1. There are 20 books arranged in a row on a shelf.

a. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15-bit sequences with exactly 6 ones.

Let the first 5 ones each represent the position of the first 5 pairs of selected/non-selected (in that order) books and the 6th represent the position of the sixth selected book. The remaining zeros represent the relative positions of the remaining (previously uncoupled) non-selected books.

b. How many ways are there to select 6 books so that no two adjacent books are selected?

Borrowing the result of part a., this answer is the same as the number of 15-bit sequences containing exactly 6 ones, by the bijection rule.
This is $\boxed{\binom{15}{6}}$, since order doesn't matter

2. Answer the following questions and provide brief justifications. Not every problem can be solved with a cute formula; you may need to fall back on case analysis, explicit enumeration, or ad hoc methods.

a. How many ways can the letters in BANANARAMA be arranged?

From the bookkeeper rule:

$$\# \text{ways} = \frac{(1+5+2+1+1)!}{1!5!2!1!1!} = \boxed{\frac{10!}{2(5!)}}$$

b. How many different paths are there from point (0,0,0) to point (12,24,36) if every step increments one coordinate and leaves the two unchanged?

There are 3 possible steps, +(1,0,0), +(0,1,0), and +(0,0,1)
The paths are sequences that end at (12,24,36), so they have 12 of the first type of step, ...

By the book keeper rule $\boxed{n_{paths} = \frac{72!}{12!24!36!}}$

c. In how many ways can $2n$ students be paired up?

\# pairings = (# of ways to choose pairs) / (# orderings of n pairs)

since it is a permutation

$$= \prod_{i=1}^{n} (\text{\# ways to choose pair } i) / n!$$

$$= \prod_{i=1}^{n} \binom{2(n-i+1)}{2} / n!$$

$$= \prod_{i=1}^{n} \frac{2(n-i+1)}{2!\,(2(n-i+1)-2)!} / n!$$

$$= \frac{1}{2!\,n!} \prod_{i=1}^{n} \frac{[2(n-i+1)]!}{(2(n-i))!}$$

$$= \boxed{\frac{(2n)!}{2!\,n!}}$$

d. How many different solutions over the natural numbers are there to the following equation $\sum_{i=1}^{8} x_i = 100$

A solution is a specification of the value of each variable x_i. Two solutions are different if different values are specified for some variable x_i.

$\#$ solutions $=$ ($\#$ solutions to $\sum_{i=1}^{8} x_i \leq 100 - \#$ sol to $\sum_{i=1}^{8} x_i \leq 99$)

There exists a bijection between the $\#$ of solutions to $\sum_{i=1}^{n} x_i \leq k$ and a $k+n$ bit number with exactly n ones. This can be illustrated by letting x_i equal the number of zeros before the ith 1.

So $\#\text{sol} = \binom{108}{8} - \binom{107}{8}$ by the bijection rule

$\boxed{\#\text{solutions} = \binom{107}{7}}$

e. In how many different ways can one choose n out of $2n$ objects where n are identical and the other n are unique.

$$\sum_{k=0}^{n} \binom{n}{k}(1)$$

← ways to choose the remaining identical objects since they're indistinguishable

ways to choose the k unique objects

sum over all combos of k unique $n-k$ identical

$$= \boxed{2^n}$$

f. How many undirected graphs are there with vertices $v_1, v_2, \ldots, v_n$ if self-loops are permitted?

Graphs are subsets of edges, so $(\#\text{graphs}) = 2^{(\#\text{edges})}$

$(\#\text{edges}) = \underbrace{\binom{n}{2}}_{\text{btwn vertices}} + n \leftarrow$ self-loops

$$\boxed{(\#\text{graphs}) = 2^{\left[\binom{n}{2} + n\right]}}$$

9. There are 15 sidewalk squares in a row. Suppose that a ball can be thrown so that it bounces on 0, 1, 2, or 3 distinct sidewalk squares. Assume the ball always moves from left to right. How many different throws are possible?

$$\# \text{ throws} = \sum_{i=0}^{3} \# \text{ of throws with } i \text{ bounces}$$
$$= \boxed{1 + \binom{15}{1} + \binom{15}{2} + \binom{15}{3}}$$
$$= 561$$

h. The working days in the next year can be numbered 1, 2, 3, ..., 300.
I'd like to avoid as many as possible.

- On even numbered days, I'll say I'm sick
- On days that are a multiple of 3, I'll say I was stuck in traffic.
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I avoid in the coming year?

$\boxed{220}$

I computed this using ps7p2h.py

3. Use the pigeonhole principle to solve the following problems.

a. Prove that among any n^2+1 points within an $n \times n$ square there must exist two points whose distance is at most $\sqrt{2}$.

Proof

Let the n^2+1 points be the pigeons

the n^2 unit squares that you can divide the $n^2 \times n^2$ square into be the pigeon-holes

And let the mapping function map each point to the subsquare it falls within (bottom left if on the boundary)

$n^2+1 > n^2$, so by the pigeonhole principle 2 points must fall within the same subsquare.

The furthest points (corners) in a subsquare are $\sqrt{2}$ apart, so there must exist two points whose distance is at most $\sqrt{2}$, namely the 2 in the same subsquare //

b. Jellybeans of 6 different flavors are stored in 5 jars. There are 11 jellybeans of each flavor. Prove that some jar contains at least 3 jellybeans of one flavor and also at least three jellybeans of some other flavor.

Proof

Let $A =$ jellybeans of the same flavor,
$B =$ jars, and
$C =$ flavors of jellybeans
$|A| > 2|B| \Rightarrow$ 3 jellybeans of the same flavor will end up in some jar by the generalized pigeonhole principle
$|C| > |B| \Rightarrow$ there will be two sets of 3 similarly flavored jelly beans in the same jar by the above line and the pigeonhole principle. //

C. Prove that among every set of 30 integers, there exist two whose difference <u>or</u> sum is a multiple of 51.

Map integer a to the set with a rem 51, where the sets are:
$\{0\}, \{1, 50\}, \dots, \{25, 26\}$
If a rem 51 = b rem 51 $\quad$ (a-b) rem 51 = 0 or 51 => difference is divisible by 51
if |a rem 51| = |51 - b rem 51| (different values in the same set) then
(a+b) rem 51 = 0 or 51 => sum is divisible by 51.
So if two values map into the same set, their sum or difference is a multiple of 51.
We have 30 values being mapped into 26 sets, so by the pigeonhole principle, there exist two values whose difference or sum is a multiple of 51.

4. Suppose you have seven dice - each a different color of the rainbow; otherwise the dice are standard, with six faces numbered 1 to 6. A 'roll' is a sequence specifying a value for each die in the rainbow order. For example, one roll is (3,1,6,1,4,5,2).
For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized product, and similar rules. Then write a simple numerical expression for the size of the set of rolls. You do not need to prove that the correspondence between the sets you describe is a bijection, and you do not need to simplify the expression you come up with.

a. For how many rolls is the value on every die different?

let A = set of numbers on a die
B = the set of dice
$|B| > |A|$ => some number must be repeated, by the pigeonhole property.

b. For how many rolls do two dice have the value 6 and the remaining dice all have different colors.

$B ::= S_2 \times$ (sequence of values of remaining dice)

colors of 6s

$\boxed{|B| = \binom{7}{2} 5!}$

c. For how many rolls do two dice have the same value and the remaining five dice all have different values?

Map a roll into an element of $B ::= S_1 \times R_2 \times P_5$ where
S_1 is the set of dice values (size 1 subsets)
R_2 is the set of all size two subsets of the 7 rainbow colors
P_5 is the set of permutations of 5 different numbers.

The repeated digit is mapped to S_1, the colors of the equal dice is mapped to R_2, and the orderings of the different dice is left to P_5.
$\therefore |A| = |B|$ since its a bijection
By the generalized product rule
$$\boxed{|B| = 6\binom{7}{2}5!}$$

d. For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice have a third value?

There is a bijection to $B ::= S_2 \times R_2 \times R_2^* \times S^*$
where R_2^* is R after the removal of 2 rainbow colors
and S^* is S after the two values are removed.

Thus $\boxed{|B| = \binom{6}{2}\binom{7}{2}\binom{5}{2}4}$ ← remaining values for 3 matching dice

color of 2nd pair

color of 1st pair

ways to pick values in the pairings

5. A 'derangement' is a permutation $(x_1, x_2, \cdots, x_n)$ of the set $\{1, 2, \ldots, 3\}$ such that $x_i \neq i \ \forall i$. The objective of this problem is to count derangements. It turns out to be easier to start by counting the permutations that are <u>not</u> derangements. Let S_i be the set of all permutations $(x_1, x_2, \cdots, x_n)$ that are not derangements because $x_i = i$. So the set of all non-derangements is:

$$\bigcup_{i=1}^{n} S_i$$

a. What is $|S_i|$?

S_i is every set $\ni x_i = i$
So the remaining $n-1$ elements can be in any order.
This makes it a permutation of $n-1$ elements, so
$$\boxed{|S_i| = (n-1)!}$$

b. What is $|S_i \cap S_j|$ where $i \neq j$?

$S_i \cap S_j$ is all remaining sets $\ni$ $x_i = i \wedge x_j = j$
So the number of sets is the number of permutations of the $n-2$ remaining elements.

$$\boxed{|S_i \cap S_j| = (n-2)!}$$

c. What is $|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}|$ where $i_1, i_2, \ldots, i_k$ are all distinct?

$|S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}|$ = (# permutations of remaining $n-k$ elements)

$= \boxed{(n-k)!}$

d. Use the inclusion-exclusion formula to express the number of non-derangements in terms of sizes of possible intersections of the sets $S_1, \dots, S_n$.

$$\boxed{\left|\bigcup_{i=1}^{n} S_i\right| = \sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \dots + (-1)^n \left|\bigcap_{i=1}^{n} S_i\right|}$$

e. How many terms in the expression in part (d) have the form $|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}|$?

It is a sum $\sum_{i_1, i_2, \ldots, i_k}$ over terms where $i \in \{1, 2, \ldots, n\}$ ^ $i_1 \neq i_2 \neq i_3 \neq \ldots \neq i_k$
Which makes it n choose k elements of the sum

$$\boxed{\binom{n}{k}}$$

f. Combine your answers to the preceding parts to prove the number of non-derangements is:

$$n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\dots\pm\frac{1}{n!}\right)$$

Conclude that the number of derangements is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots\pm\frac{1}{n!}\right)$$

from d. $\left|\bigcup_{i=1}^{n} S_i\right| = \sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \dots \pm \left|\bigcap_{i=1}^{n} S_i\right|$

from c & c

$$= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \dots \pm \binom{n}{n}(n-n)!$$

$$= \frac{n!}{\cancel{(n-1)!}\,1!}\cancel{(n-1)!} - \frac{n!}{2!\cancel{(n-2)!}}\cancel{(n-2)!} + \frac{n!}{3!\cancel{(n-3)!}}\cancel{(n-3)!} - \dots \pm \frac{n!}{n!}$$

$$= n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\dots\pm\frac{1}{n!}\right)$$

derangements = # permutations − # non derangements

$$= n! - n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\dots\pm\frac{1}{n!}\right)$$

$$= \boxed{n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots\mp\frac{1}{n!}\right)}$$

9. As n goes to infinity the number of derangements approaches a constant fraction of all permutations. What is that constant?

$$\frac{\#\text{ derangements}}{\text{all permutations}} = \frac{n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \mp \frac{1}{n!}\right)}{n!}$$

$$= \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \mp \frac{1}{n!}\right)$$

$$\lim_{n \to \infty} (\text{fract}) = e^{-1} = \boxed{\frac{1}{e}}$$