Class Exercises: 2019-10-29

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1. Two fair dice are rolled in another room, out of your sight. If the sum of the two dice values is seven, you *win*. Your friend is in the other room and can observe the dice.

   1.1. Your friend calls out that one of the dice came up six. Given this information, what is the probability that you won?

   1.2. Suppose, instead, that your friend tells you that you won. In this case, what is the probability that one of the dice came up five?

2. Sally Smart just graduated from high school. She was accepted to three reputable colleges.
   - With probability 4/12, she attends Brown.
   - With probability 5/12, she attends Dartmouth.
   - With probability 3/12, she attends Little Hoop Community College.

   Sally is either happy or unhappy in college.
   - If she attends Brown, she is happy with probability 4/12.
   - If she attends Dartmouth, she is happy with probability 7/12.
   - If she attends Little Hoop, she is happy with probability 11/12.

   2.1. What is the probability that Sally is happy in college?

   2.2. What is the probability that Sally attends Brown, given that she is happy in college?

   2.3. Show that the events "Sally attends Brown" and "Sally is happy" **are not** independent.

   2.4. Show that the events "Sally attends Dartmouth" and "Sally is happy" **are** independent.

3. The Chain Rule for probability says that if $A_1, A_2, \ldots, A_n$ are events in a probability space $(\mathscr{S}, \Pr)$, then

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \cdots \cap A_n] &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \cdots \cdot \Pr[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1, A_2] \cdot \cdots \cdot \Pr[A_n \mid A_1, A_2, \ldots, A_{n-1}] \\ &= \prod_{j=1}^{n} \Pr[A_j \mid A_1, A_2, \ldots, A_{j-1}]\,.\end{aligned}$$

   3.1. Prove this rule, i.e., prove the first equation. (The other two lines are just rewritings.)
   Hint: Don't use induction. Start with the right-hand side.

   3.2. Use this rule to answer the following question. Suppose n passengers board a flight that has n seats and they each take a seat at random, ignoring their assigned seating. The passengers board one by one. What is the probability that passengers 1 through k (inclusive) all end up in their assigned seats?

Bonus question: What do #2.3 and #2.4 teach you about independence of events?