Discrete Mathematics and Theory (2120)

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MoWeFri 1:00 - 1:50

March 22

Fundamental Theorem of Arithmetic

For all natural numbers, there exists a factorization of n suh that all factors are prime and the factorization is unique.

Greatest Common Divisor

A **common divisor** of a and b is a number that divides them both. The **greatest common divisor** of a and b is the biggest number that divides them both.

GCD Examples

- GCD(12,30)
  - 6
- GCD(60, 8)
  - 4

Note that the GCD is the "intersection" of the prime factorizations of the two numbers. So, you can find CGD as follows:

- GCD(90, 84)
  - prime factors of 84 = $2^2 * 3 * 7$
  - prime factors of 90 = $2 * 3^2 * 5$
  - GCD(90, 84) = intersection of two prime factors = $2 * 3 = 6$

Relatively Prime - aka Coprime

Two positive integers greater than 1 are **relatively prime** if and only if gcd(x ,y) = 1. Two numbers that don't have *any* factors in common besides 1.

- Prime numbers are co-prime with all other numbers
- Given the predicate "isCoprime", $C(x, y) \in {\mathbb{Z}^+}^2$ and the predicate "isPrime" P(x) with domain $\mathbb{N}$: $\forall_x \in \mathbb{Z}^+.(\forall_y \in \mathbb{Z}+.y < x \to C(x,y)) \to P(x)$
  - In english: "If your number is coprime with all other numbers less than it, then it is prime."
- 1 is coprime with every other number

Proof by Contradiction - Natural Numbers (Informally)

Theorem: $\frac{1}{2}$ is not an element of the natural numbers.

Proof, in ideas but not in a formal way:

1. Assume that $\frac{1}{2} \in \mathbb{N}$, means $\exists_x \in \mathbb{N}.\frac{1}{2} = x$
2. 1=2x
3. if two numbers are equal, they must have the same prime factorization.
4. Prime factorization of $1 = \emptyset$

5. Prime factorization of $2x = \{2, x\}$.
6. One size has zero factors, one has at least 1?? contradiction.

Proof by Contradiction - Natural Numbers (Formally)

Theorem: $\frac{1}{2}$ is not an element of the natural numbers.

1. We proceed by contradiction
2. Assume $\frac{1}{2} \in \mathbb{N}$
3. Then, $\exists_x \in \mathbb{N}.x = \frac{1}{2}$, meaning there exists a natural number, x, such that $\frac{1}{2} = x$
4. By algebra, that means $2x = 1$.
5. By the fundamental theorem of arithmetic, both sides of the equation are equal, so 1 and 2x must have the same unique prime factorization.
6. But the factors of 2x include 2, and the factors of 1 do not. Therein lies the contradiction
7. Therefore, since the assumption led to a contradiction, $\frac{1}{2} \notin \mathbb{N}$

Proof by Contradiction - Rational Numbers

Represented by $\mathbb{Q}$, which stands for quotient. That's because all rational numbers can be repreented by a quotient of two integers! AKA an improper fraction.

$$x \in \mathbb{Q} \text{ iff } x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}^+$$

1. We proceed by contradiction
2. Suppose that $\sqrt{5} \in \mathbb{Q}$
3. Then, $\exists_a \in \mathbb{Z}.\exists_b \in \mathbb{Z}^+.\sqrt{5} = \frac{a}{b}$
4. Then by algebra, $5 = \frac{a^2}{b^2}$
5. Then by algebra, $5b^2 = a^2$
6. So, the factors of $5b^2$ must be the same as the factors of a^2 by the fundamental theorem of arithmetic.
7. The factors of a^2 do not include 5, but the factors of $5b^2$ do. Therein lies the contradiction. OR since these two numbers are equal, they have to have the same unique prime factorization. However, $5b^2$ must have an **odd multiplicity** of factor 5, while a^2 has an **even multiplicity of all its factors**, since squaring an integer simply doubles the multiplicity of that integer's original factors.
8. Because the two numbers are supposed to be equal but do not have the same prime factors, there is a contradiction.
9. This contradiction means that $\sqrt{5} \notin \mathbb{Q}$

March 20 - Proof (by contradiction)

Contradiction - something that derives to false, $\perp$. **Tautology** - something that comes out to true, $\top$.

Proofs Writeup

Proof by Contradiction

Provide a counter-example showing that $f(x) = 5x$ is *not surjective* given domain and co-domain of $\mathbb{Z}$.

- $f(x) = 5x$ is not surjective because there is no x in the domain for which $f(x) = 2$ (or any non-multiple of 5)
- $\nexists_x \in \mathbb{Z}.(f(x) = 2)$.

Prove that $f(x) = 5x$ is *not surjective* given domain and co-domain of $\mathbb{Z}$.

1. We proceed by contradiction
2. Suppose f(x) *is* a surjective function
3. By definition, a function is surjecitve if its co-domain is the same set as its range
4. However, this is not the case for f(x) because there are members of its co-domain that are not part of the function's range; for example, -6 is in the co-domain but not the range.
5. Therein lies the contradiction, therefore f(x) is not a surjective function ■

Rational Numbers Proof

Represented by $\mathbb{Q}$ which stands for quotient. That's because all rational numbers can be represented by a quotient of two integers

$$x \in \mathbb{Q} \text{ iff } x = \frac{a}{b} \text{ where } a, b \in \mathbb{Z} \text{ and } b \in \mathbb{Z}^+$$

Theorem: there is no smallest rational number larger than 0.

$$\forall_{n_1} \in \mathbb{Q}^+ . \exists_{n_2} \in \mathbb{Q}^+ . n_2 < n_1$$

1. We proceed by contradiction
2. $\neg(\forall_{n_1} \in \mathbb{Q}^+ . \exists_{n_2} \in \mathbb{Q}^+ . n_2 < n_1)$
3. Applying DeMorgan's Law - $\exists_{n_1} \in \mathbb{Q}^+ . \nexists_{n_2} \in \mathbb{Q}^+ . n_2 < n_1$ - "There exists an element in the domain such that there does not exist another element that is less than the first one." ~ "there exists a smallest positive rational."
4. Suppose there does exist a smallest rational number. Let's call it a, $a = n_1$.
5. Because all the numbers are positive rationals, by assigning n_2 to be $a/2$ it will be half as small as $a = n_1$, we can assert that $n_2 < n_1$. But this contradicts our assumption, since we stated a smaller positive rational *did not exist*.
6. Our assumption led to a contradiction, since we can always divide a positive rational number by another rational to get an even smaller one, as shown above. Because our assumption led to a contradiction, our assumption must be false, and we have proven the theorem.

Number Theory

The study of integers.

Intro to Integers - what is a factor?

a is a factor of b iff b can be evenly divided by a. THat is, for some non-zero integer k, $b = ak$.

What's Prime?

A prime number is greater than 1 that is divisible only by itself and 1.

Fundamental Theorem of Arithmetic

"For all positive natural numbers, there exists a factorization of n such that all factors are prime and the factorization is unique."

Note, *multiplicity* is the number of times a prime factor appears in the factorization.

March 17 - Relations Quiz

Question 1

Consider R(x, y) defined over $\mathbb{Z} \times \mathbb{Z}$ as $x^2 = y^2$. Which of the following properties does this have (reflexive, transitive, symmetric)?

- Reflexive: yes since $x^2 = x^2$
- Transitive: yes since $x^2 = y^2$ and $y^2 = z^2$ implies $x^2 = z^2$
- Symmetric: Commutativity over the equals sign, so yes

Question 2

Consider R(x, y) defined over $\mathbb{Q} \times \mathbb{Q}$ as $xy > 2$. Which of the following properties does this have (reflexive, transitive, symmetric)?

- Reflexive: no

- Transitive: no
- Symmetric: yes

Question 3

Consider $f(x, y) = x^2 - x$ defined with domain and co-domain $\mathbb{N}$. Which of the following properties does this have (total, injective, surjective)?

- Total: yes
- Injective: no
- Surjective: no

Question 3

Consider $f(x) = x^3$ defined with domain and co-domain $\mathbb{R}$. Which of the following properties does this have (total, injective, surjective)?

- Total: yes
- Injective: yes
- Surjective: yes

Questions 4 - 6

Consider the function with domain and co-domain $\mathbb{Z}$ defined by the formula $f(x) = \lfloor 1/x \rfloor$.

Question 4

Provide a counter-example showing that this function is **not total**.

- $f(0) = \lfloor 1/0 \rfloor = \lfloor \infty \rfloor = \infty$ is not in $\mathbb{Z}$

Question 5

Provide a counter-example showing this function is **not injective**.

- f(2) and f(3) are both 0, so this function is not injective.

Question 6

Provide a counter example showing this function is **not surjective**.

- There is no x in the domain for which $f(x) = 2$

Questions 7 - 11

Consider the relation $R(x, y) : x^2 > y$ where x and y are both from $\mathbb{Z}$.

Question 7

Provide a counter-example showing this relation is **not reflexive**.

- $R(1, 1)$

Question 8

Provide a counter-example showing this relation is **not irreflexive**

- $R(-1, -1)$

Question 9

Provide a counter-example showing this relation is **not transitive**.

Question 10

provide a counter example showing that the following relation R over memebrs of the set S = {0, 1, 2} is *not antisymmetric*. R(x, y) : (x > y) or (2x = y)

- (1, 2)

provide a counter example showing that the relation R defined above is NOT transitive

- (0,0) and (0,2)

March 13, 15:

Relation

Today we'll learn that a *relation* can be defined as a subset of a cross product.

$$R(x, y) \subseteq X \times Y$$

Given this definition, which of the following is **NOT** a valid relation over $A \times B$? I.e., select which of the following are NOT subsets of $A \times B$. Recall that a cross product is a set of sequences.

Let A = {2, 3, 4} and B = {4, 5}

a. {(4, 4)}

b. {(2, 4), (3, 4), (4, 5)}

c. {(4, 4), (5, 4)}

d. $\{(a, b) | a \in A \wedge b \in B\}$

e. $\{(a, b) | a \in A \wedge b \in B \wedge b = a + 1\}$

f.

Correct answer: c is not a subset of $A \times B$ since (5, 4) is not in $A \times B$.

Relations

Math	English
$A \times B$	the cross product, $A \times B$
$R \subseteq A \times B$	R is a relation from A to B
$aRb; (a, b) \in R$	a is related to b
$aRb; (a, b) \notin R$	a is not related to b

For sets $A, B \subseteq u$, any subset of $A \times B$ is called a *relation* from A to B.

Binary (Homogenous) Relations and Notation

Let R be a binary homogenous relation on a Set A, that is:

xRy where $R \subseteq A \times A$. Note that:

$$xRy = R(x, y) := ((x, y) \in R)$$

Properties of Binary Relations: Reflexivity

Definition: A binary relation R on a set is reflexive iff:

$$\forall_x \in A. xRx$$

- "All members of the domain are related to themselves."

Some examples of this include $x = x$ or $x \leq x$.

Properties of Binary Relations: Irreflexivity

Definition: A binary relation R on a set A is *irreflexive* iff:

$$\forall_x \in A. \neg xRx$$

Some examples of this include $x \neq x$ or $x > x$ or $x < x$.

Definition: A function is irreflexive iff its complement is reflexive: $R_{irreflexive} = (R_{reflexive})^C$

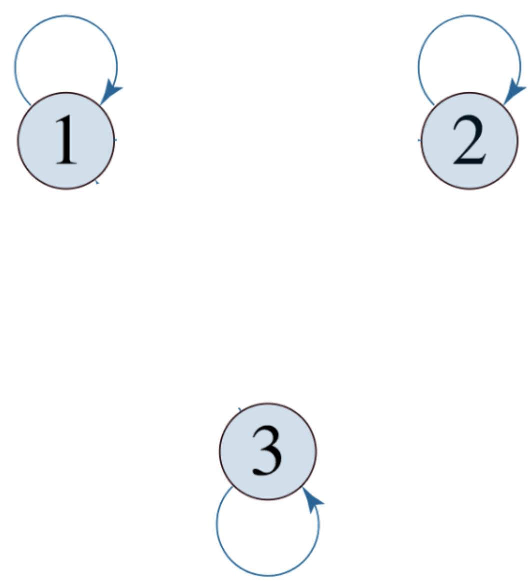
Irreflexive relations have zeroes down the main diagonal.

Proof

- A function is irreflexive iff its complement is reflexive: $R_{irreflexive} = (R_{reflexive})^C$
- consider any reflexive relation, R(x, y) over any set A.
- By definition, if R is reflexive, then all members of A must relate to themselves (define these as reflexive pairs). So, the reflexive pairs are a subset of R: $\{(x, x) \in A^2 | R(x, x)\} \subseteq R$

Consider the adjacency matrix for reflexive relations:

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \end{array}$$



Now for the irrefleive relation, we can see that the main diagonal will have 0 rather than 1.

- Consider the relation $I = R^C$. Now we will prove that I must be irreflexive. By definition, if I is irreflexive, then we know then none of the reflexive pairs will be present (since a relation R is defined as irreflexive if $\forall_x \in A. \neg xRx$.
- We know that this is the case for I(x, y) because if an ordered pair is present in R, it will not be a member of $I = R^C$ since by definition

$$I = R^C = \{(x, y) \in A^2 | \neg R(x, y)\}$$

- Following from this definition, for any subset $P \subseteq R \subseteq A^2$, we know those pairs **cannot** be in the complement ie $R^C \backslash P = R^C = I$. Now consider P to be defined as the reflexive pairs, $\{(x, x)|x \in A\}$. Finally, since removing the reflexive pairs resulted in the same set, we know that none of them are present in I. By definition, if no reflexive pairs are present in I(x,y), then the relation is irreflexive.

Properties of Binary Relations: Reflexivity over $\mathbb{R}^2$

$$\forall_x \in \mathbb{R}.xRx$$

All members of the domain are related to themselves

- x = y
- $x \geq y$
- x < y+1

Look at the desmos example below. If x = y is in the graph (shaded), then it is reflexive.

- So, for equations like $(x-1)^2 + 3 > y$, x=y is always in it. Therefore, it is a reflexive relation. If any part of x=y is not in the graph, then it is not reflexive.

Properties of Binary Relations: Symmetry

Definition: A binary relation R on a set A is *symmetric* iff:

$$\forall_{x,y} \in A.xRy \rightarrow yRx$$

Since x and y are both quantified over the same set A, it follows that $\forall_{x,y} \in A.(xRy \rightarrow yRx) \wedge (yRx \rightarrow xRy)$ then $\forall_{x,y} \in A.xRy \leftrightarrow yRx$. This is the same thing as commutativity!!

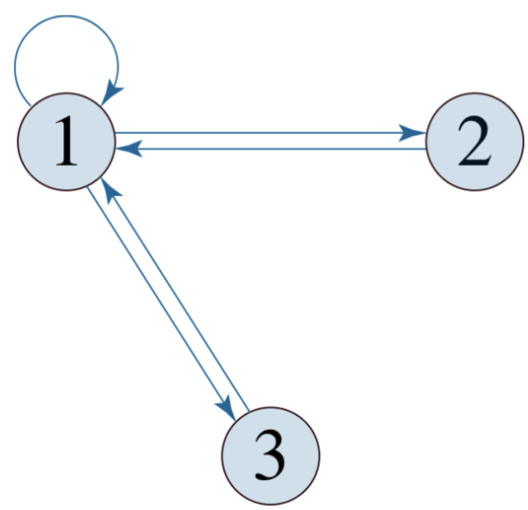
Symmetric matrices can be transposed and the result will be the same matrix.

- $A = A^T$ if and only if A is symmetric.

Symmetry Example 1 is < symmetric? NO!!

Symmetry Example 2 R = {(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)} on the set A = {1, 2, 3}

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$



Symmetry over $\mathbb{R}^2$ **Definition**: a binary relation R is symmetric iff:

$\forall_{x,y} \in \mathbb{R}.xRy \rightarrow yRx$

A symmetric relation over the reals must include its reflection over y=x and y=-x

Examples: Symmetry Example 1: Let $R \subseteq \mathbb{R}^2$ be defined as: "less than," $<$

- Not symmetric!!

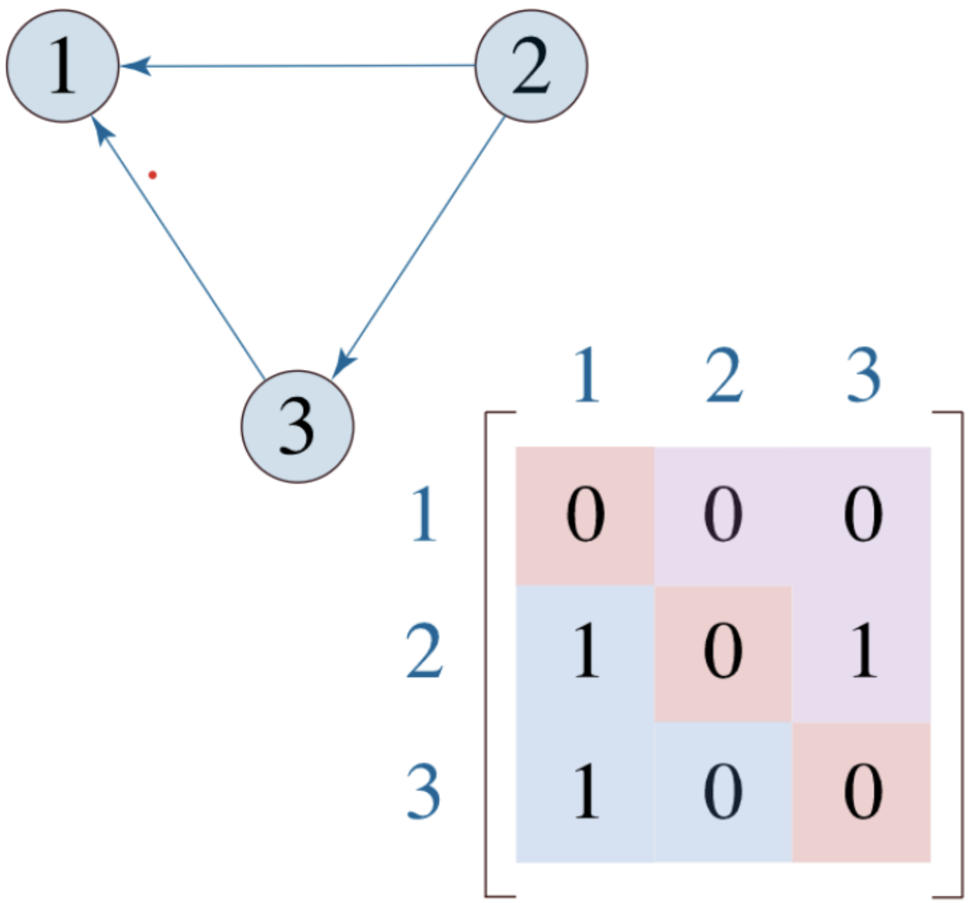
Example 2: Let R be equal to $L \subseteq P \times P$ be defined as L(x, y) means "x loves y," where P is the set of people.

Reflexivity vs. Symmetry Reflexivity relates to ITSELF vs. Symmetry is relating to something else! However, being symmetric doesn't mean you are reflexive (necessarily).

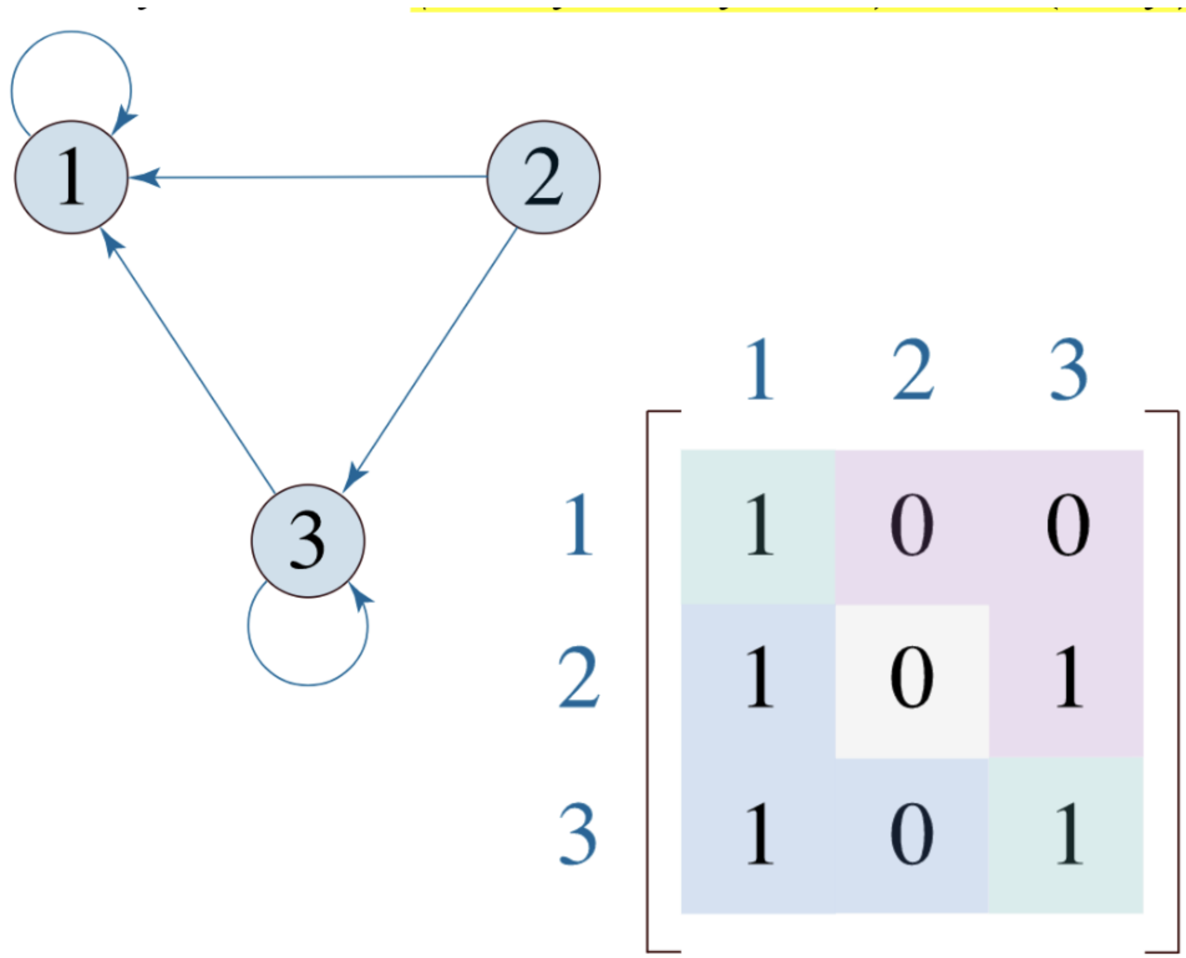
Properties of Binary Relations: Antisymmetry and Asymmetry

Asymmetry: **Definition**: A binary relation R on a set A is **asymmetric** iff $\forall_{x,y} \in A . xRy \rightarrow \neg yRx$

If it is asymmetric, it is irreflexive and thus all values on the "main diagonal" are zeroes.



Antisymmetry: **Definition**: A binary relation R on a set A is **antisymmetric** iff $\forall_{x,y} \in A (xRy \land yRx) \rightarrow x = y$



In antisymmetric relations, reflexivity is allowed but not required.

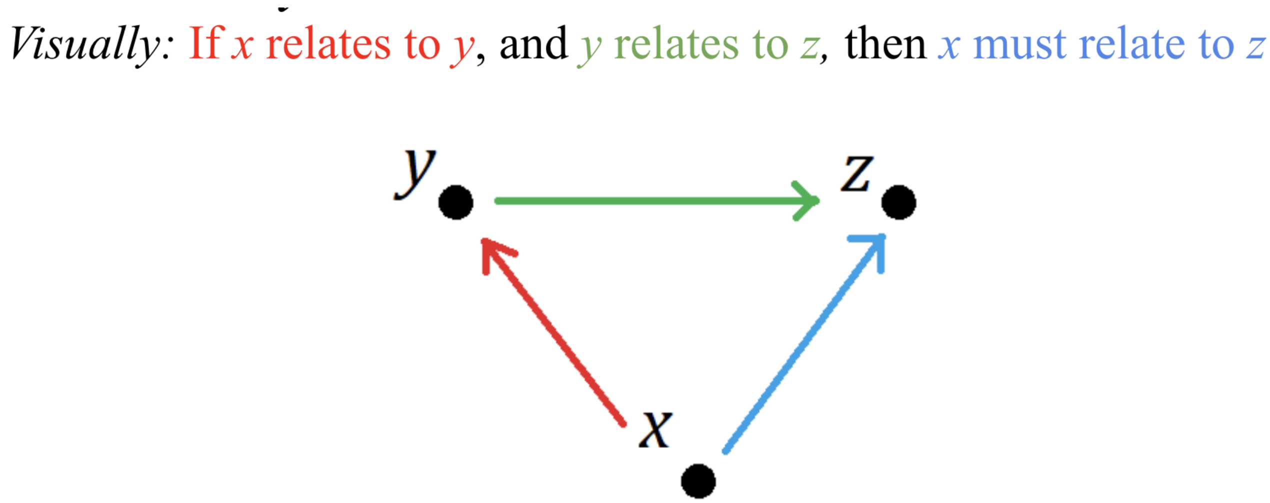
Conclusion: Asymmetry entails antisymmetry.

Examples: Less than Sign The less than sign is asymmetric and thus entails antisymmetry.

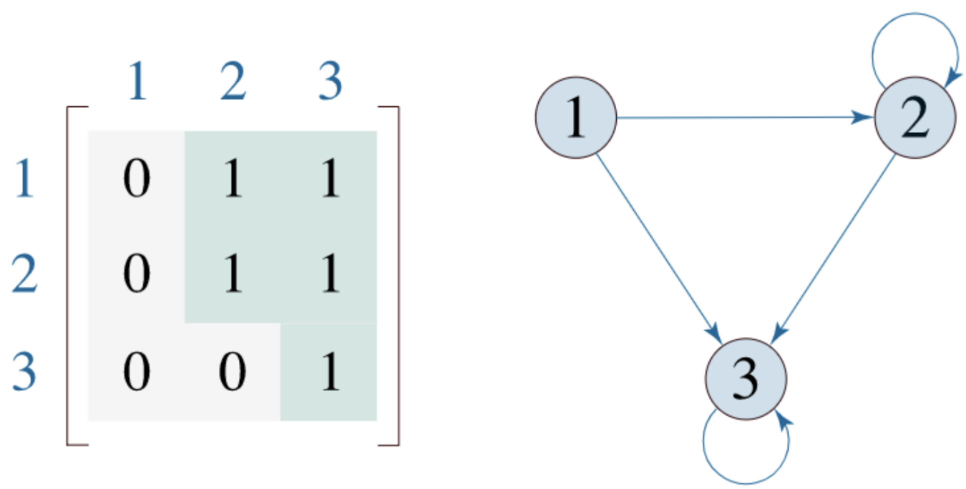
Examples: Loves If loves is asymmetric, love is not reciprocated and no one loves themselves. If loves is antisymmetric, love is not reciprocated but you can love yourself.

Transitivity

Definition: A binary relation R on a set A is transitive iff: $\forall_{x,y,z} \in A.(xRy \land yRz) \to xRz$



Note that the arrows are not going in a circle. x goes to y, y goes to z, and then x goes to z. It's hard to visualize in a matrix, easier with vectors. Below is the matrix. Note that the reflexive relationships don't change transitivity.



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{array}$$

Examples: Less than Sign. Transitive or Not?

- Transitive! If $3 > 5$ and $5 > 8$, then $3 > 8$

Equivalence Relation

Definition: A binary relation R on a set A is *an equivalence relation* iff it is *reflexive, symmetric and transitive.*

- Example: "The equals sign"

Partial Order

Definition: A binary relation R on a set A is a *partial order* iff it is *antisymmetric and transitive.*

- Example: "the less than sign $<$ or"A is a subset of B", $A \subseteq B$

Relations Vocab

- FILL IN

Function

Functions assign an element of one set to another set. Functions are a subset of relation. Functions are special, though - every A can map to *at most* one B. For example, if we consider $f_1(x) : \mathbb{R} \to \mathbb{R}$, then $f_1(x) = 1/x^2$. In this there are parts of the function that are not mapped.

$$f : A \to B$$

We are used to the notation $f(a) = b$ indicated that f assigns the element $b \in B$ to the element $a \in A$.

Codomain: A function need not be able to return every lement of its codomain.

Range: subset of the codomain

Domain: inputs

Surjective

A surjective function is a function where: $\forall_b \in B, \exists_a \in A, f(a) = b$

Examples: x^2 Is x^2 a surjective function over $\mathbb{R} \to \mathbb{R}$? NO! It is not surjective because it does not map to every element of the codomain. You can't reach the negatives!! But, x^2 is surjective over $\mathbb{R} \to [0, \infty]$.

Injective

$\forall_{a_1} \in A, \forall_{a_2} \in A, (f(a_1) = f(a_2) \to a_1 = a_2)$

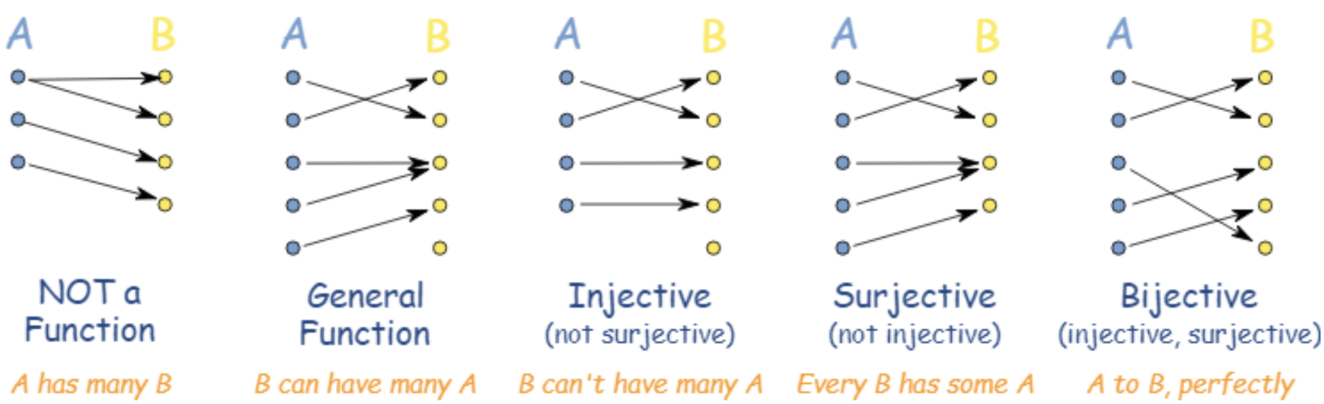
This basically means that every input maps to a different output.

Bijective

Both injective and surjective - one-to-one and every input has an output.

Summary of Functions

Here's a photo that summarizes all functions (and non-functions):



March 3:

A Note on Notation

$\forall_z.(\exists_y.A(y) \lor E(y)) \to A(z)$

- The period is a replacement for the parenthesis which tells us the *scope* of the quantifiers. Periods tell us that the scope of the quantifier is starting from it to the end of the statement or a closing parenthesis that came before the quantifier.

- You can rewrite this as:
  - $\forall_z[(\exists_y[A(y) \lor E(y)]) \to A(z)]$

Example

Domain: People. Let F = {Apollo, Britomartis, Cupid, Demeter, Bob}. L(x, y) = x loves y

What is $\forall_x \in F.\forall_y \in F.L(x, y)$ -> everyone loves everyone!

Negating Quantifiers If you see this $\nexists$, rewrite it as $\neg\exists$. Also, swap the quantifier ($\neg\forall$) to $\exists\neg(Stuff)$.

- $\neg\exists_x(F(x) \lor A(x))$ becomes $\forall_x\neg(F(x) \lor A(x))$
  - which also becomes $\forall_x.(\neg F(x) \land \neg A(x))$

DeMorgan's Practice

Rewrite: $\neg(\neg\forall_{x,y} \in F.L(x, y))$

- $\neg(\exists_{x,y} \in F.\neg L(x, y))$ -> $\forall_{x,y} \in F.L(x, y)$. Note that this was just a double negation!

Rewrite $\neg(\neg\forall_{x,y} \in F.L(x, y))$ in four different ways!!

1. $\neg\exists_x \in F.\neg\forall_y \in F.L(x, y)$
2. $\neg(\exists_x \in F.\exists_y \in F.\neg(L(x, y)))$
3. $\neg\exists_x \in F.\exists_y \in F.\neg(L(x, y))$
4. $\neg(\exists_{x,y} \in F.\neg L(x, y))$
5. $\forall_x \in F.\neg\exists_y \in F.\neg L(x, y)$
6. $\forall_{x,y} \in F\neg\neg L(x, y)$
7. $\forall_{x,y} \in F.L(x, y)$

Note that this is all one big loop - all of these are different ways to say the same thing, $\forall_{x,y} \in F.L(x, y)$.

One Trick to Avoid

Note that $\nexists_{x,y}.L(x, y) \equiv \neg\exists_x.\exists_y.L(x, y)$

Questions to Submit for extra credit

Q1: $\exists_x \in F.\exists_y \in F.\neg L(x, y)$

- There exists someone who doesn't love someone

Q2: $\exists_x \in F.\neg\exists_y \in F.L(x, y)$

- There exists someone who doesn't love anyone

Q3: $\forall_x \in F. \nexists_y \in F.\neg L(x, y)$

- $\forall_x \in F.\neg \nexists_y \in F.\neg L(x, y)$
- Everyone doesn't love someone.

Entailment

Convert this deductive argument into Propositional Logic

1. 6 is an even number (P: 6 is an even number)
2. All real numbers that are even are integers (Q: All real numbers that are even are integers)
3. 6 is an integer (R: 6 is an integer)

Predicate Logic

- $s := 6$
- E(x) = x is an even number
- Z(x) = x is an integer
- Domain: real numbers

1. E(S)
2. $\forall_x.E(x) \rightarrow Z(x)$
3. Z(s)

Universal Quantifiers and Conjunctions $\forall_x.E(x) \rightarrow Z(x)$: "All real numbers that are even are integers." Note that you must use $\rightarrow$ rather than $\wedge$.

$\exists_x.E(x) \wedge Z(x)$: "There exists a real number that is even and an integer." Note that you used the $\wedge$ rather than the $\rightarrow$. Using $\rightarrow$ is vacuously true.

Entailment

- Metalanguage:
  - Ex: comments in your code, talking in English about French, analyzing the tone, narrator, and grammar of a passager.

Feb 27

Warm-Up

Select all that are equivalent to the situation "Somebody is loved by everybody." If they aren't equivalent, give a counterexample or explain why it's incorrect.

1. $\exists_y.\forall_x(L(x,y))$
   - Correct
2. $\forall_x(\exists_y(L(x,y)))$
   - Incorrect - this can be phrased as "everybody loves at least one person."
3. $\exists_y.\forall_x L(y,x)$
   - Incorrect - this can be expressed as "there exists at least one person who loves everyone."
   - Consider the situation where someone loves everyone but everyone else doesn't love them back
4. $\forall_x.\exists_y.L(y,x)$
   - Incorrect - "everyone is receiving love"
5. $\exists_x.\forall_y L(y,x)$
   - Correct - "there exists somebody who's loved by everybody.

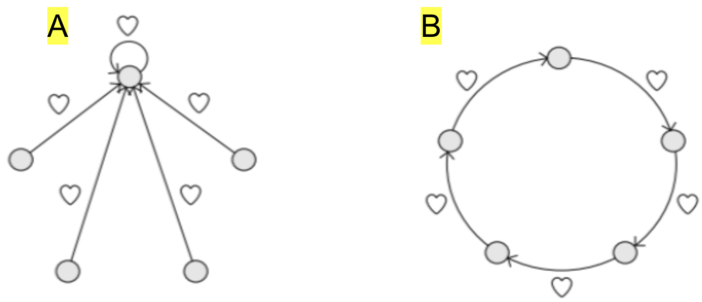
Notice how 1 and 5 are similar! They're the same statement just with "flipped" variables. Also, recall how the syntax of using . instead of parenthesis doesn't change it - they're different ways to say the same thing.

Quick Intro to Multiple Quantifiers:

Which diagram could satisfy which logical expression?

Domain: People   L(x, y) = *x loves y*

1. ∃y. ∀x (L(x,y))
2. ∀x (∃y (L(x,y))
3. ∃y. ∀x. L(y,x)
4. ∀x. ∃y. L(y,x)



1. Picture A
2. Both
3. Neither
4. Picture B

Quick Intro to Multiple Quantifiers

Switching order does not always mean logical equivalence!!

$$\exists_y \forall_x L(x,y) \not\equiv \forall_x \exists_y L(x,y)$$

However, notice how even though L(x, y) is *not* commutative the variables are commutative over the same quantifiers: $\forall_x \forall_y L(x,y) \equiv \forall_y \forall_x L(x,y)$ and if you are using the same quantifiers you can also write it as $\forall_{x,y}.L(x,y)$

interesting note: see that $\forall_x \exists_y L(x,y) \vDash \exists_y \forall_x L(x,y)$

Think about Boolean Logic with Quantifiers!!

Domain: {Ann, Bob, Chris}

- $\exists_y.\forall_x.L(x,y)$.
  - $(L(Ann, Ann) \wedge L(bob, ann) \wedge L(chris, ann)) \vee (L(ann, bob) \wedge L(bob, bob) \wedge L(chris, bob)) \vee (L(Ann, chris) \wedge L(bob, chris) \wedge L(Chris, chris))$

Examples

Domain	People
H(x)	x is happy
C(x)	x is in this class
A(x, y)	x appreciates y

1. Everyone is happy

- $\forall_k.H(k)$
2. Everyone in this class is happy
   - $\forall_x.(C(x) \to H(x))$
3. Someone is happy
   - $\exists_x.H(x)$
4. Someone in this class is happy
   - $\exists_x.(C(x) \land H(x))$
   - Note that I used the $\land$ rather than the $\to$!! This is because it is satisfied if anyone is not in the class **or** anyone is happy.
5. Not everyone is happy
   - $\neg\forall_x H(x)$
6. Only one person is happy
   - $\exists_x.H(x) \land \forall_y.(y \neq x) \to \neg H(y)$

Feb 24

Group Activity 1

Feb 22

Universal Quantifier

$\forall$ = "for all" or "given any." It expresses that a propositional function can be satisfied by *every member of the domain.*

Ambiguous Question

"Everybody does not love Chris" or "Everybody hates Chris." How cna you rephrase this?

- For all people, each one does not love Chris.
  - $\forall_x.\neg L(x, \text{Chris})$
- There does not exists one person who loves Chris.
  - $\neg(\forall_x.L(x, \text{Chris}))$
  - This can be translated as "It is not the case that across all the people everybody loves Chris."

$\exists$: Exists - Existential Quantifier

Consider the exists symbol, the complement to $\forall$. This means "there exists" or "there is at least one" or "for some." It expresses that a propositional function *can be satisfied by at least one member of the domain.*

"There does not exist one person who loves Chris."

- $\neg(\exists_x.L(x, \text{Chris}))$ or $\neg\exists_x.L(x, \text{Chris})$

You can also use $\nexists$ instead of $\neg\exists$. You can also do $\exists_x\exists_y$ as $\exists_{x,y}$

Interesting paradox

Remember that:

- $\neg\exists_x L(x, \text{Chris})$ means "There does not exist one person who loves Chris."
- $\forall_x\neg L(x, \text{Chris})$ means "For all people, each one does not love Chris."

Therefore:

$$\neg\exists_x.L(x, \text{Chris}) \equiv \forall_x.\neg L(x, \text{Chris})$$
$$\not\equiv$$
$$\exists_x.\neg L(x, \text{Chris}) \equiv \neg\forall_x.L(x, \text{Chris})$$

What does the . mean?

Note that $\exists_x.L(x) \equiv \exists_x(L(x))$. This just specifies scope.

Importance of Domains Example

Is the logical expression $\forall_x.Q(x)$ true or false with $Q(x) = (x^2 \geq x)$. False!!

- Well, I mean... what is the domain? That's the trick question! We need to specify what x is before we can evaluate this expression.

Now, here's a different question:

$\forall_x \in \mathbb{Z}(Q(x))$?

- $\top$

$\forall_x \in \mathbb{R}.Q(x)$?

- $\perp$

We can evaluate these since they have a specific domain!

$\exists$ and $\forall$

Associate "for all" with *and's* since it becomes false if just *one* truth value is false, and associate "there exists" with *or's* since it becomes true if just one truth value is true.

$\exists$ and $\forall$

What about:

- $\forall_x \in .Q(x)$
  - $\top$ - *vacuously* true.

consider it as a for loop:

```
def for_all(S)
    Q = True
    for x in S:
        Q = Q and Q(x)
    return Q
```

- $\exists_x \in .Q(x)$
  - $\perp$ - *vacuously* false

Feb 20

Review Mod1Multi2

Question 1: Can you apply simplification $((P \to Q) \to R) \vee (P \wedge Q)$?

- Yes!! Recall that simplification can go *both* ways. You could add an $\vee\perp$.

Question 2: $(P \vee R) \wedge (Q \vee R) \oplus (P \vee Q)$

- This question was taken off because there are no parenthesis to tell you which order to do things. Thus, it should either be:
  - $((P \vee R) \wedge (Q \vee R)) \oplus (P \vee Q)$
  - $(P \vee R) \wedge ((Q \vee R) \oplus (P \vee Q))$

Good to have in your back pocket

english phrase	DMT phrase
If p then q	p implies q
if p, q	p only if q
p is sufficient for q	a sufficient condition for q is p
q if p	q whenever p
q when p	q is necessary for p
a necessary condition for p is q	q follows from p
q unless $\neg p$	q provided that p

Predicates and First-Order Logic

We can only do so much with atomic propositions. To say more interesting things like "All files that are larger than 1,000 blocks are to be moved to backup provided that they have not been referenced within the last 100 days and that they are not in system files," we need more!

- Atomic propositions: just letters that reference a proposition, e.g. P = "It is sunny today" or Q = "It's Friday"
- A predicate is a proposition that has a(n) argument(s).

Predicates

Three different definitions of a predicate:

- A function that evaluates to true or false
- A proposition missing the noun(s)
- A proposition template

Determine the predicate and the arguments of the following sentence: "Sam loves Diane."

- L(x, y) = **_x loves** _y
- "Sam loves Diane" formalizes to L(Sam, Diane)
- "Diane doesn't love Sam" formalizes to $\neg L(\text{Diane}, \text{Sam})$
- "I love lucy" formalizes to $L(i, Lucy)$
- Note that $L(\text{Sam}, \text{Diane}) \not\equiv L(\text{Diane}, \text{Sam})$! It is *not* commutative. Think of this L(x, y) as a binary function. Love is directional. Diane can loves Sam without Sam loving Diane

What about the statement "Everyone loves Raymond?"

- First you have to specify what your "universe" or "domain" is. Let's say our domain is U = {A, B, C, Raymond}.
- $\forall_x \in U.L(x, \text{Raymond})$
- This is a "for all" operator which is synonymous with a repetitive "and" operator, since you could also just write it as L(A, Raymond) ∧ L(B, Raymond) ∧ L(C, Raymond) ∧ L(Raymond, Raymond)
- Also note that you don't have to specify what set the quantifier is in if the domain is specified, as it assumes it is in the domain if not specified

Feb 17

Practice before Mod1Quiz2:

Review

$\neg(A \wedge B)$ and $(\neg A \vee \neg B)$ - DeMorgan's

If you were to go from $\neg Q \vee R$ to $\neg(Q \wedge \neg R)$, make sure to also include the step of double negation in between : $\neg(Q \wedge \neg\neg R)$

Associativity Rule - when in doubt, you can always do expressions as variables.

Feb 15

Did an in-class worksheet. These are the correct answers

1: Prove $A \wedge (A \vee B) \equiv A$

symbol	Equation	Reasoning
	$A \wedge (A \vee B)$	Given
$\equiv$	$(A \vee \bot) \wedge (A \vee B)$	distributive property
$\equiv$	$A \vee (\bot \wedge B)$	simplification
$\equiv$	$A \vee \bot$	simplification
$\equiv$	A	simplification

2: Prove $(P \vee \neg P) \to P \equiv P$

symbol	equation	reasoning
	$(P \vee \neg P) \to P$	given
$\equiv$	$\neg(P \vee \neg P) \vee P$	definition of implication
$\equiv$	$(\neg P \wedge \neg\neg P) \vee P$	DeMorgan's Law
$\equiv$	$(\neg P \wedge P) \vee P$	Double Negation
$\equiv$	$\bot \vee P$	simplification
$\equiv$	P	simplification

An alternate solution is:

symbol	equation	reasoning
	$(P \vee \neg P) \to P$	given
$\equiv$	$\top \to P$	simplification
$\equiv$	$\neg\top \vee P$	definition of implication
$\equiv$	$\bot \vee P$	simplification
$\equiv$	P	simplification

3: Prove $\neg A \wedge \neg B \equiv \neg A \wedge (B \to A)$

symbol	equation	reasoning
	$\neg A \wedge \neg B$	given
$\equiv$	$(\neg A \wedge \neg B) \vee \bot$	simplification
$\equiv$	$(\neg A \wedge \neg B) \vee (\neg A \wedge A)$	simplification
$\equiv$	$\neg A \wedge (\neg B \vee A)$	Distributive Property
$\equiv$	$\neg A \wedge (B \to A)$	Definition of implication

4: Prove: $R \wedge \neg(P \to Q) \equiv P \wedge (\neg Q \wedge R)$

symbol	equation	reasoning
	$R \wedge \neg(P \to Q)$	Given
$\equiv$	$R \wedge \neg(\neg P \vee Q)$	definition of implication
$\equiv$	$R \wedge (\neg\neg P \wedge \neg Q)$	DeMorgan's Law
$\equiv$	$R \wedge (P \wedge \neg Q)$	double negation
$\equiv$	$R \wedge (\neg Q \wedge P)$	commutative
$\equiv$	$(R \wedge \neg Q) \wedge P$	associative

symbol	equation	reasoning
$\equiv$	$P \wedge (R \wedge \neg Q)$	commutative
$\equiv$	$P \wedge (\neg Q \wedge R)$	commutative

5: Prove: $(X \rightarrow Y) \wedge (\neg X \rightarrow \neg Y) \equiv X \leftrightarrow Y$

symbol	equation	reasoning
	$(X \rightarrow Y) \wedge (\neg X \rightarrow \neg Y)$	given
$\equiv$	$(X \rightarrow Y) \wedge (\neg\neg X \vee \neg Y)$	definition of implication
$\equiv$	$(X \rightarrow Y) \wedge (X \vee \neg Y)$	Double negation
$\equiv$	$(X \rightarrow Y) \wedge (\neg Y \vee X)$	commutative
$\equiv$	$(X \rightarrow Y) \wedge (Y \rightarrow X)$	definition of implication
$\equiv$	$X \leftrightarrow Y$	definition of bimplication

6: Prove $\neg(P \vee M) \rightarrow \neg M \equiv \top$

symbol	equation	reasoning
	$\neg(P \vee M) \rightarrow \neg M$	given
$\equiv$	$\neg\neg(P \vee M) \vee \neg M$	definition of implication
$\equiv$	$(P \vee M) \vee \neg M$	double negation
$\equiv$	$P \vee (M \vee \neg M)$	associativity
$\equiv$	$P \vee \top$	simplification
$\equiv$	$\top$	simplification

Feb 13

Do Now

Make a Truth Table for the expression $\neg p \wedge \neg q$. Then make a truth table for $\neg(p \wedge q)$. Are they the same?

- No! I made the truth tables in my iPad but they're not. Recall distributing a $\neg$ across parenthesis reverses or -> and, and -> or.
    - So, $\neg(P \wedge Q) \neq \neg P \wedge \neg Q$

p	q	$\neg p \wedge \neg q$		
T	F	F	$\boxed{F}$	T
F	T	T	$\boxed{F}$	F
T	T	F	$\boxed{F}$	F
F	F	T	$\boxed{T}$	T

p	q	$\neg(p \wedge q)$	
T	F	$\boxed{T}$	F
F	T	$\boxed{T}$	F
T	T	$\boxed{F}$	T
F	F	$\boxed{T}$	F

Boolean Algebra

Associative Property: You can change the order in which you perform operations and not change the outcome. So, for example, (2+3)+5=2+(3+5) is true whereas (2-3)-5 $\neq$ 2-(3-5).

For our case, we will be dealing with rules that operate over boolean values.

Which symbols are associative?

- $\neg$ - **NO**: it is a unary operator
- $\vee$ - **YES**: switching the order doesn't matter
  - $(A \vee B) \vee C \equiv A \vee (B \vee C)$
  - Think of this as a Venn Diagram - both sides are equivalent!
- $\wedge$ - **YES**: switching the order doesn't matter
  - $A \wedge (B \wedge C) \equiv A \wedge (B \wedge C)$

    - Same - think of it as a Venn Diagram.
- $\oplus$ - **YES**
- $\leftrightarrow$ - **YES**
- $\rightarrow$ - **NO**
    - Take a look at these two truth tables:
- Be careful when you use the property over different operators!! Note that expressions like $(A \wedge B) \vee C$ is not equivalent to $A \wedge (B \vee C)$!!

A	B	C	$A \rightarrow (B \rightarrow C)$	$(A \rightarrow B) \rightarrow C$
0	0	0	**1** 1	1 **0**
0	0	1	**1** 1	1 **1**
0	1	0	**1** 0	1 **0**
0	1	1	**1** 1	1 **1**
1	0	0	**1** 1	0 **1**
1	0	1	**1** 1	0 **1**
1	1	0	**0** 0	1 **0**
1	1	1	**1** 1	1 **1**

Note how we could have stopped on the first row of the $(A \rightarrow B) \rightarrow C)$ since those two rows aren't equal. If you say something is $\equiv$ then it must be true for all possibilities!

Associativity

Prove: $(P \wedge Q) \wedge (R \wedge Q) \equiv P \wedge (Q \wedge (R \vee Q))$

$(P \wedge Q) \wedge (R \vee Q)$	Given
$\equiv P \wedge (Q \wedge (R \vee Q))$	Associative

Communitive Property

Communative property is when you can swap the operands' position.

Which symbols are communative:

- $\neg$ - **NO**
- $\vee$ - **YES**
- $\wedge$ - **YES**
- $\oplus$ - **YES**
- $\leftrightarrow$ - **YES**
- $\rightarrow$ - **NO**

Here's an example of a proof using associativity and commutativity

Prove: $(P \vee Q) \vee (R \vee Q) \equiv (P \vee Q) \vee R$

$(P \vee Q) \vee (R \vee Q)$	Given
$\equiv (P \vee Q) \vee (Q \vee R)$	Comunativity
$\equiv ((P \vee Q) \vee Q) \vee R$	Associativity
$\equiv (P \vee (Q \vee Q)) \vee R$	Associativity
$\equiv (P \vee Q) \vee R$	Simplification

DeMorgan's and/or

- Or lends itself to union
- And lends itself to intersection

So, that is,

- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- That is to say that $(p \cup q)^c \equiv p^c \cap q^c$
- Same in the opposite direction!

Here's an example of a proof using DeMorgan's Laws:

De Morgan's Law Example I:

Prove $P \vee \neg(Q \wedge \neg R) \equiv P \vee (\neg Q \vee R)$

$P \vee \neg(Q \wedge \neg R)$	Given
$\equiv P \vee (\neg Q \vee \neg\neg R)$	De Morgan's Law
$\equiv P \vee (\neg Q \vee R)$	Double Negation

Distributive Property

Here's an example proof using the Distributive Property:

Distributive Law ★ Be careful w/ signs!! ★

Prove: $(P \lor \neg Q) \land (P \lor \neg R) \equiv P \lor (\neg Q \land \neg R)$

$(P \lor \neg Q) \land (P \lor \neg R)$	Given
$P \lor (\neg Q \land \neg R)$	Distributive Law

Feb 10

Do Now

Represent the problem with voting we discussed earlier as a venn diagram. *hint*: Make one set (or variable) to represent **people who are 18+** and a second intersecting set to represent **people who voted**.

How can i write that with an "if-then" statement?

- "if you voted, then you *must* be over 18."
- "if you're under 18, then you can't vote."

Truth Table Example

A	B	C	$(A \land B) \to C$	
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Boolean Algebra

Example: Prove 3(x+y) = 3x+3y. Let $x, y \in \mathbb{N}$.

Try:

- x=0 and y=0. 3(0+0) = 3(0)+3(0).
- x=1 and y=0. 3(1+0) = 3(1)+3(0).
- x=2 and y=0. 3(2+0) = 3(2)+3(0).
- ...

"Is equivlent to" $\equiv$.

Anatomy of an Equivalence Proof

Here's an example of a proof: Prove $(P \vee Q) \vee (R \vee Q) \equiv (P \vee Q) \vee R$

You must start with the left-hand side of the equation and must end with the right-hand side of the equation. You must provide justification for each step, and every expression in between must be equivalent.

sign	proposition	reasoning
	$(P \vee Q) \vee (R \vee Q)$	Given
$\equiv$	$(P \vee Q) \vee (Q \vee R)$	commutativity
$\equiv$	$((P \vee Q) \vee Q) \vee R$	associativity
$\equiv$	$(P \vee (Q \vee Q)) \vee R$	associativity
$\equiv$	$(P \vee Q) \vee R$	simplification

IMPORTANT LOGICAL RULES

Logical Rules

Equivalences

Simplifications

Simplifications have the property that they make expressions smaller, with fewer operators. The first five important ones are:

long	simplified	name of rule
$\neg\neg P$	P	double negation
$\neg\top$	$\bot$	definition of $\bot$
$P \wedge \bot$	$\bot$	simplification
$P \wedge \top$	P	simplification
$P \vee \bot$	P	simplification
$P \vee \top$	$\top$	simplification

Proof using opposite of simplificaiton Prove: $P \equiv P \wedge (P \leftrightarrow \top)$. Sneaky tactic is to switch the sides and solve.

- Start with the parenthesis. How do I simplify $p \leftrightarrow \top$?
  - $\top$: Simplification!
- Now do $P \wedge \top$
  - P: Simplification!
- Thus, $P \equiv P$

sign	proposition	rule
x	$P \wedge (P \leftrightarrow \top)$	given
$\equiv$	$P \wedge P$	simplification
$\equiv$	P	simplification.

Now you can rewrite the table to "expand" it and properly write the equation:

sign	proposition	rule
	P	given
$\equiv$	$P \land P$	simplification
$\equiv$	$P \land (P \leftrightarrow \top)$	simplification

Definition of Implication

Prove: $A \to (B \oplus A) \equiv \neg A \lor (B \oplus A)$

- think about it: $A \to B \equiv \neg A \lor B$ where $A = P$ and $B = (Q \oplus P)$

sign	proposition	rule
	$p \to (Q \oplus P)$	given
$\equiv$	$\neg p \lor (Q \oplus P)$	definition of implication

Feb 8

Reminders from Quizzes

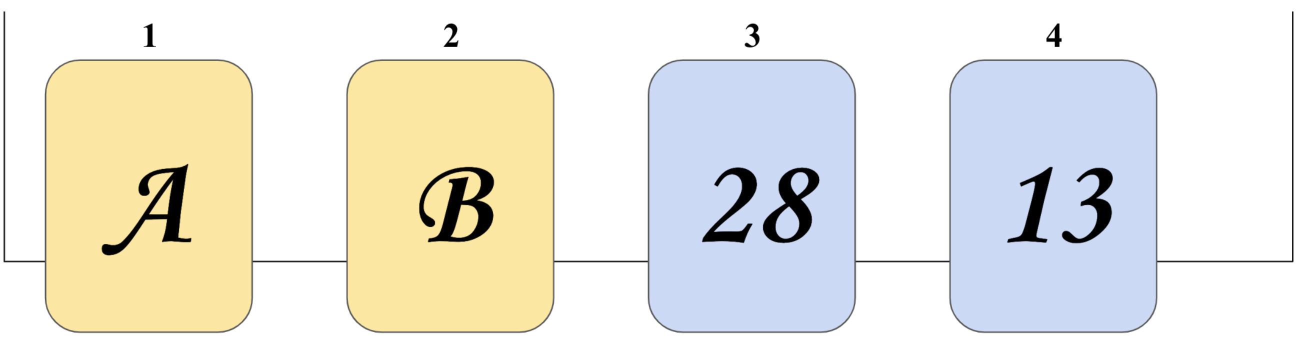
- Any powerset must be a set, i.e. $P(S) = \{\emptyset, ..., S\}$
- Sequences are in params $(\dots)$
- Sets are in curly braces $\{\dots\}$
- $\{\emptyset\} \neq \{\}$

Regrade Requests

Drop by a TA office hours *first*, then if the TA affirms that you got it right, then request points back.

Do Now - Thought Experiment

There are four cars below, each with a letter **on one side**, and a **number on the other** side.I make the unsubstantiated claim that **"if a card has a number less than 18, then there must be a vowel (A) on the other side of that card."** You are allowed to flip over **only two cards** to prove or disprove my claim.



Evaluating Propositions

A proposition: **"If a card has a number less than 18, then there must be a vowel on the other side of that card."**

How do we evaluate this proposition with this thought experiment? Use a truth table!

	A	B
n < 18	T	F
$\neg(n < 18)$	T	T

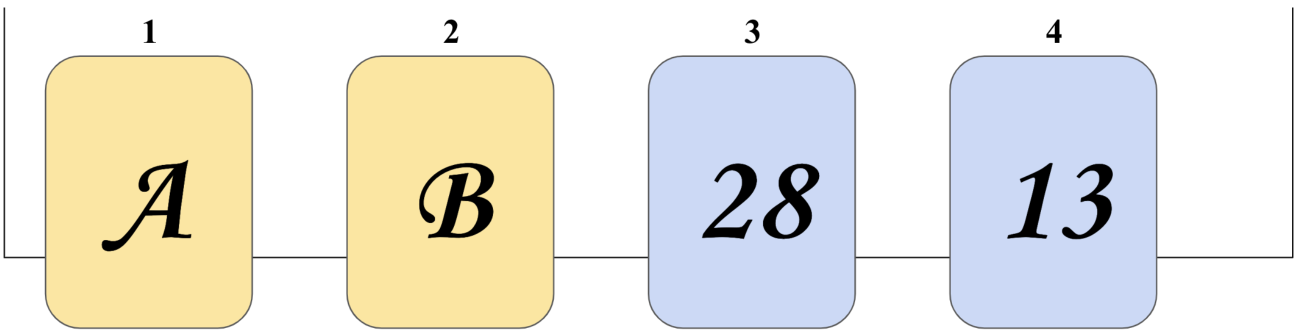
Note that: $\neg(n < 18) \equiv (n \geq 18)$

A proposition: **"Either card does *not* have a number less than 18, or it has a vowel."** Let's rephrase it as **"either the number the number is ≥ 18 or it has a vowel, or both."** Here is the associated truth table:

	A	B
n < 18	T	F
$\neg(n < 18)$	T	T

Same problem, rephrased

We now have a group of 4 people at a polling place. Some people are casting a ballot, others are not. You must be at least 18 years old to vote. Each person has an ID card – one side their age, the other with a letter. The letter A on the voter's ID card indicates they didn't vote. The letter B indicates that they did vote. Your job is to figure out if anyone voted illegally. You can flip over two cards to decide.



Which one do you flip?

- not 1, they didn't vote
- yep, flip 2 since you don't know whether or not it's right.
- Nope, not 3, you know they're a legal voter
- yep, flip 4.

So, $(n \geq 18)$ OR (isVowel) $\equiv$ if $(n < 18)$ then (isVowel)

"Implies" Operator

Truth table for "implies" operator. If it helps, think p = "the person voted" and q = "they're over 18."

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Note that $\neg p \vee q$ is the same as $p \rightarrow q$

Doin this Truth Table Her Way

Remember that $\neg$ is a unary operator, if there aren't any parenthesis (like in this example) you should be evaluating $\neg p$ first. Each operator has its own column.

Make sure to put a BOX around your answers (my answers are bolded lol)

p	q	$\neg p \vee q$
0	0	1 **1**
0	1	1 **1**
1	0	0 **0**
1	1	0 **1**

Another truth table her way. Make sure to put the final answer under the $\wedge$ symbol. the first one you evaluate is within the parenthesis, second one (final answer) is under the $\wedge$, in a box/bolded.

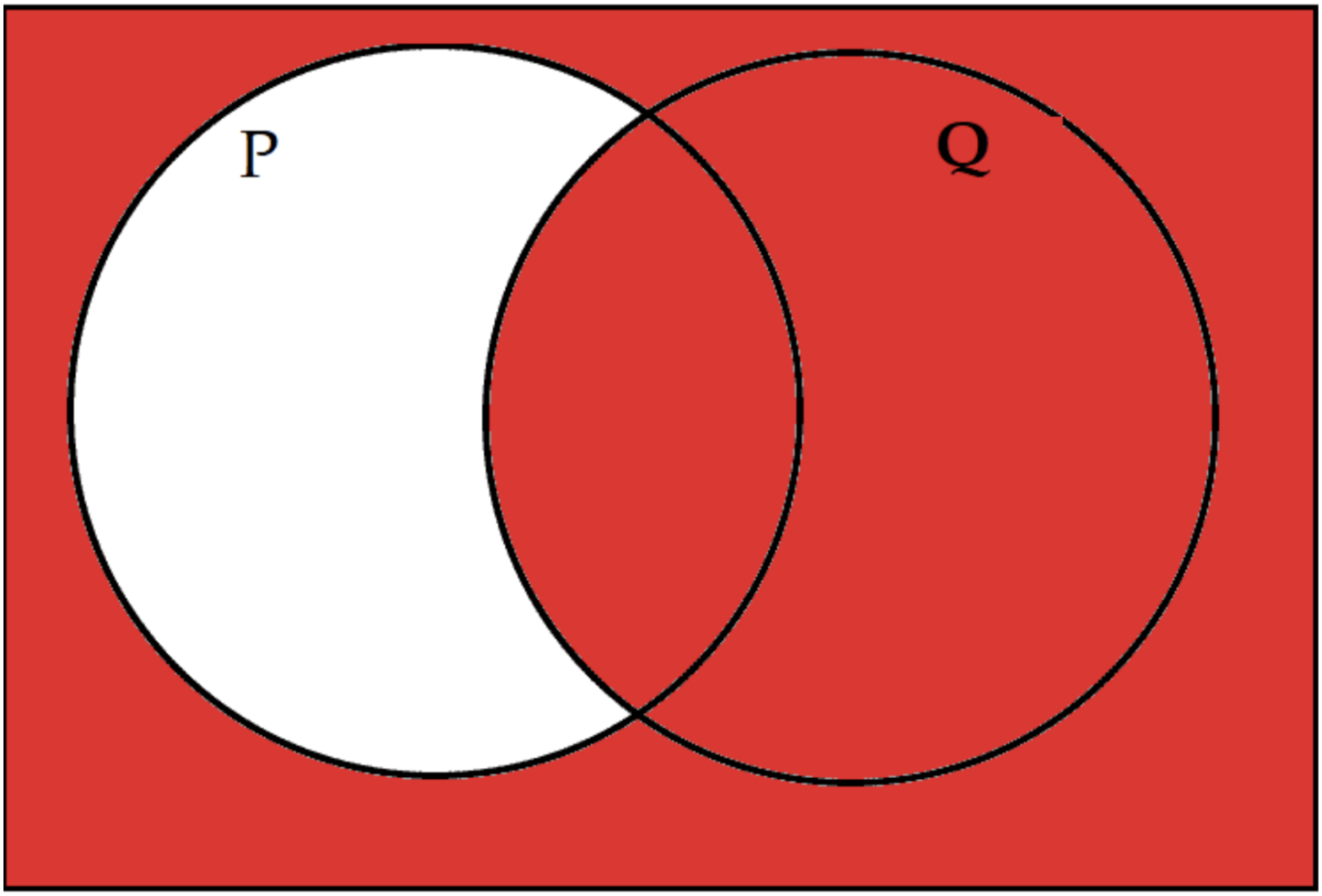
A	B	C	$(A \vee B) \wedge C$
0	0	0	-
0	0	1	-
0	1	0	-
0	1	1	-
1	0	0	-
1	0	1	-
1	1	0	-
1	1	1	-

Another way to understand implication

P = My animal is a poodle Q = it is a dog

p	q	p→q
0	0	1
0	1	1
1	0	0
1	1	1

Venn Diagram of "Implies"



Feb 6 - Conditionals

Review

Quiz Review

$\{\{x\} \times \{y\} | x \in \{-1, 0, 1, 2\} y \in \mathbb{N}\ y < x\}$

- {{(1, 0)}, {(2, 0)}, {(2,1)}}

$S = \{x - y | (x, y) \in (\{8\} \times \{3, 5\})\}$

- $\{3, 5\} \subset S$ is False.

Symbols

- $\in$ element of
- $\subset$ proper subset of
- $\subseteq$ subset of
- P(S) power set of s
- $|S|$ cardinality
- $S \times T$ S cross T
- S^2 S cross S

Propositions

A proposition is a statement that is either true or false.

Examples of a proposition	Examples of things that aren't a proposition
Jeremy got the question right	What score did you get on the quiz?
There is only one Jeremy in the class	Is Jeremy the only jeremy in the class?
Taco bell can be used as a laxative	How are you?
Something that is true or false	any imperative statement (i.e. do this, don't do this)

When dealing with propositions, we abstract away difficulties of defining, and we can give them letters (define variables), like P. So, we can say (2+2=5)=P, or ("I am a human") = Q.

True vs. False

Concept	Java/C	Python	This class	Bitwise	Name	other
true	`true`	`True`	$\top$ or 1	1	tautology	T
false	`false`	`false`	$\perp$ or 0	0	contradiction	F

The most "mathematically rigorous" way to describe True or False is: $\top$: True; and $\perp$: False. You can also use 1: True; 0: False.

Connectives

A proposition is a statement that is either true or false. We can modify, combine, and relate propositions with connectives:

$\wedge$, (logical and), $\vee$, (logical or), $\neg$, (not), $\leftrightarrow$, (iff), $\rightarrow$, (implication), $\oplus$, exclusive or.

Propositions

We can modify, combine, and relate propositions with *connectives*:

- $\vee$ is "or"
- $\wedge$ is "and"
- $\neg$ is "not"

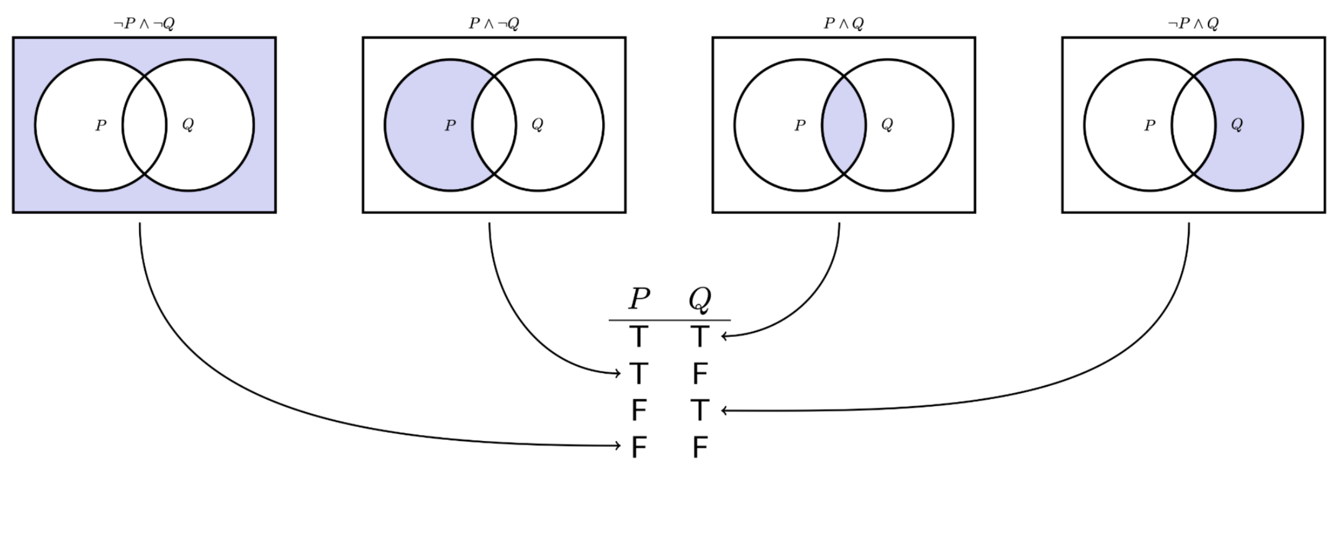
Truth Table

How to define: make a truth table!

There are two possible inputs to a "not" operator - it is either a $\top$ input or a $\perp$ input. Note that the first column, "P" is the input and $\neg P$ is the output. Notice how "not" only takes in one input, it is a "unary operator."

P	$\neg P$
0	1
1	0

Here is an image that contains all the truth tables for the truth table values for "not"



"And" Operator

Think of this as the "intersection" for example. Note how this is a "binary operator" as there are two inputs. Thus, there are four possible cases - there are *four* regions in the venn diagram!

- if you have three intersecting venn diagrams, you have 8 possible inputs.
- if you have n venn diagrams, you have 2^n inputs

p	q	$p \land q$
1	1	1
1	0	0
0	1	0
0	0	0

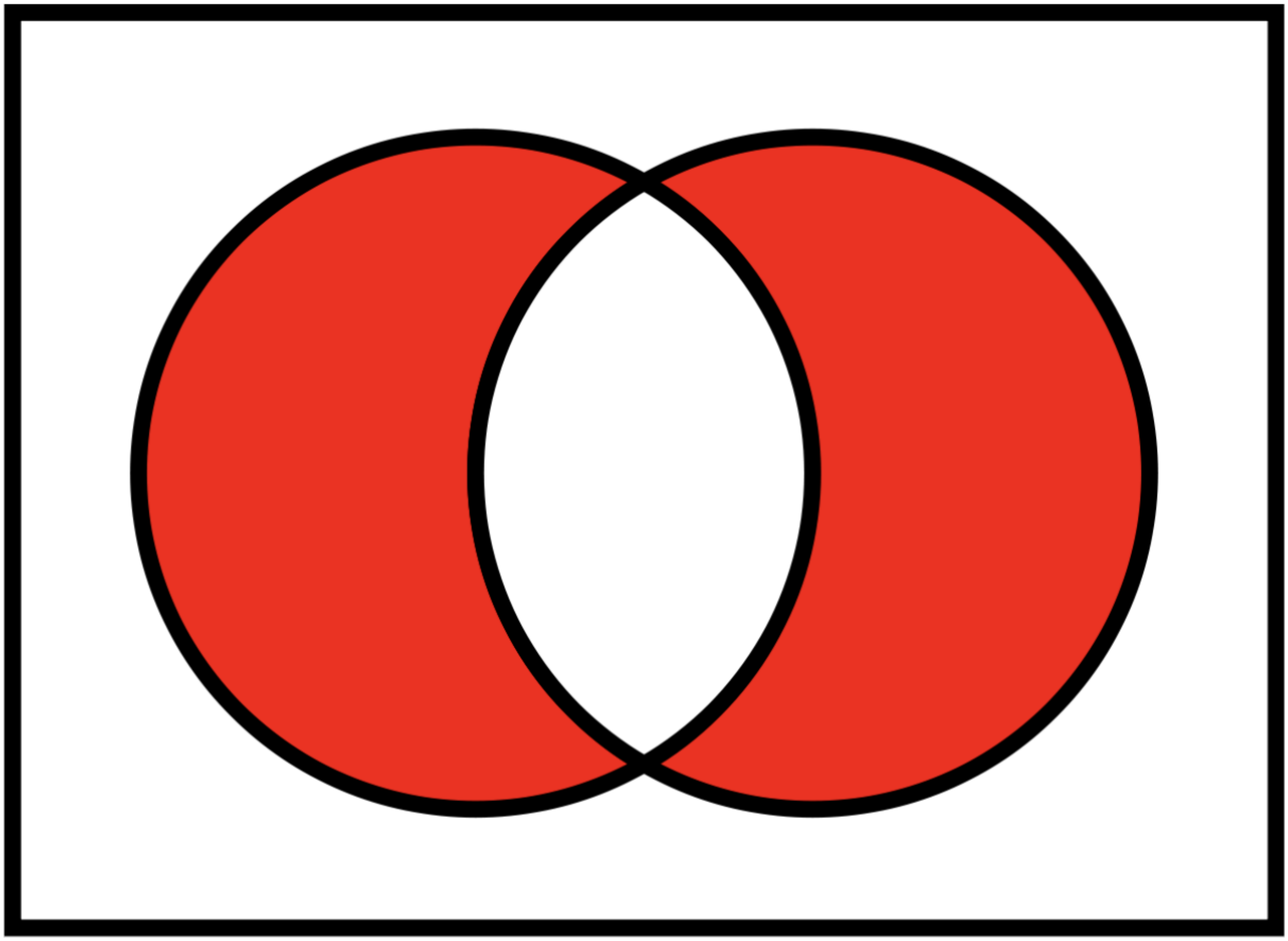
"Or" Operator

Think of this as the "union" sign, for example.

p	q	$p \lor q$
0	0	0
0	1	0
1	0	1
1	1	1

"Xor" operator

An example: * I want fries **or** a drink. - you can have both! * I want it for here **or** I want it to go. - you can only have one!! Note that this is the use of $\oplus$

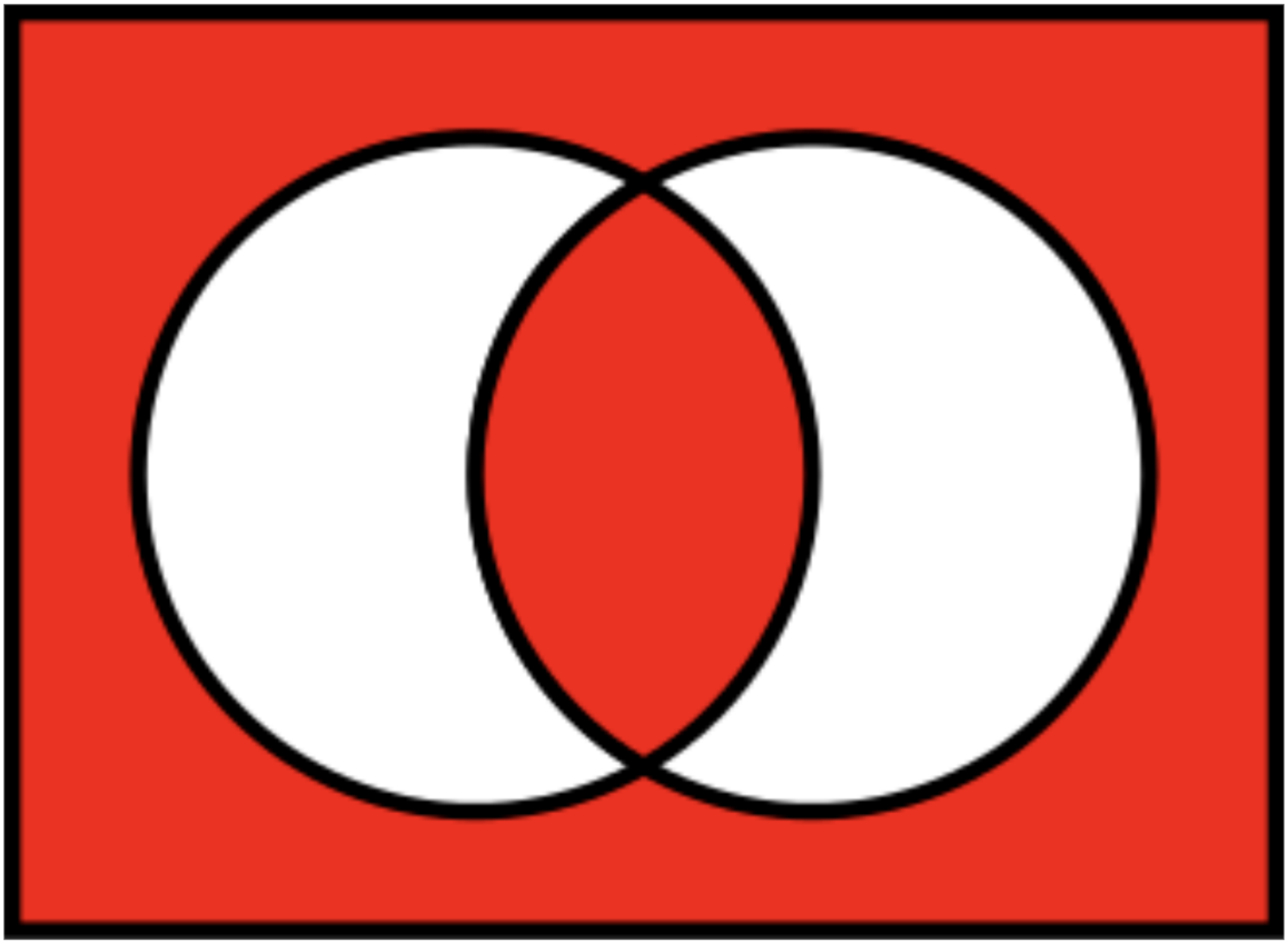


Truth Table:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

"Bi-implies" operator (iff)

This is the negation of $\oplus$.



p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

Putting Conditionals Together

p	¬	p	∨	p
0	1	0	1	x
1	0	1	1	x

How to Do Elizabeth's Truth Tables

This is the order of how to do the truth tables

p	q	$\neg(p \wedge q)$
T	T	-
T	F	-
F	T	-
F	F	-

First apply the $\wedge$ rule for the parenthesis

p	q	$\neg(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

Next apply the $\neg$ operator

p	q	$\neg(p \wedge q)$
T	T	**F** T
T	F	**T** F
F	T	**T** F
F	F	**T** F

The bolded outcome is the final answer!

Feb 3 - Quiz 1 In-Class!

Review before Quiz

Note that what's on review is *really* important.

Cartesian (Cross) Product

If $x = 3$, then what is $x \in A \times B$? = False The point is, the cartesian product returns the set of ordered pairs! Think of the cartesian product as a table:

Power Set

Recall that a power set returns a set of all possible subsets. It is a set of sets!

Other

What is $|\{\{x, y\}|x, y \in H$ | when H = {1, 2, 3}?

Think of it as a table, again! This is cartesian product of H with itself!

- | { {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3} } | = 6

What is $\{x + y|(x \in \{1, 2\}) \wedge (y \in \mathbb{N}) \wedge (y < x)\}$? = {1, 2, 3}

Which of the following contain the empty set as a number when H = {} and K = {}

- $\mathbb{N}$? No.
- P($\mathbb{Z}$)? Yes.
- $\{x|x \in (K \backslash H)\}$?
- $\{x|(x \subseteq K) \wedge (x \subseteq H)\}$? True
- $\{\{x + y, x - y\}|(x \in H) \wedge (y \in H)\}$

Feb 1 - Popular Sets, Power Set, Set-Builder Notation, Disjoint Sets

Some Popular Sets

Symbol	Set	elements
$\emptyset$	the empty set	none
$\mathbb{N}$	nonnegative integers	$0, 1, 2, 3\ldots$
$\mathbb{Z}$	integers	$\ldots 3, 2, 1, 0, 1, 2, 3\ldots$
$\mathbb{Q}$	rational numbers	$\frac{1}{2}$ 16, etc
$\mathbb{R}$	real numbers	π, e, $\sqrt{2}$
$\mathbb{C}$	complex numbers	i, $\frac{19}{2}$, etc.

A superscript restricts its set to its positive elements, for example $\mathbb{R}^+$ denotes the set of positive real numbers, and for example $\mathbb{Z}^-$ denotes the set of negative integers.

Power Sets

The set oef all subsets of a set, A, is called a *power set*, pow(A), of A. So: $B \in pow(A) \leftrightarrow B \subseteq A$

For example, the elements of pow({1, 2}) are $\emptyset$, {1}, {2}, and {1, 2}.

Questions:

- What is pow({})?
  - {$\emptyset$} (the set containing the empty set).
- What is pow({a, b, c})?
  - pow({a, b, c}) = $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$
  - note the distinction between what pow(stuff) evaluates to versus the *elements* of pow(stuff).
- What is the power set of {W, X, Y, Z}?
  - pow({W, X, Y, Z}) = {{$\emptyset$}, {W}, {X}, {Y}, {Z}, {W, X}, {W, Y}, {W, Z}...}

How can we determine a rule/pattern to determine the cardinality of a powerset?

- $|P(X)| = 2^{|X|}$

Disjoint Sets

Formal definition for disjoint sets: *two sets are disjoint if their intersection is the empty set.* An example of two sets that are **not** disjoint are {1, 2, 3} and {3, 4, 5} since they both share the element 3. However, the set {New York, Washington} and {3, 4} are disjoint.

- {1, 2} and $\emptyset$ are disjoint.
- the empty set is always disjoint with any set
- $\emptyset$ and $\emptyset$ are disjoint!

Set builder Notation

Example: $S = \{x \in A |$x is blue$\}$

- The set of all x in A "such that" property (or properties) of x that must be met in order to be an element of S.

A common breakdown of set-builder notation is with numbers: $E = \{x \in \mathbb{N} | x \geq 3\}$

- "the set of all x in the natural numbers such that x is greater than 2."

Let's formalize our set operations in "set-builder notation." Quick side note - we need to link together multiple "conditions" with "and's," "not's," and "or's."

- $\vee$ is "or." (notice similarity to $\cup$)
- $\wedge$ is "and." (notice similarity to $\cap$)
- $\neg$ is "not."

Intersection

We want to define **intersection**: $S \cap T$: the elements that belong both to S and to T.

- $S \cap T = \{x \in U | x \in T \wedge x \in S\}$
  - Note that "U" is the "universe."
- Another interesting answer: $S \cap T = \{x \in S | x \in T\}$

Union

We want to define the **union**: $S \cap T$: the elements that belong in either S or T (or both):

- $S \cup T\{x \in U | x \in T \vee x \in S\}$
- the "or" $\vee$ symbol is inclusive - includes All elements in S, T, **or** both.

Difference

We want to define **difference** $S \backslash T$:

- $S \backslash T = \{x \in U | x \in S \wedge x \notin T\}$
- "Better" answer: $S \backslash T = \{x \in U | x \in S \wedge \neg(x \in T)\}$
- Another cheeky answer: $S \backslash T = \{x \in S | x \notin T\}$

Complement

We want to define **complement**: $\bar{S}$: elements (of the universe) that don't belong to S.

- $\bar{S} = \{x \in U | x \notin S\}$
- $\bar{S} = \{x \notin S\}$

Evaluation Practice

$\{\{a, b\} | (a \in X) \wedge (b \in Y)\} = ?$

- = { {1, 2}, {1, 3}, {1, 4}, {2}, {2, 3}, {2, 4}, {3}, {3, 4} }

Jan 30 - propositions, operators, set-builder notation

Describing Sets

Listing out the elements of a set works well for sets that are small and finite. What about larger sets? Use set builder notation!!

$S = \{x \in A | xisblue\}$

- S us the set of all x in A such that…

$E = \{x \in \mathbb{N} | x > 2\}$

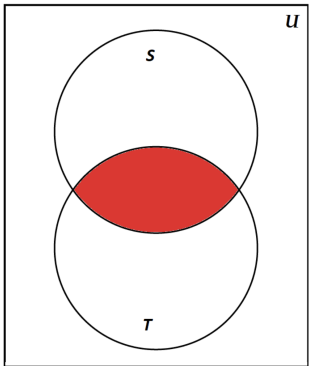
- E is {3, 4, 5, …, ∞}
- Recall that 0 is a natural number btw.
- The cardinality of E is infinity.

Jan 27 - intersection, union, difference, complement, and cartesian (cross) product

$\cap, \cup, \backslash$

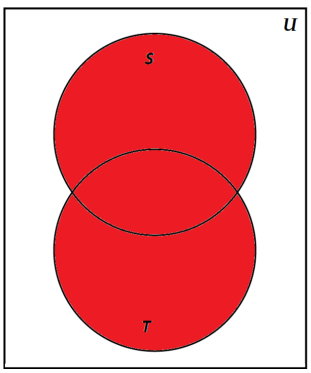
$\cap$: the intersection of two sets

The intersection of two sets S and T, denoted by $S \cap T$, that is the set containing all elements of S that also belong in T. (or equivalently, all elements of T that also belong in S)



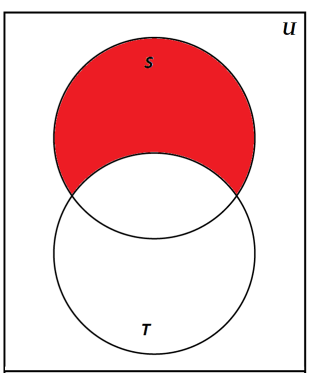
$\cup$: the union of two sets

The union of two sets S and T, denoted by $S \cup T$, is the set containing all the elements of S and also all the elements in T. (or equivalently, everything either in S or T or both)



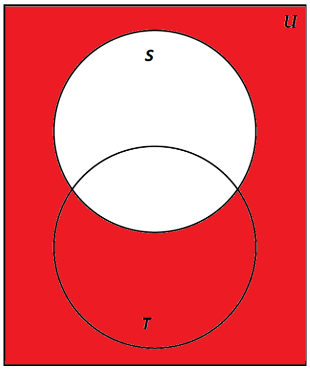
Difference $S \backslash T$: the element that belong to S but not T

Note that the difference of two sets is *not* commutative. It's like division!



Complement: $\bar{S}$ or S^C: All the elements of the universe that don't belong to S.

You have to consider the universe u, not just the "venn diagram" too. See images attached!



High Level Sets and Sequences

1. Sequences vs. Sets
2. Cartesian Product
3. Set builder notation
4. Set Operator Review

High Level: Sets vs Sequences

- Sets:
  - no duplicates
  - no order
  - has cardinality
- Sequences
  - can have duplicates
  - has order
  - has length
  - lists, arrays, ordered pairs, tuples, etc!
- Both:
  - Contain anything
  - Can have a sequence of sequences, set of sets, sequence of sets, set of sequences
  - Cannot be modified

Cartesian Products of Sets

Ordered Pair: An ordered pair is a **sequence with 2 elements.** It is a pair of objects where one element is designated as first and the other element is designated as second, denoted (a, b)

Cartesian Product: The Cartesian product of two sets A and B is denoted $A \times B$, si the set of all possible ordered pairs where the elements of A are the first and the elements of B are second. This is also called the "cross product."

Example: $\{1, 2\} \times \{3, 4, 5\} = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 3), (2, 4), (2, 5)\}$. This returns a set of sequences. The cardinality of this cross product is 6.

Example: $\{1, 2\} \times \{3, 4\} \times \{4, 5\} = \ldots$

What is $\{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$

- note sequences can have duplicates so (2, 2) is valid!
- Cardinality is 4

Cardinality of Cross Product

$|A \times B| = |A||B|$ (aka, the cardinality of the cross product of two sets is the product of the cardinality of each set)

- Weird Question $|\ \{\ \{\}\ \} \times \{1, 2, 3\}\ | =$???
  - 3 … since the cross product is $= \{\ (\{\}, 1), (\{\}, 2), (\{\}, 3)\ \}$!!
- Weird Question $|\ \{\} \times \{1, 2\}\ | =$?

- 0 ... since $|\emptyset| = 0$

- Let A= $\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$. What is $A^2 = ?$
  - $A^2 = A \times A$
  - $A^3 = A \times A \times A$
  - etc...
  - Note that A^2 is every single (x, y) coordinate that falls in the grid created by A. So, $\mathbb{R} \times \mathbb{R}$ is the coordinate plane!

Jan 25 - subsets & supersets

$\subseteq, \subset, \supseteq, \supset$

These operators compare two sets.

- $\subseteq$: subset: think about as $\leq$
- $\supset$: superset: think about as $\geq$
- $\subset$: proper subset: think about as $<$
- $\supset$: proper superset: think about as $>$

$\subseteq$

Set A is a **subset** of B, or $A \subseteq B$, if and only if **all elements of A are also in B**.

$\supseteq$

Same thing as subset, but flipped direction!

$\subset$

Set a is a **proper subset** of B, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

Consequences of this definition, $A \subset B$:

- $A \subseteq A$ (this is trivial but true)
- $|A| < |B|$

Example:

Given P={1, 2, 3}, Q={1, 3}, R={1, 3, 4}, determine whether to use subset, superset, proper subset, or proper superset to have all these equations evaluate to $\top$.

- $P \supseteq Q \wedge P \supset Q = \top$
  - (however, $\supset$ is more specific)
- $P \not\subseteq R = \top$
  - There is no answer for this one!
- $P \supseteq P \wedge P \subseteq P = \top$

Example 2:

Given P={$\emptyset$, {1, 2}, 3}, Q={1, 2}, R={}, S={3}. Determine how to evaluate to $\top$:

- $P \not\supseteq Q = \top$
  - Think about it!
- $P \supseteq R = \top$
  - Think about it, $\emptyset$ is in P! $\emptyset \subseteq$ every set!
  - And, by extension, $\emptyset \subseteq \emptyset$
- $P \supseteq S \wedge P \supset S = \top$

if you're tripped up on the first and third bullets, notice the difference and see why they're different.

Jan 23

Set Definition

A **set** is a structure that contains elements. A **set** has no other properties other than the *elements* it contains. A set can contain other sets. Duplicate values aren't allowed, and order doesn't matter! A **member** or **element** is something inside the set. A set is written with curly braces, its members separated by commas.

- {1, 3} or {dog, cat, mouse} or {3, thimble, Jules} or {{1, 2}, 1}
- Sets can be members of other sets!:
  - {{1, 2}, {3, 4}}
- Sets order doesn't matter;
  - {1, 3, 4} and {4, 3, 1} are the same set
- No duplicates!
  - {1, 3, 4, 1} will "knock" out the duplicate, should be {1, 3, 4}
  - {{1, 2}, {2, 1}} is redundant - this is not a set! It can be just written as {{1, 2}}

The **empty set** is a set with no members, which is expressed as {} or ∅ (or sometimes "null")

Cardinality is the number of elements in a set. Cardinality is denoted using $|A|$. What are the cardinality of these sets:

- $|\{1, -13, 4, -13, 1\}|$: 3
- $|\{3, 1, 2, 3, 4, 0\}|$: 3
  - note that {1, 2, 3, 4} is an element of the bigger set, *not* a subset.
- $|\emptyset|$: 0
- |{{}, {{}}, {{{}}} }|: 3

Examples of Infinite Sets

- $\mathbb{N}$: Natural numbers
  - includes 0!
- $\mathbb{Z}$: Integers
- $\mathbb{Q}$: Rational - the ratio of two integers, $\frac{a}{b}$ that is a finite or repeating decimal
- $\mathbb{R}$: Real -
  - ∞ is not a real number!!!
- $\mathbb{C}$: Complex
- $\mathbb{I}$: Imaginary

$\in$: "Element of"

checks membership of aan element

Examples:

- $2 \in \{1, 2\} = \top$
- $3 \in \{1, 2\} = \bot$
- $3 \notin \{1, 2\} = \top$
- $\{2\} \in \{1, 2\} = \bot$
- $\{2\} \in \{1, \{2\}\} = \top$
- $2 \in \{\{1, 2\}\} = \bot$
- $2 \in \{\{\}\} = \bot$
- $2 \in \{\{\{2\}\}\} = \bot$
- $\{2\} \in \{\{1, 2\}\} = \bot$
- $\{2\} \in \{\{2\}\} = \top$
- $\{2\} \in \{\{\{2\}\}\} = \bot$