Solutions for Unit 2

Solution for PS2-1.

a. $\forall y \in B$, because f is surjective, $\exists x \in A$, such that $f(x) = y$, i.e. $(y, x) \in f^{-1}$. And because f is injective, there is only one $x \in A$, such that $f(x) = y$, i.e. $(y, x) \in f^{-1}$. In conclusion, $\forall y \in B$, there is one and only one $x \in A$, such that $(y, x) \in f^{-1}$. Therefore, f^{-1} is a function.

b. **First,** we will prove that f is surjective. That is, we will prove $\forall y \in B \; \exists x \in A \, (f(x) = y)$.
Consider an arbitrary $y \in B$.
Since f^{-1} is a function, according to the definition of a function, $\exists x \in A \, ((y, x) \in f^{-1})$.
By the definition of the relation f^{-1}, this means $f(x) = y$.

Second, we will prove that f is injective. That is, we will prove $\forall x_1, x_2 \in A \, (f(x_1) = f(x_2) \implies x_1 = x_2)$.
Consider arbitrary elements $x_1, x_2 \in A$. Suppose that $f(x_1) = f(x_2)$. We will now show that $x_1 = x_2$.
Define $y = f(x_1) = f(x_2)$.
By the definition of the relation f^{-1}, we have $(y, x_1) \in f^{-1}$ and $(y, x_2) \in f^{-1}$.
Since f^{-1} is a function, for each y there must be at most one x so that $(y, x) \in f^{-1}$. It follows that $x_1 = x_2$.

Solution for PS2-2.

a. Consider an arbitrary element $y \in f(S_1 \cup S_2)$. Then $y = f(x)$ for some $x \in S_1 \cup S_2$, i.e. $x \in S_1$ or $x \in S_2$. If $x \in S_1$, $f(x) = y \in f(S_1)$, or, if $x \in S_2$, $f(x) = y \in f(S_2)$. So, in any case, $y \in f(S_1)$ or $y \in f(S_2)$. Hence $y \in f(S_1) \cup f(S_2)$. Thus,$f(S_1 \cup S_2) \subseteq f(S_1) \cup f(S_2) \cdots$

Again, consider an arbitrary element $y \in f(S_1) \cup f(S_2)$. Then $y \in f(S_1)$ or $y \in f(S_2)$. If $y \in f(S_1)$, then $y = f(x)$ for some $x \in S_1$, or, if $y \in f(S_2)$, then $y = f(x)$ for some $x \in S_2$. So, in any case, $y = f(x)$ for some $x \in S_1$ or $x \in S_2$, i.e. $x \in S_1 \cup S_2$. Hence, $y \in f(S_1 \cup S_2)$. Thus, $f(S_1) \cup f(S_2) \subseteq f(S_1 \cup S_2) \cdots$.

b. Consider an arbitrary element $x \in f^{-1}(T_1 \cup T_2)$. Then $f(x) \in T_1 \cup T_2$, i.e. $f(x) \in T_1$ or $f(x) \in T_2$. If $f(x) \in T_1$, $x \in f^{-1}(T_1)$, or, if $f(x) \in T_2$, $x \in f^{-1}(T_2)$. So, in any case, $x \in f^{-1}(T_1)$ or $x \in f^{-1}(T_2)$ i.e. $x \in f^{-1}(T_1) \cup f^{-1}(T_2)$. Thus, $f^{-1}(T_1 \cup T_2) \subseteq f^{-1}(T_1) \cup f^{-1}(T_2) \cdots$.

Again, consider an arbitrary element $x \in f^{-1}(T_1) \cup f^{-1}(T_2)$. Then $x \in f^{-1}(T_1)$ or $x \in f^{-1}(T_2)$. If $x \in f^{-1}(T_1)$, then $f(x) \in T_1$, or, if $x \in f^{-1}(T_2)$, then $f(x) \in T_2$. So, in any case, $f(x) \in T_1$ or $f(x) \in T_2$, i.e. $f(x) \in T_1 \cup T_2$. Hence, $x \in f^{-1}(T_1 \cup T_2)$. Thus, $f^{-1}(T_1) \cup f^{-1}(T_2) \subseteq f^{-1}(T_1 \cup T_2) \cdots$.

Solution for PS2-3.

a. To prove $f \circ g$ is injective, assume x_1, x_2 are such that $f(g(x_1)) = f(g(x_2))$; since f is injective, it follows $g(x_1) = g(x_2)$, and then since g is injective, $x_1 = x_2$.
To prove surjectivity, consider any $x \in C$; then since f is surjective, $\exists\, y \in B$ s.t. $f(y) = x$. Next, since g is surjective, $\exists\, z \in A$ s.t. $g(z) = y$. Overall, we've found an element $z \in A$ s.t. $(f \circ g)(z) = f(g(z)) = x$.

b. To show that $g^{-1} \circ f^{-1}$ is the inverse of $f \circ g$ (already knowing that such an inverse exists), it suffices to verify that $(f \circ g)((g^{-1} \circ f^{-1})(x)) = x$ for all $x \in C$, i.e., that $g^{-1} \circ f^{-1}$ behaves identically to the actual inverse on its domain. Verification follows since

$$(f \circ g)((g^{-1} \circ f^{-1})(x)) = f(g(g^{-1}(f^{-1}(x)))) = f(f^{-1}(x)) = x\,.$$

Solution for PS2-4.

To solve this problem, it will help to prove a little lemma first.

Lemma 1. *Let S and C be sets with $C \subseteq S$. Then $S-(S-C)=C$.*

Proof. Using the definitions of the basic set operations,

$$\begin{aligned}
S-(S-C) &= \{x : x \in S \wedge x \notin S-C\} && \triangleright \text{ definition of set difference}\\
&= \{x : x \in S \wedge \neg(x \in S \wedge x \notin C)\} && \triangleright \text{ definition of set difference}\\
&= \{x : x \in S \wedge (x \notin S \vee x \in C)\} && \triangleright \text{ de Morgan's law}\\
&= \{x : x \in S \wedge x \in C\} && \\
&= S \cap C && \triangleright \text{ definition of intersection}\\
&= C\,. && \triangleright \text{ because } C \subseteq S \qquad \square
\end{aligned}$$

First, we will prove that g is surjective. That is, we will prove $\forall B \subseteq S\ \exists A \subseteq S\ (g(A)=B)$.

Consider an arbitrary $B \subseteq S$.

Define $A := S-B$. Then, by Lemma 1, $g(A)=S-(S-B)=B$.

Thus, we have proved the existence of an A such that $g(A)=B$.

Second, we will prove that g is injective. That is, we will prove $\forall A_1,A_2 \subseteq S\ (g(A_1)=g(A_2) \implies A_1=A_2)$.

Consider arbitrary $A_1,A_2 \subseteq S$. Suppose that $g(A_1)=g(A_2)$. We will now show that $A_1=A_2$.

We have

$$\begin{aligned}
A_1 &= S-(S-A_1) && \triangleright \text{ by Lemma 1}\\
&= S-g(A_1) && \triangleright \text{ definition of } g\\
&= S-g(A_2) && \triangleright \text{ by our assumption}\\
&= S-(S-A_2) && \triangleright \text{ definition of } g\\
&= A_2\,. && \triangleright \text{ by Lemma 1}
\end{aligned}$$

Solution for PS2-5. Because S is nonempty, $\exists a \in S$. We construct h as follows:

$$h(A) = \begin{cases} A \cup \{a\}\,, & \text{if } a \notin A\\ A - \{a\}\,, & \text{if } a \in A\,. \end{cases}$$

Note that $|h(A)| = |A| \pm 1$. So, if $A \in \mathscr{P}^{\text{odd}}(S)$, then $h(A) \in \mathscr{P}^{\text{even}}(S)$. Thus, h is indeed a *function* of the form $h\colon \mathscr{P}^{\text{odd}}(S) \to \mathscr{P}^{\text{even}}(S)$.

Now, we prove that h is a *bijection*. As usual, the proof has two parts.

First, we prove that h is surjective.

Consider an arbitrary $B \in \mathscr{P}^{\text{even}}(S)$. Either $a \in B$ or $a \notin B$.

If $a \in B$, then $B-\{a\} \in \mathscr{P}^{\text{odd}}(S)$ and $h(B-\{a\}) = (B-\{a\}) \cup \{a\} = B$.

If $a \notin B$, then $B \cup \{a\} \in \mathscr{P}^{\text{odd}}(S)$ and $h(B \cup \{a\}) = (B \cup \{a\}) - \{a\} = B$.

We have shown that in either case, $\exists A \in \mathscr{P}^{\text{odd}}(S)$ such that $h(A)=B$. Therefore, h is surjective.

Second, we prove that h is injective.

Consider arbitrary sets $A_1,A_2 \in \mathscr{P}^{\text{odd}}(S)$ and suppose that $h(A_1)=h(A_2)$. We will prove that $A_1=A_2$.

For this, we will show that $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$. Actually, it suffices to prove the first of these; the second then follows by symmetry.

So, consider an arbitrary $x \in A_1$. Either $x \neq a$ or $x = a$.

- If $x \neq a$, then $x \in A_1 \cup \{a\}$ and $x \in A_1 - \{a\}$. Examining the definition of h, we see that $x \in h(A_1)$. By our assumption, $x \in h(A_2)$. So, either $x \in A_2 \cup \{a\}$ or $x \in A_2 - \{a\}$. Using $x \neq a$ again, we have $x \in A_2$.
- If $x = a$, then $a \in A_1$ and so $h(A_1) = A_1 - \{a\}$. So, $a \notin h(A_1)$. By our assumption, $a \notin h(A_2)$. Examining the definition of h, we get $h(A_2) = A_2 - \{a\}$ and $a \in A_2$. Since $x = a$, we have $x \in A_2$.

We have shown that in either case, $x \in A_2$. Thus $A_1 \subseteq A_2$. As observed earlier, this proves that h is injective.

Alternative proof of bijectivity. We could instead appeal to **??**. We will show that h is its own inverse! That is, $h^{-1} = h$. Since h is a function, this means that h^{-1} is a function, which implies that h is a bijection.

To prove that $h^{-1} = h$, we will show that $\forall A \in \mathscr{P}^{\text{odd}}(S)$ we have $h(h(A)) = A$. For this, consider an arbitrary $A \in \mathscr{P}^{\text{odd}}(S)$. Either $a \in A$ or $a \notin A$.

- If $a \in A$, then $h(h(A)) = h(A - \{a\}) = (A - \{a\}) \cup \{a\} = A$.
- If $a \notin A$, then $h(h(A)) = h(A \cup \{a\}) = (A \cup \{a\}) - \{a\} = A$.

In either case, $h(h(A)) = A$, and we are done.

Solution for PS2-6.

a. Suppose that $|A| = m$. Let $a_1, \dots, a_m$ be the elements of A.
Since f is a surjection, $(f(a_1), \dots, f(a_m))$ is a listing of *all* the elements of B, possibly with some repetitions. Therefore $|B| \leq m$.

b. Since g is an injection, the elements in the list $(g(a_1), \dots, g(a_m))$ are *distinct*. Since B contains at least these m distinct elements, $|B| \geq m$.

c. Combining parts (a) and (b), we get $|A| = |B| = m$.
Now, if $(f(a_1), \dots, f(a_m))$ has repetitions, then $|B| < m$, a contradiction. So $f(a_1), \dots, f(a_m)$ are all distinct and hence f is an injection. Therefore, f is a bijection.
Again, if the list $(g(a_1), \dots, g(a_m))$ does not cover all elements of B, then $|B| > m$, a contradiction. Hence, $(g(a_1), \dots, g(a_m))$ is a listing of all elements of B and so g is a surjection. Therefore, g is a bijection.

Solution for PS2-7. Let

$$g(n) = \begin{cases} -2n\,, & \text{if } n \leq 0 \\ 2n - 1\,, & \text{if } n > 0 \end{cases}$$

To verify that $f \circ g = \text{id}_{\mathbb{Z}}$, consider the cases where an input n is positive/negative.

$$\begin{aligned} &\text{For } n \geq 0\,, \quad f(g(n)) = f(2n-1) = (2n-1+1)/2 = n\,. \\ &\text{For } n < 0\,, \quad f(g(n)) = f(-2n) = -2n/2 = n\,. \end{aligned}$$

To verify that $g \circ f = \text{id}_{\mathbb{N}}$, consider the cases where an input m is even/odd.

$$\begin{aligned} \text{When } m \text{ is even,} \quad & g(f(m)) = g(-m/2) = -2(-m)/2 = m\,, \\ \text{When } m \text{ is odd,} \quad & g(f(m)) = g((m+1)/2) = 2 \cdot ((m+1)/2) - 1 = (m+1) - 1 = m\,. \end{aligned}$$

When $f \circ g = \text{id}_{\mathbb{Z}}$ (we say that f has g as a right inverse), f is surjective. This is because for each $x \in \mathbb{Z}$, we have the element $g(x) \in \mathbb{N}$ for which $f(g(x)) = \text{id}_{\mathbb{Z}}(x) = x$.

When $g \circ f = \text{id}_{\mathbb{N}}$ (we say that f has g as a left inverse), f is injective. This is because for all $x, x' \in \mathbb{N}$, if $f(x) = f(x')$, then applying g to both sides, $g(f(x)) = g(f(x'))$, so $x = x'$.

When both conditions hold, f is surjective and injective, hence bijective.

Solution for PS2-8. As $f(A)$ is a subset of $\mathbb{N}$, $f(A)$ is countable. Letting g be f with codomain restricted to $f(A)$, g is an injection (just like f is), and a surjection (since its codomain is its range). As g is a bijection, A is countable iff $g(A)$ is, and since $g(A) = f(A)$ is a subset of $\mathbb{N}$, it follows both $g(A)$ and A are countable.

Solution for PS2-9. Consider $f(x, y) = (x + y + 1)^2 + (x - y)$, which maps pairs in $\mathbb{N} \times \mathbb{N}$ to the set of odd positive integers. It is only necessary to show that f is injective, as then by **??** it would follow $\mathbb{N} \times \mathbb{N}$ is countable.

To do this, assume to the contrary that there are two distinct pairs (x, y) and (x', y') in $\mathbb{N} \times \mathbb{N}$, for which $f(x, y) = f(x', y')$. Expanding the definition of f and rearranging yields $(x+y+1)^2 - (x'+y'+1)^2 = (x-y)-(x'-y')$. Factoring the left hand side gives

$$(x - x' + y - y')(x + x' + y + y' + 2) = (x - x' - y + y'). \tag{1}$$

Since x, x', y, y' are all nonnegative, $(x+x'+y+y'+2) > (x-x'-y+y')$, so that in Eq. 1 either $(x-x'+y-y')$ is zero, or else the left side has a larger absolute value than the right, breaking the equality. Consequently,

$$x - x' + y - y' = 0\,, \text{ and}$$
$$x - x' - y + y' = 0\,.$$

Solving this linear system gives $x = x'$ and $y = y'$, contradicting the initial assumption that $(x, y) \neq (x', y')$.

Solution for PS2-10. Define the *weight* of a finite-length list

$$w((a_1, ..a_\ell)) = \ell + \sum_{i=1}^{\ell} |a_i|\,.$$

There are finitely many lists with a given weight. Since list weights are in $\mathbb{N}$, we can enumerate all elements of $\mathbb{N}^*$ by first listing the elements of weight 0, then those of weight 1, and so on (each element $e \in \mathbb{N}^*$ will be in the $w(e)$th enumerated group). As $\mathbb{N}^*$ can be enumerated, it is countable.