Bajari, Benkard, and Levin (Ema, 2007)

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Motivation

- Interest in estimating model of industry dynamics.
- Ex. Ericson and Pakes and McGuire (1994, RES) style models of entry and R&D
- These models can be difficult to compute.
- Many methods using conditional choice probabilities (CCP)
- Aguirregabiria and Mira (2007, EMA), Pessendorfer and Schmidt-Dengler (2008, RES), Pakes, Ostrovsky, and Berry (2007 RAND).
- Perhaps Bajari, Benkard, and Levin (2007, EMA) is the most flexible.

Notation

- The game is in discrete time with an infinite horizon
- There are N firms, denoted $i = 1, ..., N$ making decisions at times $t = 1, 2, ..., \infty$
- Conditions at time t are summarized by discrete states $\mathbf{s}_t \in S \subset \mathcal{R}^L$
- Given $\mathbf{s}_t$, firms choose actions simultaneously
- Let $a_{it} \in A_i$ denote firm i's action at time t, and $\mathbf{a}_t = (a_{1t}, ..., a_{Nt})$ the vector of time t actions

Notation

- Before choosing its action, each firm i receives a private shock ν_{it} drawn *iid* from $G_i\left(\cdot|\mathbf{s}_t\right)$ with support $\mathcal{V}_i \subset \mathcal{R}^M$
- Denote the vector of private shocks $\nu_t = (\nu_{1t}, ..., \nu_{Nt})$
- Firm i's profits are given by $\pi_i(\mathbf{a}_t, \mathbf{s}_t, \nu_{it})$ and firms share a common (& known) discount factor $\beta < 1$
- Given $\mathbf{s}_t$, firm i's expected profit (prior to seeing ν_{it}) is

$$E\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\pi_i(\mathbf{a}_\tau, \mathbf{s}_\tau, \nu_{i\tau})\middle|\mathbf{s}_t\right]$$

where the expectation is over current shocks and actions, as well as future states, actions, and shocks.

State Transitions & Equilibrium

- $\mathbf{s}_{t+1}$ is drawn from a probability distribution $P\left(\mathbf{s}_{t+1} | \mathbf{a}_t, \mathbf{s}_t\right)$
- Focus on pure strategy Markov Perfect Equilibrium
- A Markov strategy is a function $\sigma_i : S \times \mathcal{V}_i \to A_i$
- A profile of Markov strategies is a vector, $\sigma = \left(\sigma_1, ..., \sigma_N\right)$, where $\sigma : S \times \mathcal{V}_1 \times ... \times \mathcal{V}_N \to A$
- Given σ, firm i's expected profit can then be written recursively

$$V_i\left(\mathbf{s};\sigma\right) = E_\nu\left[\pi_i(\sigma\left(\mathbf{s},\nu\right), \mathbf{s}, \nu_i) + \beta \int V\left(\mathbf{s}';\sigma\right) dP\left(\mathbf{s}' | \sigma\left(\mathbf{s},\nu\right), \mathbf{s}\right) | \mathbf{s}\right]$$

- The profile σ is a MPE if, in every state, given opponent profile σ_{-i}, each firm i prefers strategy σ_i to all other alternatives σ_i'

$$\forall i, \mathbf{s} : V_i\left(\mathbf{s};\sigma\right) \geq V_i\left(\mathbf{s};\sigma_i', \sigma_{-i}\right)$$

Structural Parameters

- The structural parameters of the model are:
  - The discount factor β,
  - The profit functions $\pi_1, ..., \pi_N$,
  - The state transition probabilities P, given states and actions.
  - The distributions of private shocks $G_1, ..., G_N$.
- Treat β as known
- Estimate P directly from the observed state transitions (states and actions are observable).
- Assume the profits and shock distributions are known functions of a parameter vector $\theta : \pi_i(\mathbf{a}, \mathbf{s}, \nu_i; \theta)$ and $G_i\left(\nu_i | \mathbf{s}, \theta\right)$.
- The goal is to recover the true θ under the assumption that the data are generated by a **single** MPE.

Example: Dynamic Oligopoly

- Their main example is based on the EP framework.
- Incumbent firms are heterogeneous, each described by its state $z_{it} \in \{1, 2, ..., \bar{z}\}$; potential entrants have $z_{it} = 0$
- Incumbents can make an (observable) investment $I_{it} \geq 0$ to improve their state
- An incumbent firm i in period t earns

$$q_{it} \left(p_{it} - mc \left(q_{it}, s_t; \mu \right) \right) - C \left(I_{it}, \nu_{it}; \xi \right)$$

  where p_{it} is firm i's price, $q_{it} = q_i \left(\mathbf{s}_t, \mathbf{p}_t; \lambda \right)$ is quantity, $mc\left(\cdot\right)$ is marginal cost, and ν_{it} is a shock to the cost of investment.
- $C \left(I_{it}, \nu_{it}; \xi \right)$ is the cost of investment.

Example: Dynamic Oligopoly

- Competition is assumed to be static Nash in prices.
- Firms can also enter and exit.
- Exitors receive ϕ and entrants pay ν_{et}, an iid draw from G_e
- In equilibrium, incumbents make investment and exit decisions to maximize expected profits.
- Each incumbent i uses an investment strategy $I_i(\mathbf{s}, \nu_i)$ and exit strategy $\chi_i(\mathbf{s}, \nu_i)$ chosen to maximize expected profits.
- Entrants follow a strategy $\chi_e(\mathbf{s}, \nu_e)$ that calls for them to enter if the expected profit from doing so exceeds its entry cost.

Two-Stage Estimation Overview

- The goal of the first stage is to estimate the state transition probabilities $P(\mathbf{s}'|\mathbf{a}, \mathbf{s})$ and equilibrium policy functions $\sigma(\mathbf{s}, \nu)$
- The second stage will use the equilibrium conditions from above to estimate the structural parameters θ
- In order to obtain consistent first stage estimates, they must assume that the data are generated by a single MPE profile σ
    - This assumption has a lot of bite if the data come from multiple markets
    - It's quite weak if the data come from only a single market
    - Also rules out unobserved heterogeneity

First Stage Estimation

- Stage "0":
  - The static payoff function may be estimated "off line" in a 0^{th} stage (e.g. BLP, Olley-Pakes)
- Stage 1: Conditional Choice Probabilities: $P(a_i|s)$.
  - It's usually fairly straightforward to run the first stage: just "regress" actions on states in a "flexible manner".
  - Since these are not structural objects, you should be as flexible as possible. Why?
  - Of course, if $\mathbf{s}$ is big, you may have to be very parametric here (i.e. OLS regressions and probits).
    - In this case, your second stage estimates will be inconsistent...
  - Continuous actions are especially tricky (hard to be non-parameteric)

The Key Inversion: Choice Probabilities to Strategies

All two-step estimation methods are based on the ability to consistently estimate strategies and value functions from $P(a|s)$.

Discrete Choice:

- Assume additive separability $\pi(a,s,\nu_i) = \tilde{\pi}(a,s) + \nu_i(a_i)$, then write choice-specific value function:

$$v_i(a,s) = E_{\nu_{-i}}\left[\tilde{\pi}(a_i,\sigma_{-i}(s,\nu_{-i}),s) + \beta \int V_i(s',\sigma)dP(s'|a_i,\sigma_{-i},S)\right]$$

- Optimization implies agent i chooses a_i if $\forall a_i' \in A_i$:

$$\sigma_i(s,\nu_i) = a_i \iff v_i(a_i,s) + \nu_i(a_i) \geq v_i(a_i',s) + \nu_i(a_i')$$

- Given a parametric assumption on ν_i this identifies v_i up to location and scale.
- For example if ν_i are iid extreme value,

$$v_i(a_i',s) - v_i(a_i,s) = \ln(P(a_i'|s)) - \ln(P(a_i|s))$$

The Key Inversion: Choice Probabilities to Strategies

Continuous (or Ordered) Choice:

- Assume Monotone Choice: $\pi_i(a, s, \nu_i)$ has increasing differences in a_i, ν_i.
- Implies a unique optimal policy that is monotone in ν_i, so we can invert choice probabilities.
- Let $F_i(a_i|s) = P(\sigma_i(s, \nu_i) \leq a_i|s)$, then:

$$\sigma_i(s, \nu_i) = F_i^{-1}(G(\nu_i|s; \theta)|s)$$

- We estimate F and assume G is known up to parameters.

Estimating (Simulating) the Value Functions

- After estimating policy functions, firm's value functions are estimated by forward simulation.
- Let $V_i(\mathbf{s}, \sigma; \theta)$ denote the value function of firm i at state s assuming firm i follows the Markov strategy σ_i and rival firms follow σ_{-i}
- Then

$$V_i(\mathbf{s}, \sigma; \theta) = E\left[\sum_{t=0}^{\infty} \beta^t \pi_i(\sigma(\mathbf{s}_t, \nu_t), \mathbf{s}_t, \nu_{it}; \theta) \middle| \mathbf{s}_0 = \mathbf{s}; \theta\right]$$

  where the expectation is over current and future values of $\mathbf{s}_t$ and ν_t
- Given a first-stage estimate $\widehat{P}$ of the transition probabilities, we can simulate the value function $V_i(\mathbf{s}, \sigma; \theta)$ for **any** strategy profile σ and parameter vector θ.
- In particular, we can estimate V under the estimated policy function from the first stage.

Estimating the Value Functions

- A single simulated path of play can be obtained as follows:
  1. Starting at state $\mathbf{s}_0 = \mathbf{s}$, draw private shocks ν_{i0} from $G_i\left(\cdot|\mathbf{s}_0, \theta\right)$ for each firm i.
  2. Calculate the specified action $a_{i0} = \sigma_i\left(\mathbf{s}_0, \nu_{i0}\right)$ for each firm i, and the resulting profits $\pi_i\left(\mathbf{a}_0, \mathbf{s}_0, \nu_{i0}; \theta\right)$
  3. Draw a new state $\mathbf{s}_1$ using the estimated transition probabilities $\widehat{P}\left(\cdot|\mathbf{a}_0, \mathbf{s}_0\right)$
  4. Repeat steps 1-3 for T periods or until each firm reaches a terminal state with known payoff (e.g. exits from the market)
- Averaging firm i's discounted sum of profits over many paths yields an estimate $\widehat{V}_i\left(\mathbf{s}, \sigma; \theta\right)$, which can be obtained for any (σ, θ) pair, including both the "true" profile (which you estimated in the first stage) and any alternative you care to construct.

Special Case of Linearity

- Forward simulation yields a low cost estimate of the V's for different σ's given θ, but the procedure must be repeated for each candidate θ.
- One case is simpler.
- If the profit function is linear in the parameters θ so that

$$\pi_i\left(\mathbf{a},\mathbf{s},\nu_i;\theta\right)=\psi_i\left(\mathbf{a},\mathbf{s},\nu_i\right)\cdot\theta$$

  we can then write the value function as

$$V_i\left(\mathbf{s},\sigma;\theta\right)=E\left[\sum_{t=0}^{\infty}\beta^t\psi_i(\sigma\left(\mathbf{s}_t,\nu_t\right),\mathbf{s}_t,\nu_{it})|\mathbf{s}_0=\mathbf{s}\right]\cdot\theta=\mathbf{W}_i\left(\mathbf{s};\theta\right)\cdot\theta$$

- In this case, for any strategy profile σ, the forward simulation procedure only needs to be used once to construct each $\mathbf{W}_i$.
- You can then obtain V_i easily for any value of θ.

Second Stage Estimation: Optimality Conditions

- The first stage yields estimates of the policy functions, state transitions, and value functions.
- The second stage uses the model's equilibrium conditions

$$V_i\left(\mathbf{s};\sigma_i,\sigma_{-i};\theta\right) \geq V_i\left(\mathbf{s};\sigma_i',\sigma_{-i};\theta\right)$$

  to recover the parameters θ that "rationalize" the strategy profile σ observed in the data.
- They show how to do so for both set and point identified models
- We will focus on the point identified case here, partially identified case is similar to Holmes and CT.

Second Stage Estimation

- Denote $x = (i, s, \sigma')$ index a particular equilibrium condition, then define

$$g(x; \theta, \alpha) = V_i(\mathbf{s}; \sigma_i, \sigma_{-i}; \theta, \alpha) - V_i\left(\mathbf{s}; \sigma_i', \sigma_{-i}; \theta, \alpha\right)$$

  where α represents the first-stage parameter vector.
- The inequality defined by x is satisfied at θ, α if $g(x; \theta, \alpha) \geq 0$
- Define the function

$$Q(\theta, \alpha) \equiv \int (\min\{g(x; \theta, \alpha), 0\})^2 \, dH(x)$$

  where H is a distribution over the set $\mathcal{X}$ of inequalities.

Second Stage Estimation

- The true parameter vector θ_0 satisfies

$$Q\left(\theta_0, \alpha_0\right) = 0 = \min_{\theta \in \Theta} Q(\theta, \alpha_0)$$

  so we can estimate θ by minimizing the sample analog of $Q(\theta, \alpha_0)$
- The most straightforward way to do this is to draw firms and states at random and consider alternative policies σ_i' that are slight perturbations of the estimated policies.

Second Stage Estimation

- We can then use the above forward simulation procedure to construct analogues of each of the V_i terms and construct

$$Q_n(\theta, \alpha) \equiv \frac{1}{n_I} \sum_{k=1}^{n_I} \left(\min\left\{\widehat{g}\left(X_k; \theta, \alpha\right), 0\right\}\right)^2$$

- How? By drawing n_I different alternative policies, computing their values, finding the difference versus the optimal policy payoff, and using an MD procedure to estimate the parameters that minimize these profitable deviations.
- Assume n_I and simulation lengths increase as $n \to \infty$.
- Their estimator minimizes the objective function at $\alpha = \widehat{\alpha}_n$

$$\theta = \arg\min_{\theta \in \Theta} Q_n\left(\theta, \widehat{\alpha}_n\right)$$

- See the paper for the technical details.

Example: Dynamic Oligopoly

- Let's see how they estimate the EP model.
- First, they have to choose some parameterizations.
- They assume a logit demand system for the product market.
- There are M consumers with consumer r deriving utility U_{ri} from good i

$$U_{ri} = \gamma_0 \ln(z_i) + \gamma_1 \ln\left(y_r - p_i\right) + \varepsilon_{ri}$$

  where z_i is the quality of firm i, p_i is firm i's price, y_r is income, and ε_{ri} is an *iid* logit error
- All firms have identical constant marginal costs of production

$$mc(q_i; \mu) = \mu$$

Example: Dynamic Oligopoly

- Each period, firms choose investment levels $I_{it} \in \mathcal{R}_+$ to increase their quality in the next period.
- Firm i's investment is successful with probability

$$\frac{\rho I_{it}}{(1 + \rho I_{it})}$$

  in which case quality increases by one, otherwise it doesn't change.
- There is also an outside good, whose quality moves up by one with probability δ each period.
- Firm i's cost of investment is

$$C(I_i) = \xi \cdot I_i$$

  so there is no shock to investment (it's deterministic)

Example: Dynamic Oligopoly

- The scrap value ϕ is constant and equal for all firms.
- Each period, the potential entrant draws a private entry cost ν_{et} from a uniform distribution on $[\nu^L, \nu^H]$
- The state variable $s_t = (N_t, z_{1t}, ..., z_{Nt}, z_{out,t})$ includes the number of incumbent firms and current product qualities.
- The model parameters are $\gamma_0, \gamma_1, \mu, \xi, \phi, \nu^L, \nu^H, \rho, \delta, \beta,$ & y
- They assume that β & y are known, ρ & δ are transition parameters estimated in a first stage, $\gamma_0, \gamma_1,$ & μ are demand parameters (also estimated in a first stage), so the main (dynamic) parameters are simply $\theta = (\xi, \phi, \nu^L, \nu^H)$
- Due to the computational burden of the PM algorithm, they consider a setting in which only ≤ 3 firms can be active.
- They generated datasets of length 100-400 periods using PM.

Example: Dynamic Oligopoly

- Here are the parameters they use.

Table 2: Dynamic Oligopoly Monte Carlo Parameters

Parameter	Value	Parameter	Value
Demand:		Investment Cost:	
γ_1	1.5	ξ	1
γ_0	0.1		
M	5	Marginal Cost:	
y	6	μ	3
Investment Evolution		Entry Cost Distribution	
δ	0.7	ν^l	7
ρ	1.25	ν^h	11
Discount Factor		Scrap Value:	
β	0.925	ϕ	6

Example: Dynamic Oligopoly

- The first stage requires estimation of the state transitions and policy functions (as well as the demand and mc parameters).
- For the state transitions, they used the observed investment levels and qualities to estimate ρ and δ by MLE.
- They estimated the demand parameters by MLE as well, using quantity, price, and quality data.
- They recover μ from the static mark-up formula.
- They used local linear regressions with a normal kernel to estimate the investment, entry, and exit policies.

Example: Dynamic Oligopoly

- Given strategy profile $\sigma = (I, \chi, \chi_e)$, the incumbent value function is

$$
\begin{aligned}
V_i(s;\sigma) &= W^1(s;\sigma) + W^2(s;\sigma)\cdot\xi + W^3(s;\sigma)\cdot\phi \\
&= E\left[\sum_{t=0}^{\infty}\beta^t\widetilde{\pi}_i(s_t)|s=s_0\right] - E\left[\sum_{t=0}^{\infty}\beta^t I_i(s_t)|s=s_0\right]\cdot\xi \\
&+ E\left[\sum_{t=0}^{\infty}\beta^t\chi_i(s_t)|s=s_0\right]\cdot\phi
\end{aligned}
$$

  where the first term $\widetilde{\pi}_i(s_t)$ is the static profit of incumbent i given state s_t
- The 2^{nd} term is the expected PV of investment
- The 3^{rd} term is the expected PV of the scrap value earned upon exit.

Example: Dynamic Oligopoly

- To apply the MD estimator, they constructed alternative investment and exit policies by drawing a mean zero normal error and adding it to the estimated first stage investment and exit policies.
- They used $n_s = 2000$ simulation paths, each having length at most 80, to compute the PV W^1, W^2, W^3 terms for these alternative policies.
- They can then estimate ξ & ϕ using their MD procedure.
- It's also straightforward to estimate the entry cost distribution (parametrically or non-parametrically) - see the paper for details.

Example: Dynamic Oligopoly

Table 3: Dynamic Oligopoly With Nonparametric Entry Distribution

	Mean	SE(Real)	5%(Real)	95%(Real)	SE(Subsampling)
$n = 400,\ n_I = 500$					
ξ	1.01	0.05	0.91	1.10	0.03
ϕ	5.38	0.43	4.70	6.06	0.39
$n = 200,\ n_I = 500$					
ξ	1.01	0.08	0.89	1.14	0.05
ϕ	5.32	0.56	4.45	6.33	0.53
$n = 100,\ n_I = 300$					
ξ	1.01	0.10	0.84	1.17	0.06
ϕ	5.30	0.72	4.15	6.48	0.72

- Truth: $\xi = 1$ & $\phi = 6$
- For small sample sizes, there is a slight bias in the estimates of the exit value.
- Investment cost parameters are spot on.

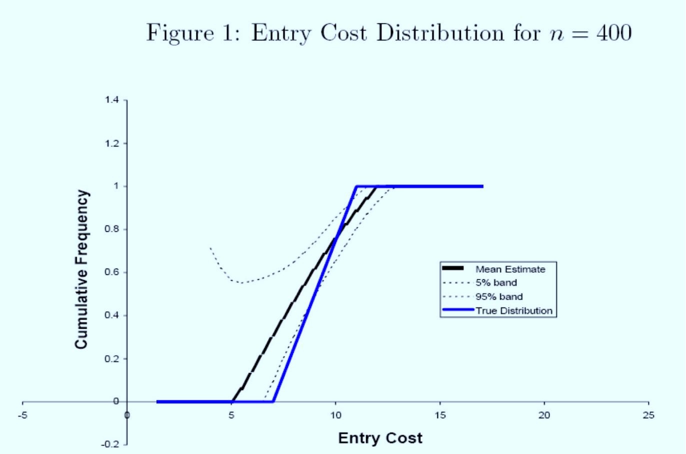
Example: Dynamic Oligopoly

Table 4: Dynamic Oligopoly With Parametric Entry Distribution

	Mean	SE(Real)	5%(Real)	95%(Real)	SE(Subsampling)
$n = 400,\ n_I = 500$					
ξ	1.01	0.06	0.92	1.10	0.04
ϕ	5.38	0.42	4.68	6.03	0.41
ν^l	6.21	1.00	4.22	7.38	0.26
ν^h	11.2	0.67	10.2	12.4	0.30
$n = 200,\ n_I = 500$					
ξ	1.01	0.07	0.89	1.13	0.05
ϕ	5.28	0.66	4.18	6.48	0.53
ν^l	6.20	1.16	3.73	7.69	0.34
ν^h	11.2	0.88	9.99	12.9	0.40
$n = 100,\ n_I = 300$					
ξ	1.01	0.10	0.84	1.17	0.06
ϕ	5.43	0.81	4.26	6.74	0.75
ν^l	6.38	1.42	3.65	8.43	0.51
ν^h	11.4	1.14	9.70	13.3	0.58

- Truth: $\xi = 1, \phi = 6, \nu^l = 7,$ & $\nu^h = 11$
- The subsampled standard errors are on average slightly smaller than the true SEs.

Example: Dynamic Oligopoly



- The entry cost distribution is recovered quite well, despite small sample size (and few entry events).

Conclusions

- Both AM & BBL are based on the same underlying idea (CCP estimation)
- As such, it's quite possible to mix and match from the two approaches
    - e.g. forward simulate the CV terms and use a MNL likelihood
- Even if you can estimate the model...can you simulate it or produce counterfactual analysis?
- If you are interested in applying this stuff, you should read *everything* you can get your hands on...
- ...and have a economically interesting and feasible application in mind.