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CQF Module 3

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Asian Option Pricing:Monte Carlo Simulation with Euler-Maruyama Scheme

1.Euler-Maruyama Price Simulation

1.1 Euler-Maruyama Scheme

The Black-Scholes equation under risk-neutral measure is:

$$dS_t = rS_t dt + \sigma S_t dWt \qquad (1)$$

r is short term rate, σ is implied volaltility. W_t is a wiener process.

The Euler discretization of (1) is :

$$S_{t+\Delta t} - S_t = rS_t\Delta t + \sigma\phi S_t \sqrt{\Delta t} \qquad (2)$$

where φ corresponds to a random variable drawn from N(0,1).

1.2 Simulation Algorithm

Inputs: S0, T − t, σ, r, ∆ t , K where K represents number of trials, and ∆ t represents unit time step size.

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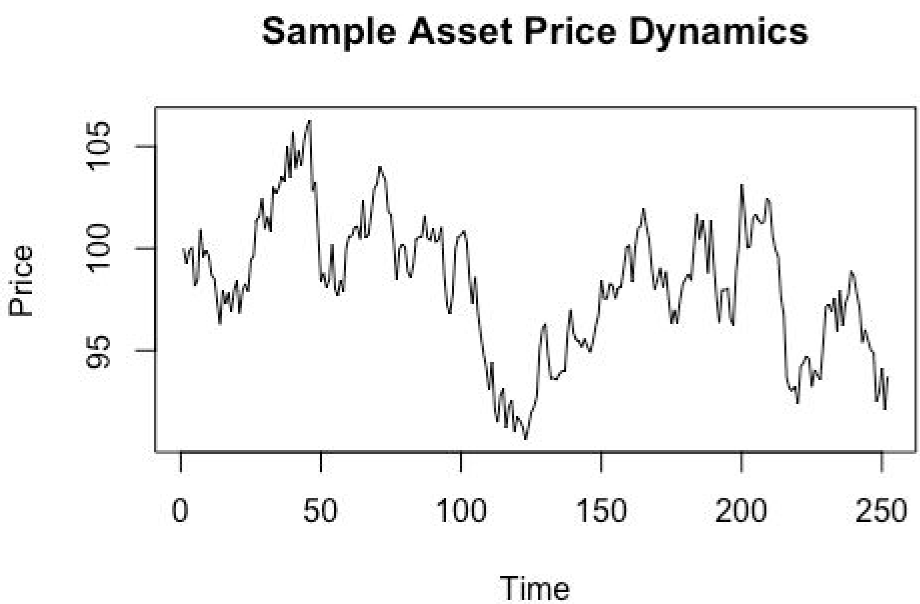
Output: S, a K*N array where K and N are defined below
 Fuction(S0, T − t, σ, r, ∆ t,K):
     N = T-t / ∆ t   //number of steps taken
      S(0,:) = S0
      For i = 1 to K:
            For j = 1 to N:
                    φ = rnorm(0,1); // generate random normal variable
                    S(i,j) = S(i,j-1) * (1+ r * ∆ t + σ * φ * √∆t );
```

Return S;

1.3 Parameter

The input parameter for this report is partially provided as: S0 = 100, E = 100, T − t = 1 (year), σ = 20%, r = 5%. Additionally, we choose K =100,200,750,1000,2500,5000,7500, 10000,20000,30000,40000,50000,60000,70000,80000,90000,10000 to represent number of paths and $\Delta t = 1 / 252$ and N = 252 so that each unit time step represent 1 trading day.

1.4 Sample Asset Price Dynamics



2. Asian Option Pricing

2.1 Sampling Method

2.1.1 Arithmetic Sampling

Define S(t) as asset price at time t, the average price over time [0,T] is defined as :

$$A_c(0,T) = \frac{1}{T}\int_0^T S(t)dt \qquad (3)$$

for continuous case. Discretely, this formula can be written as:

$$A_d(0,T) = \frac{1}{N} \sum_{i=0}^{N-1} S(i * T/N) \quad (4)$$

where N is number of samples chosen.

2.1.2 Geometric Sampling

The continuous and discrete formula for average price over time [0,T] is represented as:

$$A_c(0,T) = exp(\frac{1}{T} \int_0^T ln(S(t))dt) \quad (5)$$

$$A_d(0,T) = \left(\prod_{i=0}^{N-1} S(i * T/N) \right)^N \quad (6)$$

2.2 Strike Scheme and payoff functions

The following table includes payoff functions of asian options based on option type and strike scheme:

Option Type \ Strike Scheme	Fixed	Floating
Call	C(T) = max(A(0, T) − K, 0)	C(T) = max(S(T) − kA(0, T), 0)
Put	P(T) = max(K − A(0, T), 0)	max(kA(0, T) − S(T), 0)

*K is fixed strike price, S(T) is the price at maturity T and k is a weighting factor.For the report , we adopt the general case k = 1

The payoff function we use is:

$$V(s,t) = E(e^{-\int_t^T r d\tau} * payoff) \quad (7)$$

3. Theoretical Pricing of Fixed Asian Options

3.1 Asian Fixed Strike Arithmetic

A Levy approximation is used for the derivation of the analytical result:

$$C_{Levy} \approx S_Z N(d_1) - K_Z e^{-rT_2} N(d_2)$$
$$P_{Levy} \approx C_{Levy} - S_Z + K_Z e^{-rT_2}$$

$$d_1 = \frac{1}{\sqrt{\nu}}\left[\frac{ln(L)}{2} - ln(K_Z)\right]$$

$$d_2 = d_1 - \sqrt{\nu}$$

$$S_Z = \frac{S}{rT}(1 - e^{-rT_2})$$

$$K_Z = K - S_{Avg}\frac{T - T_2}{T}$$

$$\nu = ln(L) - 2[rT_2 + ln(S_Z)]$$

$$L = \frac{M}{T^2}$$

$$M = \frac{2S^2}{r + \sigma^2}\left[\frac{e^{(2r+\sigma^2)T_2} - 1}{2r + \sigma^2} - \frac{e^{rT_2} - 1}{r}\right]$$

*In the equation, T_2 represents time left till maturity , *T* represents ending time the averaging period. In our case , T2 = 1, T=1,

3.2 Asian Fixed Strike Geometric:

We use formulas derived by Kemna and Forst:

$$C_{geom} = S\mathrm{e}^{(b-r)(T-t)}N(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2)$$

$$P_{geom} = -S\mathrm{e}^{(b-r)(t-t)}N(-d_1) + K\mathrm{e}^{-r(T-t)}N(-d_2)$$

$$\text{where } d_1 = \frac{ln\left(\frac{S}{K}\right) + (b + \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}}$$

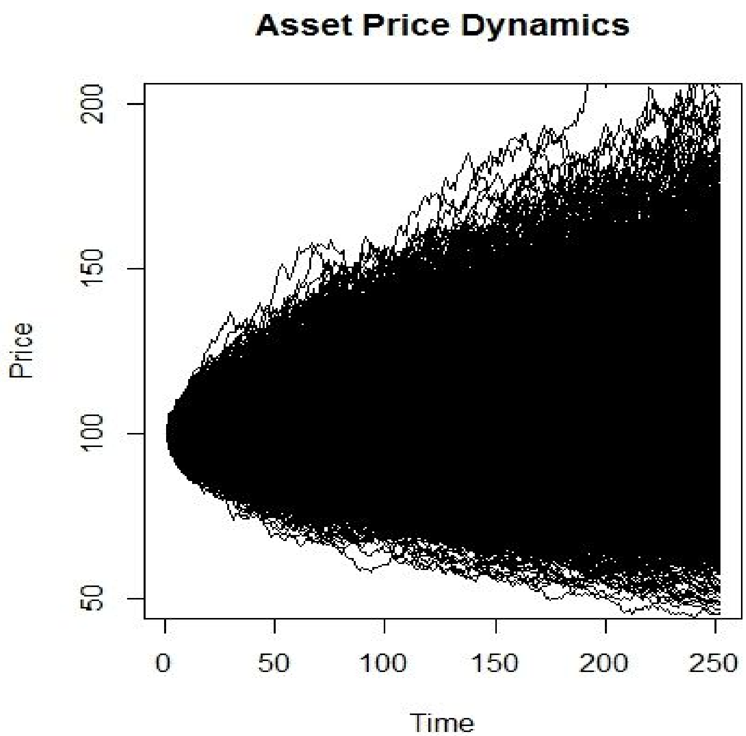
$$d_2 = \frac{ln\left(\frac{S}{K}\right) + (b - \frac{\sigma_A^2}{2})T}{\sigma_A\sqrt{T}} = d_1 - \sigma_A\sqrt{T}$$

$$\sigma_A = \frac{\sigma}{\sqrt{3}} \text{ and } b = \frac{1}{2}\left(r - \frac{\sigma^2}{6}\right)$$

In the equation, T represents time left till maturity , t represents the time at the beginning of the averaging period. In our case , T = 1, t =0

4. Result

4.1 Simulation Result



Sample graph with simulation path N = 10000

The following table contains option payoffs

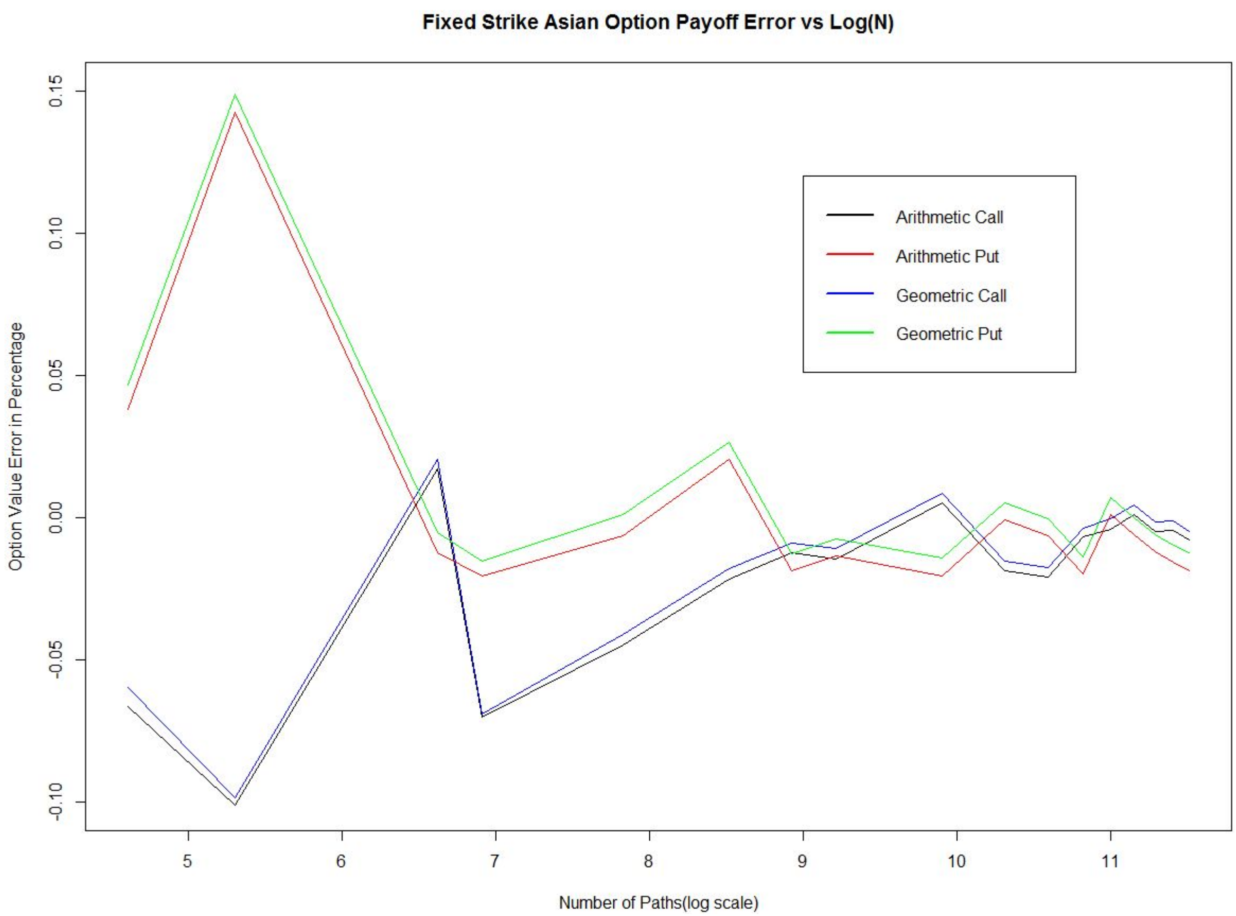
Strike Type	Fixed Strike								Floating Strike			
Avg Scheme	**Arithmetic**				**Geometric**				**Arithmetic**		**Geometric**	
Option Type	**Call**	**Error**	**Put**	**Error**	**Call**	**Error**	**Put**	**Error**	**Call**	**Put**	**Call**	**Put**
100	**5.399031**	**-6.64%**	**3.492309**	**3.79%**	**5.214999**	**-5.98%**	**3.624769**	**4.66%**	**4.926937**	**3.564289**	**5.109689**	**3.43055**
200	**5.198662**	**-10.10%**	**3.843995**	**14.25%**	**4.99945**	**-9.87%**	**3.979024**	**14.89%**	**5.652518**	**3.901972**	**5.845261**	**3.760473**

750	**5.882218**	**1.72%**	**3.323308**	**-1.23%**	**5.659602**	**2.03%**	**3.445128**	**-0.53%**	**5.667845**	**3.743219**	**5.813478**	**3.602392**
1000	**5.377146**	**-7.02%**	**3.295415**	**-2.06%**	**5.163636**	**-6.91%**	**3.409532**	**-1.55%**	**5.832659**	**3.357613**	**6.042576**	**3.239903**
2500	**5.524223**	**-4.47%**	**3.342554**	**-0.66%**	**5.319683**	**-4.09%**	**3.467569**	**0.12%**	**5.614441**	**3.637946**	**5.809914**	**3.503862**
5000	**5.65759**	**-2.17%**	**3.43345**	**2.05%**	**5.446361**	**-1.81%**	**3.554558**	**2.63%**	**5.582619**	**3.438298**	**5.785735**	**3.309077**
7500	**5.710323**	**-1.25%**	**3.301173**	**-1.89%**	**5.495885**	**-0.92%**	**3.418559**	**-1.29%**	**5.724885**	**3.419907**	**5.929544**	**3.29274**
10000	**5.698344**	**-1.46%**	**3.318806**	**-1.36%**	**5.485649**	**-1.10%**	**3.436934**	**-0.76%**	**5.887775**	**3.392912**	**6.092853**	**3.267168**
20000	**5.813182**	**0.52%**	**3.295563**	**-2.05%**	**5.594342**	**0.86%**	**3.413881**	**-1.43%**	**5.897439**	**3.419703**	**6.108225**	**3.293331**
30000	**5.673999**	**-1.88%**	**3.361869**	**-0.08%**	**5.462187**	**-1.53%**	**3.48064**	**0.50%**	**5.780645**	**3.396211**	**5.985554**	**3.270537**
40000	**5.662443**	**-2.08%**	**3.343693**	**-0.62%**	**5.448879**	**-1.77%**	**3.46225**	**-0.03%**	**5.803308**	**3.394549**	**6.00975**	**3.268869**
50000	**5.742823**	**-0.69%**	**3.298241**	**-1.97%**	**5.525618**	**-0.38%**	**3.414749**	**-1.40%**	**5.856678**	**3.367231**	**6.06613**	**3.242969**
60000	**5.759343**	**-0.41%**	**3.368243**	**0.11%**	**5.543551**	**-0.06%**	**3.488205**	**0.72%**	**5.893443**	**3.39047**	**6.102845**	**3.264116**
70000	**5.788614**	**0.10%**	**3.344389**	**-0.60%**	**5.570415**	**0.43%**	**3.463528**	**0.01%**	**5.856402**	**3.440743**	**6.065508**	**3.312511**
80000	**5.754451**	**-0.49%**	**3.324319**	**-1.20%**	**5.538087**	**-0.16%**	**3.442853**	**-0.59%**	**5.862046**	**3.370256**	**6.070967**	**3.244279**
90000	**5.75708**	**-0.45%**	**3.312333**	**-1.55%**	**5.54088**	**-0.11%**	**3.430015**	**-0.96%**	**5.831034**	**3.411632**	**6.038971**	**3.285687**
100000	**5.737638**	**-0.78%**	**3.301435**	**-1.88%**	**5.520117**	**-0.48%**	**3.419997**	**-1.25%**	**5.893934**	**3.377385**	**6.104174**	**3.251542**
Theoretical	**5.782838**	/	**3.36463**	/	**5.546819**	/	**3.46332**	/	/	/	/	/

*Error in the table refers to differences between monte carlo simulation and theoretical calculation.

4.2 Observations and Error Analysis

As required by the general assumption of black-scholes model, the payoff value of option should converge as number of paths N goes to infinity. We can observe a general converging behavior from the data listed in the table (error does not change as much after N > 10000). In general the option payoff does not fluctuate as much when N > N*, N* is a number(for the graph below it is approximately e^9) large enough. To demonstrate such behavior, the following graph provides payoff value vs iteration number for fixed strike asian options.



It is also interesting to see that most of the times (for fixed schema), the Monte-Carlo simulations tend to underestimate the option value as compared to the theoretical value. One possible reason is that the naive assumption of constant volatility can be off from reality. Models with better captured volatility could be utilized to improve the pricing such as stochastic volatility modeling.

5. Improvement

1.Variance Reduction:

The standard deviation for each type of option is :

Strike Type	Fixed Strike				Floating Strike			
Avg Scheme	Arithmetic		Geometric		Arithmetic		Geometric	
Option Type	Call	Put	Call	Put	Call	Put	Call	PutS
Standard Deviation	8.063659	5.204912	7.790177	5.347846	8.063659	5.162210	8.527668	5.012069

To reduce such deviation and make convergence faster, one can adopt the antithetic method, which follows that: E(f(x)) = $\frac{1}{2} * (E(f(x)) + E(f(-x)))$ therefore, $\sigma_{antithetic} = \sigma_{payoff}(\sqrt{1+\rho})$ where ρ is the correlation between x and -x.

2. Milstein Correction

The taylor expansion of exact solution of GBM gives:

$$S_{t+\Delta t} = S_t \left(1 + r\Delta t + \sigma\phi\sqrt{\Delta t} + \tfrac{1}{2}\sigma^2(\phi^2 - 1)\delta t\right)$$

The term $\frac{1}{2}\sigma^2(\phi^2 - 1)\delta t$ of O(δt) provides an correction on forward Euler-Maruyama scheme and provides additional accuracy to the estimation.

6.Conclusion

In the report, Euler- Maruyama scheme is utilized to price Asian option through Monte Carlo Simulation. Payoffs are evaluated through fixed/floating strike schemes and arithmetic/geometric averaging methods. The simulation results are then compared with the theoretical option values. It is concluded that options payoffs tend to be more accurate as number of simulated paths increases.To improve the simulation results, we can utilize antithetic method to reduce data variance, adopt simulation schemes such as Milstein scheme to improve accuracy or models with more complex volatility modeling techniques embedded.

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