Abstract

Abstract: find a function $x(t)$ that extremizes the functional L. Here we are going to outline the proof of the Euler Lagrange equation.

1 Theorem Statement and Proof

Theorem 1. *For a functional J such that* $J[f] = \int_{t_1}^{t_2} L(t, f(t), f'(t))$ *with an extremum on the interval* $t_1 - t_2$*, let x(t) be the function that extremizes J. Then the following equation must be satified:*

$$\frac{d}{dt} * \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

Please state theorem precisely

Proof. For a functional J with an extremum on the interval $t_1 - t_2$ such that:

$$J(a) = \int_{t_1}^{t_2} f(x(t), \dot{x}(t), t)dt,$$

we wish to find the function $x(t)$ that extremizes J. What are the bounds for the integral? Why does x even exist. Let $A(t)$ be an arbitrary function and $A(t_1) = A(t_2) = 0$. Therefore any continuous function from $t_1 - t_2$ can be represented with

$$x(t, a) = x(t) + aA(t)$$

where x(t,a) is a combination x(t), our function that extremizes J, and A(t), the "divergence" from that function, parameterized by a.

Let

$$J(a) = \int_{t_1}^{t_2} f(x(t, a), \dot{x}(t, a), t)dt,$$

be functional parameterized by a.

Note when $a = 0$, $\frac{dJ}{da}$=0.

$\frac{dL}{da} = \frac{d}{da} \int_{t_1}^{t_2} f(x(t, a), \dot{x}(t, a), t)dt$

so

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial a} dt$$

$$\frac{\partial x}{\partial a} = A(t)$$
$$\frac{\partial \dot{x}}{\partial a} = \frac{\partial x}{\partial a} \cdot \frac{d}{dt} = A'(t)$$

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) + \frac{\partial f}{\partial \dot{x}} \cdot A'(t)$$

Using Integration by parts:

$$\int \frac{\partial f}{\partial \dot{x}} \cdot A'(t)dt = \frac{\partial f}{\partial \dot{x}} A(t)|_{t_1}^{t_2} - \int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} * A(t)dt$$
$$= -\int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} * A(t)dt$$

Thus

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \cdot A(t)dt$$
$$= \int A(t) \cdot (\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}})dt$$

As $A(t)$ is arbitrary everywhere on (t_1, t_2)

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} = 0$$
$$\frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} = \frac{df}{dx}$$

proving Euler's equation. This equation must be satisfied for a function x(t) that extremizes the functional L.

2 Applications

Arc-Length

An immediate consequence of the Euler-Lagrange equation is the classical fact that the shortest distance between two points is a straight line. We will show this derivation now.

Theorem 2. *Let a and b be two real numbers with $a < b$. Let c and d be two arbitrary real numbers. Then the unique differentiable function $x(t)$ that extremizes the functional*

$$S(x) = \int_a^b \sqrt{1 + (\frac{dx}{dt})^2} dt.$$

is the linear function $x(t) = mt + y_0$ for some real numbers m and y_0.

Note that the integrand in Theorem 2 is precisely the arc-length formula from Calculus. We can solve for m and y_0 How?.

Note that we call S a functional since it takes in a function as the input and outputs a real number.

Proof. Let $x(t)$ be the function that extremizes $S(x)$. Why does this exist?

The integrand in S is

$$L(t, x, x') = \sqrt{1 + (x')^2}.$$

We know from Theorem 1 (Euler Lagrange Equation) and the fact that x minimizes S, that

$$\frac{dL}{dx} = \frac{d}{dt}\frac{dL}{dx'}.$$

Since x is not present on in $L(t, x, x')$, we have

$$\frac{dL}{dx} = 0.$$

So we have (fix these equations)

$$0 = \frac{d}{dt}\frac{dL}{dx'}$$

The Chain Rule gives:

$$= \frac{d}{dt}\frac{-\dot{x}}{\sqrt{1-(\dot{x})^2}}$$

The quotient rule

$$= \frac{-\ddot{x}\sqrt{1-(\dot{x})^2} + \dot{x}^2\ddot{x}(1-\dot{x})^{-\frac{1}{2}}}{1-(\dot{x})^2}$$

$$= \frac{\ddot{x}}{(1-\ddot{x}^2)^{\frac{3}{2}}} = 0$$

Thus $\dot{x}$ must be 0

$$\ddot{x} = 0,$$

on the interval (a, b) optional: why?. Then by integrating twice we get:

$$x = mt + y_0$$

which is the equation for a line.

□

2.1 Spring Pendulum

Proving the shortest distance between two points is a straight line.

Spring pendulum

$$U = mg(l+x)cos(\theta) + \frac{1}{2}kx^2$$

The Kinetic energy of the system is the linear kinetic energy of the spring and the tangential Kinetic energy.

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l+x)^2\dot{\theta}^2$$

These energies are combined to make the Lagraganian

$$L = K + U$$

Let f(x) be the function that extremizes the action and thus is the actual motion of the ball linearly.

To find the function f(x) the following equation must be satisfied:

$$\frac{d}{dt}\frac{dL}{d\dot{x}} = \frac{dL}{dx}$$
$$m\ddot{x} = -mgsin(\theta) + kx + m(l+x)\dot{\theta}^2$$

The RHS is the ma part of Newtons second law and the LHS is the gravitational tangential force, the spring force, and the centrifugal force.

Solving this, gives f(x).

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} = \frac{dL}{d\theta}$$

$$m(l+x)^2\ddot{\theta} = -mg(l+x)sin(\theta)$$

This equation reveals that mass $\times$tangential acceleration = torque

$\square$