Lagrange multiplier field approach to shift-symmetric theories: the φ^4 derivative theory and the crumpled–to–flat transition of membranes at two-loop order

L. Delzescaux,[1,*] C. Duclut,[2,†] D. Mouhanna,[1,‡] and M. Tissier[1,§]

[1] *Sorbonne Université, CNRS, Laboratoire de Physique Théorique de la Matière Condensée, LPTMC, 75005 Paris, France*
[2] *Laboratoire Physico-Chimie Curie, CNRS UMR 168, Institut Curie, Université PSL, Sorbonne Université, 75005, Paris, France*

We introduce a technique relying on the use of Lagrange multiplier field in order to eliminate explicit field-derivatives that plague the high orders renormalization group treatment of shift-symmetric, derivative, theories. This technique simplifies drastically the computation of fluctuations in such theories. This is illustrated by deriving the two-loop renormalization group equations and the three-loop anomalous dimension of the $\boldsymbol{\varphi}^4$ derivative theory in $D = 4 - \epsilon$, which is also relevant to describe the crumpled-to-flat transition of polymerized membranes. Some features of this transition are provided.

Introduction. Shift-symmetric theories or shift-symmetric enforced pre-existing theories are characterized, in the simplest situation, by an invariance of the kind

$$\boldsymbol{\varphi} \longrightarrow \boldsymbol{\varphi} + \mathbf{c} \tag{1}$$

where $\mathbf{c}$ is a constant vector. Such theories have been the subject of a strong activity in the past and have received a renewed interest in recent years, mainly in the context of the study of modified theories of gravity, as well as in string and brane theories. A strong motivation is that shift symmetry results in specific renormalization properties. This is for instance the case of Galileon field theory [1] that displays field and space-time shift symmetries $\varphi \longrightarrow \varphi + c + a_\mu x_\mu$ where c and the a_μ are constants. It obeys non renormalization theorems [2, 3] and [4] for a review; see also [5] for a nonperturbative treatment. In the context of Hořava-Lifshitz gravity [6] it has been shown that shift symmetry prevents the appearance of an infinite number of interactions [7]. Generalized – polynomial – shift symmetry [8–10] has also been considered, notably in connection with multicritical symmetry breaking. Very recently, shift symmetry has been considered in the context of asymptotically safe quantum gravity-matter theories where this symmetry allows generating closed renormalization group (RG) flow for the effective action, see e.g.[11]. Interestingly, motivated by these considerations, a new universality class has been discovered that gathers theories whose RG equations are projected on functions of the kinetic term [12]. Shift symmetry have also been studied in various other contexts including Horndeski gravity [13, 14], inflation [15, 16], inflation in supergravity [15] and in anti-de Sitter space [17].

Membranes. Shift symmetry is also relevant in condensed matter physics through the long distance description of both fluid and polymerized membranes, see [18, 19] for reviews. In particular, polymerized membranes have been intensively investigated these last twenty years following the discovery of graphene [20, 21] and graphene-like materials, see e.g. [22]. For a D-dimensional membrane embedded in a d-dimensional Euclidean space, the parametrization of a point $\mathbf{x} \in \mathbb{R}^D$ in the membrane is realized through the mapping $\mathbf{x} \to \mathbf{R}(\mathbf{x})$ where $\mathbf{R}$ is a field in $\mathbb{R}^d$. For obvious reasons, the energy of the membrane can only depend on variations of $\mathbf{R}$ so that the action should display a shift symmetry $\mathbf{R}(\mathbf{x}) \to \mathbf{R}(\mathbf{x}) + \mathbf{C}$ where $\mathbf{C}$ is a constant vector. The relevant action to study polymerized membranes in D dimensions is given by [23]:

$$\begin{aligned} S[\{\boldsymbol{R}\}] = \int \mathrm{d}^D x \, \bigg\{ & \frac{\kappa}{2}(\partial_\alpha^2 \mathbf{R})^2 + \frac{r}{2}(\partial_\alpha \mathbf{R})^2 + \\ & + \frac{\lambda}{8}(\partial_\alpha \boldsymbol{R}.\partial_\alpha \mathbf{R})^2 + \frac{\mu}{4}(\partial_\alpha \mathbf{R}.\partial_\beta \mathbf{R})^2 \bigg\} \end{aligned} \tag{2}$$

where Greek indices run over $1 \ldots D$ and summation over repeated indices is implicit. In Eq. (2), κ is the bending rigidity constant, r is a tension coefficient that conveys the main temperature dependence. The coefficients λ and μ, which are associated with quartic interactions, are Lamé (elasticity) coefficients that embody elasticity and shear properties of the membrane; stability requires κ, μ, and the bulk modulus $B = \lambda + 2\mu/D$ to all be positive. In agreement with shift symmetry, action (2) is expressed purely in terms of field-derivatives. It displays an invariance under the action of the Euclidean group of displacements $\mathrm{E}(d)$ that includes both rotations and translations in d dimensions: $R_\mu \longrightarrow \mathcal{R}_{\mu\nu} R_\nu + C_\mu$, where $\mathcal{R}$ is a rotation matrix and $\mathbf{C}$ a constant vector, see e.g. [24, 25]. The model (2) is directly relevant to study the crumpled-to-flat transition in polymerized membranes, see [23]. Indeed, varying the tension coefficient r, one expects a phase transition between a disordered, crumpled, phase at high temperatures and an ordered, flat, phase at low temperatures, characterized by a well-defined orientation of the membrane and, thus, a non vanishing average value of the tangent vector fields $\mathbf{t}_\alpha = \partial_\alpha \mathbf{R}$. The properties of the crumpled-to-flat transition have been studied at one-loop in the vicinity of the upper critical dimension $D = 4$ in [23], within a large-d expansion in [26, 27], by self-consistent screening approximation (SCSA) [28, 29] as well as within a nonperturbative framework in [30–35]. Note finally that, prior to its

*Electronic address: louise.delzescaux@sorbonne-universite.fr
†Electronic address: charlie.duclut@curie.fr
‡Electronic address: mouhanna@lptmc.jussieu.fr
§Electronic address: tissier@lptmc.jussieu.fr

role as long distance effective action for membranes, action (2) is nothing but the derivative version of the $O(d)$ $\boldsymbol{\varphi}^4$-model and, as such, represents the simplest but non-trivial model of shift-symmetric derivative field theory.

Starting from action (2), it is also possible to study the flat phase of membranes [26–29, 36–42] which is relevant to investigate the properties of stable graphene and graphene-like materials as well as that of biological membranes endowed with a cytoskeleton. To do that, one considers a flat configuration given by $\mathbf{R}^0 = (\mathbf{x}, \mathbf{0}_{d_c})$ where $\mathbf{0}_{d_c}$ is the null vector of dimension $d_c = d - D$, and decomposes the field $\mathbf{R}$ into $\mathbf{R}(\mathbf{x}) = [\mathbf{x} + \mathbf{u}(\mathbf{x}), \mathbf{h}(\mathbf{x})]$ where $\mathbf{u}$ and $\mathbf{h}$ represent D longitudinal, phonon, modes and $d - D$ transverse, flexural, modes, respectively. Power-counting considerations then lead to the relevant action written in terms of phonon and flexural modes [26–29, 36–42]:

$$S[\mathbf{h}, \mathbf{u}] = \int \mathrm{d}^D x \left\{ \frac{\kappa}{2} \left(\partial_\alpha^2 \mathbf{h}\right)^2 + \frac{\lambda}{2} u_{\alpha\alpha}^2 + \mu\, u_{\alpha\beta}^2 \right\}, \quad (3)$$

where $u_{\alpha\beta}$ is the strain tensor that encodes the elastic fluctuations around the flat phase configuration $\mathbf{R}^0(\mathbf{x})$: $u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \mathbf{R}.\partial_\beta \mathbf{R} - \partial_\alpha \mathbf{R}^0.\partial_\beta \mathbf{R}^0) = \frac{1}{2}(\partial_\alpha \mathbf{R}.\partial_\beta \mathbf{R} - \delta_{\alpha\beta})$. It is given by, neglecting nonlinearities in the phonon field $\mathbf{u}$:

$$u_{\alpha\beta} \simeq \frac{1}{2}\left[\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha \mathbf{h}.\partial_\beta \mathbf{h}\right] \ . \quad (4)$$

Action (3) has been investigated perturbatively in the vicinity of the upper critical dimension $D = 4$ in [27, 37, 38, 43–47]; see [48, 49] for investigation of the disordered case. It has also been studied by means of large-d expansions [26, 27, 38, 39, 50, 51], SCSA [28, 29, 40] and within a nonperturbative approach [30–32].

Scale and conformal invariance. The two models (2) and (3) and, more generally, shift-symmetric theories have recently been the subject of a special attention regarding the question of the relation between scale and conformal invariance, see [52] for a review. Indeed, it is generally believed that, at least in 2 and 4 dimensions, conformal invariance follows from scale invariance for theories displaying unitarity – in Euclidean space, reflection positivity – and Poincaré invariance [53–55]. Also, for theories that are suspected to be non-unitary – which is the case for the actions (2) and (3) – this property no longer extends straightforwardly. For instance, Riva and Cardy [56] have exhibited a model of elasticity – whose action (3) is a generalization embedded in a Euclidean d-dimensional space – which exhibits scale invariance but not conformal invariance. One notes that Riva and Cardy have considered a – non-unitary – free theory, see also [57], while one expects conformal invariance for interacting theories. Recently, Safari *et al.* [58] have investigated a large class of derivative shift-symmetric, scalar, non-unitary theories, including the one described by action (2) for $d = 1$; see also [59]. By computing the trace of the momentum-energy tensor, they have shown that these theories were displaying conformal invariance at the fixed point, in agreement with the common belief. Conversely, Mauri and Katsnelson [60] have investigated the $\boldsymbol{\varphi}^4$ vectorial, derivative theory in its flat phase described by action (3). On the same basis, they have concluded that dilatation invariance at the fixed point does not extend to full conformal invariance at the fully attractive fixed point controlling the low temperatures, flat, phase. If this is confirmed, this would constitute an uncommon example of interacting but non conformally invariant model, see [61].

Beyond one-loop order. These results, as well as almost all results derived in the context of shift-symmetric, derivative field theories, have been obtained at leading order : one-loop order in the perturbative context or using a low-derivative truncated action in the context of the nonperturbative RG [62] [79]. There is a notable exception: the φ^4 theory in its ordered phase, described by action (3), which has been investigated at high orders in perturbation theory. A characteristic of this model is that its propagators are given by [27]:

$$G_h^{ij}(p) = \langle h^i(p) h^j(-p)\rangle = \frac{\delta^{ij}}{\kappa p^4} \quad (5)$$
$$G_u^{\alpha\beta}(p) = \langle u^\alpha(p) u^\beta(-p)\rangle = \frac{P^\perp_{\alpha\beta}}{\mu p^2} + \frac{P^\parallel_{\alpha\beta}}{(\lambda + 2\mu)p^2}$$

where $P^\parallel_{\alpha\beta} = p_\alpha p_\beta/\boldsymbol{p}^2$, $P^\perp_{\alpha\beta} = \delta_{\alpha\beta} - p_\alpha p_\beta/\boldsymbol{p}^2$ while the indices i, j run over $1 \ldots d - D$ and α, β over $1 \ldots D$. One sees on these expressions that $G_u^{\alpha\beta}(p)$ involves all the coupling constants associated to the interactions. Thus, considering the two-point functions is sufficient to get the renormalization of all these couplings. This explains why this model has been investigated, although thirty years after the one-loop computation of Aronovitz and Lubensky [37], successively at two [43, 44], three [45, 47] and four loop order [46].

Clearly, it would be also very valuable to be able to extend the one-loop computation performed by Paczuski and Kardar [63] on the action (2) describing the crumpled-to-flat transition at higher orders. However, one faces here an important difficulty: the derivative nature of the interaction makes the RG treatment of action (2) extremely tedious if it is addressed by *brute force*. Indeed, the propagator of the field $\boldsymbol{R}$ is, in this case,

$$G_R^{ij}(p) = \langle R^i(p) R^j(-p)\rangle = \frac{\delta^{ij}}{\kappa p^4}. \quad (6)$$

It involves only the coupling κ and to get the renormalization of the Lamé coefficients, one has to resort to four-point functions. Also, the derivative or, in Fourier space, momentum dependence of the four-point vertices gives rise to Feynman diagrams involving a very complex structure and a kinematic of the external momenta that are extremely difficult to manage beyond one-loop order despite the use of mathematical software like Mathematica and its package LiteRed [64, 65] to reduce the loop integrals, see [66]. This is the reason why this model has been only studied at leading order in a loop-expansion.

Auxiliary fields technique. We propose here a novel treatment of derivative shift-symmetric theories that makes their RG treatment as easy as that of their non-derivative counterparts. The main – and simple – idea is that a derivative theory can be completely reparametrized in terms of auxiliary fields that represent the space-derivative of the original field(s). We basically follow the procedure first discussed by Faddeev and Popov in the framework of gauge theories [67], later applied in disordered systems [68] or dynamical theories [69–71]. Thanks to this trick, we are brought back to a theory where there are no more derivative interactions. We apply this technique to the $\boldsymbol{\varphi}^4$ derivative theory (2) and derive the RG equations for the two coupling constants and tension coefficient at two-loop order as well as the anomalous dimension at the first non-trivial – three-loop – order and discuss some properties of the crumpling-to-flat transition.

One reparametrizes the action (2) in terms of D auxiliary d-components fields $\{\boldsymbol{A}_\alpha\}$, $\alpha = 1\ldots D$ so that the partition function of the theory reads (using the notation $\boldsymbol{\varphi}$ instead of $\boldsymbol{R}$):

$$Z = \int \mathcal{D}\boldsymbol{\varphi} \prod_{\alpha=1}^{D} \mathcal{D}\boldsymbol{A}_\alpha \; \delta(\boldsymbol{A}_\alpha - \partial_\alpha\boldsymbol{\varphi})\; e^{-S[\{\boldsymbol{A}_\alpha\}]}\ .$$

The delta constraint can be raised with the help of a second set of D auxiliary d-components fields $\{\boldsymbol{B}_\beta\}$:

$$Z = \int \mathcal{D}\boldsymbol{\varphi} \prod_{\alpha,\beta=1}^{D} \mathcal{D}\boldsymbol{A}_\alpha \mathcal{D}\boldsymbol{B}_\beta\; e^{-S[\{\boldsymbol{A}_\alpha\}]} \times e^{-i\int d^Dx\, \boldsymbol{B}_\alpha.(\boldsymbol{A}_\alpha - \partial_\alpha\boldsymbol{\varphi})}\ . \tag{7}$$

In the partition function (7), the fields $\{\boldsymbol{B}_\alpha\}$ and $\{\boldsymbol{\varphi}\}$ appear only in terms quadratic in the fields. As a consequence, there is no interaction vertex with $\boldsymbol{B}$ or $\boldsymbol{\varphi}$ legs. We conclude that there are no 1PI Feynman diagrams with such fields as external legs and the $\boldsymbol{B}-\boldsymbol{\varphi}$ sector renormalizes trivially; only the auxiliary fields $\{\boldsymbol{A}_\alpha\}$ renormalize non trivially.

The matrix of second derivative of $S' = S + i\int d^Dx\, \boldsymbol{B}_\alpha.(\boldsymbol{A}_\alpha - \partial_\alpha\boldsymbol{\varphi})$ in the basis $\{\boldsymbol{X}_\alpha\} = (\{\boldsymbol{A}_\alpha\},\{\boldsymbol{B}_\alpha\},\boldsymbol{\varphi})$ is given, in Fourier space, by:

$$\frac{\delta^2 S'}{\delta X^i_\alpha \delta Y^j_\beta} = \begin{pmatrix} p_\alpha p_\beta + \delta_{\alpha\beta}\, r & i\delta_{\alpha\beta} & 0 \\ i\delta_{\alpha\beta} & 0 & -p_\alpha \\ 0 & p_\beta & 0 \end{pmatrix} \delta_{ij}\,,$$

with the indices i,j running from 1 to d. The inverse matrix provides the propagator

$$\Gamma^{(2)\,ij}_{\alpha\beta} = \begin{pmatrix} \frac{P^{\parallel}_{\alpha\beta}}{\boldsymbol{p}^2+r} & -iP^{\perp}_{\alpha\beta} & \frac{-ip_\alpha}{\boldsymbol{p}^2(\boldsymbol{p}^2+r)} \\ -iP^{\perp}_{\alpha\beta} & rP^{\perp}_{\alpha\beta} & \frac{p_\alpha}{\boldsymbol{p}^2} \\ \frac{ip_\beta}{\boldsymbol{p}^2(\boldsymbol{p}^2+r)} & -\frac{p_\beta}{\boldsymbol{p}^2} & \frac{1}{\boldsymbol{p}^2(\boldsymbol{p}^2+r)} \end{pmatrix} \delta_{ij}\,,$$

where one remarks that the propagator of the $\{\boldsymbol{A}_\alpha\}$-fields is given by $P^{\parallel}_{\alpha\beta}/(\boldsymbol{p}^2+r)$. The existence of a longitudinal propagator is the only change with respect to the standard $\boldsymbol{\varphi}^4$ theory. This is a tremendous simplification compared to the usual, brute force, treatment of derivative field theories.

Renormalization group equations at two-loop order. We have derived the two-loop order RG equations for the model (7) within the modified minimal subtraction $\overline{\text{MS}}$ scheme. The diagrammatic is the one of the $\boldsymbol{\varphi}^4$ theory while the vertices involve two coupling constants λ and μ and a tensorial algebra associated to both the vectorial – Roman – and derivative – Greek – indices. We have used *Mathematica* to perform the algebra and compute the integrals. One introduces the renormalized field $\boldsymbol{A}_{\alpha R}$ through $\boldsymbol{A}_\alpha = Z^{1/2}\boldsymbol{A}_{\alpha R}$ as well the dimensionless renormalized coupling constants λ_R and μ_R through $\lambda = k^\epsilon Z^{-2} Z_\lambda \lambda_R$ and $\mu = k^\epsilon Z^{-2} Z_\mu \mu_R$. Here, k is the running momentum scale and $\epsilon = 4 - D$. Finally, within the $\overline{\text{MS}}$ scheme, one introduces the scale $\bar{k}^2 = 4\pi e^{-\gamma_E} k^2$ where γ_E is the Euler constant. One defines the RG flow of the renormalized coupling constants at fixed bare theory $\beta_\lambda = \partial_t\lambda$, $\beta_\mu = \partial_t\mu$ and $\beta_r = \partial_t r$ with $t = \log\bar{k}$ where, for simplicity, and from now on, we omit the index R for the renormalized quantities. The running field anomalous dimension is given by $\eta = -\partial_t \log Z$. The RG functions at two-loop order are thus given by:

$$\begin{aligned}\beta_\lambda(\lambda,\mu) &= -\epsilon\lambda \\ &+ c_1\big((6d+7)\lambda^2 + 2(3d+17)\lambda\mu + (d+15)\mu^2\big) \\ &- \frac{c_1^2}{6}\big((69d+52)\lambda^3 + (54d^2-16d+541)\lambda^2\mu \\ &+ (36d^2+281d-110)\lambda\mu^2 + (6d^2+112d-95)\mu^3\big)\end{aligned} \tag{8}$$

$$\begin{aligned}\beta_\mu(\lambda,\mu) &= -\epsilon\mu + c_1\big(\lambda^2 + (d+21)\mu^2 + 10\lambda\mu\big) \\ &+ \frac{c_1^2}{12}\big((96d+55)\lambda^3 + (470d+289)\lambda^2\mu \\ &+ (146d+421)\lambda\mu^2 + (-212d+475)\mu^3\big)\end{aligned} \tag{9}$$

$$\begin{aligned}\beta_r(r,\lambda,\mu) &= -2r + 3c_1 r\big((2d+1)\lambda + (d+5)\mu\big) \\ &- \frac{3c_1^2 r}{4}\big((19d-1)\lambda^2 + 2(6d^2+5d+25)\lambda\mu \\ &+ (4d^2+41d+27)\mu^2\big)\end{aligned} \tag{10}$$

with $c_1 = 1/96\pi^2$. This last equation provides the *running* exponent $\nu(\lambda,\mu)$:

$$\begin{aligned}\nu(\lambda,\mu) &= \frac{1}{2} + \frac{3c_1}{4}\Big((2d+1)\lambda + (d+5)\mu\Big) \\ &+ \frac{3c_1^2}{16}\Big((24d^2+5d+7)\lambda^2 + 2(6d^2+61d+5)\lambda\mu \\ &+ (2d^2+19d+123)\mu^2\Big)\end{aligned} \tag{11}$$

while the *running* field renormalization $\eta(\lambda,\mu)$ at the first non-trivial – three-loop – order is given by:

$$\begin{aligned}\eta(\lambda,\mu) =& \frac{(d+2)(\lambda+2\mu)}{3(32\pi^2)^3}\times \\ &\big((2d+3)\lambda^2 + 2(d+9)\lambda\mu + (d+19)\mu^2\big)\end{aligned} \tag{12}$$

Equations (8)-(12) constitute our main results. They generalize to the next non-trivial order the expressions derived by Paczuski *et al.* [23]. Equations (8), (9) and (12) also generalize to $d \geq 1$-component vector fields those derived by Safari *et al.* [58] in the case of scalar fields. To make contact with their expressions, one has to take the limit $d \to 1$ in our RG functions (8), (9) and (12) and to form the combination $g = \lambda/2 + \mu$ whose RG flow gives that found in [58] while one finds $\eta = 5g^3/(4\pi)^6$ also in agreement with [58].

Crumpled-to-flat transition in polymerized membranes. As said, the model (2) has been investigated perturbatively at one-loop order within an ϵ-expansion in [23] and by other techniques [26–35]. In the vicinity of $D = 4$, the one-loop computation has allowed to identify a phenomenon of fluctuation induced first order phase transition below a critical value of the dimension d, approximately equal to $d_c \sim 218$. Above this critical value of d, the transition is of second order. A large amount of numerical simulations have been performed on this model in order to determine the nature of the phase transition in the physical – $D = 2, d = 3$ – case, see e.g. the contributions of Cantor and of Gomper and Kroll in [18]. These results have been controversial as they have led to conclude either to a first order transition [72, 73] or to a second order one [74]. The nonperturbative approaches, performed also directly in $D = 2$ and $d = 3$ dimensions, have also led to various kinds of transitions, second [30] or first order [33], according to the kind of truncation used.

One-loop order. Let us now discuss our predictions for the crumpled-to-flat transition. One starts with the one-loop order results that have not been explicitly given in the past literature. At leading order in ϵ, one finds four solutions in agreement with [23]. They are given by: $\lambda = (16\pi^2/3)\Omega\,\epsilon$ and $\mu = (16\pi^2/3)X[\Omega]\Omega\,\epsilon$ with $\Omega = 0, \Omega = \Omega[1,1,d], \Omega = \Omega[x_0,y_0,d]$ and $\Omega = \Omega[y_0,x_0,d]$ where the function $\Omega[x,y,d]$, as well as the parameters x_0 and y_0, are given in appendix A. One considers the case $D < 4$. The value $\Omega = 0$ identifies with the Gaussian fixed point which is always twice unstable [80] while the value $\Omega[y_0,x_0,d]$ corresponds to a once unstable one. The stability of the other fixed points depends on the value of d with respect to a critical value d_c. For $d > d_c$, the value $\Omega[1,1,d]$ corresponds to a once unstable fixed point and the value $\Omega[x_0,y_0,d]$ to a stable one. This last fixed point controls the second order, crumpled-to-flat, transition. For $d < d_c$, the fixed points associated with the values $\Omega[1,1,d]$ and $\Omega[x_0,y_0,d]$ are complex ones with conjugate coordinates. There is no longer any fully stable fixed point, and the transition is expected to be of first order. At one-loop order, the critical value d_c is given by the root of a polynomial, see appendix A, and is given by $d_c \simeq 218.20$.

Two-loop order. At two-loop order, the situation is not qualitatively modified. There are still four fixed points and, for $d > d_c$, only one, corresponding to $\Omega[x_0,y_0,d]$, is fully stable. The coordinates of the fixed point are given in appendix A. With the help of these coordinates, one can compute several physical quantities at second order in ϵ like the critical exponents ν and η. However, their full d-dependence is extremely involved and not very useful to explicit. Also, as seen from the value of d_c, in the vicinity of the upper critical dimension $D = 4$, the second order phase transition occurs at large values of the embedding dimension d. It is thus relevant to evaluate the critical quantities within a $1/d$ expansion. From the expression of $\nu(\lambda,\mu)$, Eq. (11), one finds at orders $1/d$ and ϵ^2 at the stable fixed point:

$$\nu = \frac{1}{2} + \left(\frac{1}{4} - \frac{33}{2d}\right)\epsilon + \left(\frac{1}{8} + \frac{129}{8d}\right)\epsilon^2 + \mathcal{O}\left(\frac{\epsilon}{d^2}, \frac{\epsilon^2}{d^2}, \epsilon^3\right) \tag{13}$$

that coincides exactly with the results following the $1/d$ analysis of Paczuski and Kardar [63]. In the same way, from the expression of η, Eq. (12), one finds, at the stable fixed point:

$$\eta = \frac{25}{3d}\epsilon^3 + \mathcal{O}\left(\frac{\epsilon^3}{d^2}, \epsilon^4\right) \tag{14}$$

that also coincides with the expression obtained in [63]. Expressions (13) and (14) provide strong checks of our computations, while higher order in $1/d$ or even the full expressions can be easily derived from appendix A. A quantity of strong interest is the critical value d_c at this order. It is computed by requiring that the coordinates of the stable fixed point develop an imaginary part when $d \to d_c^-$. One finds :

$$d_c(\epsilon) = 218.20 - 448.25\,\epsilon + \mathcal{O}(\epsilon^2) \tag{15}$$

where we emphasize that the correction of order ϵ would have been extremely painful to get without the present Lagrange multiplier field formalism.

As seen on expression (15), the correction of order ϵ is large and of the same order of magnitude than the dominant term. Note that this is a very generic feature of fluctuation-induced first order phase transitions, where there exists a critical number N_c of the number of components N of the order parameter above which the transition is of second order and under which it is of first order. One has, for instance, in frustrated magnets, see e.g. [75] $N_c(\epsilon) = 21.8 - 23.4\,\epsilon$ or in electroweak phase transition [76] $N_c(\epsilon) = 718 - 990.83\,\epsilon$. To get trustable predictions as for the nature of the crumpled-to-flat phase transition in the physical dimension $D = 3$, the ϵ-expansion should be easily extended to higher orders, and the series $d_c(\epsilon)$ resummed. Note however that this computation would be rather academic since in a realistic model of membranes self-avoidance is likely to destroy the crumpled phase, see e.g. [77].

Conclusion. We have introduced a technique allowing to investigate and determine the RG properties of derivative shift theories with the same level of technicality as usual – non-derivative – theories. This allowed us to derive the two-loop RG equations of the φ^4 derivative theory, also relevant to study the crumpled-to-flat transition in membranes thirty-five years after the one-loop computation of Paczuski *et al.*. This technique can be used to extend our computation at higher orders. It can also be easily generalized to a situation involving several fields, as in the flat phase of membranes, see action (3), where phonons and flexural degrees of freedom coexist, even if this case has already been studied at high (four) orders within a loop expansion. Still in the context of membranes, our approach is also relevant to investigate fluid membranes where one has moreover to deal with gauge – diffeomorphism – invariance [18, 78]. Finally, our technique can be of great interest and can be easily implemented in various contexts around that of quantum gravity, as Galileon theory, Horăva-Lifshitz gravity, string, brane theory, ... either in a perturbative or in a nonperturbative framework.

Acknowledgments

L.D. and D.M. greatly thank S. Metayer for his contribution in early investigations of the model studied here by means of a usual perturbative approach, see [66]. C. D., D.M. and M.T. thank R. Ben Alì Zinati for discussions.

Appendix A: Fixed points at one and two-loop order

At one-loop order, the four fixed points are given by:

$$\lambda_1 = \frac{16\pi^2}{3}\Omega\,\epsilon \qquad \text{and} \qquad \mu_1 = \frac{16\pi^2}{3}X[\Omega]\Omega\,\epsilon \tag{A1}$$

with $\Omega = 0$, $\Omega = \Omega[1,1,d]$, $\Omega = \Omega[x_0, y_0, d]$ and $\Omega = \Omega[y_0, x_0, d]$ and the parameters $x_0 = (-1+i\sqrt{3})/2$ and $y_0 = -(1+i\sqrt{3})/2$. The functions $\Omega[x,y,d]$ and $X[\Omega]$ are defined by:

$$\Omega[x,y,d] = \frac{1}{P_1(d)}\left(4P_2(d) + x\frac{P_3(d)}{T^{1/3}} + y\,T^{1/3}\right) \qquad \text{and} \qquad X[\Omega] = \frac{(3d+63-P_4(d)\,\Omega)}{(3d+45+P_5(d)\,\Omega)} \tag{A2}$$

where one defines $T = P_6(d) + 3\sqrt{3}\sqrt{\Delta}$ and $\Delta = -P_1(d)^2 P_7(d)^2 P_8(d)$ with $P_1(d), \dots, P_8(d)$ given by:

$$\begin{aligned}
P_1(d) =&80 + 28d - 297d^2 - 35d^3 - d^4\\
P_2(d) =&146 - 30d + 18d^2 + d^3\\
P_3(d) =&372736 - 129312d - 18960d^2 - 25493d^3 + 3042d^4 + 474d^5 + 13d^6\\
P_4(d) =&22 + 22d + d^2\\
P_5(d) =&94 + 25d + d^2\\
P_6(d) =&219152384 - 124941312d + 53895600d^2 + 14104092d^3 - 103622787d^4 - 34822215d^5 -\\
&3884853d^6 - 193509d^7 - 4365d^8 - 35d^9\\
P_7(d) =&2304 + 971d + 83d^2 + 2d^3\\
P_8(d) =&4096 - 5376d + 3765d^2 - 1981d^3 + 9d^4
\end{aligned} \tag{A3}$$

For the fixed point parametrized by $\Omega[x_0, y_0, d]$, the critical dimension d_c above which the transition is of second order is determined by finding the value of d that cancels the imaginary part of the fixed point. Here, it can be seen that it is directly determined by the sign of Δ, hence the root of the polynomial $P_8(d)$. One obtains $d_c \simeq 218.20$.

At two-loop order, the four fixed points are given by:

$$\begin{aligned}
\lambda_2 =& \frac{1}{A(d)}\Big[(485 + 829d + 96d^2)\lambda_1^4 + (4444 + 4398d + 728d^2)\lambda_1^3\mu_1 + (11038 + 4768d + 1168d^2 + 36d^3)\lambda_1^2\mu_1^2 +\\
&(2940 + 4378d + 548d^2 + 24d^3)\lambda_1\mu_1^3 + (1045 + 603d + 88d^2 + 4d^3)\mu_1^4 -\\
&32\pi^2\left((52 + 69d)\lambda_1^3 + (541 - 16d + 54d^2)\lambda_1^2\mu_1 - (110 - 281d - 36d^2)\lambda_1\mu_1^2 - (95 - 112d - 6d^2)\mu_1^3\right)\Big]\\
\mu_2 =& \frac{-1}{A(d)}\Big[(163 + 380d + 192d^2)\lambda_1^4 + (1520 + 2493d + 1072d^2)\lambda_1^3\mu_1 + (4350 + 4269d + 966d^2)\lambda_1^2\mu_1^2 +\\
&(3064 + 2715d - 154d^2)\lambda_1\mu_1^3 + (2375 - 353d - 192d^2)\mu_1^4 -\\
&16\pi^2\left((55 + 96d)\lambda_1^3 + (289 + 470d)\lambda_1^2\mu_1 + (421 + 146d)\lambda_1\mu_1^2 + (475 - 212d)\mu_1^3\right)\Big]
\end{aligned} \tag{A4}$$

where $A(d)$ is defined by:

$$A(d) = 2304\pi^2(768\pi^4 + (6+9d)\lambda_1^2 + 2P_4(d)\lambda_1\mu_1 + P_5(d)\mu_1^2 - 32\pi^2((6+3\,d)\lambda_1 + (19+2\,d)\mu_1)). \tag{A5}$$

Using these coordinates, the critical value of d_c at the ϵ-order can be computed. This is done by expanding d_c into $d_{c0} + \epsilon\, d_{c1}$ and requiring that d_{c1} takes the value under which the coordinates of the stable fixed point develop an imaginary part. One finds:

$$d_c(\epsilon) = 218.20 - 448.25\,\epsilon + \mathcal{O}(\epsilon^2) \tag{A6}$$

[1] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D **79**, 064036 (2009).
[2] M.A. Luty, M. Porrati and R. Rattazzi, JHEP **09**, 029 (2003).
[3] K. Hinterbichler, M. Trodden, and D. Wesley, Phys. Rev. D **82**, 124018 (2010).
[4] G. Goon, K. Hinterbichler, A. Joyce and M. Trodden, J. High Energ. Phys. **11**, 100 (2016).
[5] A. Codello, N.Tetradis, and O. Zanusso, JHEP **2013**, 36 (2013).
[6] P. Horava, Phys. Rev. D **79**, 084008 (2009).
[7] T. Fujimori, T. Inami, K. Izumi, and T. Kitamura, Phys. Rev. D **91**, 125007 (2015).
[8] T. Griffin, K. T. Grosvenor, P. Horava, and Z. Yan, Phys. Rev. D **88**, 101701(R) (2013).
[9] K. Hinterbichler and A. Joyce, Int. J. Mod. Phys. **23**, 1443001 (2014).
[10] T. Griffin, K. T. Grosvenor, P. Horava, and Z. Yan, Commun. Math. Phys. **985**, 101701(R) (2015).
[11] C. Laporte, A. D. Pereira, F. Saueressig and J. Wang, JHEP **12**, 001 (2021).
[12] C. Laporte, N. Locht, A. D. Pereira and F. Saueressig, Phys. Lett. B **838**, 137666 (2023).
[13] D. Traykova, E. Bellini, P.G. Ferreira, C. García-García, J. Noller and Miguel Zumalacárregui, Phys. Rev. D **104**, 083502 (2021).
[14] A. Eichhorn, R.R. Lino dos Santos and F. Wagner, JCAP **02**, 052 (2023).
[15] P. Brax and J. Martin, Phys. Rev. D **72**, 023518 (2005).
[16] B. Finelli, G. Goon, E. Pajer and L. Santoni, JCAP **2018**, 060 (2018).
[17] J. Bonifacio, K. Hinterbichler, A. Joyce, R. A. Rosen, J. High Energ. Phys. **1902**, 178 (2019).
[18] D. R. Nelson, T. Piran, and S. Weinberg, eds., *Proceedings of the Fifth Jerusalem Winter School for Theoretical Physics* (World Scientific, Singapore, 2004), 2nd ed.
[19] M. J. Bowick and A. Travesset, Phys. Rep. **344**, 255 (2001).
[20] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V.Dubonos, I. V. Gregorieva, and A. A. Firsov, Science **306**, 666 (2004).
[21] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Gregorieva, S. V. Dubonos, and A. A. Firsov, Nature (London) **438**, 197 (2005).
[22] M. I. Katsnelson, *Graphene: Carbon in Two Dimensions* (Cambridge University Press, Cambridge, U.K., 2012).
[23] M. Paczuski, M. Kardar, and D. R. Nelson, Phys. Rev. Lett. **60**, 2638 (1988).
[24] O. Zanusso, Phys. Rev. E **90**, 052110 (2014).
[25] O. Coquand, Phys. Rev B **100**, 125406 (2019).
[26] F. David and E. Guitter, Europhys. Lett. **5**, 709 (1988).
[27] J. A. Aronovitz, L. Golubovic, and T. C. Lubensky, J. Phys. (Paris) **50**, 609 (1989).
[28] P. L. Doussal and L. Radzihovsky, Phys. Rev. Lett. **69**, 1209 (1992).
[29] P. Le Doussal and L. Radzihovsky, Ann. Phys. (Amsterdam) **392**, 340 (2018).
[30] J.-P. Kownacki and D. Mouhanna, Phys. Rev. E **79**, 040101(R) (2009).
[31] F. L. Braghin and N. Hasselmann, Phys. Rev. B **82**, 035407 (2010).
[32] N. Hasselmann and F. L. Braghin, Phys. Rev. E **83**, 031137 (2011).
[33] K. Essafi, J.-P. Kownacki and D. Mouhanna, Phys. Rev. Lett. **106**, 128102 (2011).
[34] K. Essafi, J.-P. Kownacki, and D. Mouhanna, Phys. Rev. E **89**, 042101 (2014).
[35] O. Coquand, K. Essafi, J.-P. Kownacki, and D. Mouhanna, Phys. Rev E **97**, 030102(R) (2018).
[36] D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).
[37] J. A. Aronovitz and T. C. Lubensky, Phys. Rev. Lett. **60**, 2634 (1988).
[38] E. Guitter, F. David, S. Leibler, and L. Peliti, J. Phys. (Paris) **50**, 1787 (1989).
[39] I. V. Gornyi, V. Yu. Kachorovskii and A. D. Mirlin, Phys. Rev. B **92**, 155428 (2015).
[40] D. Gazit, Phys. Rev. E **80**, 041117 (2009).
[41] K. V. Zakharchenko, R. Roldán, A. Fasolino and M. I. Katsnelson, Phys. Rev. B **82**, 125435 (2010).
[42] R. Roldán, A. Fasolino, K. V. Zakharchenko, and M. I. Katsnelson, Phys. Rev. B **83**, 174104 (2011).
[43] O. Coquand, D. Mouhanna, and S. Teber, Phys. Rev. E **101**, 062104 (2020).
[44] A. Mauri and M.I. Katsnelson, Nucl. Phys. B **956**, 115040 (2020).
[45] S. Metayer, D. Mouhanna, and S. Teber, Phys. Rev. E **105**, L012603 (2022).
[46] Pikelner and Andrey, EPL **138**, 17002 (2022), 2112.07340.
[47] S. Metayer, D. Mouhanna and S. Teber, J. Phys. Conf. Ser. 2438 **1**, 012141 (2023).
[48] O. Coquand and D. Mouhanna, Phys. Rev. E **103**, 031001 (2021).
[49] S. Metayer and D. Mouhanna, Phys. Rev. E **106**, 064114 (2022).
[50] E. Guitter, F. David, S. Leibler, and L. Peliti, Phys. Rev. Lett. **61**, 2949 (1988).
[51] D. R. Saykin, I. V. Gornyi, V. Yu. Kachorovskii and I. S. Burmistrov, Ann. Phys. (N.Y.) **414**, 168108 (2020).
[52] Y. Nakayama, Phys. rept. **569**, 1 (2015).
[53] A. B. Zamolodchikov, Yad. Fiz. **44**, 821 (1986).
[54] J. Polchinski, Nucl. Phys. B **303**, 226 (1986).
[55] I. Jack and H. Osborn, Nucl. Phys. B **343**, 647 (1990).
[56] V. Riva and J. Cardy, Phys. Lett. B **622**, 339 (2005).
[57] S. El-Showk, Y. Nakayama, and S. Rychkov, Nucl. Phys. B **848**, 578 (2011).
[58] M. Safari, A. Stergiou, G. P. Vacca, and O. Zanusso, JHEP **2202**, 034 (2022).
[59] A. Tseytlin, arXiv:2212.1059 (2022).

[60] A. Mauri and M.I. Katsnelson, Nucl. Phys. B **969**, 115482 (2021).
[61] L. Delzescaux and D. Mouhanna, unpublished (2023).
[62] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, N. Wschebor, Physics Reports **1**, 910 (2021).
[63] M. Paczuski and M. Kardar, Phys. Rev. A **39**, 6086 (1989).
[64] R. N. Lee (2012), arXiv:1212.2685.
[65] R. N. Lee, J. Phys. Conf. Ser. **523**, 012059 (2014).
[66] L. Delzescaux, S. Metayer and D. Mouhanna, unpublished (2023).
[67] L. D. Faddeev and V. N. Popov, Phys. Lett. B **25**, 29 (1967).
[68] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[69] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973).
[70] H. K. Janssen, Z Physik B **23**, 377 (1976).
[71] C. De Dominicis, J. Phys., Colloq. **37**, 247 (1976).
[72] J.-P. Kownacki and H. Diep, Phys. Rev. E **66**, 066105 (2002).
[73] H. Koibuchi and N. Kusano and A. Nidaira and K. Suzuki and M. Yamada, Phys. Rev. E **69**, 066139 (2004).
[74] R. Cuerno, R. Gallardo Caballero, A. Gordillo-Guerrero, P. Monroy and J.J. Ruiz-Lorenzo, Phys. Rev. E **93**, 022111 (2016).
[75] B. Delamotte, D. Mouhanna, and M. Tissier, Phys. Rev. B **69**, 134413 (2004).
[76] P. Arnold and L. G. Yaffe, Phys. Rev. D **49**, 3003 (1994), err. ibid **55** 1114 (1997).
[77] G. Gompper and D. M. Kroll, J. Phys: Condens. Matter **9**, 8795 (1997).
[78] A. Codello and O. Zanusso, Phys. Rev. D **83**, 125021 (2011).
[79] Within the nonperturbative RG approach there is generally no recourse to expansion in powers of the usual parameters like coupling constants, temperature, inverse of the number of components. However, the action is often expanded both in powers of the field(s) and derivatives of the field(s).
[80] One omits the always relevant direction associated with the temperature.