Discrete Mathematics and Theory (2120)

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MoWeFri 1:00 - 1:50

Feb 1 - Popular Sets, Power Set, Set-Builder Notation, Disjoint Sets

Some Popular Sets

Symbol	Set	elements
$\emptyset$	the empty set	none
$\mathbb{N}$	nonnegative integers	0, 1, 2, 3...
$\mathbb{Z}$	integers	...3, 2, 1, 0, 1, 2, 3...
$\mathbb{Q}$	rational numbers	$\frac{1}{2}$ 16, etc
$\mathbb{R}$	real numbers	π, e, $\sqrt{2}$
$\mathbb{C}$	complex numbers	i, $\frac{19}{2}$, etc.

A superscript restricts its set to its positive elements, for example $\mathbb{R}^+$ denotes the set of positive real numbers, and for example $\mathbb{Z}^-$ denotes the set of negative integers.

Power Sets

The set oef all subsets of a set, A, is called a *power set*, pow(A), of A. So: $B \in pow(A) \leftrightarrow B \subseteq A$

For example, the elements of pow({1, 2}) are $\emptyset$, {1}, {2}, and {1, 2}.

Questions:

- What is pow({})?
  - {$\emptyset$} (the set containing the empty set).
- What is pow({a, b, c})?
  - pow({a, b, c}) = $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$
  - note the distinction between what pow(stuff) evaluates to versus the *elements* of pow(stuff).
- What is the power set of {W, X, Y, Z}?
  - pow({W, X, Y, Z}) = {{$\emptyset$}, {W}, {X}, {Y}, {Z}, {W, X}, {W, Y}, {W, Z}...}

How can we determine a rule/pattern to determine the cardinality of a powerset?

- $|P(X)| = 2^{|X|}$

Disjoint Sets

Formal definition for disjoint sets: *two sets are disjoint if their intersection is the empty set.* An example of two sets that are **not** disjoint are {1, 2, 3} and {3, 4, 5} since they both share the element 3. However, the set {New York, Washington} and {3, 4} are disjoint.

- {1, 2} and $\emptyset$ are disjoint.
- the empty set is always disjoint with any set
- $\emptyset$ and $\emptyset$ are disjoint!

Set builder Notation

Example: $S = \{x \in A | \text{x is blue}\}$

- The set of all x in A "such that" property (or properties) of x that must be met in order to be an element of S.

A common breakdown of set-builder notation is with numbers: $E = \{x \in \mathbb{N} | x \geq 3\}$

- "the set of all x in the natural numbers such that x is greater than 2."

Let's formalize our set operations in "set-builder notation." Quick side note - we need to link together multiple "conditions" with "and's," "not's," and "or's."

- $\vee$ is "or." (notice similarity to $\cup$)
- $\wedge$ is "and." (notice similarity to $\cap$)
- $\neg$ is "not."

We want to define **intersection**: $S \cap T$: the elements that belong both to S and to T.

- $S \cap T = \{x \in U | x \in T \wedge x \in S\}$
  - Note that "U" is the "universe."
- Another interesting answer: $S \cap T = \{x \in S | x \in T\}$

We want to define the **union**: $S \cap T$: the elements that belong in either S or T (or both):

- $S \cup T\{x \in U | x \in T \vee x \in S\}$
- the "or" $\vee$ symbol is inclusive - includes All elements in S, T, **or** both.

We want to define **difference** $S \backslash T$:

- $S \backslash T = \{x \in U | x \in S \wedge x \notin T\}$
- "Better" answer: $S \backslash T = \{x \in U | x \in S \wedge \neg(x \in T)\}$
- Another cheeky answer: $S \backslash T = \{x \in S | x \notin T\}$

Jan 30 - propositions, operators, set-builder notation

Describing Sets

Listing out the elements of a set works well for sets that are small and finite. What about larger sets? Use set builder notation!!

$S = \{x \in A | xisblue\}$ * S us the set of all x in A such that…

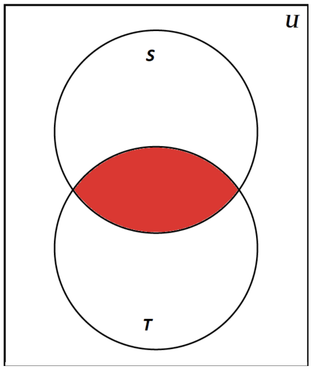
$E = \{x \in \mathbb{N} | x > 2\}$ * E is {3, 4, 5, …, ∞} * Recall that 0 is a natural number btw. * The cardinality of E is infinity.

Jan 27 - intersection, union, difference, complement, and cartesian (cross) product

$\cap, \cup, \backslash$

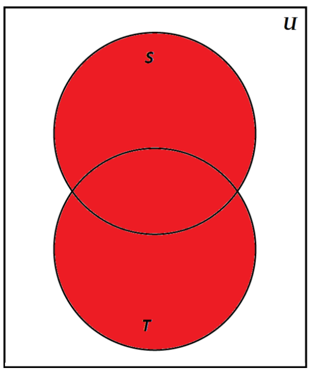
$\cap$: the intersection of two sets

The intersection of two sets S and T, denoted by $S \cap T$, that is the set containing all elements of S that also belong in T. (or equivalently, all elements of T that also belong in S)



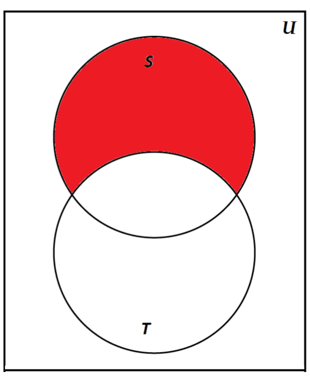
∪: the union of two sets

The union of two sets S and T, denoted by $S \cup T$, is the set containing all the elements of S and also all the elements in T. (or equivalently, everything either in S or T or both)



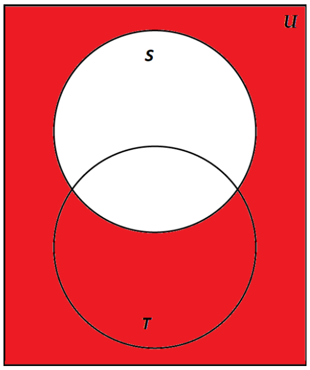
Difference $S \backslash T$: the element that belong to S but not T

Note that the difference of two sets is *not* commutative. It's like division!



Complement: $\bar{S}$ or S^C: All the elements of the universe that don't belong to S.

You have to consider the universe u, not just the "venn diagram" too. See images attached!



High Level Sets and Sequences

1. Sequences vs. Sets
2. Cartesian Product
3. Set builder notation
4. Set Operator Review

High Level: Sets vs Sequences

- Sets:
  - no duplicates
  - no order
  - has cardinality
- Sequences *can have duplicates
  - has order
  - has length
  - lists, arrays, ordered pairs, tuples, etc!
- Both:
  - Contain anything
  - Can have a sequence of sequences, set of sets, sequence of sets, set of sequences
  - Cannot be modified

Cartesian Products of Sets

Ordered Pair: An ordered pair is a **sequence with 2 elements.** It is a pair of obejcts where one element is designated as first and the other element is designated as second, denoted (a, b)

Cartesian Product: The Cartesian product of two sets A and B is denoted $A \times B$, si the set of all possible ordered pairs where the elements of A are the first and the elements of B are second. This is also called the "cross product."

Example: {1, 2} × {3, 4, 5} = {(1, 3), (1, 4), (1, 5), (2, 3), (2, 3), (2, 4), (2, 5)}. This returns a set of sequences. The cardinality of this cross product is 6.

Example: {1, 2} × {3, 4} × {4, 5} = . . .

What is {1, 2} × {2, 3} = {(1, 2), (1, 3), (2, 2), (2, 3)}

- note sequences can have duplicates so (2, 2) is valid!
- Cardinality is 4

Cardinality of Cross Product

$|A \times B| = |A||B|$ (aka, the cardinality of the cross product of two sets is the product of the cardinality of each set)

- Weird Question | { {} } × {1, 2, 3} | = ???
  - 3 . . . since the cross product is = { ({}, 1), ({}, 2), ({}, 3) } !!
- Weird Question | {} × {1, 2} | = ?
  - 0 . . . since $|\emptyset| = 0$

- Let A= $\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$. What is $A^2 = ?$
  - $A^2 = A \times A$
  - $A^3 = A \times A \times A$
  - etc…
  - Note that A^2 is every single (x, y) coordinate that falls in the grid created by A. So, $\mathbb{R} \times \mathbb{R}$ is the coordinate plane!

Jan 25 - subsets & supersets

$\subseteq, \subset, \supseteq, \supset$

These operators compare two sets.

- $\subseteq$: subset: think about as $\leq$
- $\supset$: superset: think about as $\geq$
- $\subset$: proper subset: think about as $<$
- $\supset$: proper superset: think about as $>$

$\subseteq$

Set A is a **subset** of B, or $A \subseteq B$, if and only if **all elements of A are also in B**.

$\supseteq$

Same thing as subset, but flipped direction!

$\subset$

Set a is a **proper subset** of B, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

Consequences of this definition, $A \subset B$: * $A \subseteq A$ (this is trivial but true) * $|A| < |B|$

Example:

Given P={1, 2, 3}, Q={1, 3}, R={1, 3, 4}, determine whether to use subset, superset, proper subset, or proper superset to have all these equations evaluate to $\top$.

- $P \supseteq Q \land P \supset Q = \top$
  - (however, $\supset$ is more specific)
- $P \not\subseteq R = \top$
  - There is no answer for this one!
- $P \supseteq P \land P \subseteq P = \top$

Example 2:

Given P={$\emptyset$, {1, 2}, 3}, Q={1, 2}, R={}, S={3}. Determine how to evaluate to $\top$:

- $P \not\supseteq Q = \top$
  - Think about it!
- $P \supseteq R = \top$
  - Think about it, $\emptyset$ is in P! $\emptyset \subseteq$ every set!
  - And, by extension, $\emptyset \subseteq \emptyset$
- $P \supseteq S \land P \supset S = \top$

if you're tripped up on the first and third bullets, notice the difference and see why they're different.

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Set Definition

A **set** is a structure that contains elements. A **set** has no other properties other than the *elements* it contains. A set can contain other sets. Duplicate values aren't allowed, and order doesn't matter! A **member** or **element** is something inside the set. A set is written with curly braces, its members separated by commas.

- {1, 3} or {dog, cat, mouse} or {3, thimble, Jules} or {{1, 2}, 1}
- Sets can be members of other sets!:
  - {{1, 2}, {3, 4}}
- Sets order doesn't matter;
  - {1, 3, 4} and {4, 3, 1} are the same set
- No duplicates!
  - {1, 3, 4, 1} will "knock" out the duplicate, should be {1, 3, 4}
  - {{1, 2}, {2, 1}} is redundant - this is not a set! It can be just written as {{1, 2}}

The **empty set** is a set with no members, which is expressed as {} or ∅ (or sometimes "null")

Cardinality is the number of elements in a set. Cardinality is denoted using $|A|$. What are the cardinality of these sets:

- $|\{1, -13, 4, -13, 1\}|$: 3
- $|\{3, 1, 2, 3, 4, 0\}|$: 3
  - note that {1, 2, 3, 4} is an element of the bigger set, *not* a subset.
- $|\emptyset|$: 0
- $|\{\{\}, \{\{\}\}, \{\{\{\}\}\}\ \}|$: 3

Examples of Infinite Sets

- $\mathbb{N}$: Natural numbers
  - includes 0!
- $\mathbb{Z}$: Integers
- $\mathbb{Q}$: Rational - the ratio of two integers, $\frac{a}{b}$ that is a finite or repeating decimal
- $\mathbb{R}$: Real -
  - ∞ is not a real number!!!
- $\mathbb{C}$: Complex
- $\mathbb{I}$: Imaginary

∈: "Element of"

checks membership of aan element

Examples:

- $2 \in \{1, 2\} = \top$
- $3 \in \{1, 2\} = \bot$
- $3 \notin \{1, 2\} = \top$
- $\{2\} \in \{1, 2\} = \bot$
- $\{2\} \in \{1, \{2\}\} = \top$
- $2 \in \{\{1, 2\}\} = \bot$
- $2 \in \{\{\}\} = \bot$
- $2 \in \{\{\{2\}\}\} = \bot$
- $\{2\} \in \{\{1, 2\}\} = \bot$
- $\{2\} \in \{\{2\}\} = \top$
- $\{2\} \in \{\{\{2\}\}\} = \bot$