}^n(S)$, under the affine embedding $\mathrm{Hit}^n(S) \to (\mathbb{R}^*)^{[\Gamma]}$ via trace functions of all group elements.

Taking the logarithmic limit set is a version of tropicalization which is more analytic rather than algebraic. Let $X \subset (\mathbb{C}^*)^n$ be a subvariety of a complex algebraic torus. The amoeba of X is the image of X under coordinate-wise logarithm.

$$\log(z_1, ..., z_n) := (\log|z_1|, ..., \log|z_n|)$$

The logarithmic limit set of X the polyhedral fan in $\mathbb{R}^n$ obtained by infinitely scaling down the amoeba.

$$\mathrm{trop}'(X) := \lim_{R\to\infty} \frac{\log(X)}{R}$$

In other words, $\mathrm{trop}'(X)$ consists of the points which can be obtained as a limit of $\log(z_{(i)})/R_i$ where $z_{(i)} \in X$, and R_i is a sequence of real numbers going to infinity.

The tropicalization of X is defined by a more algebraic construction using formal algebraic paths in X instead of sequences. Let $\mathcal{K} = \mathbb{C}\{\{t\}\}$ be the field of Poisseaux series, and let $X_{\mathcal{K}} \subset (\mathcal{K}^*)^n$ be the $\mathcal{K}$ points of X. The tropicalization

$\mathrm{trop}(X)$ of X is $-\overline{v(X_{\mathcal{K}})}$, the closure of the image of X under applying the coordinate-wise valuation $v : (\mathcal{K}^*)^n \to \mathbb{R}^n$. The minus sign corresponds to use of the $(\max, +)$ semifield instead of the $(\min, +)$ semifield. The logarithmic limit set turns out to coincide with the tropicalization, $\mathrm{trop}'(X) = \mathrm{trop}(X)$, [Jon16] in this setting.

Points in $\mathrm{trop}(X)$, and $\mathrm{trop}'(X)$ satisfy "tropical equations". Let $f = a_1 z^{r_1} + ... + a_k z^{r_k}$ be a polynomial function on $(\mathbb{C}^*)^n$, where $r_i \in \mathbb{Z}^n$. The tropicalization of f is the homogeneous, piecewise linear function

$$\mathrm{trop}(f)(t) := \max\{r_1 \cdot t, ..., r_k \cdot t\}$$

where $t \in \mathbb{R}^n$. If $\log(z_{(i)})/R_i$ converges to $t \in \mathbb{R}^n$ for a sequence of points $z_{(i)} \in (\mathbb{C}^*)^n$, then $\log(f)/R_i$ will converge to $\mathrm{trop}(f)(t)$ as long as a unique term realizes the maximum in $\mathrm{trop}(f)(t)$. This means that if f vanishes on X, and $t \in \mathrm{trop}'(X)$, there must be at least two terms realizing the maximum. In other words, $\mathrm{trop}(f)$ is non-differentiable at t. The polyhedral fan in $\mathbb{R}$ where $\mathrm{trop}(f)$ is non-differentiable is called a tropical hypersurface, and is denoted by $Z(\mathrm{trop}(f))$. The "fundamental theorem of tropical geometry" implies that $\mathrm{trop}(X)$ is precisely the intersection of all tropical hypersurfaces $Z(\mathrm{trop}(f))$ where f is in the ideal cutting out X.

To accommodate $\mathrm{Hit}^n(S)$, tropical geometry must be expanded to accommodate real semi-algebraic sets. A connected component of a real algebraic variety is always a semialgebraic set. Tropical geometry may have first been applied to semi-algebraic sets in [SW05]. A different approach, with character varieties in mind, was developed in [Ale13].

Tropical geometry has to be pushed in yet another direction to accommodate $\partial_{\lambda_1}\mathrm{Hit}^n(S)$. Tropicalization always depends on an embedding into an algebraic torus, and to get $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ we use the embedding $\mathrm{Hit}^n(S) \to \mathbb{R}^{[\Gamma]}$ given by traces of all group elements. Usually tropicalization is defined using a finite set of functions on X which are somehow special, but $\mathrm{Hit}^n(S)$ doesn't have any finite set of functions which is invariant under the natural symmetry, namely the mapping class group. In algebraic geometry, a well accepted version of tropicalization which does not depend on an embedding in $(\mathbb{C}^*)^n$ is the Berkovich analytification X^{an} which in a sense [Pay09], is the tropicalization of X using all functions. It is not unreasonable to use infinitely many functions for tropicalization. The construction of $\partial_{\lambda_1}\mathrm{Hit}^n(S)$ is actually much less exotic than analytification because trace functions are all non-vanishing on $\mathrm{Hit}^n(S)$.

Practically speaking, tropical geometry could be helpful for characterizing exactly which geodesic currents actually arise in the boundary of $\mathrm{Hit}^n(S)$. It could provide tools for determining an efficient collection of tropical equations for which every solution is guaranteed to show up in the logarithmic limit set.

3.1.5 Symplectic perspective on negative curvature

Many of the constructions of this paper are motivated by the symplectic perspective on negative curvature pioneered by Otal [Ota92], and with roots going back to Arnold and Hilbert. In much of this paper issues of regularity might overshadow the geometric ideas, and the fact that we work in two dimensions makes the symplectic geometry less apparent, so describe the picture here in a familiar setting. The passage from geometry to symplectic geometry is the same as always: instead of looking at a space X we look at T^*X, but the story plays out in a particular way when X is a negatively curved manifold.

Let X be a Hadamard manifold: a simply connected complete Riemannian manifold with sectional curvature bounded above by $\epsilon < 0$. The exponential map is a diffeomorphism $T_xX \to X$ for any $x \in X$. The visual boundary ∂X is the set of geodesic rays $\gamma : [0, \infty) \to X$ modulo the equivalence relation $\gamma_1 \sim \gamma_2$ if $d(\gamma_1(t), \gamma_2(t))$ is bounded. The visual boundary is naturally identified with the unit tangent sphere at any point.

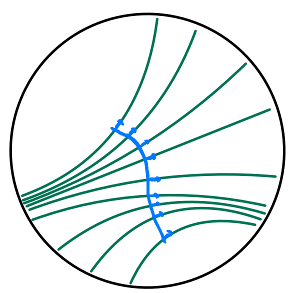
Let $\mathcal{G}$ be the space of oriented, unparametrized geodesics in X. Since X is

Hadamard, a geodesic is encoded by its visual endpoints.

$$\mathcal{G} = \partial X \times \partial X \backslash \Delta$$

We will see that $\mathcal{G}$ has a natural symplectic structure. The cotangent bundle T^*X has a cannonical symplectic structure. Under the identification of TX with T^*X by the metric, geodesic flow becomes Hamiltonian flow of the inverse metric. The space of geodesics $\mathcal{G}$ is thus symplectic reduction of T^*X.

Let U be the unit cotangent bundle of X. It is useful to view U as a principal $\mathbb{R}$ bundle over $\mathcal{G}$ where the $\mathbb{R}$ action is geodesic flow. The tautological 1-form λ on T^*X restricts to a contact form on U, which can also be viewed as a connection for this $\mathbb{R}$ bundle. This connection has a simple geometric origin back on X: a path of unit cotangent vectors (x_s, α_s) for $s \in \mathbb{R}$ is a flat section of U if $\alpha_s(x'_s) = 0$. It is also useful to view U as the space of parametrized geodesics. The connection α declares a path of parametrized geodesics $\gamma_s(t)$ to be flat if $\partial_s(\gamma_s(t))$ is perpendicular to $\partial_t(\gamma_s(t))$.



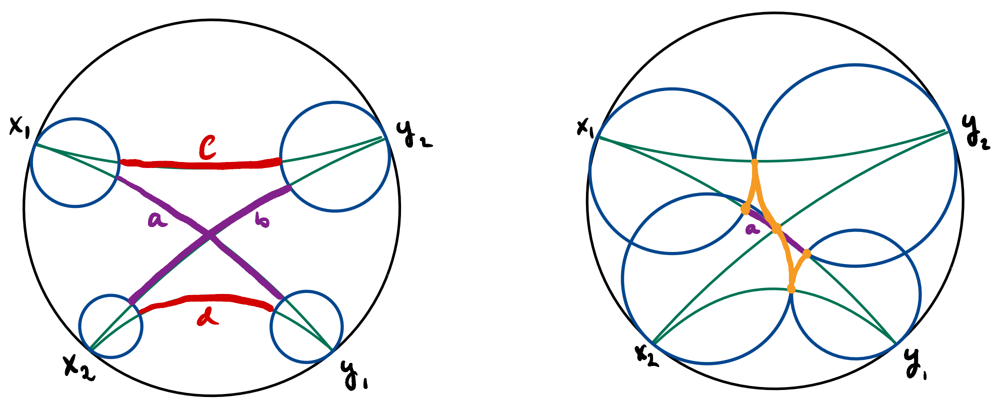
Points in X correspond to Legendrian spheres in U, namely a point corresponds to its unit cotangent sphere. More generally, the unit conormal bundle of a submanifold $Y \subset X$ is a Legendrian submanifold of U. Legendrians in U project to (possibly singular) Lagrangians in $\mathcal{G}$. The Lagrangian coming a point $p \in X$ is the sphere of geodesics passing through p, and has the special property that it projects diffeomorphically to both factors of ∂X. We call any such Lagrangian sphere monotonic, and call a Legendrian sphere in U monotonic if it projects to a monotonic Lagrangian sphere.

If we only have the contact manifold (U, λ), it is not clear if we can reconstruct the space X, but as a replacement we could consider the space L_U of all monotonic Legendrian spheres. Perhaps better, we can quotient L_U by the $\mathbb{R}$ action of geodesic flow and get the space of Lagrangian spheres X_U over which U admits flat sections. This is the idea behind definition 3.23.

The visual boundary of X has a structure called a cross ratio, which is a function on tuples $(x_1, x_2, y_1, y_2) \in \partial X^4$ with $x_i \neq y_j$, defined in terms of horospheres. Recall that a Bussmann function on X is a function of the form

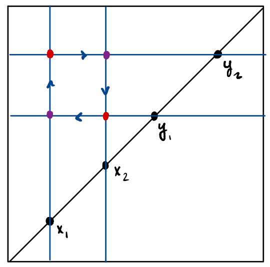
$$h(p) = \lim_{t \to \infty} d(\gamma(t), p) - t$$

where $\gamma : [0, \infty) \to X$ is a geodesic ray. A horosphere centered at $x \in \partial X$ is a level set of a Bussemann function of a geodesic ray converging to x. After choosing "cutoff" horospheres centered at x_1, x_2, y_1, y_2 we define the cross ratio to be the following combination of lengths of geodesic segments.



$$b(x_1, y_1, x_2, y_2) := a + b - c - d$$

Note that this quantity does not depend on the choice of horocycles. A horosphere S centered at x will be perpendicular to all geodesics eminaiting from (converging to) x. This means that the set of parametrized geodesics γ starting at x with $\gamma(0) \in S$ is a flat section of U. Viewing U as the unit cotangent bundle, this section is the outward (inward) conormal to S. With a judicious choice of horospheres, it becomes apparent that $b(x_1, x_2, y_1, y_2)$ is the holonomy of (U, α) around a box in $\mathcal{G}$ from (x_1, y_1) to (x_1, y_2) to (x_2, y_2) to (x_2, y_1) back to (x_1, y_1) which only moves one endpoint at a time.



Inward and outward conoromals to horospheres are flat sections over fibers of the two projections from $\mathcal{G}$ to ∂X. Existance of these flat sections implies that both these projections are Lagrangian fibrations. A symplectic manifold with two transverse Lagrangian foliations is called para-Kähler because this is the structure obtained by replacing the complex structure in the definition of Kähler with an endomorphism which squares to identity. If we choose a horosphere for every $x \in \partial X$, then we get a cut-off distance function on $\mathcal{G}$ which is a para-Kähler potential for the symplectic structure.

One last piece of this story we will use is the symplectic interpretation of distance: the length of a loop in X is the holonomy of the loop of geodesics tangent to it. Let $\gamma : [0,1] \to X$ be a differentiable path, and let $u : [0,1] \to U$ be a continuous path of unit tangent vectors to γ. Under the identification with the unit cotangent bundle, u is a path of unit covectors such that $u(t)$ evaluated on $\gamma'(t)$ is $|\gamma'(t)|$. The integral of the contact form λ along u is thus the length of γ.

Everything in this section works more generally for Finsler manifolds of negative flag curvature. The identification of unit tangent bundle with unit cotangent bundle is achieved by the legendre transform which identifies v with α if $\alpha(v) = 1$. The only subtlety to mention is that for asymmetric Finsler metrics there are two types of horospheres, one for backward endpoints of geodesics and one for forward endpoints of geodesics. In section 3.7, we will generalize this picture, in the two dimensional case, to a class of Finsler metrics whose unit balls are not necessarily properly convex. In this case the symplectic structure on $\mathcal{G}$ can become quite degenerate, and even concentrate onto a discrete or cantor subset of $\mathcal{G}$. We will develop notions of $\mathbb{R}$-bundle with connection, monotonic Legendrian sphere, and para-Kähler potential adapted to this level of generality.

3.1.6 Other compactifications and other groups

Instead of recording only the top eigenvalue, it might seem more natural to record the ordered list of all eigenvalues $(\log|\lambda_1|, ..., \log|\lambda_n|)$. This has a natural generalization to a split real lie group G, namely Weyl chamber valued translation length on the symmetric space G/K. Marked Weyl chamber length spectrum gives an map

$$\mathrm{Hit}(S, G) \to (\mathfrak{t}^+)^{[\Gamma]}$$

where $\mathfrak{t}^+$ is the positive Weyl chamber of the lie algebra of G. The corresponding compactification of $\mathrm{Hit}(S, G)$ is known as the Weyl chamber length compactification. Analogously to the spectral radius compactification, the Weyl chamber length compactification coincides with the compactification obtained from embedding $\mathrm{Hit}^n(S)$ into a space of currents, now valued in the cone spanned by positive coroots. It would be interesting to extend the construction of X_μ to these other compactifications.

3.2 Equivariant bundles and geodesic currents

In this section we will study the relationship between three objects: geodesic currents, positive holonomy functions, and equivariant bundles with positive taxi connection. There are forgetful maps relating these three objects.

$$\begin{array}{c}\text{equivariant bundles with}\\ \text{positive taxi connection}\end{array} \longrightarrow \begin{array}{c}\text{positive holonomy}\\ \text{functions}\end{array} \longrightarrow \text{geodesic currents} \tag{3.1}$$

Geodesic currents were introduced in [Bon88], and have been used used since then. The other two objects are variations on existing definitions which allow for less regularity. Positive holonomy functions are a generalization of cross ratios [Lab07]. Similarly, equivariant bundles with positive taxi-connection generalize reparametrizations of geodesic flow, which were used to study Anosov representations in [Bri+15].

Geodesic currents coming from Anosov representations, and from Finsler metrics both have natural lifts to equivariant bundles with connection. Though these bundles are more complicated objects then geodesic currents, they end up being easier to work with. We will show that any geodesic current which maps to zero in first cohomology (in a sense we will define) is the curvature of some equivariant bundle, and any two equivariant bundles with the same curvature only differ by a change in equivariance.

3.2.1 Taxi connections

We will define a notion of equivariant bundles with connection on $\mathcal{G}^\circ$. The curvature of such a bundle will be a geodesic current.

Definition 3.3. A **segment** of $\mathcal{G}$ is a subset of the form $s_{x,x';y} := [x, x'] \times \{y\}$, where $y < x \leq x' < y$, or $s_{x;y,y'} := \{x\} \times [y, y']$ where $x < y \leq y' < x$.

Let $\mathcal{G}^\circ$ be the set of points of $\mathcal{G}$ which are not fixed by any group element.

Definition 3.4. Let A be a group. A **taxi connection** F on a principal A bundle P over $\mathcal{G}$ is a G-orbit $F(s)$ of sections over every segment $s \subset \mathcal{G}$ which is compatible with restriction to subsegments. We refer to $F(s)$ as the flat sections over s. Similarly, a taxi connection on a principal A bundle over $\mathcal{G}^\circ$ is a choice of A orbit of sections over every segment $s \subset \mathcal{G}^\circ$, which is compatible with restriction.

This definition gives a notion of parallel transport along "taxi-paths" i.e. concatenations of horizontal and vertical segments. For us, A will always be an Abelian lie group, usually $\mathbb{R}$ or $\mathbb{R}^*$, though maximal tori of split real lie groups and lie algebras will also appear. Recall that principal bundles with Abelian structure group can be tensored.

Definition 3.5. Let A be an abelian group, and let P and Q be principal A bundles on a space X, the tensor product $P \otimes Q$ is the quotient of $P \times_X Q/A$ by the equivalence relation $(p \cdot a, q) \sim (p, q \cdot a)$ for $a \in A$.

If P and Q are principal A bundles with taxi connection on $\mathcal{G}$ or $\mathcal{G}^\circ$ then $P \otimes Q$ inherits a taxi connection in a straitforward way. We now define chain complexes which will help us prove things about taxi bundles.

Definition 3.6. Define the chain complex $C_2(\mathcal{G}) \xrightarrow{\partial} C_1(\mathcal{G}) \xrightarrow{\partial} C_0(\mathcal{G})$ as follows.

- Let $C_0(\mathcal{G})$ denote the free Abelian group on points in $\mathcal{G}$.
- Let $C_1(\mathcal{G})$ denote the free Abelian group on the set of segments in $\mathcal{G}$, modulo the subgroup generated by $s_{x;y,y'} + s_{x;y',y''} - s_{x;y,y''}$ for all $x < x' < x'' < y$, and $s_{x,x';y} + s_{x',x'';y} - s_{x,x'';y}$ for all $y < y' < y'' < x$.
- Let $C_2(\mathcal{G})$ denote the free Abelian group generated by boxes $r_{x,x';y,y'} := [x, x'] \times [y, y'] \subset \mathcal{G}$ modulo the subgroup generated by $r_{x,x';y,y'} + r_{x',x'';y,y'} - r_{x,x'';y,y'}$ for $x < x' < x'' < y < y'$ and $r_{x,x';y,y'} + r_{x,x';y',y''} - r_{x,x';y,y''}$ for $y < y' < y'' < x < x'$.
- Define $\partial : C_1(\mathcal{G}) \to C_0(\mathcal{G})$ by

$$\partial\, s_{x,x';y} := (x', y) - (x, y)$$

$$\partial\, s_{x;y,y'} := (x, y') - (x, y)$$

- Define $\partial : C_2(\mathcal{G}) \to C_1(\mathcal{G})$ by

$$\partial\, r_{x,x';y,y'} = s_{x;y,y'} + s_{x,x';y'} - s_{x';y,y'} - s_{x,x';y}$$

By convention we use the same notation for a segment and the corresponding generator in $C^1(\mathcal{G})$, and also denote $-s_{x,x';y}$ by $s_{x',x;y}$ and $-s_{x;y,y'}$ by $s_{x;y',y}$. We think of taxi 1-chains as integral combinations of oriented segments in $\mathcal{G}$. To talk about bundles with connection on $\mathcal{G}^\circ$ we need a restricted complex.

Definition 3.7. Let $C_*(\mathcal{G}^\circ) \subset C_*(\mathcal{G})$ be the subcomplex where

- $C_0(\mathcal{G}^\circ)$ is the free abelian group on points in $\mathcal{G}^\circ$,
- $C_1(\mathcal{G}^\circ)$ is the free abelian group on segments in $\mathcal{G}^\circ$, and
- $C_2(\mathcal{G}^\circ)$ is the free abelian group on rectangles $r_{x,x';y,y'}$ with boundary in $\mathcal{G}^\circ$.

Definition 3.8. For an Abelian group A, let $C^i(\mathcal{G}^\circ, A) := \mathrm{Hom}(C^i(\mathcal{G}), A)$ and let d be the dual differential.

A taxi connection F on the trivial bundle $\mathcal{G}^\circ \times A$ gives a cocycle $t_F \in C^1(\mathcal{G}^\circ)$. The connection F is a choice of A torsor $F(s)$ of functions $s \to A$ over each segment s, compatible with restriction. Define

$$t_F(s) = f(\partial_+ s) - f(\partial_- s)$$

where $f \in F(s)$, and $\partial_\pm s$ are the front and back endpoints of s.

Lemma 3.7. *The map $F \mapsto t_F$ from taxi connections on the trivial bundle to $C^1(\mathcal{G}^\circ, A)$. Is a bijection.*

Proof. We give an inverse. Let $t \in C^1(\mathcal{G}^\circ, A)$. For a horizontal segment $s_{x,x';y}$, define $F(s_{x,x';y})$ to be the set of functions $f : s \to A$ such that for any two points $x \leq p < q \leq x'$ we have $f((q, y)) - f((p, y)) = t(s_{p,q;y})$. The function $f((p, y)) = t(s_{x,p;y})$ will satisfy this property, so $F(s)$ is non-empty. It is easy to see that two elements of $F(s)$ must differ by a constant, and that F is compatible with restriction. Define $F(s)$ for vertical segments in the same way. $\square$

The cochain t_F corresponding to a connection F on the trivial bundle can be thought of as parallel transport. If $c \in C^1(\mathcal{G}^\circ)$ is a taxi path i.e. a sum of segments $s_1 + ... + s_k$ with $\partial_+ s_i = \partial_- s_{i+1}$ for $i = 1, ..., k-1$, then $t_F(c)$ is the parallel transport along this path. If c is a loop, i.e. $\partial_+ s_k = \partial_- s_1$, then $t_F(c)$ is the holonomy of this loop.

Lemma 3.8. *Let A be an Abelian group. Principal A-bundles with taxi connection on $\mathcal{G}^\circ$ are classified up to isomorphism by* $\mathrm{Hom}(Z_1(\mathcal{G}^\circ), A)$.

Proof. Any A-bundle admits a trivialization, and any two trivializations differ by addition of a function $f : \mathcal{G}^\circ \to A$. This means we can just study connections on the trivial bundle, quotiented by the action of addition of functions.

Let F be a taxi connection on the trivial bundle, and let $t_F \in C^1(\mathcal{G}^\circ, A)$ be the corresponding cochain. If $f : \mathcal{G}^\circ \to A$ is a function, then $F + f$ is the taxi connection obtained by change of trivialization. The cochain t_F will change in a simple way.

$$t_{F+f} = t_F + df$$

Thus A bundles with taxi connection are classified up to isomorphism by $C^1(\mathcal{G}^\circ, A)/dC^0(\mathcal{G}^\circ, A)$. Applying $\mathrm{Hom}(-, A)$ to the exact sequence

$$0 \to Z_1(\mathcal{G}^\circ) \to C_1(\mathcal{G}^\circ) \to C_0(\mathcal{G}^\circ),$$

we see $C^1(\mathcal{G}^\circ, A)/dC^0(\mathcal{G}^\circ, A) = \mathrm{Hom}(Z_1(\mathcal{G}^\circ), A)$. More concretely, the restriction of t_F to cycles doesn't depend on trivialization, and classifies bundles with connection up to isomorphism. $\square$

We refer to the element of $\mathrm{Hom}(Z_1(\mathcal{G}^\circ), A)$ corresponding to (P, F) as the holonomy function $h_{(P,F)}$ of (P, F) because when we evaluate on taxi-loops it gives the holonomy. We have shown that taxi-connections are determined by holonomy, and every holonomy function is realizable.

The notion of curvature we will use is the following:

Definition 3.9. The curvature of a taxi-connection F on an A-bundle P, $curv(F) \in C^2(\mathcal{G}^\circ, A)$, is the 2-cocycle $curv(F)$ which assigns $h_F(\partial r)$ to every rectangle r with boundary in $\mathcal{G}^\circ$.

If we trivialize P, then F is equivalent to the cochain $t_F \in C^1(\mathcal{G}^\circ, A)$, and $curv(F)$ is simply dt_F. This means that bundles with trivial curvature are classified by first cohomology of $C^*(\mathcal{G}^\circ, A)$, and curvatures which cannot be realized by bundles are classified by second cohomology.

Lemma 3.9. *The homology of $C_*(\mathcal{G}^\circ)$ is $\mathbb{Z}, \mathbb{Z}, 0$, consequently the cohomology of $C^*(\mathcal{G}^\circ)$ is $A, A, 0$.*

Note that this is the same as the homology and cohomology of $\mathcal{G}$, and can be proven directly. The group of bundles with trivial holonomy is thus A, and every curvature is realizable.

For various arguments, we will want curvature to be a measure on $\mathcal{G}$ which when integrated over a rectangle gives holonomy around the boundary. In the case of positive curvature this is indeed the case.

Lemma 3.10. *If $c \in C^2(\mathcal{G}^\circ, \mathbb{R})$ takes positive values on all rectangles, than there is a unique Borel measure on $\mathcal{G}$ which agrees with c on all rectangles with boundary in $\mathcal{G}^\circ$, and it assigns zero measure to segments in $\mathcal{G}^\circ$. Conversely every locally finite, positive, borel measure which is zero on segments in $\mathcal{G}^\circ$ gives an element of $c \in C^2(\mathcal{G}^\circ, \mathbb{R})$ by evaluation on rectangles.*

To define positivity for $A \neq \mathbb{R}$ we will assume A is endowed with a partial order, such that the monoid A_+ of elements greater than or equal to identity is isomorphic to $(\mathbb{R}_{\geq 0})^n$. For $A = \mathbb{R}$ we use the standard order, and for $A = \mathbb{R}^*$ we say $a < b$ if $|a| < |b|$ and a and b have the same sign. We let $\mathcal{C}^2(\mathcal{G}^\circ, A)_+ \subset C^2(\mathcal{G}^\circ, A)$ denote the space of cocycles such that $c(r) \in A_+$ for all rectangles r with boundary in $\mathcal{G}^\circ$, and identify these cocycles with the corresponding measures. In particular the space of invariant, positive cocycles, $C^2(\mathcal{G}^\circ, A)_+$ is naturally identified with the space of geodesic currents $\mathcal{C}(S, A)$.

3.2.2 Geodesic currents and holonomy functions

Let Now we investigate the map from Γ invariant holonomy functions to geodesic currents.

$$d : \mathrm{Hom}(Z^1(\mathcal{G}^\circ), A)_+^\Gamma \to \mathcal{C}(S, A)$$

Holonomy functions in the kernel of d, are in bijection with $H^1(\mathcal{G}^\circ, A)$. The action of Γ on $H^1(\mathcal{G}^\circ, A)$ is trivial, so $H^1(\mathcal{G}^\circ, A)$ lies inside $\mathrm{Hom}(Z^1(\mathcal{G}^\circ), A)_+^\Gamma$. This means each fiber of d is a torsor for $H^1(\mathcal{G}^\circ, A)$ which is isomorphic to A.

The cokernel of d is $\mathrm{Hom}(\Gamma, A) \simeq A^{2g}$ but this is a bit more tricky to see. Every geodesic current can be lifted to a holonomy function, but not always to a Γ-invariant holonomy function. We will construct a map $\mathcal{C}(S, A) \to \mathrm{Hom}(\Gamma, A)$ whose kernel is manifestly the image of d. This map sends a current to its poincare dual cohomology class of S. We use the signed version of Bonohon's intersection product to make this precise.

Definition 3.10. Let I_+, and I_- denote the subsets of $\mathcal{G} \times \mathcal{G}$ consisting of pairs of geodesics which intersect positively and negatively respectively.

$$I_+ := \{((x, y), (x', y')) \in \mathcal{G} \times \mathcal{G} | x' < x < y' < y\}$$

$$I_- := \{((x, y), (x', y')) \in \mathcal{G} \times \mathcal{G} | x < x' < y < y'\}$$

The intersection product of two geodesic currents μ_1, μ_2 is

$$i(\mu_1, \mu_2) := \int_{(I_- \cup I_+)/\Gamma} \mu_1 \times \mu_2$$

whereas the signed intersection product is

$$i_{sgn}(\mu_1, \mu_2) := \int_{I_+/\Gamma} \mu_1 \times \mu_2 - \int_{I_-/\Gamma} \mu_1 \times \mu_2$$

Note that i is symmetric, whereas i_{sgn} is antisymmetric. In the case where one of the arguments is $\delta_{[a]}$ for $a \in \Gamma$, the formula is a bit more explicit. Let $I_{a,+}, I_{a,-} \subset \mathcal{G}$ denote the set of geodesics intersecting (a^-, a^+) positively, and negatively respectively.

$$i(\delta_{[a]}, \mu) := \int_{(I_{a,-} \cup I_{a,+})/a} \mu$$

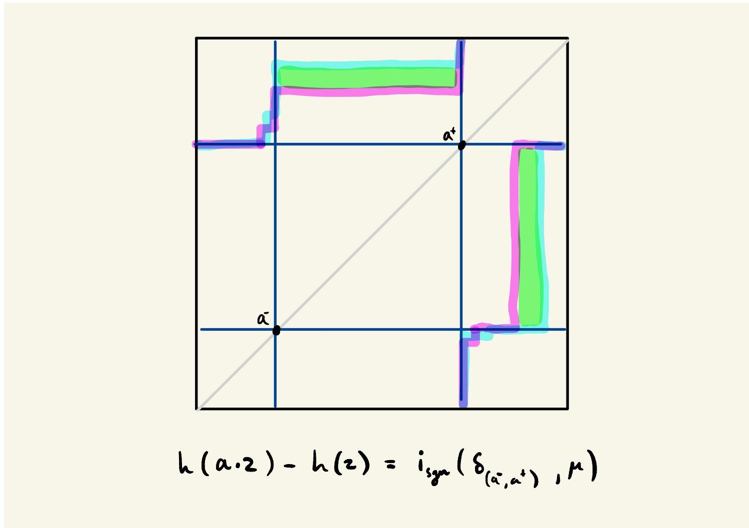
$$i_{sgn}(\delta_{[a]}, \mu) := \int_{I_{a,-}/a} \mu - \int_{I_{a,+}/a} \mu$$

In this paper, we won't use the intersection product very much because it tends to ignore the asymmetry of geodesic currents. The signed intersection product on the other hand is rarely used anywhere, because it vanishes on most geodesic currents of interest, for example those coming from Anosov representations or negatively curved Finsler metrics. The next lemma explains why this is the case.

Lemma 3.11. *For a holonomy function lifting μ to be invariant, μ must satisfy $i_{sgn}(\delta_{[a]}, \mu) = 0$ for all $a \in \Gamma$. It is enough to check this condition for a set of group elements which descend to a basis of $H^1(S, A)$.*

Proof. Let us fix an arbitrary geodesic current μ. Γ will act on the set of holonomy functions with curvature μ, which is an A torsor, so we get a homomorphism $\phi_\mu : \Gamma \to A$. Since A is abelian, ϕ_μ descends to a map $H_1(S, A) \to A$. The geodesic current can be upgraded to an invariant holonomy function only when $\phi_\mu = 0$.

Now we show that $\phi_\mu(a) = i_{sgn}(\delta_{[a]}, \mu)$. To compute $\phi_\mu(a)$ it is sufficient to compute $h(a \cdot z) - h(z)$ where h is any holonomy function with curvature μ, and z is any taxi cycle in $\mathcal{G}^\circ$ which generates the homology of $\mathcal{G}$. The following picture shows an example choice of z. The difference in holonomy is the signed measure between the two cycles. The two big rectangles will contribute $i_{sgn}(\delta_{[a]}, \mu)$ while the small rectangles contribute nothing.



$\square$

3.2.3 Holonomy functions and equivariant bundles

Now we show that the forgetful map from equivariant bundles with taxi connection to holonomy functions has kernel $H^1(S, A)$ and cokernel A.

Definition 3.11. Let $\mathcal{A}(S, A)$ denote the group of isomorphism classes of equivariant A bundle with connection on $\mathcal{G}^\circ$ which have positive holonomy.

Proposition 3.12. *There is an exact sequence.*

$$0 \to H^1(\Gamma, A) \to \mathcal{A}(S, A) \to \mathrm{Hom}(Z_1(\mathcal{G}^\circ), A)^\Gamma \to H^2(\Gamma, A) \to 0$$

Definition 3.12. Let Γ_P be the group consisting of pairs (γ, ϕ) where $\gamma \in \Gamma$, and ϕ makes the square

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d] & P \arrow[d] \\ \mathcal{G} \arrow[r, "\gamma"] & \mathcal{G} \end{tikzcd}$$

commute.

The group Γ_P is a central extension of Γ by $\mathbb{R}$, and a splitting $\Gamma \to \Gamma_P$ is precisely the data making P equivariant for Γ. Note that isomorphism classes of central extensions of Γ by $\mathbb{R}$ are classified by $H^2(\Gamma, \mathbb{R})$, which is the same as $H^2(S, \mathbb{R})$, which can be cannonically identified with $\mathbb{R}$ because S is a closed oriented surface. We denote the real number classifying a central extension $\tilde{\Gamma}$ by $\chi(\tilde{\Gamma})$ because if $B \to S$ is an oriented circle bundle, $\pi_1(B)$ will be a central extension of Γ by $\mathbb{Z}$, and $\chi(\pi_1(B))$ will be the Euler class of B.

Recall that the group structure on $H^2(G, A)$, where G is a group and A is an Abelian group, corresponds to a natural operation on central extensions.

$$[\tilde{G}_1] + [\tilde{G}_2] = [\tilde{G}_1 \underset{G}{\times} \tilde{G}_2/A]$$

Here $A \subset \tilde{G}_1 \underset{G}{\times} \tilde{G}_2$ is the subgroup $\{(a, a^{-1}) : a \in A\}$.

Proposition 3.13. *If P and Q are $\mathbb{R}$ bundles on $\mathcal{G}$ with taxi connection, with Γ-invariant curvature, then*

$$[\Gamma_{P \otimes Q}] = [\Gamma_P] + [\Gamma_Q].$$

Proof. The fiber product $\Gamma_P \underset{\Gamma}{\times} \Gamma_Q$ is the group of triples (γ, ϕ_1, ϕ_2), where $\gamma \in \Gamma$, and $\phi_1 : P \to P$, and $\phi_2 : Q \to Q$ both cover the action of γ action on $\mathcal{G}$. Quotienting this fiber product by the subgroup $\{(1, a, -a) : a \in \mathbb{R}\}$ gives $\Gamma_{P \otimes Q}$. $\square$

So we have a homomorphism from $\mathcal{H}(\mathcal{G}^\circ)$, to $H^2(S, \mathbb{R}) = \mathbb{R}$. We next check that this homomorphism is non-trivial on the subgroup of flat bundles.

Proposition 3.14. *If Q is a flat bundle on $\mathcal{G}$ with holonomy 1 around the cycle given by a full rotation, then $\chi(\Gamma_Q) = 2 - 2g$ where g is the genus of S.*

Proof. For convenience, fix a negatively curved metric on S so that we may talk about its unit tangent bundle T^1S. Note that $\tilde{\Gamma} = \pi_1(T^1S)$ is a central extension of Γ with $\chi(\tilde{\Gamma}) = \chi(S) = 2 - 2g$. The extension $\tilde{\Gamma} \to \Gamma$ commutes with the actions of $\tilde{\Gamma}$ on $\widetilde{T^1S}$, and Γ on $T^1\tilde{S}$. These actions commute with the $\mathbb{R}$ actions on $\widetilde{T^1S}$ and $T^1\tilde{S}$ by geodesic flow. The quotients $\widetilde{T^1S}/\mathbb{R} = \tilde{\mathcal{G}}$ and $T^1\tilde{S}/\mathbb{R} = \mathcal{G}$ thus inherit compatible actions of $\tilde{\Gamma}$ and Γ. We have determined that a $\mathbb{Z}$ local system with holonomy 1, namely $\tilde{\mathcal{G}} \to \mathcal{G}$, corresponds to the $\mathbb{Z}$ central extension $\tilde{\Gamma}$, with $\chi(\tilde{\Gamma}) = \chi(S)$. This will remain unchanged if we pass from $\mathbb{Z}$ to $\mathbb{R}$. $\square$

We have the following corollary.

Proposition 3.15. *If P is an $\mathbb{R}$ bundle with connection on $\mathcal{G}$ whose curvature is invariant under Γ, then there is a unique flat $\mathbb{R}$ bundle Q on $\mathcal{G}$ such that $\chi(Q \otimes P) = 0$, thus $Q \otimes P$ admits Γ equivariance.*

Proposition 3.16. *We can modify the action of Γ on P by an element of $\mathrm{Hom}(\Gamma, \mathbb{R})$, and any two equivariant singular $\mathbb{R}$-bundles with connection with curvature μ differ by such a modification.*

Proof. The action of $\mathrm{Hom}(\Gamma, \mathbb{R})$ on equivariant $\mathbb{R}$ bundles with connection comes from tensoring with trivial $\mathbb{R}$ bundles with possibly non-trivial equivariance. Let P and Q be equivariant $\mathbb{R}$ bundles with the same curvature. The $\mathbb{R}$ bundle $\mathrm{Hom}(P, Q)$ will be a flat equivariant $\mathbb{R}$ bundle. By proposition 3.14 it must have trivial holonomy, but it may be acted on non-trivially by Γ, so is described by an element of $\mathrm{Hom}(\Gamma, \mathbb{R})$. Finally note $Q = P \otimes \mathrm{Hom}(P, Q)$. $\square$

3.3 Geodesic currents and length spectra

If we have a Γ equivariant A bundle P on $\mathcal{G}$, the period $l_P(a)$ of $a \in \Gamma$ is the amount that it translates the fiber over the fixed point (a^+, a^-). Soon we will extend this notion to bundles over $\mathcal{G}^\circ$. Since P is equivariant, $l_P(a)$ will only depend on the conjugacy class of a. We call the function $l_P \in \mathbb{R}^{[\Gamma]}$ the period spectrum of P. It will follow from lemma 3.20 that if P has a positive taxi connection, then $l_P(a) + l_P(b) \geq l(ab)$ whenever the axes of a and b cross.

Definition 3.13.

$$\mathcal{L}(S,A) := \{l \in A^{[\Gamma]} : l(a)+l(b) \geq l(ab) \text{ for all } \ a,b \in \Gamma \text{ with } \ (a^- < b^- < a^+ < b^+)\}$$

We refer to elements of $\mathcal{L}(S,A)$ as length spectra. In this section we find that taking period spectrum actually gives a bijection between length spectra and equivariant bundles with connection.

$$\mathcal{A}(S,A) \xrightarrow{\simeq} \mathcal{L}(S,A)$$

3.3.1 Defining periods

By definition, an equivariant bundle with taxi connection has no fiber over (a^+, a^-), so we need to come up with an alternative definition for the period of $a \in \Gamma$.

Lemma 3.17. *Let P be an $\mathbb{R}$ bundle with taxi connection on $\mathcal{G}^\circ$. Let $a \in \Gamma$. Consider the four rays emanating from (a^-, a^+).*

$$R_N = \{(a^-, y) : y > a^+\} \qquad R_S = \{(a^-, y) : y < a^+\}$$

$$R_E = \{(x, a^+) : x > a^-\} \qquad R_W = \{(x, a^+) : x < a^-\}$$

There is a number $l(a)$, which we call the ***period*** *of a, such that if s is a flat section of P over any of these rays, we have $a \cdot s = l(a)s$.*

Proof. Since a fixes each of these rays setwise, and must send flat sections to flat sections, we get a periods for each ray, but it isn't obvious that these four periods coincide. We illustrate why sections over R_N and R_E are shifted the same amount. Let

$$p = s_{a^-,x;y} \cup s_{x;y,a^+}$$

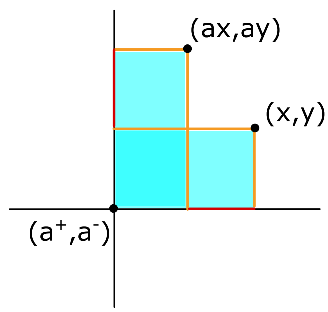
be a two segment taxi path in $\mathcal{G}^\circ$ connecting R_N to R_E. Choose a flat section σ over p. The translate $a\sigma$ will be a flat section over

$$ap = s_{a^-,ax;ay} \cup s_{ax;ay,a^+}.$$

Using σ, and $a\sigma$ to make a lift, we see that the holonomy around the figure-8 cycle obtained by joining p and ap with a segment of R_N and a segment of R_E will be the difference in the periods of a measured over R_N and R_E. This holonomy coincides with the difference in measure of two boxes with respect to the curvature of P

$$\mu_P([a^-, ax] \times [a^+, ay]) - \mu_P([a^-, x] \times [a^+, y])$$

which must vanish by invariance of the measure. $\square$



3.3.2 Reduced length spectra and currents

From an equivariant bundle with taxi connection P, one can extract the period spectrum $l_P \in \mathcal{L}(S, A)$, and the curvature $\mu_P \in \mathcal{C}_0(\Gamma, A)$. In this section we derive formulas relating l_P and μ_P. One can change the periods of an equivariant $\mathbb{R}$ bundle without changing its curvature by modifying the Γ action by an element of $\mathrm{Hom}(\Gamma, \mathbb{R})$, so we make the following definition:

Definition 3.14. The space of reduced length spectra is $\mathcal{L}(S, A)/\operatorname{Hom}(\Gamma, A)$.

We will use the well known formula 3.19 expressing $l(a) + l(a^{-1})$ as a cross ratio, along with a seemingly new formula 3.20 expressing $l_P(a) + l_P(b) - l_P(ab)$ as a cross ratio, to show that reduced period spectrum and curvature determine eachother. This equivalence between length spectra and nullhomologous currents implies the equivalence between bundles and length spectra.

$$\begin{array}{ccc} \mathcal{A}(S,A) & \xrightarrow{\quad\simeq\quad} & \mathcal{L}(S,A) \\ \downarrow & & \downarrow \\ \mathcal{C}_0(\Gamma,A) & \xrightarrow{\simeq} & \mathcal{L}(S,A)/\operatorname{Hom}(\Gamma,A) \end{array}$$

There is a general strategy for producing formulas relating periods and holonomy which we give now, though it is not strictly necessary and the reader can skip to lemmas 3.19 and 3.20.

Lemma 3.18. *Suppose $\gamma_k \cdots \gamma_1 = e$ is a relation in Γ. Then then there is a path r in $\mathcal{G}^\circ$ such that*

$$\sum_{i=1}^{k} l_P(\gamma_i) = \mathrm{hol}_P(r)$$

for any equivariant $\mathbb{R}$-connection on $\mathcal{G}$.

Proof. Choose $g = (x, y) \in \mathcal{G}^\circ$ with neither x or y fixed by any element of Γ. Choose a lift $\tilde{g}$ to the $\mathbb{R}$-bundle P. Let $g_i = \gamma_i \cdot \gamma_{i-1} \cdots \gamma_1 g$, and $\tilde{g}_i = \gamma_i \cdot \gamma_{i-1} \cdots \gamma_1 \tilde{g} g$ for $i = 0, ..., k$. Let $(x_i, y_i) = g_i$. Let r_i be the following concatenation of segments in $\mathcal{G}^\circ$.

$$(x_{i-1}, y_{i-1}) \to (x_{i-1}, \gamma_i^+) \to (x_i, \gamma_i^+) \to (x_i, y_i)$$

We next construct a discontinuous piecewise section $\tilde{r}_i$ of P over r_i with endpoints $\tilde{g}_i$ and $\tilde{g}_{i+1}$. Over $(x_{i-1}, y_{i-1}) \to (x_{i-1}, \gamma_i^+)$ use the flat section s starting at $\tilde{g}_{i-1}$, extend continuously over $(x_{i-1}, \gamma_i^+) \to (x_i, \gamma_i^+)$, but then use the section $\gamma_i(s^{-1})$ over $(x_i, \gamma_i^+) \to (x_i, y_i)$. The discontinuity of $\tilde{r}_i$ is exactly the period $l_P(\gamma_i)$.

Concatenating the sections $\tilde{r}_i$ we obtain a piecewise flat section over the closed loop r in $\mathcal{G}^\circ$ with total discontinuity

$$\sum_{i=1}^{k} l_P(\gamma_i)$$

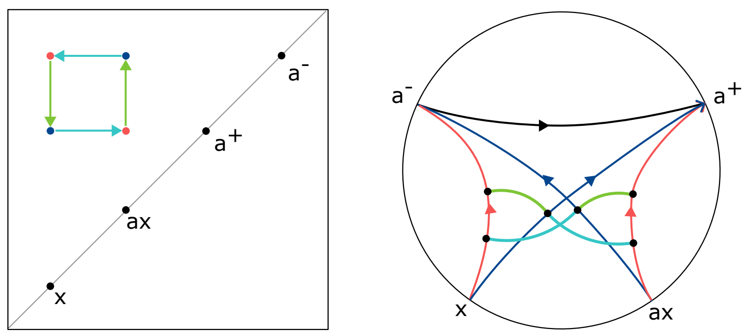
which must coincide with the holonomy around r.

□

In the proof, we chose x, and y not to be fixed points to guarentee that the resulting cycle r lay in $\mathcal{G}^\circ$. In practice, we may choose any starting point we like as long as the resulting cycle lies in $\mathcal{G}^\circ$. Applying the construction of lemma 3.18 to the simplest relation $a^{-1}a = e$, recovers the following well known fact.

Lemma 3.19. *$l(a)+l(a^{-1}) = h(x, ax; a^-, a^+)$ for any $a \in \Gamma$, and any $x \in \partial\Gamma \backslash \{a^+, a^-\}$.*

Proof. For the starting point use $g = (x, a^+)$, and choose a point p in the fiber P_g.



∎

Now we apply the strategy from lemma 3.18 to get a loop whose holonomy is $l(a) + l(b) - l(ab)$, but we deviate slightly from the algorithm so that this loop is just a single cross ratio.

Lemma 3.20. *For any $a, b \in \Gamma$ we have the following relation between periods and cross ratios.*

$$l(a) + l(b) - l(ba) = h((ba)^-, b^-; a^+, b \cdot a^+)$$

Proof. The right hand side is by definition the holonomy around a cycle with four segments. We will check that that each of these segments is in $\mathcal{G}^\circ$ so that this is well defined, but first let us complete the proof assuming this. Let $c = ba$. We subdivide so that the cycle is written with five segments.

$$(c^-, a^+) \xrightarrow{s_1} (ac^-, a^+) \xrightarrow{s_2} (b^-, a^+) \xrightarrow{s_3} (b^-, ba^+) \xrightarrow{s_4} (c^-, ba^+) \xrightarrow{s_5} (c^-, a^+)$$

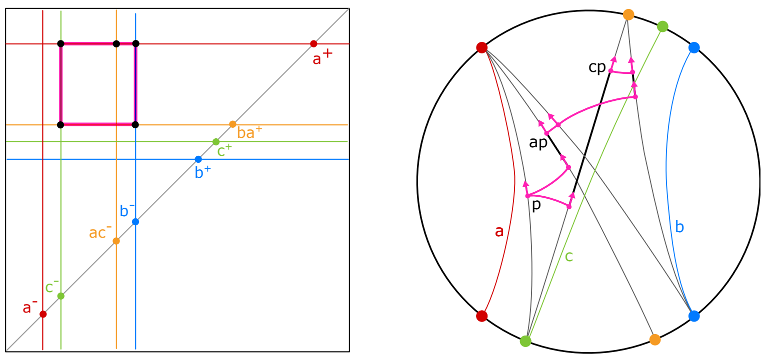
Choose a point $p \in P$ in the fiber over (c^-, a^+). Let $\tilde{s}_1$ be the flat section of P over s_1 starting at p. Let $\tilde{s}_2$ be the flat section over s_2 starting at ap. Let $\tilde{s}_3$ be the flat section over s_3 which agrees with $\tilde{s}_2$ at (b^-, a^+). Let $\tilde{s}_4 = b\tilde{s}_2$, and note that its endpoint over (c^-, ba^+) is cp. Let $\tilde{s}_5$ be the flat lift of s_5 starting at cp. The lift

$$\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3 + \tilde{s}_4 + \tilde{s}_5$$

has jumps by $l(a)$, $l(b)$ and $-l(c)$ at (ac^-, a^+), (b^-, ba^+), and (c^-, a^+) respectively, so we have

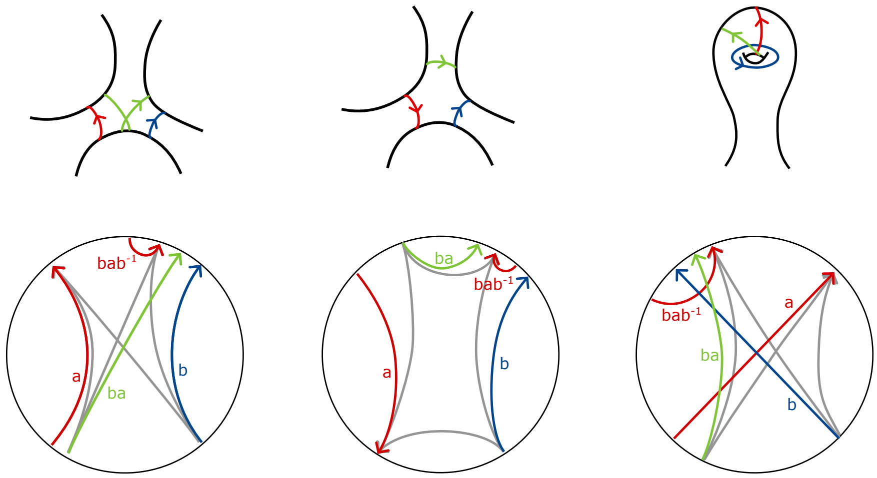
$$h((ba)^-, b^-; b \cdot a^+, a^+) + l(a) + l(b) - l(c) = 0$$

which is equivalent to the statement of the Lemma.



Now we return to the issue of showing that the segments invloved in $h((ba)^-, b^-; ba^+, a^+)$ are all in $\mathcal{G}^\circ$. There are three topological possibilities for the relative position of the fixed points of a and b.

We need to also know where the fixed points of ba and bab^{-1} are on the circle in each of these scenarios. (Note that $ba^+ = (bab^{-1})^+$.) One way to locate these fixed points is to draw the curves on the infinite volume surface $\langle a, b\rangle\backslash\tilde{S}$, which will either be a three-holed sphere, or a one-holed torus.



Here, the grey geodesics are the corners of the taxi-path which we hope lies in $\mathcal{G}^\circ$. Since this taxi-path has one coordinate being a fixed point of of a, b, ba, or bab^{-1} at all times, the only fixed points it could run into are the four geodesics in the picture. One inspects the picture to make sure this doesn't happen. $\square$

Lemma 3.20 lets us express any period as an integral combination of holonomies, and periods of a generating set.

Lemma 3.21. *For any equivariant A bundle with connection on $\mathcal{G}^\circ$, holonomy determines reduced period spectrum.*

Proof. Working with reduced length spectra is the same as choosing a standard generating set $a_1, ..., a_{2g} \in \Gamma$ and working with ordinary length spectra which satisfy $l(a_i) = l_i$ for some fixed $l_1, ..., l_{2g} \in A$.

Suppose for induction that we can express every word in $a_1, ..., a_{2g}, a_1^{-1}, ..., a_{2g}^{-1}$ of length k as a sum of l_i, and holonomies of taxi cycles in $\mathcal{G}^\circ$. Let w' be a word of length $k+1$. Suppose $w' = a_i w$. By lemma 3.20 we have

$$l(w') = l(a_i) + l(w) + h((a_i w)^-, a_i^-; w^+, a_i \cdot w^+)$$

Now suppose $w' = a_i^{-1}w$. We can first use lemma 3.20

$$l(w') = l(a_i^{-1}) + l(w) + h((a_i^{-1}w)^-, a_i^+; w^+, a_i^{-1} \cdot w^+)$$

then choose any $x \in \partial\Gamma^\circ$ not fixed by a_i and use lemma 3.19

$$l(w') = -l(a_i) + l(w) + h(x, a_i x; a_i^-, a_i^+) + h((a_i^{-1}w)^-, a_i^+; a_i^{-1}w^+, w^+)$$

$\square$

Note that we didn't need positivity of curvature to show that holonomy determines length spectrum. On the other hand, to show that reduced length spectrum determines holonomy, we will make use of positivity. By the dynamics of $SL(2,\mathbb{R})$ acting on $\mathbb{RP}^1$ we have

$$\lim_{N\to\infty} (b^N a^N)^- = a^-$$
$$\lim_{N\to\infty} b^N (a^N)^+ = b^+$$

If the points (a^-, a^+) and (b^-, b^+) have zero measure with respect to the curvature of h, then $h(x, x'; y, y')$ is continuous near $(x, x', y, y') = (a^-, b^-, a^+, a^+)$, so we have

$$\lim_{N\to\infty} l(a^N) + l(b^N) - l(b^N a^N) = h(a^-, b^-; a^+, b^+).$$

When the taxi-path is the perimeter of a box, this limit will give the measure of the interior of the box.

Fixed points of group elements are dense in $\mathcal{G}$, so knowing a measure on rectangles of the form $(a^-, b^-) \times (b^+, a^+)$ determines the measure. Curvature is thus determined by reduced length spectrum.

A positive, equivariant bundle is determined by its curvature up to changing the equivariance, but the period spectrum will clearly fix the equivariance, so we have the following.

Proposition 3.22. *Two equivariant $\mathbb{R}$ bundles with positive taxi connection with the same periods must be isomorphic.*

3.4 Rank n and tropical rank n holonomy functions

In this section we start by recalling a theorem from [Lab07] which exhibits $\mathrm{Hit}^n(S)$ as a sitting inside the space of cross ratios, but in the language of holonomy functions. We show how to associate an $\mathbb{R}^*$ bundle with taxi connection to a Hitchin representation, and we characterize exactly which $\mathbb{R}^*$ bundles with connection arise in this way. We call these rank n-holonomy functions, though they are really equivalent to Labourie's rank n cross ratios. Next we investigate the logarithmic limit cone of $\mathrm{Hit}^n(S)$ inside holonomy functions. Holonomy functions in this cone are called tropical rank n holonomy functions, and their curvatures are called tropical rank n currents. We demonstrate that tropical rank n currents have no self n-intersection. This fact was discovered by Labourie, using a slightly different framework.

3.4.1 $\mathbb{R}^*$ bundles with connection from Hitchin representations

Let Γ be the fundamental group of a closed oriented surface of genus at least 2. Let $\partial\Gamma$ denote the Gromov boundary of Γ. Recall that $\partial\Gamma$ is homeomorphic to a circle. Let V be a real vector space of dimension $n \geq 2$. Let $\rho : \Gamma \to \mathrm{SL}(V)$ be a representation in $\mathrm{Hit}^n(\Gamma)$. In particular, ρ is *projective Anosov* meaning that we have continuous equivariant limit maps

$$\xi : \partial\Gamma \to \mathbb{P}(V)$$

$$\xi^* : \partial\Gamma \to \mathbb{P}(V^*)$$

such that, for each $a \in \Gamma$, $\xi(a^+)$ is the eigenline of top eigenvalue of $\rho(a)$, and $\xi^*(a^+)$ is the eigenline of $\rho(a^{-1})^*$ of top eigenvalue. It is sometimes helpful to think of $\mathbb{P}(V^*)$ as the set of hyperplanes in $\mathbb{P}(V)$. The limit maps are transverse in the sense that, $\xi^*(x)$ contains $\xi(y)$ if and only if $x = y$. Hitchin representations are precisely the projective Anosov representations such that $V = \xi(x_1) + ... + \xi(x_n)$ for any tuple of distinct points $x_1, ..., x_n \in \partial\Gamma$. This property of ξ is known as hyperconvexity.

This pair of limit maps lets us define a function

$$B_\rho(x_1, y_1, x_2, y_2) := \frac{\langle \tilde{\xi}^*(x_1), \tilde{\xi}(y_1)\rangle \langle \tilde{\xi}^*(x_2), \tilde{\xi}(y_2)\rangle}{\langle \tilde{\xi}^*(x_1), \tilde{\xi}(y_2)\rangle \langle \tilde{\xi}^*(x_2), \tilde{\xi}(y_1)\rangle}$$

defined on tuples of points $x_1, y_1, x_2, y_2 \in \partial\Gamma$ satisfying $x_1 \neq y_2$ and $x_2 \neq y_1$ called the **cross ratio** of ρ. Here, $\tilde{\xi}$ and $\tilde{\xi}^*$ are arbitrary lifts of the limit maps to $V\backslash\{0\}$ and $V^*\backslash\{0\}$.

Definition 3.15 ([Lab07])**.** A cross ratio on $\partial\Gamma$ is a Hölder function

$$B : \{(x_1, x_2, y_1, y_2) \in \partial\Gamma^4 | x_1 \neq y_2, x_2 \neq x_1\} \to \mathbb{R}$$

satisfying

1. $B(x_1, y_1, x_2, y_2) + (x_2, y_1, x_3, y_2) = B(x_1, y_1, x_3, y_2)$
2. $B(x_1, y_1, x_2, y_2) + (x_1, y_2, x_2, y_3) = B(x_1, y_1, x_2, y_3)$
3. $B(x_1, y_1, x_2, y_2) = B(x_2, y_2, x_1, y_1)$
4. $B(x_1, y_1, x_2, y_2) = 1$ if $x_1 = x_2$ or $y_1 = y_2$
5. $B(x_1, y_1, x_2, y_2) = 0$ if $x_1 = y_1$ or $x_2 = y_2$

The space of Γ invariant cross ratios on $\partial\Gamma$ naturally embeds into the space of $\mathbb{R}^*$ valued holonomy functions $\mathrm{Hom}(Z_1(\mathcal{G}, \mathbb{R}^*))^\Gamma$, The cross ratios which are B_ρ for some $\rho \in \mathrm{Hit}^n(S)$ have a simple characterization, and are known as rank-n cross ratios.

Theorem 3.23 (Labourie [Lab07])**.** *The map $\rho \mapsto B_\rho$ embeds $\mathrm{Hit}^n(S)$ into the space of cross ratios. A cross ratio B comes from a Hitchin representation if and only if*

$$\det_{ij}(B(x_0, y_0, x_i, y_j))$$

is non-zero for any tuple of $2n+2$ distinct points $x_0, ..., x_n, y_0, ..., y_n$ and is zero for any $2n+4$ distinct points $x_0, ..., x_{n+1}, y_0, ..., y_{n+1}$.

We now realize this cross ratio as the holonomy function of a bundle. Let $\Delta \subset \mathbb{P}(V^*) \times \mathbb{P}(V)$ be the set of pairs consisting of a hyperplane containing a point. The manifold $\mathbb{P}(V^*) \times \mathbb{P}(V) \backslash \Delta$ comes with structure which we can pull back via $\xi^* \times \xi$ to $\mathcal{G}$. Namely $\mathbb{P}(V^*) \times \mathbb{P}(V) \backslash \Delta$ is the base of the principal $\mathbb{R}^*$ bundle

$$U := \{(\alpha, v) \in V^* \times V : \alpha(v) = 1\}$$

equipped with a natural connection ∇. The $\mathbb{R}^*$ action is $\lambda \cdot (\alpha, v) = (\lambda^{-1}\alpha, \lambda v)$. The connection ∇ can be described by its property that the affine subspaces

$$\{(\alpha, v) \in U : \alpha = \alpha_0\} \quad \alpha_0 \in V^* \backslash \{0\}$$

$$\{(\alpha, v) \in U : v = v_0\} \quad v_0 \in V \backslash \{0\}$$

are flat sections over the fibers of the projections of $\mathbb{P}(V^*) \times \mathbb{P}(V) \backslash \Delta$ to $\mathbb{P}(V^*)$ and $\mathbb{P}(V)$ respectively. The curvature of this connection is a symplectic form for which the projections to $\mathbb{P}(V^*)$ and $\mathbb{P}(V)$ are lagrangian fibrations. If γ is a loop in $\mathbb{P}(V^*) \times \mathbb{P}(V)$ which visits the 4 points

$$(\zeta_1^*, \zeta_1), (\zeta_1^*, \zeta_2), (\zeta_2^*, \zeta_2), (\zeta_2^*, \zeta_1) \in \mathbb{P}(V^*) \times \mathbb{P}(V)$$

via four paths in each of which only one coordinate is changing, then the holonomy of ∇ around γ is a cross ratio.

$$\mathrm{hol}_\nabla(\gamma) = \frac{\langle \zeta_1^*, \zeta_1 \rangle \langle \zeta_2^*, \zeta_2 \rangle}{\langle \zeta_1^*, \zeta_2 \rangle \langle \zeta_2^*, \zeta_1 \rangle}$$

Let P_ρ be the pullback of U by $\xi^* \times \xi$.

$$P_\rho := \{(\alpha, v) \in V^* \times V : \alpha(v) = 1, [\alpha] \in \mathrm{Im}(\xi^*), [v] \in \mathrm{Im}(\xi)\}$$

We define flat sections of P_ρ over a segment H to be the sections which are contained in a flat section of U. More concretely, for each $x \in \partial\Gamma$, and $\alpha \in \xi^*(x) \backslash \{0\}$ there is a flat section $\{x\} \times \partial\Gamma \backslash \{x\} \to U$ taking (x, y) to (α, v) where v is the element of $\xi(y)$ such that $\alpha(v) = 1$. Flat sections over horizontal lines $\partial\Gamma \backslash \{y\} \times \{y\}$ are similarly indexed by $\xi(y) \backslash \{0\}$.

Definition 3.16. A **potential** for a holonomy function $h \in \mathrm{Hom}(Z_1(\mathcal{G}), A)$ is a function $M : \mathcal{G} \to A$ such that

$$h(x_1, y_1, x_2, y_2) = \frac{M(x_1, y_1)M(x_2, y_2)}{M(x_1, y_2)M(x_2, y_1)}.$$

If h is only defined on cycles in $\mathcal{G}^\circ$, than we only ask that M be defined on $\partial\Gamma^\circ \times \partial\Gamma^\circ \backslash \Delta$.

To obtain a potential for h_ρ, simply choose lifts of the limit maps $\tilde{\xi} : C \to V$, and $\tilde{\xi}^* : C \to V^*$, and define $M(x, y) = \langle \tilde{\xi}^*(x), \tilde{\xi}(y) \rangle$. Note that when n is even there won't exist continuous lifts, but it isn't important here that the lifts be continuous. In fact, potentials are always of this form.

Lemma 3.24. *If M is a potential for a bundle with taxi connection on $\mathcal{G}$ or $\mathcal{G}^\circ$, then $M = s_2 - s_1$ where s_1 is flat along vertical segments and s_2 is flat along horizontal segments. The sections s_1, s_2 are determined up to simultaniously shifting by a constant.*

Proof. Define a connection H on the trivial bundle $\underline{A}$ on $\mathcal{G}$ as follows. If s is a vertical segment, let $H(s)$ be the set of constant functions. If s is a horizontal segment, let $H(s)$ be functions of the form $M|_s + C$. We see that

$$\mathrm{hol}_H(x_1, x_2; y_1, y_2) = M(x_1, y_1) + M(x_2, y_2) - M(x_1, y_2) - M(x_2, y_1)$$

so M is a potential for $(\underline{A}, H)$. The bundle $(\underline{A}, H)$ comes with a natural vertically flat section $s_1 = 0$, and a naturally horizontally flat section, namely $s_2 = M$, and the difference $s_2 - s_1$ is clearly M. Any other (P, H) for which M is a potential, will have the same holonomy, thus be isomorphic to $(\underline{A}, H_M)$. Choosing an isomorphism gives the desired sections of P, and different isomorphisms simultaniously shift these sections by constants. $\square$

Definition 3.17. An $\mathbb{R}^*$ valued holonomy function h is rank n if any, thus every, potential M satisfies

$$\det(M(x_i, y_j)) = 0$$

for all tuples $x_1, ..., x_{n+1}, y_1, ..., y_{n+1} \in \partial\Gamma$, and

$$\det(M(x_i, y_j)) \neq 0$$

for all tuples $x_1, ..., x_n, y_1, ..., y_n \in \partial\Gamma$ with $x_i \neq x_j$, and $y_i \neq y_j$ when $i \neq j$. By convention, $M(x, x) = 0$.

The holonomy function of P_ρ for $\rho \in \mathrm{Hit}^n(S)$ is clearly rank n, as $M(x, y) = \langle \tilde{\xi}^*(x), \tilde{\xi}(y) \rangle$ is a potential. It is clear that $n+1 \times n+1$ minors of $M(x, y)$ vanish whereas, by hyperconvexity, $n \times n$ minors do not.

Lemma 3.25. *Rank n holonomy functions in $\mathcal{H}(S, \mathbb{R}^*)$ are the same as rank n cross ratios in the sense of Labourie.*

Proof. For any holonomy function h, $h(x_0, y_0, x, y)$ is a partially defined potential for h. It is defined for $x \neq y_0$ and $y \neq x_0$. A potential m is rank n if and only if it is rank n on subsets of this form. $\square$

3.4.2 Tropical rank-n cross ratios

The notion of tropical rank-n is based on the following lemma, which is just the application of the standard tropicalization of polynomials to the determinant.

Lemma 3.26. *If A_i is a sequence of $n \times n$ matrices with $\lim \frac{\log|A_i|}{R_i} = a$ for a sequence $R_i \to \infty$, then*

$$\lim_{i\to\infty} \frac{\log|\det(A_i)|}{R_i} = \max_{\sigma \in S_n} \sum_{i=1}^{n} a(x_i, y_{\sigma(i)})$$

whenever there is a single permutation $\sigma \in S_n$ attaining the maximum. In particular, if $\det(A_i) = 0$ for all i, then two permutations must attain the maximum.

The right hand side is called the tropical determinant of the matrix a.

Definition 3.18. A holonomy function h is tropical rank n if for any potential $m(x, y)$ for h, there are two distinct permutations realizing the maximum in the tropical determinant.

$$\max_{\sigma \in S_n} \sum_{i=1}^{n} m(x_i, y_{\sigma(i)})$$

for any tuples $x_1, ..., x_{n+1}, y_1, ..., y_{n+1} \in \partial\,\Gamma^\circ$ with $x_i \neq y_j$.

Proposition 3.27. *If h_i are rank n holonomy functions and*

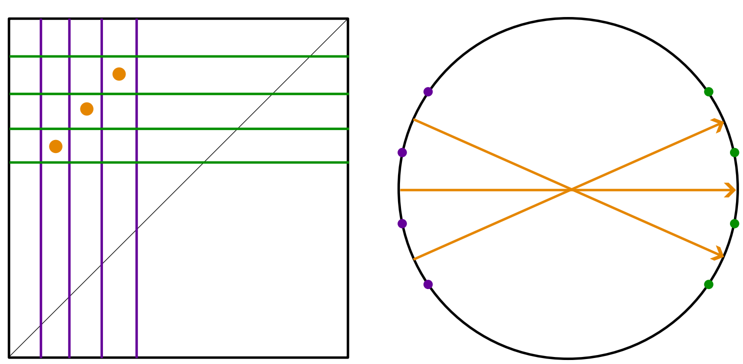
$$\lim_{i \to \infty} \frac{\log |h_i|}{R_i} = h$$

for a sequence of real numbers $R_i \to \infty$, then h must be tropical rank n.

Proof. Choose a potential m for h. Fix distinct $x_1, ..., x_{n+1}, y_1, ..., y_{n+1} \in \partial\,\Gamma$. We just have to show that $m(x_i, x_j)$ satisfies the tropical rank n condition. Further choose distinct $x_0, y_0 \in \partial\,\Gamma$. The functions $m_i^0 = \log |h_i(x_0, y_0, x, y)|/R_i$ converge to a function $m^0(x, y)$ defined for $x \neq y_0$ and $y \neq x_0$ which is tropical rank n. m^0 will only differ from $m(x, y)$ by a function of the form $f(x) + g(y)$, so $m(x, y)$ is also tropical rank n. $\square$

We do not know if every tropical rank n holonomy function arises in the boundary of $\mathrm{Hit}^n(S)$. There is a theme in

Just as the boundary of Teichmuller space consists of currents with no self intersection, boundary points of $\mathrm{Hit}^n(S)$ have no "n-intersection".



Proposition 3.28. *If μ is a tropical rank n current, then for any $x_1 < ... < x_{n+1} < y_1 < ... < y_{n+1} \in \partial\Gamma^\circ$, there must be some $i \in 1, ..., n$ such that $\mu([x_i, x_{i+1}] \times [y_i, y_{i+1}]) = 0$.*

Proof. Let h denote the holonomy function corresponding to μ and let m be any potential for h. If $\sigma, \sigma' \in S^n$ are two permutations, than the difference

$$\sum_{i=1}^{n+1} m(x_i, y_{\sigma(i)}) - m(x_i, y_{\sigma'(i)})$$

is the holonomy of a cycle. In the special case when the two permutations differ by a transposition, $\sigma' = \sigma(ij)$, the difference of the sums is

$$m(x_i, y_{\sigma(i)}) + m(x_j, y_{\sigma(j)}) - m(x_i, y_{\sigma(j)}) - m(x_j, y_{\sigma(i)}) = h(x_i, y_{\sigma(i)}, x_j, y_{\sigma(j)})$$

and in the case when $i < j$ and $\sigma(i) < \sigma(j)$ we recognize this as the measure of the box $\mu([x_i, x_j] \times [y_{\sigma(i)}, y_{\sigma(j)}])$. Thus, whenever $\sigma' = \sigma(ij)$, with $i < j$ and $\sigma(i) < \sigma(j)$,

$$\sum_{i=1}^{n+1} m(x_i, y_{\sigma(i)}) \geq m(x_i, y_{\sigma'(i)}).$$

In other words, the map

$$\sigma \mapsto \sum_{i=1}^{n+1} m(x_i, y_{\sigma(i)})$$

is weakly increasing for the reverse Bruhat order on the symmetric group. Let $X(\sigma)$ denote the number of crossings of σ, i.e. the number of pairs $i < j$ with $\sigma(i) < \sigma(j)$. The reverse Bruhat order is defined by the property that σ covers σ' iff $\sigma' = \sigma(ij)$, and $X(\sigma) = X(\sigma') + 1$. Recall that an element of a poset a covers another element b if $a > b$ and there is no c such that $a > c > b$. There is a unique maximal element for the reverse Bruhat order, namely the identity permutation.

Since m is tropical rank n, we know that at least two permutations must maximize $\sum_{i=1}^{n+1} m(x_i, y_{\sigma(i)})$. One of those must be the identity permutation, and another must be covered by identity, i.e. must be a transposition of the form $(i(i+1))$ for some $i \in \{1, ..., n\}$. The difference, $\mu([x_i, x_{i+1}] \times [y_i, y_{i+1}])$, must then be zero. $\square$

3.5 From a current to a metric space

In this section, for any nullhomologous geodesic current μ, we construct a metric space X_μ with Γ action. In the case when μ is the curvature of an equivariant bundle P with period function l, the translation length of $a \in \Gamma$ acting on X_μ is $l(a) + l(a^{-1})$. The non-symmetrized periods are encoded in a more abstract structure on X_μ which we call a relative metric. If μ is a continuous measure, than X_μ is infinite dimensional, but when μ is tropical rank n, X_μ is at most $n-1$ dimensional.

3.5.1 Holonomy zero lower submeasures

We will use a variation on the concept of an order ideal.

Definition 3.19. A subset I of a poset A is an **order ideal** if for every a in I, every element less than a is in I.

The relevant poset for us is $\mathcal{G}$ with $(x', y') < (x, y)$ if $(x < x' < y' < y)$. Order ideals of $\mathcal{G}$ arise in a natural way. Choosing a hyperbolic metric on S, $\mathcal{G}$ becomes identified with the set of geodesic half-spaces in $\tilde{S}$ which are naturally ordered by inclusion. For each $x \in \tilde{S}$, the set of half spaces not containing x is an order ideal of $\mathcal{G}$.

Definition 3.20. If μ is a measure on $\mathcal{G}$, a **lower submeasure** of μ is a measure $\nu < \mu$ such that if $(x, y) \in \operatorname{supp}(\nu)$ then ν and μ coincide on $\mathcal{G}_{<(x,y)}$.

For example, if $l \subset \mathcal{G}$ is a monotonic path in $\mathcal{G}^\circ$ which wraps around once, then define ν_l to be the measure which coincides with μ below l and is zero above l. Note that for any two loops l and l', the difference $\nu_l - \nu_{l'}$ will be a finite signed measure because μ is locally finite.

If ν is a lower submeasure, than $\bar{\nu} := \mu - \nu$ is an upper submeasure. Instead of using lower submeasures, we could use "monotone partitions" of μ: partitions $\mu = \nu + \bar{\nu}$ such that every point of $\operatorname{supp}(\nu)$ is less than or equal to every point of

$\mathrm{supp}(\bar{\nu})$. This is perhaps more natural, but we find the notation of lower submeasures less cumbersome.

Definition 3.21. A lower submeasure ν is **admissible** if $\nu - \nu_l$ is a finite measure for a monotonic loop $l \subset \mathcal{G}^\circ$.

Definition 3.22. Let μ be a nullhomologous geodesic current, and let h be the associated holonomy function. Holonomy of lower submeasures is defined by the two conditions.

- If l is a monotonic taxi loop in $\mathcal{G}^\circ$ wrapping once around, then $h(\nu_l) = h(l)$.
- If $\nu' = \nu + \epsilon$ where ϵ is a finite positive measure, then $h(\nu') = h(\nu) + |\epsilon|$.

Now we can state the definition of our geometric incarnation of geodesic currents.

Definition 3.23. If μ is a nullhomologous geodesic current, let X_μ denote the space of holonomy zero lower submeasures of μ.

The weak topology makes X_μ into a topological space, though soon we will endow it with an explicit metric. As mentioned before, a lower submeasure gives a monotone partition $\mu = \nu + \bar{\nu}$. We call $\mathrm{supp}(\nu) \cap \mathrm{supp}(\bar{\nu})$ the set of shared points.

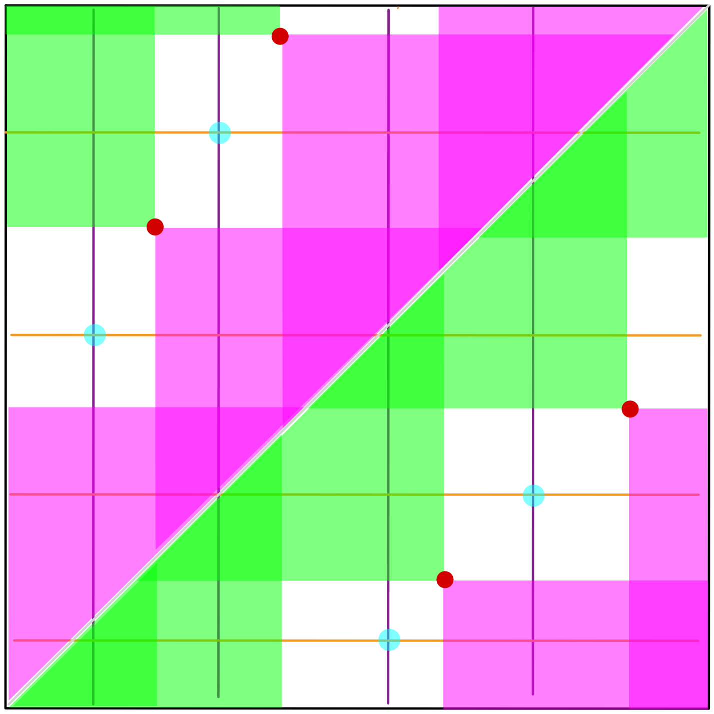
Lemma 3.29. *If the support of μ is discrete, then the set of lower submeasures with finitely many shared points is a finite dimensional cube complex.*

Proof. By evaluating a submeasure on each support point of μ, the set of submeasures is identified with a cube in $\mathbb{R}^{\mathrm{supp}(\mu)}$. For every partition $\mathrm{supp}(\mu) = L \sqcup F \sqcup U$ such that F is finite, let $C(L, F, U)$ denote the set of lower submeasures $\nu < \mu$ such that $\nu(p) = \mu(p)$ for $p \in L$, and $\nu(p) = 0$ for p in U. The set $C(L, F, U)$ is either empty, or a closed finite dimensional face of the cube of submeasures. A lower submeasure ν gives a partition $\mathrm{supp}(\mu) = L \sqcup F \sqcup U$, such that $C(L, F, U)$ is the smallest face containing ν. Every point in $C(L,F,U)$ is also a lower submeasure. The set of lower submeasures with finitely many shared points is thus a union of closed finite dimensional faces of a cube in $\mathbb{R}^{\mathrm{supp}(\mu)}$. $\square$

In some sense, this whole paper is just giving context for the following lemma.

Lemma 3.30. *Suppose μ is tropical rank n, then holonomy zero lower submeasures can have at most n shared points. Consequently, if μ is also discrete, X_μ is a polyhedral complex of dimension at most $n-1$.*

Proof. Let ν be a holonomy zero lower submeasure of μ. Let $m : \partial\Gamma^\circ \times \partial\Gamma^\circ \to \mathbb{R} \cup \{-\infty\}$ be the unimodal potential corresponding to ν. Recall that m is zero precisely on the points which are neither below ν nor above $\bar{\nu}$. Suppose $(u_1, v_1), ..., (u_n, v_n) \in \mathcal{G}$ are the shared points of ν. Choose points $(x_1, y_1), ..., (x_n, y_n)$ in the zero set of m with $u_i < x_i < u_{i+1}$ and $v_i < y_i < v_{i+1}$. This implies that if $i \neq j$, (x_i, y_j) is above or below some shared point, thus $m(x_i, x_j) < 0$. This means that the term $\sum m(x_i, y_i)$ uniquely maximizes the tropical determinant, thus μ cannot be rank $n-1$ or less. $\square$



3.5.2 The metric

Now we put a metric on X_μ. Let $\nu_1, \nu_2 \in X_\mu$. The difference $\nu_1 - \nu_2$ is a signed measure of total measure zero. We can push forward to $\partial\Gamma$ and integrate to get a function on $\partial\Gamma$.

$$f_{\nu_1,\nu_2} = \int_{[x_0,x)} (\pi_1)_*(\nu_2 - \nu_1) \in \mathrm{Fun}(\partial\Gamma, \mathbb{R})$$

Here $x_0 \in \partial\Gamma$ is a basepoint, and changing the basepoint only changes f_{ν_1,ν_2} by a constant.

Definition 3.24.

$$d(\nu_1, \nu_2) = \sup(f_{\nu_1,\nu_2}) - \inf(f_{\nu_1,\nu_2})$$

It is helpful to know that when we push forward $\nu_2 - \nu_1$ to $\partial\Gamma$, there is no cancellation:

Lemma 3.31. *Let $\epsilon = \nu_2 - \nu_1$. Let $\epsilon = \epsilon^+ - \epsilon^-$ where $\epsilon^\pm$ are positive measures. Then* $\mathrm{supp}(\epsilon^+)$ *and ϵ^- are incomparable in the sense that if $g \in \mathrm{supp}(\epsilon^+)$, then ϵ^- is zero on the set of points above or below g.*

Proof. If $g^+ \in \mathrm{supp}(\epsilon^+)$, then $g^+ \in \mathrm{supp}(\nu_2)$ and $g^+ \in \mathrm{supp}(\bar{\nu}_1)$. Consequently, $\nu_2 = \mu$ below g^+, and $\nu_1 = 0$ above g^+. This means that $\nu_2 - \nu_1$ is non-negative above and below g^+. □

Lemma 3.32. *For any $\nu_1, \nu_2 \in X_\mu$, there is a geodesic $\nu(t)$ connecting them.*

Proof. Let $\epsilon = \nu_2 - \nu_1$.

We can decompose $\partial\Gamma$ into intervals I_k such that $(\pi_1)_*(\epsilon)$ is either positive or negative on each I_k. The distance $d(\nu_1, \nu_2)$ only depends on the measures $\epsilon(\pi_1^{-1}(I_k))$, as these determine the extrema of f_{ν_1,ν_2}. We can find a path of measures $\epsilon(t)$ for $t \in [0,1]$ such that $\epsilon(t)(\pi_1^{-1}(I_k)) = t\epsilon(\pi_1^{-1}(I_k))$, $\epsilon(t)^+$ is a lower submeasure of ϵ^+, and $-\epsilon(t)^-$ is an upper submeasure of $-\epsilon^-$. The path $\nu(t) = \nu_1 + \epsilon(t)$ will be a geodesic from ν_1 to ν_2. □

We now recall the notion of translation length. Suppose ϕ is an isometry of a metric space X. The translation length of ϕ is

$$\lim_{n\to\infty} \frac{d(x, \phi^n(x))}{n}$$

for any choice of $x \in X$. If $y \in X$ is another point, we have

$$\lim_{n\to\infty} \frac{d(y, \phi^n(y))}{n} \leq \lim_{n\to\infty} \frac{2d(y,x) + d(x, \phi^n(x))}{n} = \lim_{n\to\infty} \frac{d(x, \phi^n(x))}{n}$$

so the definition doesn't depend on the choice of point. If $x \in X$ satisfies $d(x, \phi^n(x)) = nd(x, \phi(x))$ for all $n \in \mathbb{N}$, then the translation length is simply $d(x, \phi(x))$.

Proposition 3.33. *If μ is a geodesic current, and $\gamma \in \Gamma$ then a submeasure $\nu \in X_\mu$ such that $\nu = \mu$ on both $\mathcal{G}_{<(\gamma^+,\gamma^-)}$ and $\mathcal{G}_{<(\gamma^-,\gamma^+)}$ will satisfy $d(\nu, \gamma^n\nu) = nd(\nu, \gamma\nu)$ thus the translation length of γ is $d(\nu, \gamma\nu)$.*

Proof. The difference $\gamma^n\nu - \nu$ will be zero on

$$C = \mathcal{G}_{\leq(\gamma^-,\gamma^+)} \cup \mathcal{G}_{\geq(\gamma^-,\gamma^+)} \cup \mathcal{G}_{\leq(\gamma^+,\gamma^-)} \cup \mathcal{G}_{\geq(\gamma^+,\gamma^-)}$$

because both $\gamma^n\nu$ and ν are zero above the fixed points (γ^-, γ^+) and (γ^+, γ^-), agree with μ below the fixed points, and must agree with eachother at the fixed points. The complement $\mathcal{G}\backslash C$ has two components

$$D_+ = \{(x, y) : x < \gamma^- < y < \gamma^+\}$$

$$D_- = \{(x, y) : y < \gamma^- < x < \gamma^+\}$$

and γ translates D_+ upward while it translates D_- downward. This means that $\gamma^n\nu - \nu$ is always positive on D_+ while it is always negative on D_-. This means that the distance from ν to $\gamma^n\nu$ is just the integral of $\gamma^n\nu - \nu$ over D^+.

$$d(\gamma^n\nu, \nu) = \int_{D^+} \gamma^n\nu - \nu = \sum_{k=1}^{n} \int_{D^+} \gamma^k\nu - \gamma^{k-1}\nu = nd(\gamma\nu, \nu)$$

□

Together with proposition 3.19 this shows that if μ is the curvature of an $\mathbb{R}$ bundle P, the translation length of γ acting on X_μ is $l_P(\gamma) + l_P(\gamma^{-1})$.

3.5.3 A relative metric which knows periods

Since two equivariant $\mathbb{R}$ bundles with connection can have the same curvature but different periods, there cannot be any general way to endow X_μ with an asymmetric metric such that translation lengths coincide with periods. In this section we introduce a more abstract notion of asymmetric metric which we call a relative metric, and construct a relative metric on X_μ which captures periods. For cubic differential paths, when $n = 3$, there is a natural "trivialization" of the relative metric, making it into a genuine asymmetric metric, but we don't know when this happens in general.

Definition 3.25. A **relative metric** on a set X is a principal $\mathbb{R}$ bundle L on X together with a function $d : L \times L \to \mathbb{R}$ which satisfies the following three properties.

1. Homogeneity: $d(x, y + r) = r + d(x, y) = d(x - r, y)$ for all $x, y \in L$ and $r \in \mathbb{R}$,
2. Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in L$, and
3. Non-degeneracy: $d(x, y) + d(y, x) = 0$ if and only if x and y are in the same fiber of L.

If (L, d) is a relative metric on X, than the symmetrization $d(x, y) + d(y, x)$ always descends to a metric on X. If we choose a section $s : X \to L$ such that $d(s(x), s(y)) > 0$ for all $x \neq y$, then $d(s(x), s(y))$ is an asymmetric metric on X. Two different sections will give metrics which differ by a function of the form $f(y) - f(x)$.

We now construct a relative metric on X_μ. The points of this relative metric will be called "monotonic Legendrians".

Definition 3.26. A subset $M \subset \mathcal{G}$ is **monotonic** if its intersection with every fiber of each projection to $\partial \Gamma$ is a non-empty closed interval.

For example, the

Definition 3.27. Let U be a taxi bundle on $\mathcal{G}^\circ$, and $M \subset \mathcal{G}$ be a monotonic subset. A section $\phi : M \cap \mathcal{G}^\circ \to U$ is **admissible** if for any $(x, y), (x', y') \in M \cap \mathcal{G}^\circ$ with $x < x' < y < y'$, the parallel transport of $\phi(x, y)$ along

$$(x, y) \to (x, y') \to (x', y')$$

is greater than or equal to $\phi(x', y')$, and the parallel transport of $\phi(x, y)$ along

$$(x, y) \to (x', y) \to (x', y')$$

is less than or equal to $\phi(x', y')$. The pair (M, ϕ) is determined by the image $\phi(M) \subset U$, which we refer to as a **monotonic Legendrian**.

Lemma 3.34. *Let P be an equivariant $\mathbb{R}$ bundle with curvature μ. The set of monotonic Legendrians in P is a principal $\mathbb{R}$ bundle over X_μ.*

Proof. Let (S, ϕ) be a monotonic Legendrian. We define a lower submeasure $\pi(S, \phi)$ by defining the measure of a box $B = [x, x'] \times [y, y'] \in C_2(\mathcal{G}^\circ)$, to be

- 0 if B lies above S,
- $\mu(B)$ if B lies below S, and
- the difference between $\phi(x', y')$ and the parallel transport of $\phi(x, y)$ along $(x, y) \to (x', y) \to (x', y')$ if both (x, y) and (x', y') are in S.

In the other direction, if ν is a holonomy-zero lower submeasure, than we let M be the closure of the set of points which are not below support points of ν or above support points of $\bar{\nu}$. If we choose a point $s \in S$ and fix $\phi(s) \in P_s$, than there is a unique way to extend to an admissible section $\phi : S \to P$ which will give back ν under the definition above. $\square$

Definition 3.28. Let $L_1 = (S_1, \phi_1)$ and $L_2 = (S_2, \phi_2)$ be two monotonic legendrians. Let $f_{L_1,L_2} : \partial\,\Gamma^\circ \to \mathbb{R}$ be defined by $f_{L_1,L_2}(x) = \phi_2(x, y_2) - t_{x;y_1,y_2}\phi_1(x, y_1)$ where $(x, y_1) \in S_1$, $(x, y_2) \in S_2$, and $t_{x;y_1,y_2}$ denotes parallel transport along $s_{x;y_1,y_2}$. The distance from L_1 to L_2 is the supremum of this function.

$$d(L_1, L_2) := \sup f_{L_1,L_2}$$

Lemma 3.35. *$d : L_P \times L_P \to \mathbb{R}$ is homogeneous and satisfies the triangle inequality.*

Proof. The homogeneity of $d(L_1, L_2)$ is immediate. Given three monotonic legendrians, L_1, L_2, L_3 we have

$$f_{L_1,L_3} = f_{L_1,L_2} + f_{L_2,L_3}$$

so $\sup f_{L_1,L_3} \leq f_{L_1,L_2} + f_{L_2,L_3}$. $\square$

Lemma 3.36. *The supremum of f_{L_1,L_2} is always realized by a point $x \in \partial\,\Gamma$ such that there exists y with (x, y) in the intersection of the supports of L_1 and L_2, so $d(L_1, L_2)$ is the maximum of $\phi_1 - \phi_2$ over the intersection of support.*

Translation length for relative metrics is defined in exactly the same way as for ordinary metrics: we choose a point $\tilde{x} \in L$ and take the limit of $d(x, \phi^n(x))/n$ as n goes to ∞. If we choose a monotonic Legendrian with S containing (γ^+, γ^-) than it is immediate that translation length of γ acting on the relative metric space (X, L, d) is equal to the period of γ.

This relative metric symmetrizes to the ordinary metric:

Lemma 3.37. *If $L_1 = (S_1, \phi_1)$ and $L_2 = (S_2, \phi_2)$ are two monotonic legendrians in P, and ν_1 and ν_2 are the induced lower submeasures, then*

$$d(L_1, L_2) - d(L_2, L_1) = d(\nu_1, \nu_2)$$

Proof. For $x, x' \in \partial\,\Gamma^\circ$, differences $f_{L_1,L_2}(x') - f_{L_1,L_2}(x)$ are holonomies around loops surrounding a region between S_1 and S_2, thus are integrals of $(\pi_1)_*(\nu_2 - \nu_1)$ over x. This means that f_{L_1,L_2} differs from f_{x_0,ν_1,ν_2} by a constant, so we have

$$d(\nu_1, \nu_2) = \sup(f_{L_1,L_2}) - \inf(f_{L_1,L_2})$$

□

3.6 Tropical rank 2 currents

In this section we show that tropical rank 2 currents are measured laminations, and that the space of holonomy zero lower submeasures of a measured lamination is an $\mathbb{R}$-tree.

3.6.1 Symmetry

In Bonohon's origional work [Bon88], geodesic currents were defined to be invariant under the involution $(x, y) \mapsto (y, x)$ of $\mathcal{G}$. Here we call such geodesic currents **symmetric**. To show symmetry of rank 2 currents, we first show that holonomies of certain paths vanish.

Definition 3.29. For any three distinct points $x, y, z \in \partial\Gamma$, let $[x, y, z] \in Z_1(\mathcal{G})$ denote the following taxi path.

$$(x, y) \to (x, z) \to (y, z) \to (y, x) \to (z, x) \to (z, y) \to (x, y)$$

If h is a holonomy function, $h([x, y, z])$ is referred to as a **triple ratio** of h.

Lemma 3.38. *All triple ratios of rank 2 holonomy functions and tropical rank 2 holonomy functions are trivial.*

Proof. Let $M : \partial\Gamma \times \partial\Gamma \to \mathbb{R}$ be a potential for a rank 2 holonomy function $H \in \mathcal{H}(\mathbb{R}^*)$. By definition,

$$\det \begin{bmatrix} 0 & M(x,y) & M(x,z) \\ M(y,x) & 0 & M(y,z) \\ M(z,x) & M(z,y) & 0 \end{bmatrix} = 0$$

for any distinct $x, y, z \in \partial\Gamma$. This implies $M(x, y)M(y, z)M(z, x) = M(x, z)M(y, x)M(z, y)$, so the triple ratio is 1.

$$H([x, y, z]) = \frac{M(x, y)M(y, z)M(z, x)}{M(x, z)M(y, x)M(z, y)} = 1$$

Now let m be a potential for a tropical rank 2 holonomy function $h \in \mathcal{H}(\mathbb{R})$. Since there are only two sums in the tropical determinant which are not $-\infty$, they must coincide, so we again have $m(x, y) + m(y, z) + m(z, x) = m(x, z) + m(y, x) + m(z, y)$, implying that the triple ratio is zero. $\square$

Proposition 3.39. *A holonomy function is symmetric if and only if it has vanishing triple ratios.*

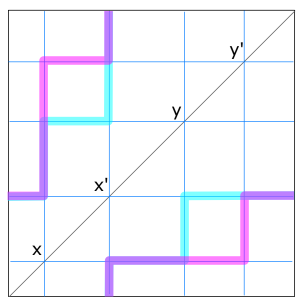
Proof. Suppose h is a symmetric holonomy function i.e. $\tau^* h = h^{-1}$. Then

$$h([x, y, z]) = h(\tau([x, y, z]))^{-1} = h([x, y, z])^{-1}$$

because the cycle $[x, y, z]$ is reversal invariant. Hence, triple ratios vanish.

Now we show that vanishing of triple ratios implies symmetry. For any distinct $x, x', y, y' \in \partial\Gamma$, the following identity of cycles holds, modulo thin cycles.

$$[x, x', y'] - [x, x', y] \simeq [x, x'; y, y'] - [y, y'; x, x']$$

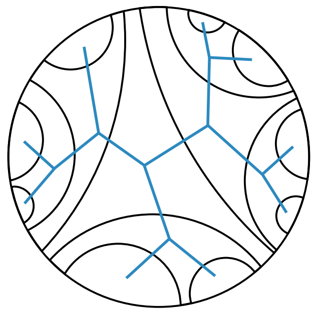


If all triple ratios of a holonomy function h vanish, than $h([x, x'; y, y']) = h([y, y'; x, x'])$. Since h is determined by its cross ratios, h must be symmetric.

$\square$

A measured lamination is a geodesic current which is symmetric and has no self-intersection. By lemma 3.28 a tropical rank 2 cross ratio has no self intersection, so must be a measured lamination.

The original construction of an $\mathbb{R}$-tree from a measured lamination in [MS91] is very intuitive, and will turn out to be the same as X_μ. Choose a hyperbolic structure on S, and identify $\mathcal{G}$ with the space of geodesics in $\tilde{S}$. Let $\tilde{L} \subset \tilde{S}$ be the union of geodesics parameterized by $\operatorname{supp}(\mu)$. Let V denote the set of components of $\tilde{S} \backslash \tilde{L}$. We define the distance between $v_1, v_2 \in V$ to be the measure of the set of geodesics in L which separate v_1 from v_2. By convention we divide by 2 to compensate for counting each geodesic with two orientations. Morgan and Shalen show that there is a unique minimal $\mathbb{R}$-tree into which V isometrically embeds. If μ is discrete, than V is the set of vertices, each leaf of $\tilde{L}$ corresponds to an edge, and the measure μ assigns to that leaf is the length of the edge.



3.6.2 Holonomy zero lower submeasures

A submeasure ν of a geodesic current μ is called symmetric if $\tau_*\nu = \bar{\nu}$, where $\bar{\nu} := \mu - \nu$.

Proposition 3.40. *A lower submeasure ν of a measured lamination μ is holonomy zero if and only if it is symmetric.*

Proof. Since μ is symmetric, the corresponding holonomy function h is symmetric, meaning $h'(z) := -h(\tau_* z)$ for any cycle z in $\mathcal{G}^\circ$. This means that $h(\tau_*\bar{\nu}) = -h(\nu)$ for all admissible lower submeasures ν. If $\nu = \tau_*\bar{\nu}$, than it must be holonomy-zero.

Conversely, suppose ν is holonomy zero. Let $\epsilon = \nu - \tau_*\bar{\nu}$. Decompose ϵ as $\epsilon^+ - \epsilon^-$ where $\epsilon^\pm$ are positive measures. We observe the following properties.

1. $|\epsilon^+| = |\epsilon^-|$.
2. $\epsilon = \tau_*\epsilon$.
3. No point in $\mathrm{supp}(\epsilon^+)$ is greater or less than any point in $\mathrm{supp}(\epsilon^-)$.

If ϵ is non-zero, than ϵ^+ and ϵ^- are both non-zero so we can find $(x, y) \in \mathrm{supp}(\epsilon^+)$ and $(x', y') \in \mathrm{supp}(\epsilon^-)$. Since ϵ is invariant under τ, we cannot have $(x', y') = (y, x)$. Since

μ has no self intersection, (y, x) and (x', y') cannot intersect. The only remaining possibility is that (x, y) or (y, x) is greater than (x', y') or (y', x'). In conclusion $\epsilon = 0$. $\square$

3.6.3 X_μ is a tree

Firstly, the metric on X_μ has a simpler form in the rank 2 case.

Lemma 3.41. *If ν_1 and ν_2 are symmetric lower submeasures of a geodesic current μ, then there is a unique geodesic from ν_1 to ν_2, and $d(\nu_1, \nu_2) = |(\nu_2 - \nu_1)^+|$.*

Proof. The difference $\epsilon = \nu_2 - \nu_1$ will satisfy the following properties:

1. $\epsilon = -\tau_* \epsilon$.

2. No point in $\mathrm{supp}(\epsilon^+)$ is greater or less than any point in $\mathrm{supp}(\epsilon^-)$.

The only allowed relative position of two geodesics (x, y) and (x', y') in $\mathrm{supp}(\epsilon^+)$ is that they are parrallel: $x < x' < y' < y$. We can find $z, z' \in \partial\,\Gamma$ such that for any $(x, y) \in \mathrm{supp}(\epsilon^+)$ we have $z < x < z' < y$. This means that ϵ^+ is totally ordered, and $\epsilon^- = -\tau_* \epsilon^+$. If we integrate $(\pi_+)_* \epsilon$ to get a function $f : \partial\,\Gamma \to \mathbb{R}$, its minimum will be attained at z', its maximum will be attained at z, and the difference $f(z') - f(z)$ will be $|\epsilon^+|$. There is a unique geodesic from ν_1 to ν_2 because ϵ^+ and ϵ^- are totally ordered. $\square$

Lemma 3.42. *X_μ is an $\mathbb{R}$-tree.*

Proof. One way to show that X_μ is an $\mathbb{R}$-tree is to show that it is a 0-hyperbolic metric space. Let $\nu_1, \nu_2, \nu_3 \in X_\mu$. We will show that the geodesics connecting any two of these points pass through

$$\nu_0 = (\nu_1 \cap \nu_2) \cup (\nu_2 \cap \nu_3) \cup (\nu_1 \cap \nu_3).$$

Here we use the notation that for two positive measures α and β, $\alpha \cap \beta$ is the biggest measure less than α and β, and $\alpha \cup \beta$ is the smallest measure greater than α and β.

The property of being a lower submeasure is closed under intersections and unions, so ν_0 is a lower submeasure. One checks that ν_0 is symmetric, thus holonomy zero. Let's check the difference between ν_0 and ν_1, and between ν_0 and ν_2.

$$\nu_0 - \nu_1 = \bar{\nu}_1 \cap \nu_2 \cap \nu_3 - \nu_1 \cap \bar{\nu}_2 \cap \bar{\nu}_3$$

$$\nu_0 - \nu_2 = \bar{\nu}_2 \cap \nu_1 \cap \nu_3 - \nu_2 \cap \bar{\nu}_1 \cap \bar{\nu}_3$$

Since these two differences are totally disjoint, there will be no canceling when we add, so the triangle inequality will be an equality for ν_1, ν_0, ν_2. This means that ν_0 lives on the geodesic from ν_1 to ν_2. The same is true of ν_1, ν_3 and ν_2, ν_3. We have shown that X_μ is 0-hyperbolic. $\square$

3.7 Currents from metrics

In this section we show how to extract a geodesic current from a Finsler metric on S which is not quite negatively curved, and not necessarily symmetric. In contrast to the negatively curved case, the current may be singular. This current will be the curvature of a bundle with connection on $\mathcal{G}^\circ$ whose periods are lengths of curves in S.

In section 3.8 we will apply the theory to Finsler metrics whose length spectra arise in $\partial_{\lambda_1}\mathrm{Hit}^3(S)$, namely triangular Finsler metrics. Triangular Finsler metrics exhibit the the eccentricities that we will have to deal with in this section: asymmetry, and non-uniqueness of geodesics, so we give the definition now.

Definition 3.30. Let μ be a cubic differential on a Riemann surface C; that is, a holomorphic section of $(T^*C)^{\otimes 3}$. We define F^Δ_μ to be

$$F^\Delta_\mu(v) := \max_{\{\alpha \in T^*_x C : \alpha^3 = \mu_x\}} 2Re(\alpha(v))$$

where $x \in C$ is a point, and $v \in T_x C$ is a tangent vector.

3.7.1 Horofunction Boundaries

In this subsection we recall Gromov's definition of horofunction boundary in the case of asymmetric metrics. Let X be a proper, geodesic, asymmetric metric space. Let $C(X)$ denote the space of continuous real valued functions with the topology of uniform convergence on compact subsets. There are natural embeddings $D_+, D_- : X \to C(S)$ given by

$$D_+(x) = d(-, x)$$

$$D_-(x) = d(x, -)$$

which are both isometric embeddings from X equipped with the metric $\min(d(x, y), d(y, x))$ to $C(X)$ equipped with the supremum norm.

Definition 3.31. The **plus horofunction compactification** $\partial^h_+ X$ of X is its closure in the quotient $C(X)/\mathbb{R}$ of continuous functions by constant functions, under the embedding D_+. Similarly, the **minus horofunction compactification** $\partial^-_h X$ is the closure of $D_-(X)$ in $C(X)/\mathbb{R}$. The plus and minus horofunction boundaries $\partial^+_h X$, and $\partial^h_- X$ are defined to be the complements of X in these two compactifications.

Plus and minus **horofunctions** are functions on X which represent points in $\partial^-_h X$, and $\partial^+_h X$ respectively. Since $\partial^-_h X$ is just $\partial^+_h X$ for the reversed metric, we will sometimes make statements only for $\partial^+_h X$.

As an example, if $X = \mathbb{C}$ with the triangular Finsler metric $F^{\Delta}_{dz^3}$, then the horofunction boundary $\partial_- X$ is a circle with a natural cell decomposition into three copies of $\mathbb{R}$, and three points. Linear functions

$$h(y) = 2Re(\zeta y) + C$$

with $\zeta^3 = 1$ will give three points in $\partial^h_-(\mathbb{C}, F^{\Delta}_{dz^3})$. There will also be horofunctions of the form

$$h(y) = \max(2Re(\zeta y) + C, 2Re(\zeta' y) + C')$$

for two distinct third roots of unity ξ, ξ' which will descend to three copies of $\mathbb{R}$ in ∂^h_-.

In general, the horofunction boundary can be quite different from the visual boundary, but in the Gromov hyperbolic case there is a close relationship.

Lemma 3.43 ([CP01])**.** *For a Gromov hyperbolic geodesic metric space X, the visual boundary ∂X is the quotient of the horofunction boundary $\partial^h X$, where two horofunctions are identified if their difference is bounded.*

The map is defined as follows: For any horofunction h, and any $p \in X$ with $h(p) = 0$, we can find a geodesic ray γ starting at p satisfying $h(\gamma(t)) = -t$. We find γ by taking geodesics from p to q_i for a sequence $q_i \in X$ converging to $[h]$. Gromov hyperbolicity guerentees these geodesics stay in a compact region. Take γ to be the limit of a convergent subsequence. Gromov hyperbolicity forces any two geodesic rays which satisfy this property, with respect to two horofunctions with bounded difference, to be bounded distance from eachother, thus represent the same point in ∂X. Note however, that γ may not converge to $[h]$ in the horofunction compactification. We will denote the projections to the visual boundary by $v_- : \partial^h_- X \to \partial X$ and $v_+ : \partial^h_+ X \to \partial X$.

In the generality we need, there is no map from the visual boundary to the horofunction boundary, but geodesic rays do always converge to horofunctions. Endpoints of geodesic rays in the horofunction boundary are called Bussmann points. For our metrics of interest, all points will be Bussmann.

Lemma 3.44. *Geodesic Rays converge in the horofunction compactification. If two rays $\gamma_1, \gamma_2 : [0, \infty) \to X$ are asymptotic, i.e.*

$$\lim_{t \to \infty} d(\gamma_1(t), \gamma_2(t + T)) = 0$$

for some T, then they converge to the same point.

Proof. The second claim is clear from the triangle inequality, but the first statement is a bit less immediate. Let $\gamma : [0, \infty) \to X$ be a geodesic, meaning $d(\gamma(t), \gamma(t')) = t' - t$ for all $0 \leq t \leq t'$. Let $x \in X$. We would like to show that the path of functions

$$h^t_\gamma(x) := d(x, \gamma(t)) - t$$

converges on all compact subsets of X as t goes to infinity. First we show that $h^t_\gamma(x)$ is decreasing in t. Let $t' \geq t$, and use the triangle inequality.

$$h^t_\gamma(x) - h^{t'}_\gamma(x) = d(x, \gamma(t)) + (t' - t) - d(x, \gamma(t')) \geq 0$$

The function $h^t_\gamma(x)$ is also bounded below as a function of t:

$$d(\gamma(0), x) + d(x, \gamma(t)) \geq t$$

$$d(\gamma(t), x) - t \geq -d(x, \gamma(0))$$

Since the path of functions $h^t_\gamma(x)$ is 1-lipshitz in x, bounded below on compact sets, and monotonically decreasing, it must converge on compact sets. □

3.7.2 A bundle with connection

We will define a pairing between minus horofunctions and plus antihorofunctions, which will allow us to construct a cross ratio, and even a bundle with taxi connection on a subset of $\partial^-_h X \times \partial^+_h X$. Later we will push-forward this bundle with connection and get a bundle with connection on $\mathcal{G}^\circ$.

Definition 3.32. Let X be a proper asymmetric geodesic metric space. The **pairing** of a minus horofunction g, and a plus horofunction h is the infimum of their sum.

$$\langle g, h\rangle := \inf_X (g + h) \in [-\infty, \infty)$$

The **cross ratio** of $[g_1], [g_2] \in \partial^h_- X$ and $[h_1], [h_2] \in \partial^h_+ X$ is the following combination of pairings.

$$b([g_1], [g_2]; [h_1], [h_2]) = \langle g_1, h_1\rangle + \langle g_2, h_2\rangle - \langle g_1, h_2\rangle - \langle g_2, h_1\rangle$$

The pairing between horofunctions is a sort of renormalized limit of distance between points.

Lemma 3.45. *Let $p \in X$ be a basepoint. Suppose x_i converges to $[g] \in \partial_h^- X$ and y_i converges to $[h] \in \partial_h^+ X$ where g and h are normalized to vanish on p. Then*

$$\lim_{i,j\to\infty} d(x_i, y_i) - d(x_i, p) - d(p, y_i) = \langle g, h \rangle$$

Proof. First note that $d(x_i, y_i)$ is the minimum of the function $d(x_i, z) + d(z, y_i)$. The set of minima is the union of all geodesics connecting x_i to y_i. Re-write the left hand side:

$$\lim_{i,j\to\infty} \inf_{z\in X} d(x_i, z) - d(x_i, p) + d(z, y_i) - d(p, y_i)$$

Gromov hyperbolicity implies that these infimums can be attained for z in some compact region of X, on which $d(x_i, z) - d(x_i, p)$ is converging uniformally to $g(z)$, and $d(z, y_i) - d(p, y_i)$ is converging uniformally to to $h(z)$. The expression becomes

$$\inf_{z\in X} h(z) + g(z) = \langle h, g \rangle$$

□

Note that $d(x_i, y_i) - d(x_i, p) - d(p, y_i)$ is just -2 times the Gromov product of x_i and y_i. If X is Gromov hyperbolic, and x_i and y_i converge in the visual boundary, then the Gromov product diverges if and only if they converge to the same point.

Lemma 3.46. *If X is Gromov hyperbolic, then $\langle g, h \rangle = -\infty$ if and only if $v([g]) = v([h])$.*

Lemma 3.45 implies a more geometric formula for the cross ratio of four horofunctions as a limit of "cross distances".

Lemma 3.47. *If $x_{1,i}, x_{2,i}$ limit to $[g_1], [g_2] \in \partial_h^- X$, and $y_{1,i}, y_{2,i}$ limit to $[h_1], [h_2] \in \partial_h^+ X$, then*

$$b([g_1], [g_2]; [h_1], [h_2]) = \lim_{i\to\infty} [d(x_{1,i}, y_{1,i}) + d(x_{2,i}, y_{2,i}) - d(x_{1,i}, y_{2,i}) - d(x_{2,i}, y_{1,i})]$$

In much the same way as for Anosov representations, we construct a bundle with connection for which these cross ratios are holonomies.

Definition 3.33. Let $\mathcal{G}^h_X$ be the subset of $\partial^h_- X \times \partial^h_+ X$ consisting of pairs which map to two distinct visual boundary points. Let U^h_X denote the set of pairs (g, h) with $\langle g, h\rangle = 0$. Endow U^h_X with the $\mathbb{R}$ action $r \cdot (g, h) = (g - r, h + r)$. It is a principal $\mathbb{R}$ bundle on $\mathcal{G}^h_X$. Endow U^h_X with the taxi connection whose horizontal and vertical flat sections are sections for which one coordinate is constant.

We can usually understand this connection using geodesics. A geodesic $\gamma : \mathbb{R} \to X$, gives a pair of horofunctions.

$$\gamma(-\infty) := \lim_{t\to-\infty} [d(\gamma(t), p) + t]$$

$$\gamma(\infty) := \lim_{t\to\infty} [d(p, \gamma(t)) - t]$$

These satisfy $\langle\gamma(-\infty), \gamma(\infty)\rangle = 0$, thus $[\gamma] := (\gamma(-\infty), \gamma(\infty))$ is a point in U^X_h. Suppose γ_1 and γ_2 are parametrized geodesics which are asymptotic in a strong sense:

$$\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) = 0$$

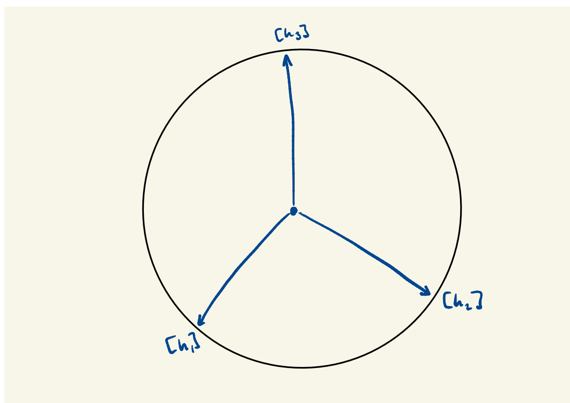
The triangle inequality implies $\gamma_1(\infty) = \gamma_2(\infty)$, so the corresponding points $[\gamma_1]$ and $[\gamma_2]$ in U_h will lie on a flat section. In the relevant examples, we can represent any point in U_h by a parametrized geodesic, but we don't know in what generality this holds.

3.7.3 Cyclic order on horofunction boundary

Let d be an invariant metric on $\tilde{S}$. We would like to push forward the curvature of U^h_d from $\mathcal{G}^d_h$ to $\mathcal{G}$ to define a geodesic current. To do this, we need the curvature of U^h_d to be a positive measure on $\mathcal{G}^d_h$. To this end, we construct cyclic orders on $\partial^-_h \tilde{S}$ and $\partial^+_h \tilde{S}$, which refine the cyclic order on $\partial\,\Gamma$, such that cross ratios are positive when expected.

From now on, let X be an asymmetric geodesic metric space homeomorphic to the disk such that every point in $\partial_h^- X$ and $\partial_h^+ X$ is the limit of a continuous path $[0, \infty) \to X$. Note that paths converging to distinct points must eventually be distinct. Further assume that any two points in X lie on a geodesic $\mathbb{R} \to X$ which is also geodesic when reversed.

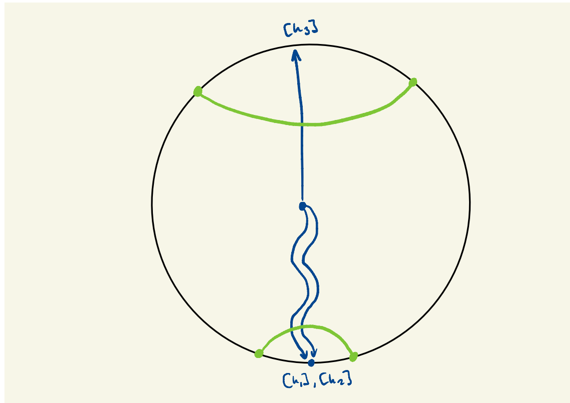
Definition 3.34. Three horofunction boundary points $[h_1], [h_2], [h_3] \in \partial_h^+ X$ are cyclically ordered if we can find three clockwise ordered rays $\gamma_1, \gamma_2, \gamma_3 : [0, \infty) \to X$ which only intersect at $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = p$, and converge to $[h_1], [h_2], [h_3]$.



This is clearly compatible with the cyclic ordering on ∂X. The next lemma shows this it is well defined.

Lemma 3.48. *Let h_1, h_2, h_3 be horofunctions on X, and suppose $v(h_1) = v(h_2) \neq v(h_3)$. Then if $h_1 \leq h_2 < h_3$ and $h_1 \leq h_2 < h_3$, we have that $h_1 = h_2$.*

Proof. $h_1 \leq h_2 \leq h_3$ implies that $h_1 - h_2$ is a decreasing function on any two way geodesic which starts between $[h_3]$ and $[h_1]$ and ends between $[h_2]$ and $[h_3]$. If also $h_2 \leq h_1 \leq h_3$, then $h_1 - h_2$ is also increasing on every such two way geodesic. A function which is constant on all such two way geodesics must be constant, so $[h_1] = [h_2]$. □



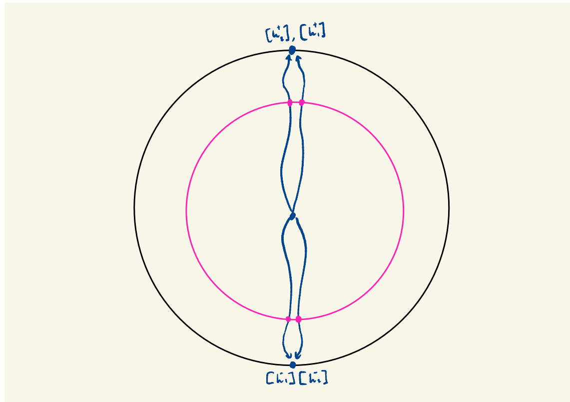
A tuple of n horofunctions is cyclically ordered if they can be reached by n cyclically ordered rays which only intersect at their starting points.

Lemma 3.49. *If* $[h_1], [h_2], [g_1], [g_2]$ *are cyclically ordered, then* $b([h_1], [h_2]; [g_1], [g_2]) \geq 0$.

Proof. Represent all four horofunctions as endpoints of cyclically ordered rays $\gamma_1^-, \gamma_2^-, \gamma_1^+, \gamma_2^+$ eminating from $p \in X$. For $r > 0$ let x_1, x_2, y_1, y_2 be the last points on each ray which is distance r from p. These points will be cyclically ordered on the boundary of the the r ball centered at p. The cross-distance

$$d(x_1, y_1) + d(x_2, y_2) - d(x_1, y_2) - d(x_2, y_1)$$

will be positive because we can find geodesics within the ball $B(p, r)$ connecting each pair of points, and the sum of the crossing geodesics' lengths is always greater than the sum of non-crossing geodesics' lengths. The cross ratio is the limit as r goes to infinity, which must also be positive. $\square$



The curvature of U_d^h thus gives a positive measure μ_d^h on $\mathcal{G}_d^h$. Pushing forward this measure to $\mathcal{G}$ we get a geodesic current μ_d. We will see in the next section that when $X = \tilde{S}$, and d is a triangular Finsler metric pulled back from S, then μ_d^h does not actually contain any more information than μ_d. The horofunction compactifications are only bigger than ∂X when there are certain types of flat strips leading to delta measures in μ_d. This is probably true for much more general Finsler metrics.

3.8 Tropical rank 3 currents

In this section we show that certain paths in $\mathrm{Hit}^3(S)$, called cubic differential rays, converge to particularly nice geodesic currents, namely currents of descending real trajectories of cubic differentials. Then we show that for such a geodesic current μ, the space X_μ is simply $\tilde{S}$. We don't know of any other geodesic currents for which this is the case.

We deduce these facts as corollaries of the main theorem of [Rei23] where it was shown that along the cubic differntial ray corresponding to α, the λ_1 spectrum approaches the length spectrum of a Finsler metric F_α^Δ. Here we show that there is a natural trivialization of the relative metric recovering the Δ-Finsler metric F_α^Δ. In [Rei23] it was conjectured that Δ-Finsler metrics are determined by their length spectra, and that these length spectra form a dense subset of $\partial_{\lambda_1} \mathrm{Hit}^3(S)$. The fact that we can directly construct the Finsler surface (S, F_α^Δ) from its horocycle current, which is in turn determined by its length spectrum affirms that Δ-Finsler surfaces are determined by length spectrum. It is still unknown whether length spectra of Δ-Finsler metrics are open and dense in $\partial_{\lambda_1} \operatorname{Hit}^3(S)$. It is also still unknown what X_μ looks like for the rest of $\partial_{\lambda_1} \operatorname{Hit}^3(S)$, though surely it is some combination of $\mathbb{R}$-tree behavior and cubic differential behavior, as observed in a related compactification [OT21b].

3.8.1 Cubic differential rays

Let C be a closed Riemann surface. Quadratic and cubic differentials on C are holomorphic sections of K^2 and K^3 respectively, where K is the cannonical bundle. In the case of Riemann surfaces K is just the holomorphic cotangent bundle. Hitchin [Hit92] defined the following family of Higgs bundles $(E, \phi_{\alpha_2,\alpha_3})$ parametrized by $(\alpha_2, \alpha_3) \in H^0(C, K^2) \times H^0(C, K^3)$.

$$E = K \oplus \underline{\mathbb{C}} \oplus K^{-1} \qquad \phi_\alpha = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 \\ 1 & 0 & \alpha_2 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving the Hitchin equation gives a cannonical flat connection on each of these Higgs bundles which preserves a real structure, giving a diffeomorphism $H^0(C, K^2) \times H^0(C, K^3) \to \operatorname{Hit}^3(C)$. Hitchin's construction works for general split real lie groups, but something special happens in the case of $\operatorname{Hit}^3(C)$. Labourie [Lab06a] and Loftin [Lof01b] showed that we can set $\alpha_2 = 0$, and instead range over all complex structures on a smooth surfaces S up to isotopy, and get a parametrization of $\operatorname{Hit}^3(S)$ by the bundle over Teichmuller space whose fibers are cubic differentials.

The Labourie-Loftin parametrization suggests that to understand the boundary of $\operatorname{Hit}^3(S)$, a good place to start is understanding how holonomies grow along paths in $\operatorname{Hit}^3(S)$ parametrized by a fixed complex structure and a ray of cubic differentials $R\alpha$, where α is a non-zero cubic differential and $R \in \mathbb{R}_{\geq 0}$. This was first investigated in [Lof07b] for the case of loops which are straight lines in the 1/3 translation structure, then in [LTW22] for general loops, and strengthened and rephrased in terms of Finsler metrics in [Rei23].

Theorem 3.50. *Let (J_i, α_i) be a sequence of pairs of complex structure with cubic differential on a smooth oriented surface S of genus at least 2, such that J_i converges uniformally to some J, and α_i/R_i^3 converges uniformally to α for some sequence of positive real numbers R_i tending to ∞. Let $\rho_i \in \mathrm{Hit}^3(S)$ be the corresponding sequence of representations. Let $[a] \in [\pi_1(S)]$. Let $F_\alpha^\Delta([a])$ denote the infimal length of loops in the free homotopy class $[a]$ with respect to the triangular Finsler metric F_α^Δ.*

$$\lim_{i\to\infty} \frac{\log|\lambda_1(\rho_i(a))|}{R_i} = F_\alpha^\Delta(a)$$

Let ρ_i be such a sequence of representations. Let l_i, and μ_i be the λ_1-spectrum, and cross ratio current of ρ_i. Let $l = \lim(l_i/R_i)$ be the limiting length spectrum. By lemma **??**, μ_i/R_i must converge to a limiting current μ which is the curvature of an equivariant taxi bundle on $\mathcal{G}^\circ$ with period spectrum l. On the other hand l is the length spectrum of a weakly convex metric on S, namely F_α^Δ, so it is the period spectrum of the generalized geodesic flow bundle $U_{(S,F_\alpha^\Delta)}$. Since equivariant taxi bundles are determined by their periods, it follows that μ is the curvature of $U_{(S,F_\alpha^\Delta)}$. We will show that μ is proportional to the current of descending real tragectories of α.

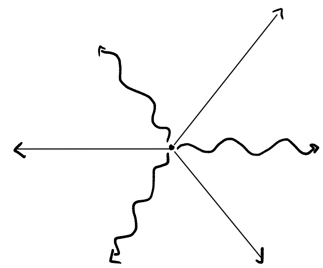
3.8.2 Cubic differential currents

To understand the geodesic current associated with the Finsler metric F_α^Δ we need to understand its geodesics. We start by describing geodesics in $\mathbb{C}$ for the metric $F_{dz^3}^\Delta$.

$$F_{dz^3}^\Delta = \max_{\zeta^3=1}[2Re(\zeta dz)]$$

A path $\gamma : \mathbb{R} \to \mathbb{C}$ is a geodesic for $F_{dz^3}^\Delta$ if there is one of these three one forms is maximal on $\gamma'(t)$ for all t. Note that there are infinitely many geodesics from 0 to

1, whereas there is a unique geodesic from 0 to -1. More generally, geodesics of the form $\gamma(t) = a - \zeta t$ where $\zeta^3 = 1$ are "rigid" in the sense that for any $t_1 < t_2$, $\gamma|_{[t_1,t_2]}$ is the unique geodesic segment from $\gamma(t_1)$ to $\gamma(t_1)$.



We can understand $F^{\Delta}_{dz^3}$ as the taxicab metric for a city that has three directions of one way streets. If you can get to some place by taking one of these streets then it is the fastest way to get there, otherwise there are multiple equally long routs. On a Riemann surface, these streets will be called descending real trajectories.

Definition 3.35. A **descending real trajectory** of a cubic differential α on a Riemann surface X is a smooth map $\gamma : (-\infty, \infty) \to X$ such that $\alpha(\gamma') = -1$. A **generalized descending trajectory** is a non-constant continuous map $\gamma : (-\infty, \infty) \to X$ such that if $\alpha(\gamma(t)) \neq 0$ then γ is differentiable at t and $\alpha(\gamma'(t)) = -1$, and also has angle at least π on both sides (in the flat metric) at zeros.

Lemma 3.51. *Generalized descending trajectories are geodesics. They are also ridgid: if γ is a generalized descending trajectory then for all $t_1 < t_2$ in $\mathbb{R}$, $\gamma|_{[t_1,t_2]}$ is the shortest path from $\gamma(t_1)$ to $\gamma(t_2)$.*

Proof. Since γ is the unique Euclidean geodesic connecting $\gamma(0)$ to $\gamma(T)$, and the triangle metric is bounded below by the Euclidean metric, any other path from $\gamma(0)$ to $\gamma(T)$ must be longer than γ with respect to F^{Δ}. $\square$

A corollary of Lemma 3.51 is that an F^{Δ}_{α}-geodesic in X cannot span a bigon with a descending trajectory with both edges oriented the same way.

We would like a local way of telling whether a path in $\tilde{S}$ is a geodesic. One might expect a result like this because $\tilde{S}$ is negatively curved to some extent. The obstruction to such a result is the same as for the taxi-cab metric: taking one right turn can be geodesic but two consecutive right turns is not, and the turns can be arbitrarily far apart.

Let X be a Riemann surface with cubic differential α. We will denote by $\Sigma \subset T^*X$ the triple branched covering of X whose points are cube roots of α. This is the spectral curve of the Higgs bundle determined by α. We will say a path $\gamma : [0, T] \to \tilde{S}$ is liftable if there is a continuous lift $\beta : \gamma \to \Sigma$ such that $2Re(\beta(\gamma'))$ is always maximal amongst the three square roots. In $(\mathbb{C}, dz^3)$, geodesics are precisely the liftable paths, and descending trajectories are the paths that admit two lifts.

If α has zeros, then a liftable path can use a zero of α to turn straight around, but after ruling this out we get our desired characterization of geodesics.

Lemma 3.52. *A path $\gamma : [0, T] \to \tilde{S}$ is a geodesic if and only if it is liftable, and is geodesic in some neighborhood of each zero.*

Proof. Suppose γ is a geodesic. It must be geodesic on the complement of its zeros, thus it is liftable on the complement of the zeros. At the zeros the three cube roots coincide, so γ is liftable. Since γ is geodesic, it must be geodesic in a neighborhood of each zero.

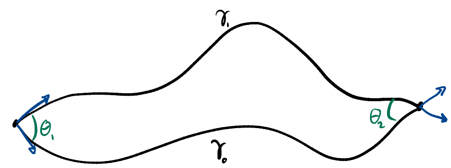
There must be some geodesic γ_0 connecting any two points p and q because $\tilde{S}$ is a complete Finsler space. Suppose γ_1 is another path which is liftable, and geodesic near zeros. We will show that γ_0 and γ_1 have the same length.

We can replace γ_0 and γ_1 with piecewise smooth paths which have the same lengths, so we can assume that they are piecewise smooth. We can further assume that γ_0 and γ_1 have disjoint interiors because otherwise we just apply the argument multiple times. Let $D \subset \tilde{S}$ be the disk bounded by γ_0 and γ_1.

The cubic differential α induces a singular Euclidean metric $|\alpha|^{2/3}$ which is flat everywhere except at zeros of α where it has cone points of angles $2\pi + 2\pi k/3$. We will apply the Gauss Bonnet theorem to show that there are no zeros of α in D. The Gauss Bonnet formula says

$$\int_D K + \int_{\partial D} \kappa = 2\pi$$

where K is the Gauss curvature, and κ is the geodesic curvature of the boundary. In our setting, the integral of Gauss curvature means the sum of cone angles which is $-2\pi/3$ times the number of zeros in D counted with multiplicity, and the integral of geodesic curvature means the total turning angle of the boundary.



Let β_0 and β_1 be lifts of γ_0 and γ_1. Let $\theta_i(t) = Arg[\beta_i(t)(\gamma_i'(t))]$. Since β_i is a maximal lift, $-\pi/3 \leq \theta_i(t) \leq \pi/3$. The turning contributions from γ_i can be expressed using θ_i.

$$T(\gamma_0) = \theta_0(1) - \theta_0(0) - 2\pi k_0/3$$

$$T(\gamma_1) = \theta_1(1) - \theta_1(0) + 2\pi k_1/3$$

Here $-2\pi k_i/3$ is the extra turning contribution from where γ_i passes through zeros. If $\gamma_i(t)$ is a zero of α, then sometimes the pair (γ_i, β_i) cannot be perturbed into the interior of D in such a way that β_i is still a maximal lift. In this situation, $\theta_i(t+\epsilon) - \theta_i(t-\epsilon)$ differes from the turning angle of γ_i in $[t-\epsilon, t+\epsilon]$ by a multiple of $2\pi/3$. The fact that γ_i are geodesic near zeros means that the turning angle corrections from zeros can only be negative for γ_0 and positive for γ_1.

The turning angles at p and q can also be expressed using β_i:

$$\theta_p = \pi - [\theta_0(1) - \theta_1(0)] - 2\pi k_p/3$$

$$\theta_q = \pi - [\theta_0(0) - \theta_1(1)] - 2\pi k_q/3$$

Here the extra factors of $2\pi/3$ come from the possibility that $\beta_0(0) \neq \beta_1(0)$ or $\beta_0(1) \neq \beta_1(1)$. This possibility can only give a negative contribution to θ_p or θ_q.

The total turning around ∂D is

$$T(\gamma_0) + \theta_q - T(\gamma_1) + \theta_p = 2\pi - (k_0 + k_1 + k_p + k_q)2\pi/3.$$

The curvature of D is non-positive, so from the Gauss Bonnet formula we conclude that the curvature of D is zero, and $k_0 = k_1 = k_p = k_q = 0$. This means that there are no zeros in D, that $\beta_0 = \beta_1$ at 0 and 1, and that β_i can both be perturbed into the interior of D, so there is a continuous lift $\bar{D} \to \Sigma$ which restricts to β_i on the boundary. Since β is a closed 1-form, we conclude that

$$\int_{\gamma_0} 2Re(\beta_0) = \int_{\gamma_1} 2Re(\beta_1)$$

so γ_0 and γ_1 have the same length.

□

Lemma 3.53. *Let $\eta : [0,1] \to \tilde{S}$ be an arc in $\tilde{S}$ on which α is purely imaginary. Assume η avoids zeros. Let γ_0 and $\gamma(1)$ be perpendicular (generalized) descending trajectories through $\eta(0)$, and $\eta(1)$ respectively. Let $\gamma_i(-\infty)$ and $\gamma_i(\infty)$ be the end-points in $\partial_h^- \tilde{S}$ and $\partial_h^+ \tilde{S}$.*

$$b(\gamma_1(-\infty), \gamma_2(-\infty); \gamma_2(\infty), \gamma_1(\infty)) = 2F_\alpha^\Delta(\eta)$$

Proof. Make a path γ_{12} by concatenating $\gamma_1|_{(-\infty,0]}$, η, and $\gamma_2|_{[0,\infty)}$, and make a path γ_{21} by concatenating $\gamma_2|_{(-\infty,0]}$, η^{-1}, and $\gamma_1|_{[0,\infty)}$. The paths γ_{12} and γ_{21} are geodesics because a constant choice of maximizing one-form of F_α can be chosen for either one. Holonomy in U_h of the sequence of geodesics γ_1, γ_{12}, γ_2, γ_{21} is $2F_\alpha^\Delta(\eta)$. More immediately it is $F_\alpha^\Delta(\eta) + F_\alpha^\Delta(\eta^{-1})$ but F_α^Δ is symmetric on η. □

Let $\mathcal{T}'(\alpha)$ be the space of parametrized descending trajectories of α with the topology of uniform convergence on compact sets. Let $\mathcal{T}$ be the quotient by reparametrization. Note that $\mathcal{T}$ maps continuously to $\mathcal{G}_h$. Define $\bar{\mathcal{T}}$ to be the closure of $\mathcal{T}$ in the space of paths. $\bar{\mathcal{T}}$ will consists of trajectories with zeros that either always turn left or always turn right. A trajectory is determined by its horofunction endpoints $([g],[h])$ as the minimum set of $g+h$, so we may view $\mathcal{T}$ and $\bar{\mathcal{T}}$ as subsets of $\mathcal{G}_h$.

Lemma 3.54. *The geodesic current associated with F_α^Δ has support $\bar{\mathcal{T}}(\alpha)$.*

Proof. It will suffice to compare the two measures in a neighborhood of each point of $\mathcal{G}_h$. Suppose $([g],[h]) \in \mathcal{T}(\alpha)$ are the endpoints of a trajectory $\gamma \in \mathcal{T}'(\alpha)$. Choosing a perpendicular segment η to a point p on γ will determine arbitrarily small boxes containing $([g],[h])$ which have positive measure by lemma 3.53, thus showing $([g],[h])$ is in $\mathrm{supp}(\mu_h)$. Since the support is closed by definition, it contains $\bar{\mathcal{T}}(\alpha)$

Suppose $([g],[h])$ is not the endpoints of a trajectory. Choose a geodesic η connecting $v([g])$ to $v([h])$. Choose any point p on η. There are three trajectories going through p. If these trajectories run into zeros, meaning that there are choices to make, choose consistant turns so that all three trajectories are in $\bar{\mathcal{T}}(\alpha)$. Since η is not a trajectory, or in the closure of trajectories, it cannot coincide with any of these three trajectories. However, it is possible that η is asymptotic to (or coincides with) one of the trajectories in the forward or backward direction. If this is the case, shift p slightly to the left or right so that all three trajectories through p cross η. There will be two trajectories γ_1 and η_2 such that the tip of η is between their tips and the tail of η is between their tails. The cross ratio $b(\gamma_1(-\infty),\gamma_2(-\infty);\gamma_1(\infty),\gamma_2(\infty))$ vanishes. To see this note that the paths

$$\gamma_{12} := (\gamma_1|_{(-\infty,0]}) \circ (\gamma_2|_{[0,\infty)})$$

$$\gamma_{21} := (\gamma_2|_{(-\infty,0]}) \circ (\gamma_1|_{[0,\infty)})$$

are geodesics, and also pass through p. We have constructed a box of zero measure containing $([g],[h])$. $\square$

In the next subsection we will often think of $\mu^h(\alpha)$ as a measure on the space of trajectories of α.

3.8.3 Lower submeasures of cubic differential currents

Let X be a simply connected Riemann surface with cubic differential α such that F_α^Δ is complete. Let μ^h be horofunction current on $\mathcal{G}^h(X)$. As with any Finsler space, points of X give lower submeasures of μ^h, and thus μ by pushforward. In fact this gives an isometric embedding

$$X \to X_\mu$$

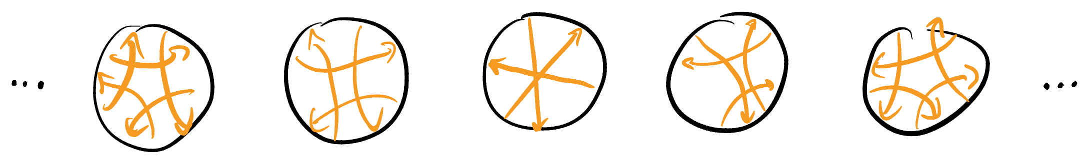
where X is equipped with the symmetrized metric. In the cubic differential case, $\nu^h(p)$ is the lower submeasure consisting of all trajectories which pass to the right of p. In this section we will show that in fact every holonomy zero lower submeasure of μ is $\nu(p)$ for some $p \in X$.

Lemma 3.55. *Let ν^h be an admissible lower submeasure of μ^h. There are finitely many maximal support points of ν^h which have a cyclic ordering $T_1, ..., T_k$ such that T_i intersects only T_{i-1} and T_{i+1}, and these intersections are points.*

Proof. The support of ν^h is a closed order ideal of $\bar{\mathcal{T}}(\alpha)$. Let M denote the set of maximal support points of ν^h. Since $\operatorname{supp}(\nu^h)$ is closed, it is generated as an order ideal by its maximal points. Because M is a set of maximal points, no trajectory in M is to the right of any other. In particular elements of M cannot share endpoints. Using the ordering of forward or backward endpoints endows M with two cyclic orders, but actually they coincide because there are no pairs of parallel trajectories in M.

M has no triple intersection because μ_h has no triple intersection. This means that $T \in M$ may intersect one element in directly in front, and one element directly behind in the cyclic order, but no more elements of M.

Actually, every trajectory in M must intersect exactly two other trajectories. To show this, we will need to use the admissability of ν. Lower submeasures coming from points, $\nu^h(p)$, are admissible. Admissability of ν^h is thus equivalent to the condition that for any $p \in \tilde{S}$, the set of trajectories in ν passing to the left of p has finite measure, and the set of trajectories $\mu^h - \nu^h$ passing to the right of p has finite measure. This will imply that the union of the right half spaces bounded by elements of M must cover the boundary circle, and the union of left half spaces bounded by elements of M must both cover the boundary circle. The combinatorial possibilities for the order of the endpoints of elements of M are:



The boundaries of half spaces give a cover of the boundary in which every interval is necessary. Since the boundary is compact, there must be finitely many intervals.

Parametrize the maximal trajectories of ν^h by $\gamma_1, ..., \gamma_n$ in cyclic order, such that $\gamma_{i+1}(0) = \gamma_i(l_i)$ for $l_i \in \mathbb{R}$.

□

Lemma 3.56. *An admissible lower submeasure ν of a cubic differential current μ is holonomy zero if and only if its maximal trajectories all intersect at a common point p, in which case $\nu = \nu(p)$.*

Proof. Recall that $\nu_h(p)$ is defined as the lower submeasure of μ_h consisting of the closure of all trajectories which pass to the right of p. The maximal support points of $\nu_h(0)$ must pass through p because if a trajectory doesn't pass through p it can be shifted to the left and still go to the right of p. Conversely, suppose we have another lower submeasure ν_h', all of whose maximal trajectories pass through p. ν_h' must be a submeasure of $\nu_h(p)$, otherwise it would have a trajectory, thus a maximal trajectory, passing to the left of p. If ν_h' is also holonomy zero, then it must coincide with $\nu_h(p)$. Now all we have to show is that if ν_h is holonomy zero than all of its trajectories pass through a common point.

Denote by $\gamma_1, ..., \gamma_k$ the unit speed parametrizations of the maximal trajectories of ν such that $\gamma_i(0)$ is the intersection point of γ_i with γ_{i-1}. Let $l_i \in \mathbb{R}$ be defined by $\gamma_i(l_i) = \gamma_{i+1}(0)$. The segments $\gamma_i|_{[0,l_i]}$ fit together in to a closed loop. We will show that the length of this loop, $l_1 + ... + l_k$, is the holonomy of ν. Let η_i be as follows.

$$\eta_i(t) = \begin{cases} \gamma_i(t) & t \leq l_i \\ \gamma_{i+1}(t - l_i) & t \geq l_i \end{cases}$$

The paths η_i are geodesic by lemma 3.52. Since γ_i parameterize the maximal trajectories of ν^h, the holonomy of ν^h is the holonomy of any monotonic taxi-path in $\mathcal{G}^h$ passing through each γ_i, so in particular the taxi-path given by the sequence of geodesics $\gamma_1, \eta_1, ..., \gamma_n, \eta_n$. The holonomy of this loop is easily seen to be $l_1 + ... + l_k$.

Suppose that $l_i > 0$ and $l_{i+1} < 0$. Then $\gamma_{i-1}, \gamma_i, \gamma_{i+1}$ would have 3-intersection giving a contradiction. We conclude that l_i are either all non-positive or non-negative. Consequently, if ν is holonomy zero, then all the l_i vanish, and all the γ_i must pass through a common point p.

$\square$

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