E-L equation proof

let x(t) be the functino that extremizes the functional L(x(t)) = $\int x(t)d$. Let A(t) be the function that represents the divigergence from the idealized path x(t). Let $A(t_1) = A(t_2) = 0$. All possible paths can be represented by x(t,a) = x(t) + aA(t) where a parameterizes the divergence of the path.

let L(a) = $\int_{t_1}^{t_2} f(x(t,a), \dot{x}(t,a), t)dt$ be functional parameterized by a.

Note when a = 0 $\frac{dL}{da}$=0.

$\frac{dL}{da} = \frac{d}{da}\int_{t_1}^{t_2} f(x(t,a), \dot{x}(t,a), t)dt$

$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial a}dt$

$$\frac{\partial x}{\partial a} = A(t)$$
$$\frac{\partial \dot{x}}{\partial a} = \frac{\partial x}{\partial a} \times \frac{d}{dt} = A'(t)$$

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) + \frac{\partial f}{\partial \dot{x}} \cdot A'(t)$$

Using Integration by parts:

$$\int \frac{\partial f}{\partial \dot{x}} \cdot A'(t)dt = \frac{\partial f}{\partial \dot{x}} A(t)\|_{t_1}^{t_2} - \int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} * A(t)dt$$
$$= -\int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} * A(t)dt$$

Thus

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} \cdot A(t)dt$$
$$= \int A(t) \cdot (\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}})dt$$

As $A(t)$ is arbitrary everywhere on the interval

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} = 0$$
$$\frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} = \frac{df}{dx}$$

proving Euler's equation. This equation must be satisfied for a function x(t) that extremizes the functional L.

Arc-Length

Find the function x(t) that extremizes the functional

$$S = \int \sqrt{1 - (\frac{dx}{dt})^2} dt$$

This is the functional that calculates the arc length of a given path.

$$\frac{d}{dt}\cdot\frac{\partial f}{\partial \dot{x}} = \frac{df}{dx}$$

$$\frac{d}{dt}\frac{-\dot{x}}{\sqrt{1-(\dot{x})^2}} = 0$$

$$\frac{-\ddot{x}\sqrt{1-(\dot{x})^2}+\dot{x}^2\ddot{x}(1-\dot{x})^{-\frac{1}{2}}}{1-(\dot{x})^2} = 0$$

$$\frac{\ddot{x}}{(1-\ddot{x}^2)^{\frac{3}{2}}} = 0$$

Thus

$$\ddot{x} = 0$$

$$\int\int \ddot{x}dt = \int\int 0\,dt$$

$$x = mt + b$$

Proving the shortest distance between two points is a straight line.

Spring pendulum

$$U = mg(l+x)cos(\theta) + \frac{1}{2}kx^2$$

The Kinetic energy of the system is the linear kinetic energy of the spring and the tangential Kinetic energy.

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l+x)^2\dot{\theta}^2$$

These energies are combined to make the Lagraganian

$$L = K + U$$

Let f(x) be the function that extremizes the action and thus is the actual motion of the ball linearly.

To find the function f(x) the following equation must be satisfied:

$$\frac{d}{dt}\frac{dL}{d\dot{x}} = \frac{dL}{dx}$$

$$m\ddot{x} = -mgsin(\theta) + kx + m(l+x)\dot{\theta}^2$$

The RHS is the ma part of Newtons second law and the LHS is the gravitational tangential force, the spring force, and the centrifugal force.

Solving this, gives f(x).

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} = \frac{dL}{d\theta}$$

$$m(l+x)^2\ddot{\theta} = -mg(l+x)sin(\theta)$$

This equation reveals that mass $\times$tangential acceleration $=$ torque