PROBLEM HW 3

(1.1) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+2b \\ 2a+b \end{bmatrix}$

$\begin{bmatrix} a+2b \\ 2a+b \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + 4ab + b^2 \geq 0$

this statement is incorrect it can be negative. an example of a positive semidefinite matrix is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(1.2) (a) looking at each partial

$\frac{\partial}{\partial \theta_{y=1}} = \frac{n_1}{\theta_{y=1}} + \frac{n_1 - n}{1-\theta_{y=1}}$, $\frac{\partial}{\partial \theta_{j|0}} = \sum_{j=1}^{d}\left[r_0^{(j)}/\theta_{j|0} + \left(\frac{r_0^{(i)} + n_1 - n}{1-\theta_{j|0}}\right)\right]$,

and $\frac{\partial}{\partial \theta_{j|1}} = \sum\left(r_1^{(i)}/\theta_{j|1} + \frac{r_1^{(i)} - n_1}{1-\theta_{j|1}}\right)$.

As observed, after one partial derivative with respect to a parameter, θ_p, only θ_p remains, all other terms drop as constants. ∴ driving with respect to another parameter, θ_q, would drop to zero, $\frac{\partial^2}{\partial \theta_p \theta_q} = 0$. QED

(b) $-f(\theta) \Rightarrow$

$\frac{\partial}{\partial \theta_{y=1}} = \frac{-n_1}{\theta_{y=1}} + \frac{n - n_1}{1-\theta_{y=1}}$ $\quad \frac{\partial}{\partial \theta_{j|1}} = \sum\left[\frac{-r_1^{(i)}}{\theta_{j|1}} + \frac{n_1 - r_1^{(i)}}{1-\theta_{j|1}}\right]$

$\frac{\partial^2}{\partial \theta_{y=1}^2} = n_1/\theta_{y=1}^2 + \frac{n-n_1}{(1-\theta_{y=1})^2}$ $\quad \frac{\partial^2}{\partial \theta_{j|1}^2} = \sum\left[\frac{r_1^{(i)}}{\theta_{j|1}^2} + \frac{n_1 - r_1^{(i)}}{(1-\theta_{j|1})^2}\right]$

$\frac{\partial}{\partial \theta_{j|0}} = \sum\left[-r_0^{(j)}/\theta_{j|0} + \frac{n - n_1 - r_0^{(j)}}{1-\theta_{j|0}}\right]$

$\frac{\partial^2}{\partial \theta_{j|0}^2} = \sum\left[r_0^{(j)}/\theta_{j|0} + \frac{n - n_1 - r_0^{(i)}}{(1-\theta_{j|0})^2}\right]$

because the Hessian is a diagonal matrix with elements equal to $\frac{\partial^2}{\partial \theta^2}$, thus we observe that

$x^T H x = \begin{bmatrix} x^{(0)} & x^{(1)} & x^{(2)} \end{bmatrix}\begin{bmatrix} \ddots & & 0 \\ & & \\ 0 & & \ddots \end{bmatrix}\begin{bmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{bmatrix}$

$= (x^{(0)})^2 \frac{\partial^2}{\partial \theta_{y=1}^2} + (x^{(1)})^2 \frac{\partial^2}{\partial \theta_{j|0}^2} + (x^{(2)})^2 \frac{\partial^2}{\partial \theta_{j|1}^2}$

because each second partial is ≥ 0, H is positive semidefinite.

(1.3) $D = \{(x_1, y_1), \ldots (x_n, y_n)\}$ where $x_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$

$$L_D(\tilde{\theta}) = \frac{1}{n} \sum_{i=1}^{n} (y_i - \tilde{\theta}^T \tilde{x}_i)^2$$

(a) $\frac{\partial}{\partial \tilde{\theta}} L_D(\tilde{\theta}) = \frac{\partial}{\partial \tilde{\theta}} \frac{1}{n} \sum_{i=1}^{n} (y_i - \tilde{\theta}^T \tilde{x}_i)^2$

$= \sum_{i=1}^{n} -2 x_i (y_i - \tilde{\theta}^T \tilde{x}_i)$

(b) $\frac{\partial^2}{\partial \tilde{\theta}} L_D(\tilde{\theta}) = \sum_{i=1}^{n} -2 \tilde{x}_i (-\tilde{x}_i^T) = \frac{2}{n} \sum \tilde{x}_i \tilde{x}^T$

$\frac{\partial}{\partial \theta_0} L_D(\theta_0) = \frac{\partial}{\partial \theta_0} \frac{1}{n} \sum \theta_0^2 = 2\theta_0 \quad \frac{\partial^2}{\partial \theta_0^2} = 2$

$\frac{\partial}{\partial \theta} L_D(\theta) = \frac{\partial}{\partial \theta} \frac{1}{n} \sum (y_i - \theta^T x_i)^2 = \frac{-2}{n} \sum x_i + x_i (y_i - \theta^T x_i)$

$\frac{\partial^2}{\partial \theta^2} = \frac{+2}{n} \sum x_i x_i^T$

HESSIAN $\Rightarrow \begin{bmatrix} \frac{\partial^2}{\partial \theta_0^2} & \cdots \\ \vdots & \frac{\partial^2}{\partial \theta^2} \cdots \\ \cdots & \frac{\partial^2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 2 & & \\ & \frac{2}{n} \sum_{i=1}^{n} x_i x_i^T & \\ & & \frac{2}{n} \sum_{i=1}^{n} \tilde{x}_i \tilde{x}_i^T \end{bmatrix}$

Programming Assignment 3: Sentiment analysis

In this programming assignment, you will implement logistic regression to predict the sentiment of reviews that come from `imdb.com` , `amazon.com` , and `yelp.com` . Make sure the notebook is in the same folder that contains `full_set.txt` .

1. Load and preprocess data

In [1]:
```
%matplotlib inline
import string
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
matplotlib.rc('xtick', labelsize=14)
matplotlib.rc('ytick', labelsize=14)
```

The data set consists of 3000 sentences, each labeled '1' (if it came from a positive review) or '0' (if it came from a negative review). To be consistent with our notation from lecture, we will change the negative review label to '-1'.

In [2]:
```
## Read in the data set.
with open("full_set.txt") as f:
    content = f.readlines()

## Remove leading and trailing white space
content = [x.strip() for x in content]

## Separate the sentences from the labels
sentences = [x.split("\t")[0] for x in content]
labels = [x.split("\t")[1] for x in content]

## Transform the labels from '0 v.s. 1' to '-1 v.s. 1'
y = np.array(labels, dtype='int8')
y = 2*y - 1

for i in range(5):
    print ('Label: ',y[i],'; ', sentences[i])
```

```
Label:  -1 ;  So there is no way for me to plug it in here in the US unless I
go by a converter.
Label:  1 ;  Good case, Excellent value.
Label:  1 ;  Great for the jawbone.
Label:  -1 ;  Tied to charger for conversations lasting more than 45 minutes.
MAJOR PROBLEMS!!
Label:  1 ;  The mic is great.
```

Preprocessing the text data

To transform this prediction problem into one amenable to linear classification, we will first need to preprocess the text data. We will do four transformations:

1. Remove punctuation and numbers.
2. Transform all words to lower-case.
3. Remove *stop words*.
4. Convert the sentences into vectors, using a bag-of-words representation.

We begin with first two steps.

In [3]:
```
## full_remove takes a string x and a list of characters removal_list
## returns x with all the characters in removal_list replaced by ' '
def full_remove(x, removal_list):
    for w in removal_list:
        x = x.replace(w, ' ')
    return x

## Remove digits
digits = [str(x) for x in range(10)]
digit_less = [full_remove(x, digits) for x in sentences]

## Remove punctuation
punc_less = [full_remove(x, list(string.punctuation)) for x in digit_less]

## Make everything lower-case
sents_lower = [x.lower() for x in punc_less]
```

Stop words

Stop words are words that are filtered out because they are believed to contain no useful information for the task at hand. These usually include articles such as 'a' and 'the', pronouns such as 'i' and 'they', and prepositions such 'to' and 'from'. We have put together a very small list of stop words, but these are by no means comprehensive. Feel free to use something different; for instance, larger lists can easily be found on the web.

In [4]:
```
## Define our stop words
stop_set = set(['the', 'a', 'an', 'i', 'he', 'she', 'they', 'to', 'of', 'it',
'from'])

## Remove stop words
sents_split = [x.split() for x in sents_lower]
sents_processed = [" ".join(list(filter(lambda a: a not in stop_set, x))) for
x in sents_split]
```

What do the sentences look like so far?

In [5]:
```
sents_processed[0:10]
```

Out[5]:
```
['so there is no way for me plug in here in us unless go by converter',
 'good case excellent value',
 'great for jawbone',
 'tied charger for conversations lasting more than minutes major problems',
 'mic is great',
 'have jiggle plug get line up right get decent volume',
 'if you have several dozen or several hundred contacts then imagine fun send
ing each them one by one',
 'if you are razr owner you must have this',
 'needless say wasted my money',
 'what waste money and time']
```

Bag of words

In order to use linear classifiers on our data set, we need to transform our textual data into numeric data. The classical way to do this is known as the *bag of words* representation. In this representation, each word is thought of as corresponding to a number in `{1, 2, ..., d}` where `d` is the size of our vocabulary. And each sentence is represented as a d-dimensional vector x, where x_i is the number of times that word i occurs in the sentence.

To do this transformation, we will make use of the `CountVectorizer` class in `scikit-learn` (Note that this is the only time you can call an external function from scikit-learn). We will cap the number of features at 4500, meaning a word will make it into our vocabulary only if it is one of the 4500 most common words in the corpus. This is often a useful step as it can weed out spelling mistakes and words which occur too infrequently to be useful.

Task P1: Once you get the bag-of-words representation, append a '1' to the beginning of each vector to allow our linear classifier to learn a bias term. What is the size of the resulting data_mat matrix?

In [6]:
```
from sklearn.feature_extraction.text import CountVectorizer

## Transform to bag of words representation.
vectorizer = CountVectorizer(analyzer = "word", tokenizer = None, preprocessor
= None, stop_words = None, max_features = 4500)
data_features = vectorizer.fit_transform(sents_processed)
print ('The original size: ',data_features.shape)
```

```
The original size:  (3000, 4500)
```

In [7]:
```
## STUDENT: YOUR CODE STARTS HERE
# Task: Append '1' to the beginning of each vector.
# Hint: You can use data_features.toarray() to transform data_features into a
 numpy array
# The output should be a numpy array named data_mat

extra_one = np.ones((3000,1), dtype=float)
data_mat = np.concatenate((extra_one, data_features.toarray()), axis=1)

## STUDENT: CODE ENDS
print ('The updated size: ',data_mat.shape)
```

```
The updated size:  (3000, 4501)
```

Training / test split

Finally, we split the data into a training set of 2500 sentences and a test set of 500 sentences (of which 250 are positive and 250 negative).

In [8]:
```
## Split the data into testing and training sets
np.random.seed(0)
test_inds = np.append(np.random.choice((np.where(y==-1))[0], 250, replace=Fals
e), np.random.choice((np.where(y==1))[0], 250, replace=False))
train_inds = list(set(range(len(labels))) - set(test_inds))

train_data = data_mat[train_inds,]
train_labels = y[train_inds]

test_data = data_mat[test_inds,]
test_labels = y[test_inds]

print("train data: ", train_data.shape)
print("test data: ", test_data.shape)
```

```
train data:  (2500, 4501)
test data:  (500, 4501)
```

2. Fitting a logistic regression model to the training data

In this section, we will implement our own logistic regression solver using gradient descent. As we have seen in the class, to learn the parameters of logistic regression, we need to perform the following optimization:

$$\tilde{\theta}_t = \underset{\tilde{\theta}}{\operatorname{argmin}}\ L_{\mathcal{D}}(\tilde{\theta}) = \underset{\tilde{\theta}}{\operatorname{argmin}} \sum_{i=1}^{n} \ln\left(1 + e^{y_i\ \tilde{\theta}_t^T \tilde{x}_i}\right)$$

where $y_i \in \{-1, +1\}$ is the label, $\tilde{\theta}$ is the vector of coefficients:

$$\tilde{\theta} = [\,\theta_0 \quad \theta_1 \quad \ldots \quad \theta_d\,]^T,$$

and $\tilde{x}$ is the "augmented" feature vector (of $d+1$ dimensions), where we stick a 1 in the front of the original features:

$$\tilde{x} = [\,1 \quad x_1 \quad \ldots \quad x_d\,]^T.$$

There is no nice, closed-form solution like with least-squares linear regression (http://en.wikipedia.org/wiki/Moore%E2%80%93Penrose_pseudoinverse) so we will use gradient descent (http://en.wikipedia.org/wiki/Gradient_descent) instead. Specifically we will use batch gradient descent which calculates the gradient from all data points in the data set. Luckily, the loss function $L_{\mathcal{D}}(\tilde{\theta})$ we want to minimize is convex (http://en.wikipedia.org/wiki/Convex_optimization) so there is only one minimum. Thus the minimum we arrive at is the global minimum.

Gradient descent is a general method and requires twice differentiability for smoothness (http://en.wikipedia.org/wiki/Smooth_function). It updates the parameters using a first-order approximation of the error surface.

$$\tilde{\theta}_{t+1} = \tilde{\theta}_t + \nabla L_{\mathcal{D}}(\tilde{\theta}_t)$$

Task P2: Derive the gradient of the loss $L_{\mathcal{D}}(\tilde{\theta})$ with respect to $\tilde{\theta}$, namely $\nabla L_{\mathcal{D}}(\tilde{\theta}_t)$. The answer should depend on data points (x_i, y_i) for $i = 1, \ldots, n$, and the model parameter $\tilde{\theta}$. Make sure you get the sign correct. Also implement the function `weight_derivative`. Print the output of the code.

In [9]:
```
def weight_derivative(weights, feature_matrix, labels):
    # Input:
    # weights: weight vector w, a numpy vector of dimension d
    # feature_matrix: numpy array of size n by d, where n is the number of dat
a points, and d is the feature dimension
    # labels: true labels y, a numpy vector of dimension d, each with value -1
or +1
    # Output:
    # Derivative of the regression cost function with respect to the weight w,
a numpy array of dimension d

    ## STUDENT: Start of code ###
    derivative_sum = 0.
    for i in range(len(labels)):
        outside = feature_matrix[i] * labels[i]
        inside = np.exp(-1 * labels[i] * np.dot(weights, feature_matrix[i]) )
        derivative_sum += outside * (1 / ( 1 + inside))


    return derivative_sum
    # End of code ###
```

In [10]:
```
# STUDENT: PRINT THE OUTPUT AND COPY IT TO THE SOLUTION FILE
my_weights = np.ones(data_mat.shape[1]) # a weight of all 1s
derivative = weight_derivative(my_weights,train_data,train_labels)

print (derivative[:10])
```

```
[ 1.23407200e+03  9.99999959e-01 -4.24835426e-18 -6.14417460e-06
  1.99987630e+00  2.99985907e+00 -1.79862100e-02  3.49776173e+01
  1.99996625e+00  1.00000000e+00]
```

Now, we can just use the same gradient descent algorithm that we wrote in assignment 2 to solve it.

In [11]:
```
def gradient_descent(feature_matrix, labels, initial_weights, step_size, toler
ance):
    # Gradient descent algorithm for logistic regression problem

    # Input:
    # feature_matrix: numpy array of size n by d, where n is the number of dat
a points, and d is the feature dimension
    # labels: true labels y, a numpy vector of dimension n
    # initial_weights: initial weight vector to start with, a numpy vector of
 dimension d
    # step_size: step size of update
    # tolerance: tolerace epsilon for stopping condition
    # Output:
    # Weights obtained after convergence

    converged = False
    weights = np.array(initial_weights) # current iterate
    i = 0
    while not converged:
        # impelementation of what the gradient descent algorithm does in every
iteration
        # Refer back to the update rule listed above: update the weight
        i += 1
        derivative = weight_derivative(weights, feature_matrix, labels)

        weights -= (step_size * derivative)

        # Compute the gradient magnitude:

        gradient_magnitude = np.sqrt(np.sum(derivative**2))

        # Check the stopping condition to decide whether you want to stop the
 iterations
        # print ("grad mag :", gradient_magnitude)
        #print ("tolerance:", tolerance)

        if gradient_magnitude < tolerance:
            converged = True

        print ("Iteration: ",i,"gradient_magnitude: ", gradient_magnitude) # f
or us to check about convergence

    return(weights)
```

Task P3: Specify the initial_weights, step_size, and tolerance for the function `gradient_descent` . Copy the outputs of the code to the solution file.

In [12]:
```
#Initialize the weights, step size and tolerance
# Start of code
#STUDENT: Specify the initial_weights, step_size, and tolerance
initial_weights =  np.zeros(4501)
step_size = 0.005
tolerance = 5
# end of code

# Use the regression_gradient_descent function to calculate the gradient decen
t and store it in the variable 'final_weights'
final_weights = gradient_descent(train_data,train_labels, initial_weights, ste
p_size, tolerance)

# end of code
print ("Here are the final weights after convergence:")
print (final_weights)
```

```
Iteration:  1 gradient_magnitude:  209.2175900826697
Iteration:  2 gradient_magnitude:  212.64421867541415
Iteration:  3 gradient_magnitude:  584.1017850068945
Iteration:  4 gradient_magnitude:  1256.785468117014
Iteration:  5 gradient_magnitude:  1536.9148922237887
Iteration:  6 gradient_magnitude:  1409.1836051491587
Iteration:  7 gradient_magnitude:  1458.5670556063646
Iteration:  8 gradient_magnitude:  1377.5557284445008
Iteration:  9 gradient_magnitude:  1300.7306897048818
Iteration:  10 gradient_magnitude:  1275.3901995059225
Iteration:  11 gradient_magnitude:  1210.8689930956264
Iteration:  12 gradient_magnitude:  1183.3122667955197
Iteration:  13 gradient_magnitude:  1143.7827656559937
Iteration:  14 gradient_magnitude:  1123.2213579344218
Iteration:  15 gradient_magnitude:  1097.5395865103742
Iteration:  16 gradient_magnitude:  1081.8619393447846
Iteration:  17 gradient_magnitude:  1061.382551300502
Iteration:  18 gradient_magnitude:  1047.2783473847555
Iteration:  19 gradient_magnitude:  1028.33031960631
Iteration:  20 gradient_magnitude:  1014.9038320157325
Iteration:  21 gradient_magnitude:  996.8935564366525
Iteration:  22 gradient_magnitude:  984.1076780231133
Iteration:  23 gradient_magnitude:  967.0239088547238
Iteration:  24 gradient_magnitude:  954.7326610353883
Iteration:  25 gradient_magnitude:  938.4105340162379
Iteration:  26 gradient_magnitude:  926.4127965970251
Iteration:  27 gradient_magnitude:  910.6622678459509
Iteration:  28 gradient_magnitude:  898.7848057155093
Iteration:  29 gradient_magnitude:  883.460324055539
Iteration:  30 gradient_magnitude:  871.5694147684499
Iteration:  31 gradient_magnitude:  856.567704891846
Iteration:  32 gradient_magnitude:  844.5740166434672
Iteration:  33 gradient_magnitude:  829.8284196486136
Iteration:  34 gradient_magnitude:  817.6958386292453
Iteration:  35 gradient_magnitude:  803.1757952524371
Iteration:  36 gradient_magnitude:  790.9223179043909
Iteration:  37 gradient_magnitude:  776.630511513173
Iteration:  38 gradient_magnitude:  764.3140257167513
Iteration:  39 gradient_magnitude:  750.2757801180821
Iteration:  40 gradient_magnitude:  737.9689018841347
Iteration:  41 gradient_magnitude:  724.2151793922668
Iteration:  42 gradient_magnitude:  711.9793910578471
Iteration:  43 gradient_magnitude:  698.5296323784795
Iteration:  44 gradient_magnitude:  686.3980477603345
Iteration:  45 gradient_magnitude:  673.2493328386929
Iteration:  46 gradient_magnitude:  661.2221773920269
Iteration:  47 gradient_magnitude:  648.3476589986032
Iteration:  48 gradient_magnitude:  636.39876527527
Iteration:  49 gradient_magnitude:  623.7539787363194
Iteration:  50 gradient_magnitude:  611.8430140899121
Iteration:  51 gradient_magnitude:  599.3758986510311
Iteration:  52 gradient_magnitude:  587.4606109236814
Iteration:  53 gradient_magnitude:  575.1208277085286
Iteration:  54 gradient_magnitude:  563.1656094899296
Iteration:  55 gradient_magnitude:  550.9104947902235
Iteration:  56 gradient_magnitude:  538.8903583561545
Iteration:  57 gradient_magnitude:  526.6871727888521
```

```
Iteration:  58 gradient_magnitude:  514.5882505547553
Iteration:  59 gradient_magnitude:  502.4139463958696
Iteration:  60 gradient_magnitude:  490.232290830791
Iteration:  61 gradient_magnitude:  478.07237403375234
Iteration:  62 gradient_magnitude:  465.81249656443856
Iteration:  63 gradient_magnitude:  453.66010817673754
Iteration:  64 gradient_magnitude:  441.33404720918537
Iteration:  65 gradient_magnitude:  429.189750161746
Iteration:  66 gradient_magnitude:  416.81690556319415
Iteration:  67 gradient_magnitude:  404.6891726768411
Iteration:  68 gradient_magnitude:  392.2967629528769
Iteration:  69 gradient_magnitude:  380.20285727512345
Iteration:  70 gradient_magnitude:  367.8266013077073
Iteration:  71 gradient_magnitude:  355.7933744864988
Iteration:  72 gradient_magnitude:  343.4778696356579
Iteration:  73 gradient_magnitude:  331.5419771529183
Iteration:  74 gradient_magnitude:  319.3402843144899
Iteration:  75 gradient_magnitude:  307.547451069085
Iteration:  76 gradient_magnitude:  295.5196197702691
Iteration:  77 gradient_magnitude:  283.9228620287215
Iteration:  78 gradient_magnitude:  272.1334663315304
Iteration:  79 gradient_magnitude:  260.79049122631613
Iteration:  80 gradient_magnitude:  249.30556587089515
Iteration:  81 gradient_magnitude:  238.27580186415088
Iteration:  82 gradient_magnitude:  227.15973799616583
Iteration:  83 gradient_magnitude:  216.50151181862836
Iteration:  84 gradient_magnitude:  205.81443658149323
Iteration:  85 gradient_magnitude:  195.58269942865454
Iteration:  86 gradient_magnitude:  185.37866330248985
Iteration:  87 gradient_magnitude:  175.6234522413175
Iteration:  88 gradient_magnitude:  165.949485027442
Iteration:  89 gradient_magnitude:  156.71508388863742
Iteration:  90 gradient_magnitude:  147.6109607604349
Iteration:  91 gradient_magnitude:  138.93557487588703
Iteration:  92 gradient_magnitude:  130.43402011517728
Iteration:  93 gradient_magnitude:  122.34972918882
Iteration:  94 gradient_magnitude:  114.47678260333875
Iteration:  95 gradient_magnitude:  107.00956998066466
Iteration:  96 gradient_magnitude:  99.7849149498983
Iteration:  97 gradient_magnitude:  92.95465087529884
Iteration:  98 gradient_magnitude:  86.39180487249278
Iteration:  99 gradient_magnitude:  80.21214450534154
Iteration:  100 gradient_magnitude:  74.3184991614723
Iteration:  101 gradient_magnitude:  68.79670932423879
Iteration:  102 gradient_magnitude:  63.57344575338471
Iteration:  103 gradient_magnitude:  58.710172824901846
Iteration:  104 gradient_magnitude:  54.15204784162417
Iteration:  105 gradient_magnitude:  49.94096659989784
Iteration:  106 gradient_magnitude:  46.03585808105781
Iteration:  107 gradient_magnitude:  42.463000360195224
Iteration:  108 gradient_magnitude:  39.190941857163686
Iteration:  109 gradient_magnitude:  36.23335805792003
Iteration:  110 gradient_magnitude:  33.5647082585785
Iteration:  111 gradient_magnitude:  31.18816887291432
Iteration:  112 gradient_magnitude:  29.080848312511723
Iteration:  113 gradient_magnitude:  27.236912176292073
Iteration:  114 gradient_magnitude:  25.63364343732374
```

```
Iteration:  115 gradient_magnitude:  24.257755625097204
Iteration:  116 gradient_magnitude:  23.08567152954206
Iteration:  117 gradient_magnitude:  22.099042437343662
Iteration:  118 gradient_magnitude:  21.274205496622386
Iteration:  119 gradient_magnitude:  20.59102839853628
Iteration:  120 gradient_magnitude:  20.027728082421643
Iteration:  121 gradient_magnitude:  19.56561037098103
Iteration:  122 gradient_magnitude:  19.186604935644556
Iteration:  123 gradient_magnitude:  18.87551562261523
Iteration:  124 gradient_magnitude:  18.61875917197336
Iteration:  125 gradient_magnitude:  18.40522000962763
Iteration:  126 gradient_magnitude:  18.22550672671518
Iteration:  127 gradient_magnitude:  18.072132886222924
Iteration:  128 gradient_magnitude:  17.939031838719774
Iteration:  129 gradient_magnitude:  17.821478528163816
Iteration:  130 gradient_magnitude:  17.715764573370677
Iteration:  131 gradient_magnitude:  17.619060008863656
Iteration:  132 gradient_magnitude:  17.529199605740764
Iteration:  133 gradient_magnitude:  17.444560587446784
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Iteration:  886 gradient_magnitude:  5.03087654763974
Iteration:  887 gradient_magnitude:  5.026503053644745
Iteration:  888 gradient_magnitude:  5.022137773824776
Iteration:  889 gradient_magnitude:  5.017780684983185
Iteration:  890 gradient_magnitude:  5.013431764008205
Iteration:  891 gradient_magnitude:  5.009090987872555
Iteration:  892 gradient_magnitude:  5.004758333633055
Iteration:  893 gradient_magnitude:  5.000433778430235
Iteration:  894 gradient_magnitude:  4.9961172994879535
Here are the final weights after convergence:
[ 0.2964133  -0.1068523   0.11042425 ...  0.         0.18507002
  0.26978528]
```

Task P4: Write the code to extract the y-intercept θ_0 and the rest of the parameters $\theta = [\,\theta_1 \quad \dots \quad \theta_d\,]^T$. Copy the code and print the outputs.

In [13]:
```
## STUDENT: CODE STARTS HERE
## Pull out the parameters (theta_0, theta) of the logistic regression model
theta0 = final_weights[0]
theta = final_weights[1:]
## STUDENT: CODE ENDS HERE
print ('y intercept: ',theta0)
print ('theta1 and theta2: ',theta[1],theta[2])
```

```
y intercept:  0.2964132952656125
theta1 and theta2:  0.11042424734280551 0.9606256296966943
```

Recall that the logistic/sigmoid function (http://en.wikipedia.org/wiki/Logistic_function) is given by

$$f(z) = \frac{1}{1 + e^{-z}}$$

Based on the logistic model, the posterior probability

$$P(y \mid \mathbf{x}) = f(y\ \tilde{\theta}^T \tilde{\mathbf{x}})$$

To make a prediction, we can simply choose the label $y \in \{-1, +1\}$ with the higher posterior probability.

Task P5: Write the code to make the prediction for a given data matrix. Report the training error and test error.

In [14]:
```
def model_predict(feature_matrix,weights):
# Prediction made by logistic regression

    # Input:
    # feature_matrix: numpy array of size n by d+1, where n is the number of d
ata points, and d+1 is the feature dimension
    #                 note we have included the dummy feature as the first col
umn of the feature_matrix
    # weights: weight vector to start with, a numpy vector of dimension d+1
    # Output:
    # labels: predicted labels, a numpy vector of dimension n

    ## STUDENT: YOUR CODE HERE
    predicted_labels = np.zeros(len(feature_matrix[0]))

    for i in range(len(feature_matrix)):
        inside = np.dot(weights, feature_matrix[i])
        predict_pos = 1 / (1 + np.exp(-1. * inside))
        predict_neg = 1 / (1 + np.exp(inside))
        predicted_labels[i] = 1 if predict_pos > predict_neg else 0

    return predicted_labels
    ## STUDENT: CODE ENDS
```

In [15]:
```
# STUDENT: copy the output of this section to the solution file

## Get predictions on training and test data
preds_train = model_predict(train_data,final_weights)
preds_test = model_predict(test_data,final_weights)

## Compute errors
errs_train = np.sum((preds_train > 0.0) != (train_labels > 0.0))
errs_test = np.sum((preds_test > 0.0) != (test_labels > 0.0))

print ("Training error: ", float(errs_train)/len(train_labels))
print ("Test error: ", float(errs_test)/len(test_labels))
```

```
Training error:  0.0004
Test error:  0.002

<ipython-input-15-92528e4315e1>:8: DeprecationWarning: elementwise comparison
failed; this will raise an error in the future.
  errs_train = np.sum((preds_train > 0.0) != (train_labels > 0.0))
<ipython-input-15-92528e4315e1>:9: DeprecationWarning: elementwise comparison
failed; this will raise an error in the future.
  errs_test = np.sum((preds_test > 0.0) != (test_labels > 0.0))
```

3. Analyzing the margin

As discussed in the lecture, the logistic regression model produces not just classifications but also conditional probability estimates.

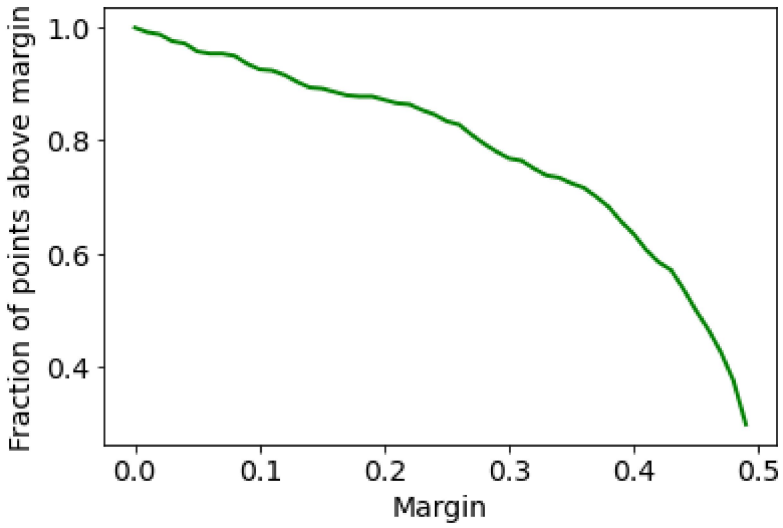
We will say that `x` has **margin** `gamma` if (according to the logistic regression model) `Pr(y=1|x) > (1/2)+gamma` or `Pr(y=1|x) < (1/2)-gamma`. For example, if `Pr(y=1|x)` is 0.7 according to the logistic regression model, then the margin is 0.2. If `Pr(y=1|x)` is 0.15 according to the logistic regression model, then the margin is 0.35.

Task P6: Implement the following function **margin_counts** that takes as input the learned weights $\tilde{\theta}$, the feature matrix (`feature_matrix`), and a value of `gamma`, and computes how many points in the data have margin at least `gamma`. Copy the code and the output plot (i.e., visualization of the test set's distribution of margin values) to the solution file.

In [16]:
```
def margin_counts(feature_matrix, weights, gamma):
## Return number of points for which Pr(y=1) lies in [0, 0.5 - gamma) or (0.5
 + gamma, 1]

    # Input:
    # feature_matrix: numpy array of size n by d+1, where n is the number of d
ata points, and d+1 is the feature dimension
    #                 note we have included the dummy feature as the first col
umn of the feature_matrix
    # weights: weight vector to start with, a numpy vector of dimension d+1
    # gamma: the margin value
    # Output:
    # number of points for which Pr(y=1) lies in [0, 0.5 - gamma) or (0.5 + ga
mma, 1]

    ## STUDENT: YOUR CODE HERE
    gamma_sum = 0
    for i in range(len(feature_matrix)):
        inside = np.dot(weights, feature_matrix[i])
        predict_pos = 1 / (1 + np.exp(-1. * inside))
        if (0 <= predict_pos < (0.5 - gamma)) or ((0.5 + gamma) < predict_pos
<= 1):
            gamma_sum += 1

    return gamma_sum

    ## STUDENT: CODE ENDS
```

We now visualize the test set's distribution of margin values.

In [17]:
```
gammas = np.arange(0,0.5,0.01)
f = np.vectorize(lambda g: margin_counts(test_data, final_weights,g))
plt.plot(gammas, f(gammas)/500.0, linewidth=2, color='green')
plt.xlabel('Margin', fontsize=14)
plt.ylabel('Fraction of points above margin', fontsize=14)
plt.show()
```



Next, we investigate a natural question: Are points `x` with larger margin more likely to be classified correctly?

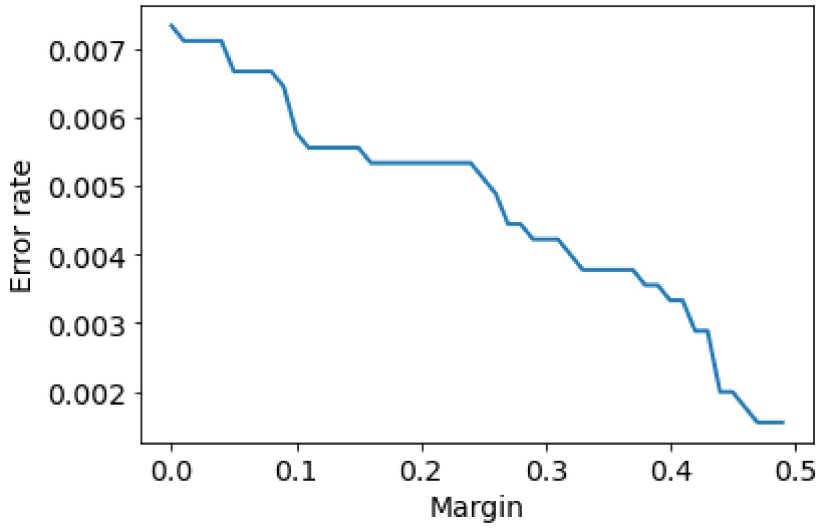
To address this, we define a function **margin_errors** that computes the fraction of points with margin at least `gamma` that are misclassified.

Task P7: Implement the function `margin_errors` that computes the fraction of points with margin at least `gamma` that are misclassified. Copy the code and the output plot (i.e., visualization of the relationship between margin and error rate) to the solution file. What do you observe from the plot?

In [18]:
```
def margin_errors(feature_matrix, labels, weights, gamma):
## Return error of predictions that lie in intervals [0, 0.5 - gamma) and (0.5
+ gamma, 1]

    # Input:
    # feature_matrix: numpy array of size n by d+1, where n is the number of d
ata points, and d+1 is the feature dimension
    #                 note we have included the dummy feature as the first col
umn of the feature_matrix
    # labels: true labels y, a numpy vector of dimension n
    # weights: weight vector to start with, a numpy vector of dimension d+1
    # gamma: the margin value
    # Output:
    # error of predictions that lie in intervals [0, 0.5 - gamma) and (0.5 + g
amma, 1]

    ## STUDENT: YOUR CODE HERE
    gamma_error = 0.
    predictions = model_predict(feature_matrix, weights)
    for i in range(len(feature_matrix)):
        inside = np.dot(weights, feature_matrix[i])
        predict_pos = 1 / (1 + np.exp(-1. * inside))
        if (0 <= predict_pos < (0.5 - gamma)) or ((0.5 + gamma) < predict_pos
<= 1):
            if predictions[i] == labels[i]:
                gamma_error += 1.

    return gamma_error / len(predictions)

    ## STUDENT: YOUR CODE ENDS
```

We now visualize the relationship between margin and error rate.

In [19]:
```
## Create grid of gamma values
gammas = np.arange(0, 0.5, 0.01)

## Compute margin_errors on test data for each value of g
f = np.vectorize(lambda g: margin_errors(test_data, test_labels,final_weights,
g))

## Plot the result
plt.plot(gammas, f(gammas), linewidth=2)
plt.ylabel('Error rate', fontsize=14)
plt.xlabel('Margin', fontsize=14)
plt.show()
```



4. Words with large influence

Finally, we attempt to partially **interpret** the logistic regression model.

Which words are most important in deciding whether a sentence is positive? As a first approximation to this, we simply take the words whose coefficients in θ have the largest positive values.

Likewise, we look at the words whose coefficients in θ have the most negative values, and we think of these as influential in negative predictions.

Task P8: Report the top 10 positive words (i.e., words with the largest positive coefficients of θ) and the top 10 negative words (i.e., words with the most negative coefficients of θ).

In [26]:
```
## Convert vocabulary into a list:
## This is a list where the i-th entry corresponds to the

vocab = np.array([z[0] for z in sorted(vectorizer.vocabulary_.items(), key=lam
bda x:x[1])])

## STUDENT: YOUR CODE HERE
sorted_words = np.argsort(final_weights)
pos_words = []
neg_words = []
for i in range(10):
    pos_words.append(vocab[sorted_words[i]])
    neg_words.append(vocab[sorted_words[-i]])

print("Top ten influential words:")
print("Positive Words")
for i in range(10):
    print(i + 1, ": ", pos_words[i])
print("Negative Words")
for i in range(10):
    print(i + 1, ": ", neg_words[i])

## STUDENT: CODE ENDS
```

```
Top ten influential words:
Positive Words
1 :  greater
2 :  loved
3 :  perfected
4 :  awful
5 :  delight
6 :  nicely
7 :  lovely
8 :  function
9 :  likes
10 :  beautifully
Negative Words
1 :  greater
2 :  disapppointment
3 :  poorly
4 :  wasted
5 :  worth
6 :  disappointment
7 :  arepas
8 :  sucks
9 :  badly
10 :  dog
```

5. (Bonus question) Classifiers that can abstain

Suppose you are building a classifier, and can tolerate an error rate of at most some value e. Unfortunately, every classifier you try has a higher error than this.

Therefore, you decide that the classifier is allowed to occasionally **abstain**: that is, to say *"don't know"*. When it actually makes a prediction, it must have error rate at most e. And subject to this constraint, it should abstain as infrequently as possible.

How would you build an abstaining classifier of this kind, starting from a logistic regression model? To get the bonus score, you need to show the following:

- A general description of the method
- Your code implementation
- A case study to show how you can use it in practice (including necessary plots)

In [21]:
```
## STUDENT: YOUR CODE HERE


## STUDENT: CODE ENDS
```