DS-GA 3001.008 Modelling time series data

2. AR(I)MA

Instructor: Cristina Savin

NYU, CNS & CDS

Recap: basic statistics of a time series

Time series $\{X_t\}$

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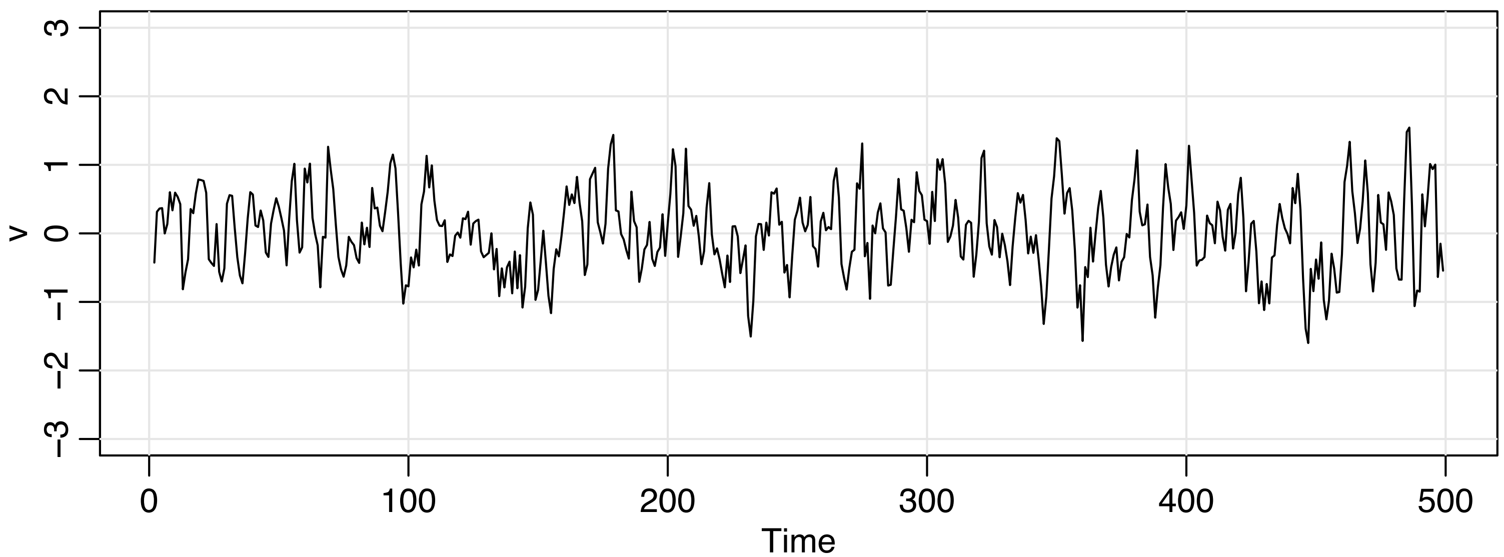
White noise $W_t \sim \mathcal{N}(0, \sigma^2)$ i.i.d.



Trivially, white noise has

$$\mu_W(t) = 0$$

$$R_W(t,u) = \begin{cases} \sigma^2, t = u \\ 0, t \neq u \end{cases}$$

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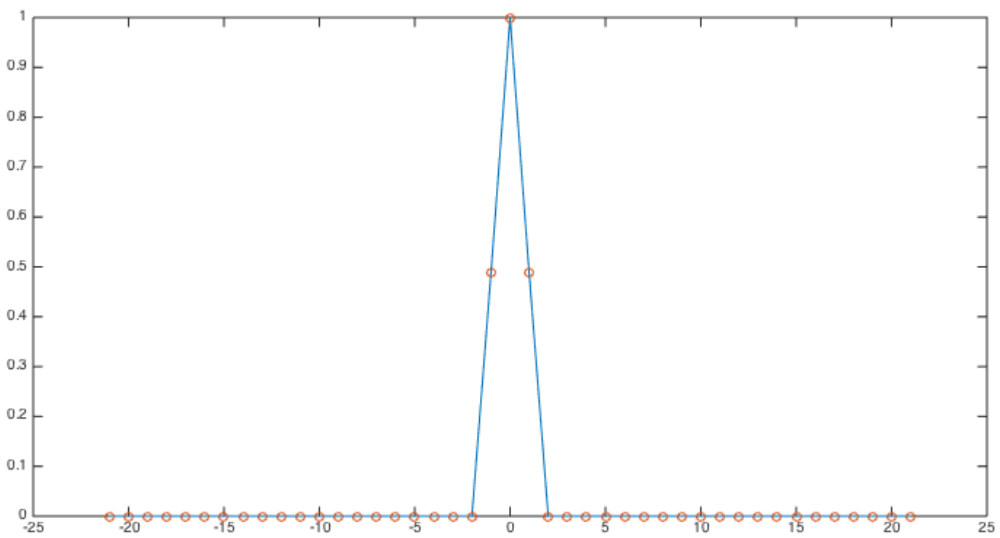

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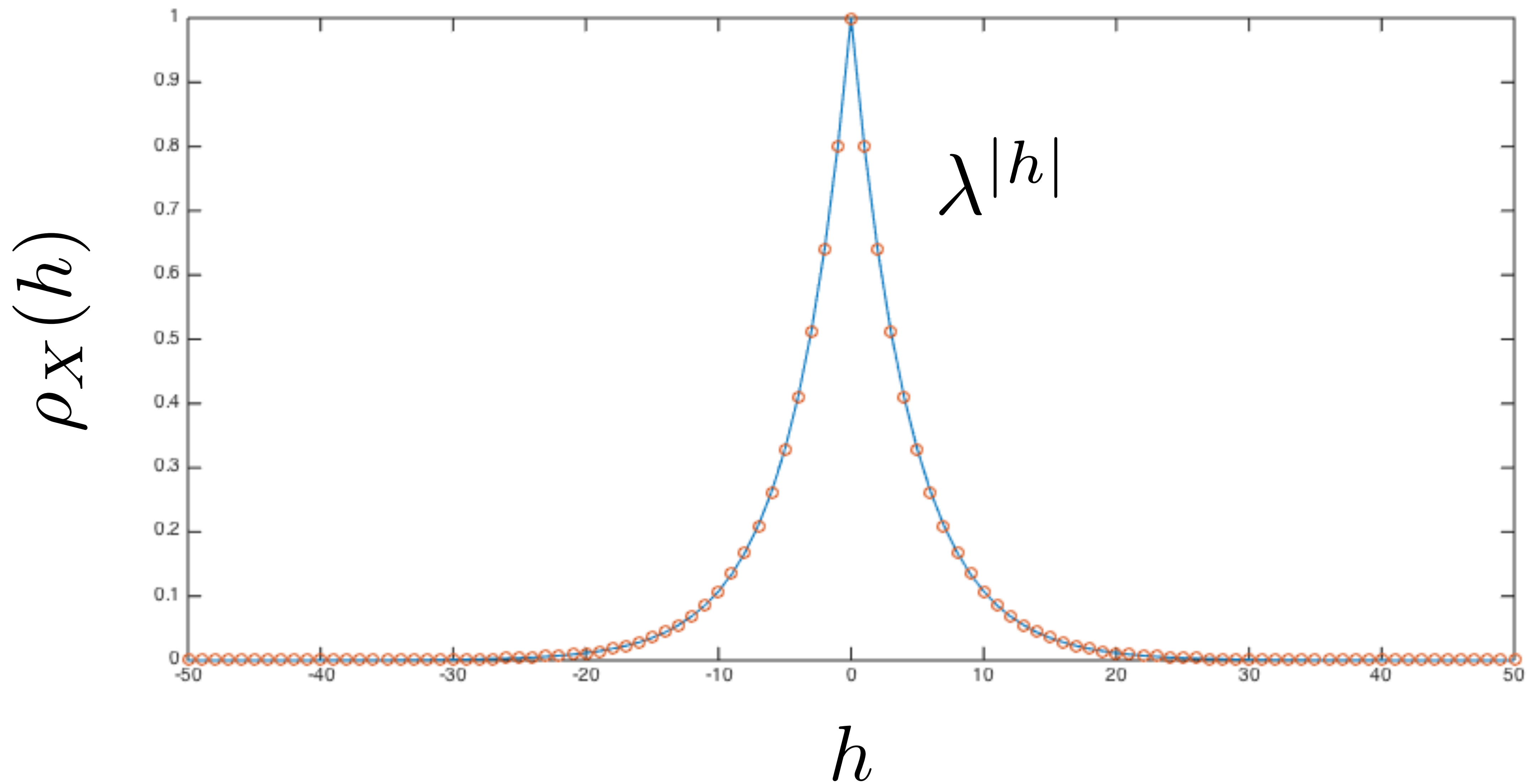
*Check other direction at home

AR(1) ACF



$\rho_X(h)$
$\lambda^{|h|}$
h
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-50
-40
-30
-20
-10
0
10
20
30
40
50

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***Useful: Cov. of linear combinations**

$$U = \sum_i a_i X_i$$

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How about the random walk?

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$$\neq \sum_k \psi_k W_{t-k}$$

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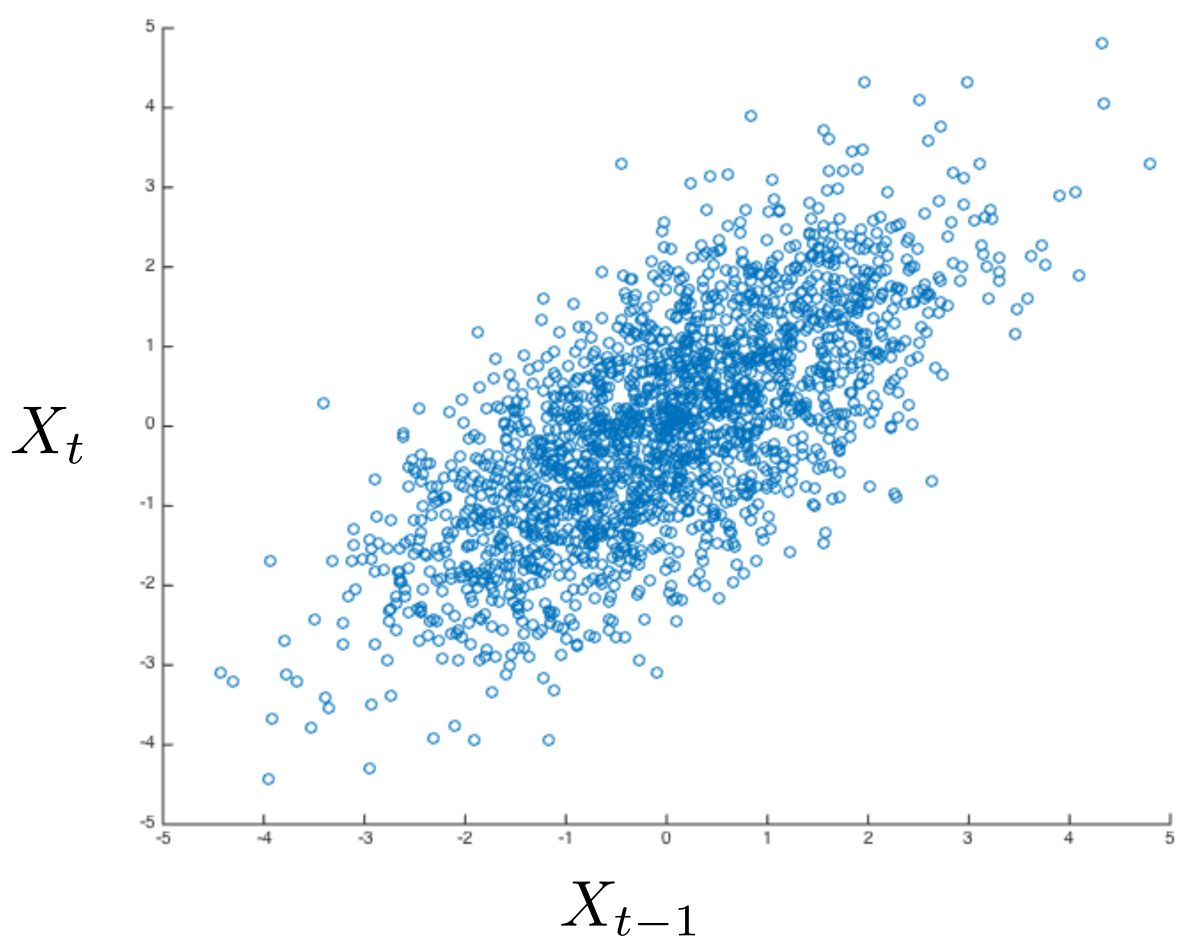

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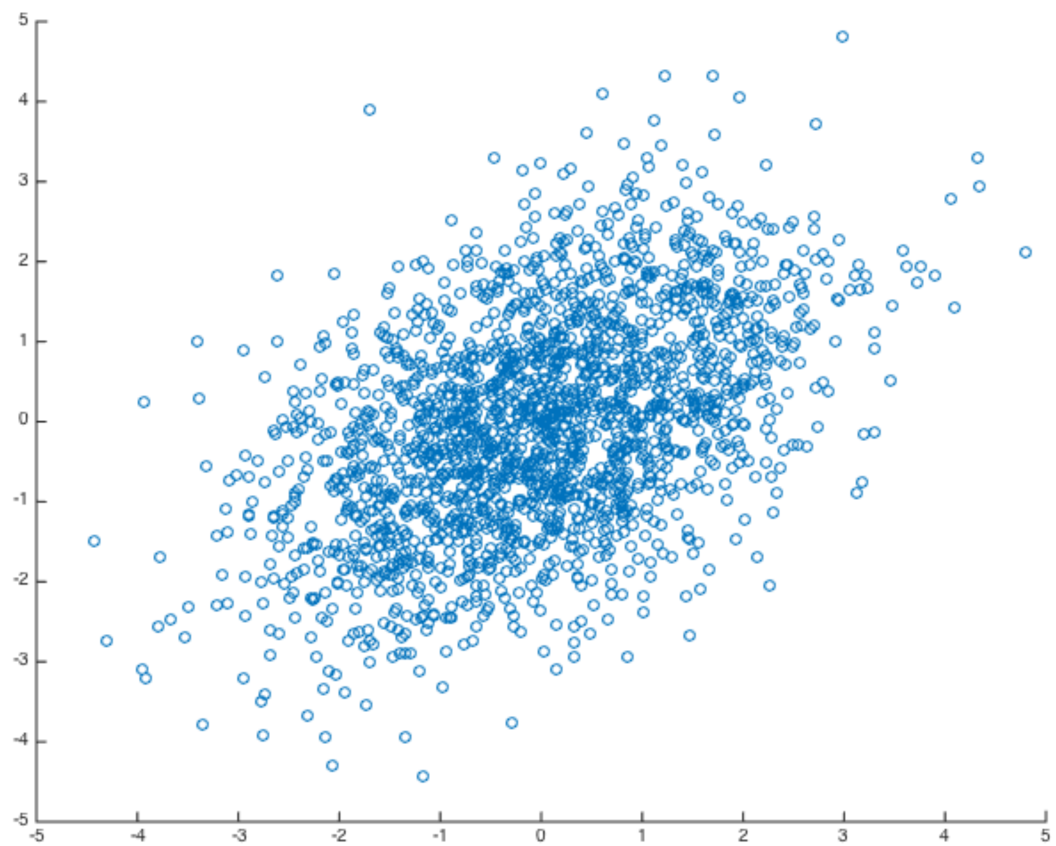

X_t

X_{t-2}

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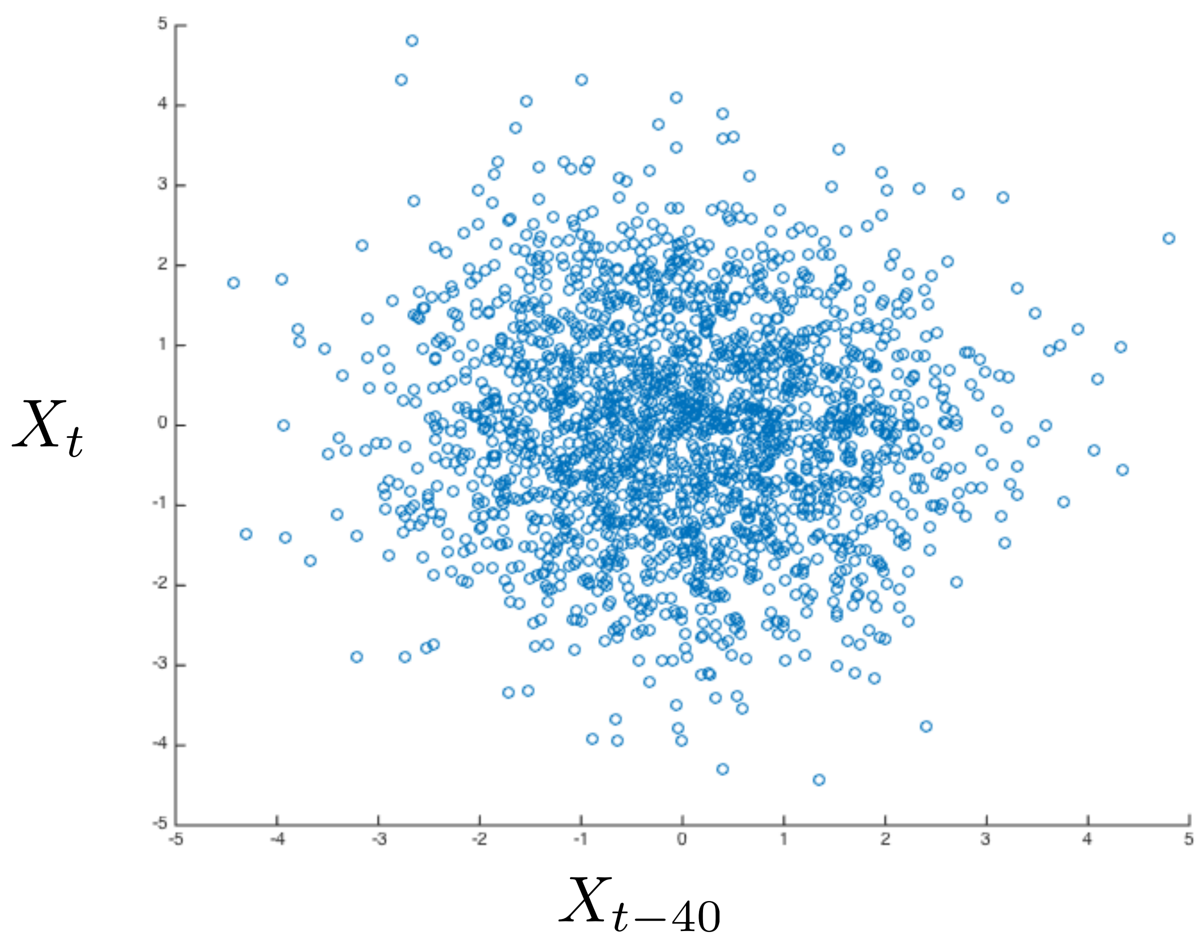

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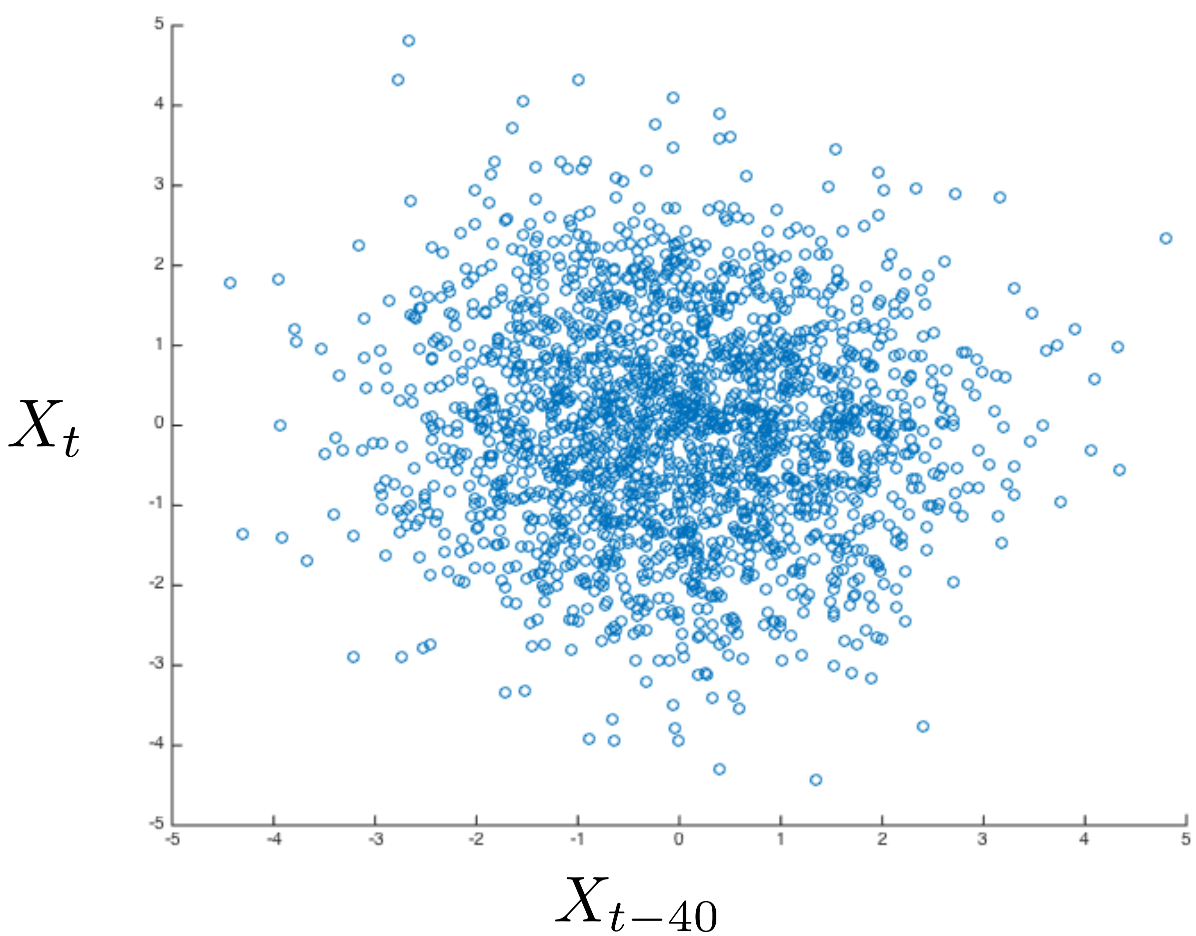

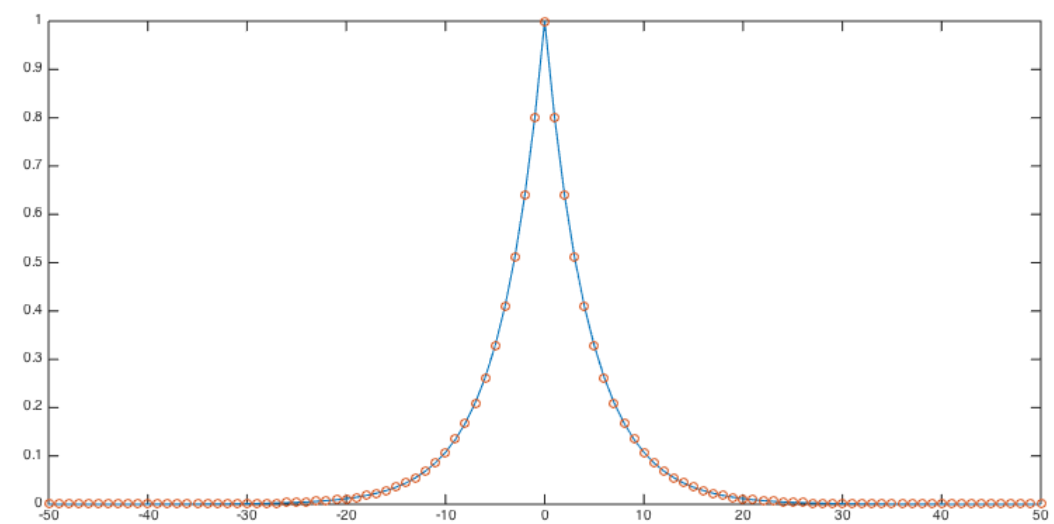

ACF determines
linear predictability

Least squares and ACF

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Least squares estimation reminder

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$$\mathcal{N}\left(\mu_{t+h} + \frac{\sigma_{t+h}\rho(t, t+h)(x_t - \mu_t)}{\sigma_t}, \sigma^2(1 - \rho(t, t+h)^2)\right)$$

Least squares and ACF

Least squares estimation reminder

$$\hat{f} = \mathrm{argmin}_f (Y - f)^2$$
$$\hat{f} = \mathbb{E}[Y|X]$$

With MSE $\mathrm{var}[Y|X]$

We can compute a least square estimate of X_{t+h} given X_t

If everything is Gaussian, then conditional expectations are easy!

If $\{X_t\}$ is jointly gaussian

$$f_X(x) = \frac{1}{2\pi^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

The pair (X_t, X_{t+h}) is also jointly gaussian, with covariance

$$\begin{pmatrix} \sigma_t^2 & \rho(t,t+h)\sigma_t\sigma_{t+h} \\ \rho(t,t+h)\sigma_t\sigma_{t+h} & \sigma_{t+h}^2 \end{pmatrix}$$

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*Gaussian identities .pdf

For a gaussian stationary process, the optimal predictor for $X_{t+h}|X_t = x_t$

takes the form:

$$f(x_t) = \mathbf{E}\left(X_{t+h}|X_t = x_t\right) = \mu + \rho_X(h)(x_t - \mu)$$

With MSE

$$\mathbf{E}\left(|X_{t+h} - f(x_t)|^2 \ , |X_t = x_t\right) = \sigma^2(1 - \rho_X(h)^2)$$

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Linear in x_t

With MSE

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$$\mathbf{E}\left(|X_{t+h} - \alpha - \beta X_t|^2\right) = E(\alpha, \beta)$$

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minimum->
derivatives zero
(check at home)

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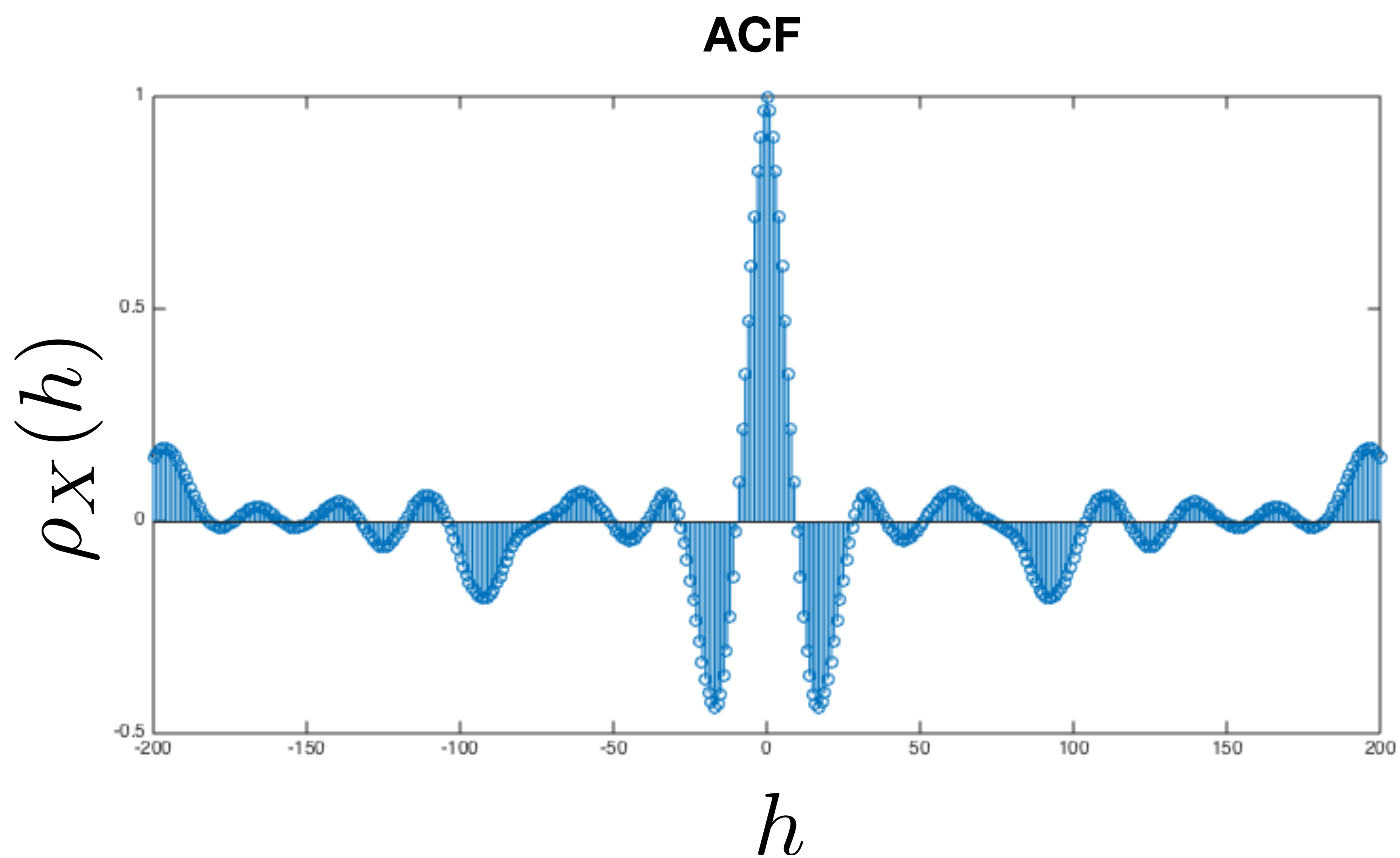
The optimal predictor
if stationary gaussian

How do we use this to model real data?



Real life looks a bit more complicated than a simple AR(1)

Can we combine the basic idea of simple linear processes to get more **expressive** power, while keeping math nice and simple?

Go back to AR(1), rewrite using backshift operator

Rewrite equation

$$X_t - \lambda X_{t-1} = W_t$$

$$(1 - \lambda B)X_t = W_t$$

$$P(B)X_t = W_t$$

Using backshift /translation operator **B**

*properties on the board

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$$B^2 X_t = BBX_t = BX_{t-1} = X_{t-2},$$
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$$X_t = \sum_{k=0} \lambda^k W_{t-k} = \underbrace{\boxed{\sum_{k=0} \lambda^k B^k}}_{Q(B)} W_t$$

$P(B) = 1 - \lambda B$ *and* $Q(B) = \sum_{k \geq 0} \lambda^k B^k$ *are related by*

$$P(B)Q(B) = 1 \ , \quad \textit{or } Q(B) = P(B)^{-1} \ .$$

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Operators **P** and **Q** behave like normal polynomials

$$\frac{1}{1 - \lambda z} = \sum_{k \geq 0} \lambda^k z^k \ , \ |\lambda| < 1, |z| \leq 1$$

$$X_t = \lambda X_{t-1} + W_t$$

What happens when $|\lambda| > 1$?

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Anti-causal : future determines the past

$$X_t = -\sum_{k=1}^{\infty} \lambda^{-k} W_{t+k}$$

Revisiting MA(1)

$$X_t = W_t + \theta W_{t-1} = (1 + \theta B)W_t = P(B)W_t$$

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essentially, we have inverted the roles of X and W

DEF: Invertible Process

A linear process $\{X_t\}$ is **invertible** if there exist $\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \dots$ with $\sum_k |\psi_k| < \infty$ and

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AR(1)

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Causal (wrt $\{W_t\}$) iff $|\lambda| < 1$.

Always invertible (wrt $\{W_t\}$).

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Increasing complexity: AR(p)

An AR(p) process $\{X_t\}$ is a stationary process satisfying

$$X_t - \lambda_1 X_{t-1} - \ldots \lambda_p X_{t-p} = W_t \ ,$$

Where $\{W_t\}$ is white noise

$$\lambda_p \neq 0$$

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$|z_k^*| \neq 1$ for all (complex) roots of P(B)

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Polynomials refresher

A polynomial of order n has n complex roots

If coeff. are real valued- pairs of conjugate roots

Stationarity and causality

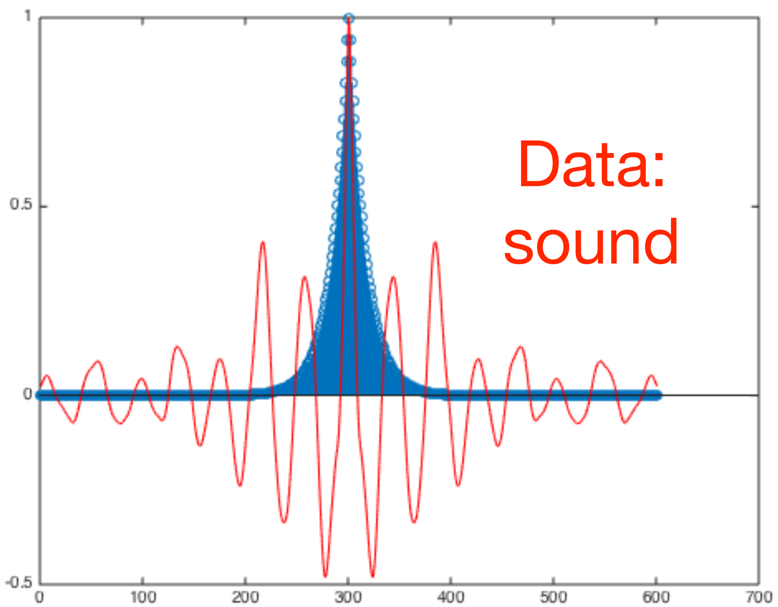
Theorem

1. *The equation* $P(B)X_t = W_t$ *has a unique stationary solution if and only if*
$$P(z) = 0 \Rightarrow |z| \neq 1 \ .$$
We call this unique solution an AR(p) process.
2. *Moreover, this process is causal if and only if*
$$P(z) = 0 \Rightarrow |z| > 1 \ .$$

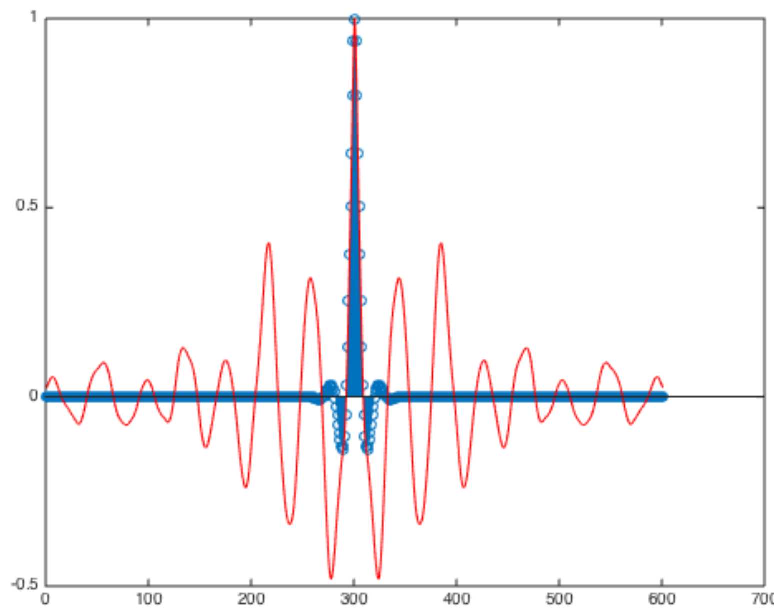
Once we have the polynomial form, recursion can be solved by traditional linear diff. eq. methods.

Increasing complexity: AR(p)

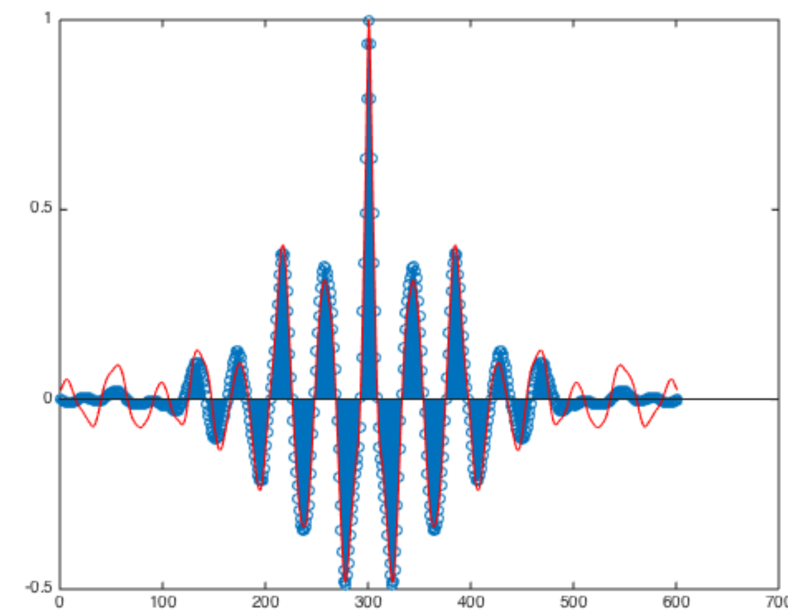
AR(1)



AR(4)



AR(16)



Increased expressive power

Increasing complexity: MA(q)

The moving average model of order q, or MA(q), is defined as

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \dots \theta_q W_{t-q} \ ,$$

Where $\{W_t\}$ is white noise

$\lambda_q \neq 0$

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$$X_t = \theta(B) W_t$$

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Where $\{W_t\}$ is white noise

$$\lambda_q \neq 0$$

$$X_t = \theta(B) W_t$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$$

Putting it all together: ARMA

An ARMA(p,q) process $\{X_t\}$ is a stationary process that satisfies

$$X_t - \lambda_1 X_{t-1} - \dots \lambda_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q} \ ,$$

Where $\{W_t\}$ is white noise

$$\lambda_p \neq 0$$

$$\lambda_q \neq 0$$

*no **common** roots

Check : causal and invertible

Putting it all together: ARMA

An ARMA(p,q) process $\{X_t\}$ is a stationary process that satisfies

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Example 3.8

$$x_t = .4x_{t-1} + .45x_{t-2} + w_t + w_{t-1} + .25w_{t-2},$$

Check : causal and invertible

Putting it all together: ARMA

An ARMA(p,q) process $\{X_t\}$ is a stationary process that satisfies

$$X_t - \lambda_1 X_{t-1} - \ldots \lambda_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q} \ ,$$

Where $\{W_t\}$ is white noise

$$\lambda_p \neq 0$$

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Special cases

AR(p) = ARMA(p, 0), ie $\theta(B) = 1$.

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For any stationary process with autocovariance R and any $k > 0$, there is an ARMA process $\{X_t\}$ such that

$$R_X(h) = R(h) \ , h \leq k \ .$$

The wonderful world of ARMA polynomials

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Where P(B) has degree p and
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Theorem

- *If P and θ have no common factors, a stationary solution to $P(B)X_t = \theta(B)W_t$ exists iff the roots of $P(z)$ avoid the unit circle: $P(z) = 0 \Rightarrow |z| \neq 1$. This is called an ARMA(p,q) process.*
- *This process is* **causal** *iff the roots of $P(z)$ are* **outside** *the unit circle: $P(z) = 0 \Rightarrow |z| > 1$.*
- *This process is* **invertible** *iff the roots of $\theta(B)$ are* **outside** *the unit circle: $\theta(z) = 0 \Rightarrow |z| > 1$.*

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Polynomial form defines the moments of the process!

So the autocorrelation $R_X(h)$ also satisfies an homogeneous recurrence:

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with initial conditions given by

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with characteristic polynomial $a(z) = a_p(z - z_1)(z - z_2) \ldots (z - z_p)$

Big picture

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*Note: proof tsa4.pdf, page 91

Big picture

Goal: solve $a_0 X_t + \cdots + a_p X_{t-p} = 0$

With initial conditions $X_1, \ldots, X_p$.

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General solution of the form*

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Coefficients adjusted from initial conditions

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Estimating ARMA parameters from data

2 methods: **Maximum likelihood**

Method of moments

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Assume: model known, and zero mean (preprocessing)

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Find parameters $\lambda_i, \; \Theta_j \; \sigma^2$

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$$\mathcal{L}(\lambda, \theta, \sigma^2) = f_{\lambda,\theta,\sigma^2}(x_1, \ldots, x_n)$$

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Jointly gaussian

$$\mathcal{L}(\lambda, \theta, \sigma^2) = (2\pi)^{-n/2} |\Gamma_n|^{-1/2} \exp\left(-\frac{1}{2}\mathbf{x}^T \Gamma_n^{-1} \mathbf{x}\right)$$

Where we've collated the data in vector $\mathbf{x} = (x_1, \ldots, x_n)$

Maximum likelihood

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Pros: efficient, works well even if model assumptions not 100% right

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In the AR(p) case, we can show the forecasting coefficients

$$X_{t+1}^{t} = \phi_{t,1} X_t + \cdots + \phi_{t,t} X_1$$

correspond exactly to the model parameters λ_i, $i = 1, \ldots, p$.

So we can regress X_t onto $X_{t-1}, \ldots, X_{t-p}$ to estimate λ_i.

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If $\{X_t\}$ is a casual AR(p) model $P(B)X_t = W_t$, it results that

$$\mathbf{E}\left(X_{t-i}\left(X_t - \sum_{j=1}^{p} \lambda_j X_{t-j}\right)\right) = \mathbf{E}\left(X_{t-i} W_t\right) \ , \ (i = 0, \ldots, p) \Leftrightarrow$$

$$\boxed{R_X(0) - \lambda^T R_p = \sigma^2 \ \text{, and } R_p = \Gamma_p \lambda} \ .$$

where $\Gamma_p = \{\gamma(k-j)\}^{p}_{j,k=1}$ is a $p \times p$ matrix, $\phi = (\phi_1, \ldots, \phi_p)'$ is a $p \times 1$ vector, and $\gamma_p = (\gamma(1), \ldots, \gamma(p))'$ is a $p \times 1$ vector.

Method of moments

Identities linking **parameters** to moments (in our case cov),
use empirical estimates,
adjust parameters to match

Yule Walker equations, with empirical estimates

$$\hat{\lambda}^T \hat{R}_p = \hat{R}_X(0) - \hat{\sigma}^2 \text{ , and } \hat{\Gamma}_p \hat{\lambda} = \hat{R}_p$$

Efficient implementation using the **Durbin-Levinson** algorithm
(you'll implement this during the lab)

What do we do about the mean? ARIMA

Integrated models for non stationary data

If linear time dependence $\mu_t = \beta_0 + \beta_1 t$

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These are tricky to use,
motivate state space models

Lab:

1. Prediction
2. Polynomials - AR/AM/ARMA
3. Method of moments

Next week: Kalman filtering

Homework 1: posted on Thu, due Febr.12