Practice for Quiz 2

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Practice for Quiz 4 - Induction

Question 1

Prove by induction that $\forall_n \in \mathbb{Z}^+. \sum_{x=1}^{n}(3x^2 - 3x + 1) = n^3$

We proceed by induction

- **Base Case**: Consider the case when $n = 1$. Then, $\sum_{x=1}^{1}(3x^2 - 3x + 1) = 3 - 3 + 1 = 1$, and $1^3 = 1$. Thus, the statement is true for $n = 1$.
- **Inductive Step**: We proceed by induction on n. Assume that the theorem holds for n, that is $\sum_{x=1}^{n}(3x^2 - 3x + 1) = n^3$. By adding the $n + 1$th term to both sides, we have:

$$\sum_{x=1}^{n}(3x^2 + 3x + 1) + (3(n+1)^2 - 3(n+1) + 1) = n^3 + (3(n+1)^2 - 3(n+1) + 1)$$

Now by simplifying, we can get:

$$\sum_{x=1}^{n+1}(3x^2 - 3x + 1) = n^3 + 3(n^2 + 2n + 1) - 3n - 3 + 1$$
$$\sum_{x=1}^{n+1}(3x^2 - 3x + 1) = n^3 + 3n^2 + 6n + 3 - 3n - 3 + 1$$
$$\sum_{x=1}^{n+1}(3x^2 - 3x + 1) = n^3 + 3n^2 + 3n + 1$$
$$\sum_{x=1}^{n+1}(3x^2 - 2x + 1) = (n+1)^3$$

By the principle of induction, we have shown that the theorem holds for all $n \in \mathbb{Z}^+$.

Question 10

Write a proof to prove by induction that $\forall_n in \mathbb{Z}^+. \sum_{i=1}^{n} 2^{-i} = 1 - 2^{-n}$.

We proceed by induction.

- **Base Case**: Consider the case when $n = 1$. Then, $\sum_{i=1}^{1} 2^{-i} = 2^{-1} = 1 - 2^{-1}$, so $\frac{1}{2} = \frac{1}{2}$. Thus, the statement is true for $n = 1$.
- **Inductive Step**: We proceed by induction on n. Assume the theorem holds for $n - 1$, that is $\sum_{i=1}^{n-1} 2^{-1} = 1 - 2^{-(n-1)}$ for some integer $n > 1$. Then:

$$2^{-n} + \sum_{i=1}^{n-1} 2^{-i} = \sum_{i=1}^{n} 2^{-i}$$
$$2^{-n} + 1 - 2^{-(n-1)} = 1 + \frac{1}{2^n} - \frac{1}{2^{n-1}} = 1 + \frac{-1}{2^n} = 1 - \frac{1}{2^n}$$

By the principle of induction, it follows that $\forall_n \in \mathbb{Z}^+. \sum_{i=1}^{n} 2^{-i} = 1 - 2^{-n}$.

Question 16

Prove by induction that $\forall_n \in \mathbb{N}.(\sum_{i=0}^{n}(2i+1)) = (n+1)^2$

We proceed by induction.

- **Base Case**: Consider the case where $n = 0$. Then, $\sum_{i=0}^{0}(2i+1) = 1 = (0+1)^2$. Thus, the statement is true for $n = 0$.
- **Inductive Step**: We proceed by induction on n. Assume that the theorem holds true for n, that is assume $\sum_{i=0}^{n}(2i+1) = (n+1)^2$ for some integer $n > 1$. Then:

$$\sum_{i=0}^{n}(2i+1) + (2(n+1)+1) = (n+1)^2 + 2n + 3$$

$$\sum_{i=0}^{n+1}(2i+1) = n^2 + 4n + 4$$

$$\sum_{i=0}^{n+1}(2i+1) = ((n+1)+1)^2$$

This means that the theorem holds for $n+1$, as well. By the principle of induction, it follows that the theorem holds for all $n \in \mathbb{N}$.

Question 17

Write a proof by induction that the following function returns $2 \cdot x$ for any non-negative integer x:

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let f(x) be computed as
    if x <= 0 then return 0
    else return 2 + f(x-1)
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We proceed by induction.

***Base Case**: Consider the case when $x = 0$. In this instance, the function returns 0. Thus, the statement is true for $x = 0$, since* $2 \cdot 0 = 0$. **Inductive Step**: We proceed by induction on x. Assume that the theorem holds for $x - 1$, for some positive integer x. Then when the function is called with x, it uses the "else" case and returns $2 + f(x-1)$; because the theorem held at $x - 1$, $f(x-1)$ is even, which means $2 + f(x-1)$ is also even so the theorem holds at x too.

By the principle of induction, it follows that the theorem holds for all $x \in \mathbb{N}$.

Question 37

Prove by induction that $\forall_n \in \mathbb{N}. \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

We proceed by induction.

- **Base Case**: Consider the case when $n = 0$. That is, consider $\sum_{i=0}^{0} i$, which is equal to 0. Then, $\frac{0(0+1)}{2} = 0$. Thus, the statement is true for $n = 0$.
- **Inductive Step**: We proceed by induction on n. Assume that the theorem holds for some n, that is, assume $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for some integer $n > 0$. Then:

$$\sum_{i=0}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$$

This means that the theorem holds for $n+1$, as well. By the principle of induction, it follows that the theorem holds for all $n \in \mathbb{N}$.

Question 38

Prove by induction that $\forall_n \in \mathbb{N}. \sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1}-1}{2^n}$

We proceed by induction.

- **Base Case**: Consider the case when $n = 0$. Then, $\sum_{x=0}^{0} \frac{1}{2^x} = \frac{2^1-1}{2^0} = \frac{1}{1} = \frac{1}{2^0} = 1$. Thus, the statement is true for $n = 0$.
- **Inductive Step**: We proceed by induction on n. Assume that the theorem holds for some n, that is assume $\sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1}-1}{2^n}$ for some arbitrary natural number $n > 0$. Then we will show that the equation must hold for $n+1$ as well.

$$\sum_{x=0}^{n} \frac{1}{2^x} + \frac{1}{2^{n+1}} = \frac{2^{n+1}-1}{2^n} + \frac{1}{2^{n+1}}$$

$$\sum_{x=0}^{n+1} \frac{1}{2^x} = 2^{-n-1}(2^{n+2}-1)$$

$$\sum_{x=0}^{n+1} \frac{1}{2^x} = \frac{2^{n+2}-1}{2^{n+1}}$$

By induction on n, we have proven that $\sum_{x=0}^{n} \frac{1}{2^x} = \frac{2^{n+1}-1}{2^n}$ for all $n \in \mathbb{N}$.