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Lecture 7 Worksheet

1. $r(t) = \langle t\sin t + \cos t, \sin t - t\cos t, \frac{2}{3}t^3 \rangle$

$$L = \int_a^b \sqrt{(f'(t))^2 + g'(t)^2 + h'(t)^2}$$

$f'(t) = \cancel{\sin t} + t\cos t \; \cancel{-\sin t}$ $\quad g'(t) = \cancel{\cos t} - \cancel{\cos t} + t\sin t$ $\quad h'(t) = 2t^2$

$$= \int_0^1 \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2t^2)^2} = \int_0^1 \sqrt{t^2\cos^2 t + t^2\sin^2 t + 4t^4}$$

$$= \int_0^1 \sqrt{t^2(\cos^2 t + \sin^2 t) + 4t^2} = \int_0^1 t\sqrt{1+4t^2}$$

$u = 1 + 4t^2 dt$

$du = 8t.$

$$\cancel{\int_0^1 t\sqrt{u}}\,du = \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\int_0^1 (u)^{1/2}\,du = \frac{1}{8}\left(\frac{2u^{3/2}}{3}\right|_0^1 = \cancel{\frac{2}{8}}$$

$$\frac{u^{3/2}}{12} = \frac{(1+4t^2)^{3/2}}{12}\Big|_0^1 = \frac{5^{3/2}}{12} - \frac{1}{12} = \frac{\sqrt{125}}{12} - \frac{1}{12} = \boxed{\frac{\sqrt{125}-1}{12}}$$

2. $r(t) = \langle \cos 4t, 2\ln(\sin(2t)), \sin(4t) \rangle$ $\quad t = \frac{\pi}{6}$ to $t = \frac{\pi}{3}$

$f'(t) = -\sin 4t \cdot 4 = -4\sin 4t$

$g'(t) = \frac{4}{\sin(2t)} \cdot \cos 2t = 4\cot$

$h'(t) = 4\cos 4t$

$$L = \int_a^b \sqrt{(4\sin 4t)^2 + \left(\frac{4\cot^2}{\sin(2t)}\right)^2 + (4\cos 4t)^2}$$

$$= \int_a^b \sqrt{16\sin^2 4t + \frac{16\cot^2 2t}{\cancel{\sin^2(2t)}} + 16\cos^2 4t}$$

$$= 4\int_a^b \sqrt{1 + \frac{1}{\sin^2(2t)}} = 4\int \cancel{\sqrt{\frac{\sin^2(2t)+1}{\sin^2(2t)}}} = 4\int_a^b \sqrt{1+\cot^2(2t)}$$

$$= 4\int_a^b \csc 2t = -\frac{4}{2}\ln|\csc(2t)+\cot(2t)| = \boxed{-2\ln|\csc(2t)+\cot(2t)|}\Big|_{\pi/6}^{\pi/3}$$

$$-2\ln\left|\csc\left(\frac{2\pi}{3}\right)+\cot\left(\frac{2\pi}{3}\right)\right| + 2\ln\left|\csc\left(\frac{\pi}{3}\right)+\cot\left(\frac{\pi}{3}\right)\right| = \boxed{0}$$

3. $\vec{r}(t) = \langle 3t - 3\sin(t), 3 - 3\cos(t) \rangle \quad t = 0 \to 2\pi$

$2\sin^2\theta = 1 - \cos(2\theta)$

$f'(t) = 3 - 3\cos t$

$g'(t) = 3\sin t$

$$L = \int_a^b \sqrt{(3-3\cos t)^2 + (3\sin t)^2}$$

$$= \int_a^b \sqrt{9 - 18\cos t + 9\cos^2 t + 9\sin^2 t}$$

$$= 3\int_a^b \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = 3\int_a^b \sqrt{2 - 2\cos t}$$

$$= 3\int_a^b \sqrt{2 - 2\cos(2 \cdot \tfrac{t}{2})} = 3\int_a^b \sqrt{2 \cdot (2\sin^2(\tfrac{t}{2}))}$$

$$= 3\int_a^b \sqrt{4\sin^2(\tfrac{t}{2})} = 3\int_a^b 2\sin(\tfrac{t}{2})$$

$$= 6 \int_0^{2\pi} -\cos(\tfrac{t}{2}) \cdot 2 = -12\cos(\tfrac{t}{2})\Big|_0^{2\pi}$$

$$= -12(\cos(\pi) - \cos 0) = -12(-1-1) = \boxed{24}$$