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BAYESIAN CALIBRATION FOR PREDICTION IN A MULTI-OUTPUT TRANSPOSITION CONTEXT: SUPPLEMENTARY MATERIAL

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This document serves as supplementary material for the manuscript "Bayesian Calibration for Prediction in a Multi-Output Transposition Context". It provides a proof of the consistency of the estimators introduced in the manuscript, specifically concerning the posterior expectation of $h(\boldsymbol{\Lambda})$ where h is a bounded function, and $\boldsymbol{\Lambda}$ denotes the calibration parameters.

KEY WORDS: *Estimator, Convergence, MCMC*

1. THEOREMS FOR ALMOST SURE EXCHANGE OF THE LIMIT AND INTEGRAL

Section 2 and Section 3 show the consistency of the estimators presented in Section 2.4 of the manuscript:

- Convergence of $\int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}}) \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}}) d\boldsymbol{\alpha}$ in Section 2,
- Convergence of $\hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda})\Big)$ given $\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}}) = \hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})$ in Section 3.

In this Section 1, the objective is to introduce theorems that we will be used for the proofs of Section 2 and Section 3.

Theorem 1. *Let $(u_L)_{L\in\mathbb{N}}$ a sequence of functions with $u_L : \mathcal{A} \longrightarrow \mathbb{R}$. If*

1. *$\mathcal{A}$ is convex*
2. *$\forall L,\ u_L$ is continuous on $\mathcal{A}$, and the first derivatives of u_L are defined*
3. *$\exists C > 0,\ \exists L_0 \in \mathbb{N}, \forall L > L_0, \forall \boldsymbol{\alpha} \in \mathcal{A}, \|\nabla u_L(\boldsymbol{\alpha})\| < C$*

Then, $(u_L)_{L>L_0}$ is uniformly equicontinuous.

Proof. Let $\epsilon > 0$. For $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in \mathcal{A}^2$, we denote by $G(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ the line segment with $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ as endpoints. The mean value theorem gives

$$\forall L > L_0, \forall(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in \mathcal{A}^2, \exists \mathbf{c} \in G(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2), \|u_L(\boldsymbol{\alpha}_2) - u_L(\boldsymbol{\alpha}_1)\| = \|\nabla u_L(\mathbf{c}) \cdot (\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1)\| \\ \leq C\|\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1\|$$

Then,

$$\|\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1\| < \frac{\epsilon}{C} \Rightarrow \ \forall L > L_0, \ \|u_L(\boldsymbol{\alpha}_2) - u_L(\boldsymbol{\alpha}_1)\| < \epsilon.$$

□

Corollary 1. *Under the same hypothesis as Theorem 1, if we additionnaly suppose that*

1. $\mathbb{Q}$ *is dense in* $\mathcal{A}$
2. $\forall \boldsymbol{\alpha} \in \mathbb{Q} \cap \mathcal{A}, \ u_L(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{} u(\boldsymbol{\alpha})$
3. u *is continuous on* $\mathcal{A}$

Then, $\forall \boldsymbol{\alpha} \in \mathcal{A}, \ u_L(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{} u(\boldsymbol{\alpha})$

Proof. Let $\boldsymbol{\alpha} \in \mathcal{A}$ and $\epsilon > 0$. From Theorem 1,

$$\forall \tilde{\boldsymbol{\alpha}} \in \mathcal{A}, \ \|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\| < \frac{\epsilon}{3C} \Rightarrow \forall L > L_0, \|u_L(\tilde{\boldsymbol{\alpha}}) - u_L(\boldsymbol{\alpha})\| < \frac{\epsilon}{3}. \tag{1}$$

From the continuity of u at $\boldsymbol{\alpha}$,

$$\exists \eta > 0, \forall \tilde{\boldsymbol{\alpha}} \in \mathcal{A}, \ \|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\| < \eta \Rightarrow \|u(\tilde{\boldsymbol{\alpha}}) - u(\boldsymbol{\alpha})\| < \frac{\epsilon}{3}. \tag{2}$$

From the convergence in $\mathbb{Q}$,

$$\forall \tilde{\boldsymbol{\alpha}} \in \mathbb{Q} \cap \mathcal{A}, \ \exists L_{\tilde{\boldsymbol{\alpha}}} \in \mathbb{N}, \ \forall L > L_{\tilde{\boldsymbol{\alpha}}}, \ \|u_L(\tilde{\boldsymbol{\alpha}}) - u(\tilde{\boldsymbol{\alpha}})\| < \frac{\epsilon}{3}. \tag{3}$$

As $\mathbb{Q}$ is dense in $\mathcal{A}$, then $\exists \tilde{\boldsymbol{\alpha}} \in \mathbb{Q} \cap \mathcal{A}, \ \|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\| < \min(\eta, \frac{\epsilon}{3C})$. From Equations 1, 2, 3, $\exists L_{\tilde{\boldsymbol{\alpha}}} \in \mathbb{N}, \forall L > \max(L_{\tilde{\boldsymbol{\alpha}}}, L_0)$,

$$\|u_L(\boldsymbol{\alpha}) - u(\boldsymbol{\alpha})\| \leq \|u_L(\boldsymbol{\alpha}) - u_L(\tilde{\boldsymbol{\alpha}})\| + \|u_L(\tilde{\boldsymbol{\alpha}}) - u(\tilde{\boldsymbol{\alpha}})\| + \|u(\tilde{\boldsymbol{\alpha}}) - u(\boldsymbol{\alpha})\| < \epsilon \tag{4}$$

□

Theorem 2. *Let* $(U_L)_{L\in\mathbb{N}}$ *a sequence such that* $\forall L \in \mathbb{N}, \{U_L(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathcal{A}\}$ *is a stochastic process, and* z *a function of* $\boldsymbol{\alpha}$*. If*

1. $\mathcal{A}$ *is convex and* $\mathbb{Q}$ *is dense in* $\mathcal{A}$
2. $\forall \boldsymbol{\alpha} \in \mathcal{A}, \mathbb{P}(U_L(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{} u(\boldsymbol{\alpha})) = 1$
3. $\mathbb{P}(\forall L \in \mathbb{N}, U_L$ *is continuous and its first derivatives are defined*$) = 1$
4. $\exists C > 0, \ \mathbb{P}(\exists L_0 \in \mathbb{N}, \forall L > L_0, \forall \boldsymbol{\alpha} \in \mathcal{A}, \|\nabla U_L(\boldsymbol{\alpha})\| < C) = 1$
5. $\exists v$ *integrable*, $\mathbb{P}(\exists L_1 \in \mathbb{N}, \forall L > L_1, \forall \boldsymbol{\alpha}, \ |U_L(\boldsymbol{\alpha}) z(\boldsymbol{\alpha})| \leq v(\boldsymbol{\alpha})) = 1$
6. u *is continuous*

Then, $\int_{\mathcal{A}} U_L(\boldsymbol{\alpha}) z(\boldsymbol{\alpha}) d\boldsymbol{\alpha} \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}} u(\boldsymbol{\alpha}) z(\boldsymbol{\alpha}) d\boldsymbol{\alpha}$.

Proof. We have

- $\mathbb{P}\left(\bigcap_{\boldsymbol{\alpha}\in\mathbb{Q}\cap\mathcal{A}}\left\{U_L(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{} u(\boldsymbol{\alpha})\right\}\right) = 1$ from hypothesis 2.
- $\mathbb{P}\left(\bigcap_{L\in\mathbb{N}}\{U_L \text{ is continuous and its first derivatives are defined}\}\right) = 1$ from hypothesis 3.
- $\exists C > 0, \mathbb{P}\left(\exists L_0 \in \mathbb{N}, \bigcap_{L>L_0}\{\forall\boldsymbol{\alpha}\in\mathcal{A}, \|\nabla U_L(\boldsymbol{\alpha})\| < C\}\right) = 1$ from hypothesis 4.

Then, from Corollary 1, $\mathbb{P}\left(\left\{\forall\boldsymbol{\alpha}\in\mathcal{A},\ U_L(\boldsymbol{\alpha})z(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{} u(\boldsymbol{\alpha})z(\boldsymbol{\alpha})\right\}\right) = 1$

Hypothesis 5 leads to

$$\mathbb{P}\left(\exists L_1\in\mathbb{N}, \bigcap_{L>L_1}\{\forall\boldsymbol{\alpha},\ |U_L(\boldsymbol{\alpha})z(\boldsymbol{\alpha})| \le v(\boldsymbol{\alpha})\}\right) = 1$$

Then the dominated convergence theorem gives

$$\mathbb{P}\left(\int_{\mathcal{A}} U_L(\boldsymbol{\alpha})z(\boldsymbol{\alpha})d\boldsymbol{\alpha} \xrightarrow[L\to\infty]{} \int_{\mathcal{A}} u(\boldsymbol{\alpha})z(\boldsymbol{\alpha})d\boldsymbol{\alpha}\right) = 1.$$

□

2. CONSISTENCY OF $\int_{\mathcal{A}}\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}$

Here the objective is to show the consistency of $\int_{\mathcal{A}}\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}$, i.e. its almost sure convergence as L tends to infinity.

We have $\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}}) = \frac{\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\alpha})p_{\mathbf{A}}(\boldsymbol{\alpha})}{\int_{\mathcal{A}}\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'}$ where $\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\mathrm{obs}} \mid \boldsymbol{\alpha})$ is defined by **??** for all $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}^\star = \boldsymbol{\alpha}_\ell^\star$ and $(\boldsymbol{\Lambda}'_k)_{k=1}^L$ i.i.d. with pdf $p_{\boldsymbol{\Lambda}}(. \mid \boldsymbol{\alpha}_\ell^\star)$.

We assume that

1. $\mathcal{A}$ is convex and $\mathbb{Q}$ is dense in $\mathcal{A}$
2. $\exists\mathcal{K}$ compact set, $\forall\boldsymbol{\alpha}\in\mathcal{A}, \mathrm{supp}(p_{\boldsymbol{\Lambda}}(. \mid \boldsymbol{\alpha})) = \mathcal{K}$.
3. h is continuous on $\mathcal{K}$, and then bounded by a constant b_0.
4. $\boldsymbol{\lambda}\mapsto p(\boldsymbol{y}_{1,\mathrm{obs}} \mid \boldsymbol{\lambda})$ is continuous on $\mathcal{K}$; it is then bounded by a constant b_1.
5. $\forall\boldsymbol{\alpha}\in\mathcal{A}, p_{\boldsymbol{\Lambda}}(. \mid \boldsymbol{\alpha})$ is continuous almost everywhere on $\mathcal{K}$.
6. $\exists b_2 > 0,\ \forall\boldsymbol{\alpha}\in\mathcal{A},\ p_{\boldsymbol{\Lambda}}(. \mid \boldsymbol{\alpha})$ is bounded by b_2 on $\mathcal{K}$.
7. $\exists b_3 > 0,\ \forall\boldsymbol{\alpha}\in\mathcal{A}, \forall\boldsymbol{\lambda}\in\mathcal{K}, p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}) > b_3$.
8. $\forall\boldsymbol{\lambda}\in\mathcal{K}, p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid .)$ is continuous, its first derivatives are defined and $\exists b_4 > 0, \forall\boldsymbol{\lambda}\in\mathcal{K}, \forall\boldsymbol{\alpha}\in\mathcal{A}, \|\nabla_{\boldsymbol{\alpha}} p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})\| < b_4$.

This proof is structured as follows:

- Show that $\int_{\mathcal{A}}\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\mathrm{obs}} \mid \boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}' \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}} p(\boldsymbol{y}_{1,\mathrm{obs}} \mid \boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'$
- Show that $\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}}) \xrightarrow[L\to\infty]{a.s.} p(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})$
- Show that $\int_{\mathcal{A}}\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha} \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}}\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})p(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}$

2.1 Convergence of $\int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'$

We have:

- $\forall \boldsymbol{\alpha} \in \mathcal{A}, \mathbb{P}\left(\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) \xrightarrow[L\to\infty]{} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha})\right) = 1$ from the strong law of large numbers.
- $\mathbb{P}\left(\forall L \in \mathbb{N}, \bigcap_{k=1}^{L} \{\boldsymbol{\Lambda}'_k \in \mathcal{K}\}\right) = 1$ and then
  - $\mathbb{P}\left(\forall L \in \mathbb{N}, \boldsymbol{\alpha} \mapsto \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) \text{ is continuous on } \mathcal{A}\right) = 1$ from the continuity of $p_{\mathbf{\Lambda}}(\boldsymbol{\lambda} \mid .)$ for $\boldsymbol{\lambda} \in \mathcal{K}$.
  - $\mathbb{P}\left(\forall L \in \mathbb{N}, \forall \boldsymbol{\alpha} \in \mathcal{A}, \forall 1 \leq a \leq r, \left|\frac{\partial \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha})}{\partial \alpha_a}\right| \leq \frac{b_1 b_4}{b_3}\right) = 1.$
  - $\mathbb{P}\left(\forall L \in \mathbb{N}, \forall \boldsymbol{\alpha} \in \mathcal{A}, \left|\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha})p_{\mathbf{A}}(\boldsymbol{\alpha})\right| \leq \frac{b_1 b_2}{b_3} p_{\mathbf{A}}(\boldsymbol{\alpha})\right) = 1.$
- $\boldsymbol{\alpha} \mapsto p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) = \int_{\mathcal{K}} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda}) p_{\mathbf{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}) d\boldsymbol{\lambda}$ is continuous on $\mathcal{A}$, as
  - $\forall \boldsymbol{\alpha} \in \mathcal{A}, \boldsymbol{\lambda} \mapsto p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda}) p_{\mathbf{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})$ is continuous almost everywhere on $\mathcal{K}$.
  - $\forall \boldsymbol{\lambda} \in \mathcal{K}, \boldsymbol{\alpha} \mapsto p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda}) p_{\mathbf{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})$ is continuous on $\mathcal{A}$.
  - $\forall \boldsymbol{\lambda} \in \mathcal{K}, \forall \boldsymbol{\alpha} \in \mathcal{A}, |p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda}) p_{\mathbf{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})| \leq b_1 b_2$, with $b_1 b_2$ integrable on $\mathcal{K}$.

Then, from Theorem 2,

$$\int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) p_{\mathbf{A}}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) p_{\mathbf{A}}(\boldsymbol{\alpha}) d\boldsymbol{\alpha}. \tag{5}$$

2.2 Convergence of $\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})$

The strong law of large numbers and Equation (5) provides, for $\boldsymbol{\alpha} \in \mathcal{A}$,

$$\begin{cases} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) p_{\mathbf{A}}(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{a.s.} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) p_{\mathbf{A}}(\boldsymbol{\alpha}), \\ \int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}'. \end{cases}$$

Then it comes

$$\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}}) = \frac{\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}) p_{\mathbf{A}}(\boldsymbol{\alpha})}{\int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}'} \xrightarrow[L\to\infty]{a.s.} p(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}}). \tag{6}$$

2.3 Convergence of $\int_{\mathcal{A}} \mathbb{E}(h(\mathbf{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}}) \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}}) d\boldsymbol{\alpha}$

As from Equation (5),

$$\int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}'$$

then $\mathbb{P}\left(\exists L_0 \in \mathbb{N}, \forall L \geq L_0, \int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}' \geq \frac{1}{2} \int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}'\right) = 1.$

In this section, we denote $U_L(\boldsymbol{\alpha}) = \frac{\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha})}{\int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha}') p_{\mathbf{A}}(\boldsymbol{\alpha}') d\boldsymbol{\alpha}'}$.

We have:

- $\forall \boldsymbol{\alpha} \in \mathcal{A}, U_L(\boldsymbol{\alpha}) \xrightarrow[L\to\infty]{a.s.} \frac{p(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha})}{\int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'}$
- $\mathbb{P}\left(\forall L \in \mathbb{N}, \bigcap_{k=1}^{L} \{\boldsymbol{\Lambda}'_k \in \mathcal{K}\}\right) = 1$ and then
  - $\mathbb{P}\left(\forall L \in \mathbb{N}, U_L \text{ is continuous on } \mathcal{A}\right) = 1$ from the continuity of $p(\boldsymbol{\lambda} \mid .)$ for $\boldsymbol{\lambda} \in \mathcal{K}$.
  - $\mathbb{P}\left(\exists L_0 \in \mathbb{N}, \forall L > L_0, \forall \boldsymbol{\alpha} \in \mathcal{A}, \forall 1 \leq a \leq r, \left|\frac{\partial U_L(\boldsymbol{\alpha})}{\partial \alpha_a}\right| \leq 2\frac{\frac{b_1 b_4}{b_3}}{\int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'}\right) = 1$.
  - $\mathbb{P}\left(\exists L_0 \in \mathbb{N}, \forall L > L_0, \forall \boldsymbol{\alpha} \in \mathcal{A},\ |\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})p_{\mathbf{A}}(\boldsymbol{\alpha})U_L(\boldsymbol{\alpha})| \leq 2b_0\, p_{\mathbf{A}}(\boldsymbol{\alpha})\frac{\frac{b_1 b_2}{b_3}}{\int_{\mathcal{A}} p(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'}\right) = 1$.
- $\boldsymbol{\alpha} \mapsto \frac{p(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha})}{\int_{\mathcal{A}} \hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}}|\boldsymbol{\alpha}')p_{\mathbf{A}}(\boldsymbol{\alpha}')d\boldsymbol{\alpha}'}$ is continuous

Then, from Theorem 2, as $\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}}) = \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})p_{\mathbf{A}}(\boldsymbol{\alpha})U_L(\boldsymbol{\alpha})$,

$$\int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})\hat{P}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})d\boldsymbol{\alpha} \xrightarrow[L\to\infty]{a.s.} \int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})p(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})d\boldsymbol{\alpha}. \quad (7)$$

3. CONSISTENCY OF $\hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda})\right)$

Here, we investigate the convergence of

$$\hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda})\right) = \frac{1}{N}\sum_{i=1}^{N} \frac{\sum_{k=1}^{M} h(\boldsymbol{\Lambda}_k)\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\mathbf{A}_i)}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\boldsymbol{\alpha}_\ell^\star)}}{\sum_{k=1}^{M} \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\mathbf{A}_i)}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\boldsymbol{\alpha}_\ell^\star)}} \quad (8)$$

for every continuous, bounded function h on $\mathcal{K}$, with $(\mathbf{A}_i)_{i=1}^N$ sampled with pdf proportional to $\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha})p_{\mathbf{A}}(\boldsymbol{\alpha})$ and $(\boldsymbol{\Lambda}_k)_{k=1}^M$ sampled with pdf proportional to $p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)$. Note that here, $\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\alpha})$ is considered known and thus is deterministic, the convergence is related to the samples $(\mathbf{A}_i)_{i=1}^N$ and $(\boldsymbol{\Lambda}_k)_{k=1}^M$. We suppose that the hypotheses of Section 2 are verified.

Let us denote

$$\hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\right) = \frac{\sum_{k=1}^{M} h(\boldsymbol{\Lambda}_k)\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\boldsymbol{\alpha}_\ell^\star)}}{\sum_{k=1}^{M} \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k|\boldsymbol{\alpha}_\ell^\star)}}.$$

Then,

$$\hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda})\right) = \frac{1}{N}\sum_{i=1}^{N} \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda}), \mathbf{A}_i\right).$$

Let us show that $\hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda})\right)$ is a consistent estimator of $\int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})d\boldsymbol{\alpha}$, with

$$\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}}) = \frac{\int_{\mathbb{R}^q} h(\boldsymbol{\lambda})p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}|\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}|\boldsymbol{\alpha}_\ell^\star)}p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)d\boldsymbol{\lambda}}{\int_{\mathbb{R}^q} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}|\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}|\boldsymbol{\alpha}_\ell^\star)}p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)d\boldsymbol{\lambda}}.$$

More precisely, we will investigate

- $\lim_{M\to\infty}\lim_{N\to\infty} \hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda})\right)$
- $\lim_{N\to\infty}\lim_{M\to\infty} \hat{E}_{N,M}^{\boldsymbol{\alpha}_\ell^\star}\left(h(\boldsymbol{\Lambda})\right)$

3.1 Almost sure $\lim_{M\to\infty}\lim_{N\to\infty}\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big)$

This proof is structured as follows:

- Show that $\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}),\boldsymbol{\alpha}\Big) \xrightarrow[M\to\infty]{a.s.} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha},\boldsymbol{y}_{1,\mathrm{obs}})$
- Show that $\int_{\mathcal{A}}\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}),\boldsymbol{\alpha}\Big)\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha}\mid\boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha} \xrightarrow[M\to\infty]{a.s.} \int_{\mathcal{A}}\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha},\boldsymbol{y}_{1,\mathrm{obs}})\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha}\mid\boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}$
- Show that $\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) \xrightarrow[N\to\infty]{a.s.} \int_{\mathcal{A}}\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}),\boldsymbol{\alpha}\Big)\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha}\mid\boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}$
- Conclude that $\mathbb{P}\left(\lim_{M\to\infty}\lim_{N\to\infty}\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) = \int_{\mathcal{A}}\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha},\boldsymbol{y}_{1,\mathrm{obs}})\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha}\mid\boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}\right) = 1$

3.1.1 *Almost sure convergence of* $\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}),\boldsymbol{\alpha}\Big)$ *to* $\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha},\boldsymbol{y}_{1,obs})$.

Let $\boldsymbol{\alpha}\in\mathcal{A}$.

$$\mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha},\boldsymbol{y}_{1,\mathrm{obs}}) = \underbrace{\frac{\int_{\mathbb{R}^q} h(\boldsymbol{\lambda})\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)}p(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)d\boldsymbol{\lambda}}{\int_{\mathbb{R}^q} p(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)d\boldsymbol{\lambda}}}_{=I_1} \times \underbrace{\frac{\int_{\mathbb{R}^q} p(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)d\boldsymbol{\lambda}}{\int_{\mathbb{R}^q} \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)}p(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)d\boldsymbol{\lambda}}}_{=\frac{1}{I_2}}.$$

As explained in [1], MCMC simulates $(\boldsymbol{\Lambda}_k)_{k=1}^M$ which is a Harris ergodic Markov Chain with invariant distribution $\frac{p(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}\mid\boldsymbol{\alpha}^\star_\ell)}{\int_{\mathbb{R}^q} p(\boldsymbol{y}_{1,\mathrm{obs}}\mid\boldsymbol{\lambda}')p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda}'\mid\boldsymbol{\alpha}^\star_\ell)d\boldsymbol{\lambda}'}$, and then

$$\begin{cases} \dfrac{1}{M}\displaystyle\sum_{k=1}^M h(\boldsymbol{\Lambda}_k)\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k\mid\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k\mid\boldsymbol{\alpha}^\star_\ell)} \xrightarrow[M\to\infty]{\text{a.s.}} I_1, \\ \dfrac{1}{M}\displaystyle\sum_{k=1}^M \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k\mid\boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k\mid\boldsymbol{\alpha}^\star_\ell)} \xrightarrow[M\to\infty]{\text{a.s.}} I_2. \end{cases} \tag{9}$$

Then it comes

$$\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}),\boldsymbol{\alpha}\Big) \xrightarrow[M\to\infty]{\text{a.s.}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha},\boldsymbol{y}_{1,\mathrm{obs}}) \tag{10}$$

3.1.2 Almost sure convergence of $\int_{\mathcal{A}} \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,obs})d\boldsymbol{\alpha}$ to $\int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,obs})\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,obs})d\boldsymbol{\alpha}$.

$\forall M \in \mathbb{N}, \forall \boldsymbol{\alpha} \in \mathcal{A}, \forall 1 \leq a \leq r,$

$$\frac{\partial \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)}{\partial \alpha_a} = \frac{1}{\left(\sum_{k=1}^M \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)}\right)^2} \times$$
$$\left(\sum_{k=1}^M h(\boldsymbol{\Lambda}_k)\frac{\frac{\partial p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{\partial \alpha_a}}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)} \sum_{k=1}^M \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)} - \sum_{k=1}^M h(\boldsymbol{\Lambda}_k)\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)} \sum_{k=1}^M \frac{\frac{\partial p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{\partial \alpha_a}}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)}\right)$$

Then, $\forall M \in \mathbb{N}$,

$$\bigcap_{k=1}^M \{\boldsymbol{\Lambda}_k \in \mathcal{K}\} \subset \left\{\forall \boldsymbol{\alpha} \in \mathcal{A}, \forall 1 \leq a \leq r, \left|\frac{\partial \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)}{\partial \alpha_a}\right| \leq 2b_0 \left|\frac{\sum_{k=1}^M \frac{\frac{\partial p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{\partial \alpha_a}}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)}}{\sum_{k=1}^M \frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\Lambda}_k \mid \boldsymbol{\alpha}_\ell^\star)}}\right| \leq 2b_0 \frac{b_2 b_4}{b_3^2}\right\}.$$

Finally, we have

- $\forall \boldsymbol{\alpha} \in \mathcal{A}, \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big) \xrightarrow[M \to \infty]{a.s.} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})$
- $\mathbb{P}\Big(\forall M \in \mathbb{N}, \bigcap_{k=1}^M \{\boldsymbol{\Lambda}_k \in \mathcal{K}\}\Big) = 1$ and then
  - $\mathbb{P}\Big(\forall M \in \mathbb{N}, \boldsymbol{\alpha} \mapsto \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)$ is continuous on $\mathcal{A}\Big) = 1$ from the continuity of $p(\boldsymbol{\lambda} \mid .)$ for $\boldsymbol{\lambda} \in \mathcal{K}$.
  - $\mathbb{P}\left(\forall M \in \mathbb{N}, \forall \boldsymbol{\alpha} \in \mathcal{A}, \forall 1 \leq a \leq r, \left|\frac{\partial \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)}{\partial \alpha_a}\right| \leq 2b_0 \frac{b_2 b_4}{b_3^2}\right) = 1.$
  - $\mathbb{P}\Big(\forall M \in \mathbb{N}, \forall \boldsymbol{\alpha} \in \mathcal{A}, \left|\hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}, \omega\Big)\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})\right| \leq b_0 \times \hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})\Big) = 1.$
- $\boldsymbol{\alpha} \mapsto \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}}) = \frac{\int_{\mathcal{K}} h(\boldsymbol{\lambda})p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)}p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)d\boldsymbol{\lambda}}{\int_{\mathcal{K}} p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)}p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)d\boldsymbol{\lambda}}$ is continuous on $\mathcal{A}$ as
  - $\forall \boldsymbol{\alpha} \in \mathcal{A}, \boldsymbol{\lambda} \mapsto p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})\frac{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})}{p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)}p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha}_\ell^\star)$ and h are continuous almost everywhere on $\mathcal{K}$.
  - $\forall \boldsymbol{\lambda} \in \mathcal{K}, \boldsymbol{\alpha} \mapsto p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})$ is continuous on $\mathcal{A}$.
  - $\forall \boldsymbol{\lambda} \in \mathcal{K}, \forall \boldsymbol{\alpha} \in \mathcal{A}, |h(\boldsymbol{\lambda})p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})| \leq b_0 b_1 b_2$, and $|p(\boldsymbol{y}_{1,\text{obs}} \mid \boldsymbol{\lambda})p_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda} \mid \boldsymbol{\alpha})| < b_1 b_2$ with $b_1 b_2$ and $b_0 b_1 b_2$ integrable on $\mathcal{K}$.

Then, from Theorem 2,

$$\int_{\mathcal{A}} \hat{E}_M^{\boldsymbol{\alpha}_\ell^\star}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})d\boldsymbol{\alpha} \xrightarrow[M \to \infty]{a.s.} \int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\text{obs}})\hat{p}_L^{\boldsymbol{\alpha}_\ell^\star}(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\text{obs}})d\boldsymbol{\alpha}$$

3.1.3 Almost sure convergence of $\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big)$ to $\int_{\mathcal{A}} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,obs})d\boldsymbol{\alpha}$ for a given M.

Again, MCMC sampling provides $(\mathbf{A}_i)_{i=1}^N$ a Harris ergodic Markov Chain with invariant distribution $\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})$. It directly comes [1]

$$\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) = \frac{1}{N}\sum_{i=1}^{N} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \mathbf{A}_i\Big) \xrightarrow[N\to\infty]{a.s.} \int_{\mathcal{A}} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha} \tag{11}$$

3.1.4 Conclusion on $\lim_{M\to\infty}\lim_{N\to\infty} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big)$.

From the previous sections, we have

$$\begin{cases} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) \xrightarrow[N\to\infty]{a.s.} \int_{\mathcal{A}} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha} \\ \int_{\mathcal{A}} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big)\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha} \xrightarrow[M\to\infty]{a.s.} \int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha} \end{cases}$$

It leads to

$$\mathbb{P}\left(\lim_{M\to\infty}\lim_{N\to\infty} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) = \int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}\right) = 1 \quad \square$$

3.2 Almost sure $\lim_{N\to\infty}\lim_{M\to\infty} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big)$

From Equation (10), we have

$$\forall \boldsymbol{\alpha} \in \mathcal{A}, \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \boldsymbol{\alpha}\Big) \xrightarrow[M\to\infty]{\text{a.s.}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})$$

It leads to

$$\hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) = \frac{1}{N}\sum_{i=1}^{N} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{M}\Big(h(\boldsymbol{\Lambda}), \mathbf{A}_i\Big) \xrightarrow[M\to\infty]{a.s.} \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \mathbf{A}_i, \boldsymbol{y}_{1,\mathrm{obs}})$$

And similarly to Equation (11),

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \mathbf{A}_i, \boldsymbol{y}_{1,\mathrm{obs}}) \xrightarrow[N\to\infty]{a.s.} \int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha},$$

Finally,

$$\mathbb{P}\left(\lim_{N\to\infty}\lim_{M\to\infty} \hat{E}^{\boldsymbol{\alpha}^\star_\ell}_{N,M}\Big(h(\boldsymbol{\Lambda})\Big) = \int_{\mathcal{A}} \mathbb{E}(h(\boldsymbol{\Lambda}) \mid \boldsymbol{\alpha}, \boldsymbol{y}_{1,\mathrm{obs}})\hat{p}^{\boldsymbol{\alpha}^\star_\ell}_L(\boldsymbol{\alpha} \mid \boldsymbol{y}_{1,\mathrm{obs}})d\boldsymbol{\alpha}\right) = 1 \quad \square$$

References

1. Vats, D., Flegal, J.M., and Jones, G.L., Strong consistency of multivariate spectral variance estimators in Markov chain Monte Carlo, *Bernoulli*, 24(3):1860 – 1909, 2018.