1. Use induction to prove that the following inequality holds for all integers $n \geq 1$ ($\mathbb{N}$): $\dfrac{\prod_{i=0}^{n}(2i+1)}{\prod_{i=0}^{n}(2i+2)} \geq \dfrac{1}{2n+2}$

Proof

We will use induction

Let $P(n)$ be the statement in the hypothesis

Base case: $P(1)$ is $\dfrac{\prod_{i=0}^{1}(2i+1)}{\prod_{i=0}^{1}(2i+2)} \geq \dfrac{1}{2+2}$

Simplifing both sides yeilds $\frac{1(3)}{2(4)} = \frac{3}{8} \geq \frac{1}{4}$, showing $P(1)$ is true

This forms a basis for induction.

Inductive step: Assume $P(n)$ is true where n is any natural number.

Then $$\frac{\prod_{i=0}^{n+1}(2i+1)}{\prod_{i=0}^{n+1}(2i+2)} = \frac{\prod_{i=0}^{n}(2i+1)}{\prod_{i=0}^{n}(2i+2)}\left(\frac{2(n+1)+1}{2(n+1)+2}\right) = \frac{\prod_{i=0}^{n}(2i+1)}{\prod_{i=0}^{n}(2i+2)}\left(\frac{2n+3}{2n+4}\right)$$

$$\geq \frac{1}{2n+2}\frac{(2n+3)}{(2n+4)} \quad \text{by induction hypothesis}$$

$$\geq \frac{1}{2n+2}\left(\frac{2n+3}{2n+4}\right)\left(\frac{2n+2}{2n+3}\right) \quad \text{since } n \geq 1$$

$$= \frac{1}{2n+4} = \frac{1}{2(n+1)+2}$$

verifying $P(n+1)$

So since $P(n) \Rightarrow P(n+1)$, by induction $P(n)$ is true $\forall n \in \mathbb{N}$ //

2. a. Let's try to use strong induction to prove that a class with $n \geq 8$ students can be divided into groups of 4 or 5. Identify the error in the bogus proof

Proof

The proof is by strong induction. Let $P(n)$ be the proposition that a recitation with n students can be divided into groups of 4 or 5 students:

$8 = 4+4$, $9 = 4+5$, $10 = 5+5$

Next, we must show that $P(8), \ldots, P(n) \Rightarrow P(n+1)$ $\forall n \geq 10$.

Assume $P(8), \ldots, P(n)$

We first form a group of 4 students.

Then we can divide the remaining $n-3$ students into groups of 4 or 5 by the assumption $P(n-3)$

Here is the error. If $n=10$, $n+1=11$, $n-3=7$ which was never shown (and in fact does not hold).

(b) Prove a correct strong induction proof that a class with $n \geq 12$ students can be divided into groups of 4 or 5.

Proof

Proof by strong induction with the same $P(n)$ as the hypothesis

Base Case: $P(0)$ is vacuously true

$P(n)$ is true for $n = 12, 13, 14, 15$ since

$12 = 3(4)$, $13 = 2(4)+5$, $14 = 4+2(5)$, $15 = 3(5)$

Inductive step: We must show $P(12), \ldots, P(n) \Rightarrow P(n+1)$ $\forall n \geq 15$

Assume $P(12), \ldots, P(n)$ are all true

Consider a class of $n+1$ students

We first form a group of 4 students, leaving $n-3$ students

Notice $n-3 \geq 12$ since $n \geq 15$

We can divide the remaining $n-3$ students into groups of 4 or 5 by $P(n-3)$ of the inductive assumption

So $P(n+1)$ is true

Thus by strong induction, for all $n \geq 12$ a class of n students can be divided into groups of 4 or 5.

3. The game of mini-nim is defined as follows: Some positive number of sticks are placed on the ground. Two players take turns removing 1, 2, or 3 sticks. The player who removes the last stick loses.

Use strong induction to show that:

The second player has a winning strategy if the number of sticks, equals $4k+1$ for some $k \in \mathbb{N}$; otherwise, the first player has a winning strategy.

Proof using strong induction $P(n)$ is the statement in the hypothesis where $n=$ the number of sticks.

Base case: $n=1$, one stick remains, so player 1 must pick it up and lose confirming $P(1)$

Inductive step: Assume $P(i)$ holds for $\forall i \leq n$ $(n \geq 1)$ and prove $P(n+1)$

Case 1: $n = 4k$: $n+1 = 4k+1$

Player 1 removes 1, 2, or 3 sticks then Player 2 removes 3, 2, or 1 respectively as a response, netting 4 removed sticks now $4k+1-4 = 4(k-1)+1$ sticks remain

from the inductive hypothesis $P(n-4)$, P2 wins since if $n \geq 5$ $\exists i \in \mathbb{N} \ni n-4 = 4(i)+1$, and $n=1$ was handled in the base case

Case 2: $n = 4k+1$: $n+1 = 4k+2$

Player 1 removes 1 stick. Now a stack of $4k+1$ sits in front of P2, and by $P(n)$ through the inductive assumption P1 will win since the players switch roles (P2 picks the next stick) with the new stack.

Case 3: $n = 4k+2$: $n+1 = 4k+3$

Player 1 removes 2 sticks, $4k+1$ sticks remain $P(n-1) \Rightarrow$ P1 wins by an argument similar to case 2

case 4: $n = 4k+3$: $n+1 = 4k+4$

P_1 removes 3, $4k+1$ remain for player 2, $P(n-2) \Rightarrow P_1$ wins

So by strong induction $P(n)$ holds $\forall n \in \mathbb{N}, n \geq 1$. //

4. Consider the equivalent way of viewing the subset take-away game from the in-class problem on Friday, Week 2: for a fixed, finite set, A, let S initially be all the proper subsets of A. Players can alternately choose a set $B \in S$ and remove B and all sets that contain B from S; they continue playing on the updated S. The player that chooses the last set in S wins.

(a) Use the well-ordering property to show that, in any game, one of the players must have a winning strategy.

Proof by contradiction

Assume S is the smallest collection ∍ neither player has a winning strategy.

This directly implies that there is no move player 1 can make to win since he has no winning strategy.

It also means ∃ a move ∍ player 2 is not given a winning strategy, since if P2 has a winning strategy regardless of P1's move, P2 would already have a winning strategy on S.

So there must exist a move by P1 such that in the next turn neither player has a winning strategy on the updated S

$\Rightarrow\Leftarrow$, this is a contradiction since the updated S is smaller than S, and S was assumed to be the smallest collection of its sort

So there is no smallest set S ∍ neither player has a winning strategy

By the well ordering property there is no set and by extension, game ∍ neither player has a winning strategy.

So in every game a player has a winning strategy

b. If the whole set A is a possible move in the game, explain why the first player must have a winning strategy.

Proof

Case 1 The collection of proper subsets of A has a winning strategy for the second player:
Then player 1 removes A, since no other sets in S contain the entire set the proper subset of A remains, and player 2 moves next on that, leaving the winning strategy for player 1

Case 2: $S/\{A\}$ has a winning strategy for the first player:
Every valid move in $S/\{A\}$ is also valid in S, and removes A from S, since it contains every proper subset of A.
So player 1 makes the same move as they would if the game were $S/\{A\}$ following their winning strategy, and are in the same situation as if the game were on $S/\{A\}$, leaving player 1 a winning strategy.

By the hypothesis of 4.a. these cases are exhaustive, so player 1 always has a winning strategy. //