Solutions for Unit 8

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Solution for PS8-1.

a. $27 \times 37 = 999$, by the product principle.

b. $52 \times 51 = 2652$, by the generalized product principle.

c. $\lfloor 1000/3 \rfloor + \lfloor 1000/5 \rfloor - \lfloor 1000/15 \rfloor = 467$, by the generalized sum principle.

Solution for PS8-2.

a. Let C be the set of ways in which we can answer a single question on the test. Then the set of choices for the entire test is C^{10}. So we apply the product principle.

(a) In this case, $|C| = 4$, so the number of ways is $4^{10} = 1\,048\,576$.

(b) Here $|C| = 5$—one of the choices is to leave an answer blank—so the number of ways is $5^{10} = 9\,765\,625$.

b. Consider natural numbers below 10^9, padded up to nine digits by adding zeros as needed. Let

$$\begin{aligned}
S &= \{0, 1, 2, \ldots, 10^9 - 1\}\,, \\
A &= \{n \in S : n\text{'s padded decimal representation contains a '1'}\}\,, \\
B &= \{n \in S : n\text{'s padded decimal representation does not contain a '1'}\}\,.
\end{aligned}$$

Every number in B can be thought of as a sequence of 9 characters, with each character drawn from $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$, a set of size 9. By the product principle, $|B| = 9^9$.

Since $A \cap B = \varnothing$, by the sum principle, $|S| = |A| + |B|$. Therefore, $|A| = |S| - |B| = 10^9 - 9^9$.

The question asks about numbers in the set $(S - \{0\}) \cup \{10^9\}$. The number 0 does not contain the digit '1', so there's nothing to take away, but the number 10^9 does contain the digit '1', so we need to add one to this figure. This gives a final answer of

$$10^9 - 9^9 + 1 = 612\,579\,512\,.$$

Solution for PS8-3. Let a_k denote the sum of all multiples of k between 1 and 1000. Repeatedly using the basic summation formula $1 + 2 + \cdots + n = n(n+1)/2$, we obtain

$$\begin{aligned}
a_3 &= 3 + 6 + \cdots + 999 = 3(1 + 2 + \cdots + 333) = 3 \times \frac{333 \times 334}{2} = 166\,833\,; \\
a_5 &= 5 + 10 + \cdots + 1000 = 5(1 + 2 + \cdots + 200) = 5 \times \frac{200 \times 201}{2} = 100\,500\,; \\
a_{15} &= 15 + 30 + \cdots + 990 = 15(1 + 2 + \cdots + 66) = 15 \times \frac{66 \times 67}{2} = 33\,165\,.
\end{aligned}$$

The desired answer is $a_3 + a_5 - a_{15} = 166833 + 100500 - 33165 = 234168$. We subtracted a_{15} because numbers which are multiple of both 3 and 5—i.e., multiples of $\operatorname{lcm}(3,5) = 15$—are included twice when we write $a_3 + a_5$.

Solution for PS8-4. Using the sum principle, it's $62^6 + 62^7 + 62^8 - 52^6 - 52^7 - 52^8 = 167\,410\,949\,583\,040$.

Solution for PS8-5. Let $s(n) := n^6 + n^7 + n^8$; this is the number of six-to-eight character strings where each character is drawn from a set of size n.

By the generalized sum principle, the desired answer is $s(62) - s(52) - 2s(36) + s(10) + 2s(26)$.

Solution for PS8-6.

a. For any arbitrary choices of x and y, we can always choose z in such a way that $x + y + z$ is even. More precisely, if $x + y$ is odd then we choose z to be odd, else we choose z to be even. Thus, choosing a 3-tuple $(x, y, z) \in T$ can be seen as making the following sequence of choices:

- Choose x freely from D. There are 10 choices.
- Choose y freely from D. There are 10 choices.

- Choose z from $\{1,3,5,7,9\}$ if $x+y$ is odd; else choose z from $\{0,2,4,6,8\}$. There are exactly 5 choices in each case.

Thus, by the generalized product principle, $|T| = 10 \times 10 \times 5 = 500$.

b. Let the n digits be $x_1, x_2, \ldots, x_n$ (from left to right). Then we want to find number of choices for $x_1, \ldots, x_n$ such that $1 \le x_1 \le 9$; for each $i \in \{2,3,\ldots,n\}$, $0 \le x_i \le 9$; and $x_1 + x_2 + \cdots + x_n$ is even.

Now, for any arbitrary choices of $x_1, \ldots x_{n-1}$, we can always choose x_n in such a way that $x_1 + x_2 + \cdots + x_n$ is even. More precisely, if $x_1 + \cdots + x_{n-1}$ is odd, then we choose x_n to be odd, else we choose x_n to be even. Hence for each list of choices of $x_1, \ldots x_{n-1}$, there are exactly 5 choices of x_n such that $x_1 + x_2 + \cdots + x_n$ is even. By the generalized product principle, the total number of choices for all the digits is

$$9 \times 10^{n-2} \times 5 = 45 \times 10^{n-2},$$

since there are 9 choices for x_1, and 10 choices for each of $x_2, \ldots x_{n-1}$.

Alternate Solution for PS8-6.

a. Let $A = \{1,3,5,7,9\}$ be the set of odd digits and let $B = \{0,2,4,6,8\}$ be the set of even digits. We break T into four pairwise disjoint subsets, each of which is a Cartesian product. So the product principle applies to each subset.

- $T_1 = \{(x,y,z) \in D^3 : x, y, \text{ and } z \text{ are even}\} = B \times B \times B$. Then $|T_1| = |B| \times |B| \times |B| = 5 \times 5 \times 5 = 125$.
- $T_2 = \{(x,y,z) \in D^3 : x \text{ and } y \text{ are odd}, z \text{ is even}\} = A \times A \times B$. Then $|T_2| = |A| \times |A| \times |B| = 125$.
- $T_3 = \{(x,y,z) \in D^3 : x \text{ and } z \text{ are odd}, y \text{ is even}\} = A \times B \times A$. Then $|T_3| = 125$ as well.
- $T_4 = \{(x,y,z) \in D^3 : y \text{ and } z \text{ are odd}, x \text{ is even}\} = B \times A \times A$. Then $|T_4| = 125$ as well.

b. Same as above.

Solution for PS8-7.

a. $4 \times 3 \times 2 \times 1 = 4! = 24$, by the generalized product principle.

b. $n!$, by the generalized product principle.

Solution for PS8-8.

a. If we color one of the 1s red, and the other black, then we have four distinct symbols, leading to $4! = 24$ permutations.

Now consider the function f that maps a colored permutation to an ordinary (uncolored) permutation of $(1,1,4,9)$ by "removing the colors." This f is a 2-to-1 correspondence. So, by the division principle, the number of permutations of $(1,1,4,9)$ is $24/2 = 12$.

b. We can either reason directly, or follow the steps of Problem 15.27 ("The Tao of BOOKKEEPER") from the [LLM] book. Then we apply similar logic to the word "CONDESCENDENCE". This 14-letter word has 3 Cs, 3 Ns, 2 Ds, 4 Es, 1 O, and 1 S. Therefore, it has

$$\frac{14!}{3! \cdot 3! \cdot 2! \cdot 4! \cdot 1! \cdot 1!} = 50\,450\,400$$

anagrams.

Solution for PS8-9.

a. 2^{mn}, since each relation is just a subset of $A \times B$.

b. n^m, by the generalized product principle (choosing a function means making a sequence of $|A|$ choices).

c. $n(n-1)\cdots(n-m+1)$, by the generalized product principle. Note that this equals 0 when $m > n$.

d. If $m = n$ then $n!$, else 0.

Solution for PS8-10. For a palindrome of length n, where n is even, we can arbitrarily choose the first $n/2$ bits and then the last $n/2$ bits are automatically fixed. Thus, there are 2 choices for each of the first $n/2$ bits, and then 1 choice for each remaining bit. This leads to $2^{n/2}$ possible palindromes.

If n is odd, then we can arbitrarily choose the first $(n+1)/2$ bits and then the last $(n-1)/2$ bits are automatically fixed. As before, this leads to $2^{(n+1)/2}$ possible palindromes.

We can combine these two cases and say that the number of n-bit palindromes for any natural number n is $2^{\lceil n/2 \rceil}$.

Solution for PS8-11.

a. $\binom{13}{10}$. This is pretty much the definition of "n choose k."

b. $13 \times 12 \times \cdots \times 4 = \dfrac{13!}{3!}$, by the generalized product principle.

c. $\binom{13}{10} - \binom{11}{10}$, subtracting off sets of 10 players chosen solely from the 11 students.

Solution for PS8-12.

a. $\binom{10}{4}$, since this amounts to choosing the 4 locations (out of 10) where the 1s will occur.

b. $\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4}$, by applying the previous observation four times.

c. $\binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}$, along similar lines.

d. $\binom{10}{5}$, along similar lines.

Solution for PS8-13.

a. 5040.

b. 1440.

c. 240.

d. 2640.