

1. Let $n \geq 2$ be an integer. We choose a random integer $X \in \mathbb{Z}_n$ uniformly. Let $Y = \gcd(X, n)$. Determine $\text{Ex}[Y]$ in each of the following cases.
 - 1.1. $n = 7$.
 - 1.2. $n = 9$.
 - 1.3. $n = p^2$, where p is a prime.

2. We roll two fair dice, a *red* die and a *blue* die, and they show numbers X and Y , respectively (these are therefore random variables). Let $W = X + Y$.
 - 2.1. Compute $\text{Ex}[X^2 \mid X \text{ is a perfect square}]$.
 - 2.2. Show that $\text{Ex}[WX] \neq \text{Ex}[W] \text{Ex}[X]$.

3. We roll 24 fair dice, and they show numbers X_1, \dots, X_{24} .
 - 3.1. How many sixes do we expect to see? In other words, compute $\text{Ex}[|\{j : X_j = 6\}|]$.
 - 3.2. Even though each die is fair, they have been connected together by very thin weightless threads and this causes X_1, \dots, X_{24} to be correlated in some unknown way. (For example, it may be that whenever X_1 is even, X_2 is more likely to be even than odd; or that whenever X_8 is a prime number, X_{20} is sure to be prime; or both of the above.) How does this affect your answer above?

4. The below problems have special significance in Computer Science. They model the process of inserting keys into a hash table.
 - 4.1. Two people that have the same birthday are said to form a *calendrical bond*. Assuming that the n students in a class have birthdays distributed uniformly among the d days in a year, and birthdays are mutually independent, what is the expected number of calendrical bonds among students in the class? Derive a formula in terms of n and d , then apply it to our CS30 class, using $n = 57$ and $d = 365$.

Hint: The random variable of interest here is a sum of $\binom{n}{2}$ indicator RVs.
 - 4.2. There are n bins, initially all empty. Then n balls are thrown randomly (uniformly) and independently into the bins: “uniformly” means that each ball is equally likely to go into each of the bins. What is the expected number of bins that remain empty after this process?

Hint: Again use a sum of appropriate indicator RVs.