Problem Set 8

1. Find the coefficients of

a. x^{10} in $(x+(1/x))^{100}$

$$x^{10} = x^{55}\left(\frac{1}{x}\right)^{45}$$

So the coefficient of x^{10} is $\boxed{\binom{100}{55}}$

b. x^k in $(x^2-(1/x))^n$

$x^k = (x^2)^a \left(\frac{1}{x}\right)^b$

$a+b=n \Rightarrow b=n-a$ $\Rightarrow b=\frac{2n-k}{3}$

$2a-b=k \Rightarrow 3a-n=k \Rightarrow a=\frac{n+k}{3}$

Factoring in the negative coefficient on $1/x$ with a $(-1)^b$ term

So the coefficient is $\boxed{\binom{n}{(n+k)/3}(-1)^{(2n-k)/3}}$

2. Suppose a generalized World Series between the Sox and the Cardinals involved $2n+1$ games. As usual, the generalized Series will stop when one team has won more than half the possible games.

a. Suppose that when the Sox finally win the series, the Cards have managed to win exactly r games ($r \leq n$). How many possible win-loss patterns are possible for the Sox to win the series this way? Express your answer as a binomial coefficient.

Let's observe the win-loss patterns from the perspective of the Sox

Then (# ways for Sox to win the series given $r \leq n$ losses)

= (# ways to arrange $n+r+1$ series of w's and l's with r l's, where the $(n+r+1)$th value is a W (since the Sox win the series))

= (# ways to arrange $n+r$ series of w's and l's with r l's)

$= \boxed{\binom{n+r}{r}}$

b. How many possible win-loss patterns are possible for the Sox to win the GSeries when the Cards win at most r games? Express your answer as a binomial coefficient.

Arrange all possible $r+n+1$ games in any order, but after the final sox win, the remaining Cardinal winning games go unplayed.

$$\boxed{\binom{r+n+1}{r}}$$

c. Give a combinatorial proof that $\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$

Proof

Let $A ::=$ the sequences of $n+r+1$ games described in part b. r is constant.
Let $B ::=$ the sequences of $i+n+1$ games where the sox win the final game, and cards win i games as described in part a, where $0 \le i \le r$
A bijection $A \to B$ exists where the remaining games after the final sox win in each sequence of A are removed to map to a sequence of B.
Notice $|B| = \sum_{i=0}^{r} \binom{n+i}{i}$ as argued in part a.
By the bijection rule $|A| = |B|$
So $\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$. //

d. Verify $\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$.

Proof

Induction hypothesis $P(r)$: for $(r \geq 0) \in \mathbb{Z}$, $(n \geq r) \in \mathbb{Z}$, $\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$

Base Case. $r=0$ $\sum_{i=0}^{0} \binom{n+i}{i} = \binom{n}{0} = \binom{n+r+1}{0}$ $P(0)$ is true forming a basis for induction

Inductive Step: Assume $P(r)$ is true where $(r \geq 0) \in \mathbb{Z}$

Notice
$$\sum_{i=0}^{r+1} \binom{n+i}{i} = \sum_{i=0}^{r} \binom{n+i}{i} + \binom{n+r+1}{r+1}$$
$$= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \qquad \text{by induction hypothesis}$$
$$= \frac{(n+r+1)!}{r!(n+1)!} + \frac{(n+r+1)!}{(r+1)!(n)!} \qquad \text{def of } \binom{n}{k}$$
$$= (n+r+1)!\left(\frac{(r+1)!n! + (n+1)!r!}{r!(n+1)!n!(r+1)!}\right)$$
$$= (n+r+1)!\left(\frac{\cancel{n!}\cancel{r!}(r+n+1+1)}{\cancel{r!}\cancel{n!}(n+1)!(r+1)!}\right)$$
$$= \frac{(n+r+2)!}{(n+1)!(r+1)!} \qquad \text{def of !}$$
$$= \boxed{\frac{(n+(r+1)+1)!}{(n+1)!(r+1)!}} \qquad \text{proving } P(r+1)$$

So it follows by induction that $P(r)$ is true for all $r \in \{0\} \cup \mathbb{N}$ //

3.a. Let a_n be the number of length n ternary strings (strings of the digits 0, 1, and 2) that contain two consecutive digits that are the same.
Find a recurrence formula for a_n.

For a given n there are 3^n length strings composed of 0,1,2
a_n ternary and $3^n - a_n$ non-ternary.
The number of ternary strings a_n can be expressed as a sum of two types, those included in a_{n-1} if the last digit is removed (I) and those that aren't, but are still ternary in $\{0,1,2\}^n$ (II)
For type I, every string included in a_{n-1} is a valid string in a_n, so we can append any of the 3 values to the end $\longrightarrow 3a_{n-1} = |I|$
For type II, you simply repeat the last digit of a previously invalid length $n-1$ string to make it a valid length n string $\longrightarrow |II| = (3^{n-1} - a_{n-1})(1)$

non-ternary $n-1$ — only can append that value

So a recurrence formula is: $a_n = 3a_{n-1} + (3^{n-1} - a_{n-1})$

$$\boxed{a_n = 3^{n-1} + 2a_{n-1}}$$

b. Show that $\frac{-x}{1-2x} + \frac{x}{(1-3x)(1-2x)}$ is a closed form for the generating function for the sequence $a_0, a_1, \ldots$

$a_n = 2a_{n-1} + 3^{n-1}$

$\searrow \langle 0, 1, 3, 9, \ldots \rangle \to \underline{x} + \underline{3x^2} + 9x^3 + \ldots \to x\left(\sum_{i=0}^{n} (3x)^i\right)$

$= \frac{x}{1-3x}$

$\downarrow$

$2xA(x) \leftarrow$ last term

$\nwarrow$ right shift

$A(x) \overset{?}{=} 2xA(x) + \frac{x}{1-3x}$

for $n \leq 2$ the coeff of $\nwarrow$ is 0

$A(x) = 2xA(x) + \frac{x}{1-3x} - \underline{x}$ so we include this term to get the expected result

$A(x)(1-2x) = \frac{x}{1-3x} - x$

$A(x) = \frac{-x}{(1-2x)} + \frac{x}{(1-3x)(1-2x)}$ //

c. Find $r, s \in \mathbb{R} \ni \frac{1}{(1-2x)(1-3x)} = \frac{r}{1-2x} + \frac{s}{1-3x}$

Using partial fraction decomposition

$1 = r(1-3x) + s(1-2x)$

$1 = (r+s) + x(-3r-2s)$

correlating constants and coefficients

$r+s=1 \Rightarrow r=1-s$

$-3r-2s=0$

$-3(1-s)-2s=0 \Rightarrow \boxed{s=3}$

$\Rightarrow \boxed{r=1-s=-2}$

d. Use the previous results to write a closed form for the nth term of the sequence

$$A(x) = \frac{-x}{1-2x} + \frac{x}{(1-3x)(1-2x)} \quad \text{from b.}$$

$$= \frac{3x^2}{(1-3x)(1-2x)}$$

scale by coefficients

$$= 3x^2\left(\frac{3}{(1-3x)} + \frac{-2}{(1-2x)}\right) \quad \text{from c}$$

geoseries of powers of 3

geo-series of powers of 2

right shift twice

Scale by 3

Combining with the product and sum rules

$$a_n = 3(3\,3^{n-2} - 2\,2^{n-2})$$

$$\boxed{a_n = 3(3^{n-1} - 2^{n-1})}$$

4. Suppose there are four kinds of doughnuts: plain, chocolate, glazed, and butterscotch. Write generating functions for the number of ways to select the flavors of n doughnuts, subject to the following constraints.

a. Each flavor occurs an odd number of times.

For an individual flavor

$$\langle 0,1,0,1,0,\ldots\rangle \longrightarrow x+x^3+x^5+\ldots$$
$$= x(1+x^2+x^4+\ldots)$$
$$= x(1+(x^2)^1+(x^2)^2+\ldots)$$
$$= x\left(\frac{1}{1-x^2}\right) \qquad \text{geometric series}$$
$$= \frac{x}{1-x^2}$$

Since there are 4 flavors each with the same constraint, by the convolution rule, the generating function is:

$$\boxed{\left(\frac{x}{1-x^2}\right)^4}$$

b. Each flavor occurs a multiple of 3 times.

$$\boxed{\left(\frac{1}{1-x^3}\right)^4}$$ ← 4 flavors, by convolution rule.

↑ selecting a multiple of 3 for a flavor

C. There are no chocolate doughnuts and at most 1 glazed doughnut

Chocolate $\langle 0 \rangle \longrightarrow 1$

Glazed $\langle 0, 1 \rangle \longrightarrow 1 + x$

Plain $\langle 1, 1, \ldots \rangle \longrightarrow \frac{1}{1-x}$

Butterscotch $\langle 1, 1, \ldots \rangle \longrightarrow \frac{1}{1-x}$

By the convolution rule, the generating function is

$$\boxed{\frac{1+x}{(1-x)^2}}$$

d. There are 1, 3, or 11 chocolate doughnuts, and 2, 4, or 5 glazed.

Chocolate $\langle 0, 1, 0, 1, 0, \ldots, 1, 0, 0, \ldots \rangle \rightarrow x + x^3 + x^{11}$

Glazed $\longrightarrow x^2 + x^4 + x^5$

Unconstrained flavors (2) are each $\langle 1, 1, \ldots \rangle \rightarrow \frac{1}{1-x}$

By convolution rule the generating function is

$$\boxed{\frac{(x + x^3 + x^{11})(x^2 + x^4 + x^5)}{(1-x)^2}}$$

e. Each flavor occurs at least 10x.

For an individual flavor $\langle 0, \dots, 0, 1, 1, \dots \rangle \to x^{10} + x^{11} + \dots$

$$= x^{10}(1 + x + x^2 + \dots)$$
$$= x^{10}\left(\frac{1}{1-x}\right) = \frac{x^{10}}{1-x}$$

For 4 flavors $\left(\frac{x^{10}}{1-x}\right)^4 = \boxed{\frac{x^{40}}{(1-x)^4}}$