DS-GA 3001.008 Modelling time series data

L11. Time Series in the Frequency Domain

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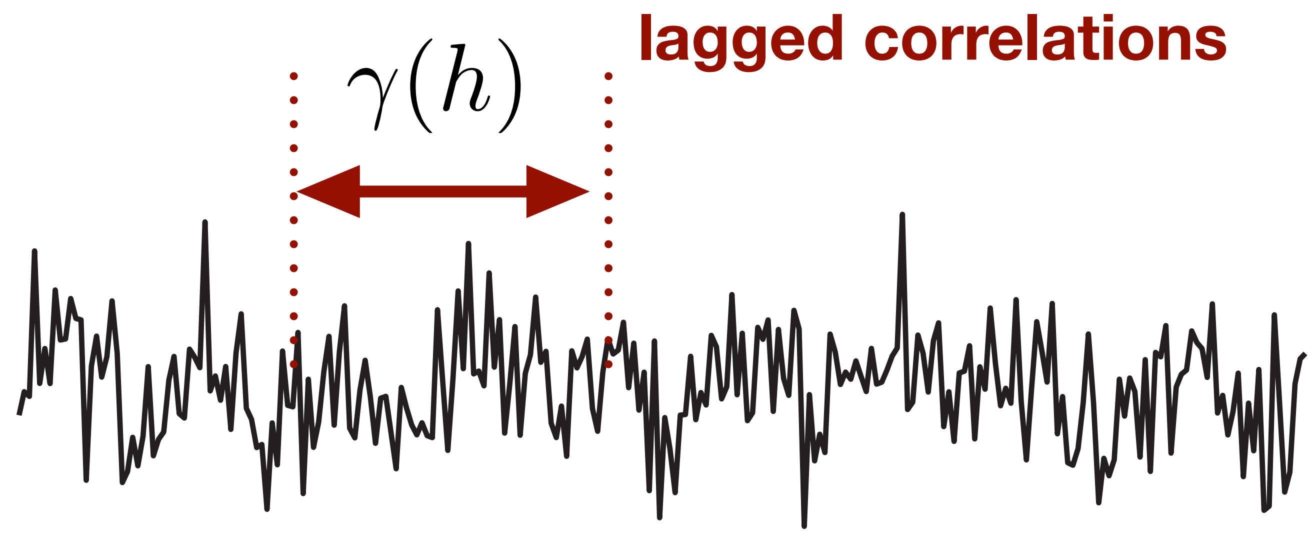
NYU, CNS & CDS

Stationary time series

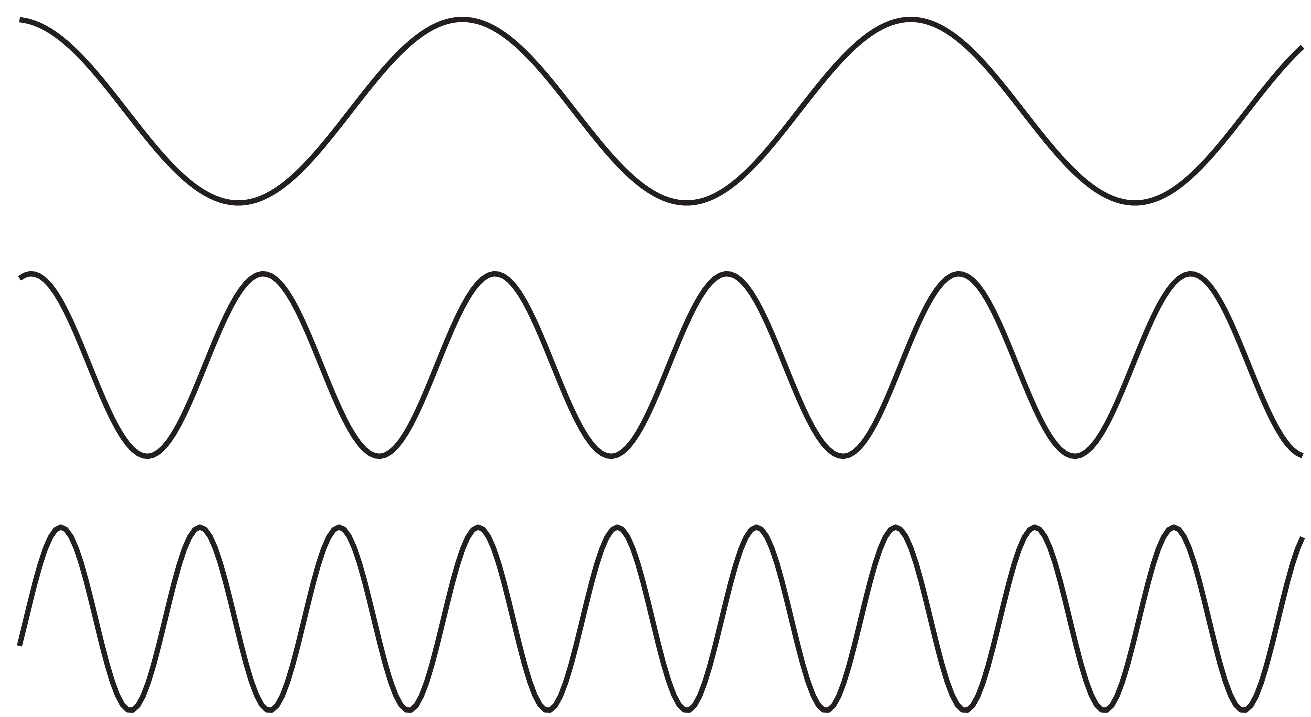


frequency domain

$a_1 \times$

$a_2 \times$

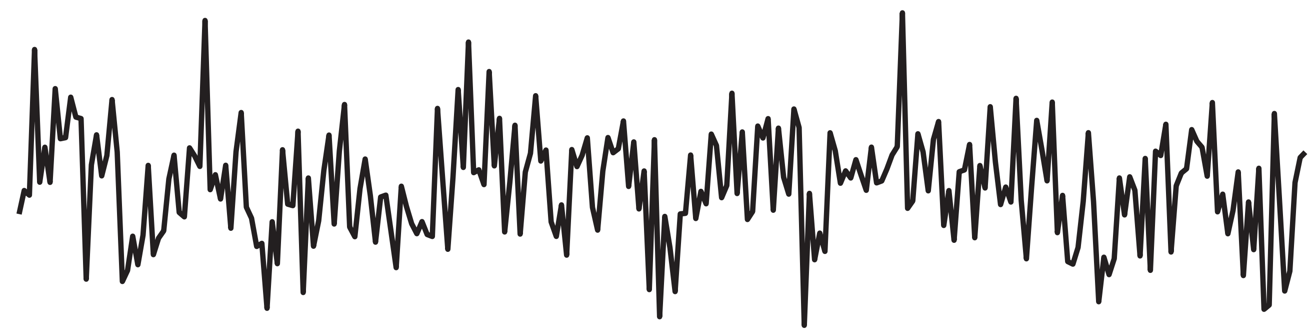
$a_3 \times$



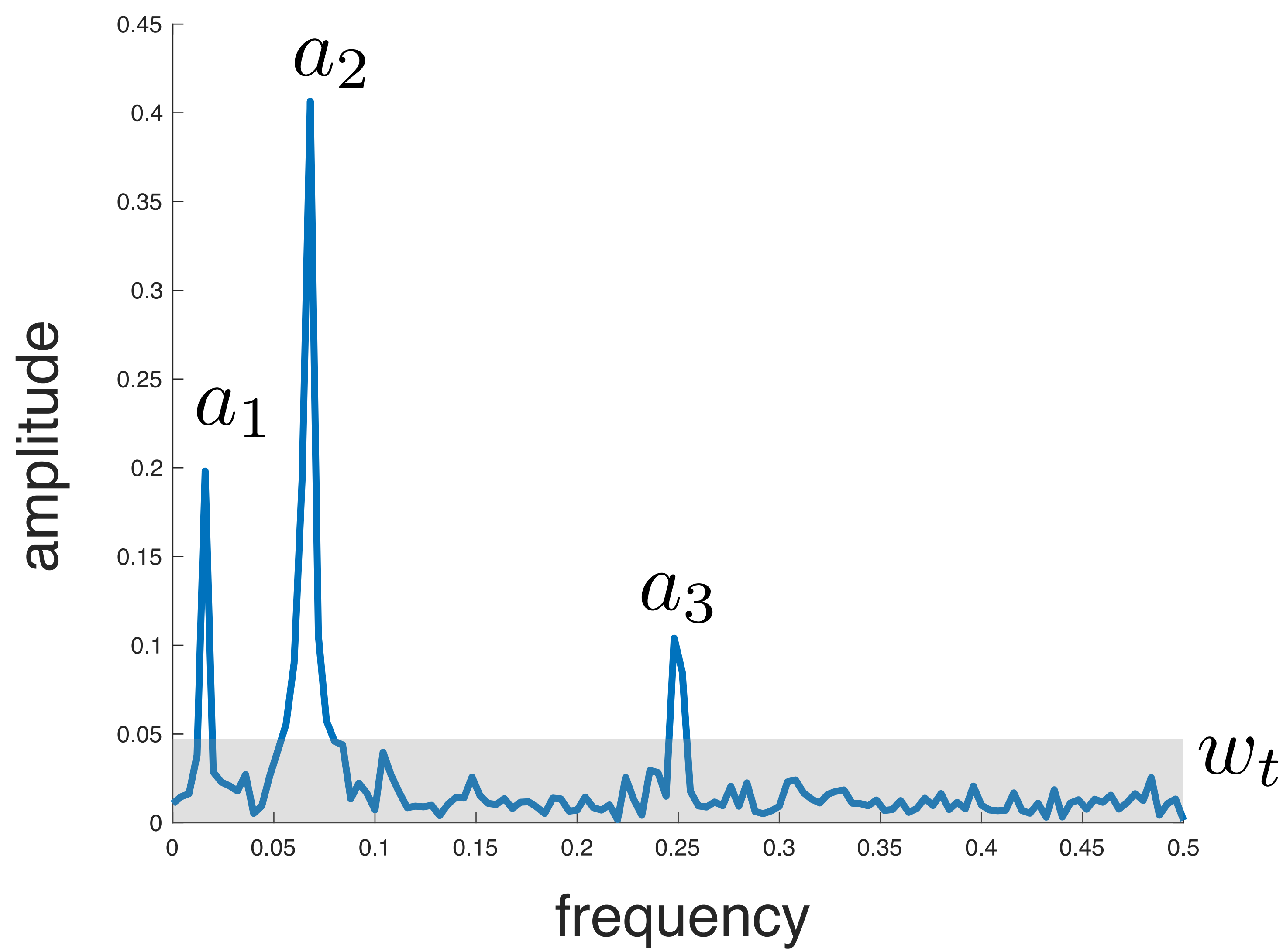
+ white noise

The time-lag vs. periodic view may be useful for different things

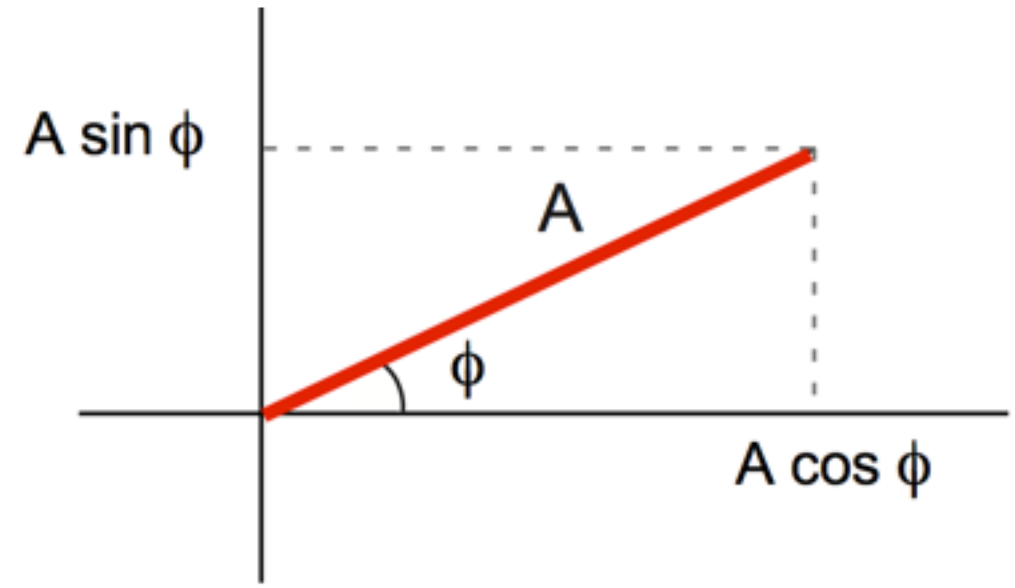
Stationary time series

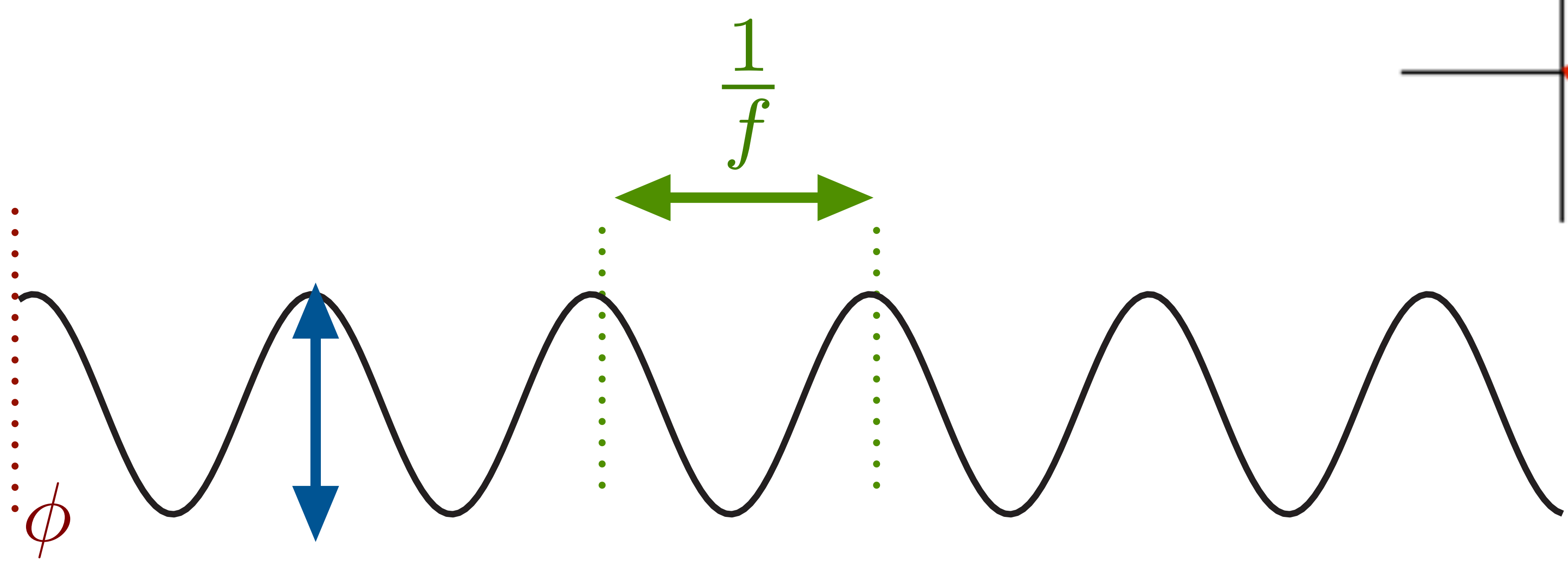


frequency domain



Reminder basics frequency analysis





$$x_t = A\cos(2\pi\omega t + \phi)$$

amplitude frequency phase

$$x_t = U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t),$$

details on the board

Link to time lagged interpretation

$$\begin{aligned}
\gamma_x(h) = \operatorname{cov}(x_{t+h}, x_t) &= \operatorname{cov}(U_1 c_{t+h} + U_2 s_{t+h}, U_1 c_t + U_2 s_t) \\
&= \operatorname{cov}(U_1 c_{t+h}, U_1 c_t) + \operatorname{cov}(U_1 c_{t+h}, U_2 s_t) \\
&\quad + \operatorname{cov}(U_2 s_{t+h}, U_1 c_t) + \operatorname{cov}(U_2 s_{t+h}, U_2 s_t) \\
&= \sigma^2 c_{t+h} c_t + 0 + 0 + \sigma^2 s_{t+h} s_t = \sigma^2 \cos(2\pi\omega h),
\end{aligned}$$

A general way to represent periodic signals

$$x_t = \sum_{k=1}^{q} \left[U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t) \right],$$

$$\gamma_x(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k h),$$

$$x_t = a_0 + \sum_{j=1}^{(n-1)/2} \left[a_j \cos(2\pi t\, j/n) + b_j \sin(2\pi t\, j/n) \right],$$

Frequency analysis $$a_j = \frac{2}{n} \sum_{t=1}^{n} x_t \cos(2\pi t j/n) \quad \text{and} \quad b_j = \frac{2}{n} \sum_{t=1}^{n} x_t \sin(2\pi t j/n).$$

Scaled periodogram $$P(j/n) = a_j^2 + b_j^2, = \tfrac{4}{n}\, |d(j/n)|^2.$$

computed efficiently using **discrete Fourier transform DFT**

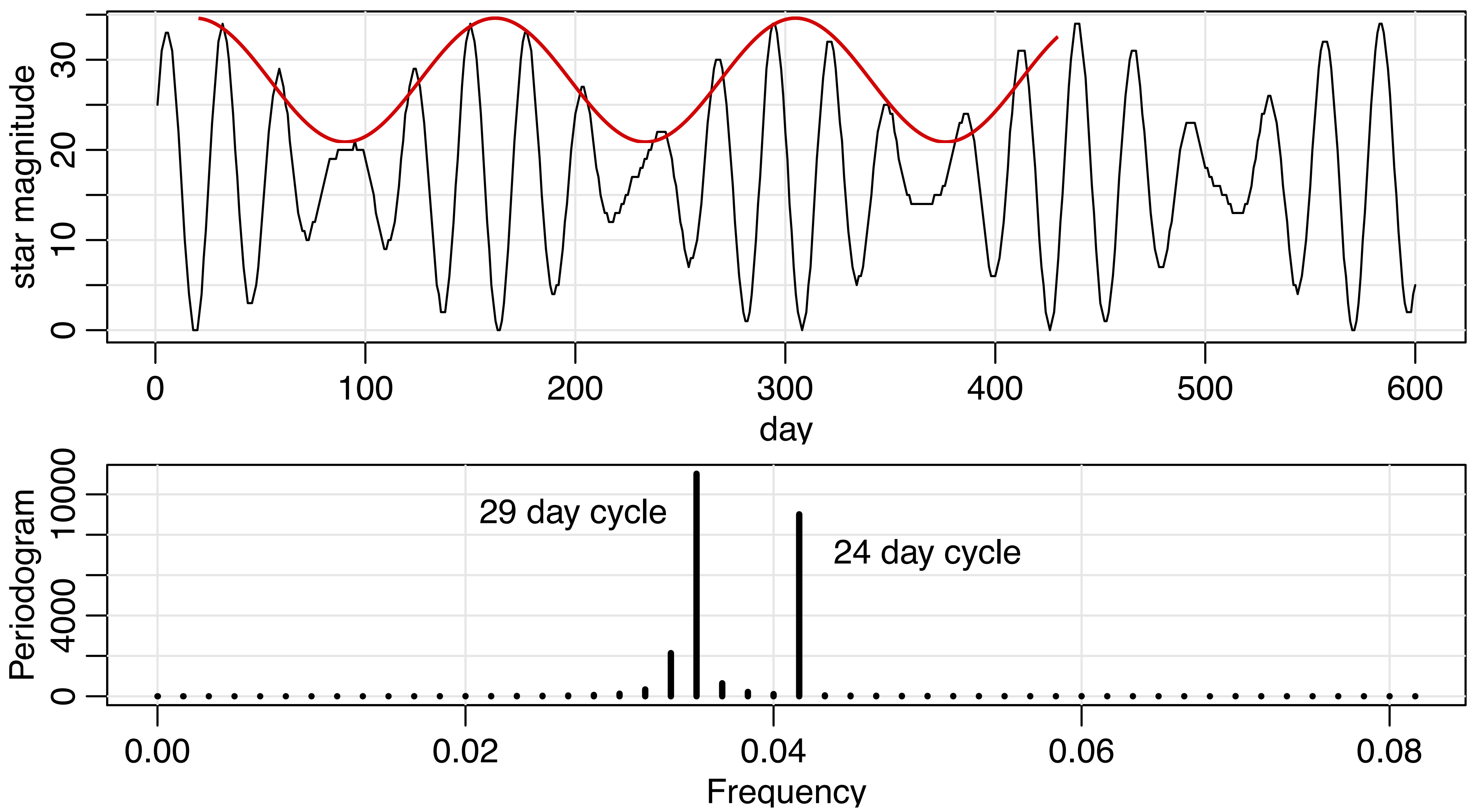
$$\begin{aligned} d(j/n) &= n^{-1/2} \sum_{t=1}^{n} x_t \exp(-2\pi i t j/n) \\ &= n^{-1/2} \left(\sum_{t=1}^{n} x_t \cos(2\pi t j/n) - i \sum_{t=1}^{n} x_t \sin(2\pi t j/n) \right), \end{aligned}$$

$$\mathcal{O}(n \log(n))$$

Note 1: reminder complex numbers (on the board)

Note 2: discrete/continuous time**; discrete/**continuous frequencies

amplitude modulated signal



Spectral density

$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t)$$

U_1 and U_2 are uncorrelated zero-mean random variables with equal variance σ^2.

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i\omega_0 h}$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i\omega h}\, dF(\omega)$$

spectral distribution function $$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2 & \omega \ge \omega_0. \end{cases}$$

$$dF(\omega) = f(\omega)\, d\omega,$$

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty, \tag{4.15}$$

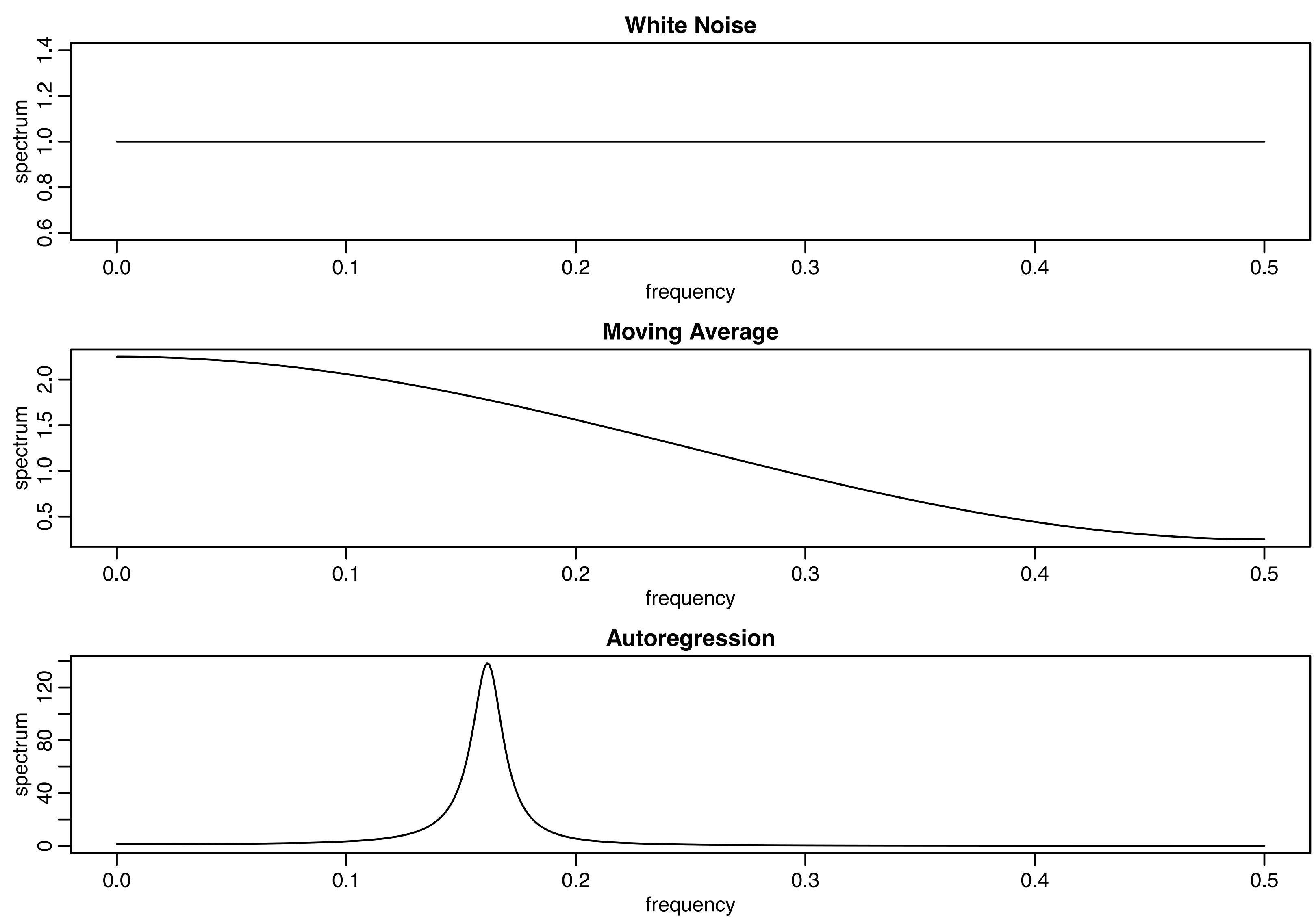
then it has the representation

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{2\pi i\omega h} f(\omega)\, d\omega \quad h = 0, \pm 1, \pm 2, \ldots \tag{4.16}$$

as the inverse transform of the spectral density,

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)\mathrm{e}^{-2\pi i\omega h} \quad -1/2 \leq \omega \leq 1/2. \tag{4.17}$$

Example: white noise $f_w(\omega) = \sigma_w^2$ **flat spectrum**



check at home for AR(1), MA(1)

Filters

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty.$$ **Impulse response**

$$A(\omega) = \sum_{j=-\infty}^{\infty} a_j \, e^{-2\pi i \omega j},$$ **frequency response**

$$f_y(\omega) = |A(\omega)|^2 \, f_x(\omega),$$

proof on the board

Eigen-spectrum of covariance matrix

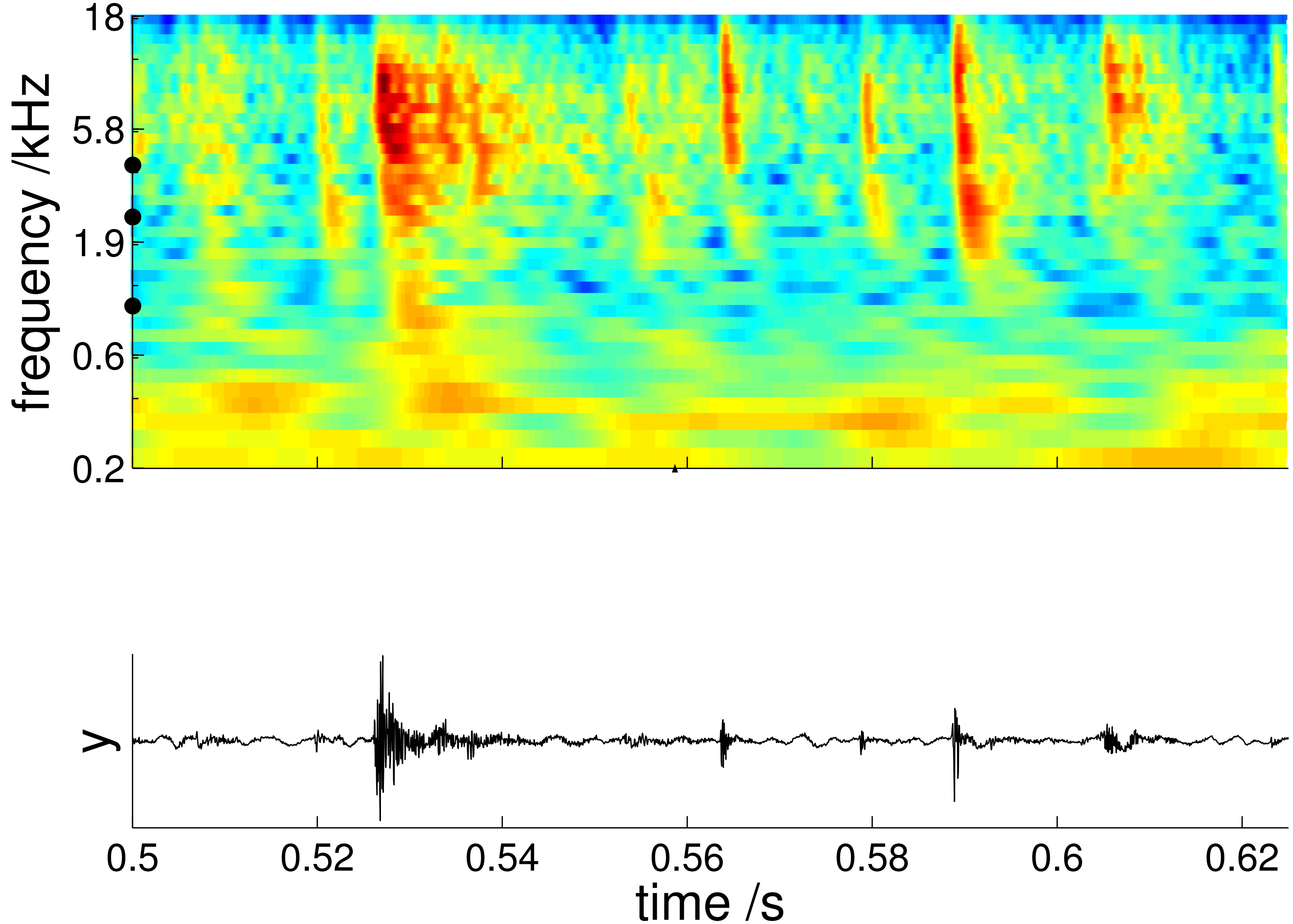
$$\mathrm{cov}(X) = \mathit{\Gamma}_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix}.$$

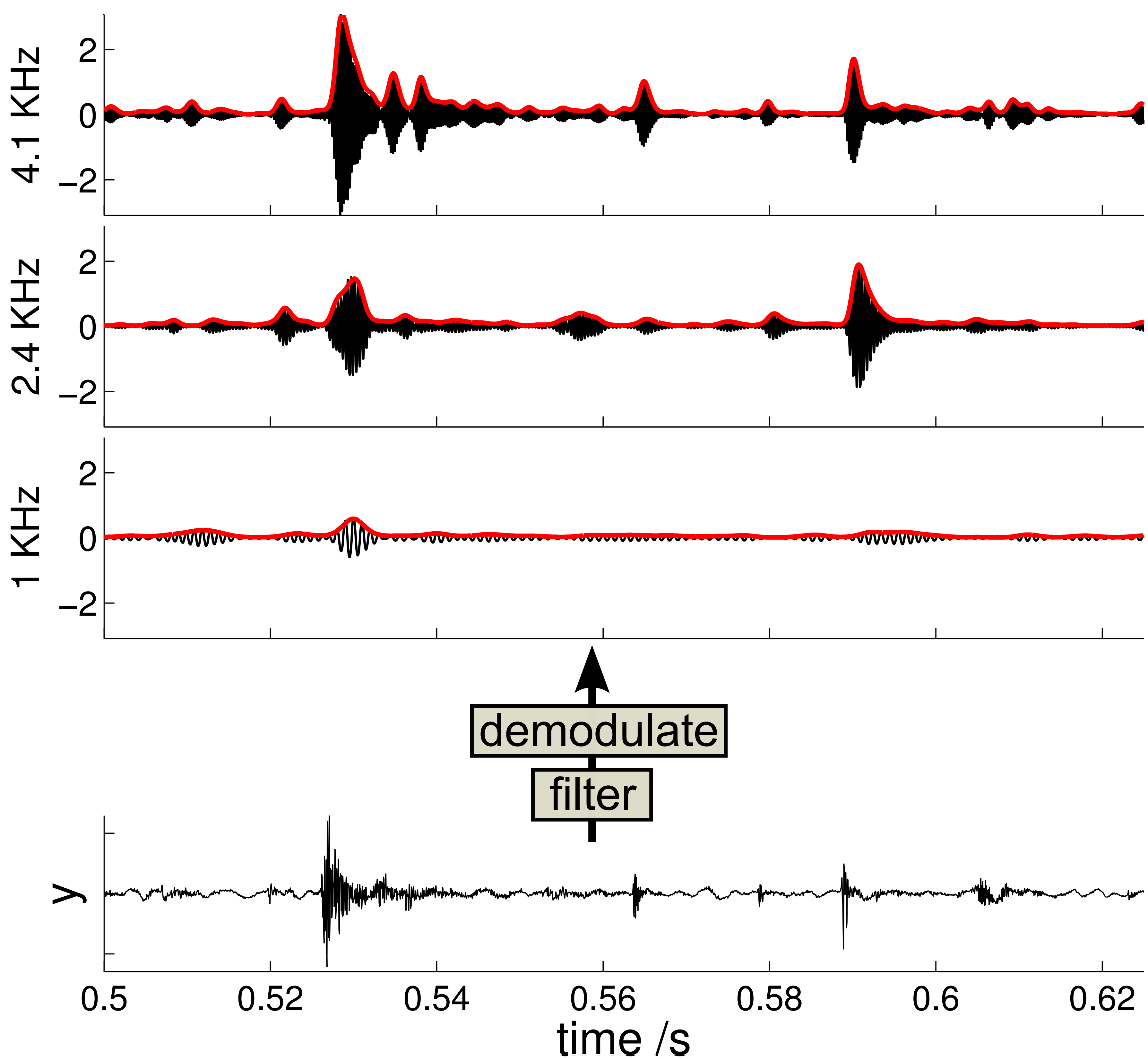
For n sufficiently large, the eigenvalues of $\mathit{\Gamma}_n$ are

$$\lambda_j \approx f(\omega_j) = \sum_{h=-\infty}^{\infty} \gamma(h)\, \mathrm{e}^{-2\pi i h j/n},$$

compute spectrum via PCA

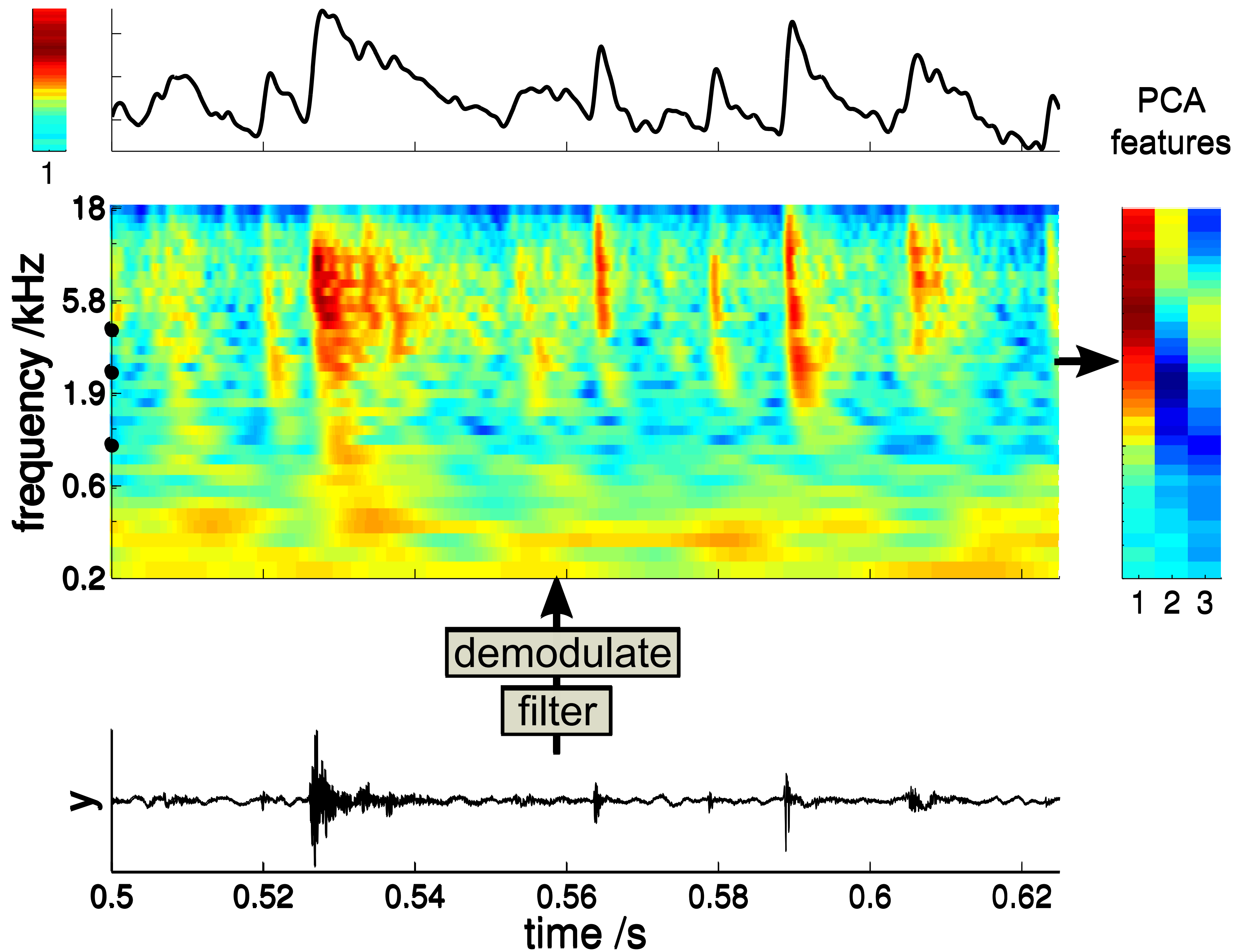
The interesting cases are not stationary





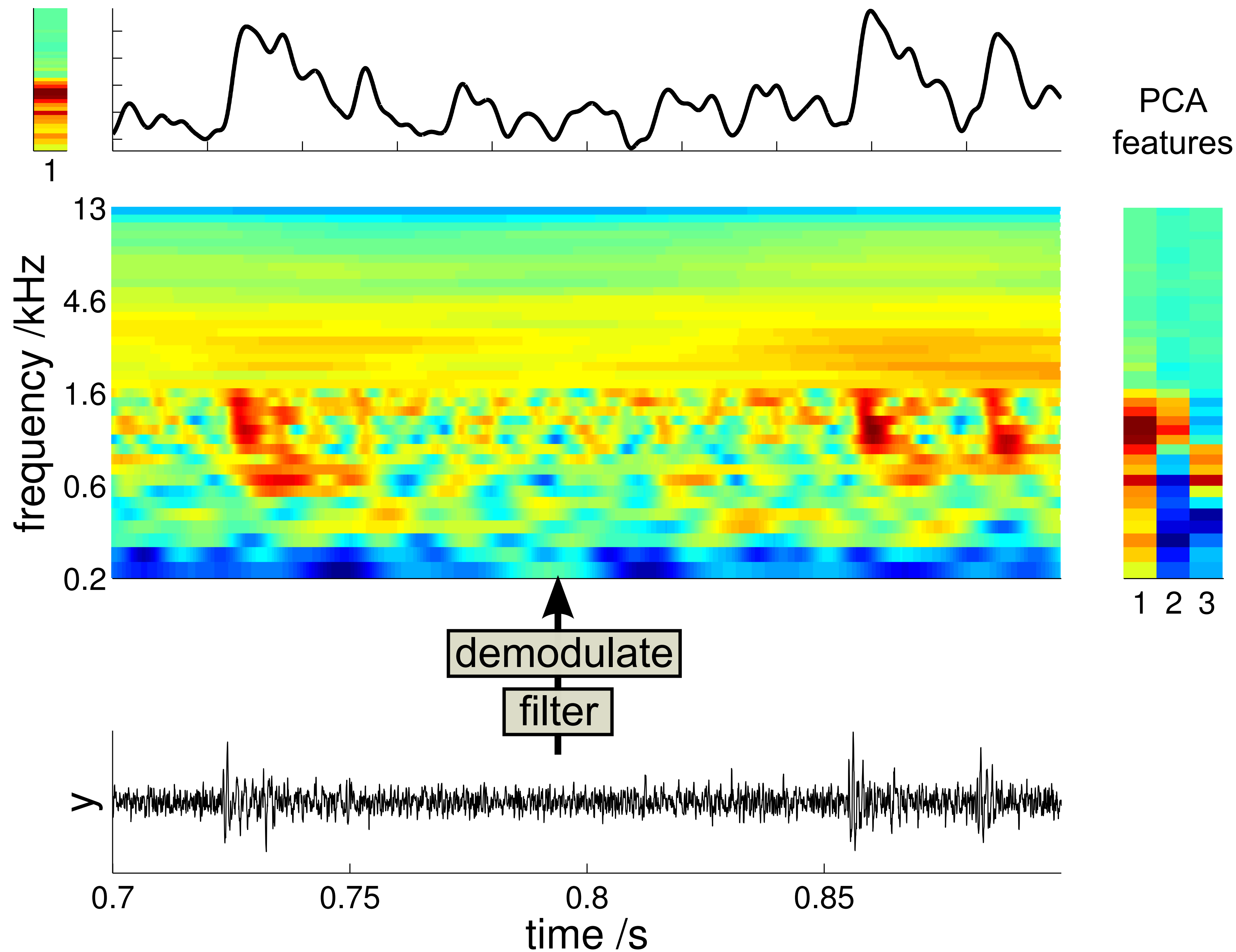
4.1 KHz
2.4 KHz
1 KHz
2
0
−2
demodulate
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

Fire



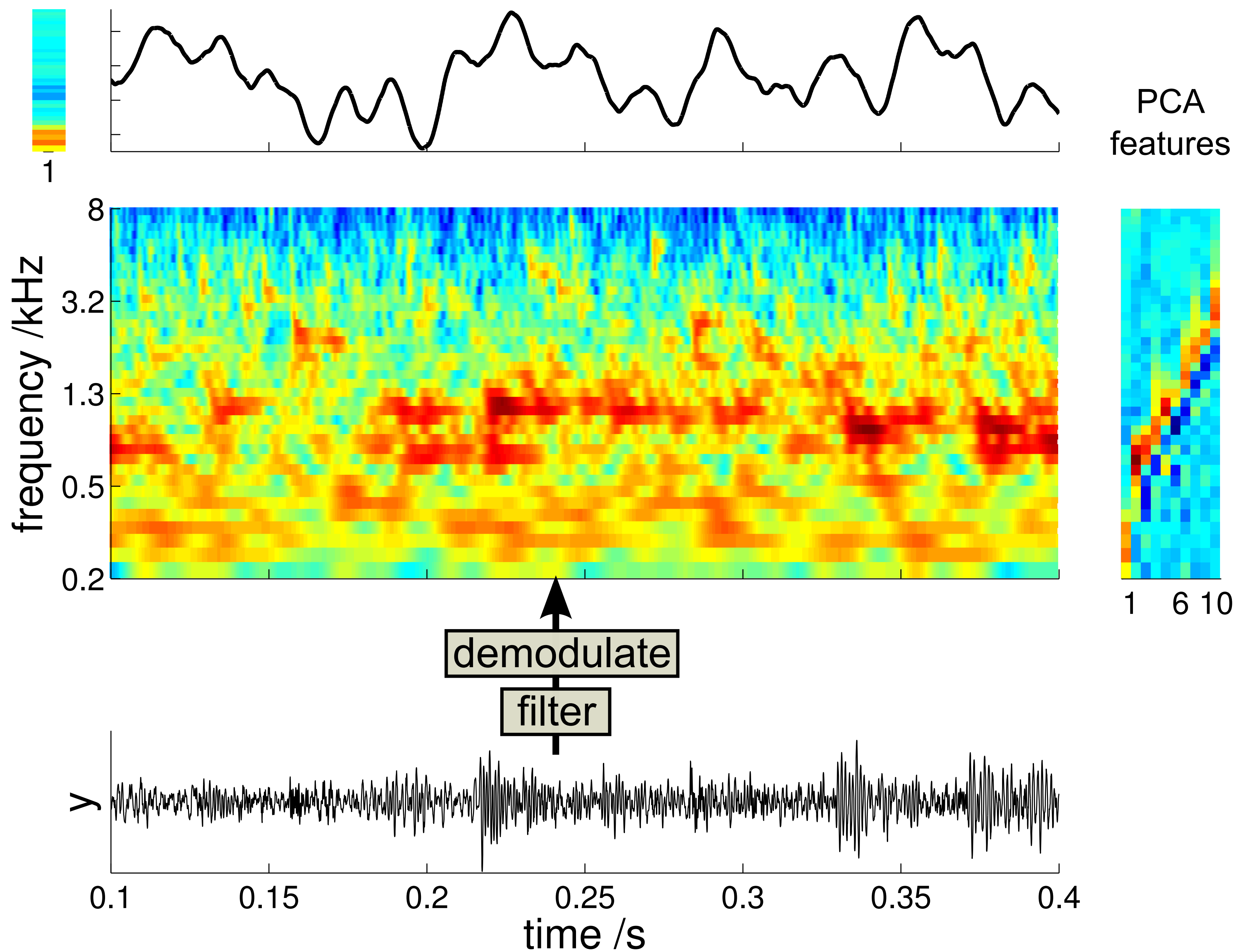
1
PCA
features
18
5.8
1.9
0.6
0.2
frequency /kHz
1
2
3
demodulate
filter
y
0.5
0.52
0.54
0.56
0.58
0.6
0.62
time /s

Wind



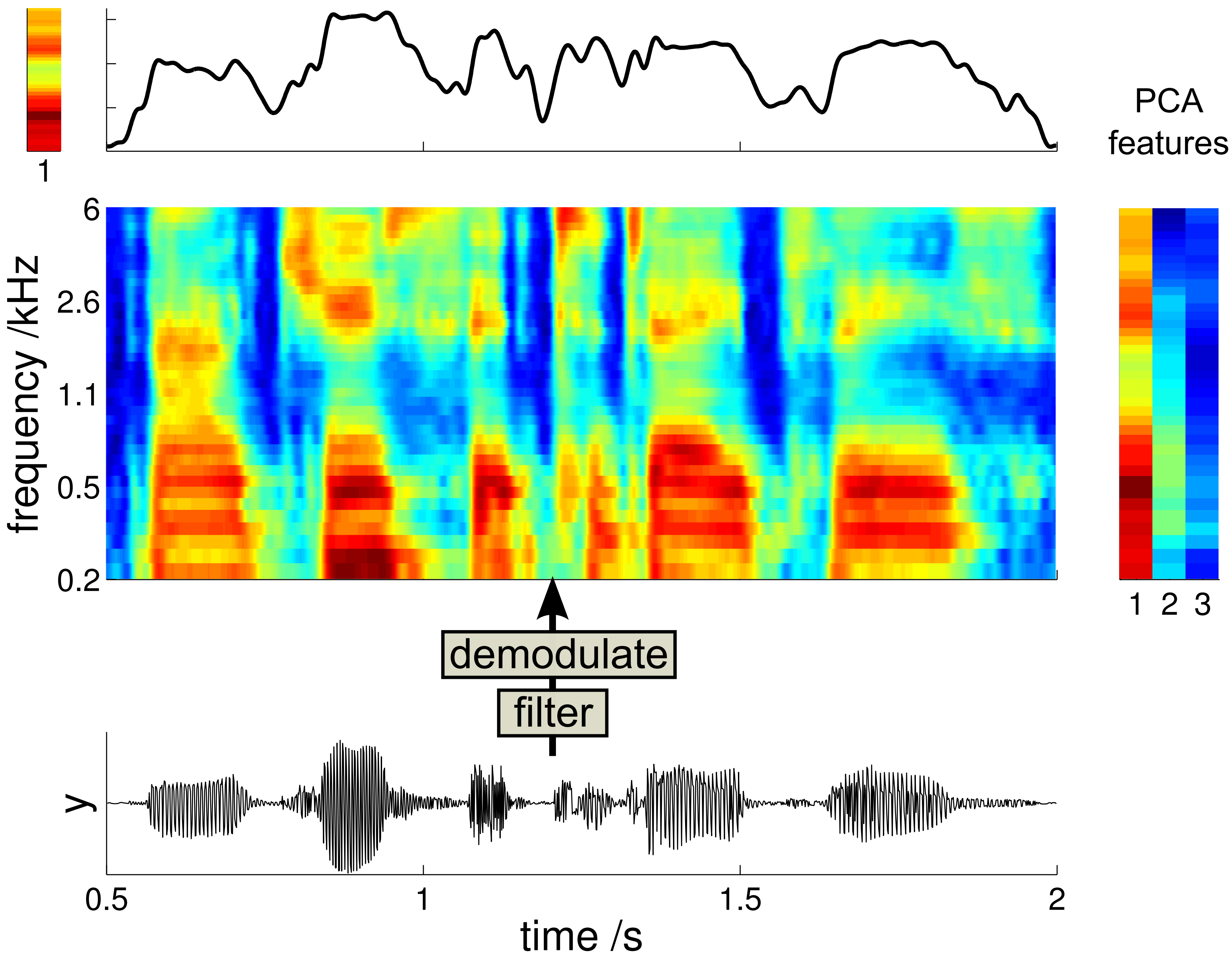
1
PCA
features
13
4.6
1.6
0.6
0.2
frequency /kHz
1 2 3
demodulate
filter
y
0.7
0.75
0.8
0.85
time /s

Water



1
PCA
features
8
3.2
1.3
0.5
0.2
frequency /kHz
1
6
10
demodulate
filter
y
0.1
0.15
0.2
0.25
0.3
0.35
0.4
time /s

Speech



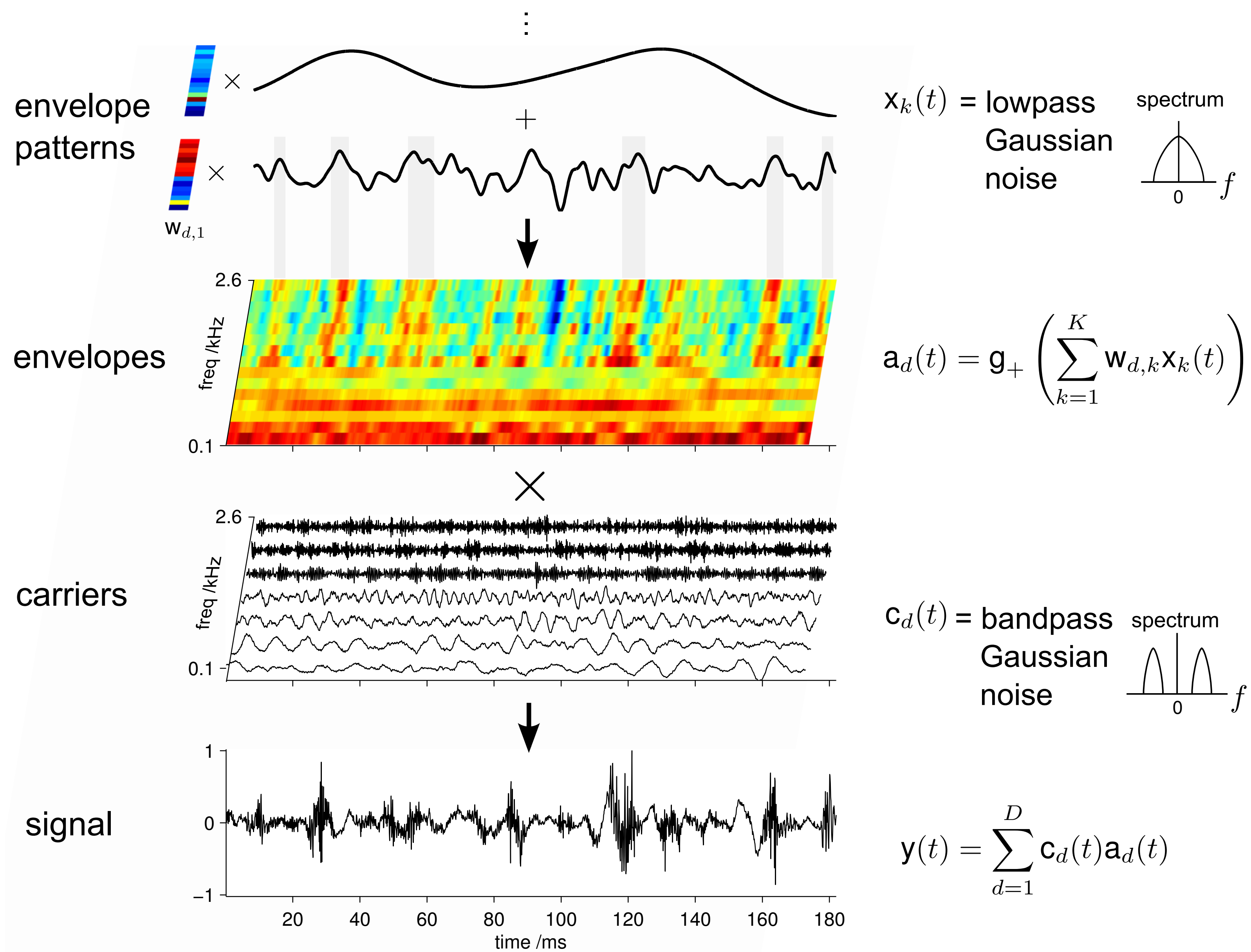
1
PCA
features
6
2.6
1.1
0.5
0.2
frequency /kHz
1 2 3
demodulate
filter
y
0.5
1
1.5
2
time /s

Spectral analysis reveals a lot of potentially useful statistical regularities in real-world data

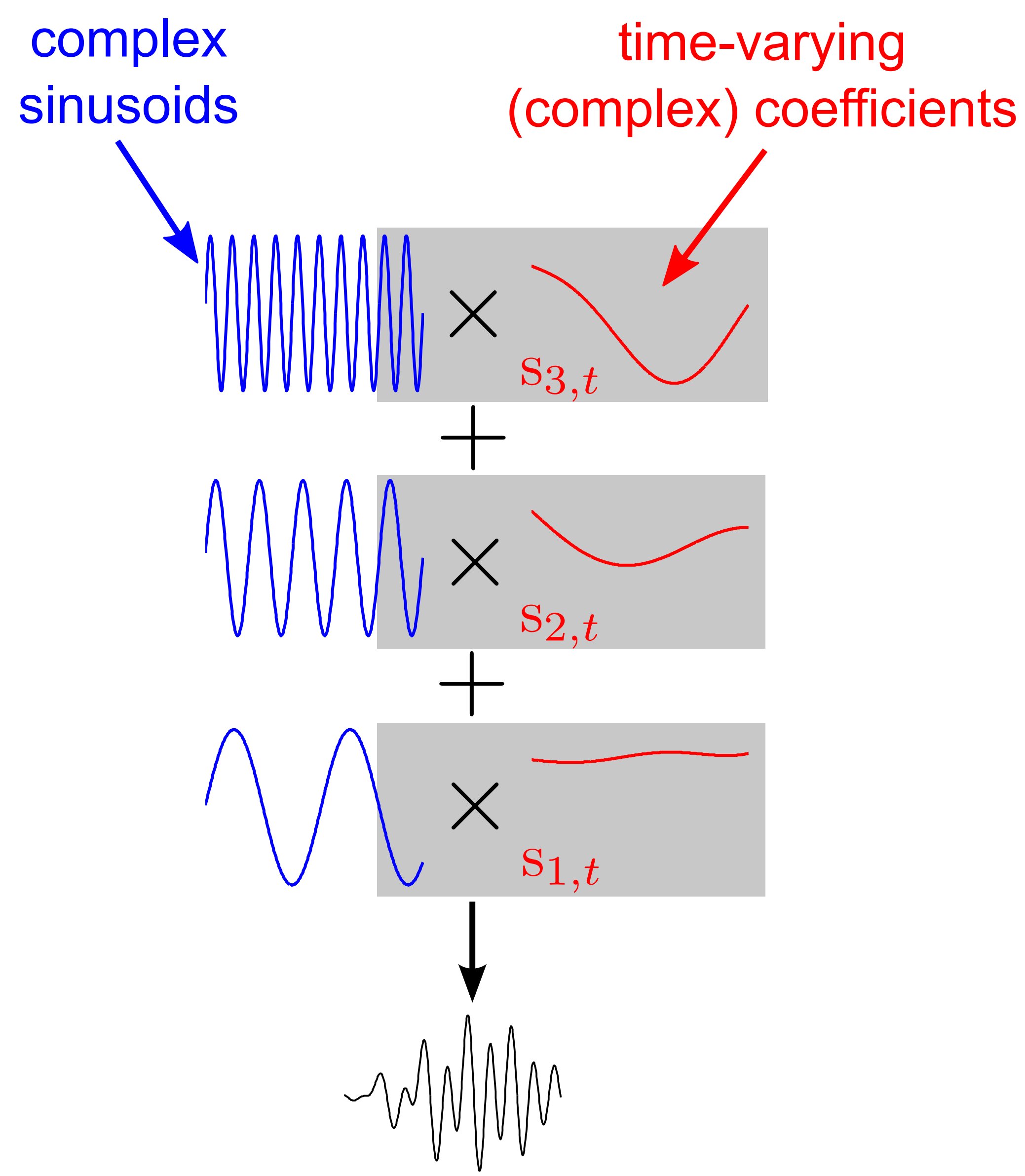
energy in sub-bands (power-spectrum)
patterns of co-modulation
time-scale of the modulation
depth of the modulation (sparsity)

We can build probabilistic models that capture such phenomena and fit them to data

Statistical Model



Generative model in complex domain

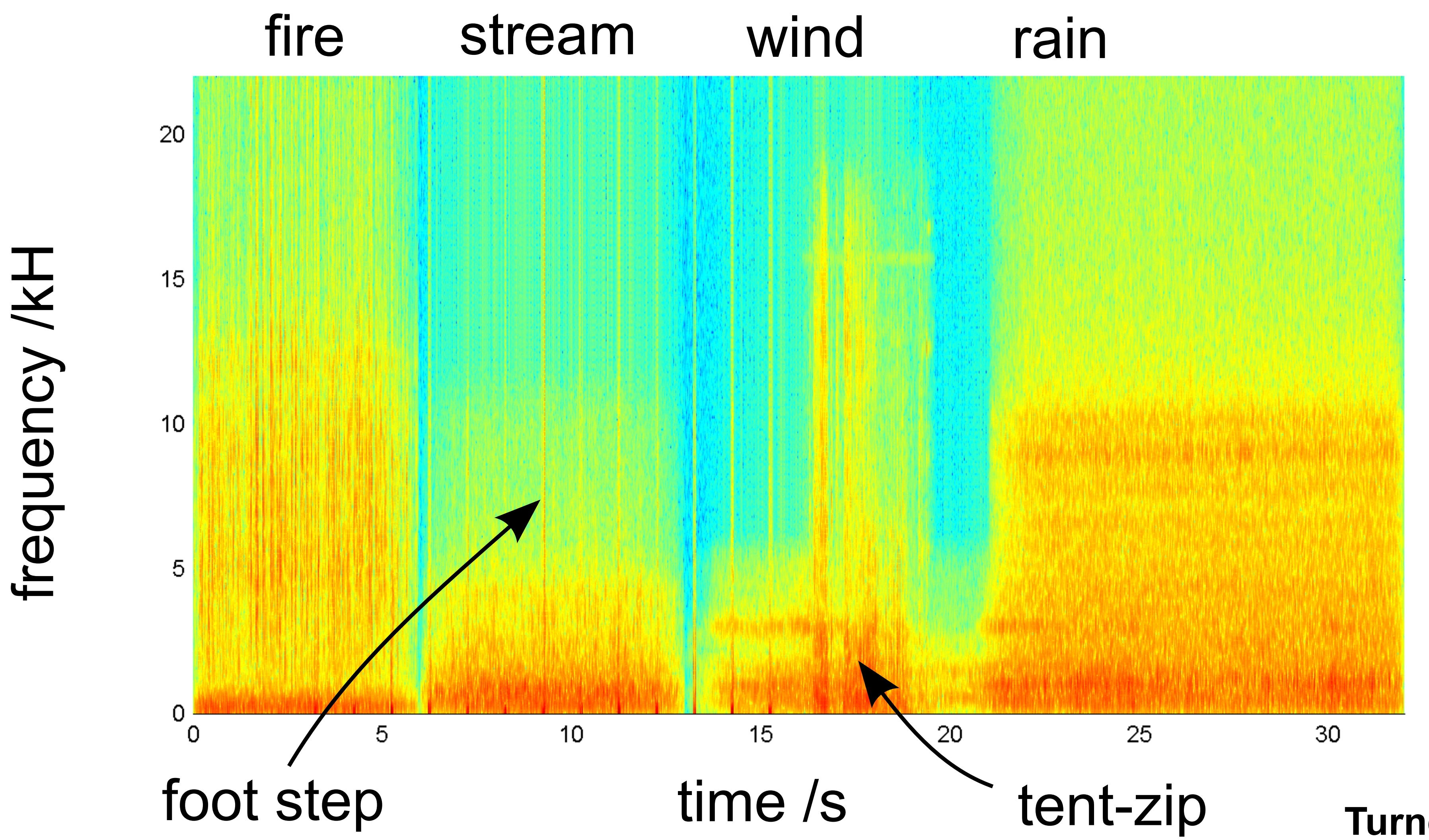


$$\mathrm{y}_t = \sum_d \Re\left(\mathrm{e}^{i\omega_d t}\mathrm{s}_{d,t}\right) + \sigma_{\mathrm{y}}\eta_t$$

$$\Re\left(\mathrm{s}_{d,t}\right) \sim \mathcal{GP}(0, \Gamma)$$

$$\Im\left(\mathrm{s}_{d,t}\right) \sim \mathcal{GP}(0, \Gamma)$$

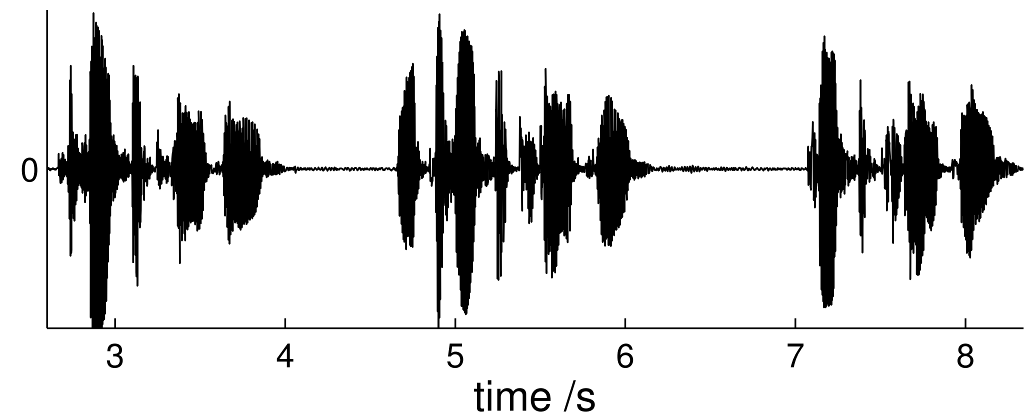
Sound Generation Demo



Turner, 2010

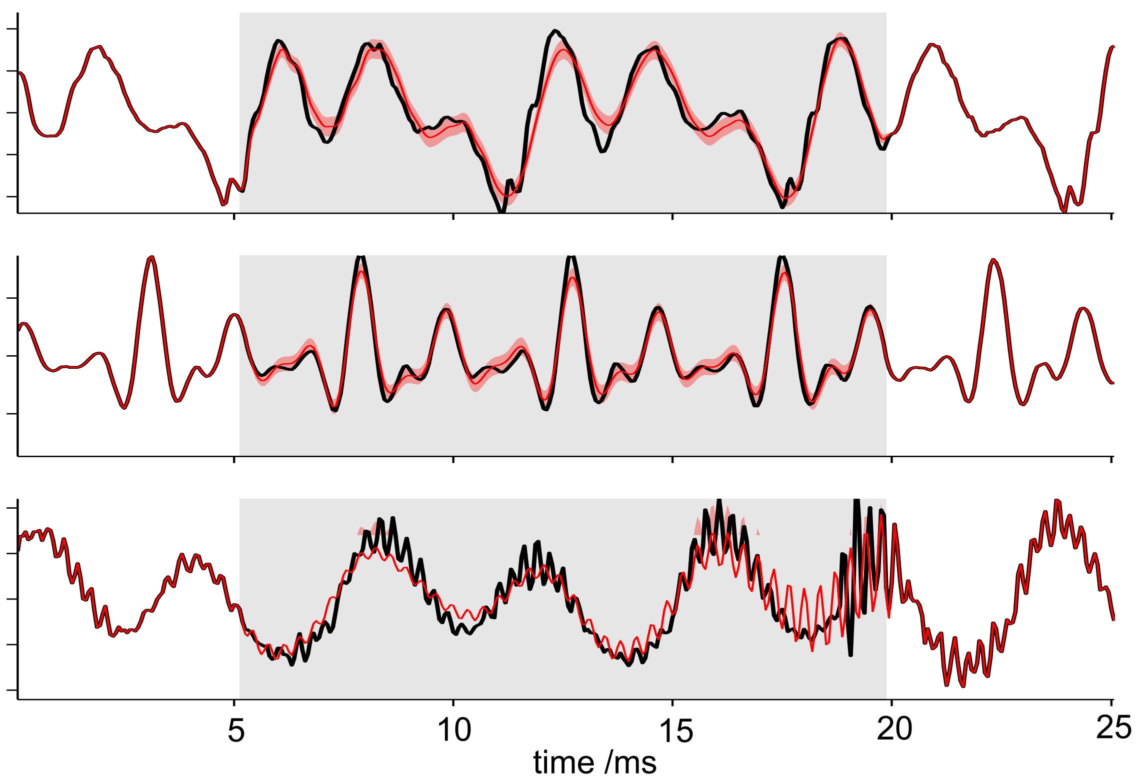
Application: speech imputation

audio
time-series
data



$y(t)$
$T = 10^5$-10^7
datapoints

reconstruction
using a GP
model



robustness
adaptation

Probabilistic signal processing

Classical signal processing

fast methods
important variables

see extra reading if interested