ρ_{Boat} Design Report

Charlie Weiss and Nathan Yee

I. Introduction/Executive Summary

WE have been tasked with the design of a boat that meets certain performance requirements. We must create a boat that floats, floats flat (the deck of the boat must be parallel to the surface of the water when fully loaded), and has an angle of vanishing stability between 120 and 140 degrees. It must be built out of 1' X 2' X 2 inch EPS board and adhesive, have a 92 gram aluminum mast 50 cm long, and carry a payload of two soda cans. We may use up to an additional 300 grams of ballast if we wish.

In this report, you will find five sections following this one. Section II, Background and Terminology, introduces key concepts and vocabulary for boat design. It describes how the shape of a boat is defined and its important physical features, as well as key concepts related to how boats move.

Section III, Design Considerations, explains how these concepts are used to design our boat. It explains how the boat's centroids, and the forces that act on those locations, affect each of our design requirements.

Section IV, Proposed Design and Justification, proposes the simple boat design we have created to meet the requirements. Here you will find that our boat is defined by an extruded parabola with a center of mass a little more than halfway down and a mass of 1171 grams. Through mathematical analysis, we prove that the boat floats, floats flat, and has an angle of vanishing stability of 128.6 degrees. This meets all the design requirements.

Section V, Performance, tells you how our boat design performed in reality compared to our analysis. You will see that the boat performed as we expected to in terms of floating and floating flat, and within margin of error for the AVS.

II. Background and Terminology

What are boats and how do they work?

There are many properties of boats we must be familiar with before we are able to design one. These have to do with the shape of the boat, its position in the water, and important angles.

A boat has specific vocabulary to refer to its parts. What one considers the front is called the "bow" and the back is the "stern." If you were standing on the boat looking to the bow, your right would be "starboard" and your left would be "port." The outer shell of the boat is the "hull" and the top of it the "deck." Figure 1 depicts these visually.

In general, boats are defined by three sets of contour lines, or sections. Those that layer from bow to stern are called "buttocks." This means that, were you to stand in front of the bow of the boat and look at it, and shine a light from behind the stern, the silhouette you would see is a buttocks section. By the same principle, the sections that layer from port to starboard are "stations" and those from the bottom of the hull to the deck are "waterlines." These are also depicted for a standard boat in Figure 2. Individual sections can be manipulated the change the shape of the boat.

C. Weiss and N. Yee are with Olin College of Engineering

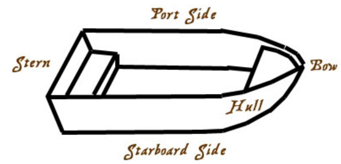

Fig. 1. Diagram for a typical boat, depicting the bow, stern, port side, starboard side, and hull. Note that this diagram does not include the deck, which on larger boat is the platform encasing the top of the boat. Credit due to: http://s105.photobucket.com/user/eeeeurgh/media/boat.jpg.html

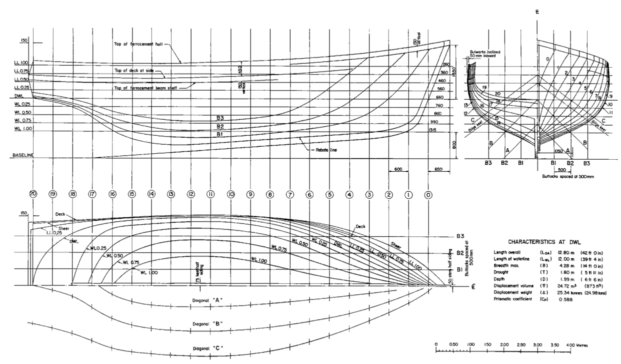

Fig. 2. Section lines for a standard boat. In the top left are stations viewed from the side, top right are buttocks viewed from the front, and the bottom left are waterlines from a birds-eye view. Credit due to: http://www.fao.org/docrep/003/v9468e/v9468e05.htm

Two important physical features of boats are their center of mass (COM) and center of buoyancy (COB). The center of mass is the point where the mass is equal on all sides. It is a useful measurement, because the distributed weight of the boat can be modeled as acting at that one point.

The center of buoyancy is similar to the center of mass. It is the point where the buoyant force on the boat (the reactive force of water pressure that keeps it floating) is evenly distributed on all sides of the boat. Incidentally, its location is the same as that of the center of mass of the water displaced by the boat, so the two are interchangeable.

An important feature of a boat is its stability, which is described using many terms. It is said to "heel" when it leans from starboard to port, at an angle which we call the "heel angle." As it does so, the "waterline," where the surface of the water is positioned on the boat, changes. This change in the position of the water displaced changes where the center of buoyancy is located, which creates a "righting arm." The righting arm ($\vec{r}$) is the horizontal distance between the center of mass and center of buoyancy. It is directly proportional to the "righting moment," the force which either pushes the boat to its original position, or causes it to capsize. The righting arm helps us determine something called the "angle of vanishing stability (AVS)," which is the heel angle past which the boat capsizes.

Many boats contain a structure that runs lengthwise along the bottom of the hull, called a "keel." These are extremely common on sailboats, for two main reasons. A weighted keel helps lower the center of mass of a boat without changing the center of buoyancy that much, which helps with the stability of the boat, as we shall see later. The structure of the keel also increases the amount of water drag acting against the boat, which, due to the position of forces and reactionary forces on the boat, actually helps propel it forward. While we will not be investigating this ourselves, it is worth looking into.

III. Design Considerations

There are three considerations we must design for:

1) The boat should float
2) The boat float flat
3) The boat have an angle of vanishing stability between 120 and 140 degrees

The boat should float

To understand how a boat is floats in water, we examine all forces acting on it. We start with the simplest possible case: a point object boat that experiences force of gravity ($\vec{F_g}$) and force of buoyancy ($\vec{F_B}$) in opposite directions. The relative magnitudes of these forces lead to three possible states:

1) The boat is sinking: $\vec{F_g} > \vec{F_B}$
2) The boat is floating: $\vec{F_g} = \vec{F_B}$
3) The boat is rising: $\vec{F_g} < \vec{F_B}$

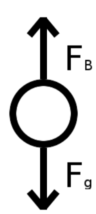

Fig. 3. Free body diagram for the boat represented as a point object. This free body diagram depicts the state in which the boat is floating, $\vec{F_B} = \vec{F_g}$

Figure 3 is a free body diagram for the second case, in which the boat floats.

To determine whether the boat is sinking, floating, or rising, we must determine both $\vec{F_g}$ and $\vec{F_B}$.

$\vec{F_g}$ is a constant:

$$\vec{F_g} = m\vec{g}$$

where m is the mass of the boat and $\vec{g}$ is gravitational acceleration on Earth.

$\vec{F_B}$ is determined by the density of the fluid, the amount of displaced fluid, and gravitational acceleration:

$$\vec{F_B} = \rho V \vec{g}$$

where ρ is the density of water, V is the displaced volume of water, and $\vec{g}$ is the gravitational acceleration on Earth.

The boat should float flat

To determine whether or not a boat floats flat, we need to see how it behaves when it is floating flat, or the heel angle (θ) $= 0°$, and how it behaves when it is tilted, $0° < \theta < AVS$. We want our boat to accomplish two things:

1) When the boat is flat, $\theta = 0°$, it should be in static equilibrium.
2) When the boat is tilted, $0° < \theta < AVS$, it should right itself.

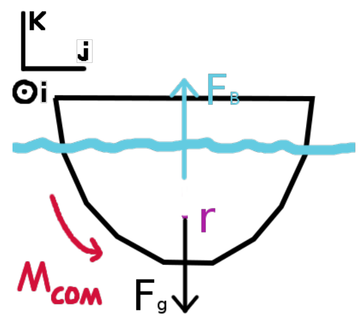

Fig. 4. Free body diagram for $\theta = 0°$. The location of the waterline is reactionary such that $\vec{F_B} = \vec{F_g}$. $\vec{M}_{COM}$ is the moment about the center of mass. The small purple dot represents righting arm equal to zero, $\vec{r} = 0$.

Figure 4 depicts a flat boat, $\theta = 0°$. We observe that the $\vec{F_g}$ and $\vec{F_B}$ are equal magnitude and directly above and below each other. The relative locations between the forces imply that $\vec{r} = 0$. As a result, $\vec{M}_{COM} = 0$ so the boat will not tend to tilt in any direction. This boat is in static equilibrium.

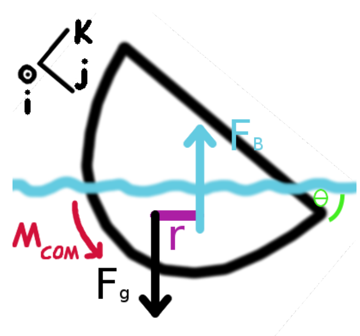

Fig. 5. Free Body Diagram for $\theta = 40°$. The placement of the waterline is reactionary such that $\vec{F_B} = \vec{F_g}$. $\vec{M}_{COM}$ is the moment about the center of mass. The purple line depicts the righting arm for a moment about the center of mass

Figure 5 depicts a tilted boat $\theta = 40°$. In this case, $\vec{F_g}$ and $\vec{F_B}$ are offset in both the $\hat{\imath}$ and $\hat{\jmath}$ directions. As a result, we must calculate both the righting arm ($\vec{r}$) and the sign of the moment about the center of mass.

First, we calculate the centers of mass and buoyancy. The general equation for this is:

$$\vec{r}_{centroid} = \frac{\sum \vec{r}_i m_i}{\sum m_i}$$

where $\vec{r}_{centroid}$ is the position of the centroid, $\vec{r}_i$ is the position of a certain point i and m_i is the mass at that point.

Since we neglect the weight of foam, the COM is determined entirely based on the location of the payload, mast, and ballast:

$$\vec{r}_{COM} = \frac{M_{Mast} r_{Mast} + M_{Cans} r_{Cans} + M_{Ballast} r_{Ballast}}{M_{total}}$$

where $\vec{r}_{COM}$ is the position of the center of mass, M_{Mast}, M_{Cans}, $M_{Ballast}$, and M_{total} are the masses of the mast, cans, ballast, and total mass, respectively, and r_{Mast}, r_{Cans}, and $r_{Ballast}$ are their respective positions.

The center of buoyancy is a slightly more involved calculation, because of the shape of the water displaced. Assuming the shape of the hull to be defined by a function $z = |y|^n - 1$ and the waterline by $z = y\tan(\theta) + d$ where θ is the heel angle and d is the depth of the waterline, the calculation becomes:

$$\vec{r}_{COB} = \frac{\int_{y_1}^{y_2} \int_{|y|^n - 1}^{y\tan(\theta) + d} (y\hat{i} + z\hat{j}) \rho \, dz dy}{\int_{\mathrm{y1}}^{\mathrm{y2}} \int_{|y|^n - 1}^{y\tan(\theta) + d} \rho \, dz dy}$$

where $\vec{r}_{COB}$ is the position of the center of buoyancy, ρ is the density of water, y_1 is the first intercept of the waterline and the hull and y_2 is the second intercept. Note that this integral will change if the waterline intercepts the deck instead of the hull the second time.

Next, we calculate $\vec{r}$. This can be done by projecting the vector between the center of mass and center of buoyancy onto a unit vector in the direction of the waterline:

$$\vec{r} = (\vec{r}_{COB} - \vec{r}_{COM}) \cdot (\sin(\theta)\hat{j} + \cos(\theta)\hat{k})$$

where $\vec{r}$ is the righting moment and θ is the heel angle.

Finally, we do a cross product of the righting arm and $\vec{F}_B$ to find the moments about the center of mass:

$$\vec{M}_{com} = \vec{r} \times \vec{F}_B$$

where $\vec{M}_{com}$ is the moment about the center of mass, and $\vec{r}$ is the righting arm for $\vec{F}_B$.

Using the right hand rule, we can determine the sign of the moment. If the sign of the moment and the sign of θ are the same, the boat will right itself until it is flat.

The boat must have an AVS between 120° and 140°

The angle of vanishing stability is the heel angle at which a boat capsizes. Mathematically, this heel angle must meet two conditions:

1) $\vec{M}_{COM} = 0$ at AVS
2) $\frac{\partial \vec{M}_{COM}}{\partial \theta} < 0$

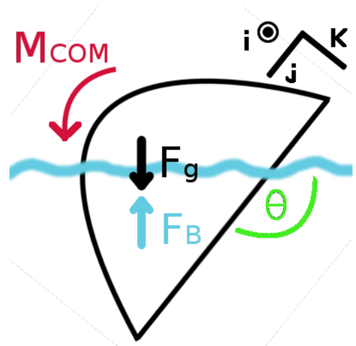

Fig. 6. Free body diagram for $\theta = AVS$. The location of the waterline is reactionary such that $\vec{F}_B = \vec{F}_g$. $\vec{M}_{COM}$ is the moment about the center of mass. The $\vec{M}_{COM} = 0$ at $\theta = AVS$ becausE $\vec{r} = 0$.

First, start by calculating $\vec{M}_{COM} = 0$. We can do this by finding when $\vec{r} = 0$:

$$0 = (\vec{r}_{COB} - \vec{r}_{COM}) \cdot (sin\theta\hat{j} + cos\theta_k)$$

which is equal to zero when the vectors are perpendicular to each other. Therefore, $\vec{F}_g$ and $\vec{F}_B$ must be above and below each other relative to the waterline. This also implies that increasing or decreasing the heel angle would make the righting arm negative or positive respectively. In other words, $\frac{\partial \vec{M}_{COM}}{\partial \theta} < 0$.

IV. Proposed Design and Justification

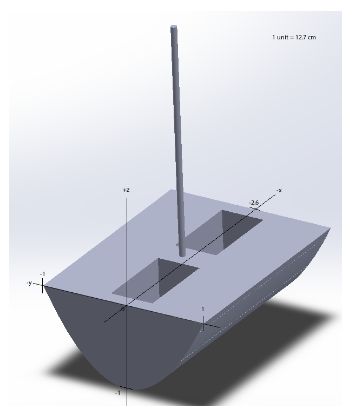

Fig. 7. Visual representation of our boat design, defined by $z = y^2 - 1$ for $y \in [-1, 1]$ extruded along the x axis, where 1 unit on the graph represents 12.7 cm in the actual boat.

Given these design considerations, the boat we propose is a simple extruded parabola defined by $z = y^2 - 1$ for $y \in [-1, 1]$, 2.6093 units in length, with a scale of 1 unit = 12.7 cm, as depicted in Figure 7. The center of mass is located at $(0\hat{i} + 0\hat{j} - 0.6975\hat{k})12.7$ cm.

To see whether or not the boat floats, we must compare the magnitudes of the $\vec{F}_g$ and the $\vec{F}_B$. We will start by calculating F_g:

$$F_g = M_{boat}g$$
$$M_{boat} = M_{Mast} + 2M_{Can} + M_{Ballast}$$
$$M_{boat} = 92 + 2(389.5) + 300$$
$$M_{boat} = 1171$$
$$\vec{F}_g = 1171g$$

Now, we will calculate F_B. At the surface of the water, F_B is either less than or equal to F_g. To calculate the upper bound for F_B, we maximize displaced liquid by calculating as if the entire boat is submerged under water. First calculate the boat's volume:

$$V = (12.7^3)(2.6093)\int_{-1}^{1} y^2 dy$$
$$V = 7126.5$$

Next, calculate range of possible values of F_B:

$$F_B \leq \rho V g$$
$$F_B \leq 1(7126.5)g$$
$$F_B \leq 7126.5g$$

The boat is at the surface. Therefore F_B can't be greater than F_g:

$$F_B = 1171g$$

In this case, F_B is equal to F_g. This means that our design will float at the surface of the water.

To see whether or not the boat floats flat, we must meet two requirements:

1) When the boat is flat, $\theta = 0°$, it should be in static equilibrium.
2) When the boat is tilted, $0° < \theta < AVS$, it should right itself.

First, based on the previous set of calculations, we know that the sum of forces is equal to 0. We also know that for a heel angle of $\theta = 0°$, $\vec{M}_{COM} = 0$. Therefore, our boat is in static equilibrium.

Second, we need to understand the moment's behavior given heel angles between 0° and the AVS. A heel angle in this range will create a positive moment. Since the sign of the angle and the sign of the moment are the same, the moment will make the boat right itself.

Our proposed design is both in static equilibrium at $\theta = 0°$ and wants to restore itself when displaced by a heel angle $0° < \theta < AVS$. Therefore, the boat wants to float flat.

Finally, we must check the boat's angle of vanishing stability. Figure 8 shows a graph of $\vec{M}_{COM}$ vs heel angles of the boat.

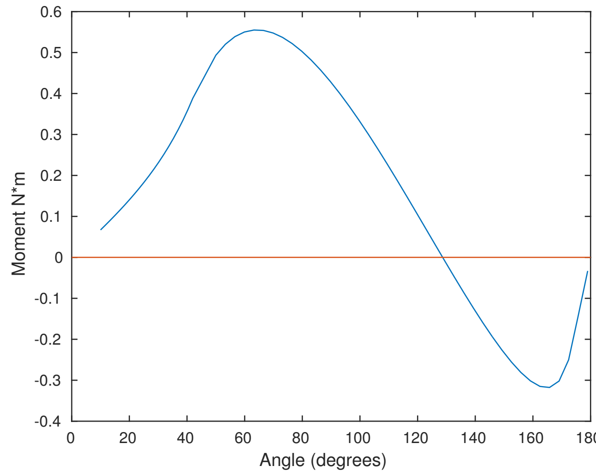

Fig. 8. Graph of $\vec{M}_{COM}$ vs. heel angles. The AVS is defined as the point at which $\vec{M}_{COM} = 0$ and $\frac{\partial \vec{M}_{COM}}{\partial \theta} < 0$. The AVS of this graph is at 128.6 deg.

As it shows, the moment increases to a maximum around 60 degrees, decreases past 0 to a minimum around 160 degrees, and increases back to zero at 180. Recall that the AVS is the angle at which $\vec{M}_{COM} = 0$ and $\frac{\partial \vec{M}_{COM}}{\partial \theta} < 0$. This means the point at which it decreases past 0 is the AVS, which in this case is 128.6 degrees. The AVS is well within our target range.

V. Performance

The boat performed almost exactly as we expected. It met the first two requirements with ease, both floating and floating flat. However, there is a discrepancy between the measured AVS (125°) and calculated AVS (128.6°).

This discrepancy is within an approximately 3% margin of error. We can attribute the difference to minor fabrication errors (the boat is not a perfectly extruded parabola) that could alter the location of the center of buoyancy, and small differences in mass (the mast we used was a few grams heavier, and the cans a few grams lighter) that altered the position of the center of mass. These factors help explain why there exists a discrepancy between calculated and measured AVS.

VI. Conclusion

Using the concepts of center of mass and center of buoyancy, we were able to create a boat that fits the design requirements. Our boat design is just one of many that fit these design requirements. We could change the design of our boat by manipulating the shape of the hull and location of the center of mass to optimize for different parameters, such as stability or speed. We could also extend the use of the concepts to analysis in 3D space.