Agent-based models: understanding the economy from the bottom up

By Arthur Turrell of the Bank's Advanced Analytics Division.[1]

- Agent-based modelling has long enjoyed success in the natural sciences, providing insights into everything from cancer to the eventual fate of the Universe.
- It is suited to modelling complex systems such as the economy, particularly those in which different agents' interactions combine to produce unexpected outcomes.
- In economics, agent-based models have shown how business cycles occur, how the statistics observed in financial markets (such as 'fat tails') arise, and how they can be a useful tool in formulating policy.

Overview

Agent-based models explain the behaviour of a system by simulating the behaviour of each individual 'agent' within it. These agents and the systems they inhabit could be the consumers in an economy, fish within a shoal, particles in a gas, or even galaxies in the Universe.

The strength of these models is that they show how even very simple behaviours can combine from the 'bottom up' to recreate the more complex behaviours observed in the real world. An example would be how the decisions of each individual fish create the seemingly organised and unpredictable movements of the shoal.

This 'bottom-up' approach is in contrast to models which are 'top down', and which presume how agents' behaviours will combine together, sometimes by assuming that all agents are identical. The different approaches have different strengths.

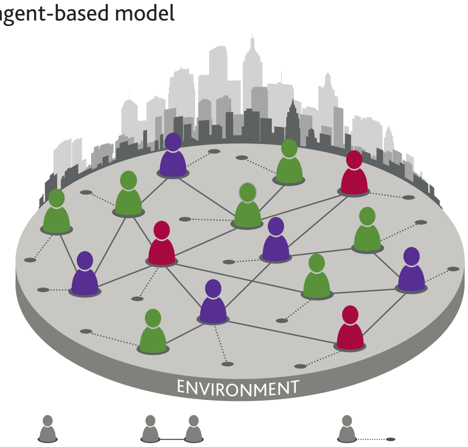
The agent-based approach to problem-solving began in the physical sciences but has now spread to many other disciplines including biology, ecology, computer science and epidemiology. In recent years, agent-based models have become more common in economics, including at the Bank of England.

There are challenges to their use, including the need for advanced programming skills, the need to carefully interpret their results, and how to best select the appropriate behaviours for the agents. In particular, there are not always obvious criteria for choosing which behaviours are the most realistic. These issues have been a barrier to their more widespread adoption in economics.

Despite being less widely used, agent-based models have produced many important insights in economics, including how the statistics observed in financial markets arise, and how business cycles occur.

Recently, the Bank of England has developed agent-based models of two markets: corporate bonds and housing. The increased availability of data and computational power mean that agent-based modelling looks set to gain importance as a tool for both understanding the economy, and for exploring the consequences of policy actions.

Summary figure Schematic of the typical elements of an agent-based model



(1) The author would like to thank David Bholat and Chris Cai for their help in producing this article.

Introduction

This article considers the strengths of agent-based modelling and the ways that it can be used to help central banks understand the economy. These models provide a complement to more traditional economic modelling which was criticised following the Great Recession.(1)

Agent-based models have different strengths and weaknesses to other approaches in economics. They have advantages in describing how the different actions and properties of individual agents combine to drive the overall behaviour of systems.

This article explains the motivation behind developing models and how they can be used to better understand the world, before discussing more details of what makes agent-based models different. It then describes how these models were first developed and used in disciplines outside of economics.

The general strengths and weaknesses of agent-based models are discussed, with examples of how their advantages have been used to improve the understanding of certain markets.

The penultimate section is an overview of how agent-based models are being applied in economics in general and at the Bank of England specifically. Finally, some thought is given to how this line of modelling might be used in the future.

Agent-based models are called by different names in different disciplines, including Monte Carlo simulations (in the physical sciences), individual-based models (in biology and ecology), agent-based computational economics models (in economics) and multi-agent systems (in computer science and logistics). This article uses 'agent-based model' to refer to any model in which the interactions and behaviours of a large number of heterogeneous agents are simulated, manually or by a computer.

Modelling

Modelling is ubiquitous across academic disciplines, governments and the private sector. Most models attempt to isolate the underlying causes of the behaviour of systems, removing extraneous detail and focusing on what matters to the hypothesis, question or policy under consideration. Models may be as simple as thought experiments but, in quantitative subjects, often involve mathematics and simulation.

Using an artificial and simplified version of the world allows researchers and policymakers to explore what might happen in certain scenarios. In macroeconomics, data may be scarce and experiments can rarely, if ever, be performed in the real world, and this makes models especially useful. Different models are good for answering different questions and so a wide range of them are required.

All models should be able to reproduce, as much as possible, the real world observables they seek to explain. It also helps if they are easy to use and interpret, and if they can explain the phenomena as simply as possible. Over time, models which explain reality more adequately and more concisely are favoured, replacing those which explain it less well.

Agent-based models are suited to studying problems in which the combination of the interactions of many agents drives the overall behaviour of the system. They solve problems from the 'bottom up' rather than through rules imposed from the 'top down'. Typically, creating an agent-based model requires knowledge of mathematics, statistics, and computer science, as well as the discipline in which it is being applied.

These models get their name because they involve simulating a large number of 'agents'. Each agent is a self-contained unit which follows its own behavioural rules. Most often, this is achieved within a computer simulation but it need not be.

Agents could represent the consumers in an economy, fish within a shoal, particles in a gas, or even galaxies in the Universe. The behaviours or rules that agents follow depend on the question of interest. Some models have many different types of agent, for instance firms, workers and governments. These may themselves differ; for instance, each worker might have a different productivity, each firm a different size.

Agent-based models have illuminated a surprisingly wide range of subjects including military planning and battlefield analysis, operational research, computer science, biology, ecology, epidemiology, economics, social sciences and the physical sciences.

With them, such varied problems as the separation of Brazil nuts from other mixed nuts, the behaviour of traders in the stock market, the flocking of birds, the fall of ancient civilisations, the spread of disease, and the eventual fate of the Universe have been examined.

In a few cases, results which could only have been obtained through agent-based modelling have significantly changed the course of history: calculations of the way that particles are transported through a pile of fissile material in a nuclear reactor or weapon would be prohibitively difficult to do in any other way.

Their use in economics comes with some particular nuances which are explored in this article. In the natural sciences, agent behaviours are typically much more constrained than in

(1) See Haldane (2016).

economics. Because of this, agent-based models in economics typically produce insights rather than quantitative forecasts. They are typically qualitative rather than quantitative, and they are good for determining what scenarios might occur rather than exactly what will occur.

An agent-based model in economics would not normally be appropriate for forecasting the price of a particular asset, for example. But it can give an idea of what actions by traders might move the prices of assets, or why the supply of an asset is much more volatile than the demand for it. Other phenomena which they have explained in different contexts include cycles, bubbles, clustered volatility, fire sales of assets, and the onset of 'bear' and 'bull' markets.(1) Despite being more suited to insights than quantitative prediction, there have been some successful examples of forecasting with agent-based models in economics; for example, for the demand for electricity or the repayment rate of mortgages.(2)

The Great Recession profoundly challenged the economics profession, particularly economic modelling. It demonstrated that the economy is complex, and not always at a stable equilibrium. The after-effects of the crisis are still being felt. Established tools, such as 'dynamic stochastic general equilibrium' models(3) have been criticised(4) for not having enough to say about the dynamics of crises. Agent-based models are one response to the challenge and this article explores their potential in aiding our understanding of the economy.

The remainder of this section illustrates how these models have developed and what uses they have previously been put to. Then, the article turns to a specific example of agent-based modelling in economics which demonstrates some of their general features.

The origins of agent-based modelling

The initial spur for developing agent-based models came in the 1930s when physicist Enrico Fermi was trying to solve problems involving the transport of neutrons, a sub-atomic particle, through matter. Neutron transport was difficult to model as each step in a neutron's journey is probabilistic: there is a chance the particle will directly interact with, scatter off, or pass-by other particles. Previous methods had tried to capture the aggregate behaviour of all the neutrons at once, but the immense number of different possibilities for neutron paths through matter made the problem very challenging.

Fermi developed a new method to solve these problems in which he treated the neutrons individually, using a mechanical adding machine to perform the computations for each individual neutron in turn. The technique involved generating random numbers and comparing them to the probabilities derived from theory. If the probability of a neutron colliding were 0.8, and he generated a random number smaller than 0.8, he allowed a 'simulated' neutron to collide. Similar techniques were used to find the outgoing direction of the neutron after the collision. By doing this repeatedly, and for a large number of simulated neutrons, Fermi could build up a picture of the real way that neutrons would pass through matter. Fermi took great delight in astonishing his colleagues with the accuracy of his predictions without, initially, revealing his trick of treating the neutrons like agents.(5)

The agent-based techniques Fermi and colleagues developed went on to play an important role in the development of nuclear weapons and nuclear power. At around the same time that Fermi was developing his technique, the first electronic computers were becoming available at the world's leading scientific institutions. Computing power remains key to agent-based modelling today, with some of the world's supercomputers being harnessed for ever more detailed simulations.(6)

By 1947 scientists had developed a name for this technique which reflected its probabilistic nature: the Monte Carlo method. The story goes that the name was inspired by Stanislaw Ulam's uncle, who would often ask to borrow money by saying he 'just had to go to Monte Carlo'. In 1949, Metropolis and Ulam published a paper together entitled *The Monte Carlo Method*(7) which explained the many uses of the new technique of using random numbers to tackle problems. Not all of these were agent-based models but all relied on using artificially generated random numbers to solve problems. This more general Monte Carlo technique has been applied very widely, for instance to calculating solutions to equations with many parameters, to the management of risk and catastrophes, and to investments in finance.

The more general Monte Carlo method has the strength that it can very efficiently explore a large number of possibilities. For instance, the usual way for Fermi's neutron problem to have been treated would have been to create a grid of every single possibility and then fill in what happens for each of them. This means that even implausibly unlikely scenarios are computed. Monte Carlo instead focuses on the most likely outcomes. This property can make the difference to whether a particular problem is solvable or not. The Monte Carlo method can also deal with distributions, for instance across income, which are not described by a normal distribution.(8)

(1) See Zeeman (1974).
(2) See Geanakoplos *et al* (2012).
(3) A class of models in which markets are assumed to simultaneously clear; see Burgess *et al* (2013).
(4) See Mankiw (2006); Haldane (2016); Ascari, Fagiolo and Roventini (2015).
(5) See Metropolis (1987).
(6) Lawrence Livermore National Laboratory (2013), 'Record simulations conducted on Lawrence Livermore supercomputer', available at https://www.llnl.gov/news/record-simulations-conducted-lawrence-livermore-supercomputer.
(7) See Metropolis and Ulam (1949).
(8) Normal distributions are what physical properties, such as height, tend to follow and they have a well-known mathematical description. They are also known as Gaussian distributions or 'bell-curves'. Many properties such as wealth, income, or firm size do not follow a normal distribution.

These are strengths which agent-based models inherit from the more general technique. However, this article focuses solely on Monte Carlo simulation, also known as agent-based modelling. This seeks to describe a system of interacting agents and the evolution of that system, rather than calculate the solution to a single equation. In this article, Monte Carlo simulation and agent-based modelling are used synonymously.

Agent-based modelling has been used across a wide range of disciplines, as discussed in the box on page 177. Before embarking on the strengths and weaknesses of agent-based models, a simple example is presented which illustrates some of their general features. This example is an early and influential model by economics Nobel Memorial prize laureate Thomas Schelling.

An early agent-based model in economics

In the late 1960s and early 70s, Thomas Schelling[1] developed a model that seeks to understand the effects of agents' preferences about where they live. The description below does not exactly follow Schelling's original example, but retains its most salient features.

Imagine there are two species named Econs and Humans who co-exist. Econs are always rational. Humans are emotional and sometimes make mistakes. Although Econs and Humans peacefully co-exist and live in the same city, they each have a slight preference for living closer to the same species.

This propensity to want to be near others of their type can be characterised by a number *f*, which can be thought of as their strength of preference. It represents the fraction of neighbours that they ideally wish to be of the same species, with an *f* of 1 meaning that they will only be happy if all of their neighbours are of the same species.

If agents of either type are unhappy, they can choose to move house and, at random, are given a new property. Over time, more and more agents will be happy with where they live and stop moving. Using simulations, **Figure 1** shows an initial neighbourhood of Humans and Econs which is mixed.

What happens if the Humans and Econs are now allowed to move around until almost all of them are 'happy' according to the value of *f*? **Figure 2** shows one example simulation with *f*=25%; the agents remain generally inter-mixed. What is surprising is how quickly the mix of Humans and Econs becomes segregated as *f* increases. **Figure 3** shows an example of a final distribution with *f*=26%.

The agents are now clearly segregated, even though the change in their preferences was very small from **Figure 2**. This is an example of a 'tipping point', also known in the physical sciences as a 'phase transition'. It is a sudden, emergent change in the overall system. Tipping points like this can occur in systems which are coupled together by their agents; here, a small change in *f* can mean that almost everyone has to move in order to satisfy their new preferences. As a result, the neighbourhood can look very different after the change in *f*.

Figure 1 The initial distribution of Humans and Econs

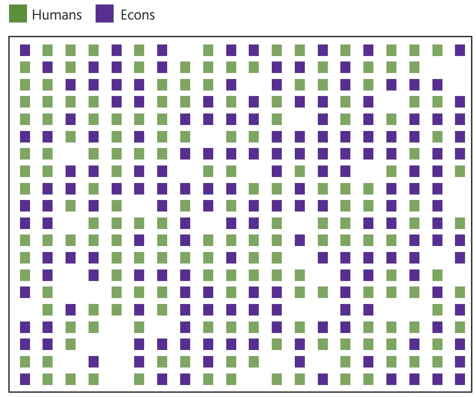


Figure 2 The final distribution of Humans and Econs with *f*=25%

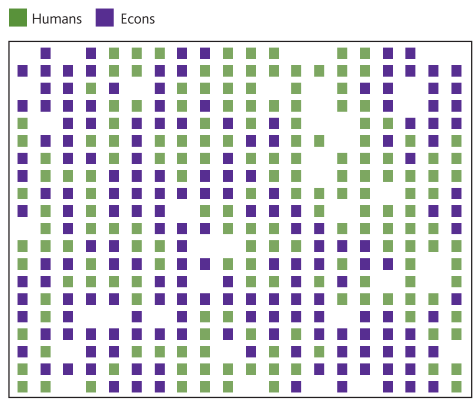
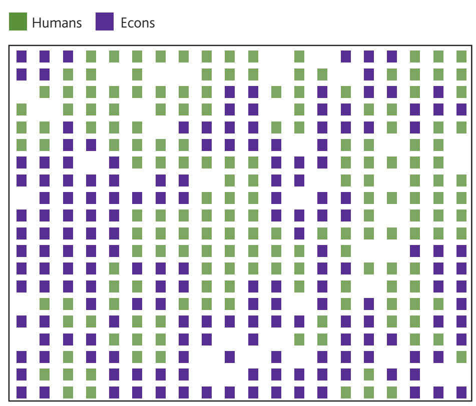


Figure 3 The final distribution of Humans and Econs with *f*=26%



(1) See Schelling (1969, 1971); for an example see www.jeromecukier.net/projects/models/segregate.html.

Agent-based modelling across disciplines

Agent-based modelling soon became a very popular technique in the physical sciences;[1] an early paper has received over 30,000 citations by other researchers.[2] Deep insights have emerged, for instance that the structure of the Universe may be flat rather than curved,[3] meaning that the Universe is unlikely to end in a 'Big Crunch' with all matter concentrated at a single point in space.

The range of applications in the physical sciences today is broad indeed and includes plasma physics,[4] particle-based cancer therapies, materials science, crystallisation, magnetism and nuclear fusion.[5] One unexpected application, published with the title *Why the Brazil Nuts Are on Top*,[6] simulated how mixed nuts segregate over time in a bag, research which has had important implications for industries dealing with particulate matter such as pharmaceuticals and manufacturing.

Computer science has also seen heavy use of the technique. The polymath John von Neumann is best known in economics for his work on game theory but he was also a formidable computer scientist. In a 1948 lecture he describes his idea for 'cellular automata'. These are artificial models of 'cells', which look like filled squares on a board (similar to the Schelling model discussed in this article).[7] Von Neumann found that, contrary to intuition, very simple rules for the cell agents gave rise to puzzlingly complex behaviour which looks life-like. This is a so-called emergent behaviour because it is almost impossible to predict based on the rules which the individual agents follow.

The military were early adopters of agent-based modelling. War games had been played with board and dice for many years, but there was a switch to computational agents in the 1960s. As well as being used to understand the dynamics of past battles, such as how optimal the searching operations of the British Army were in detecting German U-boats in the Bay of Biscay,[8] they are today used by the world's largest armies to formulate military strategy. In recent times, the automated behaviour of agents in war models has spilled over into reality with the development of autonomous weapons and vehicles.

This trend is also reflected in civil applications of autonomous robots. Most recently, there has been much work on multiple autonomous robotics systems in which the agents are robots that need to decide how to move about in the real world, for instance in driverless cars. It is extremely useful to be able model the behaviour of autonomous robots before trying them out in the field. For some tasks, centralised control of all robots is not feasible so designing individual behaviours which produce the overall desired outcome is important.

Biologists and ecologist began simulating the behaviour of organisms within an environment using agent-based models in the 1980s. These models were extended to include more complex phenomena, such as agent-environment interactions. An example is the research on marine organisms which includes behaviours such as swimming, feeding, being preyed upon, and organisms' interactions with ocean currents. One of the many applications of this sort of model could be examining how organisms fair with higher ocean temperatures, or higher concentrations of CO_2.[9]

Successes of these models include predicting the spawning locations of fish, explaining how the trout in Lake Michigan, USA, became contaminated, and describing how socially learned behaviours lead to distinct cultural groups among mammals. In these cases the advantage of an agent-based model was to be able to include many different agents and agent behaviours in a single model which could be run millions of times to determine the most probable outcomes.

Recent applications have focused on conservation and migration, including the management of forests, the timing of animal migrations, and the population pressures on endangered species. For instance, one model is of the endangered tiger population in Nepal; it uses the observed behaviour of individual tigers to explore future conservation scenarios.[10]

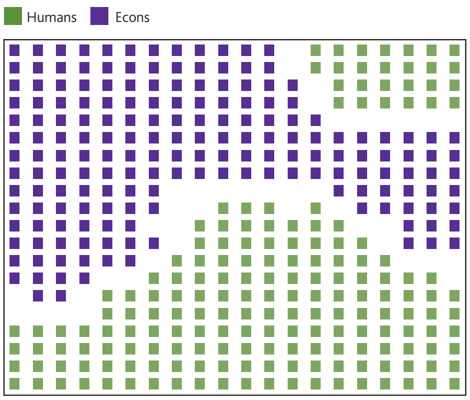
An important health-related application of agent-based models is in epidemiology — the study of the spread of diseases. A good understanding of the complex dynamics of epidemics could save millions of lives. Agent-based models have allowed country-specific information such as geographical data, commuting patterns, age distributions and other census information to be taken into account.[11]

In biology, agent-based models are now being used to simulate entire animal cells, cancers, bacteria and even the effects of new drugs on patients. Other applications include air traffic control (in which agents represent aircraft and co-ordinate to minimise fuel use), transportation systems (matching agents to destinations), crowd control in emergency situations, shopping patterns, predicting land use patterns[12] and non-player agents in computer games.

(1) See Haldane (2016) for more.
(2) See Metropolis *et al* (1953).
(3) See Davis *et al* (1985).
(4) See Turrell, Sherlock and Rose (2015).
(5) See Turrell (2013).
(6) See Rosato *et al* (1987).
(7) See von Neumann (1951); an example may be found at www.jeromecukier.net/projects/models/ca.html.
(8) See Carl (2015); Champagne and Hill (2009).
(9) See Werner *et al* (2001).
(10) See Carter *et al* (2015).
(11) See Degli Atti *et al* (2008).
(12) See Heppenstall *et al* (2011).

Figure 4 shows what can happen when all agents have very strong preferences.

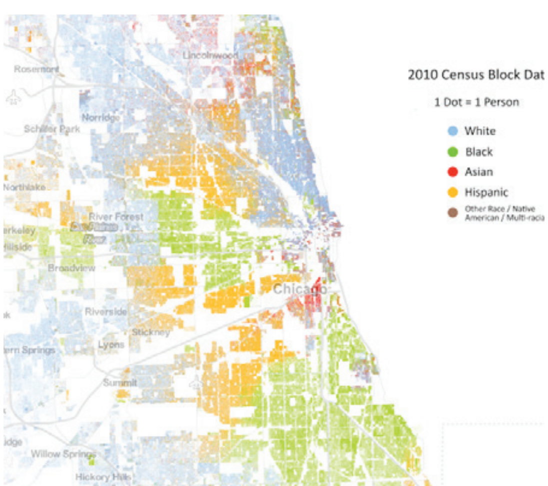
Figure 4 The final distribution of Humans and Econs with f=70%



Schelling's model is very simple but it shows how even mild changes in preferences can lead to significant changes at the macro-level.

Figure 5 shows that strikingly similar patterns may be seen in the US Census data for Chicago. However, there are many differences between the simple model presented and the real world; preferences may not be symmetric across groups as is assumed in the basic Schelling model, there are many practical barriers to moving (such as budget constraints) and what was labelled as 'preference' is likely to be much richer in reality, reflecting the complicated socio-economic historical relationships between groups.

Figure 5 US Census data for Chicago showing segregation[(a)]



Source: http://demographics.coopercenter.org/DotMap/.

(a) Nate Silver, 538.com; see http://fivethirtyeight.com/features/the-most-diverse-cities-are-often-the-most-segregated/.

What is agent-based modelling good (and bad) for?

The previous section gave a specific example of the kinds of insights which come out of agent-based models. Across many of their applications, there are a set of recurring strengths and weaknesses which are explored in this section. Particular reference is made to the applications of these models to problems in economics and the social sciences.

Strengths

Emergent behaviour

The single most powerful feature of agent-based modelling is that the individual actions of the agents combine to produce macroscopic[(1)] behaviour.

A good example of this is the herding of sheep or the flocking of birds.[(2)] Individual behaviours combine to produce an effect which looks organised even when the rules for each agent are incredibly simple.[(3)] Traffic jams are another familiar and unwelcome example; models and experiments have shown that jams can result even when there is no impediment to traffic.[(4)] People can herd in their economic actions and expectations too, for instance in their expectations about inflation. This has direct consequences for the economy.

The most important example of emergent behaviour in economics is Adam Smith's metaphor of the invisible hand: how the self-interested actions of real agents in the economy combine to produce socially optimal outcomes. One of the strengths of agent-based modelling is that this invisible hand is made visible and its workings may be examined. This is in contrast to some other model approaches in which the actions of many individuals are assumed to lead to a particular outcome, often using a single representative agent. This simplification is valid in some cases but not all combinations of behaviours can be represented by the actions of a single agent.[(5)]

Heterogeneity

As individual agents are modelled, it is possible to explore the consequences of agents being heterogeneous; that is agents being different in some way, perhaps by income, preferences, education or productivity.

Incorporating heterogeneity allows for the modelling of much richer behaviour. Inequality is a good example — aggregate wealth can increase but if it is only a small fraction of the population driving this phenomenon it would suggest very

(1) At the large scale, in this case the size of the system which the agents inhabit.
(2) See Macy and Willer (2002).
(3) An example may be found at www.tjansson.dk/2012/11/yabi-yet-another-boids-implementation-simulation-of-flocking-animals/.
(4) An example of an agent-based model of traffic jams can be run in your internet browser at www.traffic-simulation.de/.
(5) This is an example of the 'fallacy of composition'.

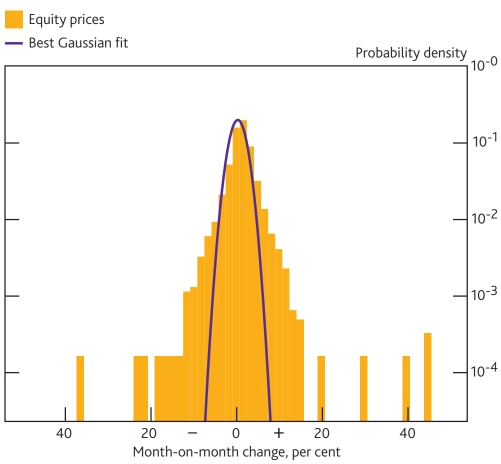
different underlying economic reasons than if the entire population were becoming wealthier.

Stylised facts

Perhaps the greatest success of agent-based models in economics has been explaining the stylised facts observed in asset markets.(1)(2) There are a number of phenomena observed empirically in the markets for assets such as bonds or equities which are not explained by traditional economic theory. Some of the two most widely seen across markets are:

- Clustered volatility, in which the standard deviation of returns on an asset exhibits trends which are 'clustered' in time.(3)
- 'Fat tails', in which extreme events such as large price changes occur more frequently than would be expected if a normal (or Gaussian) distribution was assumed and far beyond what would be expected if traders were behaving rationally. An example is shown in **Chart 1**.

Chart 1 Not normal: changes in the price of equities have a fat-tailed distribution (1709–2016)



Sources: Hills, Thomas and Dimsdale (2016) and Bank calculations.

The *Santa Fe Artificial Stock Market*(4) is a good example of how trading activities can affect aggregate market statistics. This model is similar in many respects to a traditional economic model but with the important distinction that agents have heterogeneous expectations of returns. This makes it impossible for one agent to know what other agents' expectations are, and thus impossible to form an unambiguous and rational expectation of price. Instead, agents adaptively learn using a range of expectation models. An evolutionary process occurs on agents' strategies; they are constantly updated in light of the actual path of the market. Interestingly, the model produces two regimes — one which looks likes the rational expectations world with a market price at the fundamental price of the financial product, and one in which clustered volatility and 'fat tails' occur. Rapid evolution of strategies causes the non-rational markets which are more like those observed in reality.

Many agent-based models have shown that chartist or trend-based trading (in which traders follow the trend direction of prices), and also leverage,(5) can contribute to price overshoots, and that this can drive clustered volatility, high trading volumes and fat tails.(6)

Realistic behaviours

The generation of realistic behaviour, based on observed behaviour, can be a strength of agent-based models. Research in behavioural economics has shown that people often use heuristics(7) when making decisions and that they are not fully rational. These behaviours can be collectively described as 'bounded rationality'. An example is that people react more negatively to loss than they do positively to gain, a phenomenon known as loss aversion. There are several models which explore what happens when purely rational options are not available or are too costly, or when agents' environments change over time.(8)

Exploring the possibilities

One of the advantages of agent-based modelling is that it can very efficiently explore a large number of possibilities. Probabilistic rules applied to each individual agent in turn can be a simpler way of exploring scenarios than working out how the entire population of agents should behave together. An example is in the transport of particles, and was the prompt for Fermi to originally develop the Monte Carlo method. Another is in epidemiology, which can similarly be modelled either with sets of equations which try to summarise all behaviour at once, or an agent-based model. The equation approach quickly becomes complicated as more and more agent properties that are relevant to disease (such as health, age, and even commuting pattern) are introduced. Or agent-based models can be applied to conflict in which a relatively inflexible set of equations modelling the rate of change of size of armies can be replaced with agent-based models which can capture the full heterogeneity of combatants in modern warfare.

(1) See Hong and Stein (1999).
(2) See Cutler, Poterba and Summers (1989); Lux and Marchesi (1999, 2000).
(3) See Cont (2007).
(4) See LeBaron, Arthur and Palmer (1999).
(5) See Thurner, Farmer and Geanakoplos (2012).
(6) See Tesfatsion (2002).
(7) Heuristics are 'rules of thumb' for making decisions which simplify the process but may not always give the optimal decision in all circumstances.
(8) See Gode and Sunder (1993); Farmer, Patelli and Zovko (2005); Challet and Zhang (1997); Hommes (2006); Axelrod (2006).

Complexity, non-linearity and multiple equilibria

Another strength of agent-based models is that they can describe complex systems. Complex systems are characterised by having many interconnected parts, having variables which can change dramatically and which can demonstrate self-organisation. Additionally, complex systems undergo sudden, dramatic transitions, sometimes called phase transitions. Recent work on agent-based models of the macroeconomy has described phase transitions between low and high unemployment.[1] The economy displays many of the characteristics associated with complex systems.

Weaknesses

Too much freedom?

While it is true that agent-based modelling has many strengths, it also presents challenges.

In many ways, the greatest strength — the flexibility to model such a vast range of scenarios — is also the greatest weakness. The sheer extent of choice in constructing agent-based models as compared to more traditional economic models means that modellers face the problem of selecting the right components for the problem at hand. Simulation results can vary dramatically depending on which assumptions are used, so modellers must take great care in choosing them. Further work is needed to develop objective means for choosing the most appropriate assumptions.

The Lucas critique

The huge range of behaviours available for agents means that agent-based models can be vulnerable to the Lucas critique. This critique centres around the fact that agents' choices may not follow historically observed relationships when policy interventions are made which are premised on those relationships. In principle, agent behaviours can be designed to respond to changing circumstances but there is a trade-off between creating agents that will always follow their optimal course of action and building simple, understandable models. This is why fully rational behaviour is useful as an approximation: it gives an unambiguous and relatively simple set of rules for how agents can always act in their own best interests.

Each agent-based model should be as Lucas-critique proof as possible but often the most interesting behaviours — such as bounded rationality — are the ones which are the most difficult to make robust to the critique.

Difficult to generalise

The proliferation of choices in constructing agent-based models leads to another weakness, which is that they tend to be bespoke. For instance, the agent-based model which tells us how bonds are traded is unlikely to be very helpful for answering questions about the housing market.

Calibration and interpretation

Calibration involves adjusting the model to fit with the known facts, for instance initialising it with empirical data. Validation is checking that the output of the model is reasonable given what is known, and perhaps cross-checking it with other models or variations in the assumptions. If a model contains many different options in how it is constructed, it can spuriously reproduce data that look similar to empirical data. This is known as overfitting, and it is a problem for all models. Calibration is even more difficult in agent-based models, however, because they typically produce stylised facts rather than quantitative forecasts and there are various ways the agreement with empirically observed stylised facts could be assessed.

The results of agent-based models can also be difficult to communicate because they must be presented alongside the assumptions used to create them. Although true of all models to some extent, this problem is less acute with models based on historical data alone as they use common statistical techniques. Nor is it the case with models based on rational expectations; if agents can only act perfectly rationally then there is a strong constraint which is easily understood across all similar models. When using agent-based modelling to inform the choices of policymakers, this weakness can be a barrier.

Finally, it can be difficult to understand how changing model inputs affect the model output. This is an unavoidable feature of complex systems, and of the real world itself. It is in contrast to more analytically tractable models in which the effects of changing a parameter may have a much clearer economic interpretation. Although every agent-based model does have a unique mathematical representation, the equations would be difficult to transcribe from a computer programme.

Despite these challenges, agent-based models provide an important tool for understanding the world, and one which has delivered in an astonishing range of scenarios.

We now turn to how these models can improve our understanding of the economy as a whole, a topic which is especially relevant to central banks.

Macroeconomic agent-based modelling

The macroeconomy is characterised by having business cycles: fluctuations in the growth of GDP around its long-term trend. In standard macroeconomic modelling, these fluctuations are the result of unexplained ('exogenous') shocks.[2] For instance, inflation would be forced to suddenly change but not as a

(1) See Gualdi *et al* (2015).
(2) A shock could be described as an unexpected and discontinuous change in a variable.

consequence of any emergent phenomena within the model. These are called exogenous shocks.

In the real world, the fluctuations in GDP are likely to be endogenous — that is, generated by the economy itself. Agent-based models provide a way of making these fluctuations endogenous, and therefore also provide a route to understanding their causes and what policies might affect them. The Great Recession is a compelling example of a reason why policymakers need to understand what drives these fluctuations.

Several agent-based models have been putting together the different elements and agents which are required to realistically reproduce the stylised facts of the economy of an entire country, or even several countries interlinked by trade. These elements include the entry and exit of firms,(1) endogenous innovation, monetary policy and fiscal policy.

One of these macroeconomic models especially addresses monetary policy.(2) In it, there are firms, consumers, and prices which are changed according to simple expectations. Business cycles similar to those observed in reality are generated. A simulated experiment shows that an active monetary policy, formulated according to a simple rule,(3) reduces the size of the fluctuations in GDP relative to having a static policy rule. This occurs because firms' demand for credit falls when the central bank raises interest rates according to the rule.

Another model(4) features an economy composed of heterogeneous capital and consumption-good firms, a labour force, banks, a government and a central bank. Capital-good firms perform research and produce heterogeneous machine tools. Consumption-good firms invest in new machines and produce a homogeneous consumption good. Consumption-good firms finance their production and investments primarily with their liquid assets and, if required, bank credit. Capital-good firms produce using cash advanced by their consumers, rather than using banks.

By incorporating the financial sector, this model is able to reproduce many features seen in empirical macroeconomics, including the cycles in GDP, investment and consumption, as well as the volatility of these three variables relative to each other. Banking crises are also an emergent phenomena of the model; as high production and investment levels raise firms' debt, the firms' net worth decreases, increasing their credit risk. Banks then ration credit and force firms to curb production and investment, with the potential to trigger a recession. Bank failures emerge from the accumulation of loan losses on banks' balance sheets. The model allows for a better understanding of the chain of events which lead to banking crises. Confidence that these events do reflect the real world situation can be gained from the reproduction of stylised facts such as the distribution of banking crisis durations being very close to the empirical one.

Experiments on monetary policy with this model suggest that a dual mandate to target both inflation and unemployment result in a higher average growth in GDP with lower volatility than targeting inflation alone achieves.

Other macroeconomic agent-based models(5) have looked at how interest rates are set,(6)(7) at how liquidity traps can be endogenously(8) generated(9) and at the effects of unconventional monetary policy.(10)

The cost in complexity of these macro models is balanced by the insights which are generated. They can reproduce an impressive list of macroeconomic stylised facts: business cycles; the procyclicality of productivity, nominal wages, firms' debt, bank profits and inflation; the countercyclicality of unemployment,(11) prices, mark-ups, and loan losses; and the appearance of fat tails in the distribution of output growth. Alternative and complementary models, such as dynamic stochastic general equilibrium models, have not been able to generate all of these phenomena endogenously.(12) These models thereby aid the understanding of how complex macro-level phenomena emerge from underlying micro-level phenomena.

These strong empirical credentials lend confidence to the conclusions of policy experiments undertaken with agent-based models. The flexibility of the models means that extremely fine-tuned regulation can be 'tested' using them. Examples might include the effect of regulation on liquidity and profitability, the interaction of micro and macroprudential policies, or how credit networks can give rise to business cycles and financial crises.(13) As an example, the NASDAQ stock exchange has used an agent-based model to design regulation which eliminates loopholes that could be abused by its users.(14)

The next section discusses two agent-based models which have been developed in the Bank and which attempt to replicate the stylised facts observed in two different markets: corporate bonds and housing.

(1) See Eliasson (1991).
(2) See Gatti and Desiderio (2015).
(3) The Taylor rule; see Taylor (1993).
(4) See Dosi *et al* (2015); this model incorporates aspects of Keynesian, Schumpeterian and Minskian economics.
(5) See Dilaver, Jump and Levine (2016).
(6) See Delli Gatti *et al* (2005).
(7) See Raberto, Teglio and Cincotti (2008).
(8) Having an internal cause rather than being the result of an external shock.
(9) See Oeffner (2008).
(10) See Cincotti, Raberto and Teglio (2010).
(11) See Gualdi *et al* (2015).
(12) See Ascari, Fagiolo and Roventini (2015).
(13) See Gatti, Gaffeo and Gallegati (2010).
(14) See Bonabeau (2002).

Agent-based modelling in the Bank of England

Trading in corporate bonds by open-ended mutual funds(1)

An agent-based model designed to capture some of the dynamics of trading in corporate bonds by open-ended mutual funds was developed within the Bank. The model aimed to be as parsimonious as possible while reproducing realistic behaviour for the market.

The assets under the management of corporate bond funds have more than doubled since the 2008 financial crisis. At the same time, concerns about the fragility of fixed-income markets have grown. Despite the market being larger, there are worries that there has been a reduction in market liquidity, so that large orders have more of an effect on prices.

The dynamics of this market have important implications for financial stability. Overshooting during adjustments in the price of corporate bonds may unnecessarily reduce the ability of some companies to service refinanced debt, threatening their solvency. Some firms may also be deterred from raising new financing. In extremis, this could cause an impairment of the supply of credit to the real economy.

A stylised picture of the model can be seen in **Figure 6**. There are a representative pool of investors, a single market maker through which all trades are made, and three distinct types of fund. The model endogenously reproduces one of the important stylised facts observed in the corporate bond market: the distribution of daily log-price returns. This is shown in **Chart 2**, where the empirical observations are from a US investment-grade corporate bond index. The very tail ends of the distribution do not match the empirical data; a situation which could potentially be improved by sacrificing some of the parsimony of the model. However, the overall fit of the distribution is a good match to data.

The model looks at how investors redeeming the corporate bonds held for them by open-ended mutual funds can cause feedback loops in which bond prices fall further.

For example, if interest rates were to rise, existing corporate bonds might become less attractive and redemptions could take place as some investors pull their wealth out. Investors who redeem their bonds first get a good price when funds sell them, but poor liquidity may cause bond prices to fall. These price falls could prompt remaining investors to redeem their bonds too, so that funds have to sell off more bonds and prices fall even further. A feedback loop of redemptions takes place in which wealth is destroyed and those who make the initial redemptions enjoy a first-mover advantage. This is similar to the chain of events which caused bank runs before the advent of deposit insurance.

Figure 6 Schematic of an agent-based model of the corporate bond market

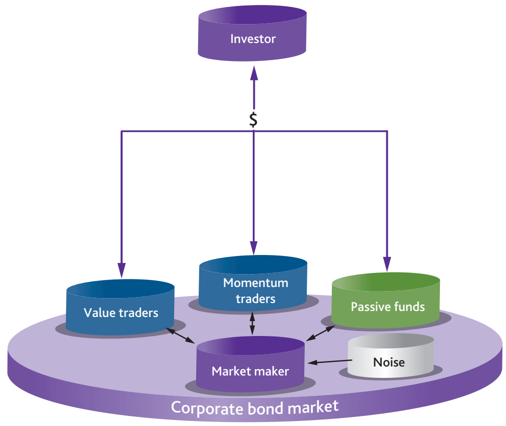
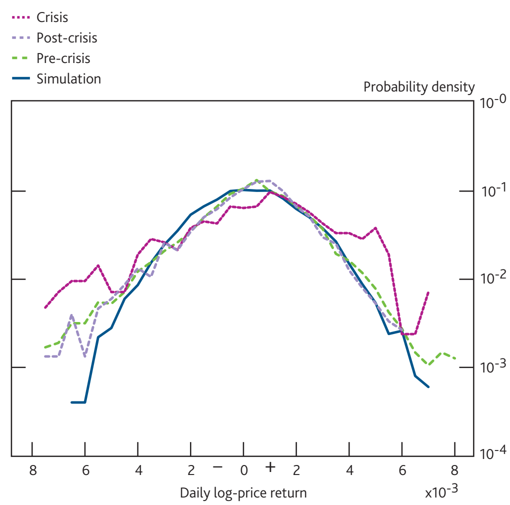


Chart 2 Reproducing stylised facts: the distribution of daily log-price returns

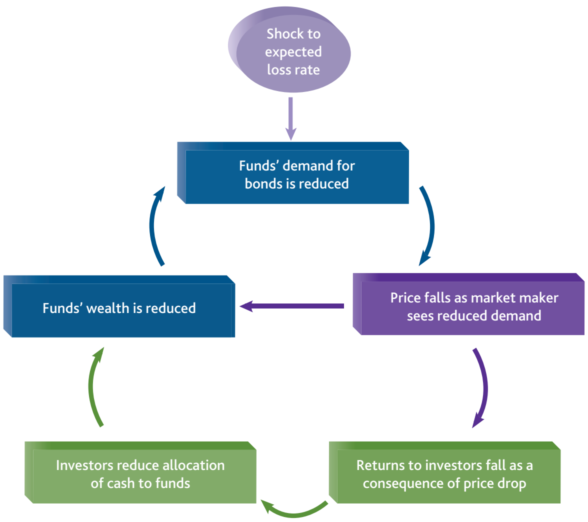


Simulated experiments were undertaken with this model. The aim was to understand how changes in the behaviours of traders in bonds might affect the extent of dislocation in price and yield following a shock.

The shock used was to funds' expectations about the fraction of companies that will fail in a given year, also known as the expected loss rate. This creates a negative feedback loop in the price for bonds. If funds expect more companies to fail, they are likely to demand higher yields from bonds to compensate them for this — so a sudden change in the expected loss rate pushes down on prices. The aftermath of this is seen in overshoots in both price and yield before they settle down their values. The steps in the feedback loop are shown in **Figure 7**.

(1) See Braun-Munzinger, Liu and Turrell (2016).

Figure 7 Capturing non-linear relationships: the feedback loop following a shock to funds' expected loss rate



Given the shock, and its known effects in 'normal' times, scenarios in which the trading behaviour was different could be explored.

In one of these scenarios, the effect of an increase in the fraction of funds with passive trading strategies was explored. This led to a surprising result; there is a tail risk of severe overshoots when the presence of passive investment funds increases. Another finding is that if all funds were to increase the time window over which redemptions were made by investors, the extent of price dislocation would be significantly reduced in times of crisis. This is because the feedback loop shown in **Figure 7** is stronger when investors withdraw an amount of wealth on a single day as opposed to withdrawing it over a longer time period.

An agent-based model of the UK housing market(1)

The housing market has often been a source of financial stress and crisis, looking across a wide range of countries and a wide span of time. The market exhibits clear and significant cyclicality. Capturing these cyclical dynamics is not straightforward and one potential reason is that the housing market comprises many types of participant — for instance renters, first-time buyers, and buy-to-let landlords. These agents are heterogeneous by income, gearing and location, so they have different incentives. The combination of their actions gives rise to the cyclical dynamics.

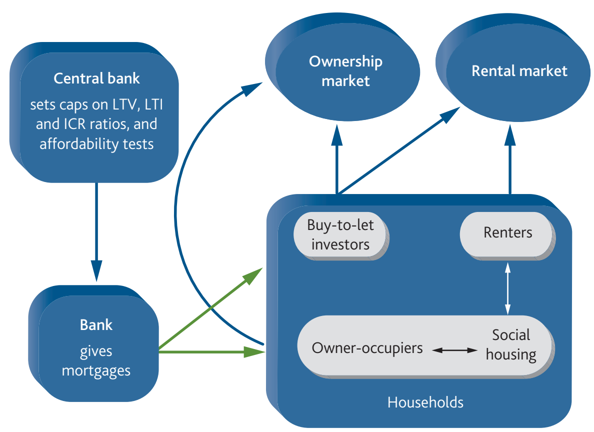
In addition to a banking sector (mortgage lender) and a central bank, the model comprises households of three types:

- renters who decide whether to attempt to buy a house when their rental contract ends and, if so, how much to bid;
- owner-occupiers who decide whether to sell their house and buy a new one and, if so, how much to bid/ask for the property; and
- buy-to-let investors who decide whether to sell their rental property and/or buy a new one and, if so, how much to bid/ask for the property. They also decide whether to rent out a property and, if so, how much rent to charge.

The behavioural rules of thumb that households follow when making these decisions are based on factors such as their expected rental payments, house price appreciation and mortgage cost. These households differ not only by type, but also by age, income, bank balances, rental payment and other properties.

An important feature of the model is that it includes an explicit banking sector, which provides mortgage credit to households and which sets the terms and conditions available to borrowers in the mortgage market. The banking sector's lending decisions are themselves subject to regulation by a central bank, which sets loan to income (LTI), loan to value (LTV), and interest cover ratio policies (ICR). The various different agents, and their interlinkages, are shown in **Figure 8**.

Figure 8 Agents and interactions in the housing market model



This model was calibrated using housing market data sources and household surveys so that the agents in the model have characteristics which match those in the UK population over a particular period of time. It includes behavioural characteristics such as how often and by how much the price of a house is reduced if it remains unsold.

(1) See Baptista *et al* (2016).

One of the key features of the model is that it is able to endogenously generate real-world house price cycles generated by the model in **Chart 3**. It also reproduces key aspects of the UK housing market, such as the empirical distribution of the share of loans by loan to income band based on the Product Sales Database (PSD) of UK mortgages. This is shown in **Chart 4**.

Chart 3 A benchmark run of the model showing boom and bust cycles in the house price index

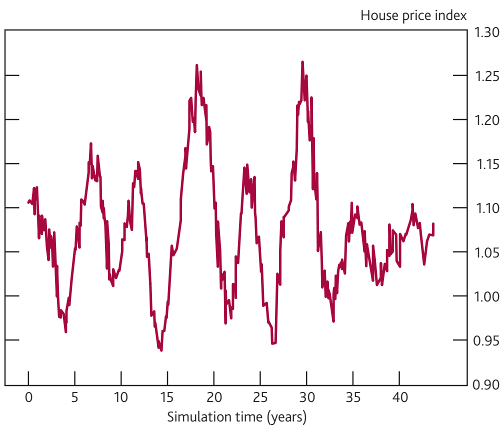
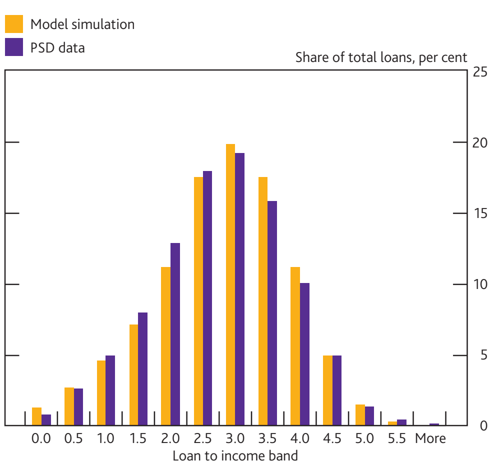


Chart 4 The model can reproduce the loan to income distribution for the United Kingdom



The model was used to look at several scenarios for the housing market. In one, a larger buy-to-let market was found to cause much larger swings in house prices during cycles. In another, the effect of a macroprudential policy that limits lenders to making a maximum of 15% of its mortgages at loan to income ratios greater than 3.5 was explored (the pre-existing Bank policy is a ratio of 4.5). With this limit in place, the endogenous cycles in house price generated by the model are dampened.

The future for agent-based modelling in economics

Agent-based models in economics thrive on data. More data means fewer assumptions need to be made, and more of the structure of the model can be based on what is observed. Fortunately for modellers, the amount of microeconomic data which is available is increasing considerably and opportunities to refine models are likely to be in ready supply.

The relentless improvements in computer processing power also provide new opportunities to model the economy on ever more granular scales, and with more complex behaviours.

In the future, the ability to incorporate realistic behaviours could be exploited further as artificial intelligence matures. The technique known as machine learning is widely used by tech companies to, for instance, suggest the films that a consumer might like to watch.(1) In the future, these techniques could be used to 'train' an artificial agent to behave like a real consumer or firm. Advanced artificial intelligence could make agent-based models more Lucas-critique proof by having agents respond realistically to new circumstances. A glimpse into what is possible in this respect was given recently when an artificial intelligence repeatedly beat a trained fighter pilot in an air-to-air combat simulation(2) — an extremely demanding scenario for a computer. Recent progress by computers at various games, including *Go* and *Jeopardy!* add weight to this argument.

There are less exotic ways in which progress might be achieved, for instance with the development of a standard set of tools for calibrating and validating agent-based models. Another way to meet some of the criticism levelled at these models is for researchers to investigate the most parsimonious models possible which still reproduce observed stylised facts.

With respect to central banks, there are three particularly promising areas of development for agent-based modelling. The first is the ongoing application of macroeconomic agent-based models to monetary policy. Several models which explicitly include central banks have now been established and are on hand to examine policy questions. The second is in modelling the banking and financial sector, to determine how financial stress is transmitted through the system as a whole. Third, researching the potential impact of the introduction of a central bank digital currency could be explored using an agent-based model. As recently described,(3) some specifications for a central bank issued digital currency could have consequences for financial stability. An agent-based model may be suited for analysing the effects of

(1) See Friedman, Hastie and Tibshirani (2001).
(2) See Ernest *et al* (2016).
(3) See the Bank of England's research questions on central bank digital currencies, available at www.bankofengland.co.uk/research/Documents/onebank/cbdc.pdf.

the different possible specifications of a central bank digital currency.

Conclusion

Since their inception, agent-based models have brought a wealth of insights to those problems to which they are suited. They have found use in an astonishingly diverse range of subjects; from tigers to traders, from particles to people.

This includes complex systems generally, but particularly those in which outcomes emerge from individuals' choices or agents are heterogeneous. This has obvious, useful applications to modelling the economy and many examples have demonstrated how agent-based models in economics can aid the understanding of empirically observed phenomena.

A 'bottom-up' perspective on the economy is a compelling complement(1) to the 'top-down' approaches which are already widely used.

The agent-based approach comes with difficulties of its own, especially around the assumptions and validity, but the future holds promise in these respects, partly due to ever improving techniques but also because of the availability of rich data sets to calibrate the models against.

Agent-based models have an important place in understanding, and even designing, markets, and provide a unique platform for augmenting policymakers' judgements about the economy.

(1) See Fagiolo and Roventini (2012).

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