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Problem Set #1: Propositions & Proofs

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1. A real number $r$ is called sensible if $\exists a,b \in \mathbb{Z}^+ \ni$ $\sqrt{a/b} = r$. Prove that $\sqrt[3]{2}$ is not sensible.

Proof

Proof by contradiction, assume for purposes of contradiction that $\sqrt[3]{2}$ is sensible.

Then $\exists a,b \in \mathbb{Z}^+ \ni \quad \sqrt[3]{2} = \sqrt{a/b}$ by the def of sensible

Squaring both sides $\sqrt[3]{4} = a/b \quad \Rightarrow \sqrt[3]{4}$ is rational

Then $\exists m,n \in \mathbb{Z}^+ \ni \frac{m}{n} = a/b$ and $m/n$ share no common factors

So $4 = \frac{m^3}{n^3} \Rightarrow 4n^3 = m^3 \Rightarrow m^3$ is even

Suppose $m$ is odd

then $m^3$ would be odd since the product of any number of odd numbers is itself odd.

$\Rightarrow\Leftarrow$ since $m^3$ is even, so $m$ is even

Then $\exists z \in \mathbb{Z}^+ \ni m = 2z$

So $4n^3 = m^3 = (2z)^3 = 8z^3$

$\Rightarrow n^3 = 2z^3$ which implies $n^3$, and subsequently $n$, as shown above are even.

However this leads to a contradiction, since if $m \wedge n$ are even they share a common factor. $\Rightarrow\Leftarrow$

So, by contradiction, $\sqrt[3]{2}$ is <u>not</u> sensible.//

2. Translate the following into a predicate formula.

There is a student who has e-mailed exactly two other people in the class, besides possibly herself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are equality and $E(x,y)$ meaning that "x has sent e-mail to y."

Let $S$ be the set of students in the class.

$\exists x \in S \ni \exists y_1, y_2 \in S \setminus \{x\} \ni y_1 \neq y_2 \wedge E(x,y_1) \wedge E(x,y_2) \wedge$
$\forall z \in S \setminus \{x, y_1, y_2\}, \neg E(x,z)$

3. Express each of the following predicates and propositions in formal logic notation. The domain of discourse is $\mathbb{N}$.

a. $n$ is the sum of three perfect squares.

$\exists x, y, z \ni x \cdot x + y \cdot y + z \cdot z = n$

b. $x > 1$.

$\exists w \ni (1 + w = x) \wedge (w \neq 0)$

c. $n$ is a prime number.

$(n > 1) \wedge \neg(\exists x \ni \exists y \ni xy = n \wedge x \neq 1 \wedge y \neq 1)$

d. $n$ is the product of two distinct primes

$\exists x, y \ni (x \cdot y = n) \wedge \neg(x = y) \wedge (x \text{ is prime}) \wedge (y \text{ is prime})$

e. There is no largest prime number.

$\forall x,\ x \text{ is prime} \Rightarrow \exists y > x \ni y \text{ is prime}$

f. (Goldbach Conjecture). Every even natural number $n > 2$ can be expressed as the sum of two primes.

$\forall n,\ (\exists m \ni (m + m = n) \wedge \exists y \ni y = 1 \wedge n > y + y) \Rightarrow (\exists x, y \ni x + y = n \wedge x \text{ is prime} \wedge y \text{ is prime})$

9. (Bertrand's Postulate) If $n>1$, then there is always at least one prime $p$ such that $n<p<2n$.

$n>1 \Rightarrow \exists p \ni p$ is prime $\wedge$ $n<p$ $\wedge$ $p<n+n$

4. If a set, $A$, is finite, then $|A| < 2^{|A|} = |\mathcal{P}(A)|$, and so there is no surjection from set $A$ to its powerset. Show that this is still true if $A$ is infinite.
Hint: Remember Russell's paradox and consider $\{x \in A \mid x \notin f(x)\}$ where $f$ is such a surjection

Proof Consider a set $A$
Assume $\exists$ a surjection $f: A \to \mathcal{P}(A)$
Let $B ::= \{x \in A \mid x \notin f(x)\} \Rightarrow x \in B \Leftrightarrow x \notin f(x), \forall x \in A$
Notice $B \subseteq A \Rightarrow B \in \mathcal{P}(A)$
Since $f$ is a surjection $\exists\, b \in A \ni f(b) = B$
so $x \in f(b) \Leftrightarrow x \notin f(x)$ since $f(b) = B$, $\forall x \in A$
However, if we let $x = b$ this statement becomes $b \in f(b) \Leftrightarrow b \notin f(b)$
$\Rightarrow\!\cdot\!\Leftarrow$ this is a contradiction, so no such surjection from $A \to \mathcal{P}(A)$ exists //

5. a. Prove that $\exists z. [P(z) \wedge Q(z)] \rightarrow [\exists x. P(x) \wedge \exists y. Q(y)]$ is valid

Proof

Assume $\exists z. [P(z) \wedge Q(z)]$

So for some $z_0$ in the domain we have $P(z_0) \wedge Q(z_0)$

Then $P(z_0)$ is true, so $\exists x$, namely $x = z_0 \ni P(x)$

Similarly $Q(z_0)$ is true $\Rightarrow \exists y. Q(y)$

Then $[\exists x. P(x) \wedge \exists y. Q(y)]$

Thus $\exists z. [P(z) \wedge Q(z)] \rightarrow [\exists x. P(x) \wedge \exists y. Q(y)]$ //

b. Prove the converse is not valid by describing a counter model

$[\exists x. P(x) \wedge \exists y. Q(y)] \not\Rightarrow \exists z. [P(z) \wedge Q(z)]$

Proof by counter example

Let the domain be $\mathbb{N}$, $P(x)$ be $x = 1$ and $Q(x)$ be $x = 2$

Then $\exists x \in \mathbb{N}$, namely $x = 1 \ni P(x)$ holds

Similarly $\exists y \in \mathbb{N} \ni Q(x)$ holds $(y = 2)$

So the antecedent is true in this senorio

Suppose $\exists z. P(z) \wedge Q(z)$

Then $z = 1 \wedge z = 2 \Rightarrow 1 = 2 \Rightarrow \cdot \Leftarrow$ contradiction

So the consequent is false

thus the statement is false //

6. a. Give an example where the following result fails:

False Theorem: For sets $A, B, C$, and $D$, let

$$L ::== (A \cup C) \times (B \cup D)$$

$$R ::== (A \times B) \cup (C \times D)$$

Then $L = R$

Let $A = \{0\}$, $B = \{a\}$, $C = \{9\}$, $D = \{z\}$

Then $L = \{0, 9\} \times \{a, z\}$

$L = \{(0,a), (0,z), (9,a), (9,z)\}$

and $R = \{(0,a)\} \cup \{(9,z)\} = \{(0,a), (9,z)\} \neq L$

b. Identify the mistake in the following proof of the False Theorem

Bogus Proof: Since $L$ and $R$ are both sets of pairs it's sufficient

√ to prove that $(x,y) \in L \iff (x,y) \in R \;\; \forall x, y$

The proof will be a chain of iff implications

√ $(x,y) \in L$ iff

√ $x \in A \cup C \;\wedge\; y \in B \cup D$ iff

√ $(x \in A \vee x \in C) \wedge (y \in B \vee y \in D)$ iff

X $(x \in A$ and $y \in B)$ or else $(x \in C$ and $y \in D)$ iff

$x \in C \wedge y \in B$ and $x \in A \wedge y \in D$ are also valid senarios

c) Fix this proof to show $R \subseteq L$

Proof

$(x,y) \in L$ $\Leftrightarrow$

$x \in A \cup C$ and $y \in B \cup D$ $\Leftrightarrow$

$(x \in A \vee x \in C) \wedge (y \in B \vee y \in D)$ if

$(x \in A \wedge y \in B)$ or else $(x \in C \wedge y \in D)$ $\Leftrightarrow$

$(x,y) \in A \times B \quad \vee \quad (x,y) \in C \times D$ $\Leftrightarrow$

$(x,y) \in (A \times B) \cup (C \times D) = R$

so $R \subseteq L$ //