Assignment 1: Matrices and Linear Algebra

INTRODUCTION

Matrices are powerful tools with many different applications, especially when solving physical problems. This experiment will demonstrate using matrices to solve a set of simultaneous linear equations as can be seen in Eq.1 for a N dimensional problem.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{1}$$

Where the A matrix and b vector are known, and the x vector is unknown. We can find **x** by using the inverse of A where

$$\begin{aligned} \mathrm{A}^{-1}\mathrm{Ax} &= \mathrm{A}^{-1}\mathrm{b} \\ \mathbf{x} &= \mathrm{A}^{-1}\mathbf{b} \end{aligned} \tag{2}$$

As we know a matrix, A, multiplied by its inverse gives an identity matrix of the same dimension To evaluate the inverse of A, Cramers rule is used, Eq.3,

$$\mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}}\mathrm{C}^T \tag{3}$$

where C is the matrix of cofactors of A. Found by eliminating the column and row of the corresponding element, and finding the determinant of the resulting matrix (or minor), where C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{4}$$

The determinant for N = 2 and N = 3 is found using the known solutions

$$\begin{aligned} |A_{2x2}| &= a_{11}a_{22} - a_{12}a_{21} \\ |A_{3x3}| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{aligned} \tag{5}$$

For higher order matrices, Laplaces expansion is used, where the matrix can be expanded along a row or column using the determinants of the minor matrices

TASK 1

The routine created using Cramers rule gives the following errors using the equation below to find the difference from the expected value.

$$A^{-1}A - I = 0 \tag{6}$$

The maximum absolute from the resulting matrix is found, and the following graph can be shown.

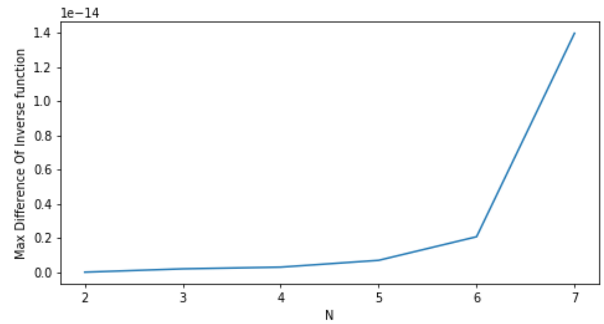

FIG. 1. Graph showing how Error scales with N, the size of the matrix.

This graph shows an exponential increase in error as order increases. This is expected as for each increase in order significantly increases the number or calculations. The number of calculations to find the cofactor matrix scales in $O(N!)$

The computation time was then looked at, resulting in the following.

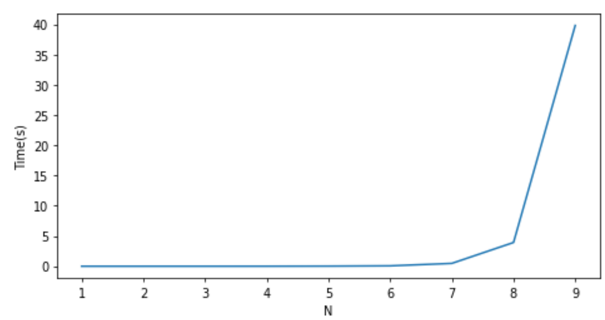

FIG. 2. Graph showing how computation time (s) scales with N, the size of the matrix.

As expected, the computation time graph follows the same trend as that of the errors, this is because the errors and the computational time are dependent on the number of operations.

Cramers rule breaks down when a singular matrix is given, a singular matrix gives a determinant of zero and hence Cramers rule returns an undefined number.

TASK 2

In this section two other methods for solving Eq.1 are compared to the routine in Task 1. LU (Lower-Upper)

Decomposition decomposes the matrix A into two triangular matrices L and U as described in the Assignment task. This then gives us two equations to solve,

$$\begin{aligned} \mathrm{L}\mathbf{y} &= \mathbf{b} \\ \mathrm{U}\mathbf{x} &= \mathbf{y} \end{aligned} \tag{7}$$

The second method is SV (Singular Value) decomposition, which expresses the matrix A as

$$A = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{*} \tag{8}$$

where $\mathbf{U}$ is a mxm unitary matrix, $\mathbf{V}^{*}$ is the conjugate transpose of the nxn unitary matrix $\mathbf{V}$ and $\mathbf{\Sigma}$ is a mxn diagonal matrix with real, positive diagonal elements. This can be used to find the inverse of A, using

$$A^{-1} = \mathbf{V}\Sigma^{+}\mathbf{U}^{*} \tag{9}$$

where Σ^{+} is the pseudo-inverse of Σ. The inverse is then used as in equation 2 to find the required solution.

The speed (time taken) of these different methods in solving a random simultaneous equation was compared, to Task 1 in FIG(!!!!!!!) below.

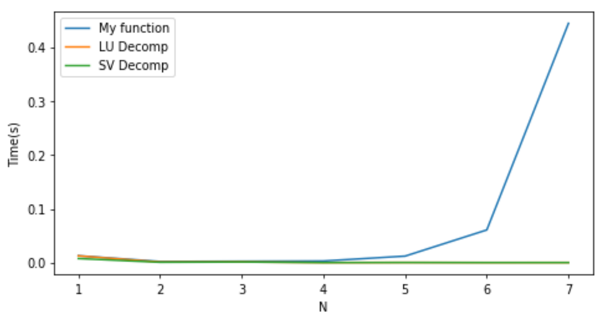

FIG. 3. Graph showing how computation time (s) scales with N, the size of the matrix, for the three different methods

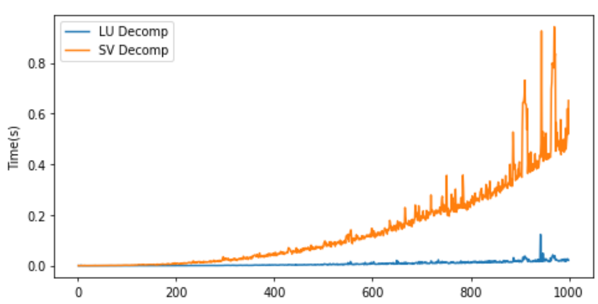

FIG. 4. Graph showing how computation time (s) scales with N, the size of the matrix, for the LU and SU decomposition methods

From this we see That SVD and LU decomposition solve the equation in significantly less time than Cramers rule as N increases to higher values. From FIG(!!!!!!) we can can see when comparing LU and SVD alone for much larger values of N, that SVD and LUD lines diverge, with SVD increasing at a faster rate $O(N^3)$. This is because LUD is a less complex calculation than SVD and has a lower time complexity.

The behaviour of the methods near a singularity was then investigated

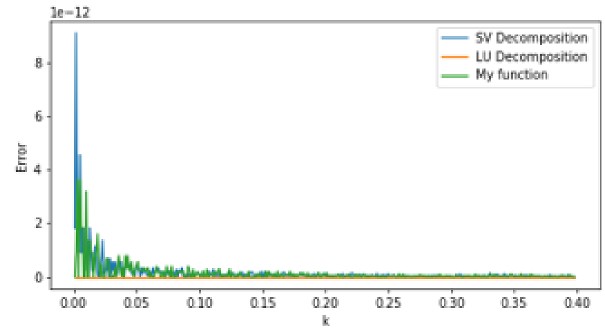

FIG. 5. Graph showing how computation time (s) scales with N, the size of the matrix, for the LU and SU decomposition methods

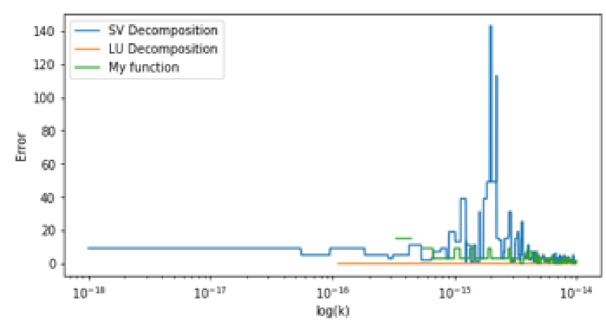

FIG. 6. Graph showing how computation time (s) scales with N, the size of the matrix, for the LU and SU decomposition methods

FIG. 5 shows that as we approach zero for SVD and Cramers rule the error increases sharply, this is expected as the floating point error increases at lower values. In FIG. 6 log(k) is plotted for even smaller values of k. From this we see below a certain value LU and Cramers rule no longer produce values, this is due to the singularity being treated as an actual singularity and the process resulting in an undefined number (i.e determinant = 0). SVD does not use this and therefore should produce results up until the floating point limit of the python console, 53 - bit.

TASK 3

In this section, A trapeze artist is moved across the stage attached to three wires, using an equation analagous to Eq.1, the tension in the wires can be modelled at any position the can move to. Initially the problem is simpler allowing the artist only to move in one plane, eliminating the third wire. Using a force diagram and trigonometric relations, a set of simultaneous equations is formed. Using Eq.2 this is easily solved.

$$\begin{pmatrix} -sin(\alpha) & sin(\theta) \\ cos(\alpha) & cos(\theta) \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ mg \end{pmatrix} \tag{10}$$

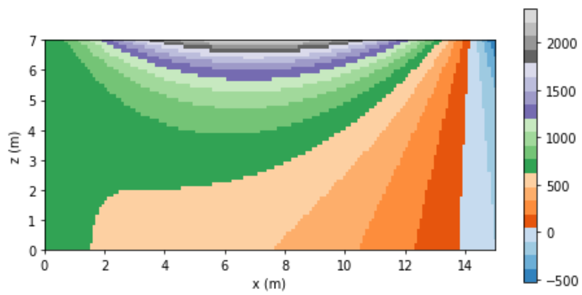

FIG. 7. Intensity plot where intensity shows the Tension in wire 1

From this graph we can see how the tension changes with position. We find the maximum tension is 2366 N at an (x,z) position (7.6,6.9). This method can then be extrapolated to 3d, using a force diagram. You now get a 3x3 angle matrix. This gives the following graph for z = 7, where the maximum tension is found, for the rear wire.

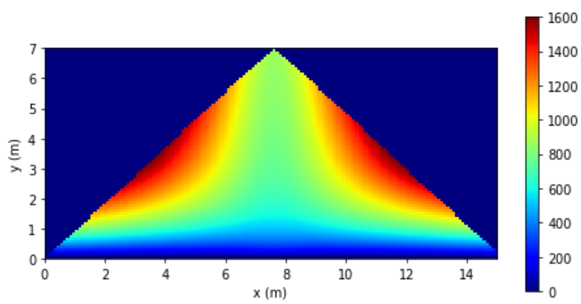

FIG. 8. Intensity plot where intensity shows the Tension in wire 1, taken at z = 7

From this we find the Maximum tension is found at (x,y,z) = (11.5,3.7,7).

CONCLUSION

In conclusion, Cramer's rule was shown to be a reasonable method to find the inverse matrix and solve a set of simultaneous equations. Though computational time and error increases greatly with matrix size compared to SVD and LUD methods. We saw that SVD increases with matrix size faster than LUD due to computational complexity. It was demonstrated that, close to a singularity Cramers rule and SVD had large errors, comparatively to LUD which had constant error for all values of k. Finally it was shown linear algebra methods can be used to solve a 'real' world problem analytically.