Practice for Quiz 2

Charlie Meyer

MoWeFri 1:00 - 1:50

Pratice 1-2: Boolean Algebra

This includes both the pratice from the worksheet Elizabeth handed out in-class on Wednesday, review from the Discord Review session on Thursday, and as many practice problems I could do from 1-2.

Worksheet

1: Prove $A \wedge (A \vee B) \equiv A$

symbol	Equation	Reasoning
	$A \wedge (A \vee B)$	Given
$\equiv$	$(A \vee \perp) \wedge (A \vee B)$	distributive property
$\equiv$	$A \vee (\perp \wedge B)$	simplification
$\equiv$	$A \vee \perp$	simplification
$\equiv$	A	simplification

2: Prove $(P \vee \neg p) \rightarrow P \equiv P$

symbol	equation	reasoning
	$(P \vee \neg P) \rightarrow P$	given
$\equiv$	$\neg(P \vee \neg P) \vee P$	definition of implication
$\equiv$	$(\neg P \wedge \neg\neg P) \vee P$	DeMorgan's Law
$\equiv$	$(\neg P \wedge P) \vee P$	Double Negation
$\equiv$	$\perp \vee P$	simplification
$\equiv$	P	simplification

An alternate solution for (2) is:

symbol	equation	reasoning
	$(P \vee \neg P) \rightarrow P$	given
$\equiv$	$\top \rightarrow P$	simplification
$\equiv$	$\neg\top \vee P$	definition of implication
$\equiv$	$\perp \vee P$	simplification
$\equiv$	P	simplification

3: Prove $\neg A \wedge \neg B \equiv \neg A \wedge (B \rightarrow A)$

symbol	equation	reasoning
	$\neg A \wedge \neg B$	given
$\equiv$	$(\neg A \wedge \neg B) \vee \perp$	simplification
$\equiv$	$(\neg A \wedge \neg B) \vee (\neg A \wedge A)$	simplification
$\equiv$	$\neg A \wedge (\neg B \vee A)$	Distributive Property
$\equiv$	$\neg A \wedge (B \rightarrow A)$	Definition of implication

4: Prove: $R \wedge \neg(P \rightarrow Q) \equiv P \wedge (\neg Q \wedge R)$

symbol	equation	reasoning
	$R \wedge \neg(P \to Q)$	Given
$\equiv$	$R \wedge \neg(\neg P \vee Q)$	definition of implication
$\equiv$	$R \wedge (\neg\neg P \wedge \neg Q)$	DeMorgan's Law
$\equiv$	$R \wedge (P \wedge \neg Q)$	double negation
$\equiv$	$R \wedge (\neg Q \wedge P)$	commutative
$\equiv$	$(R \wedge \neg Q) \wedge P$	associative
$\equiv$	$P \wedge (R \wedge \neg Q)$	commutative
$\equiv$	$P \wedge (\neg Q \wedge R)$	commutative

5: Prove: $(X \to Y) \wedge (\neg X \to \neg Y) \equiv X \leftrightarrow Y$

symbol	equation	reasoning
	$(X \to Y) \wedge (\neg X \to \neg Y)$	given
$\equiv$	$(X \to Y) \wedge (\neg\neg X \vee \neg Y)$	definition of implication
$\equiv$	$(X \to Y) \wedge (X \vee \neg Y)$	Double negation
$\equiv$	$(X \to Y) \wedge (\neg Y \vee X)$	commutative
$\equiv$	$(X \to Y) \wedge (Y \to X)$	definition of implication
$\equiv$	$X \leftrightarrow Y$	definition of bimplication

6: Prove $\neg(P \vee M) \to \neg M \equiv \top$

symbol	equation	reasoning
	$\neg(P \vee M) \to \neg M$	given
$\equiv$	$\neg\neg(P \vee M) \vee \neg M$	definition of implication
$\equiv$	$(P \vee M) \vee \neg M$	double negation
$\equiv$	$P \vee (M \vee \neg M)$	associativity
$\equiv$	$P \vee \top$	simplification
$\equiv$	$\top$	simplification

Discord Review Session

Prove: $A \oplus B \equiv \neg(A \leftrightarrow B)$

operator	expression	reached by
	$A \oplus B$	given
$\equiv$	$(A \vee B) \wedge \neg(A \wedge B)$	definition of XOR
$\equiv$	$(A \vee B) \wedge (\neg A \vee \neg B)$	DeMorgan's
$\equiv$	$(\neg\neg A \wedge \neg B) \vee \neg(\neg B \vee A)$	double negation
$\equiv$	$\neg(\neg A \vee B) \vee \neg(\neg B \vee A)$	DeMorgan's Law
$\equiv$	$\neg(\neg A \vee B) \vee \neg(B \to A)$	definition of implication
$\equiv$	$\neg(A \to B) \vee \neg(B \to A)$	definition of implication
$\equiv$	$\neg((A \to B) \wedge (B \to A))$	DeMorgan's Law
$\equiv$	$\neg(A \leftrightarrow B)$	definition of biimplication

Another Solution (not sure if it's legal though)

- Use Definition of xor: $A \oplus B \equiv \neg(A \to B)$. The purpose of the question above was to show another way to do that, though.

Practice 1-2: Boolean Algebra

Question 1:

Prove $P \to Q \equiv \neg Q \to \neg P$

operator	expression	reached by
	$P \to Q$	given
$\equiv$	$\neg P \vee Q$	definition of implication

operator	expression	reached by
$\equiv$	$Q \vee \neg P$	commutativity
$\equiv$	$\neg\neg Q \vee \neg P$	double negation
$\equiv$	$\neg Q \rightarrow \neg P$	definition of implication

Question 2

Prove $\neg(P \wedge Q \wedge R) \equiv (\neg P \vee \neg Q \vee \neg R)$

operator	expression	reached by
	$\neg(P \wedge Q \wedge R)$	given
$\equiv$	$\neg((P \wedge Q) \wedge R)$	Associativity
$\equiv$	$(\neg(P \wedge Q) \vee \neg R)$	DeMorgan's Law
$\equiv$	$((\neg P \vee \neg Q) \vee \neg R)$	DeMorgan's Law
$\equiv$	$(\neg P \vee \neg Q \vee \neg R)$	associativity

In the solution on the website, their first step was to do $\neg(P \wedge (Q \wedge R))$ but I did $\neg((P \wedge Q) \wedge R)$. Doesn't matter.

Question 3

Prove $P \wedge (P \rightarrow Q) \equiv P \wedge Q$

operator	expression	reached by
	$P \wedge (P \rightarrow Q)$	given
$\equiv$	$P \wedge (\neg P \vee Q)$	definition of implication
$\equiv$	$(P \wedge \neg P) \vee (P \wedge Q)$	Distributive
$\equiv$	$(\bot) \vee (P \wedge Q)$	simplification
$\equiv$	$(P \wedge Q)$	simplification
$\equiv$	$P \wedge Q$	simplification

The answer on the website has the stripping of parenthesis around $(\bot)$ and therefore does one less step than me but same difference.

Question 4

Prove $\neg(P \wedge Q) \equiv (Q \rightarrow (\neg P))$

operator	expression	reached by
	$\neg(P \wedge Q)$	given
$\equiv$	$(\neg P) \vee (\neg Q)$	DeMorgan's Law
$\equiv$	$(\neg Q) \vee (\neg P)$	commutative property
$\equiv$	$Q \rightarrow (\neg P)$	definition of implication

Question 5

Prove $(P \wedge \neg Q) \equiv \neg(P \rightarrow Q)$

operator	expression	reached by
	$(P \wedge \neg Q)$	given
$\equiv$	$(\neg\neg P \wedge \neg Q)$	double negation
$\equiv$	$\neg(\neg P \vee Q)$	DeMorgan's Law
$\equiv$	$\neg(P \rightarrow Q)$	definition of implication

Cringe double negation but whatever don't forget to do that. It's easy to overlook that step when you're DeMorgans-ing.

Question 6

Prove $P \to (A \vee Q) \equiv (P \wedge \neg A) \to Q$

operator	expression	rule used
	$P \to (A \vee Q)$	given
$\equiv$	$\neg P \vee (A \vee Q)$	definition of negation
$\equiv$	$(\neg P \vee A) \vee Q$	associativity
$\equiv$	$\neg\neg(\neg P \vee A) \vee Q$	double negation
$\equiv$	$\neg(P \wedge \neg A) \vee Q$	DeMorgan
$\equiv$	$(P \wedge \neg A) \to Q$	definition of implication

The answer online took extra steps but I think my answer is cleaner (and still valid).

Question 7

Did this over discord, see discord review section

Question 8.

Prove $(P \to Q) \equiv \neg(\neg Q \to \neg P)$

operator	expression	rule
	$(P \to Q)$	given
$\equiv$	$\neg P \vee Q$	definition of implication
$\equiv$	$(\neg P \vee Q)$	associativity
this	cannot	be proven

Working backwards... $\neg(\neg\neg Q \vee \neg P)$

$\neg(Q \vee \neg P)$

$\neg Q \wedge P$

As you can see there is no way to work backwards to get $(\neg P \vee Q)$ from $(\neg Q \wedge P)$ therefore it is unsolvable.

Another way to see that it is unsolvable: If P and Q are both $\perp$, the left-hand side is $\top$ and the right-hand side is $\perp$. So they're not $\equiv$.

Question 9

Prove $A \to (B \to C) \equiv (A \wedge B) \to C$

operator	expression	rule
	$A \to (B \to C)$	given
$\equiv$	$A \to (\neg B \vee C)$	DeMorgan's
$\equiv$	$\neg A \vee (\neg B \vee C)$	DeMorgan's
$\equiv$	$(\neg A \vee \neg B) \vee C$	Associativity
$\equiv$	$\neg(A \wedge B) \vee C$	DeMorgan's
$\equiv$	$(A \wedge B) \to C$	Definition of implication

Question 10

For more practice, prove

$A \oplus B \oplus C \equiv (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C) \vee (A \wedge B \wedge C)$, or prove any equivalence rule using other equivalence rules, or prove $\forall_x$ chapter 15 exercise C or $\forall_x$ chapter 17 exercise B or all exercises in $\forall_x$ chapter 19$

Questions 11-21

"The Team wins or I am sad. If the team wins, then I go to a movie. My dog is quiet. If i am sad, then my dog barks."

- W: the team wins
- S: I am sad
- M: I go to a movie

- B: my dog barks

We can express the passage above as: $(W \vee S) \wedge (W \to M) \wedge \neg B \wedge (S \to B)$.

The following proof is one way to show that:

- my dog doesn't bark
- the team won
- I am not sad, and
- I went to the movies

expression	rule used
$(W \vee S) \wedge (W \to M) \wedge \neg B \wedge (S \to B)$	given
$(W \vee S) \wedge (\neg W \vee M) \wedge \neg B \wedge (S \to B)$	definition of implication
$(W \vee S) \wedge (\neg W \vee M) \wedge \neg B \wedge (\neg S \vee B)$	definition of implication
$(W \vee S) \wedge (\neg W \vee M) \wedge (\neg B \wedge (\neg S \vee B))$	associativity
$(W \vee S) \wedge (\neg W \vee M) \wedge ((\neg B \wedge \neg S) \vee (\neg B \wedge B))$	distributive
$(W \vee S) \wedge (\neg W \vee M) \wedge ((\neg B \wedge \neg S) \vee \bot)$	simplification
$(W \vee S) \wedge (\neg W \vee M) \wedge (\neg B \wedge \neg S)$	simplification
$(W \vee S) \wedge (\neg W \vee M) \wedge \neg B \wedge \neg S$	associativity
$\neg S \wedge (W \vee S) \wedge (\neg W \vee M) \wedge \neg B$	commutativity
$(\neg S \wedge (W \vee S)) \wedge (\neg W \vee M) \wedge \neg B$	associativity
$((\neg S \wedge W) \vee (\neg S \wedge S)) \wedge (\neg W \vee M) \wedge \neg B$	distributive
$((\neg S \wedge W) \vee \bot) \wedge (\neg W \vee M) \wedge \neg B$	simplification
$((\neg S \wedge W)) \wedge (\neg W \vee M) \wedge \neg B$	simplification
$\neg S \wedge W \wedge (\neg W \vee M) \wedge \neg B$	associativity
$\neg S \wedge \neg B \wedge W \wedge (\neg W \vee M)$	commutativity
$\neg S \wedge \neg B \wedge (W \wedge (\neg W \vee M))$	associativity
$\neg S \wedge \neg B((W \wedge \neg W) \vee (W \wedge M))$	distributive
$\neg S \wedge \neg B \wedge (\bot \vee (W \wedge M))$	simplification
$\neg S \wedge \neg B \wedge ((W \wedge M))$	simplification
$\neg S \wedge \neg B \wedge W \wedge M$	associativity

Note: removing redundant parenthesis like $((W \wedge M))$ to $W \wedge M$ is associativity, *not* simplification.

Questions 22 - 24

Some lists of loical axioms will include the contrapositive, for example $(P \to Q) \equiv ((\neg Q) \to (\neg P))$. We can derive the contrapositive from other rules as follows:

Expression	Reached by
$P \to Q$	given
$\neg P \vee Q$	definition of implication
$Q \vee \neg P$	commutativity
$\neg\neg Q \vee \neg P$	double negation
$(\neg Q) \to (\neg P)$	definition of implication

Question 25:

Given the expression $(P \wedge Q) \vee (O \wedge \neg Q)$, which **two** of the following can be reached by a single application of a rule listed on our equivalences?

- $((P \wedge Q) \vee P) \wedge ((P \wedge Q) \vee \neg Q)$ - distributive
- $P \wedge (Q \vee \neg Q)$ - Distributive (backwards)

Questions 26-30:

Consider the following proof that "if you challenge me to a game, I'll play and win. But I can't win against you" means that you won't challenge me. We can formalize this passage into several propositions, as follows:

- C: if you challenge me to game
- P: I'll play
- W: I'll win

We can symbolize this passage as: $(C \to (F \land W)) \land \neg W$

expression	rule used
$(C \to (F \land W)) \land \neg W$	given
$(\neg C \lor (P \land W)) \land \neg W$	definition of implication
$\neg W \land (\neg C \lor (P \land W))$	commutativity
$(\neg W \land \neg C) \lor (\neg W \land (P \land W))$	distributive
$(\neg W \land \neg C) \lor ((P \land W) \land \neg W)$	commutative
$(\neg W \land \neg C) \lor (P \land (W \land \neg W))$	associativity
$(\neg W \land \neg C) \lor (P \land \bot)$	simplification
$(\neg W \land \neg C) \lor \top$	simplification
$\neg W \land \neg C$	simplification
$\neg C \land \neg W$	simplification
$\neg C$	entailment rule $A \land B \vDash A$

Question 31

Which of the following statements are tautologies? (A tautology is an expression that always evaluates to true):

- $(P \land \neg P) \to Q$ - Tautology
- $(P \lor \neg P) \to Q$ - Not a tautology
- $(P \land Q) \to (P \lor Q)$ - Tautology

Question 32

Select the correct formalism of "If I were rich and famous you wouldn't treat me like this!" Assume each parenthesized english statement becomes one symbol.

- ((I'm rich) $\land$ (I'm famous)) $\to \neg$ (you treat me like this)
- ((I'm rich) $\land$ (im famous)) $\to$ (you wouldn't treat me like this)

Question 33

Starting from $(A \land \neg B) \lor (B \land \neg C)$, which of the following can be reached by only one application of the distributive rule?

- $((A \land \neg B) \lor B) \land ((A \land \neg B) \lor \neg C)$

Question 34

I have an expression consisting of a disjunction of several conjunctions. I want to get some of the terms of the conjunctions to cancel out. To do this, I should first:

- Know what a disjunction and a conjunction is:
  - disjunction is $\lor$ and conjunction is $\land$
- Use the distributive law to change it into a conjunction of disjunctions OR use the distributive law to factor out common terms.
  - The second option is probably better tho :grin: