Practice for Quiz 2

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MoWeFri 1:00 - 1:50

Practice for Quiz 4 - Counting

Review from unfinished slides problems

Question 1

A local US telephone number has 7 digits and cannot start with 0, 1, or the three digits 555. How many such telephone numbers are possible?

- The first digit has 8 possibilities (the digits 2 through 9) and the other six digits each have 10 possibilities.
- However, we have included the numbers that start with 555 - each of these has 10 choices for the last four digits.
- Subtracting the disallowed yields: $8 \times 10^6 - 10^4 = 7999900$.
- *Addendum*: Another method is to count all 10^7 numbers and subract the 10^6 starting with 0, the 10^6 starting with 1, and the 10^4 starting with 555. This yields the same answer.

Question 2

How many sequences are there with exactly 8 characters taken from the 26 lower-case ASCII letters and 10 ASCII digits, **with no digits repeated twice back-to-back**?

- $36 * 35^7$

Question 3

Exactly 8 characters taken from the 26 lower-case ASCII letters and 10 ASCII digits, with no repeated characters (neither abcdaefg nor abcddefg are allowed).

- $\frac{36!}{(36-8)!}$.

Question 4

How many 21-element subsets of a 31 element set are there?

- $\binom{31}{21}$

Question 5

How many unique shufflings of "alabama" are there?

- $\frac{7!}{4!}$

Counting Problems from Website

Question 1

Prove that $|P(S)| = 2^{|S|}$.

We proceed by induction on the cardinality of S.

- **Base case**: $|S|S = 0$.
  - In this case, $S = \emptyset$ and $P(S) = \{\emptyset\}$. Thus, $|P(S)| = 1 = 2^0$.
- **Inductive step**: Suppose $|S| > 0$ and all sets of cardinality $|S| - 1$ have $2^{|S|-1}$ elements tin their powerset. Pick an arbitrary member x of S and define $T = S - \{x\}$. For every member y of $P(T)$, we know that $P(S)$ has two members: y and $y \cup \{x\}$. We also know that:
  - Those members are distinct because one contains x and the other does not.
  - All members of P(S) can be generated in this way
  - no single member of P(S) is generated from two distinct y

Thus, $|P(S)| = 2 * |P(T)|$. Because $|T| = |S| - 1, |P(T)| = 2^{|S|-1}$. Thus, $|P(S)| = 2\{\{|S|\}\}$.

Question 2

Prove by induction that the number of distinct k member subsets of an n member set is denoted $\binom{n}{k}$ and is equal to $\frac{n!}{k!(n-k)!}$.

- I won't write this out. We aren't being tested on this at the moment, you can read the proof yourself on the website.

Question 3

Prove that the number of permutations of a sequence with n distinct elements is $n!$.

We proceed by induction on n.

- **Base case**: Consider the case when $n = 0$. By definition, $0! = 1$. Further, consider the case where $n = 1$. $1! = 1$. Thus, we have shown that $0! = 1! = 1$.
- **Inductive step**: Assume that a $n - 1$ element sequence with distinct elements has $(n - 1)!$ permutations. Then we construct and count all permutations of an n element sequence, S_n, as follows:
  1. Consider the $n - 1$ element sequence S_{n-1} defined as all elements of S_n except for the last.
  2. Create the $(n - 1)!$ permutations of S_{n-1}
  3. From each permutation of S_{n-1}, generate n permutations of S_n where the _i_th permutation generated from s is s with the last element of S_n in the _i_th spot.

Because all elements of S_n are unique, resulting permutations are distinct. We generated n permutations for each of $(n - 1)!$ sub-permutations, for a total of $n \times (n - 1)! = n!$ permutations. By the principle of induction, it holds that the number of permutations of any sequence of n distinct elements is $n!$.

Question 4

Prove that for all the finite sets S, $|S^k| = |S|^k$.

We proceed by induction on k.

- **Base Case**: Consider when $k = 0$. Then $S^k = \{\emptyset\}$, and therefore the cardinality is 1.
- **Inductive Step**: Assume that $|S^{k=1}| = |S|^{k-1}$. Then we can enumerate the elements of S^k as follows:
  - For each element x of S^{k-1}, create $|S|$ sequences of length k; each starts with a different element of S and then is followed by the elements of x in order. This results in $|S||S^{k-1}| = |S|^k$ elements in total.

By the principle of induction, it follows that $|S^k| = |S|^k$.

Question 5

Assume that a "digit" is an integer between 0 and 9, inclusive. Choose the corret answer:

- there are more length-5 sequences of digits than cardinality-5 sets of digits.
- there are fewer length-5 sequences of digits than cardinality-5 sets of digits
- there are the same number of length-5 sequences of digits and cardinality-5 sets of digits.

Answer:

- To calculate the amount of length-5 sequences of digits, we can get that by multiplying 10 by itself 5 times. This gives us 10ˆ5.
- to calculate the amount of cardinality-5 sets of digits, we can get this by doing $\binom{10}{5}$. This gives us 252.
- We can see that there are more length-5 sequences of digits than cardinality-5 sets of digits.