Abstract

Abstract: find a function $x(t)$ that extremizes the functional L. Here we are going to outline the proof of the Euler Lagrange equation.

1 Theorem Statement and Proof

Theorem 1. *Let J be the functional defined by*

$$J[f] := \int_{t_1}^{t_2} L(t, f(t), f'(t))$$

with an extremum on the interval $[t_1, t_2]$. Let $x(t)$ be a $C^1[t_1, t_2]$ function that extremizes J. Then the following equation must be satified:

$$\frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

Recall that $C^1[a, b]$ represents the set of all functions that have a continuous derivative on (a, b). Recall from Calculus every differentiable function is continuous.

Proof. By assumption in the Theorem 1, let $x \in C^1[t_1, t_2]$ be a function that extremizes J. We wish to show that

$$\frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

Let $A(t)$ be an arbitrary function and $A(t_1) = A(t_2) = 0$. Therefore any function in $C^1[t_1, t_2]$ can be represented with

$$x(t, a) = x(t) + aA(t)$$

where x(t,a) is a combination x(t), our function that extremizes J, and A(t), the "divergence" from that function, parameterized by a.

Let

$$J(a) = \int_{t_1}^{t_2} f(x(t, a), \dot{x}(t, a), t)dt,$$

be functional parameterized by a.

Use sentences

Note when $a = 0$, x(t,a) becomes x(t). Thus when $a = 0$ J has an extremum by defintion of x(t) and

$$\frac{dJ}{da}|_{a=0} = 0.$$

Why? is this equation true?

$\frac{dJ}{da} = \frac{d}{da}\int_{t_1}^{t_2} f(x(t,a), \dot{x}(t,a), t)dt$

So, by the chain rule, we have

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial a}dt.$$

Solving for $\frac{\partial x}{\partial a}$ and $\frac{\partial \dot{x}}{\partial a}$ we get

$$\frac{\partial x}{\partial a} = A(t)$$
$$\frac{\partial \dot{x}}{\partial a} = \frac{\partial x}{\partial a} \cdot \frac{d}{dt} = A'(t)$$

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) + \frac{\partial f}{\partial \dot{x}} \cdot A'(t).$$

Using Integration by parts:

$$\int \frac{\partial f}{\partial \dot{x}} \cdot A'(t)dt = \frac{\partial f}{\partial \dot{x}}A(t)|_{t_1}^{t_2} - \int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} * A(t)dt$$
$$= -\int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} * A(t)dt$$

Thus the equation becomes,

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \cdot A(t)dt$$
$$= \int A(t) \cdot (\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}})dt$$

As $A(t)$ is arbitrary, by the fundamental lemma of calculus of variations:

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} = 0$$
$$\frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} = \frac{df}{dx}$$

proving Euler's equation. So we have

$$\int_{t_1}^{t_2} f(t)A(t) = 0.$$

Also, $A(t)$ is (basically) arbitrary. Why must $f(t) = 0$?? This equation must be satisfied for a function x(t) that extremizes the functional L.

2 Applications

Arc-Length

An immediate consequence of the Euler-Lagrange equation is the classical fact that the shortest distance between two points is a straight line. We will show this derivation now.

Theorem 2. *Let a and b be two real numbers with $a < b$. Let c and d be two arbitrary real numbers. Then the unique differentiable function $x(t)$, with $x(a) = c$ and $x(b) = d$, that extremizes the functional*

$$S(x) = \int_a^b \sqrt{1 + (\frac{dx}{dt})^2} dt.$$

is the linear function $x(t) = mt + y_0$ for some real numbers m and y_0.

Note that we call S a functional since it takes in a function as the input and outputs a real number.

Proof. Let $x(t)$ be the function that extremizes $S(x)$. Why does this exist? We can see from various statments of the Euler Lagrange that IF a minimizer exists, then it must satisfy the Euler Lagrange equation. This does not prove that a minimizer exists. Furthermore, even if we find a solution to the Euler Lagrange equation, it's not clear that this is a minimizer. (for instance, think of $y = x^3$, which has a point where the derivative is 0 but is neither a max or min.)

The integrand in S can be represented by a functional L,

$$L(t, x, x') = \sqrt{1 + (\dot{x})^2}.$$

We know from Theorem 1 (Euler Lagrange Equation) and the fact that x minimizes S, that

$$\frac{dL}{dx} = \frac{d}{dt}\frac{dL}{d\dot{x}}.$$

Since x is not present on in $L(t, x, \dot{x})$, we have

$$\frac{dL}{dx} = 0.$$

So we have

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{dL}{d\dot{x}} \\ &= \frac{d}{dt}\frac{\dot{x}}{\sqrt{1+(\dot{x})^2}} \\ &= \frac{\ddot{x}\sqrt{1+(\dot{x})^2} - \dot{x}^2\ddot{x}(1+\dot{x})^{-\frac{1}{2}}}{1+(\dot{x})^2} \\ &= \frac{\ddot{x}}{(1+\ddot{x}^2)^{\frac{3}{2}}}. \end{aligned}$$

In the second inequality we used the quotient rule from Calculus and in the third line we used the chain rule. Thus $\ddot{x}$ must be 0

$$\ddot{x} = 0,$$

on the interval (a, b). By a consequence of Rolle's theorem, $\ddot{x} = 0$, $\dot{x}$ is a constant, say m. Therefore x has a constant slope and

$$x(t) = mt + y_0,$$

which is the equation for a line.

$\square$

We can solve for m and y_0, To solve for m we use the rise over run for the interval

$$m := \frac{x(a) - x(b)}{a - b},$$

then we can solve for y_0, given the point $(a, x(a))$ and m.

Thus, the shortest distance between two points is a straight line.

Spring pendulum

$$U = mg(l + x)\cos(\theta) + \frac{1}{2}kx^2$$

The Kinetic energy of the system is the linear kinetic energy of the spring and the tangential Kinetic energy.

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l + x)^2\dot{\theta}^2$$

These energies are combined to make the Lagraganian

$$L = K + U$$

Let f(x) be the function that extremizes the action and thus is the actual motion of the ball linearly.

To find the function f(x) the following equation must be satisfied:

$$\frac{d}{dt}\frac{dL}{d\dot{x}} = \frac{dL}{dx}$$
$$m\ddot{x} = -mgsin(\theta) + kx + m(l + x)\dot{\theta}^2$$

The RHS is the ma part of Newtons second law and the LHS is the gravitational tangential force, the spring force, and the centrifugal force.

Solving this, gives f(x).

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} = \frac{dL}{d\theta}$$

$$m(l + x)^2\ddot{\theta} = -mg(l + x)sin(\theta)$$

This equation reveals that mass $\times$tangential acceleration = torque

$\square$