Solutions for Unit 10

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Solution for PS10-1.

a. If we pick just one kind of candy (two pieces of it), there are 6 choices.
If we pick two distinct kinds of candy, there are $\binom{6}{2}$ choices.
Together, there are $6+\binom{6}{2}=21$ choices.

b. We break this down into three disjoint cases and add up the results.

- All three pieces of candy are of the same kind: $\binom{6}{1}$ choices.
- We pick exactly two kinds of candy: this gives us $\binom{6}{2}$ to choose the two kinds, following which we have to choose which of the two kinds we're going to pick two pieces of. By the generalized product principle, there are $\binom{6}{2}\cdot 2$ choices overall.
- We pick three distinct kinds of candy: $\binom{6}{3}$ choices.

Solution for PS10-2.

a. With books ▯ and separator ♦, one configuration is ▯▯♦▯▯▯▯▯♦♦▯▯▯▯▯▯♦▯▯♦. If the candy types are ordered and denoted A,B,C,D,E,F, then number of each type can be determined by counting the number of books between a given pair of separators. The configuration can be written as AA♦$BBBBB$♦♦$DDDDDD$♦EE♦, so that 2 pieces of A,5 pieces of type B, 6 pieces of D, and 2 of E are chosen.
A second layout is ▯♦ ▯ ▯♦ ▯ ▯ ▯ ♦ ▯ ▯ ▯ ▯♦ ▯ ▯ ▯ ♦ ▯ ▯, producing $1,2,3,4,3,2$ counts of types A,B,C,D,E,F, respectively.

b. $\binom{20}{5}$, the number of ways to place 5 separators in a list of 20 items (books + separators).

c. Applying the book/separator model, there are t books, and $n-1$ separators. The final count is $\binom{t+n-1}{n-1}$.

Solution for PS10-3.

a. Use $f(x_1,x_2,\ldots,x_k)=\underbrace{000\cdots0}_{x_1}\ 1\ \underbrace{000\cdots0}_{x_2}\ 1\ \cdots\cdots\underbrace{000\cdots0}_{x_k}\ 1\ \underbrace{000\cdots0}_{n-(x_1+\cdots+x_k)}$.

b. Use $g(y_1,y_2,\ldots,y_k)=(y_1,y_2-y_1,y_3-y_2,\ldots,y_k-y_{k-1})$.

c. $|S_{n,k}|=|L_{n,k}|=\binom{n+k}{k}$.

Solution for PS10-4.

a.

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n\cdot(n-1)!}{k\cdot(k-1)!(n-k)!}=\frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}=\frac{n}{k}\binom{n-1}{k-1}$$

b.

$$\binom{n}{m}\binom{m}{k}\cdot 1=\frac{n!}{m!(n-m)!}\cdot\frac{m!}{k!(m-k)!}\cdot\frac{(n-k)!}{(n-k)!}=\frac{n!}{k!(n-k!)}\cdot\frac{(n-k)!}{(m-k)!(n-k-(m-k))!)}=\binom{n}{k}\binom{n-k}{m-k}$$

Solution for PS10-5.

a. Let $x=1$, $y=1$. By the binomial theorem,

$$\begin{aligned}2^n=(1+1)^n=(x+y)^n&=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\\&=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k=\sum_{k=0}^{n}\binom{n}{k}.\end{aligned}$$

b. Let $x = -1$, $y = 1$. Applying the binomial theorem,

$$\begin{aligned} 0 = (1-1)^n = (x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \end{aligned}$$

Solution for PS10-6.

a. By the binomial theorem,

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

b. Starting with the binomial theorem on $(1+1)^{n-1}$,

$$\begin{aligned} (1+1)^{n-1} &= \sum_{k=0}^{n-1} \binom{n-1}{k} \\ n2^{n-1} &= \sum_{k=0}^{n-1} n \binom{n-1}{k} && \text{multiplying by } n \\ n2^{n-1} &= \sum_{k=0}^{n-1} (k+1) \frac{n}{k+1} \binom{n-1}{k} \\ n2^{n-1} &= \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} && \text{by } \textbf{\textit{PS10-4 a}} \\ n2^{n-1} &= \sum_{k=1}^{n} k \binom{n}{k} && \text{reindexing} \\ n2^{n-1} &= \sum_{k=0}^{n} k \binom{n}{k} && \text{adding a term at } k=0 \text{ whose value is } 0 \end{aligned}$$

We have derived the statement to be proven.

Solution for PS10-7.

a. Both sides of the equation equal the number of subsets of an arbitrary n-element set S. The total number of subsets is well known to be 2^n. The sum on the left hand side, counts, for each $0 \le k \le n$, the number of subsets of S containing k elements, and sums the result. Since all subsets of S are counted exactly once on the left hand side, the left hand side counts all subsets of S, and equals the right hand side. □

b. The equation is equivalent to the claim that the number of odd subsets of a given n-element set equals the number of even subsets. This is true by the previous homework problem ***PS2-5*** $^{\textbf{\textit{HW}}}$. □

c. There is a well-known committee-chairperson fable. Both expressions indicate the number of ways to pick a k person committee from n candidates, with a single designated chairperson. Using Generalized Product Principle, $k\binom{n}{k}$ is the number of ways ($\binom{n}{k}$) to pick a committee, multiplied by the number of ways to then pick a chairperson from that committee (k). $n\binom{n-1}{k-1}$ is the number of ways to pick a chairperson (n) times the number of ways to pick everyone else in the committee ($\binom{n-1}{k-1}$). □

d. A generalization of the committee-chairperson story. Given a set A of n elements, both sides express the number of ways to pick an m-element subset B of A, and a k-element subset C of B. The $\binom{n}{m}\binom{m}{k}$ is the number of ways to pick the subset B as a subset of A first, and then pick C as a subset of B; the $\binom{n}{k}\binom{n-k}{m-k}$ is the number of ways to pick C as a subset of A, and then pick $B - C$ as a subset of $A - C$. □

e. For a given set S with n elements, 3^n is the number of functions from S to $0, 1, 2$. For each such function f, we can define a function $g : R(f) \to 0, 1$ on the set $R(f) = \{x \in S : f(x) \in \{1, 2\}\}$ by $g(x) = f(x)$. Counting the number of functions g, we find that for each size k of the domain for g, there are $\binom{n}{k}$ possible k-element domains that are subsets of S, and 2^k possible functions.

Viewed in terms of the hint, there are two methods to find the number of possible sets A,B for which $B \subseteq A \subseteq \{1, 2, \ldots, n\}$. First, for each $i \in \{1, 2, \ldots, n\}$, there are three possible states relative to A and B: either $i \notin A$, $i \in B$, or $i \in A - B$. From this one can derive a total count of 3^n. An argument, partitioning by the size of A, as in the previous paragraph will also give $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$.

f. Both sides indicate the number of ways to pick a committee (with at least one person) from a set of n candidates. One could first pick a chairperson (n choices), and then pick the rest of the committee (2^{n-1} choices). Alternatively, one could, for each size of a committee, first pick a committee ($\binom{n}{k}$ choices), and then pick a chairperson (k choices); adding the product over all possible committee sizes gives $\sum_{k=0}^{n} k\binom{n}{k}$.

Solution for PS10-8.

a. For each k, we prove the identity by induction over n. Let $P_k(n)$ be the proposition that the equation

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

is true. We will show by induction that $P_k(n)$ is true for all $n \geq k$.

Base case. This is $P_k(k)$, which is true since

$$\sum_{m=k}^{k} \binom{m}{k} = \binom{k}{k} = 1 = \binom{k+1}{k+1}$$

Induction step. We seek to prove $P_k(n)$, for $n > k$, assuming that $P_k(n-1)$ is true. Since by $P_k(n-1)$,

$$\sum_{m=k}^{n-1} \binom{m}{k} = \binom{n}{k+1},$$

adding $\binom{n}{k}$ to both sides produces

$$\begin{aligned}
\sum_{m=k}^{n-1} \binom{m}{k} + \binom{n}{k} &= \binom{n}{k+1} + \binom{n}{k} \\
\sum_{m=k}^{n} \binom{m}{k} &= \binom{n}{k+1} + \binom{n}{k} \\
\sum_{m=k}^{n} \binom{m}{k} &= \binom{n+1}{k+1},
\end{aligned}$$

where the last line follows by Pascal's identity. □

b. Let $V = \{1, 2, \ldots, n+1\}$. Then the number of ways to pick a $k+1$ element set $J \subset V$ is $\binom{n+1}{k+1}$. One can also pick a such a subset J by first picking the *smallest* element i_{min} of J, and then the remainder of J, ($J - \{i_{min}\}$, a set of k elements), for which there are $\binom{n+1-i_min in}{k}$ choices. As the possible values of i_{min} range from 1 to $n+1-k$, combining the number of sets produced for each value of i_{min} gives the sum $\sum_{m=1}^{n+1-k} \binom{n+1-m}{k} = \sum_{m=k}^{n} \binom{m}{k}$. □

Solution for PS10-9. If p is a prime, $0 < k < p$,

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}.$$

Since $p! = p\cdot(p-1)\cdots 2\cdot 1$, $p|p!$. On the other hand, since $k < p$, $p \nmid i$, for all $1 \leq i \leq k$, so by iterated applications of the contrapositive of Euclid's lemma, $p \nmid k!$. Similarly, as $p-k < p$, $p \nmid (p-k)!$.

Let $a = n!$, $b = k!(p-k)!$. Since we know $\frac{a}{b} \in \mathbb{Z}$, there is some constant $c \in \mathbb{Z}$ for which $a = bc$. Since $p \mid a$, and $p \nmid b$, by Euclid's lemma, $p \mid c$. Defining $d = \frac{c}{p} \in \mathbb{Z}$, we find $\frac{a}{b} = \frac{pdb}{b} = pd$. Since p is a factor of the integer product pd, it is a factor of $\frac{a}{b} = \binom{p}{k}$.