Problem Set 9

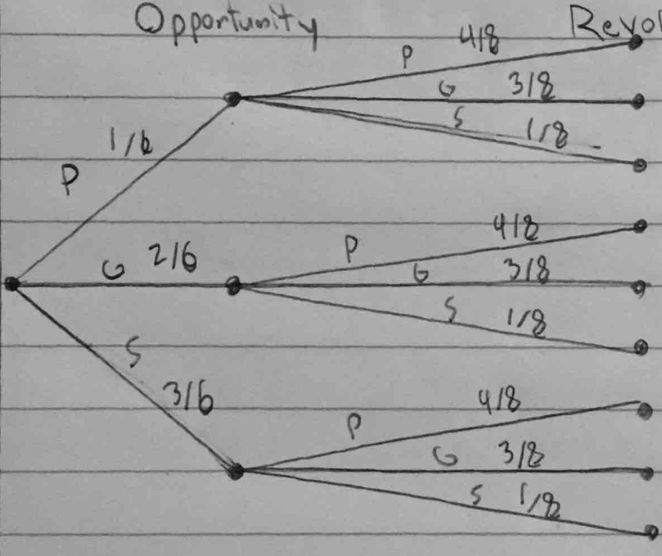
1. Professor Plum, Mr. Green, and Miss Scarlet are all plotting to shoot Colonel Mustard. If one of these three has both an opportunity and the revolver, then that person shoots Colonel Mustard. Otherwise, Colonel Mustard escapes. Exactly one of the three has an opportunity with the following probabilities:

$$Pr(P \text{ has opportunity}) = 1/6$$
$$Pr(G \text{ has opportunity}) = 2/6$$
$$Pr(S \text{ has opportunity}) = 3/6$$

Exactly one has the revolver with the following probabilities, regardless of who has an opportunity:

$$Pr(P \text{ has revolver}) = 4/8$$
$$Pr(G \text{ has revolver}) = 3/8$$
$$Pr(S \text{ has revolver}) = 1/8$$

a. Draw a tree diagram for this problem. Indicate edge & outcome probabilities



Outcome	CM shot?	Probability
(P,P)	×	1/12
(P,G)		1/16
(P,S)		1/18
(G,P)		1/6
(G,G)		1/8
(G,S)		1/24
(S,P)		1/4
(S,G)		3/16
(S,S)		1/16

b. What is the probability that Colonel Mustard is shot?

Denote outcomes by (i,j) $i,j \in \{P,G,S\}$ where i is the opportunity &
j is the revolver

$$
\begin{aligned}
P(M \text{ shot}) &= \sum P(\{(i,i) \mid i \in \{P,G,S\}\}) \\
&= \sum_{i \in \{P,G,S\}} P(i,i) \\
&= P(P,P) + P(G,G) + P(S,S) \\
&= \frac{1}{12} + \frac{1}{8} + \frac{1}{16} \\
&= \boxed{\frac{13}{48}}
\end{aligned}
$$

c. What is the probability that Colonel Mustard is shot, given that Miss Scarlet does not have the revolver?

Let M be the event that CM is shot $M = \{(i,i) \mid i \in \{P,G,S\}\}$

Let $/SR$ be the event that S does not have the revolver

$S = \{(i,j) \mid i \in \{P,G,S\};$
$j \in \{P,G\}\}$

$$P(M \mid /SR) = \frac{P(M \cap \{/SR\})}{P(/SR)}$$

$$= \frac{(P(P,P) + P(G,G))}{(P(P,P) + P(P,G) + P(G,P) + P(G,G) + P(S,P) + P(S,G))}$$

$$= \frac{\left(\frac{4}{48} + \frac{6}{48}\right)}{\left(\frac{4}{48} + \frac{3}{48} + \frac{8}{48} + \frac{6}{48} + \frac{12}{48} + \frac{9}{48}\right)} = \boxed{\frac{5}{21}}$$

d. What is the probability that Mr. Green had an opportunity, given that Colonel Mustard was shot?

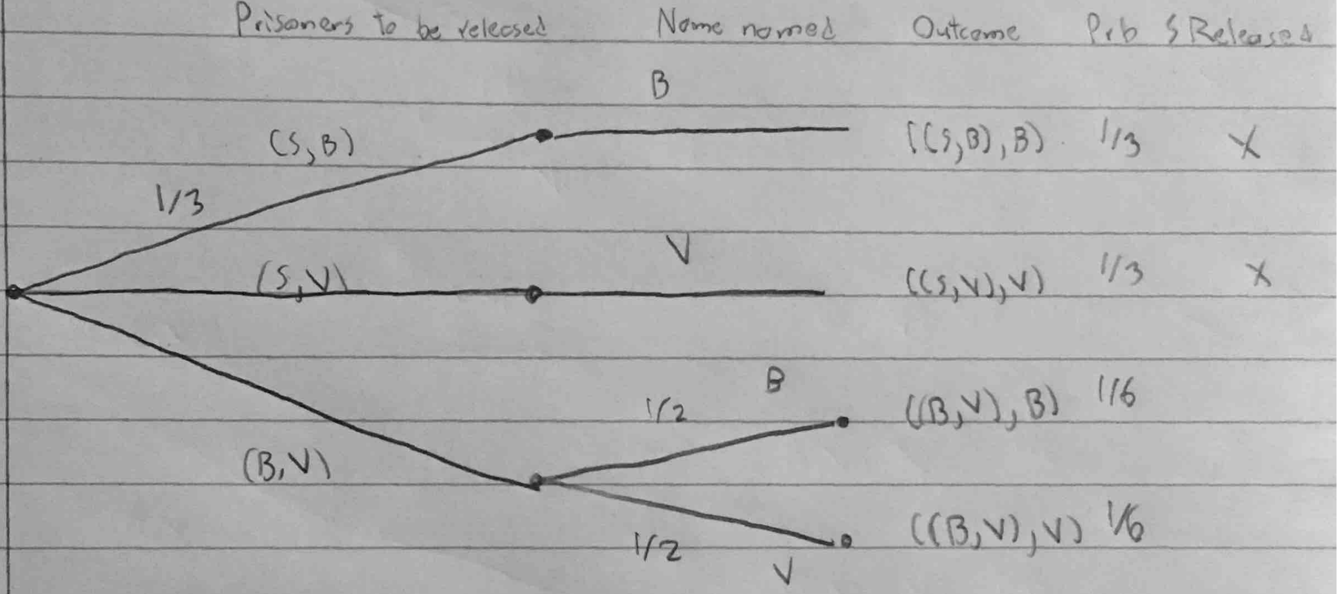
$$P(G\ \text{opportunity} \mid M\ \text{shot}) = \frac{P((G\ \text{opportunity}) \cap (M\ \text{shot}))}{P(M\ \text{shot})}$$

$$= \frac{P(G,G)}{13/48}$$

$$= \frac{6/48}{13/48}$$

$$= \boxed{\frac{6}{13}}$$

2. There are 3 prisoners: V, S, B. 2 of the 3 will be released at random, chosen uniformly, but the names will not be immediatley released. S figures his probability of release is 2/3. A guard offers to tell S the name of 1 of the prisoners to be released, but S declines. He figures if the guard says B, it will drop his release probability to 1/2 since he or V are now equally likely. Using a tree diagram & the 4-step method, evaluate S's conclusion



$\boxed{\text{S is wrong.}}$ In 2/4 possible outcomes, when a name is named he will be released, but the probabilities are weighted $\Rightarrow P(S) = 2/3$ either way

3. You shuffle a deck of cards and deal your friend a 5-card hand.

a. Suppose your friend says, "I have an ace of spades." What is the probability that she has another ace.

3rd card ace   4th card ace   5th card ace

$$P(2\,A \mid A \text{ of } \spadesuit) = \frac{P(A/\spadesuit \cap A\spadesuit)}{P(A\spadesuit)}$$

Ace of spades

$$= \frac{\frac{1}{52}\left(\frac{3}{51} + \frac{48}{51}\left(\left(\frac{3}{50}\right) + \frac{47}{50}\left(\left(\frac{3}{49}\right) + \frac{46}{49}\left(\frac{3}{48}\right)\right)\right)\right)}{\frac{1}{52}}$$

$$= \boxed{\frac{922}{4165} \sim 0.22}$$

b. Suppose your friend says, "I have an ace." What is the probability that she has another ace.

$$P(\geq 2A \mid \geq 1A) = \frac{P(\geq 2A \cap P \geq 1A)}{P(\geq 1A)} = \frac{P(\geq 2A)}{P(\geq 1A)}$$

ways to choose 2 A ← ways to choose remaining hands

$$P(\geq 1A) = \frac{\overbrace{\binom{4}{1}\binom{48}{4}}^{\#\text{ hands with 1 ace}} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\underbrace{\binom{52}{5}}_{\#\text{ total possible hands}}}$$

$$P(\geq 2A) = \frac{\sum_{i=2}^{4} (\#\text{ hands with exactly } i \text{ A})}{(\#\text{ total possible hands})}$$

$$= \left[\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}\right] \Big/ \left[\binom{52}{5}\right]$$

So $P(\geq 2A \mid P \geq 1A) = \boxed{\dfrac{\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}{\binom{4}{1}\binom{48}{4} + \binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}}} \approx 0.123$

c. Are your answers to (a) and (b) the same? Explain why.

No.

The condition of ace of spades rules out significantly more possibilities, increasing the conditional probability.

4. Finalphobia is a rare disease

- A person selected uniformly at random has finalphobia with probability $1/100$
- A person with finalphobia has shaky hands with probability $9/10$.
- A person with finalphobia has shaky hands with probability $1/20$.

What is the probability that a person selected uniformly at random has finalphobia given that he or she has shaky hands?

$$P(F|S) = \frac{P(F\cap S)}{P(S)} = \frac{P(F)P(S|F)}{\left((P(S|F)P(F) + P(S|/F)P(/F)\right))} = \boxed{\frac{18}{117}} \sim 0.15$$

$$P(S|F) = \frac{P(S\cap F)}{P(F)} \Rightarrow P(S\cap F) = P(F)P(S|F)$$

$$P(S) = P(S|F)(P(F)) + P(S|/F)P(/F)$$

$$P(F) = \frac{1}{100},\ P(S|F) = \frac{9}{10},\ P(S|/F) = \frac{1}{20},\ P(/F) = \frac{99}{100}$$

5. $P(L) = \frac{1}{6}$, $P(P) = \frac{1}{4}$ (these events represent success.

a. If at least one of them succeeds, what becomes the probability of L

Represent events as ordered pairs. 0 represents non-success, (L, P) is both events are successful.

$$P(L \mid \geq 1 \text{ succeeds}) = \frac{P(L \cap \geq 1 \text{ succeeds})}{P(\geq 1 \text{ succeeds})}$$

$$= \frac{P((L,0) \cup (L,P))}{P((L,0) \cup (0,P) \cup (L,P))}$$

$$= \frac{P(L)P(\neg P) + P(L)P(P)}{(P(L)P(\neg P) + P(\neg L)P(P) + P(P)P(L))}$$

$$= \frac{\frac{1}{6}\left(\frac{3}{4}\right) + \frac{5}{6}\left(\frac{1}{4}\right)}{\left(\frac{1}{6}\left(\frac{3}{4}\right) + \frac{5}{6}\left(\frac{1}{4}\right) + \frac{1}{6}\left(\frac{1}{4}\right)\right)} = \boxed{\frac{4}{9}}$$

b. If at most one of them succeeds, what is the probability that Jelani becomes president?

Let M be the event that ≤ 1 succeeds

Then $M = \{(\bar{P} \cap \bar{L}) \cup (\bar{P} \cap L) \cup (P \cap \bar{L})\}$

$$P(P \mid M) = \frac{P(P \cap M)}{P(M)} = \frac{P(P \cap \bar{L})}{[P(\bar{P} \cap \bar{L}) + P(\bar{P} \cap L) + P(P \cap \bar{L})]}$$

$$= \frac{(\frac{1}{4})(\frac{5}{6})}{[(\frac{3}{4})(\frac{5}{6}) + \frac{1}{6}(\frac{3}{4}) + \frac{5}{6}(\frac{1}{4})]} = \boxed{\frac{5}{23}}$$

C. If exactly one of them succeeds, what is the probability that it is Sayon?

Let E be the event that exactly one succeeds

$E = \{(\bar{L} \cap P) \cup (L \cap \bar{P})\}$

$$P(L|E) = \frac{P(L \cap E)}{P(E)} = \frac{P(L \cap \bar{P})}{P((\bar{L} \cap P) \cup (L \cap \bar{P}))}$$

$$= \frac{P(L)P(\bar{P})}{P(\bar{L})P(P) + P(L)P(\bar{P})}$$ since they are independent

$$= \frac{\frac{1}{6}\left(\frac{3}{4}\right)}{\left[\frac{5}{6}\left(\frac{1}{4}\right) + \left(\frac{1}{6}\right)\left(\frac{3}{4}\right)\right]}$$

$$= \boxed{\frac{3}{8}}$$

6. Suppose n balls are thrown randomly into n boxes, so each ball lands in each box with uniform probability. Also, suppose the outcome of each throw is independent of all the other throws.

a. Let X_i be an indicator random variable whose value is 1 if box i is empty and 0 otherwise. Write a simple closed form expression for the probability distribution of X_i. Are $X_1, X_2, \ldots, X_n$ independent random variables.

$$P(X_i = 1) = \underbrace{\left(1 - \tfrac{1}{n}\right)}_{\text{P empty after 1 throw}}{}^{\overset{\text{after } n \text{ throws}}{n}} = \boxed{\left(\frac{n-1}{n}\right)^n}$$

They are $\boxed{\text{not independent}}$

If n balls are thrown, there must exist an empty box.

So $P(X_1 = X_2 = \ldots = X_n = 1) = 0 < \prod_{i=1}^{n} P(X_i = 1)$

Thus the X_i's are dependent

b. Show that $P(\text{at least } k \text{ balls fall into the first box}) \leq \binom{n}{k}\left(\frac{1}{n}\right)^k$

$$P(\text{a specific set of } k \text{ balls fall into the first box}) = \prod_{i=1}^{k} P(\text{ball in 1st box})$$
$$= \left(\frac{1}{n}\right)^k$$

There are $\binom{n}{k}$ such subsets of k balls

By the union bound inequality

$$\boxed{P(\geq k \text{ balls in the first box}) \leq \left(\frac{1}{n}\right)^k \binom{n}{k}}$$

c. Let R be the maximum numbers of balls that land in each of the boxes. Conclude from the previous parts that $P(R \geq k) \leq \frac{n}{k!}$

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\begin{aligned}
P(R \geq k) &= P(\text{at least } k \text{ balls land in } \underline{\text{some}} \text{ box}) \\
&\geq n\, P(\geq k \text{ balls in a particular box}) \quad \text{by part b. and the Union Bound Inequality} \\
&= n \binom{n}{k} \left(\frac{1}{n}\right)^k \\
&= \frac{n\, n!}{(n-k)!\, k!} \left(\frac{1}{n}\right)^k \\
&= \frac{n}{k!} \left(\frac{n!}{(n-k)!\, n^k} \right) \\
&= \frac{n}{k!} \Bigg(\underbrace{\frac{n(n-1)(n-2)\cdots(n-k-1)}{(n)(n)(n)\cdots(n)}}_{\leq 1} \Bigg) \\
&\leq \frac{n}{k!} \quad //
\end{aligned}
$$

d. Conclude that $\lim_{n\to\infty} P(R \geq n^{\epsilon}) = 0 \quad \forall \epsilon > 0.$

$$P(R \geq n^{\epsilon}) \leq \frac{n}{(n^{\epsilon})!} \sim \frac{n}{\sqrt{2\pi n^{\epsilon}}\left(\frac{n^{\epsilon}}{e}\right)^{n^{\epsilon}}}$$ by Stirling's Approximation

$$\leq \frac{n e^{n^{\epsilon}}}{n^{n^{\epsilon}}} \leq \frac{n}{n^{n^{\epsilon}}} = n^{-(n^{\epsilon}-1)} \to 0$$ in the lim $n \to \infty$ //

7. (an open ended discussion question) Consider a set S, consisting of 77 twenty-one digit numbers. We can use the pigeonhole principle to prove that two distinct subsets of numbers in S have the same sum, but actually finding two such subsets can be difficult. Naively, we could sum the elements in all 2^{77} subsets and find two that match, but this is a huge computational task. Recall the birthday principle: If there are d days in a year and $\sqrt{2d}$ people in a room, then the probability that two share a birthday is about $1 - 1/e = 0.632\ldots$ How could the birthday principle be used to help you find two distinct subsets of S with the same sum using significantly fewer than 2^{77} operations — say only a trillion operations?
What assumptions must you make?

The potential sums are all in $\{0, 1, 2, \ldots, 77 \cdot 10^{21}\}$
Assume the sums are uniformly distributed among that set and independent (not true). We have 2^{77} sums and $77 \cdot 10^{21}$ sum values.
2^{77} is too many options. Using the birthday principle if we consider $\sqrt{2(77 \cdot 10^{21})}$ random subsets, then we have a good chance of finding a matching pair.

$$\sqrt{2(77 \cdot 10^{21})} \leq \sqrt{10^3 \cdot 10^{21}} = \sqrt{10^{24}}$$
$$= 10^{12} \text{, as expected.}$$