

Abstract

Abstract: find a function $x(t)$ that extremizes the functional L . Here we are going to outline the proof of the Euler Lagrange equation.

1 Theorem Statement and Proof

Theorem 1. *Let J be the functional defined by*

$$J[f] := \int_{t_1}^{t_2} L(t, f(t), f'(t))$$

with an extremum on the interval $[t_1, t_2]$. Let $x(t)$ be a $C^1[t_1, t_2]$ function that extremizes J . Then the following equation must be satisfied:

$$\frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

Recall that $C^1[a, b]$ represents the set of all functions that have a continuous derivative on (a, b) . Recall from Calculus every differentiable function is continuous.

Proof. By assumption in the Theorem 1, let $x \in C^1[t_1, t_2]$ be a function that extremizes J . We wish to show that

$$\frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}.$$

Let $A(t)$ be an arbitrary function and $A(t_1) = A(t_2) = 0$. Therefore any function in $C^1[t_1, t_2]$ can be represented with

$$x(t, a) = x(t) + aA(t)$$

where $x(t, a)$ is a combination $x(t)$, our function that extremizes J , and $A(t)$, the "divergence" from that function, parameterized by a .

Let

$$J(a) = \int_{t_1}^{t_2} f(x(t, a), \dot{x}(t, a), t) dt,$$

be functional parameterized by a .

Use sentences

Note when $a = 0$, $x(t,a)$ becomes $x(t)$. Thus when $a = 0$ J has an extremum by definition of $x(t)$ and

$$\frac{dJ}{da}|_{a=0} = 0.$$

Why? is this equation true?

$$\frac{dJ}{da} = \frac{d}{da} \int_{t_1}^{t_2} f(x(t,a), \dot{x}(t,a), t) dt$$

So, by the chain rule, we have

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \frac{\partial x}{\partial a} + \frac{\partial f}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial a} dt.$$

Solving for $\frac{\partial x}{\partial a}$ and $\frac{\partial \dot{x}}{\partial a}$ we get

$$\begin{aligned} \frac{\partial x}{\partial a} &= A(t) \\ \frac{\partial \dot{x}}{\partial a} &= \frac{\partial x}{\partial a} \cdot \frac{d}{dt} = A'(t) \end{aligned}$$

$$0 = \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) + \frac{\partial f}{\partial \dot{x}} \cdot A'(t).$$

Using Integration by parts:

$$\begin{aligned} \int \frac{\partial f}{\partial \dot{x}} \cdot A'(t) dt &= \frac{\partial f}{\partial \dot{x}} A(t) \Big|_{t_1}^{t_2} - \int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} \cdot A(t) dt \\ &= - \int \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} \cdot A(t) dt \end{aligned}$$

Thus the equation becomes,

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \frac{\partial f}{\partial x} \cdot A(t) - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} \cdot A(t) dt \\ &= \int A(t) \cdot \left(\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} \right) dt \end{aligned}$$

As $A(t)$ is arbitrary, by the fundamental lemma of calculus of variations:

$$\begin{aligned}\frac{\partial f}{\partial x} - \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} &= 0 \\ \frac{d}{dt} \cdot \frac{\partial f}{\partial \dot{x}} &= \frac{df}{dx}\end{aligned}$$

proving Euler's equation. So we have

$$\int_{t_1}^{t_2} f(t)A(t) dt = 0.$$

Also, $A(t)$ is (basically) arbitrary. Why must $f(t) = 0$? This equation must be satisfied for a function $x(t)$ that extremizes the functional L .

2 Applications

Arc-Length

An immediate consequence of the Euler-Lagrange equation is the classical fact that the shortest distance between two points is a straight line. We will show this derivation now.

Theorem 2. *Let a and b be two real numbers with $a < b$. Let c and d be two arbitrary real numbers. Then the unique differentiable function $x(t)$ that extremizes the functional*

$$S(x) = \int_a^b \sqrt{1 + \left(\frac{dx}{dt}\right)^2} dt.$$

is the linear function $x(t) = mt + y_0$ for some real numbers m and y_0 .

Note that we call S a functional since it takes in a function as the input and outputs a real number.

Proof. Let $x(t)$ be the function that extremizes $S(x)$. Why does this exist? We can see from various statements of the Euler Lagrange that IF a minimizer exists, then it must satisfy the Euler Lagrange equation. This does not prove that a minimizer exists. Furthermore, even if we find a solution to the Euler Lagrange equation, it's not clear that this is a minimizer. (for instance, think

of $y = x^3$, which has a point where the derivative is 0 but is neither a max or min.)

The integrand in S can be represented by a functional L ,

$$L(t, x, x') = \sqrt{1 + (x')^2}.$$

We know from Theorem 1 (Euler Lagrange Equation) and the fact that x minimizes S , that

$$\frac{dL}{dx} = \frac{d}{dt} \frac{dL}{dx'}.$$

Since x is not present on in $L(t, x, x')$, we have

$$\frac{dL}{dx} = 0.$$

So we have (fix these equations: should be a plus coming from the arc length formula)

$$\begin{aligned} 0 &= \frac{d}{dt} \frac{dL}{dx'} \\ &= \frac{d}{dt} \frac{\dot{x}}{\sqrt{1 + (\dot{x})^2}} \\ &= \frac{\ddot{x} \sqrt{1 + (\dot{x})^2} - \dot{x}^2 \ddot{x} (1 + \dot{x})^{-\frac{1}{2}}}{1 + (\dot{x})^2} \\ &= \frac{\ddot{x}}{(1 + \dot{x}^2)^{\frac{3}{2}}}. \end{aligned}$$

In the second inequality we used the quotient rule from Calculus and in the third line we used the chain rule. Thus \ddot{x} must be 0

$$\ddot{x} = 0,$$

on the interval (a, b) . Why is it true that a function with $x'' = 0$ must be a line? When $\ddot{x} = 0$, \dot{x} is a constant. therefore x has a constant slope. Then by integrating twice we get:

$$x(t) = mt + y_0$$

which is the equation for a line.

□

We can solve for m and y_0 , To solve for M we could do rise over run for the interval $\frac{x(a)-x(b)}{a-b}$ then we can solve for y_0 , given the point $(a, x(a))$ and m .
 Proving the shortest distance between two points is a straight line.
 Spring pendulum

$$U = mg(l+x)\cos(\theta) + \frac{1}{2}kx^2$$

The Kinetic energy of the system is the linear kinetic energy of the spring and the tangential Kinetic energy.

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(l+x)^2\dot{\theta}^2$$

These energies are combined to make the Lagrangian

$$L = K + U$$

Let $f(x)$ be the function that extremizes the action and thus is the actual motion of the ball linearly.

To find the function $f(x)$ the following equation must be satisfied:

$$\begin{aligned} \frac{d}{dt} \frac{dL}{d\dot{x}} &= \frac{dL}{dx} \\ m\ddot{x} &= -mg\sin(\theta) + kx + m(l+x)\dot{\theta}^2 \end{aligned}$$

The RHS is the main part of Newton's second law and the LHS is the gravitational tangential force, the spring force, and the centrifugal force.

Solving this, gives $f(x)$.

$$\frac{d}{dt} \frac{dL}{d\dot{\theta}} = \frac{dL}{d\theta}$$

$$m(l+x)^2\ddot{\theta} = -mg(l+x)\sin(\theta)$$

This equation reveals that mass \times tangential acceleration = torque

□