Passive odd viscoelasticity

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Active chiral viscoelastic materials exhibit elastic responses perpendicular to the applied stresses, referred to as odd elasticity. We use a covariant formulation of viscoelasticity combined with an entropy production analysis to show that odd elasticity is not only present in active systems but also in broad classes of passive chiral viscoelastic fluids. In addition, we demonstrate that linear viscoelastic chiral solids do require activity in order to manifest odd elastic responses. In order to model the phenomenon of passive odd viscoelasticity we propose a chiral extension of Jeffreys model. We apply our covariant formalism in order to derive the dispersion relations of hydrodynamic modes and obtain clear imprints of odd viscoelastic behavior.

I. INTRODUCTION

Mechanical responses to the applied external stresses are key in the understanding of materials. Such responses can take a simple elastic form for materials, whose microscopic composition consists of a lattice of atoms, or a more complex viscoelastic behavior for substances with a compound mesoscopic structure. Such media include polymers [1], metamaterials [2, 3] and biomatter [4, 5]. Viscoelastic behavior manifests itself through energy dissipation when external stress is applied and then removed. This can be understood as a symptom of a nonvanishing viscosity and thus a fluid-like characterization of a viscoelastic material. As a consequence, a proper description of viscoelastic responses requires a combination of elastic and viscous components in the defining constitutive laws. In general, this is a complicated task with many unknown variables, which requires a phenomenological treatment. The non-equilibrium, dissipative nature of viscoelasticity suggests that it is a transient phenomenon to an equilibrium state that can be either a solid or a fluid. As a result we can consider two distinct classes of materials: viscoelastic solids and viscoelastic fluids. If one assumes that the relation between stress and strain is linear it is possible to write down constitutive relations, tor these two classes, parameterized by response coefficients.

In its simplest incarnation, a linear model for solids is known as the Kelvin-Voigt model and for fluids as the Maxwell model. They are usually visualized in one dimension in terms of connected springs and dashpots. The Kelvin-Voigt solid is modelled by a spring and a damper connected in parallel and the Maxwell fluid is modelled by a spring and a damper connected in series. Although these models are at the core of rheological descriptions of viscoelasticity they are often not capable of capturing experimental stress-strain responses in more complex materials. The Maxwell model does not describe a progressive deformation of a material under constant stress (creep), and the Kelvin-Voigt model does not describe stress relaxation. One way to improve this is to account for higher-order relaxations in the constitutive relations. This can be visualized by extending our basic Kelvin-Voigt and Maxwell models by additional springs and dashpots. Viscoelastic models that consist of two springs and one dashpot are called standard linear solids or Zener models [1] and models built from two dashpots and one spring are called standard linear fluids or Jeffreys models [6]. One-dimensional models can be generalized to higher dimensions, in which the responses are controlled by tensors of coefficients that respect symmetries of a given system. Therefore for a class of materials that break chirality new parity-breaking transport coefficients have to be taken into account even for the simple Kelvin-Voigt and Maxwell models [7].

Chirality is an asymmetry under mirror imaging. This asymmetry plays an important role in various biological systems [8–13], metamaterials [3, 14], condensed matter [15, 16] and high-energy physics [17]. An effective hydrodynamic theory of systems with broken chiral symmetry in two dimensions is distinguished by a new transport coefficient called Hall viscosity [18–28]. Hall viscosity is a non-dissipative transport coefficient that can appear in passive systems, i.e. systems that obey the laws of thermodynamics. However, it has been shown that in active systems an analogous elastic coefficient that breaks mechanical reciprocity and requires activity can be present [29]. Moreover, in the context of active viscoelastic media, modelled by either the Kelvin-Voigt model or the Maxwell model, it was recently argued that such a breaking will be controlled by an interplay of Hall viscosity and an odd elastic coefficient [7]. Imprints of odd viscoelas-

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ticity were recently investigated in an experiment with starfish embryos [30]. The aim of the present work is to investigate viscoelasticity of chiral systems beyond the simplest Kelvin-Voigt and Maxwell models. The main result of our analysis is a phenomenon of passive odd viscoelasticity. We show under which conditions activity is necessary for a non-zero effect and elucidate viscoelastic fluids, in which odd elasticity does not necessarily break mechanical reciprocity. As a result it should be present in chiral passive systems.

Our study is based on irreversible thermodynamics [31–33]. In this setting, the existence of an entropy current whose divergence is enforced to be positive semi-definite embodies a local version of the second law of thermodynamics. This imposes non-trivial constraints on the various transport coefficients. This approach can be incorporated in the study of viscoelastic fluids [34–36] but traditional studies are based on broken symmetry variables (i.e. Goldstone fields of spontaneous broken translations) and only describe Kelvin-Voigt-type models [36]. In order to embed more complex rheology models, such as Maxwell and Jeffreys model, into the framework of viscoleastic hydrodynamics, we follow the approach of [37, 38], in which metric degrees of freedom that describe the evolution of the lattice structure of the material are taken into account. In [37, 38] the case of parity-even systems in three dimensions was studied in detail, in the relativistic context. In this paper, we generalize this approach to chiral systems in two dimensions and extend it to systems with Galilean symmetry. We explicitly demonstrate that parity-odd transport, under appropriate entropic restrictions, can be incorporated into a passive chiral version of Jeffreys model.

This paper is organised as follows. In sec. II we give a brief overview of our main results and introduce the passive chiral Jeffreys model. In sec. III we study this model in detail, in particular the conservation laws and the propagation of shear elastic waves. In sec. IV we perform a technical analysis of the entropy current and derive the constraints on transport coefficients, constitutive relations and rheology equations from the second law of thermodynamics. In sec. V we discuss some implications of our results and future research avenues. We also provide app. A in which we show how the chiral models we look at can be obtained from material diagrams following the laws of electric circuits.

II. SUMMARY OF RESULTS

The purpose of this paper is to show that, taking into account entropy constraints, parity-breaking elastic effects can be incorporated in to an odd extension of Jeffreys model [39–42]. The goal of this section is to briefly outline some of the details of this model and the consequences of our analysis.

Typically, viscoelastic models are defined via phenomenological equations specifying the time evolution of stresses τ_{ij} in terms of the time evolution of strains $\mathcal{E}_{ij}$, where $i, j = 1, 2$ are spatial cartesian indices. In order to describe both even and odd contributions it is useful to define the following tensors

$$\eta_{ijkl} = \frac{1}{2}\delta_{k(i}\delta_{j)l} - \frac{1}{2}\delta_{ij}\delta_{kl} \quad , \tag{II1}$$

$$\eta^*_{ijkl} = -\frac{1}{4}(\epsilon_{ik}\delta_{jl} + \epsilon_{il}\delta_{jk} + \epsilon_{jk}\delta_{il} + \epsilon_{jl}\delta_{ik}) \quad , \tag{II2}$$

where δ_{kl} is the Kronecker delta and ϵ_{kl} is the Levi-Civita tensor. Eq. II1 introduces the traceless projector while eq. II2 is a parity-odd tensor that enables the description of both odd elasticity and odd viscosity in two dimensions. In eq. II2 and in the rest of this paper we use the upperscript $*$ to denote parity-odd contributions.

In order to characterise the models that we will soon introduce, it is useful to decompose the non-equilibrium stress into a bulk stress τ and a shear $\tau_{\langle ij\rangle}$ part. In particular the stress is given by $\tau_{ij} = \tau\delta_{ij} + \tau_{\langle ij\rangle}$, where we have introduced the short-hand notation $A_{\langle ij\rangle} = \eta_{ij}{}^{kl}A_{kl}$ and $A = A_k{}^k$, holding for any two-tensor A_{ij}. As in traditional textbooks, it is possible to introduce the phenomenological model of interest, which we refer to as the *odd Jeffreys model*, by means of material diagrams (one for the bulk sector and one for the shear sector). The bulk sector diagram is depicted in fig. 1 and the model's

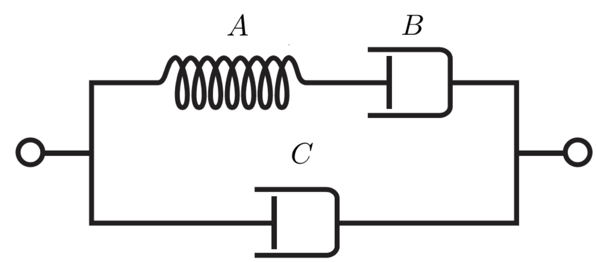

FIG. 1. Diagram corresponding to the full bulk sector given in section IV D.

characteristic equation is given by

$$\dot{\tau} + \sigma\tau = \alpha\ddot{\mathcal{E}} + \tilde{\beta}\dot{\mathcal{E}} \quad , \tag{II3}$$

where we have introduced the notation $\frac{d}{dt}A_{ij} = \dot{A}_{ij}$ for any two-tensor A_{ij}. The coefficients in eq. II3 are functions of A, B, and C introduced in fig. 1. We refer the reader to the app. A for the explicit formulae. We note that the bulk sector as determined by eq. II3 does not contain parity-odd terms. However, the shear sector, for which the material diagram is depicted in fig. 2 is given by the characteristic equation

$$\begin{aligned}&\dot{\tau}_{\langle ij\rangle} + (\chi\eta_{ijl}{}^{kl} + \chi^*\eta^*_{ij}{}^{kl})\tau_{kl} \\ &= (\gamma\eta_{ij}{}^{kl} + \gamma^*\eta^*_{ij}{}^{kl})\ddot{\mathcal{E}}_{kl} + (\zeta\eta_{ij}{}^{kl} + \zeta^*\eta^*_{ij}{}^{kl})\dot{\mathcal{E}}_{kl} \quad ,\end{aligned} \tag{II4}$$

contains several parity-odd contributions. These contributions, as we will show, leave a clear imprint on the

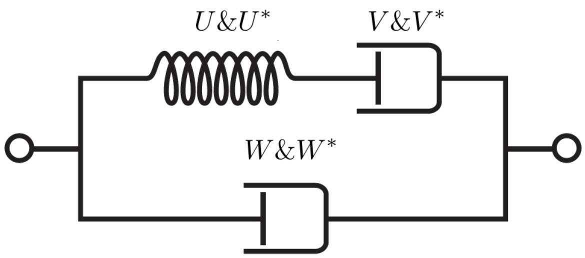

FIG. 2. Diagram corresponding to the shear sector. The "&" refers to a parallel connection of an odd and an even component of the same type.

propagation of elastic shear of waves by splitting the spectra of excitations and by e.g. modifying the speed of wave propagation. Analogous to (II3), all transport coefficients in the shear sector are functions of V, V^*, U, U^*, W, W^* introduced in fig. 2 (see app. A for details).

The model equations (II3) and (III12) can be obtained from the diagrammatic representations presented in figs. 1 and 2 following analogous rules to those used in the context of electric circuits. In this case the bulk coefficients $\sigma, \alpha, \tilde{\beta}$ in (II3) and the shear coefficients $\chi, \chi^*, \gamma, \gamma^*, \zeta, \zeta^*$ are arbitrary. However, the main goal of this paper is to embed these models into a formal hydrodynamic framework of viscoelastic fluids. Within this setup, requiring the second law of thermodynamics to be satisfied leads to nontrivial constraints and relations among the various coefficients.

The fact that the odd Jeffreys model allows us to incorporate odd elastic terms in the sense of [7, 29] while still satisfying the second law is a non-trivial result. In fact, when requiring entropy positivity for the two models studied in [7, 29], in particular the *odd Kelvin-Voigt model* given by the model equation

$$\tau_{\langle ij\rangle} = (\phi\eta_{ij}{}^{kl} + \phi^*\eta^*_{ij}{}^{kl})\mathcal{E}_{kl} + (\psi\eta_{ij}{}^{kl} + \psi^*\eta^*_{ij}{}^{kl})\dot{\mathcal{E}}_{kl} \quad , \tag{II5}$$

and the *odd Maxwell model*, which is equivalent to the shear sector of Jeffreys model (III12) but with γ and γ^* set to zero

$$\dot{\tau}_{\langle ij\rangle} + (\chi\eta_{ijl}{}^{kl} + \chi^*\eta^*_{ij}{}^{kl})\tau_{kl} = (\zeta\eta_{ij}{}^{kl} + \zeta^*\eta^*_{ij}{}^{kl})\dot{\mathcal{E}}_{kl} \quad , \tag{II6}$$

the parity-odd elastic terms are not entropically allowed.[1] Specifically, the coefficients ϕ^* of the odd Kelvin-Voigt model and ζ^* of the odd Maxwell model are required to vanish, leaving behind only non-zero parity-odd contributions due to odd viscosity. Therefore, the two simpler models introduced in [7, 29] can only describe active systems and not passive ones.

[1] We provide diagrammatic constructions of these models in app. A.

III. THE ODD JEFFREYS MODEL

In this section we elucidate the necessary details of the odd Jeffreys model discussed in the previous section. In particular, we embed the model in a simple hydrodynamic framework and give spatially covariant expressions for the characteristic equations and conservation laws. We also introduce a useful definition of strain and study the shear elastic wave spectra.

A. Strain

In order to provide a spatially covariant formulation of the model in the previous section we begin by introducing a covariant definition of the strain $\mathcal{E}_{ij}$. To this aim we consider a fluid particle at point ξ^i and time t in fluid/material space and a set of scalar fields $X^a(\xi^i, t)$ that map the point ξ^i at time t to a point in the physical space [43]. We focus on flat spaces endowed with the Euclidean metric δ_{ab} where $a, b, c, ...$ are spatial indices in the physical space. With this set of scalar fields and the background metric, we introduce an induced metric that describes the distances between fluid particles as they move in time, which can vary due to applied stress

$$g_{ij} = \delta_{ab}\, \partial_i X^a(\xi^k, t)\partial_j X^b(\xi^k, t) \quad . \tag{III1}$$

The strain is then defined as the difference between this metric and a reference metric $g^{(0)}_{ij}$ that models the distances between fluid particles when the fluid is at rest, specifically

$$\mathcal{E}_{ij} = \frac{1}{2}(g_{ij} - g^{(0)}_{ij}) \quad . \tag{III2}$$

In order to track the evolution of the strain we define a covariant derivative of the form

$$\frac{D}{Dt}A = \dot{A} + \mathcal{L}_N A \quad , \tag{III3}$$

for any tensor A, where we have omitted the indices. Here $\mathcal{L}_N$ is the Lie derivative along the vector N^i describing the movement of the fluid particle with respect to the coordinate (frame) choice in fluid space. For a change in time δt the particle's coordinate ξ^i moves with respect to the frame choice, i.e the coordinate of a particle initially located at ξ^i changes to $\xi^i(t) + N^i(\xi^k, t)\delta t$ in a time step δt (see figure 3).

Given the definition (III3) we require the reference metric introduced in (III2) to be covariantly conserved

$$\frac{D}{Dt}g^{(0)}_{ij} = 0 \quad . \tag{III4}$$

We thus view the reference metric just as an auxiliary metric that allows us to compare deformations at a given time t with the original state of the material while simultaneously making a straightforward connection with the

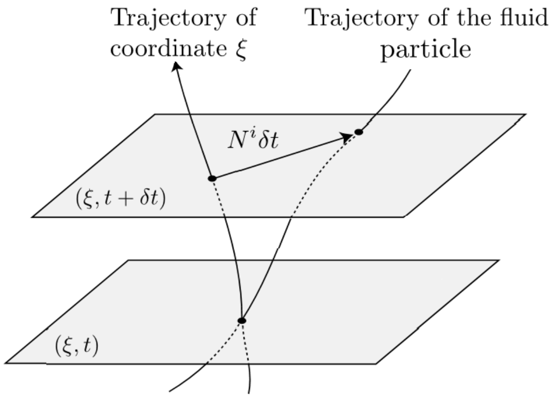

FIG. 3. The vector N^i describes the motion of the fluid particles with respect to the coordinate frame.

notion of strain commonly used in standard textbooks. Using (III4) we obtain the following identity for the strain rate

$$K_{ij} \equiv \frac{D}{Dt}\mathcal{E}_{ij} = \frac{1}{2}\frac{D}{Dt}g_{ij} \quad . \tag{III5}$$

We will use these definitions to recast the model equations in a covariant form.

B. Covariant model equations

A description of the motion of a fluid can be obtained by attaching a set of coordinate labels X^a to each element of the fluid. The freedom in the choice of these labels corresponds to different lab frames [44]. The introduction of the vector N^i in (III3) allows for frame choices that simplify our analysis. As we are working with infinitesimal strains, one possible choice, known as the lab frame, is given by $X^a = X^a_{(0)} + w^a$ with $\dot{X}^a = \dot{w}^a = \delta u^a$ as well as $N^i = u^i_{(0)}$. Here, $X^a_{(0)}$ is defined such that $g_{ij}|_{X^a = X^a_{(0)}} = g^{(0)}_{ij}$ and $u^i = u^i_{(0)} + \delta u^i$ is the fluid velocity, where $u^i_{(0)}$ denotes the velocity of the fluid in its initial state which obeys $\nabla_i u^j_{(0)} = 0$ such that eq. III4 is satisfied. In turn, w^a is a small displacement vector and $u^a = u^i\partial_i X^a$.[2] This leads to

$$K_{ij} = \frac{1}{2}\nabla_{(i}\delta u_{j)} = \frac{1}{2}\nabla_{(i}u_{j)} \quad . \tag{III6}$$

The covariant derivative introduced in (III6) is defined in the usual way

$$\begin{aligned} \nabla_i A_j^{\ k} &= \partial_i A_j^{\ k} - \Gamma^l_{ij}A_l^{\ k} + \Gamma^k_{il}A_j^{\ l} \quad , \\ \Gamma^l_{ij} &= \frac{1}{2}g^{lk}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \quad , \end{aligned} \tag{III7}$$

[2] Alternatively, one can work with the comoving frame in which $N^i = 0$ and $\dot{X}^a = u^a$. These two choices correspond to Eulerian and Lagrangian formulations of fluid dynamics.

for any tensor $A_j^{\ k}$ where Γ^l_{ij} is the Christoffel connection build from the induced metric g_{ij}. With these structures in hand, we now covariantize the definitions of section II, in particular

$$\eta_{ijkl} = \frac{1}{2}g_{k(i}g_{j)l} - \frac{1}{2}g_{ij}g_{kl} \quad , \tag{III8}$$

$$\eta^*_{ijkl} = -\frac{1}{4}(\varepsilon_{ik}g_{jl} + \varepsilon_{il}g_{jk} + \varepsilon_{jk}g_{il} + \varepsilon_{jl}g_{ik}) \quad , \tag{III9}$$

$$A_{\langle ij\rangle} = \eta_{ij}^{\ \ kl}A_{kl} \ , \ A = g^{ij}A_{kl} \quad . \tag{III10}$$

It is then straightforward to obtain the covariant form of the bulk (II3) and shear (III12) equations for the odd Jeffreys model which now take the form

$$\frac{D}{Dt}\tau + \sigma\tau = \alpha\frac{D}{Dt}K + \tilde{\beta}K \quad , \tag{III11}$$

$$\begin{aligned} &\frac{D}{Dt}\tau_{\langle ij\rangle} + (\chi\eta_{ijl}^{\ \ \ kl} + \chi^*\eta^{*\ kl}_{ij})\tau_{kl} \\ &= (\gamma\eta_{ij}^{\ \ kl} + \gamma^*\eta^{*\ kl}_{ij})\frac{D}{Dt}K_{kl} + (\zeta\eta_{ij}^{\ \ kl} + \zeta^*\eta^{*\ kl}_{ij})K_{kl} \quad . \end{aligned} \tag{III12}$$

C. Conservation laws and the second law of thermodynamics

The simplest way to embed Jeffreys model in a framework of viscoelasticity is to treat the stress τ_{ij} as dynamical degrees whose evolution is determined by (III11)-(III12). Furthermore, one supplements the system with additional conservation laws that determine the evolution of hydrodynamic fields. In particular, the evolution of the fluid velocity u^i introduced in (III6), the temperature T and the chemical potential μ associated with particle number are respectively determined by

$$\rho\dot{u}^i + \rho u^j\nabla_j u^i + \nabla_j t^{ij} = 0 \quad , \tag{III13}$$

$$\dot{\epsilon} + \nabla_j(\epsilon u^j) + \nabla_j(j^j_\epsilon + u_i t^{ij}) = 0 \quad , \tag{III14}$$

$$\dot{\rho} + \nabla_j(\rho u^j) = 0 \quad . \tag{III15}$$

Here ϵ and ρ are the energy and mass density respectively, which are functions of T, μ. We note that one can find different sign conventions for the stress in the literature. We furthermore defined

$$t^{ij} = pg^{ij} + \tau^{ij} \ , \ \epsilon = \epsilon_0 + \frac{1}{2}\rho u^2 \quad , \tag{III16}$$

where p is the equilibrium pressure, ϵ_0 is the part of the energy density that excludes the kinetic energy and $u^2 = u_i u^i$. Both p and ϵ_0 are functions of T and μ. In addition, we introduced the heat current j^j_ϵ, which contains gradient corrections that do not play a role in the analysis that we will carry out in this section. In sec. IV we show how we can explicitly derive the form of j^j_ϵ. The conservation laws presented in (III13)-(III15) are typical of systems with Galilean symmetry.

D. Modes

Given that the hydrodynamic system has now been specified, we will study the elastic shear wave spectra in a simplified context. In particular, we will not consider energy or mass density fluctuations and we will focus only on the shear sector of the model for which the defining equation is given by (III12).

As earlier advertised in sec. II, the coefficients appearing in (III12) are not arbitrary due to constraints arising from the second law of thermodynamics as we will show in the next section. Nevertheless, it is possible to make a choice of coefficients that simplifies the model and is still compatible with the second law. In particular, we take the same limit as in [7], which corresponds to a very slow relaxation of the strain

$$\chi \to 0 \ , \ \chi^* \to 0 \quad . \tag{III17}$$

In addition we set $\zeta^* = 0$ and require $\gamma < 0$ as well as $\zeta < 0$. We now consider equilibrium states with vanishing velocity $u^i_{(0)} = 0$ and vanishing strain (i.e. $g_{ij} = \delta_{ij}$). Using the momentum conservation equation (III13) and acting with a time derivative leads to

$$\begin{aligned}\rho_0 \partial_t^2 \delta u_j &= -\partial_i \partial_t \delta\tau_{ij} \\ &= -\partial_i \left\{ (\gamma {\eta_{ij}}^{kl} + \gamma^* {\eta^*_{ij}}^{kl}) \partial_t K_{kl} + \zeta {\eta_{ij}}^{kl} K_{kl} \right\} \quad .\end{aligned} \tag{III18}$$

Similarly to [7] we contract (III18) with $\partial_i^* \equiv \epsilon_{ij}\partial^j$ and with ∂_i as to obtain a set of four equations. Contracting with ∂_i^* leads to

$$\rho_0 \partial_t^2 \Omega = -\frac{\gamma}{2}\partial_t \partial_i^2 \Omega + \frac{\gamma^*}{2}\partial_t \partial_i^2 \Theta - \frac{\zeta}{2}\partial_i^2 \Omega \quad , \tag{III19}$$

where we have defined $\Theta = \partial_i \delta u^i$ and $\Omega = \epsilon^{ij}\partial_i \delta u_j$. Contracting with ∂_i instead leads to

$$\rho_0 \partial_t^2 \Theta = -\frac{\zeta}{2}\partial_i^2 \Theta - \frac{\gamma}{2}\partial_t \partial_i^2 \Theta - \frac{\gamma^*}{2}\partial_t \partial_i^2 \Omega \quad . \tag{III20}$$

Using eqs. III20 and III19 and considering plane waves of the form $\delta u_i \sim e^{-i\omega t + ikx}$ with frequency ω and wave number k we obtain the characteristic matrix

$$M = \begin{pmatrix} -\frac{1}{2} i\gamma k^2 \omega + \frac{\zeta k^2}{2} + \rho_0 \omega^2 & -\frac{1}{2} i\gamma^* k^2 \omega \\ \frac{1}{2} i\gamma^* k^2 \omega & -\frac{1}{2} i\gamma k^2 \omega + \frac{\zeta k^2}{2} + \rho_0 \omega^2 \end{pmatrix} . \tag{III21}$$

Requiring $\det(M) = 0$ leads to a characteristic equation with four roots given by

$$\omega^{(s,s')} = \frac{k\left(k(s\gamma^* + i\gamma) + s'\sqrt{-8\zeta\rho_0 - k^2(\gamma + is\gamma^*)^2}\right)}{4\rho_0} \ , \tag{III22}$$

where $s, s' \in \{-1, 1\}$. Defining $s\gamma^* \equiv \nu$ eq. (III22) can be written as

$$\omega_\pm = \frac{k\left(k(\nu + i\gamma) \pm \sqrt{-8\zeta\rho_0 - k^2(\gamma + i\nu)^2}\right)}{4\rho_0} \ , \tag{III23}$$

Expanding for small k yields

$$\omega_\pm = \pm \frac{\sqrt{-\zeta}k}{\sqrt{2}\sqrt{\rho_0}} + \frac{ik^2(\gamma + i\nu)}{4\rho_0} + \mathcal{O}(k^3) \quad . \tag{III24}$$

We plot the real and imaginary part of the modes of eq. III23 as a function of $-\zeta$ and ν in fig. 4. From these plots, and already as indicated in (III24), we observe that these modes have always negative imaginary parts and their decay is controlled by the dissipative coefficient γ. This is a consequence of the second law of thermodynamics and is unlike the odd viscoelastic models considered in [7] in the context of active systems, in which modes with positive imaginary part are also present. Another feature of the modes presented in fig. 4 is that the real part of the dispersion relation acquires a correction due to the presence of the odd coefficient γ^*. This correction implies that there are two modes with distinct real parts depending on the value of s. This is a clear imprint of the odd nature of the Jeffreys model we introduced.

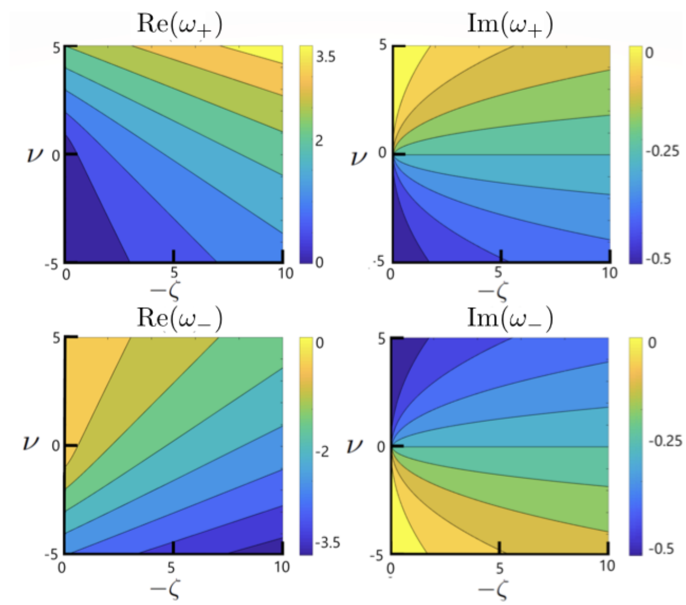

FIG. 4. Color plot of Re($\omega_\pm$) and Im($\omega_\pm$) as a function of $-\zeta$ and ν.

IV. ENTROPY ANALYSIS

In the previous sections we treated the stresses τ_{ij} as dynamical degrees of freedom and stipulated its dynamics by means of phenomenological material diagrams. In this section we derive Jeffreys model from the second law of thermodynamics. In particular, we derive the constitutive relations, rheology equations and entropy constraints that arise from implementing the second law of thermodynamics (IV2) following the framework of [37]. Using these general constitutive relations we demonstrate how to obtain the odd Jeffreys model that we introduced and studied in the previous sections. We also show how other simpler models (Kelvin-Voigt and Maxwell) arise as limiting cases of our general analysis, showing that parity-odd elastic terms are not compatible with entropy constraints, thus forbidding a passive incarnation of these models.

A. Intrinsic metric

Traditional hydrodynamic frameworks that implement the second law of thermodynamics in the context of viscoelastic fluids are based on introducing Goldstone fields of spontaneously broken translation symmetry [34–36]. However, as demonstrated in [36], these formulations only allow for Kelvin-Voigt-type models and not, for instance, Jeffreys model. As such, we work with the formulation of [37, 38, 45] in which metric degrees of freedom are introduced.

Within this framework, a different notion of strain that also describes plastic deformations can be defined by introducing an evolving intrinsic metric $\tilde{g}_{ij}(\xi^k, t)$ [37, 38]. This metric measures the deformation of the distances between fluid particles as they are experienced due to bond configurations of the material. The difference between the induced metric and the intrinsic metric is a form of strain that is distinct from III2, namely

$$\kappa_{ij} = \frac{1}{2}(g_{ij} - \tilde{g}_{ij}) \quad . \tag{IV1}$$

When a material is deformed and a strain is created, this intrinsic metric evolves as to cause (IV1) to vanish. This can be understood as the bond configuration of the material, once the system is fully relaxed, adapting to the induced metric after the applied deformation. This change of the intrinsic metric is called a plastic deformation. The intrinsic metric $\tilde{g}_{ij}$ prior to any deformation is equal to the reference metric $g^{(0)}_{ij}$ introduced in (III2).

B. Thermodynamics and the second law

In this framework, the hydrodynamic fields consist of the fluid velocity u^i, temperature T, chemical potential μ and the intrinsic metric $\tilde{g}_{ij}$. The evolution of u^i, T, μ is again determined by the conservation laws (III13)-(III15) while the dynamics of $\tilde{g}_{ij}$ can be derived by supplementing the system with the second law of thermodynamics

$$\dot{s} + \nabla_j(s u^j) + \nabla_j j^j_s = \Delta \quad , \tag{IV2}$$

where s is the entropy density, $j^j_s = j^j_\epsilon/T$ includes gradient corrections that will be determined[3] and Δ is required to satisfy $\Delta \geq 0$. The entropy density s and the remaining thermodynamic quantities satisfy the usual first law and Euler relations

$$d\epsilon_0 = T ds + \mu d\rho \quad , \quad \epsilon_0 + p = Ts + \rho\mu \quad , \tag{IV3}$$

characteristic of a system with Galilean symmetry. Instead of treating the stresses τ_{ij} as dynamical degrees of freedom, the second law (IV2) allows one to derive constitutive relations for τ_{ij} in terms of u^i, T, μ, κ_{ij} and rheological equations for the evolution of κ_{ij}. When combined, Jeffreys model is obtained as we will show in the next sections.

C. Constraints from entropy production

We now focus on the second law of thermodynamics (IV2) and use it to derive constitutive relations and rheology equations. Using the conservation laws (III13)-(III15) together with the thermodynamic relations (IV3) we explicitly evaluate the left hand side of (IV2) and obtain [46]

$$\Delta = -\tau^{ij}\frac{K_{ij}}{T} - \frac{1}{T^2} j^i_\epsilon \partial_i T - \frac{\lambda_1}{T}\kappa^{\langle ij\rangle}\frac{D}{Dt}\kappa_{\langle ij\rangle} - \frac{\lambda_2}{T}\kappa\frac{D}{Dt}\kappa \ , \tag{IV4}$$

where λ_1 and λ_2 are positive coefficients that correspond to elastic moduli.[4] All terms on the right hand side of eq. IV4 should be understood as being gradient suppressed. In particular, we are assuming the following power counting scheme[5]

$$\kappa_{ij}, K_{ij}, \partial_i T = \mathcal{O}(\partial) \quad , \tag{IV5}$$

which implies that we are treating the strain, gradients of the fluid velocity and temperature to be of the same relative order.

Imposing $\Delta \geq 0$ as required by the second law allows us to determine the constitutive relations for τ_{ij} and j^i_ϵ as well as the form of the rheology equations for the shear $\frac{D}{Dt}\kappa_{\langle ij\rangle}$ and bulk $\frac{D}{Dt}\kappa$ sectors.[6] This analysis can be split

[3] The identity $j^j_s = j^j_\epsilon/T$ follows from a convenient choice of hydrodynamic frame. A consequence of the same choice of hydrodynamic frame is the absence of gradient corrections to the mass conservation equation (III15).

[4] In [46] it was shown that these two coefficients may be related to the F and G coefficients that we introduce below. We have not demonstrated this relationship here and since it does not affect the main results of this work, we treat them as free parameters.

[5] Gradients of the chemical potential $\partial_i\mu$ are also of order $\mathcal{O}(\partial)$. However, they do not explicitly appear in our analysis due to the choice of hydrodynamic frame.

[6] Note that while $\kappa_{ij} \sim \mathcal{O}(\partial)$, terms of the form $\kappa\frac{D}{Dt}\kappa$ are of order $\mathcal{O}(\partial^2)$ [37].

into parity-even and parity-odd contributions as well as in scalar, vector and traceless tensor contributions to the constitutive relations. In the parity-even sector we obtain the following contributions[7]

$$j^i_\epsilon = -\frac{L}{T^2}\partial^i T \quad , \tag{IV6}$$

$$\begin{pmatrix} \tau_{\langle ij\rangle} \\ -\frac{\lambda_1}{T}\frac{D}{Dt}\kappa_{\langle ij\rangle} \end{pmatrix} = \left\{ \begin{pmatrix} 0 & -G \\ G & 0 \end{pmatrix} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \right\} \begin{pmatrix} -\frac{1}{T}K_{\langle ij\rangle} \\ \kappa_{\langle ij\rangle} \end{pmatrix} , \tag{IV7}$$

$$\begin{pmatrix} \tau \\ -\frac{\lambda_2}{T}\frac{D}{Dt}\kappa \end{pmatrix} = \left\{ \begin{pmatrix} 0 & -F \\ F & 0 \end{pmatrix} + \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \right\} \begin{pmatrix} -\frac{K}{T} \\ \kappa \end{pmatrix} , \tag{IV8}$$

where the coefficients $L, G, K_1, K_2, F, M_1, M_2$ are functions of T and μ. In order for the condition $\Delta \geq 0$ to be satisfied, we need to require $L \geq 0$ and the K-matrix (with coefficients K_1, K_2) as well as the M-matrix (with coefficients M_1, M_2) to be positive semi-definite, i.e. all matrix eigenvalues must be positive semi-definite.[8] The other coefficients F, G are arbitrary and denote non-dissipative contributions to the constitutive relations [49]. In turn, the parity-odd contributions take the form

$$j^i_\epsilon = -\frac{\beta}{T^2}\varepsilon^i_{\ j}\partial^j T \quad , \tag{IV9}$$

$$\begin{pmatrix} \tau_{\langle ij\rangle} \\ -\frac{\lambda_1}{T}\frac{D}{Dt}\kappa_{\langle ij\rangle} \end{pmatrix} = \eta^{*\ kl}_{ij} \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{pmatrix} \begin{pmatrix} -\frac{1}{T}K_{kl} \\ \kappa_{kl} \end{pmatrix} \quad , \tag{IV10}$$

where $\beta, \alpha_1, \alpha_2, \alpha_3$ are arbitrary functions of T and μ and characterise non-dissipative transport.

D. Viscoelastic models

Given the constitutive relations and rheology equations derived in sec. IV C we are in a position of deriving the models that we discussed throughout this paper. Using (IV7)-(IV10), we find the constitutive relations

$$\begin{aligned} \tau_{\langle ij\rangle} &= -(K_1\eta_{ij}^{\ \ kl} + \alpha_1\eta^{*\ kl}_{ij})\frac{1}{T}K_{kl} \\ &\quad + (-G\eta_{ij}^{\ \ kl} + \alpha_3\eta^{*\ kl}_{ij})\kappa_{kl} \quad , \\ \tau &= -M_1\frac{1}{T}K + -F\kappa \quad , \end{aligned} \tag{IV11}$$

as well as the rheology equations

$$\begin{aligned} -\frac{\lambda_2}{T}\frac{D}{Dt}\kappa_{\langle ij\rangle} &= -(G\eta_{ij}^{\ \ kl} + \alpha_3\eta^{*\ kl}_{ij})\frac{1}{T}K_{kl} \\ &\quad + (K_2\eta_{ij}^{\ \ kl} + \alpha_2\eta^{*\ kl}_{ij})\kappa_{kl} \quad , \\ -\frac{\lambda_2}{T}\frac{D}{Dt}\kappa &= -F\frac{1}{T}K + M_2\kappa \quad . \end{aligned} \tag{IV12}$$

This rheology equation describes the evolution of the strain (IV1), including the evolution of the intrinsic metric $\tilde{g}_{ij}$. The intrinsic metric is a dynamical degree of freedom which makes the system non-Markovian and therefore difficult to deal with. The most obvious way to simplify the set of equations (IV11)-(IV12) is by taking the elastic limit. This is achieved by requiring the intrinsic metric to satisfy

$$\frac{D}{Dt}\tilde{g}_{ij} = 0 \quad . \tag{IV13}$$

In this context, $\tilde{g}_{ij} = g^{(0)}_{ij}$, where $g^{(0)}_{ij}$ was introduced in (III2), and the notions of strain we introduced in sec. III A coincide, namely $\mathcal{E}_{ij} = \kappa_{ij}$. The condition (IV13) fixes almost all coefficients appearing in IV12. In fact, using eqs. III3 and IV1 we find

$$\begin{aligned} &K_2 = \alpha_3 = \alpha_2 = M_2 = 0 \quad , \\ &G = \lambda_1 \quad , \quad F = \lambda_2 \quad . \end{aligned} \tag{IV14}$$

We note that (IV13) forces the coefficients M_2, K_2 to vanish. Imposing IV14 in the constitutive relations (IV11) leads to

$$\begin{aligned} \tau_{\langle ij\rangle} &= -(K_1\eta_{ij}^{\ \ kl} + \alpha_1\eta^{*\ kl}_{ij})\frac{1}{T}K_{kl} - \lambda_1\eta_{ij}^{\ \ kl}\kappa_{kl} \quad , \\ \tau &= -M_1\frac{1}{T}K - \lambda_2\kappa \quad . \end{aligned} \tag{IV15}$$

Comparing this with the defining equation of the shear sector of the odd Kelvin-Voigt model (II5) we identify

$$\phi = \lambda_1 \quad , \quad \phi^* = 0 \quad , \quad \psi = -K_1 \quad , \quad \psi^* = -\alpha_1 \quad . \tag{IV16}$$

As advertised in sec. II, the entropy constraints require that $\phi^* = 0$ and so no odd elastic contributions to the constitutive relations for the stresses are allowed. The coefficient ψ^* is allowed and characterises odd-viscosity. As noted in [7, 29] a non-zero ϕ^* coefficient can be added to the model in the case of an active system but here we note that this cannot be the case in a passive one.

Having discussed this simpler case in which the condition (IV13) is enforced, we turn to the most general case. It is convenient to begin by focusing on the bulk sector of (IV11) and (IV12) involving τ and κ as this sector does not contain parity-odd contributions. Acting with D/Dt on the bulk sector of eq. IV11 and using the bulk sector of (IV12) leads to

$$\frac{D}{Dt}\tau + \frac{M_2 T}{\lambda_2}\tau = -\frac{M_1}{T}\frac{D}{Dt}K + \frac{-M_1M_2 - F^2}{\lambda_2}K \quad . \tag{IV17}$$

[7] In both the parity-even and parity-odd sectors we have not included non-canonical terms besides the elastic terms. These potential additional terms correspond to hydrostatic transport as discussed in [47] for the Galilean case and more extensively in [33, 48] for the relativistic case.

[8] Time-reversal invariance has been taken into account while defining the K and M matrices [46, 49], leading to the vanishing of the off-diagonal components. We leave a detailed analysis based on Kubo formulae for future research.

This equation corresponds to the bulk sector of Jeffreys model [39–42]. Indeed, comparing eq. (IV17) with eq. (II3) leads to

$$\sigma = \frac{M_2 T}{\lambda_2} \ , \ \tilde{\beta} = \frac{-M_1 M_2 - F^2}{\lambda_2} \ , \ \alpha = -\frac{M_1}{T} \quad . \tag{IV18}$$

We now consider the shear sector of (IV11) and (IV12) which involves parity-odd terms. In order to find an equation of the form (III12) it is useful to make use of the following identities

$$\eta_{ijkl}\eta^{*klmn} = \eta^{*\ mn}_{ij} \quad , \tag{IV19}$$

$$\eta_{ijkl}\eta^{klmn} = \eta_{ij}^{\ \ mn} \quad , \tag{IV20}$$

$$\eta^{*}_{ijkl}\eta^{*klmn} = -\eta_{ij}^{\ \ mn} \quad , \tag{IV21}$$

in order to find the following equation

$$\eta^{ij}_{\ \ mn} = \frac{C_1 \eta^{ij}_{\ \ kl} - C_2 \eta^{*ij}_{\ \ \ kl}}{C_1^2 + C_2^2}(C_1 \eta^{kl}_{\ \ mn} + C_2 \eta^{*kl}_{\ \ \ mn}) \quad , \tag{IV22}$$

which holds for any coefficient C_1 and C_2. With this in mind, we follow the same steps as in the bulk sector, and eventually find

$$\begin{aligned} \frac{D}{Dt}\tau_{\langle ij\rangle} = \Big\{ &- G\eta_{ij}^{\ \ kl} + \alpha_3 \eta^{*\ kl}_{ij} \Big\} \\ &\cdot \Bigg[- \Gamma(\Omega \eta_{kl}^{\ \ mn} + \Omega^* \eta^{*\ mn}_{kl})\frac{1}{T}\frac{D}{Dt}K_{mn} \\ &- \frac{T\Gamma}{\lambda_1}(K_2 \eta_{kl}^{\ \ mn} + \alpha_2 \eta^{*\ mn}_{kl}) \\ &\cdot \Big\{ - G\eta_{mn}^{\ \ op}\tau_{op} - \alpha_3 \eta^{*\ op}_{mn}\tau_{op} \\ &+ (\Omega \eta_{mn}^{\ \ op} + \Omega^* \eta^{*\ op}_{mn})\frac{1}{T}K_{op} \Big\} \\ &+ \frac{T}{\lambda_1}\Big\{ G\eta_{kl}^{\ \ mn} + \alpha_3 \eta^{*\ mn}_{kl} \Big\}\frac{1}{T}K_{mn} \Bigg], \end{aligned} \tag{IV23}$$

where we defined the coefficients

$$\begin{aligned} \Gamma &= \frac{1}{G^2 + \alpha_3^2} \quad , \\ \Omega &= G K_1 + \alpha_3 \alpha_1 \quad , \\ \Omega^* &= \alpha_1 G - K_1 \alpha_3 \quad . \end{aligned} \tag{IV24}$$

Comparing (IV23) with the characteristic equation of the shear sector of Jeffreys model (III12) we identify the following coefficients

$$\begin{aligned} \chi &= \frac{T}{\lambda_1}K_2 \ , \ \chi^* = \frac{T}{\lambda_1}\alpha_2 \quad , \\ \gamma &= -\frac{K_1}{T} \ , \ \gamma^* = -\frac{\alpha_1}{T} \quad , \\ \zeta &= -\frac{1}{\lambda_1}(-\alpha_1\alpha_2 + \alpha_3^2 + G^2 + K_1 K_2) \quad , \\ \zeta^* &= -\frac{1}{\lambda_1}(\alpha_2 K_1 + \alpha_1 K_2) \quad . \end{aligned} \tag{IV25}$$

This is the general form of Jeffreys model compatible with entropy constraints. In sec. III D we studied the modes of a simplified case in which $K_2 = \alpha_2 = 0$ which implies $\chi = \chi^* = \zeta^* = 0$.

Another interesting limiting case of Jeffreys model with coefficients (IV25) is obtained by setting $\alpha_1 = K_1 = 0$. This leads to $\gamma = \gamma^* = \zeta^* = 0$. Comparing this case with the characteristic equation of the odd Maxwell model (II6) leads to

$$\chi = \frac{T}{\lambda_1}K_2 \ , \ \chi^* = \frac{T}{\lambda_1}\alpha_2 \ , \ \zeta = -\frac{1}{\lambda_1}(\alpha_3^2 + G^2) \ . \tag{IV26}$$

Also, as advertised in sec. II the coefficient $\zeta^* = 0$ vanishes for the odd Maxwell model due to entropy constraints.

V. DISCUSSION

We have demonstrated that even though odd elasticity can only exist in active systems, odd viscoelasticity, which contains transient odd elasticity can exist in equilibrium. In this setting, parity-odd elastic terms are analogous to transport coefficients such as Hall viscosity and Hall conductance studied in the context of quantum matter. Our motivation, however, has been driven by the relevance of odd elastic responses in biological systems and metamaterials. In addition, we have shown that such responses leave clear imprints in the linear spectrum of fluctuations. The specific model we discussed - Jeffreys model - is a simple model that respects the second law of thermodynamics and could be realised using metamaterials.

Metamaterials are artificially engineered structures, in which the properties of their constituents can be appropriately designed. One example of metamaterials consists of colloidal suspensions. It has been demonstrated that odd transport coefficients can be probed in a colloidal suspension of rotating particles suspended throughout a substance of larger molecules [26]. Such active suspensions require a constant transfer of angular momentum provided by an external magnetic field, which presents an experimental challenge. Our analysis suggests that, since activity is not necessary to probe odd transport coefficients, a passive colloidal suspension of chiral objects such as granular particles [8] or helical nanoribbons [50] is enough to see imprints of both odd viscosity and odd elasticity.

Parity-odd elastic responses are also relevant for chiral systems in quantum matter and high-energy physics in which the system may exhibit Lorentzian rather than Galilean symmetry. In fact, the method by which we first obtained some of the results presented in this paper was to first consider parity-odd responses in a higher-dimensional relativistic theory and later dimensionally reduce to arrive at a hydrodynamic theory with Galilean symmetry as in [47]. These details will be given in another publication.

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Appendix A: Diagrams

In this appendix we consider a diagrammatic representation of the models that we introduced in section II. This can be achieved by working with diagrams of the type presented in fig. 1 in terms of electric circuits. Within this setting, the stress plays the role of the electric current, i.e. when drawing components in series, the stress going through the components is the same for every component. Similarly, the stress is divided when the components are connected in parallel. Several components can be considered in such circuits but the ones that we focus on are the spring and the dashpot. A spring plays the role akin to a resistance, which gives an amount of stress that is proportional to the strain, and the strain thus plays the role of potential. The dashpot also gives stress, but now it is induced by the time-derivative of the strain.

To begin with we consider the circuit that we introduced in fig. 1 which represents the bulk sector of Jeffreys model. Following the circuit rules the characteristic equations are

$$\begin{aligned} &\tau = \tau_{(1)} + \tau_{(2)} \quad , \quad \mathcal{E} = \mathcal{E}_{(1)} + \mathcal{E}_{(2)} \quad , \\ &\tau_{(1)} = A\mathcal{E}_{(1)} \quad , \quad \tau_{(1)} = BK_{(2)} \quad , \quad \tau_{(2)} = CK \quad , \end{aligned} \tag{A1}$$

where A, B, C are arbitrary coefficients associated with each component in the diagram of fig. 1. We note that while the total stress τ and the total strain $\mathcal{E}$ are physical quantities, the interpretation of the individual stresses $\tau_{(1)}, \tau_{(2)}$ and individual strains $\mathcal{E}_{(1)}, \mathcal{E}_{(2)}$ is not always clear and such quantities should be thought of as auxiliary quantities. Given (A1) we act with D/Dt on $\tau_{(1)}$ and use the remaining identities to find

$$\frac{D}{Dt}\tau + \frac{A}{B}\tau = C\frac{D}{Dt}K + \left(A + \frac{AC}{B}\right)K \quad , \tag{A2}$$

which when compared with eq. III11 we identify

$$\sigma = \frac{A}{B} \quad , \quad \alpha = C \ , \quad \tilde{\beta} = A + \frac{AC}{B} \quad . \tag{A3}$$

Before addressing the shear sector of Jeffreys model, we consider the simpler case of the passive Kelvin-Voigt model represented in fig. 5, from which we can immediately extract the constitutive equation

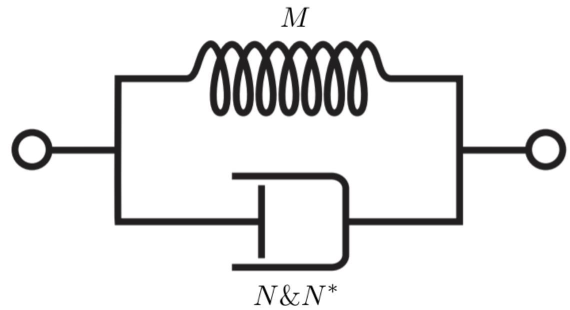

FIG. 5. Material diagram corresponding to the passive odd Kelvin-Voigt model.

$$\tau_{\langle ij\rangle} = (N\eta_{ij}{}^{kl} + N^*\eta^{*}_{ij}{}^{kl})K_{kl} + M\eta_{ij}{}^{kl}\mathcal{E}_{kl} \quad . \tag{A4}$$

Comparing this constitutive equation with eq. (II5) we identify

$$\psi = N \quad , \quad \psi^* = N^* \quad , \quad \phi = M \quad , \quad \phi^* = 0 \quad . \tag{A5}$$

Moving on to the shear sector of Jeffreys model represented in fig. 2, the circuit equations take the more intricate form

$$\begin{aligned} \mathcal{E}_{\langle ij\rangle} &= \mathcal{E}^{(1)}_{\langle ij\rangle} + \mathcal{E}^{(2)}_{\langle ij\rangle} \quad , \\ \tau_{\langle ij\rangle} &= \tau^{(1)}_{\langle ij\rangle} + \tau^{(2)}_{\langle ij\rangle} \quad , \\ \tau^{(1)}_{\langle ij\rangle} &= (U\eta_{ij}{}^{kl} + U^*\eta^{*}_{ij}{}^{kl})\mathcal{E}^{(1)}_{kl} \quad , \\ \tau^{(1)}_{\langle ij\rangle} &= (V\eta_{ij}{}^{kl} + V^*\eta^{*}_{ij}{}^{kl})K^{(2)}_{kl} \quad , \\ \tau^{(2)}_{\langle ij\rangle} &= (W\eta_{ij}{}^{kl} + W^*\eta^{*}_{ij}{}^{kl})K_{kl} \quad . \end{aligned} \tag{A6}$$

As in the bulk sector, we can manipulate (A6) in order to find the rheology equation

$$\begin{aligned} &\frac{D}{Dt}\tau_{\langle ij\rangle} = (W\eta_{ij}{}^{kl} + W^*\eta^{*}_{ij}{}^{kl})\frac{D}{Dt}K_{kl} \\ &+ (U\eta_{ij}{}^{kl} + U^*\eta^{*}_{ij}{}^{kl})\big[K_{kl} - \Theta(V\eta_{kl}{}^{mn} - V^*\eta^{*}_{kl}{}^{mn}) \\ &\cdot (\tau_{mn} - W\eta_{mn}{}^{op}K_{op} + W^*\eta^{*}_{mn}{}^{op}K_{op})\big] \quad , \end{aligned} \tag{A7}$$

where, for convenience, we have defined

$$\Theta = \frac{1}{V^2 + V^{*2}} \quad . \tag{A8}$$

Comparing eq. (A7) with (III12) one readily identifies

$$\begin{aligned} &\chi = \Theta(UV + U^*V^*) \ , \ \chi^* = \Theta(-UV^* + U^*V) \quad , \\ &\gamma = W \ , \ \gamma^* = W^* \quad , \\ &\zeta = U + \Theta(UVW - U^*VW^* + U^*V^*W + UV^*W^*) \ , \\ &\zeta^* = U^* + \Theta(U^*VW + UVW^* - UV^*W + U^*V^*W^*) \ . \end{aligned} \tag{A9}$$

FIG. 6. Diagram corresponding to the passive odd Maxwell model. The "&" refers to a parallel connection of an odd and an even component of the same type.

It is also possible to invert the relations in (A9) and although we do not do this here explicitly as it is a very heavy operation, we have verified numerically that there are always real solutions. It is interesting to note that eq. A9 implies that χ^* can only be non-zero if both even and odd components are non-vanishing, whereas all other components can be non-zero if only odd or only even components are non-zero.

A limiting case of this model can be attained if we take W, W^*, U^* to be zero. In this case we obtain the diagram given in fig. 6. This diagram corresponds to the passive odd Maxwell model introduced in eq. (II6) with the following coefficients

$$\begin{aligned}&\chi = \frac{UV}{V^2+V^{*2}} \ , \ \chi^* = -\frac{UV^*}{V^2+V^{*2}} \ , \ \gamma = 0 \ , \\ &\gamma^* = 0 \ , \ \zeta = U \ , \ \zeta^* = 0 \ .\end{aligned} \tag{A10}$$

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