Problem Set 4

1. For functions $f: A \to B$ and $g: B \to C$, the composition of g and f, written $g \circ f$, is the function $h: A \to C$ where

$$h(a) ::= g(f(a))$$

a. Prove that if f and g are bijections, then so is $g \circ f$

Proof

f and g are injective by def of bijective

Consider $x, y \in A \ni g \circ f(x) = g \circ f(y)$

then $g(f(x)) = g(f(y))$ from the def of composition of functions

$\Rightarrow f(x) = f(y)$ since g is injective

$\Rightarrow x = y$ since f is injective

$\Rightarrow g \circ f$ is injective by def of injective

f and g are surjective by def of bijective

Consider $c \in C$

$\exists\, b \in B \ni g(b) = c$ since g is surjective

$\exists\, a \in A \ni f(a) = b$ since f is surjective

$\Rightarrow \exists\, a \in A \ni g \circ f(a) = g(f(a)) = c \Rightarrow g \circ f$ is surjective by def

So from the def of bijective, $g \circ f$ is bijective //

b. Prove that if $f: A \to B$ is a bijection, then $\exists$ a bijection, $e: B \to A$ $\ni$ $e \circ f = I_A$ where $I_A: A \to A$ and $I_A(a) ::= a$ for all $a \in A$

Proof

$f: A \to B$ is a bijection $\Rightarrow$ each element of B is mapped to exactly once

$\Rightarrow |A| = |B|$

Define e, $\forall b \in B$, $b = f(a) \Rightarrow e(b) = a \Rightarrow e \circ f(a) = a$

the condition $b = f(a)$ is always satisfied since f is a surjection

the condition $e(b) = a$ is possible since $|A| = |B|$

Consider $x, y \in B \ni e(x) = e(y)$

from the definition of e, $\exists a, b \in A \ni f(a) = x \wedge f(b) = y$

$a = b \Rightarrow f(a) = f(x) \Rightarrow$ since f is a function

$\Rightarrow x = y$, so e is injective

Consider $a \in A$

Then $\exists b \in B \ni$ then $\exists a \ni f(a) = b$ since f is surjective

$\Rightarrow e(b) = a$ from the definition of e

$\Rightarrow$ e is surjective

So e is bijective by definition

Consider $a \in A$

$e(f(a)) = a \Rightarrow e \circ f = I_A$ //

c. Prove that graph isomorphism is an equivalence relation

Proof

Let $G=(V,E)$, $G'=(V',E')$ and $G''=(V'',E'')$ be graphs defined by their vertexes V and edges E

Show $\cong$ is reflexive

$\forall G$, define the isomorphic mapping between vertexes $f = I_V$ and edges $g = I_E$. (existence verified by part b). These are bijective functions $\Rightarrow$ 1-to-1 correspondence, so by the definition of $\cong$

$G \cong G \ \forall G \Rightarrow \cong$ is reflexive

Show $\cong$ is symmetric

Assume $G \cong G'$

From the definition of $\cong$ $\exists f: V \to V' \wedge g: E \to E' \ni f \wedge g$ are bijective $\ni u \in V$ is an vertex of $e \in E \Rightarrow f(u) \in V'$ is a vertex of $g(e) \in E'$

Bijective functions have bijective inverse functions $\Rightarrow \exists f^{-1}, g^{-1}$

$f(u) \in V'$ is a vertex of $g(e) E' \Rightarrow f^{-1}(f(u)) = u \in V$ is an end point of $g^{-1}(g(e)) \in E$

So f^{-1}, g^{-1} form a 1-to-1 correspondence between G' and G

$G' \cong G \Rightarrow \cong$ is symmetric

Show $\cong$ is transitive

Suppose $\exists$ bijections $f: V \to V'$, $g: E \to E'$, $f': V' \to V''$, $g': E' \to E''$ that maintain endpoint relations

$f' \circ f: V \to V''$ and $g' \circ g: E \to E''$ are bijections as shown in a

$u \in V$ is a vertex of $e \in E \Rightarrow f(u) \in V'$ is a vertex of $g(e) \in E'$

$\Rightarrow f'(f(u)) \in V''$ is an end point of $g'(g(e)) \in E''$

Thus $f' \circ f(u) \in V'' \wedge g' \circ g(e) \in E''$, preserving end point resolutions

since $G \cong G'$

Since $G' \cong G''$

Thus $G \cong G''$, so $\cong$ is transitive

From the definition of an equivalence relation, graph isomorphism is an equivalence relation. //

2. The proof of the Handshake Theorem in Week 5 Notes is a little more informal than is desirable in the beginning of 6.042. Rewrite the proof more carefully as an induction on the number of edges in a graph

Handshake Theorem: The sum of the number of degrees of the vertices in a graph equals twice the number of edges

Proof

We will use induction.

Let $P(n) =: \sum_{v \in V} deg(v) = 2n$ where $n = |E|$

Base case: $P(1)$

Consider a graph with 1 edge, $G(V, E)$

The edge, by definition, touches two vertexes, call these V_1, V_2

$\sum_{v \in V} deg(v) = deg(V_1) + deg(V_2) = 1 + 1 = 2$ since V_1 and V_2 are touched by the one and only edge, showing $P(1)$ is true

This forms a basis for induction

Inductive step: for $k \in \mathbb{N}$, assume $P(k)$ is true

Consider a graph with $k+1$ edges, $G(V, E)$

Case 1: $\exists\, v_1, v_2 \in V \ni deg(v_1) = deg(v_2) = 1 \wedge$ an edge connects v_1 to v_2

These vertexes are only connected to each other, so we can seperate them from the sum $\sum_{v \in V} deg(v) = \sum_{v \in V / \{v_1, v_2\}} deg(v) + \sum_{\{v_1, v_2\}} deg(v)$

$\sum_{v \in V} deg(v) = 2k + 2$ from our inductive hypothesis and the logic of our base case

Case 2: $\forall\, v_1 \in V$, $deg(v_1) > 1$

Every vertex is touched by more than one edge

If we remove an edge $\sum_{v \in V} deg(v) = 2k$ from our inductive hypothesis

The edge we removed necessarily connected two vertices with $deg \geq 2$, or two non zero contributions to the total sum.

the edge connecting these vertices, v_1 and v_2 will add one degree to each term since the edge touches both of them once.

So if we replace the removed edge we add two to the sum

$\sum_{v \in V} deg(v) = 2k+2$

Case 3: $\exists$ an edge such that, the edge connects a degree 1 vertex v_1 to a degree > 1 vertex v_2

If we remove this edge we have $2k$ degrees, $deg(v_1) = 0$, $deg(v_2) =$ some $a > 0$

When we add it back $deg(v_1) = 1$, $deg(v_2) = a+1$, incrementing the total by 2

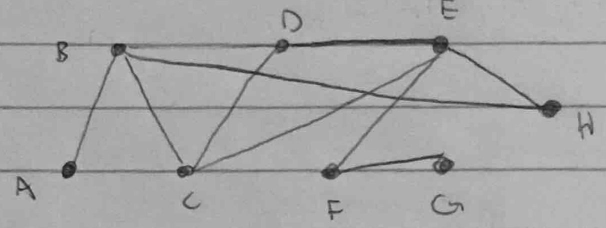
$\sum_{v \in V} deg(v) = 2k+2$

So $\sum_{v \in V} deg(v) = 2(k+1)$, verifying $P(k+1)$

So $P(k) \Rightarrow P(k+1)$, by induction $P(n)$ is true $\forall n \in \mathbb{N}$ //

3. The <u>distance</u> between two vertices in a graph is the length of the shortest path between them. The diameter of a graph is the distance between the two vertices that are furthest apart

a. What is the diameter of the following graph? Briefly explain your answer.



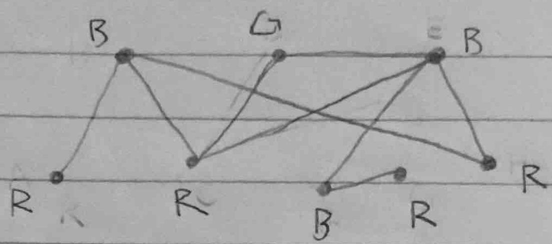
5. A→G are the furthest vertices, and are separated by a distance of 5.

b. What is the chromatic number of this graph? Prove it.

The graph has the chromatic number 3.

Proof

Let k be the chromatic number



The figure above shows the graph is 3 colorable.
This can be verified by noticing all adjacent vertices have different colors.

So $k \leq 3$

Now we will show the graph is not bipartite

The cycle CDEC has length 3

a graph G is bipartite $\Leftrightarrow$ it has no odd length cycles

So the graph is not bipartite $\Rightarrow k$

Thus $k > 2$

$2 < k \leq 3 \Rightarrow k = 3$

The graph's chromatic number is 3. //

c. Suppose every vertex in a graph is within a distance n of a vertex, v. Prove that the diameter of the graph is at most $2n$.

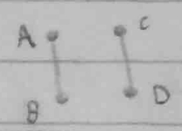
Proof

Consider a graph G which has the property: every vertex is G is within a distance n of a vertex v.
Call the furthest two vertices of G, A and B, and their distance, which is by definition the diameter of the graph, d.
One could construct a path $P = A, \ldots, v$ that has a length $\leq n$ from the properties of G.
Similarly a path $Q = v, \ldots, B$ could be constructed with length $\leq n$
Thus a path of length $\leq n + n = 2n$ could be constructed by joining P to Q end to end at v.
This is not necessarily the shortest path, but it is a path, so $d \leq 2n$. //

4. If a graph is connected, then every vertex must be adjacent to some other vertex. Is the converse of this true? If every vertex is adjacent to some other vertex, then is the graph connected? The answer is no.

a. Give a minimal example of a graph in which every vertex is adjacent to at least one other vertex, but the graph is not connected.

G



$V = \{A, B, C, D\}$

$E = \{\{A, B\}, \{C, D\}\}$

$\forall v \in V$ is adjacent to some $(u \neq v) \in V$ since there exists an edge from each one

The vertices $\{A, D\}$ are not connected because there is no path of edges joining them.

The graph G is not connected since not every pair of vertices is connected.

b. So something is wrong with the following proof. Exactly where is the first mistake in the proof?

<u>False Theorem</u> 4.1. If every vertex in a graph is adjacent to another vertex, then the graph is connected.

<u>Proof</u> by Induction

Let $P(n)$ be the predicate that if every vertex in an n-vertex graph is adjacent to another vertex, then the graph is connected. In the base case, $P(1)$ is trivially true because there is only one vertex.

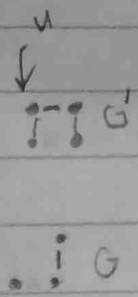
In the inductive step, we assume $P(n)$ to prove $P(n+1)$. Start with an $n+1$ vertex graph, G', in which every vertex is adjacent to another vertex. Now take some vertex v away from the graph and let G be the remaining graph. By assumption v is adjacent in G' to one of the n vertices of G; call that one u.

Now we must show that for every pair of distinct vertices x_1 and x_2 in G', there is a path between them. If both x_1 and x_2 are vertices of G, then since G has n vertices, we may assume by induction it is connected

↑

Here is the first error. The predicate is <u>every vertex in</u> an n-vertex graph <u>is adjacent to another</u> vertex $\Rightarrow$ the graph is connected.

While G' met the precedent's terms, when we removed a vertex to form G there was <u>no</u> gaurentee that every vertex would remain adjacent to another (an example can easily be constructed, see the left margin) so we cannot apply the induction hypothesis to G.



5.a. Show that every planar graph has a node of degree at most 5.

Consider a planar graph G with v vertices and e edges.
Let H be the largest connected subgraph of G.
Let v' and e' be the number of vertices in H respectively.

Case 1 $(v' \leq 2)$:

Then the max number of edges any vertex in H can have is 1.

Case 2 $(v' > 2)$:

$e' \leq 3v' - 6$ by lemma 5.2. since H is connected and planar

$e' = \frac{1}{2} \sum_{i=1}^{v'} deg(n_i) \leq 3v' - 6$ from the handshake theorem

rewriting $v'(min(deg(n_i)) \leq \sum_{i=1}^{v'} deg(n_i) \leq 6v' - 12 < 6v'$

so $min(deg(n_i)) < 6v'$

so $\exists n \in V' \ni deg(n) < 6$ and $V' \subseteq V$

Thus every planar graph has a node of degree at most 5.

b. Conclude that <u>any</u> planar graph can be colored with six colors.

<u>Proof</u> by strong induction on the number of vertices

<u>Induction hypothesis</u> $P(n)$: any planar graph with n vertices can be colored with 6 colors

<u>Base Case</u> ($n \leq 6$) Each vertex can be assigned a different color and the number of colors will total ≤ 6. So $P(k)$ holds for $k \in \{1, 2, 3, 4, 5, 6\}$

<u>Inductive step</u>: Assume $P(n)$ is true for some $n > 6 \in \mathbb{N}$

Let G be an $(n+1)$-vertex planar graph

Remove the vertex with degree ≤ 5, whose existence is justified by 5.a., and its associated edges to form G'. Call this vertex v.

G' is an n-vertex planar graph.

From our induction hypothesis we can conclude that G' is six colorable.

If we form G from reinserting v and its edges into G', we can color v one of the six colors from G', because v is adjacent to at most 5 nodes, and v will be a different color from its adjacent nodes.

So $P(n) \Rightarrow P(n+1)$

Thus by induction $\forall n \in \mathbb{N}$, $P(n)$ is true

So <u>any</u> planar graph can be colored with 6 colors.