HW05

October 11, 2018

View in Colaboratory

1 HW05 MLMATH

1.1 1.

1.2 2.

```
In [0]: from scipy.io import loadmat
        from scipy.special import legendre
        from scipy.integrate import quad
        import numpy as np
        from matplotlib import pyplot as plt
        % matplotlib inline
        from tqdm import tqdm
```

1.2.1 (a)

Import data using scipy

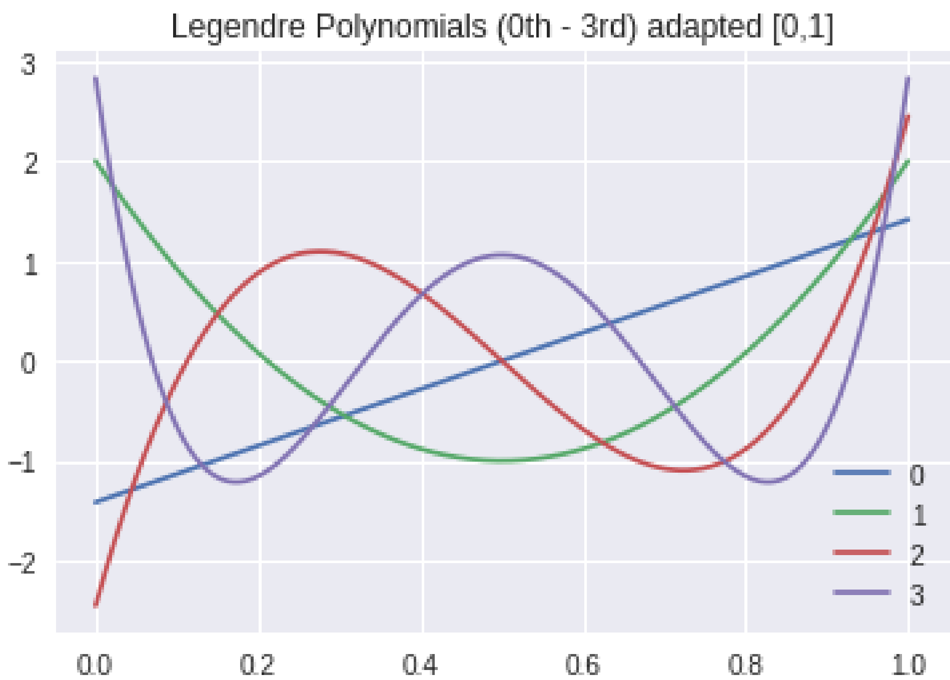
```
In [0]: data = loadmat('hw5p2_clusterdata.mat')
        T, y = data['T'], data['y']
```

Python version of the lpoly function

```
In [0]: lpoly = lambda p,z: np.sqrt(2)*np.sqrt((2*p+2)/2)*legendre(p+1)(2*z-1)
```

Defined the Legendre Polynomials over range [0,1]

```
In [4]: t = np.linspace(0,1,500)
        for n in range(4):
            plt.plot(t,lpoly(n,t), label=n)
        plt.legend()
        plt.title('Legendre Polynomials (0th - 3rd) adapted [0,1]')
        plt.show()
```



Setting up the least squares problem with the matrix A

```
In [5]: A = np.array([lpoly(n,T) for n in np.arange(0,4)]).squeeze().T
        w_hat = np.linalg.inv(A.T@A)@A.T@y
        y_hat = A@w_hat
        y_0 = A@np.ones(4)
        sample_error = lambda y, y_hat:  np.sqrt(((y-y_hat)**2).sum())
        print_error = lambda name, y, y_hat:  print('Sample Error for {}:{:.3f}'
                                               .format(name,sample_error(y,y_hat) ))
        print_error('w=1', y, y_0)
        print_error('LS Solution', y, y_hat)
        plt.scatter(T, y_0, label='w=1')
        plt.scatter(T, y_hat, label='w_hat')
        plt.scatter(T, y, label='Data')
        plt.legend()
```

```
Sample Error for w=1:289.583
Sample Error for LS Solution:5.907
```

```
Out[5]: <matplotlib.legend.Legend at 0x7f0781fc7ef0>
```

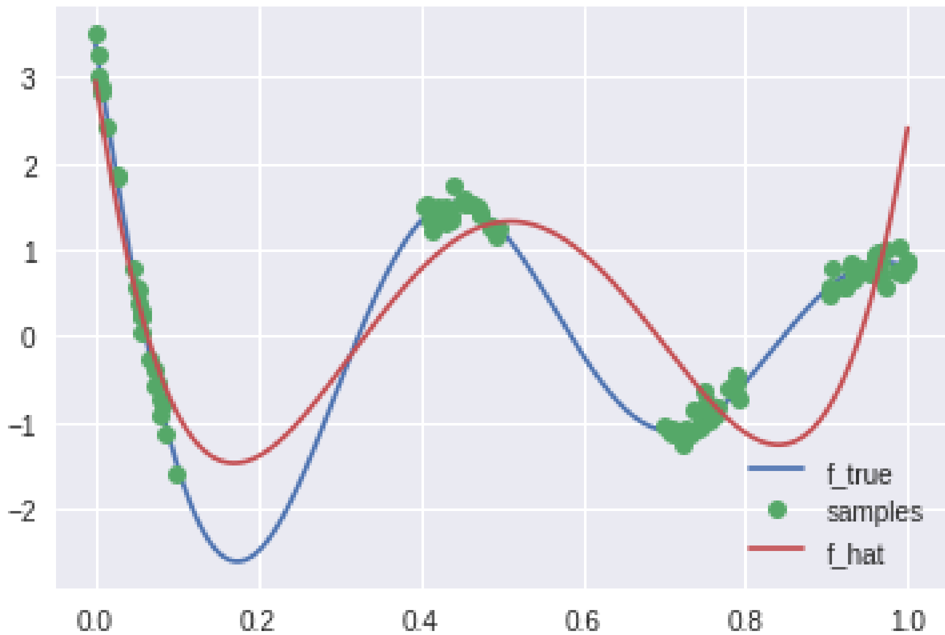
1.2.2 (b)

$f_{true}(t) = \frac{sin(12(t+0.2))}{t+0.2}$

```
In [6]: f_true = lambda t: np.sin(12*(t+0.2))/(t+0.2)
        f_hat = lambda t: np.array([lpoly(n,t) for n in np.arange(0,4)]).squeeze().T@w_hat
        gen_error = lambda f_1, f_2: np.sqrt(quad(lambda t: np.abs(f_1(t)-f_2(t))**2,0,1)[0])
        print('Squared Generalization Error: {:.3f}'.format(gen_error(f_true, f_hat)))
        plt.plot(t, f_true(t), label='f_true')
        plt.plot(T, y, 'o', label='samples')
        plt.plot(t, f_hat(t), label='f_hat')
        plt.legend()
```

```
Squared Generalization Error: 0.819
```

```
Out[6]: <matplotlib.legend.Legend at 0x7f0781f4b7f0>
```

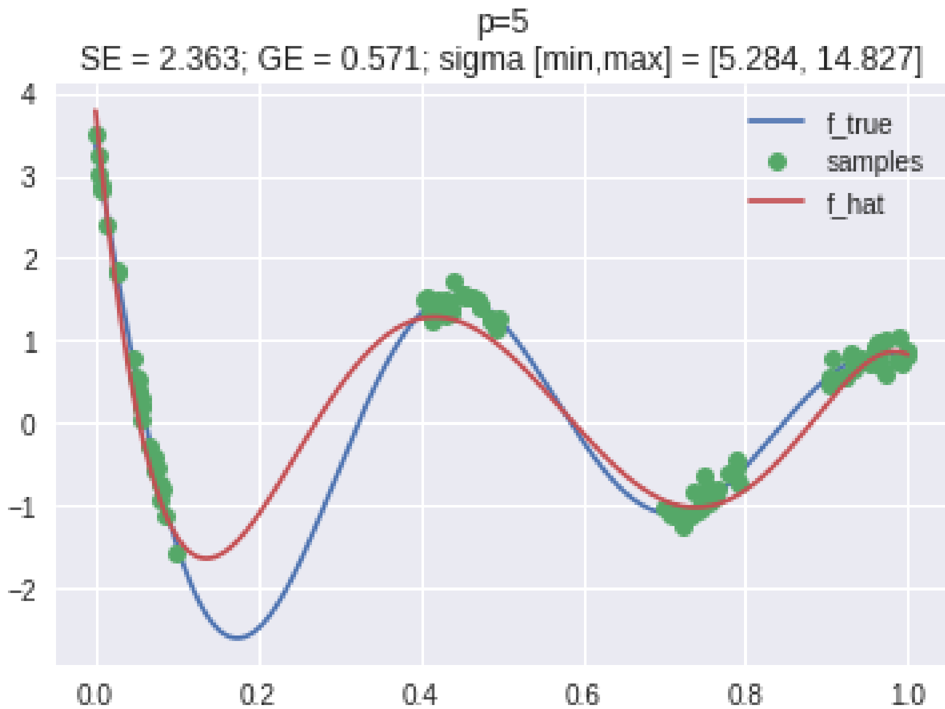


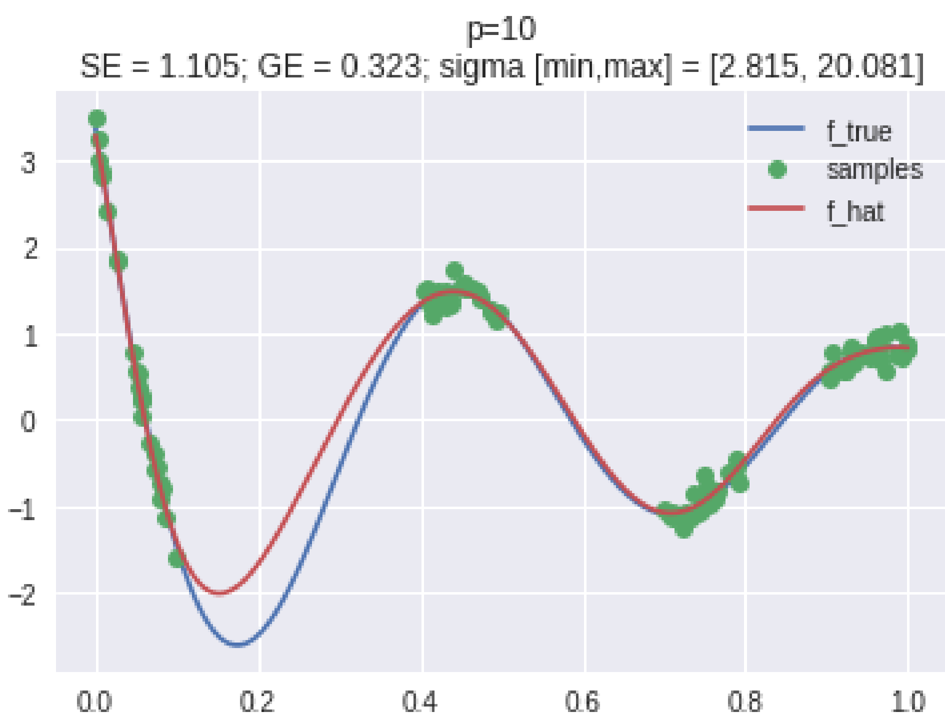
1.2.3 (c)

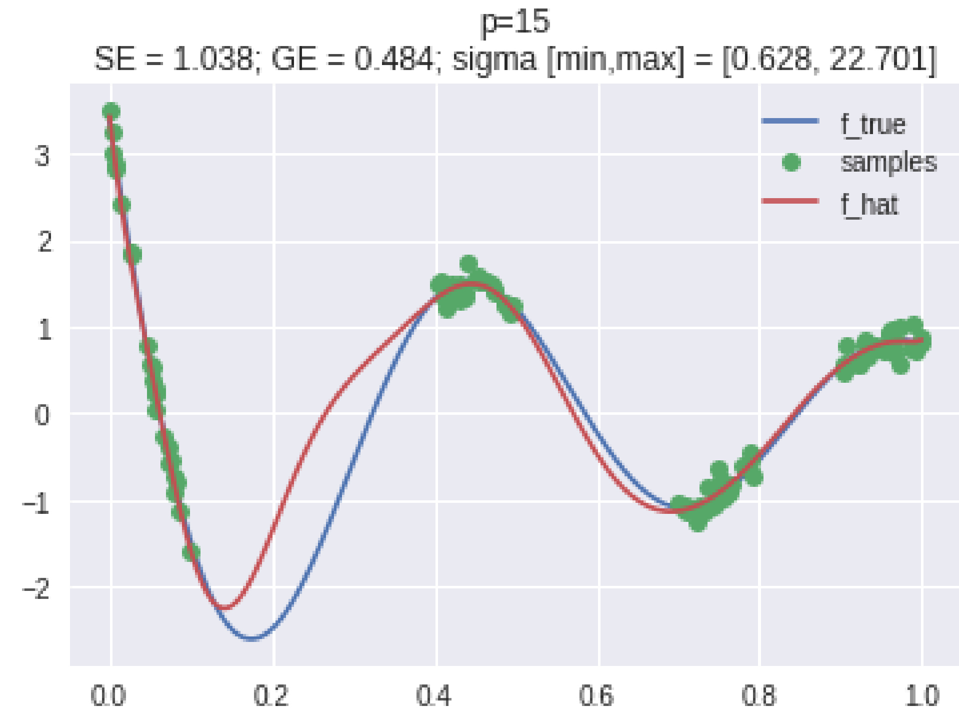
```
In [7]: p_list = np.array([5,10,15,20,25])
        se_l = []
        ge_l = []
        for i, p in enumerate(p_list):
            A = np.array([lpoly(n,T) for n in np.arange(0,p)]).squeeze().T
            u, sv, v = np.linalg.svd(A, full_matrices=False)
            simat = np.diag(1/sv)
            A_ih = v.T@simat@u.T
            w_hat = A_ih@y
            y_hat = A@w_hat
            f_hat = lambda t: np.array([lpoly(n,t) for n in np.arange(0,p)]).squeeze().T@w_hat
            se = sample_error(y_hat,y)
            ge = gen_error(f_hat,f_true)
            se_l.append(se)
            ge_l.append(ge)
            plt.figure()
            plt.plot(t, f_true(t), label='f_true')
            plt.plot(T, y, 'o', label='samples')
            plt.plot(t, f_hat(t), label='f_hat')
            plt.title('p={}\nSE = {:.3f}; GE = {:.3f}; sigma [min,max] = [{:.3f}, {:.3f}]'.form
            plt.legend()

        plt.figure()
        plt.plot(p_list, se_l, label='Sample Error')
```

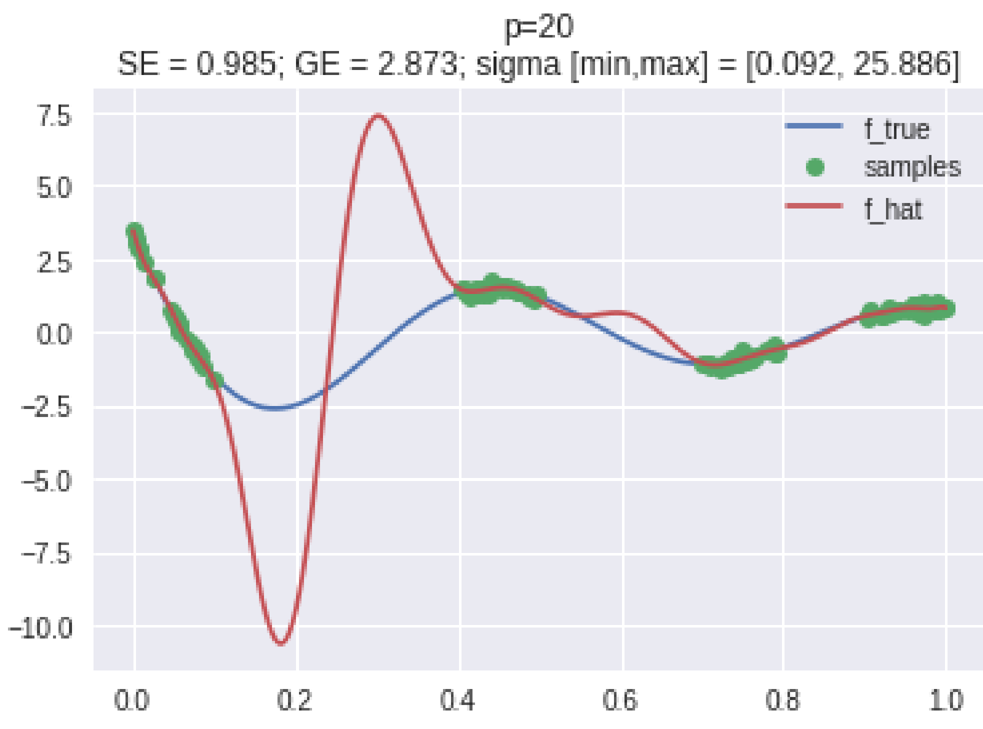
```
plt.plot(p_list, ge_l, label='Generalization Error')
plt.title('Sample and Generalization Error for p = {}'.format(p_list))
```

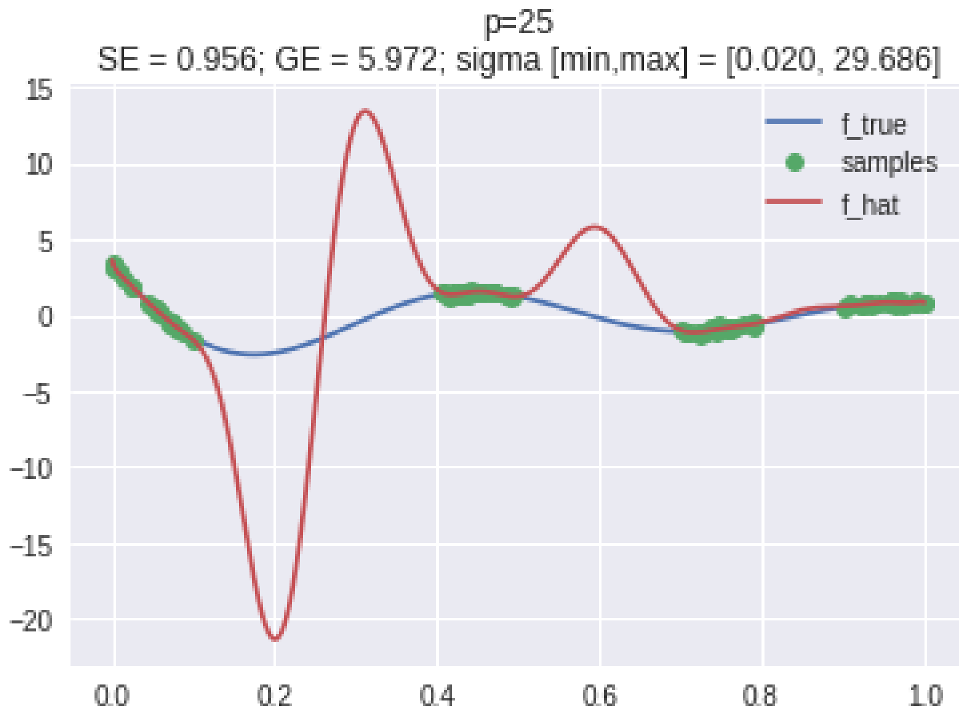
Out[7]: Text(0.5,1,'Sample and Generalization Error for p = [5 10 15 20 25]')

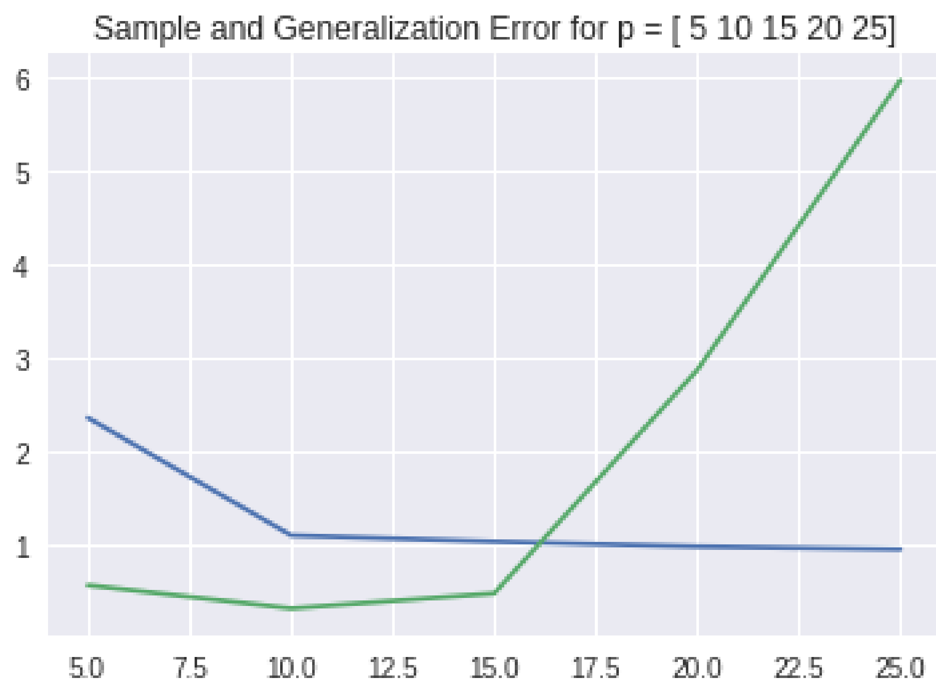





p=15
SE = 1.038; GE = 0.484; sigma [min,max] = [0.628, 22.701]
f_true
samples
f_hat
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0


p=20
SE = 0.985; GE = 2.873; sigma [min,max] = [0.092, 25.886]
f_true
samples
f_hat
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
0.0
0.2
0.4
0.6
0.8
1.0


p=25
SE = 0.956; GE = 5.972; sigma [min,max] = [0.020, 29.686]
f_true
samples
f_hat
15
10
5
0
−5
−10
−15
−20
0.0
0.2
0.4
0.6
0.8
1.0

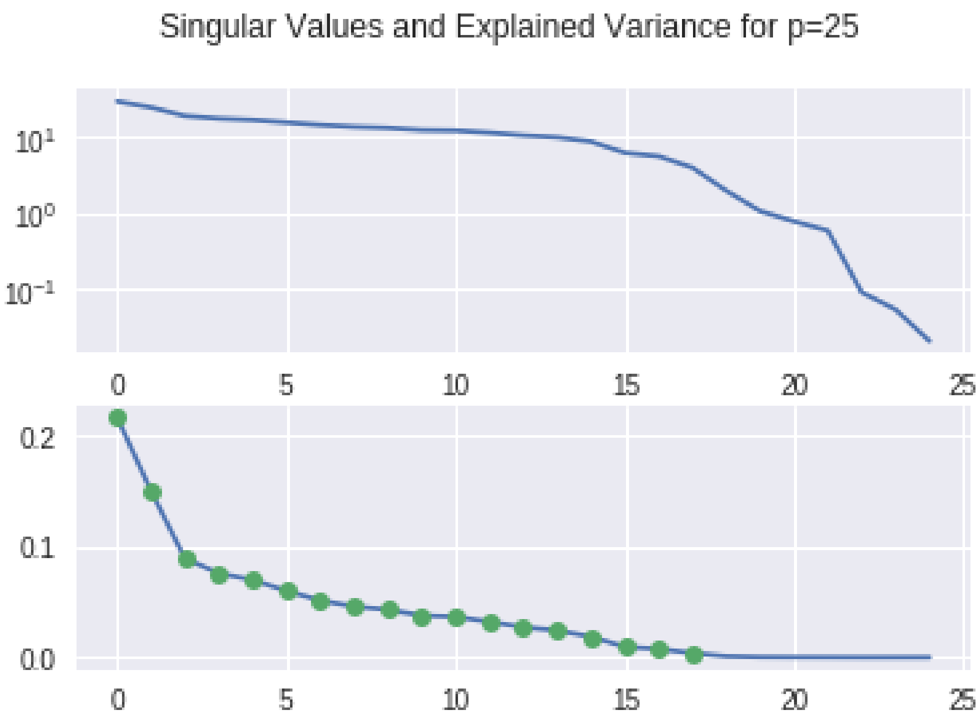

Sample and Generalization Error for p = [5 10 15 20 25]
6
5
4
3
2
1
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
25.0

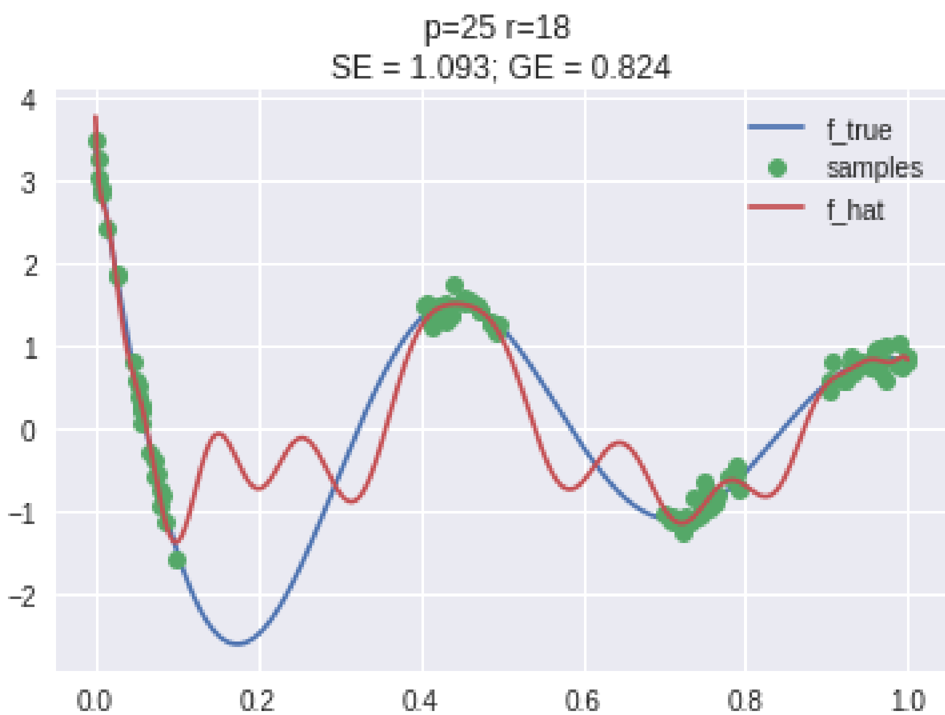
The generalization error for least squares falls apart at $p = 20$ because the smallest singular value $\sigma_{20} = 0.015$ causes the noise error to blow up. Sample error however, is solving to fit the points in y with least squares, which always has a minimum solution. The concept demonstrated by the difference between generalization error and sample error is that of overfitting. Though the solution to least squares is convex, allowing for solutions to be determined at some point additional bases actually causes the process to overfit to the samples and the generalization error out of control. In the case of the Legendre Polynomials, additional bases adds higher-order polynomial terms that cause $\hat{f}(t)$ to have extra peaks and troughs beyond that of $f_{true}(t)$. These additional troughs and peaks will provide additional benefit for reducing sample error, but cause the generalization error to explode.

###(d)

In [8]:
```
p = 25
A = lambda t_: np.array([lpoly(n,t_) for n in np.arange(0,p)]).squeeze().T
u, sv, v = np.linalg.svd(A(T), full_matrices=False)
fig, ax = plt.subplots(2)
explained_var = sv**2/(sv**2).sum()
idx = explained_var > 0.001
ax[0].semilogy(sv)
ax[1].plot(explained_var)
ax[1].plot(explained_var[idx], 'o')
fig.suptitle('Singular Values and Explained Variance for p=25')
simat = np.diag(1/sv[idx])
A_ih = v[idx,:].T@simat@u[:,idx].T
w_hat = A_ih@y
y_hat = A(T)@w_hat
f_hat = lambda t: np.array([lpoly(n,t) for n in np.arange(0,p)]).squeeze().T@w_hat
se = sample_error(y_hat,y)
ge = gen_error(f_hat,f_true)
se_l.append(se)
ge_l.append(ge)
plt.figure()
plt.plot(t, f_true(t), label='f_true')
plt.plot(T, y, 'o', label='samples')
plt.plot(t, f_hat(t), label='f_hat')
plt.title('p={} r={} \nSE = {:.3f}; GE = {:.3f}'.format(p,idx.sum(), se, ge))
plt.legend()
```

Out[8]:
```
<matplotlib.legend.Legend at 0x7f0781b95cc0>
```


Singular Values and Explained Variance for p=25
10^1
10^0
10^{-1}
0
5
10
15
20
25
0.2
0.1
0.0
0
5
10
15
20
25


p=25 r=18
SE = 1.093; GE = 0.824
f_true
samples
f_hat
4
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0

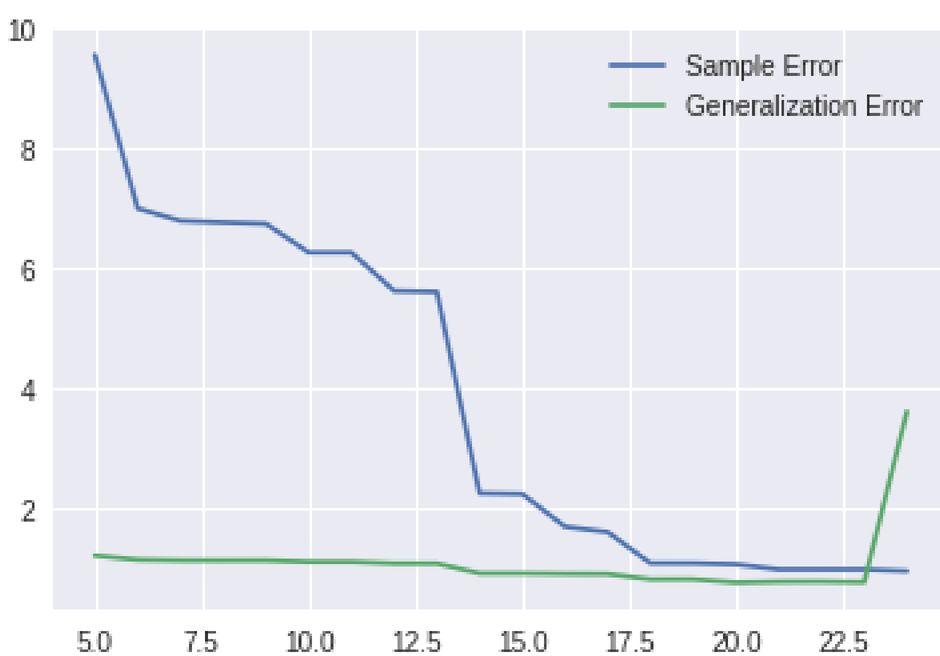
The choice of singular values was based on the explained variance of each signular value, $ev_{\sigma_r} = \sigma_r^2 / \sum_n^R \sigma_n^2$. More specifically:

$$\Sigma' = \{\sigma_r : ev_{\sigma_r} > 0.001, r = 1 \dots R\}$$

1.2.4 (e)

In [9]:
```
u, sv, v = np.linalg.svd(A(T), full_matrices=False)
R_l = np.arange(5,25,1)
se_l = []
ge_l = []
for idx in R_l:
    simat = np.diag(1/sv[:idx])
    A_ih = v[:idx,:].T@simat@u[:,:idx].T
    w_hat = A_ih@y
    y_hat = A(T)@w_hat
    f_hat = lambda t: np.array([lpoly(n,t) for n in np.arange(0,p)]).squeeze().T@w_hat
    se = sample_error(y_hat,y)
    ge = gen_error(f_hat,f_true)
    se_l.append(se)
    ge_l.append(ge)

plt.plot(R_l, se_l, label='Sample Error')
plt.plot(R_l, ge_l, label='Generalization Error')
plt.legend()
```

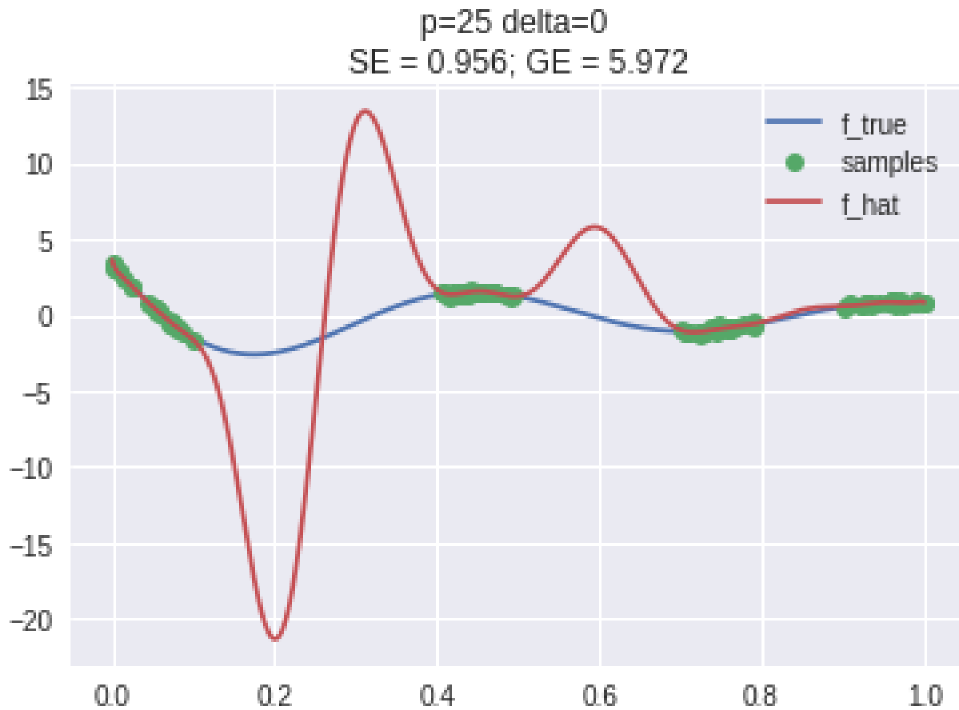
Out[9]: <matplotlib.legend.Legend at 0x7f0781cafc50>

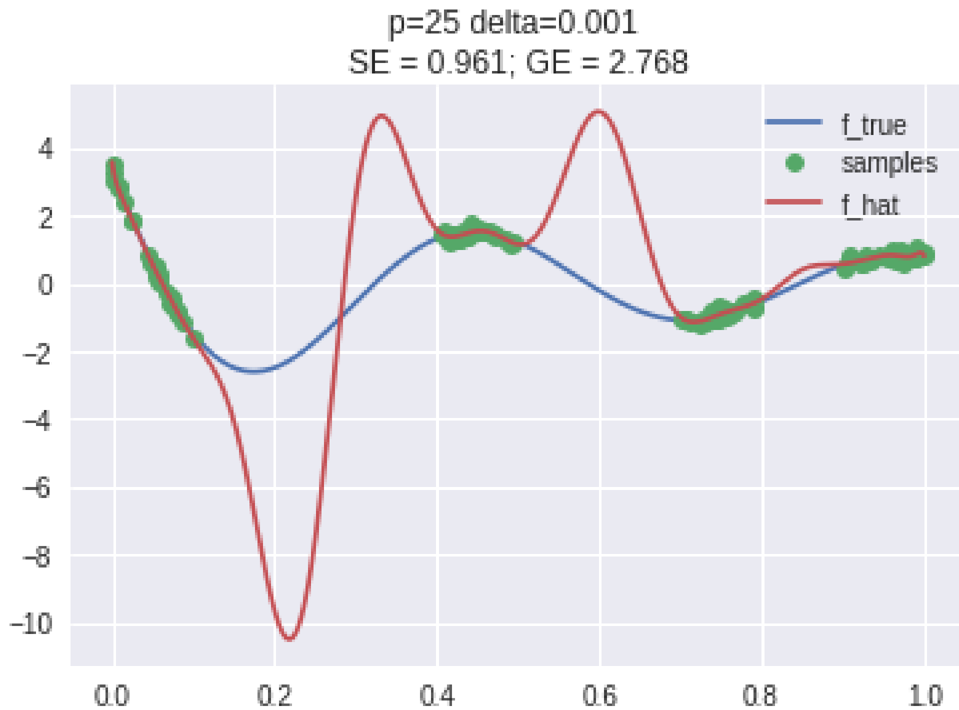


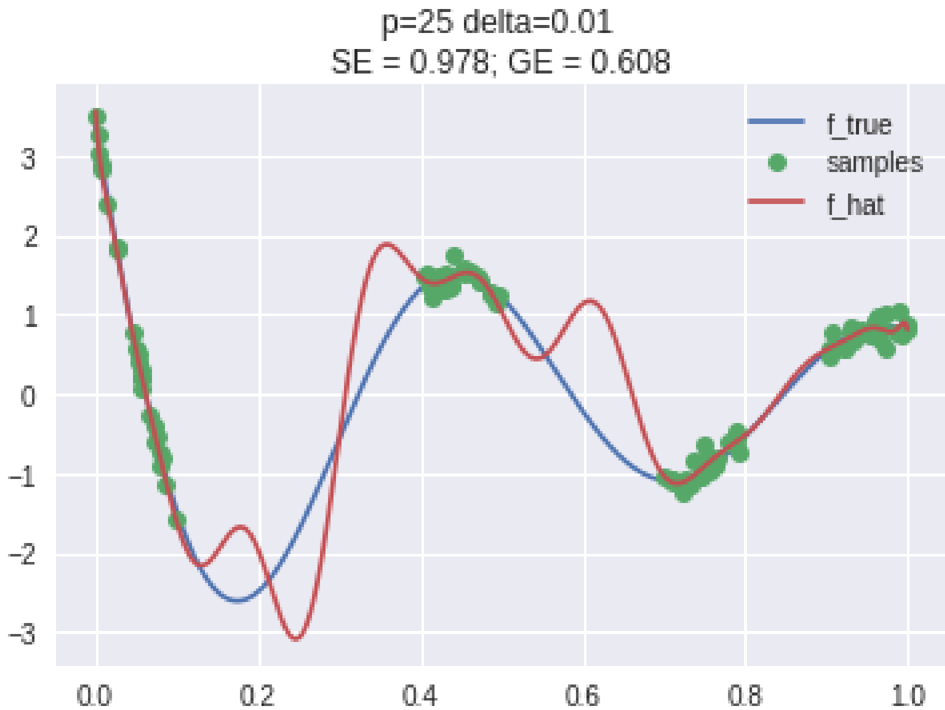
Noise error will depend on only on Σ' as previously defined. When $5 \leq R' \leq 17$, the Truncation Error is evident in the large Sample Error, while the Noise Error is suppressed as evident by the low Generalization Error. When $R' > 22$ the Noise Error explodes as evident by the Generalization Error, while the Truncation Error is suppressed.

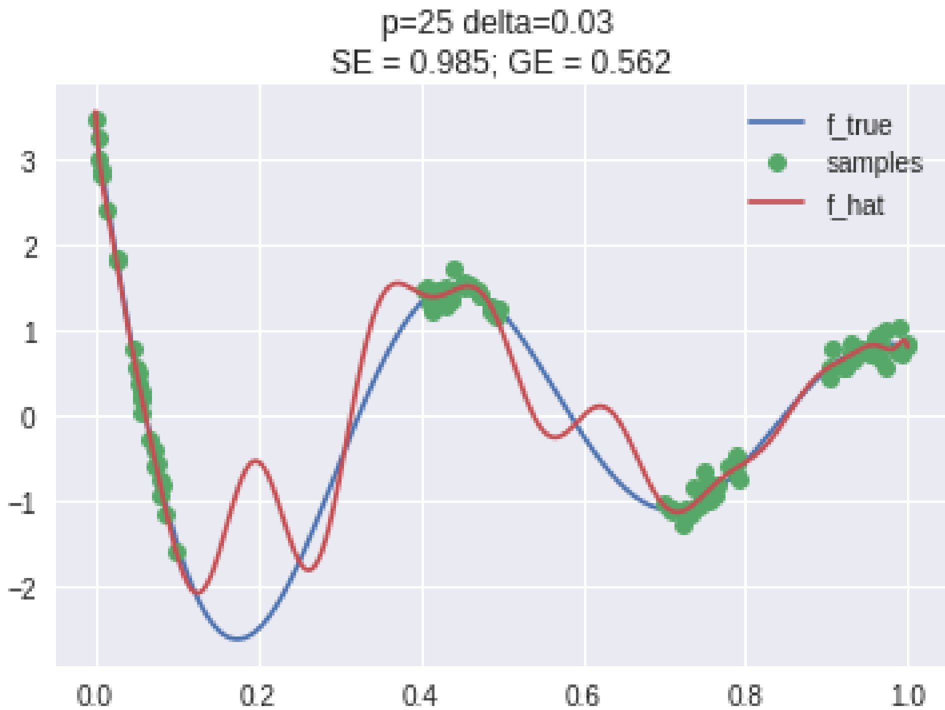
1.2.5 (f)

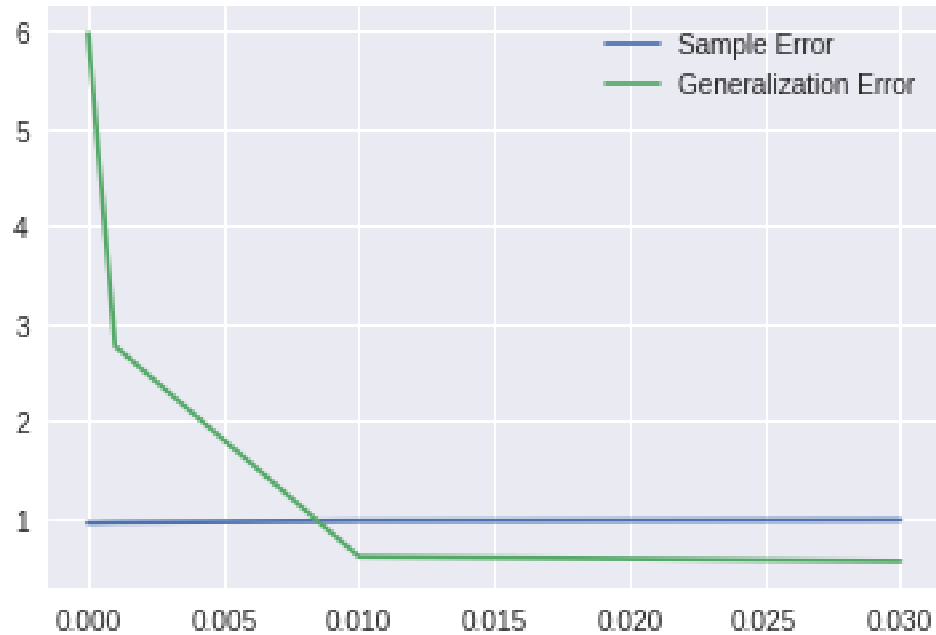
```
In [17]: p = 25
         d_l = [0, 1e-3, 1e-2, 3e-2]
         se_l = []
         ge_l = []
         for delta in d_l:
             u, sv, v = np.linalg.svd(A(T), full_matrices=False)
             simat = np.diag(sv/(sv**2 + delta))
             A_ih = v.T@simat@u.T
             w_hat = A_ih@y
             y_hat = A(T)@w_hat
             f_hat = lambda t: np.array([lpoly(n,t) for n in np.arange(0,p)]).squeeze().T@w_ha
             se = sample_error(y_hat,y)
             ge = gen_error(f_hat,f_true)
             se_l.append(se)
             ge_l.append(ge)
             plt.figure()
             plt.plot(t, f_true(t), label='f_true')
             plt.plot(T, y, 'o', label='samples')
             plt.plot(t, f_hat(t), label='f_hat')
             plt.title('p={} delta={} \nSE = {:.3f}; GE = {:.3f}'.format(p, delta, se, ge))
             plt.legend()
         plt.figure()
         plt.plot(d_l, se_l, label='Sample Error')
         plt.plot(d_l, ge_l, label='Generalization Error')
         plt.legend()

Out[17]: <matplotlib.legend.Legend at 0x7f07815974e0>
```


p=25 delta=0
SE = 0.956; GE = 5.972
f_true
samples
f_hat


p=25 delta=0.001
SE = 0.961; GE = 2.768
f_true
samples
f_hat


p=25 delta=0.01
SE = 0.978; GE = 0.608
f_true
samples
f_hat
3
2
1
0
-1
-2
-3
0.0
0.2
0.4
0.6
0.8
1.0


p=25 delta=0.03
SE = 0.985; GE = 0.562
f_true
samples
f_hat
3
2
1
0
-1
-2
0.0
0.2
0.4
0.6
0.8
1.0

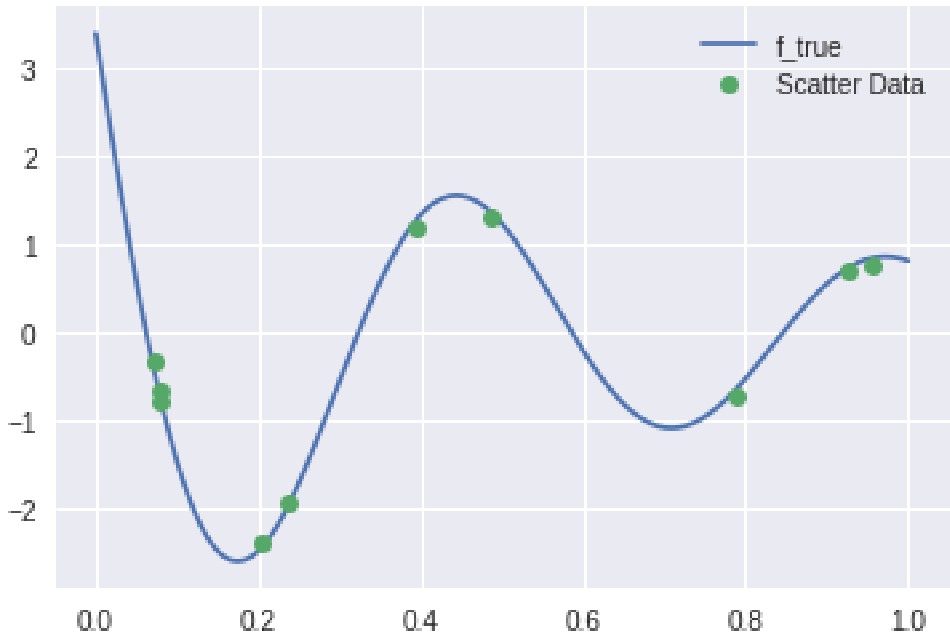


As δ increases the Generalization Error also decreases because the unstable singular values are stabalized. However, when δ is increased beyond a point the Sample and Generalization Error begin to increase. The Sample Error is not impacted nearly as much as the Generaliztion Error.

1.3 3.

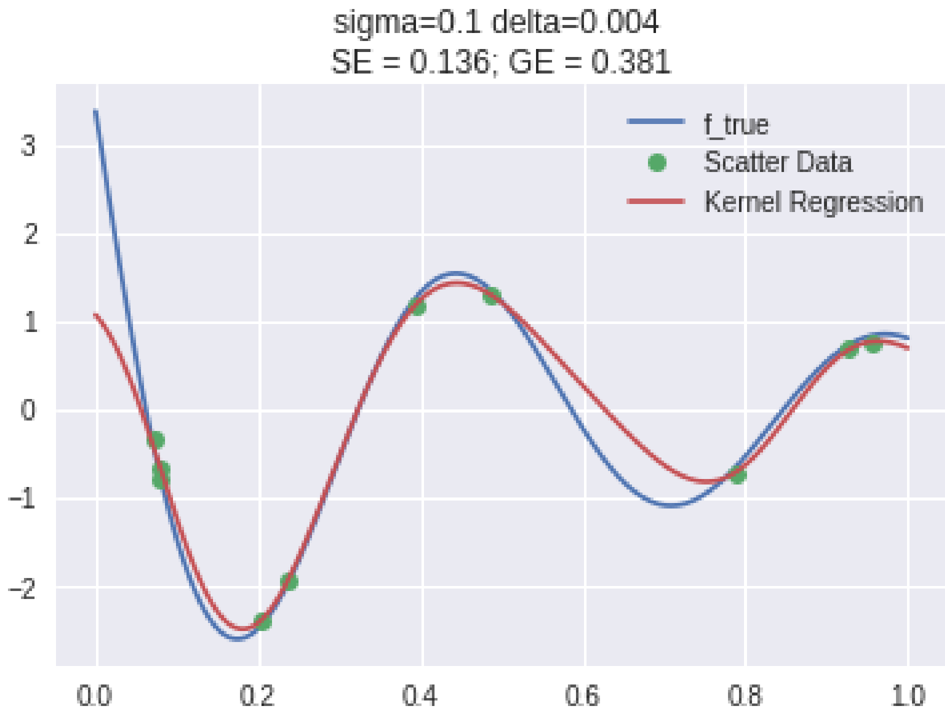
1.3.1 (a)

```
In [24]: data = loadmat('hw5p3_scatterdata.mat')
         T, y = data['T'], data['y']
         plt.plot(t, f_true(t), label='f_true')
         plt.plot(T,y,'o', label='Scatter Data')
         plt.legend()

Out[24]: <matplotlib.legend.Legend at 0x7f0781764358>
```



In [64]:
```
sigma = 1/10
delta = 0.004
k = lambda s, t, sigma:  np.exp(-np.abs(t-s)**2/(2*sigma**2))
K = np.empty((T.shape[0], T.shape[0]))

for n,t in enumerate(T):
    for m,s in enumerate(T):
        K[n,m] = k(s,t,sigma)

alpha_hat = np.linalg.inv(K+delta*np.eye(T.shape[0]))@y
f_hat = lambda t: np.array([k(t, tm, sigma) for tm in T]).T@alpha_hat
t = np.linspace(0,1,500)
se = sample_error(f_hat(T),y)
ge = gen_error(f_hat,f_true)
plt.plot(t, f_true(t), label='f_true')
plt.plot(T, y,'o', label='Scatter Data')
plt.plot(t, f_hat(t), label='Kernel Regression')
plt.title('sigma={} delta={} \nSE = {:.3f}; GE = {:.3f}'.format(sigma, delta, se, ge)
plt.legend()
```

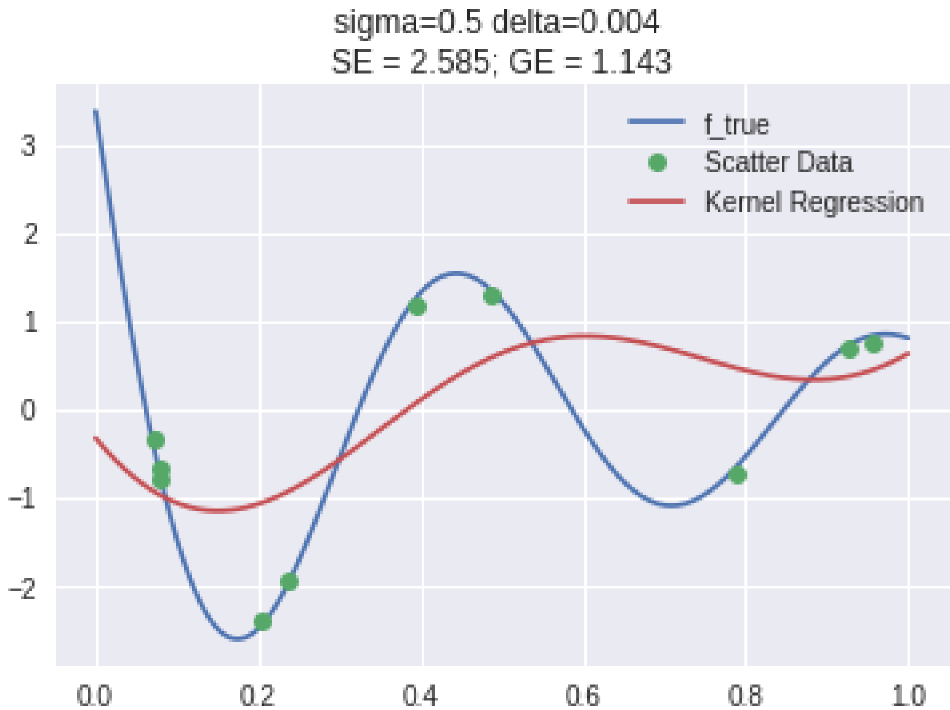
Out[64]:
```
<matplotlib.legend.Legend at 0x7f07808492e8>
```

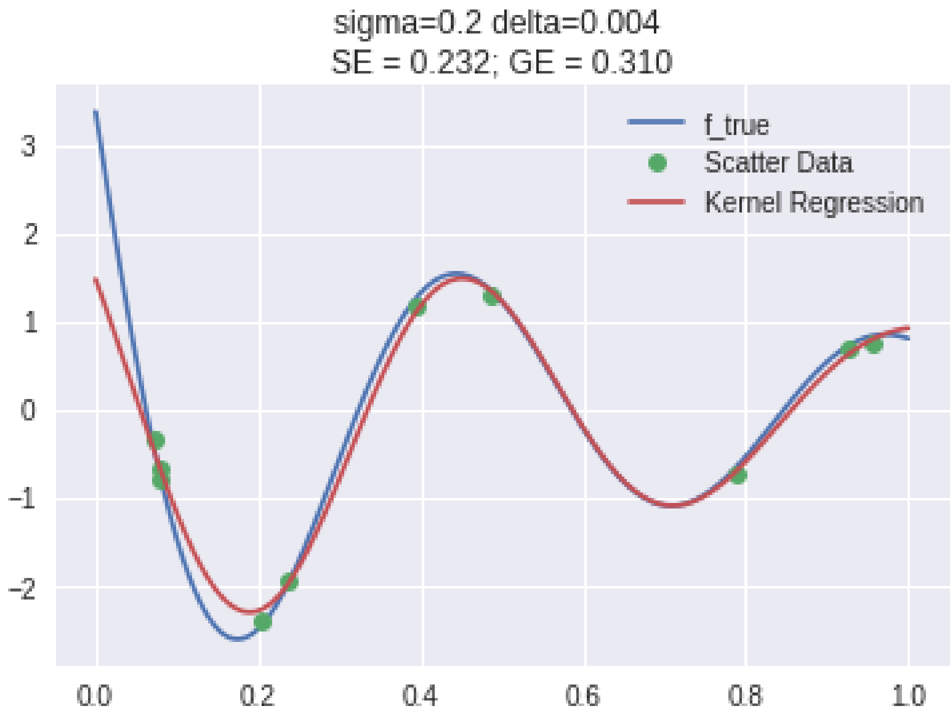


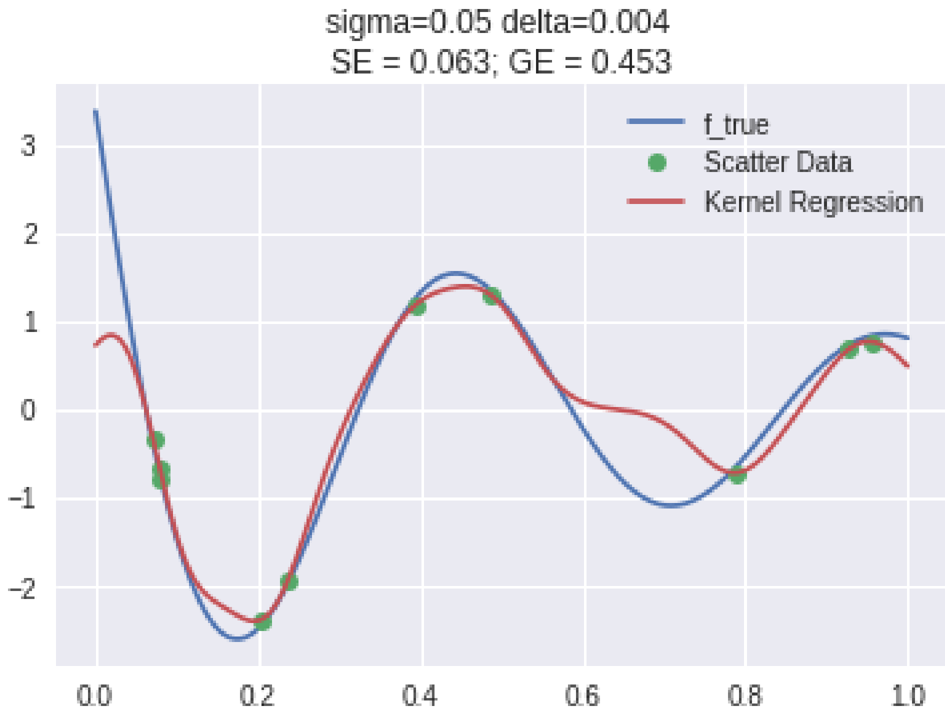
This is a good choice of delta because .....

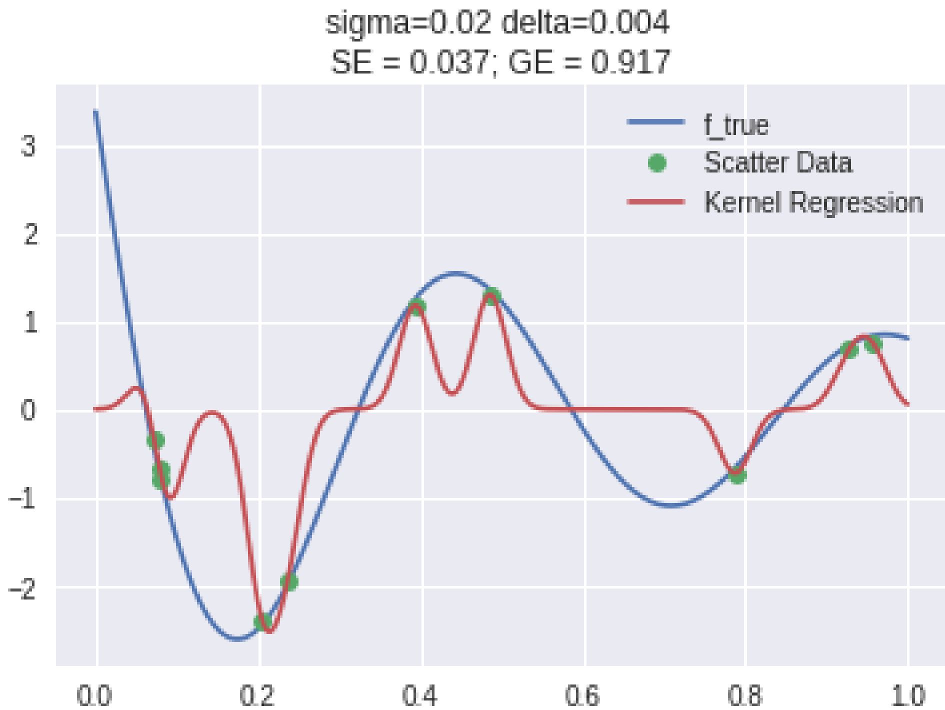
1.3.2 (b)

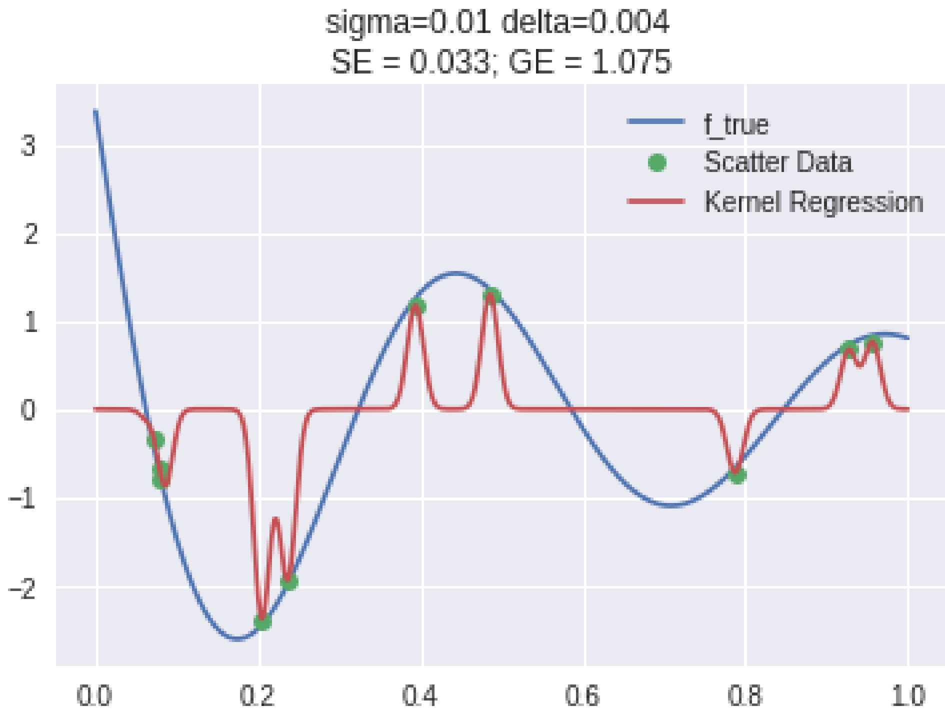
```
In [65]: s_l = np.array([1/2, 1/5, 1/20, 1/50, 1/100, 1/200])
         delta = 0.004
         for sigma in s_l:
             K = np.empty((T.shape[0], T.shape[0]))
             for n,t in enumerate(T):
                 for m,s in enumerate(T):
                     K[n,m] = k(s,t,sigma)

             alpha_hat = np.linalg.inv(K+delta*np.eye(T.shape[0]))@y
             f_hat = lambda t: np.array([k(t, tm, sigma) for tm in T]).T@alpha_hat
             t = np.linspace(0,1,500)
             se = sample_error(f_hat(T),y)
             ge = gen_error(f_hat,f_true)
             plt.figure()
             plt.plot(t, f_true(t), label='f_true')
             plt.plot(T, y,'o', label='Scatter Data')
             plt.plot(t, f_hat(t), label='Kernel Regression')
             plt.title('sigma={} delta={} \nSE = {:.3f}; GE = {:.3f}'.format(sigma, delta, se,
             plt.legend()
```

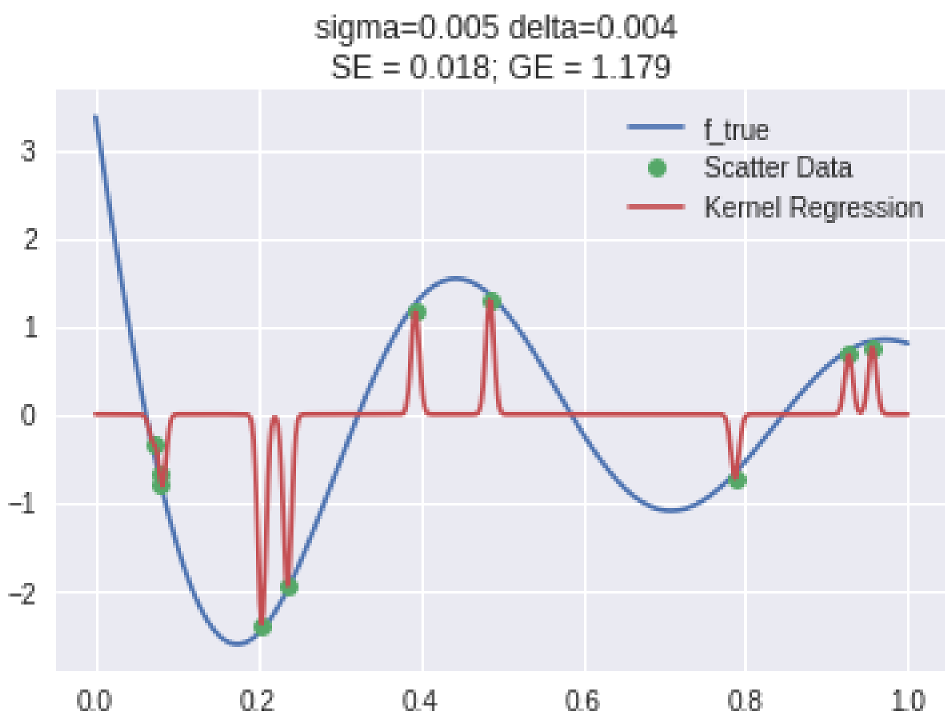

sigma=0.5 delta=0.004
SE = 2.585; GE = 1.143
f_true
Scatter Data
Kernel Regression
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0


sigma=0.2 delta=0.004
SE = 0.232; GE = 0.310
f_true
Scatter Data
Kernel Regression
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0


sigma=0.05 delta=0.004
SE = 0.063; GE = 0.453
f_true
Scatter Data
Kernel Regression
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0


sigma=0.02 delta=0.004
SE = 0.037; GE = 0.917
f_true
Scatter Data
Kernel Regression
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0


sigma=0.01 delta=0.004
SE = 0.033; GE = 1.075
f_true
Scatter Data
Kernel Regression
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0


sigma=0.005 delta=0.004
SE = 0.018; GE = 1.179
f_true
Scatter Data
Kernel Regression
3
2
1
0
−1
−2
0.0
0.2
0.4
0.6
0.8
1.0

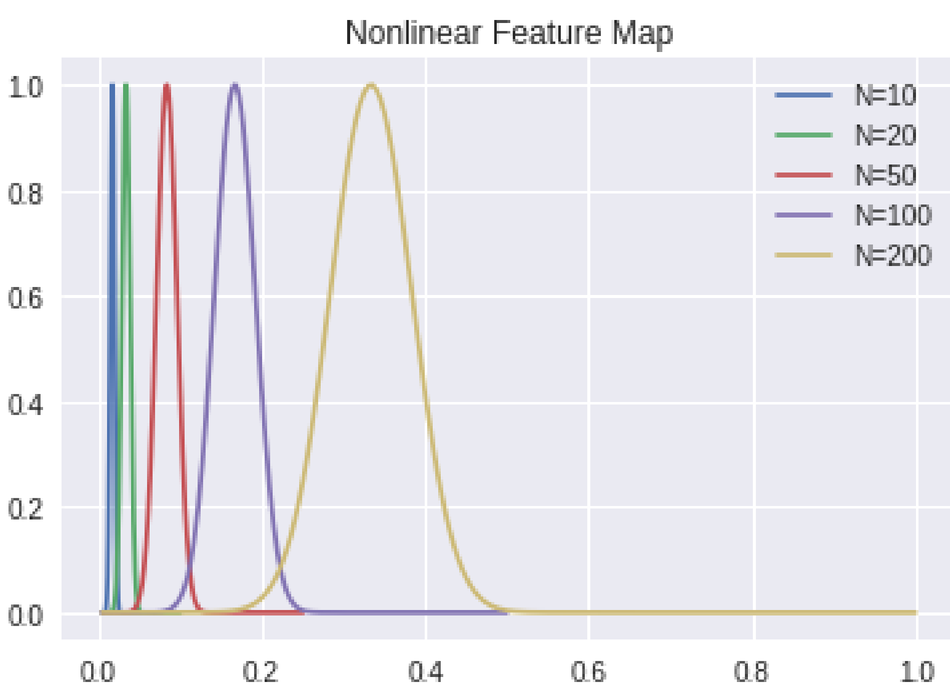
The Radial Basis Function's σ term modifies the width of the bump function. When σ is small the width is small and would require there to be more samples in order to have lower Generalization Error.

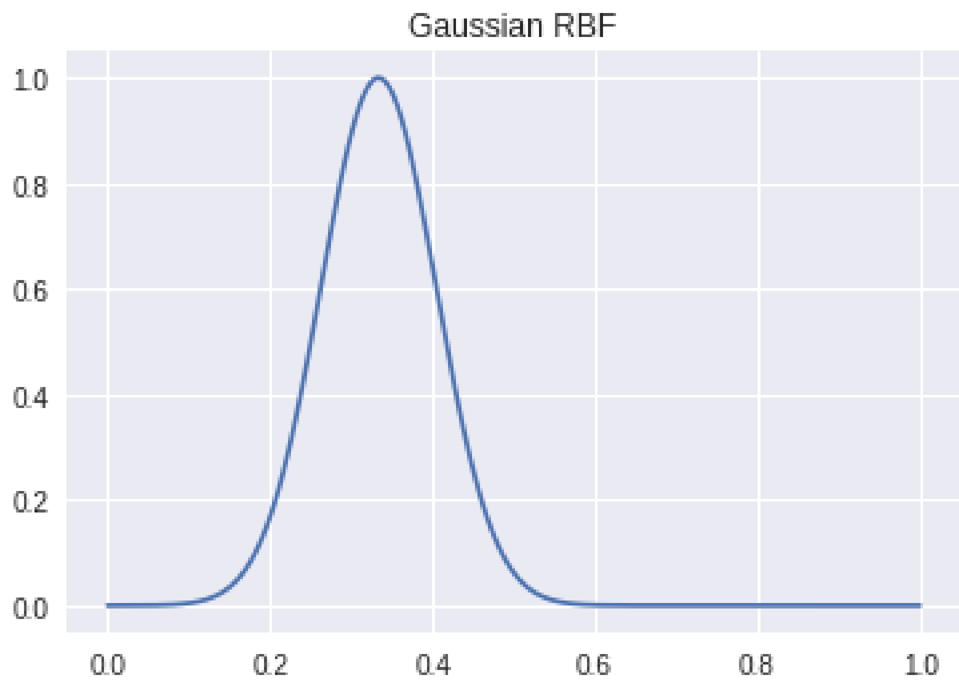
1.4 4.

1.4.1 (a)

In [85]:
```
g = lambda t: np.exp(-200*t**2)
psi = lambda N, k, t: g(t-k/N)
t = 1/3
plt.figure()
for N in [10, 20, 50, 100, 200]:
    k = np.linspace(1,N,200)
    plt.plot(k/200, psi(N, k, t), label='N={}'.format(N))
plt.title('Nonlinear Feature Map')
plt.legend()

phi = lambda s, t: np.exp(-100*np.abs(s-t)**2)
plt.figure()
s = np.linspace(0,1,200)
plt.plot(s, phi(s, t))
plt.title('Gaussian RBF')
```

Out[85]:
```
Text(0.5,1,'Gaussian RBF')
```




Gaussian RBF
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0