Solutions to Problem Set 1

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PS1-1
Set-builder to roster notation.

a. $\{x : x \text{ is a multiple of 7 and } 0 < x < 50\}$.
Solution. $\{7, 14, 21, 28, 35, 42, 49\}$.

b. $\{x + y : x \in \mathbb{N}, y \in \mathbb{N}, \text{ and } xy = 12\}$.
Solution. $\{7, 8, 13\}$.

c. $\{S : S \subseteq \{1, 2, 3, 4\} \text{ and } |S| \text{ is odd}\}$.
Solution. $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

PS1-2 ^{HW}
More set-builder to roster notation.

a. $\{x^3 : x \in \mathbb{Z} \text{ and } x^2 < 20\}$ [2 points]
Solution. $\{-64, -27, -8, -1, 0, 1, 8, 27, 64\}$.

b. $\{x \in \mathbb{R} : x = x^2\}$. [2 points]
Solution. $\{0, 1\}$.

c. $\{S : \{1, 2\} \subseteq S \subseteq \{1, 2, 3, 4\}\}$ [2 points]
Solution. $\{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$.

d. $\{S \subseteq \{1, 2, 3, 4\} : S \text{ is disjoint from } \{2, 3\}\}$ [2 points]
Solution. $\{\varnothing, \{1\}, \{4\}, \{1, 4\}\}$.

PS1-3
Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{0, 1, 5, 6, 9\}$.

a. What is $A \cup B$? What is $(A \cup B) \cup C$?
Solution. $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$; $(A \cup B) \cup C = \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10\}$.

b. What is $B \cup C$? What is $A \cup (B \cup C)$?
Solution. $B \cup C = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}$; $A \cup (B \cup C) = \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10\}$.

c. What is $A \cap B \cap C$?
Solution. $\{6\}$.

d. Verify by direct computation that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Solution. We computed $A \cup B$ above. Using that, $(A \cup B) \cap C = \{1, 5, 6\}$.
Further, $A \cap C = \{1, 5, 6\}$ and $B \cap C = \{6\}$. So, $(A \cap C) \cup (B \cap C) = \{1, 5, 6\}$.
Hence, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

e. What is $A - B$? What is $B - C$?
Solution. $A - B = \{1, 3, 5\}$; $B - C = \{2, 4, 8, 10\}$.

f. What is $(A - B) - C$? What is $A - (B - C)$?
Solution. $(A - B) - C = \{3\}$; $A - (B - C) = \{1, 3, 5, 6\}$.

g. Verify by direct computation that $(A - B) - C = A - (B \cup C)$.
Solution. We already know $(A - B) - C = \{3\}$.
Further, $B \cup C = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}$ and so, $A - (B \cup C) = \{3\}$.
Hence, $(A - B) - C = A - (B \cup C)$.

h. Verify by direct computation that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.
Solution. We already know $A - (B - C) = \{1, 3, 5, 6\}$.
Further, $A - B = \{1, 3, 5\}$ and $A \cap B \cap C = \{6\}$. So, $(A - B) \cup (A \cap B \cap C) = \{1, 3, 5, 6\}$.
Hence, $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

i. What is $(A \cap B) \times (B - C)$?

Solution. $\{(2,2),(2,4),(2,8),(2,10),\ (4,2),(4,4),(4,8),(4,10),\ (6,2),(6,4),(6,8),(6,10)\}$.

j. Verify by direct computation that $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.

Solution. We already computed $A \cup B \cup C = \{0,1,2,3,4,5,6,8,9,10\}$ above.

Now, $A - B = \{1,3,5\}$; $B - C = \{2,4,8,10\}$; $C - A = \{0,9\}$; $A \cap B \cap C = \{6\}$.

So, $(A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C) = \{0,1,2,3,4,5,6,8,9,10\}$.

Hence, $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.

PS1-4

Let A, B, and C be arbitrary sets. Prove each of the following.

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Solution. Consider an arbitrary element $(x,y) \in A \times (B \cup C)$.

Then $x \in A$ and $y \in B \cup C$, i.e., $y \in B$ or $y \in C$. Thus, $(x,y) \in A \times B$ or $(x,y) \in A \times C$.

So, $(x,y) \in (A \times B) \cup (A \times C)$. Hence, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. ... **(i)**

Again, consider any $(x,y) \in (A \times B) \cup (A \times C)$.

Then $(x,y) \in A \times B$ or $(x,y) \in A \times C$. Thus, $x \in A$ and $y \in B$ or $y \in C$, i.e. $y \in B \cup C$.

So, $(x,y) \in A \times (B \cup C)$. Hence, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. ... **(ii)**

Thus, from **(i)** and **(ii)**, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

b. $(A - C) \cap (C - B) = \varnothing$.

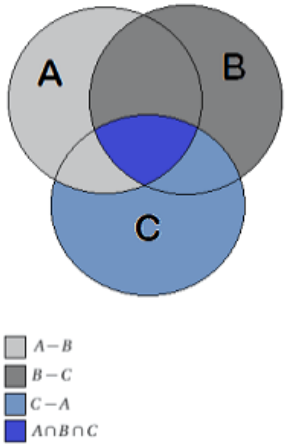
Solution. Consider an arbitrary $x \in A - C$. Then $x \in A$ and $x \notin C$.

Now, $C - B = \{y : y \in C \text{ and } y \notin B\}$. Thus $x \notin C - B$.

Hence, $(A - C) \cap (C - B) = \varnothing$.

c. $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$. ◁ It may help to draw a Venn diagram.

Solution. The following Venn diagram can help us in proving this.



Part (i) Consider any $x \in (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.

Then $x \in A - B$ or $x \in B - C$ or $x \in C - A$ or $x \in A \cap B \cap C$.

In the first three cases, we have (respectively) $x \in A$ or $x \in B$ or $x \in C$. In the fourth case, we have all three things: $x \in A$, $x \in B$, and $x \in C$.

Thus, in any case, at least one of the following holds: $x \in A$, $x \in B$, or $x \in C$. Hence, $x \in A \cup B \cup C =$ LHS.

This proves that RHS $\subseteq$ LHS.

Part (ii) Now consider any $x \in A \cup B \cup C$. Then $x \in A$ or $x \in B$ or $x \in C$.

Case 1. x belongs to all three of A, B, and C.

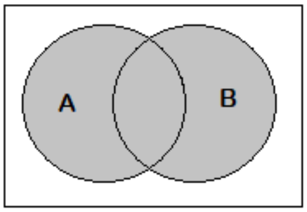
Then, $x \in A \cap B \cap C$. Therefore, $x \in (A-B) \cup (B-C) \cup (C-A) \cup (A \cap B \cap C)$.
Case 2. There is at least one set among A, B, and C to which x does not belong.
Without loss of generality, suppose that $x \notin B$. We now have two subcases.

Case 2.1. x doesn't belong to A.
In this case, $x \notin A$ and $x \notin B$, so we must have $x \in C$. So $x \in C - A$.
Therefore, $x \in (A-B) \cup (B-C) \cup (C-A) \cup (A \cap B \cap C)$.
Case 2.2. x does belong to A.
In this case, $x \in A$ and $x \notin B$, so we must have $x \in A - B$.
Therefore, $x \in (A-B) \cup (B-C) \cup (C-A) \cup (A \cap B \cap C)$.

Thus, in every case, $x \in$ RHS, so LHS $\subseteq$ RHS.

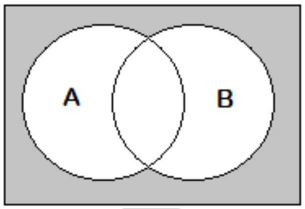
PS1-5 ^{HW}
Proofs of set equalities.

a. Proof that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. [4 points]
Solution.
The Venn diagram for $A \cup B$ is given by the following figure:



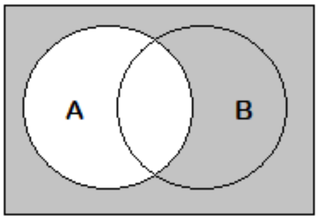
$A \cup B$

Hence, the Venn diagram for $\overline{A \cup B}$ is:

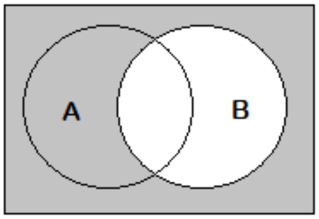


$\overline{A \cup B}$

Again, the Venn diagrams for $\overline{A}$ and $\overline{B}$ are:

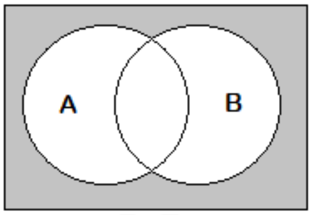


$\overline{A}$



$\overline{B}$

Hence, the Venn diagram for $\overline{A} \cap \overline{B}$ is given by:



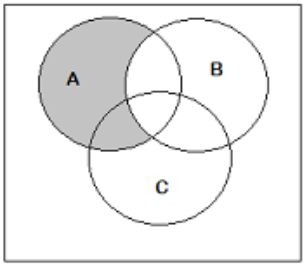
$\overline{A} \cap \overline{B}$.

So we see that the diagrams for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ are the same. Hence $\overline{A \cup B} = \overline{A} \cap \overline{B}$

b. Similarly, prove that $(A - B) - C = (A - C) - (B - C)$. [4 points]

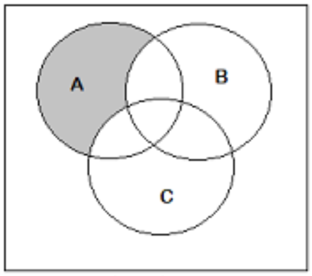
Solution.

The Venn diagram for $A - B$ is given by:



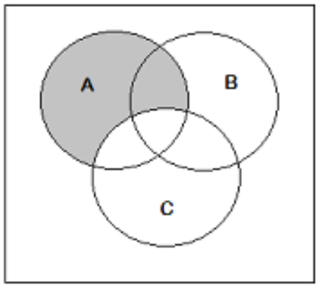
$A - B$

So, the Venn diagram for $(A - B) - C$ is:

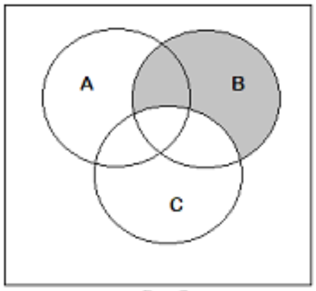


$(A - B) - C$

Again, the Venn diagrams for $A - C$ and $B - C$ are:

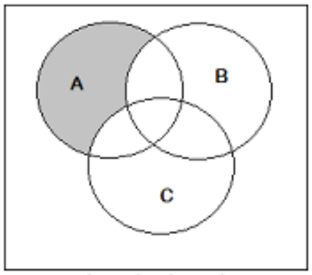


$A - C$



$B - C$

Thus, the Venn diagram for $(A - C) - (B - C)$ is given by:



$(A-C)-(B-C)$

So we see that the Venn diagrams for $(A-B)-C$ and $(A-C)-(B-C)$ are the same. Hence, $(A-B)-C = (A-C)-(B-C)$.

c. Algebra-style proof that $(A \cap B) \cup (A \cap \overline{B}) = A$. [4 points]

Solution. Consider an arbitrary $x \in A$.
If $x \in B$, then $x \in A \cap B$.
Otherwise, $x \notin B$, so $x \in \overline{B}$ and so $x \in A \cap \overline{B}$.
Combining the above two conclusions, $x \in (A \cap B) \cup (A \cap \overline{B})$.
Thus, $A \subseteq (A \cap B) \cup (A \cap \overline{B})$. ... **(i)**

Now consider an arbitrary $x \in (A \cap B) \cup (A \cap \overline{B})$.
Then $x \in A \cap B$ or $x \in A \cap \overline{B}$.
In the former case, $x \in A$ and $x \in B$. In the latter case, $x \in A$ and $x \in \overline{B}$.
In either case, $x \in A$.
Thus, $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$. ... **(ii)**

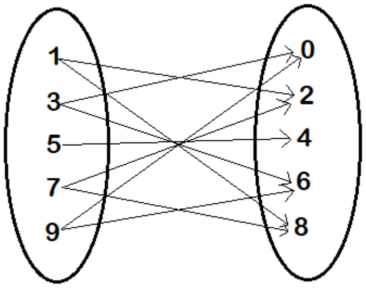
Combining **(i)** and **(ii)**, $(A \cap B) \cup (A \cap \overline{B}) = A$.

PS1-6 ^{HW}

The "completes" relation. [3 points]

Solution. As a set, the relation is $\{(1,2),(1,8),(3,0),(3,6),(5,4),(7,2),(7,8),(9,0),(9,6)\}$.

Below is a pictorial representation.



PS1-7

Are these relations (a) symmetric; (b) transitive?

a. The relation "divides", on $\mathbb{N}$ ("m divides n" means "n/m is an integer").

Solution. This relation is
(a) NOT symmetric [Reason: 1 divides 2, but 2 does not divide 1.]
(b) transitive

b. The relation “is disjoint from”, on $\mathscr{P}(\mathbb{Z})$.

Solution. This relation is
(a) symmetric
(b) NOT transitive [Reason: $\{1\}$ disj from $\{2\}$, and $\{2\}$ disj from $\{1\}$, but $\{1\}$ is not disjoint from $\{1\}$.]

c. The relation “is no larger than”, on $\mathscr{P}(\mathbb{Z})$. We say that A is no larger than B when one of the following holds:

- A and B are both finite sets, and $|A| \le |B|$.
- A is a finite set and B is an infinite set.
- A and B are both infinite sets.

Solution. This relation is:
(a) NOT symmetric [Reason: $(\{1\},\{1,2\}) \in$ “is no larger than”, but $(\{1,2\},\{1\}) \notin$ “is no larger than”.]
(b) transitive

PS1-8 ^{HW}

Same instructions as the previous problem, ***PS1-7***.

a. The relation “is a subset of”, on $\mathscr{P}(\mathbb{Z})$. [4 points]

Solution. This relation is:
(a) NOT symmetric [Reason: $\{1\}$ is a subset of $\{1,2\}$ but $\{1,2\}$ is not a subset of $\{1\}$.]
(b) transitive

b. $\{(m,n) \in \mathbb{N} \times \mathbb{N}$: the sum of the digits of m equals the sum of the digits of $n\}$. [4 points]

Solution. This relation is (a) symmetric, (b) transitive.

c. The relation “overlapped” on the set of all US presidents. [4 points]

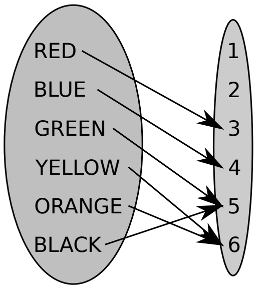
Solution. This relation is:
(a) symmetric
(b) NOT transitive [Reason: George Washington overlapped Thomas Jefferson and Thomas Jefferson overlapped Abraham Lincoln, but Washington did not overlap Lincoln.]

PS1-9

Let $S = \{$“RED”, “BLUE”, “GREEN”, “YELLOW”, “ORANGE”, “BLACK”$\}$ and $T = \{1,2,3,4,5,6\}$. Consider the function len: $S \to T$ given by len(s) = the length of the string s (as in the Python programming language).

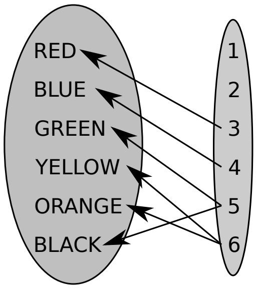
a. Describe the “len” function pictorially, using arrows, as done in class.

Solution.



b. Reverse the directions of all the arrows in your picture. Does this new picture represent a function $g: T \to S$. If not, why not?

Solution.



No: functions associate exactly one value (output) with each argument (input). In the picture above, there are multiple arrows leaving elements 5 and 6. Also, there are no arrows leaving elements 1 and 2.

PS1-10 Prove: if $S_1, S_2 \subseteq A$, then $f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$.

Solution. Consider an arbitrary element $y \in f(S_1 \cup S_2)$.

Then $y = f(x)$ for some $x \in S_1 \cup S_2$, i.e., $x \in S_1$ or $x \in S_2$. In the former case, $y = f(x) \in f(S_1)$. In the latter case, $y = f(x) \in f(S_2)$. Overall, $y \in f(S_1) \cup f(S_2)$. Thus, LHS $\subseteq$ RHS.

Next, consider an arbitrary element $y \in f(S_1) \cup f(S_2)$. Then $y \in f(S_1)$ or $y \in f(S_2)$.

In the former case, $y = f(x)$ for some $x \in S_1 \subseteq S_1 \cup S_2$. In the latter case, $y = f(x)$ for some $x \in S_2 \subseteq S_1 \cup S_2$. Thus, in each case, $y \in f(S_1 \cup S_2)$. Thus, RHS $\subseteq$ LHS.

PS1-11

The functions $f, g \colon \mathbb{R} \to \mathbb{R}$ are given by the formulas $f(x) = x^2 + 1$ and $g(x) = x + 2$. Find $f \circ g$ and $g \circ f$.

Solution. $(f \circ g)(x) = x^2 + 4x + 5; \quad (g \circ f)(x) = x^2 + 3$

PS1-12 ^{HW}

The functions $f, \mathrm{id} \colon \mathbb{R} \to \mathbb{R}$ are given by the formulas $f(x) = x^3 + 7$ and $\mathrm{id}(x) = x$.

a. Find a function $g \colon \mathbb{R} \to \mathbb{R}$ such that $f \circ g = \mathrm{id}$. [2 points]

Solution. Pick an arbitrary $x \in \mathbb{R}$ and let $y = g(x)$.
Then $f(y) = f(g(x)) = x$, since $f \circ g = \mathrm{id}$. Thus, $y^3 + 7 = x$, which implies $y = \sqrt[3]{x-7}$.
We conclude that the required function $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = \sqrt[3]{x-7}$.
(Note that this g is a well-defined function from $\mathbb{R}$ to $\mathbb{R}$ since every real number has a unique real cube root.)

b. For the function g you found above, find $g \circ f$. [2 points]

Solution. We compute: $(g \circ f)(x) = g(f(x)) = \sqrt[3]{f(x) - 7} = \sqrt[3]{x^3 + 7 - 7} = x$.
Hence, $g \circ f = \mathrm{id}$.

CS 30
Fall 2019
Discrete Mathematics

Solutions for Unit 2

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Solution for PS2-1.

a. $\forall y \in B$, because f is surjective, $\exists x \in A$, such that $f(x) = y$, i.e. $(y, x) \in f^{-1}$. And because f is injective, there is only one $x \in A$, such that $f(x) = y$, i.e. $(y, x) \in f^{-1}$. In conclusion, $\forall y \in B$, there is one and only one $x \in A$, such that $(y, x) \in f^{-1}$. Therefore, f^{-1} is a function.

b. **First,** we will prove that f is surjective. That is, we will prove $\forall y \in B\ \exists x \in A\ (f(x) = y)$.
Consider an arbitrary $y \in B$.
Since f^{-1} is a function, according to the definition of a function, $\exists x \in A\ ((y, x) \in f^{-1})$.
By the definition of the relation f^{-1}, this means $f(x) = y$.

Second, we will prove that f is injective. That is, we will prove $\forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \implies x_1 = x_2)$.
Consider arbitrary elements $x_1, x_2 \in A$. Suppose that $f(x_1) = f(x_2)$. We will now show that $x_1 = x_2$.
Define $y = f(x_1) = f(x_2)$.
By the definition of the relation f^{-1}, we have $(y, x_1) \in f^{-1}$ and $(y, x_2) \in f^{-1}$.
Since f^{-1} is a function, for each y there must be at most one x so that $(y, x) \in f^{-1}$. It follows that $x_1 = x_2$.

Solution for PS2-2.

a. Consider an arbitrary element $y \in f(S_1 \cup S_2)$. Then $y = f(x)$ for some $x \in S_1 \cup S_2$, i.e. $x \in S_1$ or $x \in S_2$. If $x \in S_1$, $f(x) = y \in f(S_1)$, or, if $x \in S_2$, $f(x) = y \in f(S_2)$. So, in any case, $y \in f(S_1)$ or $y \in f(S_2)$. Hence $y \in f(S_1) \cup f(S_2)$. Thus,$f(S_1 \cup S_2) \subseteq f(S_1) \cup f(S_2) \cdots$

Again, consider an arbitrary element $y \in f(S_1) \cup f(S_2)$. Then $y \in f(S_1)$ or $y \in f(S_2)$. If $y \in f(S_1)$, then $y = f(x)$ for some $x \in S_1$, or, if $y \in f(S_2)$, then $y = f(x)$ for some $x \in S_2$. So, in any case, $y = f(x)$ for some $x \in S_1$ or $x \in S_2$, i.e. $x \in S_1 \cup S_2$. Hence, $y \in f(S_1 \cup S_2)$. Thus, $f(S_1) \cup f(S_2) \subseteq f(S_1 \cup S_2) \cdots$.

b. Consider an arbitrary element $x \in f^{-1}(T_1 \cup T_2)$. Then $f(x) \in T_1 \cup T_2$, i.e. $f(x) \in T_1$ or $f(x) \in T_2$. If $f(x) \in T_1$, $x \in f^{-1}(T_1)$, or, if $f(x) \in T_2$, $x \in f^{-1}(T_2)$. So, in any case, $x \in f^{-1}(T_1)$ or $x \in f^{-1}(T_2)$ i.e. $x \in f^{-1}(T_1) \cup f^{-1}(T_2)$. Thus, $f^{-1}(T_1 \cup T_2) \subseteq f^{-1}(T_1) \cup f^{-1}(T_2) \cdots$.

Again, consider an arbitrary element $x \in f^{-1}(T_1) \cup f^{-1}(T_2)$. Then $x \in f^{-1}(T_1)$ or $x \in f^{-1}(T_2)$. If $x \in f^{-1}(T_1)$, then $f(x) \in T_1$, or, if $x \in f^{-1}(T_2)$, then $f(x) \in T_2$. So, in any case, $f(x) \in T_1$ or $f(x) \in T_2$, i.e. $f(x) \in T_1 \cup T_2$. Hence, $x \in f^{-1}(T_1 \cup T_2)$. Thus, $f^{-1}(T_1) \cup f^{-1}(T_2) \subseteq f^{-1}(T_1 \cup T_2) \cdots$.

Solution for PS2-3.

a. To prove $f \circ g$ is injective, assume x_1, x_2 are such that $f(g(x_1)) = f(g(x_2))$; since f is injective, it follows $g(x_1) = g(x_2)$, and then since g is injective, $x_1 = x_2$.
To prove surjectivity, consider any $x \in C$; then since f is surjective, $\exists\, y \in B$ s.t. $f(y) = x$. Next, since g is surjective, $\exists\, z \in A$ s.t. $g(z) = y$. Overall, we've found an element $z \in A$ s.t. $(f \circ g)(z) = f(g(z)) = x$.

b. To show that $g^{-1} \circ f^{-1}$ is the inverse of $f \circ g$ (already knowing that such an inverse exists), it suffices to verify that $(f \circ g)((g^{-1} \circ f^{-1})(x)) = x$ for all $x \in C$, i.e., that $g^{-1} \circ f^{-1}$ behaves identically to the actual inverse on its domain. Verification follows since

$$(f \circ g)((g^{-1} \circ f^{-1})(x)) = f(g(g^{-1}(f^{-1}(x)))) = f(f^{-1}(x)) = x\,.$$

Solution for PS2-4.

To solve this problem, it will help to prove a little lemma first.

Lemma 1. *Let S and C be sets with $C \subseteq S$. Then $S - (S - C) = C$.*

Proof. Using the definitions of the basic set operations,

$$\begin{aligned}
S - (S - C) &= \{x : x \in S \wedge x \notin S - C\} && \triangleright \text{ definition of set difference} \\
&= \{x : x \in S \wedge \neg(x \in S \wedge x \notin C)\} && \triangleright \text{ definition of set difference} \\
&= \{x : x \in S \wedge (x \notin S \vee x \in C)\} && \triangleright \text{ de Morgan's law} \\
&= \{x : x \in S \wedge x \in C\} && \\
&= S \cap C && \triangleright \text{ definition of intersection} \\
&= C\,. && \triangleright \text{ because } C \subseteq S \qquad \square
\end{aligned}$$

First, we will prove that g is surjective. That is, we will prove $\forall B \subseteq S\ \exists A \subseteq S\ (g(A) = B)$.

Consider an arbitrary $B \subseteq S$.

Define $A := S - B$. Then, by Lemma 1, $g(A) = S - (S - B) = B$.

Thus, we have proved the existence of an A such that $g(A) = B$.

Second, we will prove that g is injective. That is, we will prove $\forall A_1, A_2 \subseteq S\ (g(A_1) = g(A_2) \implies A_1 = A_2)$.

Consider arbitrary $A_1, A_2 \subseteq S$. Suppose that $g(A_1) = g(A_2)$. We will now show that $A_1 = A_2$.

We have

$$\begin{aligned}
A_1 &= S - (S - A_1) && \triangleright \text{ by Lemma 1} \\
&= S - g(A_1) && \triangleright \text{ definition of } g \\
&= S - g(A_2) && \triangleright \text{ by our assumption} \\
&= S - (S - A_2) && \triangleright \text{ definition of } g \\
&= A_2\,. && \triangleright \text{ by Lemma 1}
\end{aligned}$$

Solution for PS2-5. Because S is nonempty, $\exists a \in S$. We construct h as follows:

$$h(A) = \begin{cases} A \cup \{a\}\,, & \text{if } a \notin A \\ A - \{a\}\,, & \text{if } a \in A\,. \end{cases}$$

Note that $|h(A)| = |A| \pm 1$. So, if $A \in \mathscr{P}^{\text{odd}}(S)$, then $h(A) \in \mathscr{P}^{\text{even}}(S)$. Thus, h is indeed a *function* of the form $h \colon \mathscr{P}^{\text{odd}}(S) \to \mathscr{P}^{\text{even}}(S)$.

Now, we prove that h is a *bijection*. As usual, the proof has two parts.

First, we prove that h is surjective.

Consider an arbitrary $B \in \mathscr{P}^{\text{even}}(S)$. Either $a \in B$ or $a \notin B$.

If $a \in B$, then $B - \{a\} \in \mathscr{P}^{\text{odd}}(S)$ and $h(B - \{a\}) = (B - \{a\}) \cup \{a\} = B$.

If $a \notin B$, then $B \cup \{a\} \in \mathscr{P}^{\text{odd}}(S)$ and $h(B \cup \{a\}) = (B \cup \{a\}) - \{a\} = B$.

We have shown that in either case, $\exists A \in \mathscr{P}^{\text{odd}}(S)$ such that $h(A) = B$. Therefore, h is surjective.

Second, we prove that h is injective.

Consider arbitrary sets $A_1, A_2 \in \mathscr{P}^{\text{odd}}(S)$ and suppose that $h(A_1) = h(A_2)$. We will prove that $A_1 = A_2$.

For this, we will show that $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$. Actually, it suffices to prove the first of these; the second then follows by symmetry.

So, consider an arbitrary $x \in A_1$. Either $x \neq a$ or $x = a$.

- If $x \neq a$, then $x \in A_1 \cup \{a\}$ and $x \in A_1 - \{a\}$. Examining the definition of h, we see that $x \in h(A_1)$. By our assumption, $x \in h(A_2)$. So, either $x \in A_2 \cup \{a\}$ or $x \in A_2 - \{a\}$. Using $x \neq a$ again, we have $x \in A_2$.
- If $x = a$, then $a \in A_1$ and so $h(A_1) = A_1 - \{a\}$. So, $a \notin h(A_1)$. By our assumption, $a \notin h(A_2)$. Examining the definition of h, we get $h(A_2) = A_2 - \{a\}$ and $a \in A_2$. Since $x = a$, we have $x \in A_2$.

We have shown that in either case, $x \in A_2$. Thus $A_1 \subseteq A_2$. As observed earlier, this proves that h is injective.

Alternative proof of bijectivity. We could instead appeal to **??**. We will show that h is its own inverse! That is, $h^{-1} = h$. Since h is a function, this means that h^{-1} is a function, which implies that h is a bijection.

To prove that $h^{-1} = h$, we will show that $\forall A \in \mathscr{P}^{\text{odd}}(S)$ we have $h(h(A)) = A$. For this, consider an arbitrary $A \in \mathscr{P}^{\text{odd}}(S)$. Either $a \in A$ or $a \notin A$.

- If $a \in A$, then $h(h(A)) = h(A - \{a\}) = (A - \{a\}) \cup \{a\} = A$.
- If $a \notin A$, then $h(h(A)) = h(A \cup \{a\}) = (A \cup \{a\}) - \{a\} = A$.

In either case, $h(h(A)) = A$, and we are done.

Solution for PS2-6.

a. Suppose that $|A| = m$. Let $a_1, \ldots, a_m$ be the elements of A.
Since f is a surjection, $(f(a_1), \ldots, f(a_m))$ is a listing of *all* the elements of B, possibly with some repetitions. Therefore $|B| \leq m$.

b. Since g is an injection, the elements in the list $(g(a_1), \ldots, g(a_m))$ are *distinct*. Since B contains at least these m distinct elements, $|B| \geq m$.

c. Combining parts (a) and (b), we get $|A| = |B| = m$.
Now, if $(f(a_1), \ldots, f(a_m))$ has repetitions, then $|B| < m$, a contradiction. So $f(a_1), \ldots, f(a_m)$ are all distinct and hence f is an injection. Therefore, f is a bijection.
Again, if the list $(g(a_1), \ldots, g(a_m))$ does not cover all elements of B, then $|B| > m$, a contradiction. Hence, $(g(a_1), \ldots, g(a_m))$ is a listing of all elements of B and so g is a surjection. Therefore, g is a bijection.

Solution for PS2-7. Let

$$g(n) = \begin{cases} -2n\,, & \text{if } n \leq 0 \\ 2n - 1\,, & \text{if } n > 0 \end{cases}$$

To verify that $f \circ g = \mathrm{id}_{\mathbb{Z}}$, consider the cases where an input n is positive/negative.

$$\begin{aligned} \text{For } n \geq 0\,, \quad & f(g(n)) = f(2n-1) = (2n-1+1)/2 = n\,. \\ \text{For } n < 0\,, \quad & f(g(n)) = f(-2n) = -2n/2 = n\,. \end{aligned}$$

To verify that $g \circ f = \mathrm{id}_{\mathbb{N}}$, consider the cases where an input m is even/odd.

$$\begin{aligned} \text{When } m \text{ is even,} \quad & g(f(m)) = g(-m/2) = -2(-m)/2 = m\,, \\ \text{When } m \text{ is odd,} \quad & g(f(m)) = g((m+1)/2) = 2 \cdot ((m+1)/2) - 1 = (m+1) - 1 = m\,. \end{aligned}$$

When $f \circ g = \mathrm{id}_{\mathbb{Z}}$ (we say that f has g as a right inverse), f is surjective. This is because for each $x \in \mathbb{Z}$, we have the element $g(x) \in \mathbb{N}$ for which $f(g(x)) = \mathrm{id}_{\mathbb{Z}}(x) = x$.

When $g \circ f = \mathrm{id}_{\mathbb{N}}$ (we say that f has g as a left inverse), f is injective. This is because for all $x, x' \in \mathbb{N}$, if $f(x) = f(x')$, then applying g to both sides, $g(f(x)) = g(f(x'))$, so $x = x'$.

When both conditions hold, f is surjective and injective, hence bijective.

Solution for PS2-8. As $f(A)$ is a subset of $\mathbb{N}$, $f(A)$ is countable. Letting g be f with codomain restricted to $f(A)$, g is an injection (just like f is), and a surjection (since its codomain is its range). As g is a bijection, A is countable iff $g(A)$ is, and since $g(A) = f(A)$ is a subset of $\mathbb{N}$, it follows both $g(A)$ and A are countable.

Solution for PS2-9. Consider $f(x, y) = (x + y + 1)^2 + (x - y)$, which maps pairs in $\mathbb{N} \times \mathbb{N}$ to the set of odd positive integers. It is only necessary to show that f is injective, as then by **??** it would follow $\mathbb{N} \times \mathbb{N}$ is countable.

To do this, assume to the contrary that there are two distinct pairs (x, y) and (x', y') in $\mathbb{N}\times\mathbb{N}$, for which $f(x, y) = f(x', y')$. Expanding the definition of f and rearranging yields $(x+y+1)^2-(x'+y'+1)^2 = (x-y)-(x'-y')$. Factoring the left hand side gives

$$(x - x' + y - y')(x + x' + y + y' + 2) = (x - x' - y + y'). \qquad (1)$$

Since x, x', y, y' are all nonnegative, $(x+x'+y+y'+2) > (x-x'-y+y')$, so that in Eq. 1 either $(x-x'+y-y')$ is zero, or else the left side has a larger absolute value than the right, breaking the equality. Consequently,

$$\begin{aligned} x - x' + y - y' &= 0\,, \text{ and} \\ x - x' - y + y' &= 0\,. \end{aligned}$$

Solving this linear system gives $x = x'$ and $y = y'$, contradicting the initial assumption that $(x, y) \neq (x', y')$.

Solution for PS2-10. Define the *weight* of a finite-length list

$$w((a_1, ..a_\ell)) = \ell + \sum_{i=1}^{\ell} |a_i|\,.$$

There are finitely many lists with a given weight. Since list weights are in $\mathbb{N}$, we can enumerate all elements of $\mathbb{N}^*$ by first listing the elements of weight 0, then those of weight 1, and so on (each element $e \in \mathbb{N}^*$ will be in the $w(e)$th enumerated group). As $\mathbb{N}^*$ can be enumerated, it is countable.

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Solutions for Unit 3

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Solution for PS3-1.

a. $(2m)+(2n) = 2(m+n)$, which is even.

b. $(2m+1)+(2n+1) = 2(m+n+1)$, which is even.

c. $(2m+1)+(2n) = 2(m+n)+1$, which is odd.

d. $(2m)\cdot(n) = 2(mn)$, which is even.

e. $(2m+1)\cdot(2n+1) = 2(2mn+m+n)+1$, which is odd.

Solution for PS3-2.

a. Since $3^b = 3^d$, we can simplify the equation to $2^a = 2^c$, which makes $a = c$. Thus, $(a,b) = (c,d)$.

b. Rearranging the equation, $2^{a-c} = 3^{d-b}$. So let $p = a-c, q = d-b$. By assumption, $q > 0$, so $q \in \mathbb{N}$.

c. Each time we multiply an odd integer by three, the result must be odd. Since 3^q is obtained by starting with one (an odd integer) and multiplying by three q times, the final result must be odd.

d. We have that $2^p = 3^q$, which is odd. If $p < 0$, $2p$ is not even an integer. If $p > 0$, $2^p = 2\times 2^{p-1}$, which is two times an integer, so it is even. The only way that 2^p can be odd is if $p = 0$.

e. Since $p = a-c = 0$, we have $a = c$. Thus, $2^a = 2^c$ and the original equation simplifies to $3^b = 3^d$, which implies $b = d$. Thus, $(a,b) = (c,d)$.

f. We have shown that $f(a,b) = f(c,d) \implies (a,b) = (c,d)$, so by definition, f is injective.

Solution for PS3-3. If A is countable, then we know there exists and injection $g : A \to \mathbb{N}$ by the definition of countability. If B is a subset of A we can define a function $f : B \to \mathbb{N}$ with $f(x) = g(x)$. To prove that B is countable we will show that f is an injection.

Let x_1 and x_2 be any two elements of B. If $f(x_1) = f(x_2)$, then $g(x_1) = g(x_2)$. But g is injective, so this implies $x_1 = x_2$. Therefore f is also injective.

Solution for PS3-4.

a. If A and B are both countable then there are injections $g : A \to \mathbb{N}$ and $h : B \to \mathbb{N}$. We can define a function $f : (A\cup B) \to \mathbb{N}$ by the following:

$$f(x) = \begin{cases} 2g(x)+1\,, & \text{if } x \in A\,, \\ 2h(x)\,, & \text{otherwise (i.e., } x \in (B-A))\,. \end{cases}$$

We can show this function is an injection. Let x_1, x_2 be elements of $A\cup B$ with $f(x_1) = f(x_2)$. Either $f(x_1)$ will be odd, or it will be even.

If $f(x_1)$ is odd, then $f(x_1) = 2g(x_1)+1$ (note that $2h(x)$ can never be odd because $h(x)$ is a natural number). Similarly, $f(x_2)$ is odd (it is equal to $f(x_1$ after all) so we have $f(x_2) = 2g(x_2)+1$. This means $2g(x_1)+1 = 2g(x_2)+1$ which implies $g(x_1) = g(x_2)$. But we know g is an injective function, so $x_1 = x_2$.

If $f(x_1)$ is even, then $f(x_1) = 2h(x_1)$ (again note that $2g(x)+1$ can never be even because $g(x)$ is a natural number). We also know$f(x_2$ is even so it equals $2h(x_2)$, therefore $2h(x_1) = 2h(x_2)$. This in turn implies $h(x_1) = h(x_2)$. We know h is an injection so $x_1 = x_2$.

So in either case we know if $f(x_1) = f(x_2)$ then $x_1 = x_2$. Therefore f is an injective function from $A\cup B$ to $\mathbb{N}$ and there $A\cup B$ must be countable.

b. If A and B are both countable then there are injections $g : A \to \mathbb{N}$ and $h : B \to \mathbb{N}$. As in part a, we can define a function $f : (A\times B) \to \mathbb{N}$ by $f(a,b) = 2^{g(a)}3^{h(b)}$. As before we can prove this is an injection.

Let $f(a_1,b_1) = f(a_2,b_2)$, so $2^{g(a_1)}3^{h(b_1)} = 2^{g(a_2)}3^{h(b_2)}$. From the uniqueness of prime factorization we know that this implies $g(a_1) = g(a_2)$ and $h(a_1) = h(a_2)$. But g and h are both injections so $a_1 = a_2$ and $b_1 = b_2$. This means the ordered pairs (a_1,b_1) and (a_2,b_2) are equal to each other. Therefore f is an injective function and $A\times B$ is countable.

Solution for PS3-5. Let's repeatedly invoke the fact "A and B countable $\implies A \times B$ countable".

Using it with $A = \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N}$ is countable.

Now, using it with $A = \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

Next, using it with $A = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

Finally, using it with $A = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

Solution for PS3-6. Since A is infinite, there exists some element $a_0 \in A$.

Since A is infinite, it has at least two elements, so there exists some element $a_1 \in A$ distinct from a_0.

Since A is infinite, it has at least three elements, so there exists some element $a_3 \in A$ distinct from a_0, a_1.

Since A is infinite, it has at least four elements, so there exists some element $a_4 \in A$ distinct from a_0, a_1, a_2.

Proceeding in this fashion, for each $n \in \mathbb{N}$ we have an element a_n distinct from all elements a_m where $m < n$. Now define a function $f : A \to \mathbb{N}$ as follows.

$$f(x) = \begin{cases} n\,, & \text{if } x = a_n \text{ for some } n \in \mathbb{N}\,, \\ 0\,, & \text{otherwise (i.e., } x \text{ is not in the list } (a_0, a_1, a_2, \ldots))\,. \end{cases}$$

This function is surjective because, given any $n \in \mathbb{N}$, there exists the element $a_n \in A$ for which $f(a_n) = n$.

Solution for PS3-7. Since S is finite, given any particular length ℓ, there are only a finite number of ℓ-length strings in S^*. (To be precise, there are $|S|^\ell$ such strings, though we don't need this fact.) Therefore, we can list all the elements of S^* as follows:

> empty string, followed by
> all strings of length 1 in some arbitrary order, followed by
> all strings of length 2 in some arbitrary order, followed by
> all strings of length 3 in some arbitrary order, followed by ......

This listing implicitly defines a bijection $f : \mathbb{N} \to S^*$, proving that S^* is countable.

Solution for PS3-8. Every Python program is just a string over a certain alphabet: say, the alphabet of all Unicode characters.

Thus, the set of all Python programs is a subset of a countable set, so it is itself a countable set.

Solution for PS3-9. Let $I = (0, 1)$. We define $f(x, y)$ as follows, for $(x, y) \in I \times I$. Let

$$x = 0.a_1a_2a_3\cdots\,,$$
$$y = 0.b_1b_2b_3\cdots$$

be the unique decimal representations of x and y, as defined in class. Now construct the number

$$z = 0.a_1b_1a_2b_2a_3b_3\cdots\,.$$

The sequence of digits in this definition of z has infinitely many non-9s, so it is a legit decimal representation of a real number in I. Set $f(x, y) = z$.

Students: You should write up a formal proof that f is indeed an injection.

Solution for PS3-10. Consider the function $g : (0, 1] \to (0, 1)$ defined by:

$$g(x) = \begin{cases} \frac{x}{2}\,, & \text{if } \exists n \in \mathbb{Z} \text{ such that } x = 2^{-n}\,, \\ x\,, & \text{otherwise}\,. \end{cases}$$

To prove this is a bijection we will construct an inverse function. Define $h(x) : (0,1) \to (0,1]$ to be:

$$h(x) = \begin{cases} 2x\,, & \text{if } \exists n \in \mathbb{Z} \text{ such that } x = 2^{-n}\,, \\ x\,, & \text{otherwise}\,. \end{cases}$$

To prove that g and h are inverses of each other (which, in turn, shows that g is a bijection) we must show that $g \circ h = \mathrm{id}_{(0,1)}$ and $h \circ g = \mathrm{id}_{(0,1]}$.

To show $g \circ h = \mathrm{id}_{(0,1)}$, let x be any element of $(0,1)$. We must show $g(h(x)) = x$. If $x = 2^{-n}$ for some integer n, then $g(h(x)) = g(2x)$, but $2x = 2^{1-n}$ and $(n-1)$ is an integer, so $g(2x) = \frac{2x}{2} = x$. If instead $x \neq 2^{-n}$ for every integer n, we have $g(h(x)) = g(x) = x$. In either case $g(h(x)) = x$ so $g \circ h = \mathrm{id}_{(0,1)}$.

Similarly To show $h \circ g = \mathrm{id}_{(0,1]}$, we let $x \in (0,1]$. Once again we have two cases either $x = 2^{-n}$ for some integer n, or $x \neq 2^{-n}$ for every integer n. In the former case, $h(g(x)) = h(\frac{x}{2})$. However $\frac{x}{2} = 2^{-(n+1)}$ and $(n+1)$ is an integer so $h(\frac{x}{2}) = 2\frac{x}{2} = x$. In the latter case $h(g(x)) = h(x) = x$. Therefore $h(g(x))$ is always x and $h \circ g = \mathrm{id}_{(0,1]}$

So g and h are inverse functions of each other. Because g is a function with an inverse function it must be a bijection.

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Solutions for Unit 4

Solution for PS4-1.

Number	Positive divisors	Prime?
12	$1, 2, 3, 4, 6, 12$	No
15	$1, 3, 5, 15$	No
29	$1, 29$	Yes
64	$1, 2, 4, 8, 16, 32, 64$	No
72	$1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72$	No
73	$1, 73$	Yes
75	$1, 3, 5, 15, 25, 75$	No

Solution for PS4-2. We claim that $n \in \mathbb{N}^+$ has an odd number of positive divisors iff n is a perfect square. Informally, this is because the positive divisors of n can be grouped into pairs of the form $\{d, n/d\}$, so there must be an even number of such divisors in total *except* when one of these "pairs" is in fact a singleton set, which happens when $d = n/d$ for some d, i.e., $n = d^2$.

Here is the formal proof. Let

$$\begin{aligned} A &= \{d \in \mathbb{N}^+ : d \mid n \text{ and } d < \sqrt{n}\}\,, \\ B &= \{d \in \mathbb{N}^+ : d \mid n \text{ and } d > \sqrt{n}\}\,, \\ D &= \{d \in \mathbb{N}^+ : d \mid n\}\,. \end{aligned}$$

The function $f : A \to B$ given by $f(a) = n/a$ is a bijection from A to B, because its inverse is the function $g : B \to A$ given by $g(b) = n/b$. Therefore, $|A| = |B|$. If $\sqrt{n}$ is not an integer, then $D = A \cup B$, so $|D| = |A| + |B| = 2|A|$, which is even. If $\sqrt{n}$ is an integer, then $D = A \cup B \cup \{\sqrt{n}\}$, so $|D| = |A| + |B| + 1 = 2|A| + 1$, which is odd. This completes the proof of our claim.

Using our claim, the desired set of integers is $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$.

Solution for PS4-3. For $\mathbb{Z}_{11}$:

$\otimes$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

For $\mathbb{Z}_{12}$:

$\otimes$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	1	6	11	4	9	2	7
6	0	6	0	6	0	6	0	6	0	6	0	6
7	0	7	2	9	4	11	6	1	8	3	10	5
8	0	8	4	0	8	4	0	8	4	0	8	4
9	0	9	6	3	0	9	6	3	0	9	6	3
10	0	10	8	6	4	2	0	10	8	6	4	2
11	0	11	10	9	8	7	6	5	4	3	2	1

Observe that in the table for $\mathbb{Z}_{11}$, the zero entries are confined to the zero row and the zero column, whereas in the table for $\mathbb{Z}_{12}$, there are some zero entries in addition to these "obvious" ones. In fact, for each $x \in \mathbb{Z}_{12}$ that is not relatively prime to 12, there is at least one zero entry in the row/column for x.

Solution for PS4-4. Using the definition of congruence, $d \mid a-b$ and $d \mid x-y$.

Therefore, $d \mid (a-b)+(x-y) = (a+x)-(b+y)$, i.e., $a+x \equiv b+y \pmod{d}$.

Further, $d \mid (a-b)x$ and $d \mid b(x-y)$. Therefore, $d \mid (a-b)x + b(x-y) = ax - by$, i.e., $ax \equiv by \pmod{d}$.

Solution for PS4-5. Using the result of ***PS4-4***, if $a \equiv b \pmod{d}$, then $a^2 \equiv b^2 \pmod{d}$. Using the same result again, $a^3 \equiv b^3 \pmod{d}$. Proceeding in this way, $a^n \equiv b^n \pmod{d}$ for all $n \in \mathbb{N}^+$.

Now take $d = a-b$. Trivially, $d \mid a-b$, so $a \equiv b \pmod{d}$. Therefore, $a^n \equiv b^n \pmod{d}$, which means $d \mid a^n - b^n$.

Solution for PS4-6. Take $d = a+b$. Then $a \equiv -b \pmod{d}$.

As in ***PS4-5***, $a^n \equiv (-b)^n \pmod{d}$. Since n is odd, $(-1)^n = -1$. Therefore, $(-b)^n = (-1)^n b^n = -b^n$. We conclude that $a^n \equiv -b^n \pmod{d}$, so $d \mid a^n + b^n$.

Solution for PS4-7. Start by observing that $2^4 = 16 \equiv -1 \pmod{17}$. Therefore, powers of 2^4 are going to be easy to figure out. We use this to simplify:

$$2^{2019} = 2^{4\times 504+3} = 16^{504} \times 2^3 \equiv (-1)^{504} \times 8 = 8 \pmod{17}.$$

Solution for PS4-8. Let n^2 be a perfect square. To reason about the last digit of n^2, we consider arithmetic modulo 10. To reduce the number of cases to consider, we further use $9 \equiv -1 \pmod{10}$ and so on.

- If $n \equiv 0 \pmod{10}$, then $n^2 \equiv 0 \pmod{10}$.
- If $n \equiv \pm 1 \pmod{10}$, then $n^2 \equiv 1 \pmod{10}$.
- If $n \equiv \pm 2 \pmod{10}$, then $n^2 \equiv 4 \pmod{10}$.
- If $n \equiv \pm 3 \pmod{10}$, then $n^2 \equiv 9 \pmod{10}$.
- If $n \equiv \pm 4 \pmod{10}$, then $n^2 \equiv 16 \equiv 6 \pmod{10}$.
- If $n \equiv 5 \pmod{10}$, then $n^2 \equiv 25 \equiv 5 \pmod{10}$.

It follows that $n^2 \not\equiv 7 \pmod{10}$.

Alternate Solution for PS4-8. Let n^2 be a perfect square. If the last digit of n^2 is 7, then $n^2 \equiv 2 \pmod{5}$. However, the following exhaustive list of cases shows that this is not possible.

- If $n \equiv 0 \pmod 5$, then $n^2 \equiv 0 \pmod 5$.
- If $n \equiv \pm 1 \pmod 5$, then $n^2 \equiv 1 \pmod 5$.
- If $n \equiv \pm 2 \pmod 5$, then $n^2 \equiv 4 \pmod 5$.

Solution for PS4-9. Given that decimal representation, the value of n is given by

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^2 a_2 + 10 a_1 + a_0 = \sum_{j=0}^{k} 10^j a_j \,.$$

Now $10 \equiv 1 \pmod 3$, therefore $10^j \equiv 1^j = 1 \pmod 3$ for each j. Using this in the above equation, we obtain $n \equiv \sum_{j=0}^{k} a_j \pmod 3$, as desired.

Solution for PS4-10. Let $n = m(m+1)(m+2)$ be the product of three consecutive integers. Then, modulo 3, the integers m, $m+1$, and $m+1$ are congruent to 0, 1, and 2 in some order. Therefore $n \equiv 0 \times 1 \times 2 = 0 \pmod 3$, i.e., $n = 3k$ for some integer k.

At least one of m, $m+1$, and $m+2$ is even. Therefore, n is even. If k were odd, then $n = 3k$ would have been odd. Therefore, k is even. Let $k = 2\ell$ for some integer ℓ. Then $n = 3 \cdot 2\ell = 6\ell$, which is divisible by 6.

Note: It's not enough to say that n is divisible by 2 and by 3, *therefore* it is divisible by 6. That "therefore" needs to be justified.

Solutions for Unit 5

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Solution for PS5-1.

a. By repeated application of “Theorem 6” from the lecture notes,

$$\begin{aligned}\gcd(13631, 8213) &= \gcd(8213, 13631 \bmod 8213)\\ &= \gcd(5418, 8213 \bmod 5418)\\ &= \gcd(2795, 5418 \bmod 2795)\\ &= \gcd(2623, 2795 \bmod 2623)\\ &= \gcd(172, 2623 \bmod 172)\\ &= \gcd(43, 172 \bmod 43)\\ &= \gcd(43, 0) = 43\,.\end{aligned}$$

b. Using the Extended Euclidean Algorithm,

$$\begin{aligned}43 &= 1 \cdot 43 + 0 \cdot 0 && \text{Base case, } k_0 = 1;\ \ell_0 = 0\\ &= 0 \cdot 172 + 1 \cdot 43 && k_1 = \ell_1 = 0;\ \ell_1 = k_0 - \lfloor a_1/b_1 \rfloor \cdot \ell_0 = 1 - \lfloor 172/43 \rfloor \cdot 0\\ &= 1 \cdot 2623 - 15 \cdot 172 && k_2 = 1;\ \ell_2 = 0 - \lfloor 2623/172 \rfloor \cdot 1\\ &= -15 \cdot 2795 + 16 \cdot 2623 && k_3 = -15;\ \ell_3 = 1 - \lfloor 2795/2623 \rfloor \cdot (-15)\\ &= 16 \cdot 5418 - 31 \cdot 2795 && k_4 = 16;\ \ell_4 = -15 - \lfloor 5418/2795 \rfloor \cdot 16\\ &= -31 \cdot 8213 + 47 \cdot 5418 && k_5 = -31;\ \ell_5 = 16 - \lfloor 8213/5418 \rfloor \cdot -31\\ &= 47 \cdot 13631 - 78 \cdot 8213 && k_6 = 47;\ \ell_6 = -31 - \lfloor 13631/8213 \rfloor \cdot 47\end{aligned}$$

we find $k = 47$, $\ell = -78$.

c. Consider $k = 47 - 8213/43 = -144$, and $\ell = -78 + 13631/43 = 239$.

Solution for PS5-2.

a. False. Take $a = 2$, $b = 6$, and $c = 3$.

b. True. This is because $d \mid c \iff d^n \mid c^n$. The reverse implication isn’t straightforward, but can be proved using the Unique Factorization Theorem.

c. True. If $d = \gcd(b, c)$, then clearly ad is a common divisor of ab and ac. On the other hand, by LCT, $\exists\, k, \ell \in \mathbb{Z}$ such that $d = kb + \ell c$, so $ad = kab + \ell ac$, whence $\gcd(ab, ac) \mid ad$.

d. False. Take $a = 2$, $b = 4$.

e. True. If $ka + \ell b = 1$, then $\ell^2 b^2 = (1 - ka)^2 = k^2a^2 - 2ka + 1$. Therefore, $(2k - k^2a)a + \ell^2 b^2 = 1$.

f. True. The given condition implies $g := \gcd(a, b) \geq 3$. Clearly, $\gcd(a^2, b^2) \geq g \geq 3$. Had 2 been an IntLC of a^2 and b^2, we would have had $\gcd(a^2, b^2) \mid 2$.

Solution for PS5-3.

a. The code doesn’t handle negative numbers correctly. Calling egcd$(9, -6)$ returns $(-3, 1, 2)$. However, by definition $\gcd(9, -6) = 3$, not -3.

b.

```
def egcd(a, b):
    if b < 0:
        g, k, l = egcd(a, -b)
        return (g, k, -l)
    elif a < 0:
        g, k, l = egcd(-a, b)
        return (g, -k, l)
    elif b == 0:
        return (a, 1, 0)
    else:
        g, k, l = egcd(b, a % b)
        return (g, l, k - (a // b) * l)
```

Solution for PS5-4.

a. Assume that a good call $\text{egcd}(a,b)$ is made, so that $0 \neq a \geq b \geq 0$. If it makes an immediate recursive call, $b \neq 0$. The new call is $\text{egcd}(b,r)$, where $r = a \bmod b$. By definition of the "mod" operation, $r < b$. Therefore, $0 \neq b \geq r \geq 0$, i.e., the new call is good. Furthermore, the size of the new call is

$$b + r < b + b \leq b + a = s\,.$$

b. Consider the sequence of sizes of all recursive calls that result from an initial good call to egcd. By the above results, these recursive calls are all good, so the sizes are all natural numbers, and the sequence is decreasing. The sequence cannot be infinite (the sizes cannot decrease forever), so it eventually terminates, i.e., there is eventually a call to egcd that does not result in further recursion.

Solution for PS5-6.

a. Let $m = \text{lcm}(a,b)$. Being a positive multiple of a, m must equal ax for some $x \in \mathbb{N}^+$. Similarly, $m = by$ for some $y \in \mathbb{N}^+$. Now suppose that $\gcd(x,y) = z > 1$. Then x/z and y/z are both integers and so

$$\frac{ax}{z} = \frac{by}{z} = \frac{m}{z} < m$$

is a common multiple of a and b that is smaller than m, a contradiction.

b. By LCT, $\exists\, k,\ell \in \mathbb{Z}$ such that $1 = kx + \ell y$. Therefore, $a/y = kax/y + \ell ay/y = kb + \ell a$.

c. Since $d := \gcd(a,b)$ must divide every IntLC of a and b, in particular, $d \mid a/y$.

d. Since $a/y \in \mathbb{Z}$, it is obviously a divisor of a. Further, $b/(a/y) = by/a = x \in \mathbb{Z}$, so a/y divides b as well.

e. Since a/y is one particular common divisor of a and b, the *greatest* common divisor $d \geq a/y$. But we also showed that $d \mid a/y$. Therefore, $d = a/y$.

f. A simple calculation: $\gcd(a,b) \cdot \text{lcm}(a,b) = dm = (a/y)(by) = ab$.

Solution for PS5-7.

a. The only divisors of p are 1 and itself. Of these, only 1 is a divisor of b, since $0 < b < p$. Therefore, $\gcd(b,p) = 1$. By the Inverse Existence Theorem, b has an inverse modulo p.

b. Let $a = b^{-1}$. We claim that the function f_a is the inverse of f_b. Indeed,

$$f_a(f_b(x)) = f_a(bx \bmod p) = (a(bx \bmod p) \bmod p) = abx \bmod p = x\,,$$

so $f_a \circ f_b = \text{id}_{\mathbb{Z}_p}$. Similarly, $f_b \circ f_a = \text{id}_{\mathbb{Z}_p}$. By results from earlier in the course, f_b is a bijection.

Solution for PS5-8. The fact to be shown follows from its contrapositive, the statement that if n is not even, then n^2 is not even. This statement is precisely *PS3-1e*, the fact that the product of two odd numbers is odd.

a. Assume, to the contrary, that $\sqrt{2} = u/v$, in lowest terms, for some $u,v \in \mathbb{N}^+$. Then $u^2 = 2v^2$, which is even. Therefore, u is even. Let $u = 2w$, where $w \in \mathbb{N}^+$. We get

$$(2w)^2 = 2v^2\,, \quad \text{i.e., } v^2 = 2w^2\,,$$

which is even. Therefore, v is even. Since u and v are both even, the expression u/v is not in lowest terms, a contradiction.

b. Suppose that $p \mid n^2 = n \cdot n$. Applying Euclid's Lemma (the "consequently" portion), we directly get $p \mid n$.

c. Let p be an arbitrary prime. Assume, to the contrary, that $\sqrt{p} = u/v$, in lowest terms, for some $u, v \in \mathbb{N}^+$. Then $u^2 = pv^2$, so $p \mid u^2$, so $p \mid u$ (by the previous part). Let $u = pw$, where $w \in \mathbb{N}^+$. We get

$$(pw)^2 = pv^2\,, \quad \text{i.e., } v^2 = pw^2\,,$$

which means $p \mid v^2$. Therefore, $p \mid v$. Since p divides both u and v, the expression u/v is not in lowest terms, a contradiction.

d. Suppose that $p \mid n^2 = n \cdot n$. Applying Euclid's lemma (the "consequently" portion), we directly get $p \mid n$.

e. Let p be an arbitrary prime and $a \in \mathbb{Z}$. By repeatedly using Euclid's Lemma, we get that

$$\begin{aligned} p \mid a^n = a \cdot a^{n-1} &\implies \text{either } p \mid a \text{ or } p \mid a^{n-1} = a \cdot a^{n-2} \\ &\implies \text{either } p \mid a \text{ or } p \mid a^{n-2} = a \cdot a^{n-3} \\ &\implies \cdots \\ &\implies \text{either } p \mid a \text{ or } p \mid a\,. \end{aligned}$$

In short, $p \mid a^n \implies p \mid a$.

Now let n be an arbitrary integer ≥ 2. Assume, to the contrary, that $p^{1/n} = u/v$, in lowest terms, for some $u, v \in \mathbb{N}^+$. Then $u^n = pv^n$, so $p \mid u^n$, so $p \mid u$ (by the above). Let $u = pw$, where $w \in \mathbb{N}^+$. We get

$$(pw)^n = pv^n\,, \quad \text{i.e., } v^n = p^{n-1}w^n\,.$$

Since $n \geq 2$, this means $p \mid v^n$. Therefore, $p \mid v$. Since p divides both u and v, the expression u/v is not in lowest terms, a contradiction.

Solution for PS5-9.

a. The numbers 1 and $p-1$ (which are distinct because $p \geq 3$) are self-inverses, because $1^2 \equiv 1 \pmod{p}$ and $(p-1)^2 \equiv (-1)^2 \equiv 1 \pmod{p}$. To show that there are no others, suppose $a \in \mathbb{Z}_p$ is a self-inverse, so $a^2 \equiv 1 \pmod{p}$. By the definition of congruence,

$$p \mid a^2 - 1 = (a-1)(a+1)\,.$$

By Euclid's Lemma, either $p \mid a-1$ or $p \mid a+1$, i.e., either $a \equiv 1 \pmod{p}$ or $a \equiv -1 \pmod{p}$.

b. We can solve this by exhaustive case analysis, but let's be cleverer. Reasoning as above, 1 and 14 are clearly self-inverses. Further, if $a^2 \equiv 1 \pmod{15}$, then

$$15 \mid a^2 - 1 = (a-1)(a+1) \quad \implies \quad 5 \mid (a-1)(a+1)\,.$$

So $a \equiv \pm 1 \pmod{5}$. Besides 1 and 14, the only other values in $\mathbb{Z}_{15}$ satisfying this are $4, 6, 9$, and 11. But by analogous reasoning, $a \equiv \pm 1 \pmod{3}$, so we can eliminate 6 and 9. Finally, $4^2 \equiv 11^2 \equiv 1 \pmod{15}$.

Thus, there are exactly four self-inverses modulo 15: they are $1, 4, 11$, and 14.

Solutions for Unit 6

Solution for PS6-1. When $\gcd(a, m) \neq 1$, the number a has no inverse modulo m, nor can any positive power of a be congruent to 1 modulo m.

I used the 'egcd' Python function from the lecture notes to compute GCDs and, as a result, inverses when they exist. For the powers, I used some trial and error.

number	**inverse**	**power congruent to 1**
1	1	$1^1 \equiv 1$
10	$\nexists$	$\nexists$
13	7	$13^4 \equiv 1$
19	19	$19^2 \equiv 1$
27	$\nexists$	$\nexists$
29	29	$29^2 \equiv 1$

Solution for PS6-2. Let $n = q(p-1)+r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < p-1$. By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod{p}$. Therefore,

$$a^n = (a^{p-1})^q \cdot a^r \equiv 1^q a^r = a^{n \bmod (p-1)} \pmod{p}\,.$$

Solution for PS6-3. Suppose that $a, b \in \mathbb{Z}_m^*$ and $c = ab \bmod m$. We must show that $c \in \mathbb{Z}_m^*$.

By the Inverse Existence Theorem, a^{-1} and b^{-1} exist. Therefore,

$$c \cdot (b^{-1}a^{-1}) = abb^{-1}a^{-1} \equiv 1 \pmod{m}\,,$$

so c^{-1} exists. By the Inverse Existence Theorem, $\gcd(c, m) = 1$, so $c \in \mathbb{Z}_m^*$.

Solution for PS6-4.

a. First, check that f_a is indeed a function from $\mathbb{Z}_m^*$ to $\mathbb{Z}_m^*$. Of course, for any $m \in \mathbb{Z}_m^*$, $ax \bmod m$ is a unique defined value in $\mathbb{Z}_m$; the only question is whether it lies in $\mathbb{Z}_m^*$. By the result of ***PS6-3***, it does.

To prove that f_a is a *bijection*, we demonstrate that it has an inverse function. By the Inverse Existence Theorem, $\exists\, b \in \mathbb{Z}_m^*$ such that $ab \equiv 1 \pmod{m}$. Now, for all $x \in \mathbb{Z}_m^*$,

$$f_b(f_a(x)) = b(ax \bmod m) \bmod m = bax \bmod m = x\,,$$

so $f_b \circ f_a = \text{id}$. Similarly, $f_a \circ f_b = \text{id}$. This completes the proof.

b. Let $L = (b_1, b_2, \ldots, b_{\phi(m)})$ be a list of all the elements of $\mathbb{Z}_m^*$. By the previous part, the list

$$L' = (ab_1 \bmod m, ab_2 \bmod m, \ldots, ab_{\phi(m)} \bmod m)$$

consists of the same elements as L, but perhaps in a different order. Comparing the products of the elements in each list,

$$b_1 b_2 \cdots b_{\phi(m)} \equiv a^{\phi(m)} b_1 b_2 \cdots b_{\phi(m)} \pmod{m}\,.$$

By the Inverse Existence Theorem, each b_i has an inverse b_i^{-1}. Multiplying both sides by $b_1^{-1} b_2^{-1} \cdots b_{\phi(m)}^{-1}$ gives $1 \equiv a^{\phi(m)} \pmod{m}$.

c. When m is a prime, every nonzero number in $\mathbb{Z}_m$ is coprime to m, so $\mathbb{Z}_m^* = \{1, 2, \ldots, m-1\}$ and $\phi(m) = m-1$. The congruence now reads $a^{m-1} \equiv 1 \pmod{m}$, which is exactly Fermat's Little Theorem.

Solution for PS6-5.

a. By definition, $P_{m,a}$ is an infinite sequence, but all its elements lie in the finite set $\mathbb{Z}_m^*$. Therefore, there must be a repetition in the sequence. Let $i < j$ be two positions such that $a^i \bmod m = a^j \bmod m$. By the Inverse Existence Theorem, a has an inverse b modulo m. So,

$$a^i \equiv a^j \pmod{m} \implies b^i a^i \equiv b^i a^j \pmod{m} \implies 1 \equiv a^{j-i} \pmod{m}.$$

Thus, 1 reappears in the sequence at position $j - i$.

b. Let k be the smallest positive index at which 1 appears in $P_{m,a}$. Then $a^k \equiv 1 \pmod{m}$. For any index $\ell > k$, let $\ell = qk + r$ with $q, r \in \mathbb{N}$ and $0 \le r < k$. Then

$$a^\ell = (a^k)^q \cdot a^r \equiv 1^q a^r = a^r \pmod{m}.$$

Therefore, $P_{m,a}$ consists of the block $(a^0 \bmod m, \ldots, a^{k-1} \bmod m)$ repeated infinitely often.

c. Let's look more closely at the argument in Part ***a.*** If we consider the first $m + 1$ elements in the sequence, there must already be a repetition because the elements come from a set of size $\le m$. Therefore, we can enforce $0 \le i < j \le m$ in that argument.

Thus, 1 reappears at position $j - i \le m$. So the value of k in the previous part—which is the period—is $\le m$.

d. Look even more closely at the argument above. The elements in the sequence in fact come from the set $\mathbb{Z}_m^*$, whose cardinality is $\phi(m) \le m - 1$. Therefore, $k \le m - 1$. In particular, the period cannot be m.

Note: With a little more effort, you can in fact show that $k \mid \phi(m)$, so the period must be a divisor of $\phi(m)$.

Alternate Solution for PS6-5.

a. Since $a \in \mathbb{Z}_m^*$, by Euler's Theorem, $a^\phi(m) \equiv 1 \pmod{m}$. Since $\phi(m) > 0$, we see that 1 reappears in the sequence at position $\phi(m)$.

b. Same as above.

c. The period is clearly at most $\phi(m)$. Since $\mathbb{Z}_m^* \subset \mathbb{Z}_m$, it follows that $\phi(m) < m$. So the period is $\le m$.

d. Of course, we've in fact shown that the period is $< m$. In particular, it can't be m.

Solution for PS6-6. Consider an arbitrary $a \in \mathbb{Z}_{pq}$. The positive divisors of pq are $1, p, q$, and pq. So $\gcd(a, pq)$ must be one of these four numbers. Let's count how many numbers a lead to each of these GCDs.

Case 1. $\gcd(a, pq) = 1$. Then $a \in \mathbb{Z}_{pq}^*$. By definition, there are $\phi(pq)$ such numbers a.

Case 2: $\gcd(a, pq) = pq$. Since $a < pq$, this means $a = pq$, i.e., there is exactly one possibility for a.

Case 3: $\gcd(a, pq) = p$. Then $p \mid a$ and $a < pq$, so $a \in \{p, 2p, 3p, \ldots, (q-1)p\}$, i.e., $q - 1$ possibilities for a.

Case 4: $\gcd(a, pq) = q$. Analogously, there are $p - 1$ possibilities for a in this case.

Since there is no overlap between the cases and there are $|\mathbb{Z}_{pq}| = pq$ total possibilities for a, we obtain

$$pq = \phi(pq) + 1 + (q-1) + (p-1).$$

Solving for $\phi(pq)$ gives $\phi(pq) = pq - p - q + 1 = (p-1)(q-1)$.

Solution for PS6-7.

a. We work modulo p. Imagine drawing an arrow from each $a \in \mathbb{Z}_p^*$ to a^{-1}. Then the arrow from a^{-1} will point to $(a^{-1})^{-1} = a$. We can then pair off a and a^{-1}. If we consider any other element $b \notin \{a, a^{-1}\}$, then $\{b, b^{-1}\}$ will be another pair disjoint from $\{a, a^{-1}\}$.

There is a catch: a might equal a^{-1} sometimes! But by ***PS5-9***^{HW}, this only happens for $a = 1$ and $a = p - 1$. So the argument above works for all $a \in S$.

b. Consider the modulo-p product Q of all numbers in S. We can rearrange the product to place each $a \in S$ adjacent to its partner $a^{-1} \in S$. The product within each pair is 1 modulo p. Therefore, so is the overall product, i.e., $Q \equiv 1 \pmod{p}$. Therefore,

$$(p-1)! = 1 \times Q \times (p-1) \equiv 1 \times 1 \times (-1) \equiv -1 \pmod{p}.$$

c. By the definition of congruence, the last statement above can be rewritten as $p \mid (p-1)! + 1$.

Solution for PS6-8. Since m is composite, we can write $m = ab$ where $2 \le a \le m-1$ and $2 \le b \le m-1$. Consider the list of factors $L = (1, 2, \ldots, m-1)$ whose product equals $(m-1)!$. Three cases arise.

Case 1: $a \neq b$. In this case both a and b appear in L. Therefore $m = ab \mid (m-1)!$, whence $m \nmid (m-1)! + 1$.

Case 2: $a = b > 2$. In this case, $m = a^2 > 2a$, so a and $2a$ both appear in L. Thus, $m = a^2 \mid (m-1)!$, as before.

Case 3: $a = b = 2$. Then $m = 4$ and we check directly that $4 \nmid 3! + 1 = 7$.

Solution for PS6-9.

a. By the GCD Linear Combination Theorem (LCT), $\exists\, k, \ell \in \mathbb{Z}$ such that $\gcd(a,b) = ka + \ell b$. By ***PS5-6***^{HW},

$$\frac{n}{\operatorname{lcm}(a,b)} = \frac{n \cdot \gcd(a,b)}{ab} = \frac{n(ka + \ell b)}{ab} = \frac{kn}{b} + \frac{\ell n}{a} \in \mathbb{Z},$$

since $b \mid n$ and $a \mid n$.

b. From the given info,

- $\gcd(p_1, p_2) = 1$, so n is divisible by $\operatorname{lcm}(p_1, p_2) = p_1 p_2$;
- $\gcd(p_1 p_2, p_3) = 1$, so n is divisible by $\operatorname{lcm}(p_1 p_2, p_3) = p_1 p_2 p_3$;
- and so on.

Note: Once we study mathematical induction, we'll learn a better way to write this type of proof formally.

Solution for PS6-10. We first work out the factorization $2730 = 2 \times 3 \times 5 \times 7 \times 13$. By the previous result, it suffices to show that each of these prime factors divides $n^{13} - n$.

Consider each $p \in \{2, 3, 5, 7, 13\}$. If $p \mid n$ then $p \mid n^{13}$ as well, so $p \mid n^{13} - n$. Otherwise, if $p \nmid n$, we apply Fermat's Little Theorem:

- For $p = 2$, we have $n^{13} \equiv 1^{13} \equiv 1 \equiv n \pmod{2}$.
- For $p = 3$, we have $n^{13} = (n^2)^6 \cdot n \equiv 1^6 \cdot n \equiv n \pmod{3}$.
- For $p = 5$, we have $n^{13} = (n^4)^3 \cdot n \equiv 1^3 \cdot n \equiv n \pmod{5}$.
- For $p = 7$, we have $n^{13} = (n^6)^2 \cdot n \equiv 1^2 \cdot n \equiv n \pmod{7}$.
- For $p = 13$, we have $n^{13} \equiv n \pmod{13}$.

Solutions for Unit 7

Solution for PS7-1.

a. If $M = 0$, clearly $M' = 0$. Else, since $ed \equiv 1 \pmod{P-1}$, write $ed = k(P-1)+1$ where $k \in \mathbb{N}$. We compute

$$M' \equiv C^d \equiv M^{ed} = (M^{P-1})^k \cdot M \equiv 1^k M = M \pmod{P},$$

where the last congruence is because of Fermat's Little Theorem. Since $M, M' \in \mathbb{Z}_P$, $M = M'$.

b. Dr. Speedy's "cryptosystem" is not secure at all! Anyone can use the Extended GCD Algorithm to compute d from e and P (which are both public) and then cheerfully decrypt any message to Bob that they can intercept.

Solution for PS7-2. Suppose that $N = pq$, where p and q are distinct primes. Using the result of a previous class exercise, $\phi(N) = \phi(pq) = (p-1)(q-1) = N - p - q + 1$.

Therefore, $q = N - \phi(N) + 1 - p$. Substituting this expression for q in $N = pq$, we obtain

$$p(N - \phi(N) + 1 - p) = N\,,$$
$$\text{i.e.,}\quad p^2 + (\phi(N) - N - 1)p + N = 0\,.$$

Run algorithm $\mathscr{A}$ to obtain $\phi(N)$. We now know all the coefficients in this quadratic equation for p. Solving it, we obtain p. Then we obtain $q = N/p$.

Solution for PS7-3. One can verify that the Decryption Theorem for RSA still holds with 10-prime RSA, so Dr. Tricksy's idea is not immediately flawed.

For regular, 2-prime RSA, with public key (N_2, e_2), and private key d, the time needed to perform encryption and decryption is given by the time needed for modular exponentiation. This mainly depends on the number of bits needed to express N_2 and e_2. Given the value of N_2, the value of e_2 is essentially unrestrained; it must only be coprime to $\phi(N)$. For 10-prime RSA, and a modulus N_{10} with the same number of bits as N_2, because only a small fraction of numbers are not coprime to N_{10}, one can pick an encryption exponent $e_1 0$ almost exactly equal to e_2. As a result, for the same key size, encryption and decryption procedures do not differ significantly between 2-prime and 10-prime RSA.

On the other hand, for the same key size (number of bits of N_2 and N_{10}), the time to factor the modulus can differ significantly for for 10-prime RSA. Here we consider the brute force factoring algorithm, that checks divisibility by every number $\leq \sqrt{n}$. Better algorithms, like the General Number Field Sieve or the Elliptic-Curve Factorization Method, are more efficient, but their runtime analysis is more complicated. Write the factors of N_2 and N_{10} in increasing order, so that $N_2 = p_1 \cdot p_2$, and $N_{10} = q_1 \cdot q_2 \cdot q_3 \cdots q_{10}$, with $p_1 < p_2$, and $q_1 < q_2 \ldots < q_{10}$. Then brute force factorization of N_2 requires time $O(p_1)$ to discover the smaller factor; while factoring N_{10} requires time $O(q_1 + q_2 + \ldots + q_9) = O(q_9)$ to discover the 9 smallest factors.

To make breaking 2-prime RSA difficult, p_1 and p_2 are chosen roughly equal in size, so that $p_1 \approx \sqrt{N_2}$, and it takes $O(\sqrt{N_2})$ time to brute-force factor N_2. To make breaking 10-prime RSA hard, q_9 should be as large as possible. This can be done by setting $q_1 = 2$, $q_2 = 3$, $q_3 = 5$, and picking q_9 and q_{10} both $\approx \sqrt{N_{10}}$, in which case the time to factor N_{10} is the a constant multiple of that for N_2. There isn't any reason to prefer 10-prime RSA over 2-prime RSA.

If the prime factors of N_{10} are all roughly the same size, then $q_1 \approx q_2 \approx \ldots q_{10} \approx N_{10}^{1/10}$, and it only takes $O(N_{10}^{1/10})$ time to break Dr. Tricksy's method, which is far less than $O(\sqrt{N_2})$.

Solution for PS7-4.

a. When we call `modpow_bad(a, k, n)`, it results in k calls to `modmult`. Since k is a 1024-bit integer, the value of k could be as large as $2^{1024} - 1$ and is typically greater than 2^{1000}. Even if each call to `modmult` takes just one nanosecond, the time spent by `modpow_bad` would be orders of magnitude greater than the age of the universe!

b. By repeated squaring, compute $a \to a^2 \to a^4 \to a^8 \to a^{16} \to a^{32} \to a^{64}$ (all computations modulo n). Then combine the results like this: $a \cdot a^2 \cdot a^{16} \cdot a^{64} = a^{1+2+16+64} = a^{83}$ (again, modulo n).

c. If n is even, let $n = 2k$. Then $x^n = x^{2k} = (x^2)^k = (x^2)^{\lfloor n/2 \rfloor}$.
If n is odd, let $n = 2k+1$. Then $x^n = x^{2k+1} = x \cdot (x^2)^k = x \cdot (x^2)^{\lfloor n/2 \rfloor}$.

d. The equation in the previous part shows how raising to the power n can be reduced to raising to the power $\lfloor n/2 \rfloor$, provided we first compute the square. We can translate this directly into the following recursive implementation.

```
def modpow(a, k, n):
    """compute a**k modulo n quickly, assuming k >= 0, n > 0"""
    if k == 0:
        return 1
    square = modmult(a, a, n)
    temp = modpow(square, k // 2, n)
    if k % 2 == 0:
        return temp
    else:
        return modmult(a, temp, n)
```

e. When I ran the above code on the given numbers, it "instantly" returned 4808550559.

Solution for PS7-5.

a. Since $\gcd(m,n) = 1$, by ***PS5-6***^{HW}, $\operatorname{lcm}(m,n) = mn$. Therefore, by ***PS6-9***, if $m \mid s$ and $n \mid s$, then $mn \mid s$. Now, to prove what's asked for, take $s = x - y$.

b. Suppose that $f(x) = f(y)$. Then $(x \bmod m, x \bmod n) = (y \bmod m, y \bmod n)$. In other words, $x \equiv y \pmod{m}$ and $x \equiv y \pmod{n}$. Therefore, by the previous part, $x \equiv y \pmod{mn}$. Since both x and y belong to $\mathbb{Z}_{mn}$, this forces $x = y$.

c. Since f is injective, $|\text{Range}(f)| = |\text{Domain}(f)| = |\mathbb{Z}_{mn}| = mn$. However, $|\text{Codomain}(f)| = |\mathbb{Z}_m \times \mathbb{Z}_n| = mn$ as well. Therefore, $\text{Range}(f) = \text{Codomain}(f)$, which makes f surjective.

d. By the previous two parts, f is bijective. Therefore, for each $(a,b) \in \mathbb{Z}_m \times \mathbb{Z}_n$, there is one and only one value $x \in \mathbb{Z}_{mn}$ such that $f(x) = (a,b)$. This value, which is precisely $f^{-1}(a,b)$, is the unique solution to the system of congruences.

Solution for PS7-6.

a. We already know from ***PS7-5*** that f is injective on the full domain $\mathbb{Z}_{mn}$. Therefore, we only need to show that $x \in \mathbb{Z}_{mn}^* \iff f(x) \in \mathbb{Z}_m^* \times \mathbb{Z}_n^*$. This is logically equivalent to $x \notin \mathbb{Z}_{mn}^* \iff f(x) \notin \mathbb{Z}_m^* \times \mathbb{Z}_n^*$, so we'll prove the latter instead.

Suppose that $x \notin \mathbb{Z}_{mn}^*$. Then $\gcd(x, mn) = d > 1$. Let p be a prime divisor of d. Then $p \mid x$ and $p \mid mn$. By Euclid's Lemma, either $p \mid m$ or $p \mid n$. Assume WLOG that $p \mid m$. Let $f(x) = (a,b)$. Then

$$\gcd(a,m) = \gcd(x \bmod m, m) = \gcd(x,m) \geq p > 1\,,$$

whence $a \notin \mathbb{Z}_m^*$, implying $f(x) \notin \mathbb{Z}_m^* \times \mathbb{Z}_n^*$.

On the other hand, suppose that $f(x) = (a,b) \notin \mathbb{Z}_m^* \times \mathbb{Z}_n^*$. Assume WLOG that $a \notin \mathbb{Z}_m^*$. Then $\gcd(a,m) = d > 1$. Therefore,

$$\gcd(x, mn) \geq \gcd(x,m) = \gcd(x \bmod m, m) = \gcd(a,m) = d > 1\,,$$

whence $x \notin \mathbb{Z}_{mn}^*$.

b. The existence of a bijection from finite set A to finite set B implies $|A| = |B|$. Therefore, $\phi(mn) = \phi(m)\phi(n)$. In the language of Unit 8, we are using the bijection principle and the product principle.

Solution for PS7-7.

a. A square root of unity modulo m is exactly the same thing as a self-inverse modulo m. If m were prime, as shown in ***PS5-9***^{HW}, it would have had only two such square roots: 1 and $m-1$.

b. By definition, $b^2 \equiv 1 \pmod{m}$, so $m \mid b^2 - 1 = (b-1)(b+1)$. Let p be a prime divisor of m. Then $p \mid (b-1)(b+1)$, so by Euclid's Lemma, either $p \mid b-1$ or $p \mid b+1$. Since $b \neq 1$, $b \neq m-1$, and $m \geq 3$, we have $1 \leq b-1 < b+1 < m$.

Suppose that $p \mid b-1$. Then $\gcd(m, b-1) \geq p > 1$. Also, $\gcd(m, b-1) \leq b-1 < m$. Therefore $\gcd(m, b-1)$ is a nontrivial divisor of m.

Suppose that $p \mid b+1$. Then $\gcd(m, b+1) \geq p > 1$. Also, $\gcd(m, b+1) \leq b+1 < m$. Therefore $\gcd(m, b+1)$ is a nontrivial divisor of m.

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Solutions for Unit 8

Solution for PS8-1.

a. $27 \times 37 = 999$, by the product principle.

b. $52 \times 51 = 2652$, by the generalized product principle.

c. $\lfloor 1000/3 \rfloor + \lfloor 1000/5 \rfloor - \lfloor 1000/15 \rfloor = 467$, by the generalized sum principle.

Solution for PS8-2.

a. Let C be the set of ways in which we can answer a single question on the test. Then the set of choices for the entire test is C^{10}. So we apply the product principle.

(a) In this case, $|C| = 4$, so the number of ways is $4^{10} = 1\,048\,576$.

(b) Here $|C| = 5$—one of the choices is to leave an answer blank—so the number of ways is $5^{10} = 9\,765\,625$.

b. Consider natural numbers below 10^9, padded up to nine digits by adding zeros as needed. Let

$$
\begin{aligned}
S &= \{0, 1, 2, \ldots, 10^9 - 1\}\,,\\
A &= \{n \in S : n\text{'s padded decimal representation contains a `1'}\}\,,\\
B &= \{n \in S : n\text{'s padded decimal representation does not contain a `1'}\}\,.
\end{aligned}
$$

Every number in B can be thought of as a sequence of 9 characters, with each character drawn from $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$, a set of size 9. By the product principle, $|B| = 9^9$.

Since $A \cap B = \varnothing$, by the sum principle, $|S| = |A| + |B|$. Therefore, $|A| = |S| - |B| = 10^9 - 9^9$.

The question asks about numbers in the set $(S - \{0\}) \cup \{10^9\}$. The number 0 does not contain the digit '1', so there's nothing to take away, but the number 10^9 does contain the digit '1', so we need to add one to this figure. This gives a final answer of

$$10^9 - 9^9 + 1 = 612\,579\,512\,.$$

Solution for PS8-3. Let a_k denote the sum of all multiples of k between 1 and 1000. Repeatedly using the basic summation formula $1 + 2 + \cdots + n = n(n+1)/2$, we obtain

$$
\begin{aligned}
a_3 &= 3 + 6 + \cdots + 999 = 3(1 + 2 + \cdots + 333) = 3 \times \frac{333 \times 334}{2} = 166\,833\,;\\
a_5 &= 5 + 10 + \cdots + 1000 = 5(1 + 2 + \cdots + 200) = 5 \times \frac{200 \times 201}{2} = 100\,500\,;\\
a_{15} &= 15 + 30 + \cdots + 990 = 15(1 + 2 + \cdots + 66) = 15 \times \frac{66 \times 67}{2} = 33\,165\,.
\end{aligned}
$$

The desired answer is $a_3 + a_5 - a_{15} = 166833 + 100500 - 33165 = 234168$. We subtracted a_{15} because numbers which are multiple of both 3 and 5—i.e., multiples of $\operatorname{lcm}(3,5) = 15$—are included twice when we write $a_3 + a_5$.

Solution for PS8-4. Using the sum principle, it's $62^6 + 62^7 + 62^8 - 52^6 - 52^7 - 52^8 = 167\,410\,949\,583\,040$.

Solution for PS8-5. Let $s(n) := n^6 + n^7 + n^8$; this is the number of six-to-eight character strings where each character is drawn from a set of size n.

By the generalized sum principle, the desired answer is $s(62) - s(52) - 2s(36) + s(10) + 2s(26)$.

Solution for PS8-6.

a. For any arbitrary choices of x and y, we can always choose z in such a way that $x + y + z$ is even. More precisely, if $x + y$ is odd then we choose z to be odd, else we choose z to be even. Thus, choosing a 3-tuple $(x, y, z) \in T$ can be seen as making the following sequence of choices:

- Choose x freely from D. There are 10 choices.
- Choose y freely from D. There are 10 choices.

- Choose z from $\{1,3,5,7,9\}$ if $x+y$ is odd; else choose z from $\{0,2,4,6,8\}$. There are exactly 5 choices in each case.

Thus, by the generalized product principle, $|T| = 10 \times 10 \times 5 = 500$.

b. Let the n digits be $x_1, x_2, \ldots, x_n$ (from left to right). Then we want to find number of choices for $x_1, \ldots, x_n$ such that $1 \le x_1 \le 9$; for each $i \in \{2,3,\ldots,n\}$, $0 \le x_i \le 9$; and $x_1 + x_2 + \cdots + x_n$ is even.

Now, for any arbitrary choices of $x_1, \ldots x_{n-1}$, we can always choose x_n in such a way that $x_1 + x_2 + \cdots + x_n$ is even. More precisely, if $x_1 + \cdots + x_{n-1}$ is odd, then we choose x_n to be odd, else we choose x_n to be even. Hence for each list of choices of $x_1, \ldots x_{n-1}$, there are exactly 5 choices of x_n such that $x_1 + x_2 + \cdots + x_n$ is even. By the generalized product principle, the total number of choices for all the digits is

$$9 \times 10^{n-2} \times 5 = 45 \times 10^{n-2},$$

since there are 9 choices for x_1, and 10 choices for each of $x_2, \ldots x_{n-1}$.

Alternate Solution for PS8-6.

a. Let $A = \{1,3,5,7,9\}$ be the set of odd digits and let $B = \{0,2,4,6,8\}$ be the set of even digits. We break T into four pairwise disjoint subsets, each of which is a Cartesian product. So the product principle applies to each subset.

- $T_1 = \{(x,y,z) \in D^3 : x, y, \text{ and } z \text{ are even}\} = B \times B \times B$. Then $|T_1| = |B| \times |B| \times |B| = 5 \times 5 \times 5 = 125$.
- $T_2 = \{(x,y,z) \in D^3 : x \text{ and } y \text{ are odd, } z \text{ is even}\} = A \times A \times B$. Then $|T_2| = |A| \times |A| \times |B| = 125$.
- $T_3 = \{(x,y,z) \in D^3 : x \text{ and } z \text{ are odd, } y \text{ is even}\} = A \times B \times A$. Then $|T_3| = 125$ as well.
- $T_4 = \{(x,y,z) \in D^3 : y \text{ and } z \text{ are odd, } x \text{ is even}\} = B \times A \times A$. Then $|T_4| = 125$ as well.

b. Same as above.

Solution for PS8-7.

a. $4 \times 3 \times 2 \times 1 = 4! = 24$, by the generalized product principle.

b. $n!$, by the generalized product principle.

Solution for PS8-8.

a. If we color one of the 1s red, and the other black, then we have four distinct symbols, leading to $4! = 24$ permutations.

Now consider the function f that maps a colored permutation to an ordinary (uncolored) permutation of $(1,1,4,9)$ by "removing the colors." This f is a 2-to-1 correspondence. So, by the division principle, the number of permutations of $(1,1,4,9)$ is $24/2 = 12$.

b. We can either reason directly, or follow the steps of Problem 15.27 ("The Tao of BOOKKEEPER") from the [LLM] book. Then we apply similar logic to the word "CONDESCENDENCE". This 14-letter word has 3 Cs, 3 Ns, 2 Ds, 4 Es, 1 O, and 1 S. Therefore, it has

$$\frac{14!}{3! \cdot 3! \cdot 2! \cdot 4! \cdot 1! \cdot 1!} = 50\,450\,400$$

anagrams.

Solution for PS8-9.

a. 2^{mn}, since each relation is just a subset of $A \times B$.

b. n^m, by the generalized product principle (choosing a function means making a sequence of $|A|$ choices).

c. $n(n-1)\cdots(n-m+1)$, by the generalized product principle. Note that this equals 0 when $m > n$.

d. If $m = n$ then $n!$, else 0.

Solution for PS8-10. For a palindrome of length n, where n is even, we can arbitrarily choose the first $n/2$ bits and then the last $n/2$ bits are automatically fixed. Thus, there are 2 choices for each of the first $n/2$ bits, and then 1 choice for each remaining bit. This leads to $2^{n/2}$ possible palindromes.

If n is odd, then we can arbitrarily choose the first $(n+1)/2$ bits and then the last $(n-1)/2$ bits are automatically fixed. As before, this leads to $2^{(n+1)/2}$ possible palindromes.

We can combine these two cases and say that the number of n-bit palindromes for any natural number n is $2^{\lceil n/2 \rceil}$.

Solution for PS8-11.

a. $\binom{13}{10}$. This is pretty much the definition of "n choose k."

b. $13 \times 12 \times \cdots \times 4 = \dfrac{13!}{3!}$, by the generalized product principle.

c. $\binom{13}{10} - \binom{11}{10}$, subtracting off sets of 10 players chosen solely from the 11 students.

Solution for PS8-12.

a. $\binom{10}{4}$, since this amounts to choosing the 4 locations (out of 10) where the 1s will occur.

b. $\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4}$, by applying the previous observation four times.

c. $\binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}$, along similar lines.

d. $\binom{10}{5}$, along similar lines.

Solution for PS8-13.

a. 5040.

b. 1440.

c. 240.

d. 2640.

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Solutions for Unit 9

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Solution for PS9-1. Let $P(n)$ be the statement " $\sum_{i=1}^{n}(2i-1) = n^2$. " We shall prove by induction on n that $\forall\, n \in \mathbb{N} : P(n)$.

Base case. $P(0)$ states "$0 = 0^2$." This is obviously true.

Induction step. Assume that $P(k)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \left(\sum_{i=1}^{k}(2i-1)\right) + (2(k+1)-1) \\
&= k^2 + (2(k+1)-1) && \triangleleft\ \textit{by the induction hypothesis} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2 .
\end{aligned}$$

Therefore, $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-2. Let $P(n)$ be the given statement. We shall prove it for all $n \in \mathbb{N}$ by induction on n.

Base case. $P(0)$ states "$0 = 1! - 1$." This is obviously true.

Induction step. Assume that $P(k)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} j \cdot j! &= \left(\sum_{j=1}^{k+1} j \cdot j!\right) + (k+1)\cdot(k+1)! \\
&= (k+1)! - 1 + (k+1)\cdot(k+1)! && \triangleleft\ \textit{by the induction hypothesis} \\
&= (k+1)! \cdot (1 + (k+1)) - 1 \\
&= (k+2)! - 1 .
\end{aligned}$$

Therefore, $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-3. Fix a *particular*, though arbitrary, real number $x \in \mathbb{R} - \{1\}$.

(This is an important step! From here on, for the rest of this proof, x is no longer a variable.)

Let $P_x(n)$ be the following statement:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1} .$$

We shall prove by induction on n that $\forall\, n \in \mathbb{N} :\ P_x(n)$.

Base case. $P_x(0)$ states "$1 = (x-1)/(x-1)$." This is obviously true.

Induction step. Assume that $P_x(k)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
1 + x + x^2 + \cdots + x^{k+1} &= \left(1 + x + x^2 + \cdots + x^k\right) + x^{k+1} \\
&= \frac{x^{k+1}-1}{x-1} + x^{k+1} && \triangleleft\ \textit{by the induction hypothesis} \\
&= \frac{x^{k+1} - 1 + x^{k+2} - x^{k+1}}{x-1} \\
&= \frac{x^{k+2}-1}{x-1} + x^{k+1} .
\end{aligned}$$

Therefore, $P_x(k+1)$ is true. We have shown that $P_x(k) \implies P_x(k+1)$.

Thus, by the principle of mathematical induction, we've proved that $\forall\, n \in \mathbb{N} :\ P_x(n)$.

Since we did this for an arbitrary choice of x, we've in fact shown that $\forall\, x \in \mathbb{R} - \{1\}\ \forall\, n \in \mathbb{N} : P_x(n)$.

Solution for PS9-4. Let $P(n)$ be the statement "$3 \mid n^3 + 2n$." We shall prove it for all $n \in \mathbb{N}$ by induction on n.

Base case. $P(0)$ states "$3 \mid 0^3 + 2 \times 0$," i.e., "$3 \mid 0$." This is obviously true.

Induction step. Assume that $P(k)$ is true for some $k \geq 0$. This implies $k^3 + 2k = 3m$, for some $m \in \mathbb{N}$. Notice that

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2(k+1) = (k^3 + 2k) + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)\,.$$

Therefore, $3 \mid (k+1)^3 + 2(k+1)$, i.e., $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-5. Let $P(n)$ be the statement "$5 \mid 8^n - 3^n$." We shall prove it for all $n \in \mathbb{N}$ by induction on n.

Base case. $P(0)$ states "$5 \mid 8^0 - 3^0$," i.e., "$5 \mid 1 - 1 = 0$." This is obviously true.

Induction step. Assume that $P(k)$ is true for some $k \geq 0$. This implies $8^k - 3^k = 5m$, for some $m \in \mathbb{N}$. Notice that

$$8^{k+1} - 3^{k+1} = 8 \cdot 8^k - 3 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k = 5 \cdot 8^k + 3 \cdot 5m = 5(8^k + 3m)\,.$$

Therefore, $5 \mid 8^{k+1} - 3^{k+1}$, i.e., $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-6.

a. Let $P(n)$ denote the statement " $2^n \geq n^3$ ". We shall prove that $P(n)$ holds for all $n \geq 10$, by induction on n.

Base case ($n = 10$). The following shows that the base case, i.e., $P(10)$, holds.

$$2^{10} = 1024 > 1000 = 10^3\,.$$

Induction step ($n \geq 10$). Assume that $P(k)$ is true for some $k \geq 10$. Then

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&\geq 2k^3 && \text{by the induction hypothesis} \\
&= k^3 + k^3 \\
&\geq k^3 + 10k^2 && \text{since } k \geq 10 \\
&\geq k^3 + 3k^2 + 7k && \text{since } k^2 \geq k \\
&\geq k^3 + 3k^2 + 3k + 1 && \text{since } 4k \geq 1 \\
&= (k+1)^3\,,
\end{aligned}$$

which is $P(k+1)$. This proves that $P(k) \implies P(k+1)$ for all $k \geq 10$.

Thus, by the principle of mathematical induction, the proof is complete.

b. Consider the following for $n > 1$.

$$\begin{aligned}
\frac{1}{n} - \frac{1}{(n+1)^2} &= \frac{(n+1)^2 - n}{n(n+1)^2} = \frac{n^2 + 2n + 1 - n}{n(n+1)^2} \\
&= \frac{n^2 + n + 1}{n(n+1)^2} \\
&> \frac{n^2 + n}{n(n+1)^2} && \text{this is strictly greater because numerator is 1 less here,} \\
&= \frac{n(n+1)}{n(n+1)^2} \\
&= \frac{1}{n+1}\,.
\end{aligned}$$

Hence, $\frac{1}{n} - \frac{1}{(n+1)^2} > \frac{1}{n+1}$, and multiplying by -1 we get the reverse inequality,

$$-\left(\frac{1}{n} - \frac{1}{(n+1)^2}\right) < -\frac{1}{n+1}\,. \tag{1}$$

Let $P(n)$ denote the statement

$$\text{“} \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}\,. \text{”}$$

We shall prove that $P(n)$ holds for all $n > 1$, by induction on n.

Base case $(n = 2)$. The following shows that the base case, i.e., $P(2)$, holds.

$$\sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} = \frac{5}{4} < \frac{6}{4} = 2 - \frac{1}{2}\,.$$

Induction step $(n \geq 2)$. Assume $P(k)$ is true for some $k \geq 2$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{by induction hypothesis,} \\
&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) && \text{rearranging,} \\
&< 2 - \frac{1}{k+1} && \text{by (1),}
\end{aligned}$$

which is $P(k+1)$. This proves that $P(k) \implies P(k+1)$ for all $k \geq 2$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-7. For this problem, we're given that the basic sum principle holds. Let $P(n)$ be the statement of the extended sum principle, i.e., the statement

“If the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.”

We shall prove it for all $n \geq 2$ by induction on n.

Base case. $P(2)$ is the basic sum principle, which does hold (we're given this).

Induction step. Assume that $P(k)$ is true for some $k \geq 2$. Towards proving $P(k+1)$, consider arbitrary pairwise disjoint sets $A_1, A_2, \ldots, A_{k+1}$. Let $C = A_1 \cup A_2 \cup \cdots \cup A_{k+1}$ and $B = A_1 \cup A_2 \cup \cdots \cup A_k$.

Since A_{k+1} has no elements in common with any of $A_1, A_2, \ldots, A_k$, it has no elements in common with their union either. In other words, $B \cap A_{k+1} = \varnothing$. Therefore,

$$\begin{aligned}
|C| &= |B \cup A_{k+1}| \\
&= |B| + |A_{k+1}| \qquad = (k+1)! - 1 + (k+1) \cdot (k+1)! \qquad \triangleleft \ \textit{by the induction hypothesis} \\
&= (k+1)! \cdot (1 + (k+1)) - 1 \\
&= (k+2)! - 1\,.
\end{aligned}$$

Since $P(k)$ is true (this is the induction hypothesis), B is a countable set. Now $C = B \times A_{k+1}$ and we have proved earlier that the Cartesian product of two countable sets is countable. Therefore, C is countable, establishing $P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-8. Let $P(n)$ be the statement

"For all sets $A_1, A_2, \ldots, A_n$, if each A_i is countable, then so is $A_1 \times A_2 \times \cdots \times A_n$."

We shall prove it for all $n \in \mathbb{N}^+$ by induction on n.

Base case. $P(1)$ states "For all sets A_1, if A_1 is countable, then so is A_1." This is obviously true.

Induction step. Assume that $P(k)$ is true for some $k \in \mathbb{N}^+$. Towards proving $P(k+1)$, consider arbitrary sets $A_1, A_2, \ldots, A_{k+1}$ such that each A_i is countable, and let $C = A_1 \times A_2 \times \cdots \times A_{k+1}$. Let $B = A_1 \times A_2 \times \cdots \times A_k$. Since $P(k)$ is true (this is the induction hypothesis), B is a countable set. Now $C = B \times A_{k+1}$ and we have proved earlier that the Cartesian product of two countable sets is countable. Therefore, C is countable, establishing $P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

Solution for PS9-9. Let $P(n)$ be the statement "n can be written as a sum of one or more distinct powers of 2." We shall prove it for all $n \in \mathbb{N}^+$ by induction on n.

Base case. $P(1)$ states "1 can be written as a sum of one or more distinct powers of 2."

This is true: we simply write $1 = 2^0$.

Induction step. Assume that $P(m)$ is true for all m with $1 \le m < k$. We shall now prove $P(k)$.

Let 2^r (where $r \in \mathbb{N}$) be the largest power of 2 that is $\le k$. In other words, $2^r \le k < 2^{r+1}$. Two cases arise.

Case 1. We have $k = 2^r$. In this case, k can be written as the "sum" of a single power of 2, namely 2^r. Therefore, $P(k)$ holds.

Case 2. We have $k > 2^r$. In this case, let $\ell = k - 2^r \in \mathbb{N}^+$. By the induction hypothesis, $P(\ell)$ is true, so we can write $\ell = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_1 < a_2 < \cdots < a_s$. Therefore,

$$k = \ell + 2^r = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s} + 2^s\,.$$

We're almost there, but we have to show that 2^s is distinct from all the other powers of 2 appearing above. But $k < 2^{r+1}$, so $2^{a_s} \le \ell = k - 2^r < 2^r$. It follows that $a_s < r$, which proves the distinctness. Therefore, $P(k)$ holds.

We have now proved that $P(1) \wedge P(2) \wedge \cdots \wedge P(k-1) \implies P(k)$.

Thus, by the (strong version of the) principle of mathematical induction, the proof is complete.

Alternate Solution for PS9-9. We can do the induction step differently. Let's jump right in to that part of the proof. We're trying to prove $P(k)$.

Case 1. k is even. By the induction hypothesis, $P(k/2)$ is true, so we can write $k/2 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_1 < a_2 < \cdots < a_s$. Therefore,

$$k = 2^{1+a_1} + 2^{1+a_2} + \cdots + 2^{1+a_s}$$

and these powers of 2 are in ascending order, so they are distinct. Therefore, $P(k)$ holds.

Case 2. k is odd. By the induction hypothesis, $P((k-1)/2)$ is true. This means that we can write $(k-1)/2 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_1 < a_2 < \cdots < a_s$. Therefore,

$$k = 1 + 2\left(\frac{k-1}{2}\right) = 2^0 + 2^{1+a_1} + 2^{1+a_2} + \cdots + 2^{1+a_s}\,.$$

Again, these powers of 2 are in ascending order, so they are distinct. Therefore, $P(k)$ holds.

The rest of the proof is the same as before.

Solution for PS9-10. For all integers $r \in \mathbb{N}^+$, let $Q(r)$ be the statement that for all rectangular bars of chocolate with r tiles, the number of snap operations used to break the bar into individual tiles must be exactly $r-1$. The proof is by strong induction on $Q(r)$.

Base Case $Q(1) = 0$, because a bar with one tile is already broken into individual tiles, and because the bar can not longer be broken into two distinct parts.

Induction Step For any $t \geq 2$, assume that for all $1 \leq k < t$, $Q(k)$ is true. To prove that $Q(t)$ follows, consider any rectangular bar with t tiles. Any snap operation divides the rectangle into two rectangular fragments with a and b tiles, respectively, so that $a + b = t$. The number of snaps used to reduce the bar into individual tiles is one plus the number of snaps required to resolve each of the two fragments:

$$Q(t) = 1 + Q(a) + Q(b) = 1 + (a-1) + (b-1) = a + b - 1 = t - 1.$$

This is always the same value, no matter how the bar is partitioned into two parts.

By strong induction, $Q(t) = t - 1$ for all rectangular chocolate bars. As an $m \times n$ bar has mn tiles, it uses exactly $mn - 1$ snaps to break up into individual tiles.

(The proof only uses the fact that each snap partitions the bar—a collection of tiles—into two (disjoint) smaller collections. The shape of the bar and manner of snapping do not matter.)

Solution for PS9-11. Let $P(m)$ denote the statement "if a tournament has a cycle of length m, then it has a cycle of length 3". We shall prove that for all $m \geq 3$ $(P(m))$, by induction on m.

Base case ($m = 3$). $P(3)$ is a statement of the form "if X, then X", so it is trivially true.

Assume $P(k)$ is true for some $k \geq 3$. We're going to prove $P(k+1)$.

First note that $P(k+1)$ is implicitly a "for all" statement. So, to prove it, we consider an arbitrary tournament that has a cycle of length $k+1$, consisting of the players $\{p_1, p_2, p_3, \ldots, p_k, p_{k+1}\}$, where p_1 beats p_2, who beats p_3, $\ldots$, who beats p_{k+1}, who beats p_1. Now consider first 3 players in the cycle: $\{p_1, p_2, p_3\}$. Two cases arise.

Case 1: p_3 beats p_1. Then we have a cycle of length 3, consisting of the players $\{p_1, p_2, p_3\}$, where p_1 beats p_2, who beats p_3, who beats p_1.

Case 2: p_1 beats p_3. Then our tournament has a cycle of length k, consisting of the players $\{p_1, p_3, p_4, \ldots, p_k, p_{k+1}\}$, where p_1 beats p_3, who beats p_4, $\ldots$, who beats p_{k+1}, who beats p_1. By the induction hypothesis, the existence of this cycle implies that the tournament has a cycle of length 3.

In both cases we concluded that our tournament has a cycle of length 3. Therefore $P(k+1)$ is true. This shows that $P(k) \implies P(k+1)$.

Thus, by mathematical induction, it follows that $\forall m \geq 3$ $(p(m))$. □

Solutions for Unit 10

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Solution for PS10-1.

a. If we pick just one kind of candy (two pieces of it), there are 6 choices.
If we pick two distinct kinds of candy, there are $\binom{6}{2}$ choices.

Together, there are $6+\binom{6}{2}=21$ choices.

b. We break this down into three disjoint cases and add up the results.

- All three pieces of candy are of the same kind: $\binom{6}{1}$ choices.
- We pick exactly two kinds of candy: this gives us $\binom{6}{2}$ to choose the two kinds, following which we have to choose which of the two kinds we're going to pick two pieces of. By the generalized product principle, there are $\binom{6}{2}\cdot 2$ choices overall.
- We pick three distinct kinds of candy: $\binom{6}{3}$ choices.

Solution for PS10-2.

a. With books ▯ and separator ♦, one configuration is ▯▯♦▯▯▯▯▯♦♦▯▯▯▯▯▯♦▯▯♦. If the candy types are ordered and denoted A,B,C,D,E,F, then number of each type can be determined by counting the number of books between a given pair of separators. The configuration can be written as AA♦$BBBBB$♦♦$DDDDDD$♦EE♦, so that 2 pieces of A,5 pieces of type B, 6 pieces of D, and 2 of E are chosen.

A second layout is ▯♦ ▯ ▯♦ ▯ ▯ ▯ ♦ ▯ ▯ ▯ ▯♦ ▯ ▯ ▯ ♦ ▯ ▯, producing $1,2,3,4,3,2$ counts of types A,B,C,D,E,F, respectively.

b. $\binom{20}{5}$, the number of ways to place 5 separators in a list of 20 items (books + separators).

c. Applying the book/separator model, there are t books, and $n-1$ separators. The final count is $\binom{t+n-1}{n-1}$.

Solution for PS10-3.

a. Use $f(x_1,x_2,\ldots,x_k)=\underbrace{000\cdots0}_{x_1}\,1\,\underbrace{000\cdots0}_{x_2}\,1\,\cdots\cdots\underbrace{000\cdots0}_{x_k}\,1\,\underbrace{000\cdots0}_{n-(x_1+\cdots+x_k)}$.

b. Use $g(y_1,y_2,\ldots,y_k)=(y_1,y_2-y_1,y_3-y_2,\ldots,y_k-y_{k-1})$.

c. $|S_{n,k}|=|L_{n,k}|=\begin{pmatrix}n+k\\k\end{pmatrix}$.

Solution for PS10-4.

a.

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n\cdot(n-1)!}{k\cdot(k-1)!(n-k)!}=\frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}=\frac{n}{k}\binom{n-1}{k-1}$$

b.

$$\binom{n}{m}\binom{m}{k}\cdot 1=\frac{n!}{m!(n-m)!}\cdot\frac{m!}{k!(m-k)!}\cdot\frac{(n-k)!}{(n-k)!}=\frac{n!}{k!(n-k!)}\cdot\frac{(n-k)!}{(m-k)!(n-k-(m-k))!)}=\binom{n}{k}\binom{n-k}{m-k}$$

Solution for PS10-5.

a. Let $x=1$, $y=1$. By the binomial theorem,

$$\begin{aligned}2^n=(1+1)^n=(x+y)^n&=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\\&=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k=\sum_{k=0}^{n}\binom{n}{k}.\end{aligned}$$

b. Let $x = -1$, $y = 1$. Applying the binomial theorem,

$$\begin{aligned} 0 = (1-1)^n = (x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}. \end{aligned}$$

Solution for PS10-6.

a. By the binomial theorem,

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

b. Starting with the binomial theorem on $(1+1)^{n-1}$,

$$\begin{aligned}
(1+1)^{n-1} &= \sum_{k=0}^{n-1} \binom{n-1}{k} \\
n2^{n-1} &= \sum_{k=0}^{n-1} n \binom{n-1}{k} && \text{multiplying by } n \\
n2^{n-1} &= \sum_{k=0}^{n-1} (k+1) \frac{n}{k+1} \binom{n-1}{k} \\
n2^{n-1} &= \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} && \text{by } \textbf{\textit{PS10-4 a}} \\
n2^{n-1} &= \sum_{k=1}^{n} k \binom{n}{k} && \text{reindexing} \\
n2^{n-1} &= \sum_{k=0}^{n} k \binom{n}{k} && \text{adding a term at } k = 0 \text{ whose value is } 0
\end{aligned}$$

We have derived the statement to be proven.

Solution for PS10-7.

a. Both sides of the equation equal the number of subsets of an arbitrary n-element set S. The total number of subsets is well known to be 2^n. The sum on the left hand side, counts, for each $0 \le k \le n$, the number of subsets of S containing k elements, and sums the result. Since all subsets of S are counted exactly once on the left hand side, the left hand side counts all subsets of S, and equals the right hand side. □

b. The equation is equivalent to the claim that the number of odd subsets of a given n-element set equals the number of even subsets. This is true by the previous homework problem ***PS2-5*** $^{\textit{\textbf{HW}}}$. □

c. There is a well-known committee-chairperson fable. Both expressions indicate the number of ways to pick a k person committee from n candidates, with a single designated chairperson. Using Generalized Product Principle, $k\binom{n}{k}$ is the number of ways ($\binom{n}{k}$) to pick a committee, multiplied by the number of ways to then pick a chairperson from that committee (k). $n\binom{n-1}{k-1}$ is the number of ways to pick a chairperson (n) times the number of ways to pick everyone else in the committee ($\binom{n-1}{k-1}$). □

d. A generalization of the committee-chairperson story. Given a set A of n elements, both sides express the number of ways to pick an m-element subset B of A, and a k-element subset C of B. The $\binom{n}{m}\binom{m}{k}$ is the number of ways to pick the subset B as a subset of A first, and then pick C as a subset of B; the $\binom{n}{k}\binom{n-k}{m-k}$ is the number of ways to pick C as a subset of A, and then pick $B - C$ as a subset of $A - C$. □

e. For a given set S with n elements, 3^n is the number of functions from S to $0,1,2$. For each such function f, we can define a function $g : R(f) \to 0,1$ on the set $R(f) = \{x \in S : f(x) \in \{1,2\}\}$ by $g(x) = f(x)$. Counting the number of functions g, we find that for each size k of the domain for g, there are $\binom{n}{k}$ possible k-element domains that are subsets of S, and 2^k possible functions.

Viewed in terms of the hint, there are two methods to find the number of possible sets A,B for which $B \subseteq A \subseteq \{1,2,\ldots,n\}$. First, for each $i \in \{1,2,\ldots,n\}$, there are three possible states relative to A and B: either $i \notin A$, $i \in B$, or $i \in A - B$. From this one can derive a total count of 3^n. An argument, partitioning by the size of A, as in the previous paragraph will also give $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$.

f. Both sides indicate the number of ways to pick a committee (with at least one person) from a set of n candidates. One could first pick a chairperson (n choices), and then pick the rest of the committee (2^{n-1} choices). Alternatively, one could, for each size of a committee, first pick a committee ($\binom{n}{k}$ choices), and then pick a chairperson (k choices); adding the product over all possible committee sizes gives $\sum_{k=0}^{n} k\binom{n}{k}$.

Solution for PS10-8.

a. For each k, we prove the identity by induction over n. Let $P_k(n)$ be the proposition that the equation

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

is true. We will show by induction that $P_k(n)$ is true for all $n \geq k$.

Base case. This is $P_k(k)$, which is true since

$$\sum_{m=k}^{k} \binom{m}{k} = \binom{k}{k} = 1 = \binom{k+1}{k+1}$$

Induction step. We seek to prove $P_k(n)$, for $n > k$, assuming that $P_k(n-1)$ is true. Since by $P_k(n-1)$,

$$\sum_{m=k}^{n-1} \binom{m}{k} = \binom{n}{k+1},$$

adding $\binom{n}{k}$ to both sides produces

$$\begin{aligned}\sum_{m=k}^{n-1} \binom{m}{k} + \binom{n}{k} &= \binom{n}{k+1} + \binom{n}{k}\\ \sum_{m=k}^{n} \binom{m}{k} &= \binom{n}{k+1} + \binom{n}{k}\\ \sum_{m=k}^{n} \binom{m}{k} &= \binom{n+1}{k+1},\end{aligned}$$

where the last line follows by Pascal's identity. □

b. Let $V = \{1,2,\ldots,n+1\}$. Then the number of ways to pick a $k+1$ element set $J \subset V$ is $\binom{n+1}{k+1}$. One can also pick a such a subset J by first picking the *smallest* element i_{min} of J, and then the remainder of J, ($J - \{i_{min}\}$, a set of k elements), for which there are $\binom{n+1-i_m in}{k}$ choices. As the possible values of i_{min} range from 1 to $n+1-k$, combining the number of sets produced for each value of i_{min} gives the sum $\sum_{m=1}^{n+1-k} \binom{n+1-m}{k} = \sum_{m=k}^{n} \binom{m}{k}$. □

Solution for PS10-9. If p is a prime, $0 < k < p$,

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}.$$

Since $p! = p\cdot(p-1)\cdots2\cdot1$, $p|p!$. On the other hand, since $k < p$, $p \nmid i$, for all $1 \leq i \leq k$, so by iterated applications of the contrapositive of Euclid's lemma, $p \nmid k!$. Similarly, as $p - k < p$, $p \nmid (p-k)!$.

Let $a = n!$, $b = k!(p-k)!$. Since we know $\frac{a}{b} \in \mathbb{Z}$, there is some constant $c \in \mathbb{Z}$ for which $a = bc$. Since $p \mid a$, and $p \nmid b$, by Euclid's lemma, $p \mid c$. Defining $d = \frac{c}{p} \in \mathbb{Z}$, we find $\frac{a}{b} = \frac{pdb}{b} = pd$. Since p is a factor of the integer product pd, it is a factor of $\frac{a}{b} = \binom{p}{k}$.

Solutions for Unit 11

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Solution for PS11-1.

a. For this problem, let $D = \{0,1,2,\ldots,9\}$ and $P = \{1,2,\ldots,9\}$.

- *Sample space:* $S = \{(d_1,d_2,\ldots,d_k) : \text{each } d_i \text{ is a digit}\} = D^k$.
- *Events of interest:* $E = \{(d_1,d_2,\ldots,d_k) \in S : \text{each } d_i \neq 0\} = P^k$.
- *Outcome probabilities:* Each outcome is S is given to be equally likely (see the words "independently and uniformly at random" in the problem statement), so the probability of each outcome is $1/|S|$.
- *Event Probabilities:* Because of the uniformity, $\Pr[E] = |E|/|S| = |P^k|/|D^k| = (9/10)^k$.

b. Zephyr, please fill in details.
Final answer $= \binom{90}{10}/\binom{100}{10}$.

c.
- *Sample space:* $S = \{(a,b) : a \neq b \text{ and } 1 \leq a,b \leq 5\}$.
- *Events of interest:* $E = \{(2,1),(3,1),(4,1),(5,1),\ (1,3),(2,3),(4,3),(5,3),\ (1,5),(2,5),(3,5),(4,5)\}$.
- *Outcome probabilities:* Uniform, by the given info.
- *Event Probabilities:* $\Pr[E] = |E|/|S| = 12/(5\times 4) = 3/5$.

d. Zephyr, please fill in details.
Final answer $= 1/n$.

e.
- *Sample space:* $S = \{\mathrm{H},\mathrm{T}\}^n = \{\text{length } n \text{ sequences of letters H or T}\}$.
- *Events of interest:*

$$E = \{\mathrm{HHH\cdots H},\ \mathrm{THH\cdots H},\ \mathrm{TTH\cdots H},\ \ldots,\ \mathrm{TT\cdots TH},\ \mathrm{TTT\cdots T}\}$$
$$= \{\mathrm{T}^i\mathrm{H}^{n-i} : 0 \leq i \leq n\}\,.$$

- *Outcome probabilities:* Uniform, as each character in a string of heads or tails has equal probability to be H or T. The probability of any specific string is the probability that n independent coin flips produce the string, namely $(1/2)^n$, and is the same for all strings.
- *Event Probabilities:* $\Pr[E] = |E|/|S| = (n+1)/2^n$.

f. Let D be the set of all cards in the deck, so that $|D| = 52$, S be the set of 13 spades, and H be the initial hand of five cards.

- *Sample space:* $\Omega = \{\{a,b\} \subset D - H\}$, the set of groups of two cards drawn from the set of cards not currently in the hand. As the cards are drawn from "the rest of the deck", the two discarded cards are not eligible to be selected.
- *Events of interest:* $E = \{\{a,b\} \subset S - H\}$, the set of spades that are not currently in hand.
- *Outcome probabilities:* As the cards are selected uniformly at random, the probabilities corresponding to each set of two cards are also uniform.
- *Event Probabilities:* $\Pr[E] = |E|/|S| = \binom{|S-H|}{2}/\binom{|D-H|}{2} = \binom{10}{2}/\binom{47}{2}$.

Solution for PS11-2.

a. Let D be the standard set of 52 cards.

- *Sample space:* $S = \{H \subseteq D : |H| = 5\}$.
- *Events of interest:* $E = \{H \in S : H \text{ is a full house}\}$.
- *Outcome probabilities:* Uniform, according to the given information.
- *Event Probabilities:* Because of the uniformity, $\Pr[E] = |E|/|S|$. Clearly, $|S| = \binom{52}{5}$. To count $|E|$, break down the processing of choosing five cards to create a full house as follows:

  Step 1. Choose the rank of the triplet; there are 13 choices.
  Step 2. Choose the rank of the doublet; there are 12 choices, given the previous choice.
  Step 3. Choose the suits of the three cards in the triplet; there are $\binom{4}{3} = 4$ choices.
  Step 4. Choose the suits of the two cards in the doublet; there are $\binom{4}{2} = 6$ choices.

By the generalized product principle, $|E| = 13 \times 12 \times 4 \times 6$.
Therefore, $\Pr[E] = 3744/\binom{52}{5} \approx 0.00144057623049219 68 \approx 0.14\%$.

b. Let B be the standard bag of 100 Scrabble tiles and let $C \subseteq B$ be the set of non-blank tiles: $|C| = 98$.

- *Sample space:* $S = \{R \subseteq B : |R| = 7\}$.
- *Events of interest:* $E = \{R \in S : R \nsubseteq C\}$ is the event that our rack doesn't consist only of non-blanks, i.e., that our rack contains a blank. We'll work instead with the complement $\overline{E} = \{R \subseteq C : |R| = 7\}$.
- *Outcome probabilities:* Uniform, according to the given information.
- *Event Probabilities:* Because of the uniformity,

$$\Pr[E] = \frac{|E|}{|S|} = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{\binom{98}{7}}{\binom{100}{7}} = 1 - \frac{93 \times 92}{100 \times 99} \approx 0.1357575758 \approx 13.58\%\,.$$

c. The answer is the same as before.

The easiest way to see this by using a different sample space that makes a certain symmetry clear. First, let's give each of the 100 Scrabble tiles a unique index and assume that tiles #99 and #100 are the two blanks. Now let's model the experiment as choosing 14 tiles from the bag: the first seven for your opponent and the next seven for you.

- *Sample space:* $S = \{(x_1, \ldots, x_{14}) :$ each $x_i \in \{1, \ldots, 100\}$ and $x_i \neq x_j$ for $i \neq j\}$.
- *Events of interest:* $F = \{(x_1, \ldots, x_{14}) \in S : x_i \geq 99$ for some $i \in \{8, \ldots, 14\}\}$. Let's also consider another event $G = \{(x_1, \ldots, x_{14}) \in S : x_i \geq 99$ for some $i \in \{1, \ldots, 7\}\}$; we'll soon see why.
- *Outcome probabilities:* Uniform, according to the given information.
- *Event Probabilities:* We want to compute $\Pr[F]$. In ***PS11-2 b***, we computed $\Pr[G]$ (using a different sample space to model the experiment). But now consider the function $f : F \to G$ given by

$$f(x_1, \ldots, x_7, x_8, \ldots, x_{14}) = f(x_8, \ldots, x_{14}, x_1, \ldots, x_7)\,.$$

It is a bijection from F to G (notice that $f^{-1} = f$), proving that $|F| = |G|$. Therefore,

$$\Pr[F] = |F|/|S| = |G|/|S| = \Pr[G] \approx 13.58\%\,.$$

Solution for PS11-3.

Step 1: *Define the sample space.*

Let W denote a Boston Red Sox win in a particular game, and L denote a Boston Red Sox loss.

Then, $S = \{$WW, WLW, LL, LWL, LWW,WLL$\}$.

Step 2: *Define events of interest.*

Part (a): Let A be the event that a total of 3 games are played.

So, $A = \{$WLW, LWL, LWW, WLL$\}$.

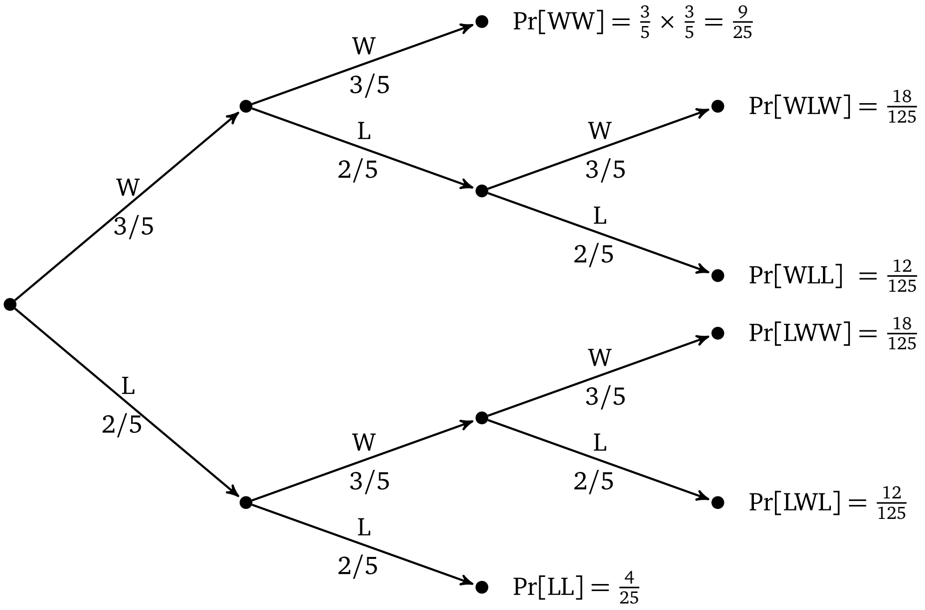
Part (b): Let B be the event that the winner of the series loses the first game.

So, $B = \{$LWW, WLL$\}$.

Part (c): Let C be the event that the *correct* team (which is obviously Red Sox) wins the series.

So, $C = \{$WW, WLW, LWW$\}$.

Step 3: *Figure out outcome probabilities.*


Pr[WW] = $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$
W
3/5
Pr[WLW] = $\frac{18}{125}$
L
2/5
W
3/5
W
3/5
L
2/5
Pr[WLL] = $\frac{12}{125}$
Pr[LWW] = $\frac{18}{125}$
W
3/5
L
2/5
W
3/5
L
2/5
Pr[LWL] = $\frac{12}{125}$
L
2/5
Pr[LL] = $\frac{4}{25}$

Sample space is set of leaves of the tree in the figure.

Step 4: *Compute event probabilities.*

$$\begin{aligned}
\Pr[A] &= \Pr[\{\text{WLW, LWL, LWW, WLL}\}] = \frac{18}{125} + \frac{12}{125} + \frac{18}{125} + \frac{12}{125} = \frac{12}{25}\,, \\
\Pr[B] &= \Pr[\{\text{LWW, WLL}\}] = \frac{18}{125} + \frac{12}{125} = \frac{6}{25}\,, \\
\Pr[C] &= \Pr[\{\text{WW,WLW, LWW}\}] = \frac{9}{25} + \frac{18}{125} + \frac{18}{125} = \frac{45+18+18}{125} = \frac{81}{125}\,.
\end{aligned}$$

Solution for PS11-4. **Part (i)**. Rolling total is 8.

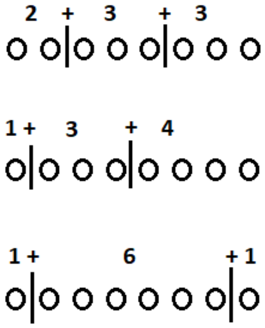
(a) When two dice are rolled, the sample space $S = \{(x, y) : 1 \le x \le 6; 1 \le y \le 6\}$
Our event of interest $A = \{(x, y) : x + y = 8; 1 \le x \le 6; 1 \le y \le 6\} = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
Each outcome is equally likely. Hence, $Pr[A] = |A|/|S| = 5/36$

(b) When three dice are rolled, the sample space $S = \{(x, y, z) : 1 \le x \le 6; 1 \le y \le 6; 1 \le z \le 6\}$
Our event of interest $B = \{(x, y, z) : x + y + z = 8; 1 \le x \le 6; 1 \le y \le 6; 1 \le z \le 6\}$
Each outcome is equally likely. Hence, $Pr[B] = |B|/|S| = |B|/216$.
So we need to find $|B|$. We shall check the possible unordered outcomes and then find the number of ways each of them can be permuted to get the number of ordered outcomes (x, y, z).
$(1,1,6) \longrightarrow 3!/2! = 3$ ways.
$(1,2,5) \longrightarrow 3! = 6$ ways.
$(1,3,4) \longrightarrow 3! = 6$ ways.
$(2,2,4) \longrightarrow 3!/2! = 3$ ways.
$(2,3,3) \longrightarrow 3!/2! = 3$ ways.
Hence, $|B| = 3 + 6 + 6 + 3 + 3 = 21$. Hence $Pr[B] = 21/216 = 7/72$.

Alternate Solution 1: Let $x = 1 + x_1$ and $y = 1 + x_2$. Then, $x_1, x_2 \ge 0$ and $x + y + z = 8 \Rightarrow x_1 + x_2 + z = 6$. Since $z \ge 1$, we get $x_1 + x_2 \le 5$. Note that this ensures $0 \le x_1, x_2 \le 5$ and hence, $x, y, z \le 6$.
Thus, to count $|B|$, it is enough to find the number of possible non-negative integer solutions to the inequality $x_1 + x_2 \le 5$.

This is exactly $|S_{5,2}|$ as defined in ***PS6-4***. So, by Part **c** of ***PS6-4***, $|S_{5,2}| = \binom{5+2}{2} = 21$.

Alternate Solution 2: Finding the value of $|B|$ is same as finding the number of possible positive integer solutions to the equation $x + y + z = 8$. (Note that x, y, z being positive and summing up to 8 ensures that $x, y, z \leq 6$.) This is same as the number of ways of partitioning 8 identical objects into 3 groups. So this is same as arranging the 8 objects in a row and finding the number of ways of placing partition markers in any 2 gaps between the elements (so that it is partitioned into $2+1=3$ groups). There are 7 gaps between the 8 elements and we choose any 2 gaps to place the markers. So this can be done in $\binom{7}{2} = 21$ ways.



Therefore, $Pr[A] = 5/36 = 10/72 > 7/72 = Pr[B]$.

Hence it is more likely to get a total of 8 when two dice are rolled than when three dice are rolled.

Part (ii). Rolling total is 9.

This is similar to **Part(i)** and we get that when two dice are rolled, the probability is 1/9 and when three dice are rolled, it is 25/216. So it is more likely to get a rolling total of 9 when three dice are rolled than when two dice are rolled.

Note: For **Part(ii)**, if you use one of the alternate methods mentioned in **Part(i)**, the equations will no longer ensure that $x, y, z \leq 6$. However, it will ensure that $x, y, z \leq 7$. So you have to eliminate the 3 cases $(1,1,7), (1,7,1), (7,1,1)$ in the end to get the correct number.

Solution for PS11-5.

Step 1: *Define the sample space.*

Let's think from the perspective of the first player. In each *round* (two tosses), let W denote the first player wins; let L denote the first player loses; let T denote neither player wins (tie). Then

$$S = \{W, L, TW, TL, TTW, TTL, \ldots\}$$

Step 2: *Define events of interest.*

At the end of the game, let E_W denote the first player wins, E_L denote the first player loses, E_T denote neither player wins. Note that the game will not stop until the winner is determined. Then

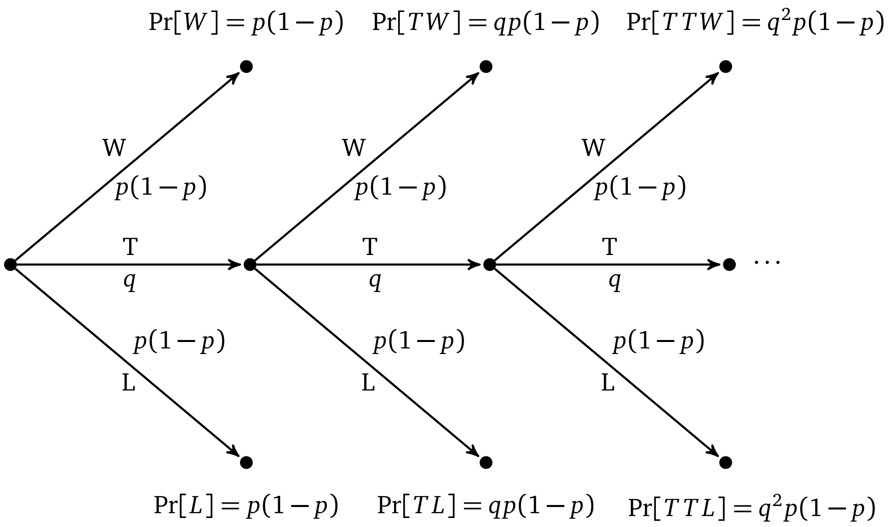
$$E_W = \{W, TW, TTW, \ldots\}$$
$$E_L = \{L, TL, TTL, \ldots\}$$
$$E_T = \phi$$

Step 3: *Figure out outcome probabilities.*

In each *round*,

$$\Pr[W] = p(1-p)$$
$$\Pr[L] = p(1-p)$$

$$Pr[T] = p^2 + (1-p)^2 = q \text{ (Say)}$$



Step 4: *Compute event probabilities.*

$$\Pr[E_W] = \Pr[\{W, TW, TTW, \ldots\}] = \sum_{i=0}^{\infty} q^i p(1-p)$$

$$\Pr[E_L] = \Pr[\{L, TL, TTL, \ldots\}] = \sum_{i=0}^{\infty} q^i p(1-p)$$

$$\Pr[E_T] = \Pr[\phi] = 0$$

We can obtain the probabilities by summing the infinite series. But let's use some neat trick here. Let $s = \Pr[E_W]$. We can observe that $\Pr[E_W] = \Pr[E_L]$, so, $s = \Pr[E_L]$. And because $E_W \cap E_L = \varnothing$, we can apply *Disjoint Sum Rule* here. So, $Pr[E_W \cup E_L] = \Pr[E_W] + \Pr[E_L] = 2s$. Besides, we know that $E_W \cup E_L = S$ is a certain event. So,

$$Pr[E_W \cup E_L] = \Pr[E_W] + \Pr[E_L] = 2s = 1$$

So,

$$s = \frac{1}{2}$$

Alternate Solution: We can see that the tree is repeating itself. In the beginning of every new round, the probability that the first player wins is always s, regardless of previous results. So, we can obtain an equation as follows:

$$s = \underbrace{p(1-p)}_{\text{wins in the first round}} + \underbrace{qs}_{\text{wins in the other rounds}}$$

$$s = p(1-p) + (p^2 + (1-p)^2)s$$

$$2p(p-1)s = p(p-1)$$

Because $0 < p < 1$

$$s = \frac{1}{2}$$

Solution for PS11-6. Let's first clarify what a strategy is. A strategy is a *plan* for a game, which tells you what to do ("take" or "skip") under *all* circumstances in the course of a game until you reach the end of the game. Under a strategy, there is a probability to win the game. So we can define a function $q(n,k,S)$ as the probability of winning when we have n cards with k black ones and we use strategy S. If we let S_0 denote the strategy "take the top card", then

$$q(n,k,S_0)=\frac{k}{n}$$

Then let's define our predicate. Let $P(n)$ denote the statement "for any k such that $0 \le k \le n$ and any strategy S, $q(n,k,S) \le q(n,k,S_0)=k/n$." We shall prove that for all $n \ge 1$ $(P(n))$, by induction on n.

Base case: $(n=1)$. As we have only one card, there is only one strategy, which is "take the top card". So, for $k=0,1$ and any strategy S, $q(n,k,S) \le q(n,k,S_0)=k/n$, i.e., $p(1)$ holds.

Induction step: $(n \ge 1)$.

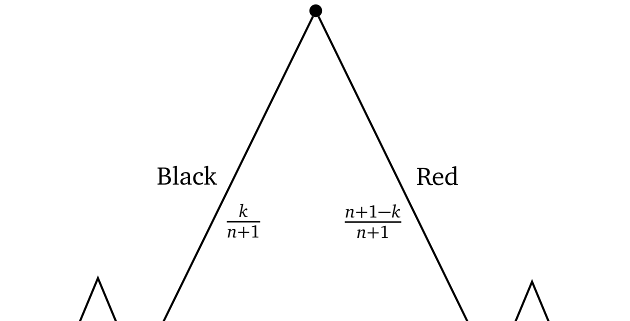
Assume $P(n)$. Consider $P(n+1)$.

if $k=0$, whatever your strategy S is, $q(n+1,k,S)$=0, because there are no black cards, which means we will never win the game. So, $q(n+1,k,S) \le q(n+1,k,S_0)=k/(n+1)$.

if $k=n+1$, whatever your strategy S is, $q(n+1,k,S)$=1, because there are all black cards, which means we will always win the game. So, $q(n+1,k,S) \le q(n+1,k,S_0)=k/(n+1)$.

if $1 \le k \le n$, consider any other strategy but S_0. We shall skip the first card, otherwise it is S_0. After the first card is revealed: if it turns out to be black (in a probability of $k/(n+1)$), we know that $k-1$ black cards remain in the rest n cards, and let S_{black} denote the following sub-strategy; if it turns out to be red (in a probability of $(n+1-k)/(n+1)$), we know that k black cards remain in the rest n cards, and let S_{red} denote the following sub-strategy. So,

$$\begin{aligned}
q(n+1,k,S) &= \frac{k}{n+1}\cdot q(n,k-1,S_{\text{black}}) + \frac{n+1-k}{n+1}\cdot q(n,k,S_{\text{red}}) \\
&\le \frac{k}{n+1}\cdot\frac{k-1}{n} + \frac{n+1-k}{n+1}\cdot\frac{k}{n} && \text{(by assumption } p(n)) \\
&= \frac{k^2-k+nk+k-k^2}{n(n+1)} \\
&= \frac{k}{n+1} \\
&= q(n+1,k,S_0)
\end{aligned}$$



So we proved that

$$q(n+1,k,S) \le q(n+1,k,S_0)$$

This is exactly $P(n+1)$. So we have shown that $P(n) \implies P(n+1)$. This completes the proof by induction.

Solution for PS11-7.

a. $\Pr[A] = \Pr[(A-B)\cup(A\cap B)] = \Pr[A-B] + \Pr[A\cap B]$

b. $1 = \Pr[S] = \Pr[A] + \Pr[\overline{A}]$, where S is the sample space.

c. $\Pr[A\cup B] = \Pr[(A-B)\cup B] = \Pr[A-B] + \Pr[B]$; now use Part **a**.

d. Use Part **c** and the fact that $\Pr[A\cap B] \ge 0$.

e. $\Pr[B] = \Pr[A\cup(B-A)] = \Pr[A] + \Pr[B-A] \ge \Pr[A]$.

Selected Solutions for Unit 12

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Solution for PS12-1. There is a tempting intuitive approach to the first question: "Given the value of one die, the other die still equally likely to be any of the six possible values, so the probability of hitting the exact value required to win is $1/6$." This is incorrect!

Instead, let us do our usual four steps. Let $D = \{1, 2, 3, 4, 5, 6\}$ be the possible values shown by one die.

- *Sample space:* $S = D \times D$.
- *Events of interest:*

$$\begin{aligned} W &= \{(x, y) \in S : x + y = 7\} \text{ is the event that you won;} \\ E_6 &= \{(x, y) \in S : x = 6 \lor y = 6\} \text{ is the event that one of the dice came up six;} \\ E_5 &= \{(x, y) \in S : x = 5 \lor y = 5\} \text{ is the event that one of the dice came up five.} \end{aligned}$$

- *Outcome probabilities:* Uniform on S, since the dice are fair.
- *Event Probabilities:* The two parts of the problem are asking for $\Pr[W \mid E_6]$ and $\Pr[E_5 \mid W]$, respectively.

a. Using the definition of conditional probability,

$$\Pr[W \mid E_6] = \frac{\Pr[W \cap E_6]}{\Pr[E_6]} = \frac{|W \cap E_6|/|S|}{|E_6|/|S|} = \frac{|\{(1,6),(6,1)\}|}{|\{(1,6),\ldots,(5,6),(6,6),(6,5),\ldots,(6,1)\}|} = \frac{2}{11}\,.$$

b. Similarly,

$$\Pr[E_5 \mid W] = \frac{\Pr[E_5 \cap W]}{\Pr[W]} = \frac{|E_5 \cap W|/|S|}{|W|/|S|} = \frac{|\{(2,5),(5,2)\}|}{|\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}|} = \frac{1}{3}\,.$$

Solution for PS12-2.

a. $4/9$.

b. $5/23$.

Solution for PS12-5.

a.

$$\begin{aligned} &\Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \cdots \cdot \Pr[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}] \\ &= \Pr[A_1] \cdot \frac{\Pr[A_1 \cap A_2]}{\Pr[A_1]} \cdot \frac{\Pr[A_1 \cap A_2 \cap A_3]}{\Pr[A_1 \cap A_2]} \cdot \frac{\Pr[A_1 \cap A_2 \cap A_3 \cap A_4]}{\Pr[A_1 \cap A_2 \cap A_3]} \cdot \cdots \cdot \frac{\Pr[A_1 \cap A_2 \cap \cdots \cap A_n]}{\Pr[A_1 \cap A_2 \cap \cdots \cap A_{n-1}]} \\ &= \Pr[A_1 \cap A_2 \cap \cdots \cap A_n] \end{aligned}$$

b. For $i = 1, 2, \ldots, n$, let A_i be the event that the ith passenger (to board the flight) sits on his assigned seat.

$$\begin{aligned} \Pr[A_1 \cap A_2 \cap \cdots \cap A_k] &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \cdots \cdot \Pr[A_k \mid A_1 \cap A_2 \cap \cdots \cap A_{k-1}] \\ &= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \cdots \cdot \frac{1}{n-k+1} \\ &= \frac{(n-k)!}{n!} \end{aligned}$$

Solution for PS12-7. We'll denote the outcomes using two-letter strings: the first letter will be one of B (for Brown), D (for Dartmouth), or L (for Little Hoop); the second letter will be one of H (for happy) or U (for unhappy). Draw a tree diagram showing these outcomes: the root will have three children and each of those children will have two leaf children.

- *Sample space:* $S = \{\text{BH, BU, DH, DU, LH, LU}\}$.
- *Events of interest:*

$$\begin{aligned}\text{Happy} &= \{\text{BH, DH, LH}\}\,; \\ \text{Brown} &= \{\text{BH, BU}\}\,; \\ \text{Dartmouth} &= \{\text{DH, DU}\}\,.\end{aligned}$$

- *Outcome probabilities:* Label the tree diagram using the given numbers, then compute:

$$\begin{aligned}\Pr[\text{BH}] &= \frac{4}{12}\cdot\frac{4}{12} = \frac{16}{144}\,; & \Pr[\text{BU}] &= \frac{4}{12}\cdot\frac{8}{12} = \frac{32}{144}\,; \\ \Pr[\text{DH}] &= \frac{5}{12}\cdot\frac{7}{12} = \frac{35}{144}\,; & \Pr[\text{DU}] &= \frac{5}{12}\cdot\frac{5}{12} = \frac{25}{144}\,; \\ \Pr[\text{LH}] &= \frac{3}{12}\cdot\frac{11}{12} = \frac{33}{144}\,; & \Pr[\text{LU}] &= \frac{3}{12}\cdot\frac{1}{12} = \frac{3}{144}\,.\end{aligned}$$

- *Event Probabilities:* Computed below.

a. $\Pr[\text{Happy}] = \Pr[\text{BH,DH,LH}] = 84/144 = 7/12$.

b. $\Pr[\text{Brown} \mid \text{Happy}] = \dfrac{\Pr[\text{BH}]}{\Pr[\text{Happy}]} = \dfrac{16/144}{7/12} = \dfrac{4}{21}$.

c. Observe that $\Pr[\text{Brown} \mid \text{Happy}] \neq \Pr[\text{Brown}]$.

d. Observe that $\Pr[\text{Happy} \mid \text{Dartmouth}] = \Pr[\text{Happy}]$.

Solution for PS12-8. Uniform probability space on $\{1,2,3,4,5,6\}$; take $A = \{2,4,5,6\}$, $B = \{2,4,5\}$, $C = \{1,2,3\}$.

Solution for PS12-10.

a. Just as in the original Monty Hall game, $\Pr[\text{GP}] = 1/3$, because the prize is equally likely to be behind any particular door.

b. If the contestant does not pick the prize door, then the prize is behind one of the two remaining doors, both equally likely. When Carol picks a random door from among these two, she reveals the prize with probability $1/2$. Therefore, $\Pr[\text{OP} \mid \overline{\text{GP}}] = 1/2$.

c. We use the law of total probability:

$$\Pr[\text{OP}] = \Pr[\text{OP} \mid \text{GP}]\cdot\Pr[\text{GP}] + \Pr[\text{OP} \mid \overline{\text{GP}}]\cdot\Pr[\overline{\text{GP}}] = 0 \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}\,.$$

d. In each round of the game, the probability that Carol will open the prize door (causing it to continue for at least one more round) is precisely $\Pr[\text{OP}]$, which we just calculated to be $1/3$. Since the rounds are independent,

$$\begin{aligned}\Pr[\text{game continues at least to round } n+1] &= \prod_{i=1}^{n} \Pr[\text{Carol opens prize door in } i\text{th round}] \\ &= \prod_{i=1}^{n} \Pr[\text{OP}] = \frac{1}{3^n}\,.\end{aligned}$$

Thus, the probability that the game will continue forever is $\lim_{n\to\infty} 1/3^n = 0$.

e. These probabilities are as follows.

i) When GP occurs, the contestant has chosen the prize door and the strategy of sticking with the choice is going to win. So $\Pr[W \mid \text{GP}] = 1$.

ii) When $\overline{\mathrm{GP}} \cap \mathrm{OP}$ occurs, the game gets restarted, which means we're back to square one. The probability of winning is again $\Pr[W]$. Thus, $\Pr[W \mid \overline{\mathrm{GP}} \cap \mathrm{OP}] = \Pr[W] = w$.

iii) When $\overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}$ occurs, the initial guess was wrong, the game ends in the first round, and by sticking with his initial choice, the contestant is guaranteed to lose. So, $\Pr[W \mid \overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] = 0$.

f. Using the law of total probability, we have

$$\begin{aligned} w &= \Pr[W] \\ &= \Pr[\mathrm{GP}] \cdot \Pr[W \mid \mathrm{GP}] + \Pr[\overline{\mathrm{GP}} \cap \mathrm{OP}] \cdot \Pr[W \mid \overline{\mathrm{GP}} \cap \mathrm{OP}] + \Pr[\overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] \cdot \Pr[W \mid \overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] \\ &= \frac{1}{3} \cdot 1 + \Pr[\mathrm{OP} \mid \overline{\mathrm{GP}}] \cdot \Pr[\overline{\mathrm{GP}}] \cdot w + \Pr[\overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] \cdot 0 \\ &= \frac{1}{3} + \frac{1}{2}\left(1 - \frac{1}{3}\right) w + 0 \\ &= \frac{1+w}{3}\,. \end{aligned}$$

Solving this equation gives us the answer: $w = 1/2$.

g. In the modified game, there are three possibilities for each outcome of the overall random experiment.

i) The contestant would win by using a "stick" strategy.

ii) The contestant would win by using a "switch" strategy.

iii) The game simply continues forever.

Because of the third possibility, we can't *immediately* conclude that Pr[win using "switch" strategy] $= 1 - \Pr[W]$. Instead, we conclude that this probability equals $1 - \Pr[W] -$ Pr[game continues forever].

However, we have computed Pr[game continues forever] $= 0$, so in fact the desired probability is still equal to $1 - \Pr[W]$, i.e., the conclusion is still sound.

Selected Solutions for Unit 13

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Solution for PS13-1. In each case, the sample space is $\mathbb{Z}_n$ and the probability function is $\Pr[j] = 1/n$ for all $j \in \mathbb{Z}_n$.

- ***a.*** $\operatorname{Ex}[Y] = \sum_{j=0}^{n-1} Y(j)\Pr[j] = (1/n)\sum_{j=0}^{n-1}\gcd(j,n) = (1/7)\left(7+\sum_{j=1}^{6} 1\right) = 13/7$.
- ***b.*** In this case, $\gcd(j,9)$ equals 9 for one value of j (namely, $j=0$), equals 3 for two values of j (namely, $j=3$ and $j=6$), and equals 1 for the remaining six values of j. Therefore, $\operatorname{Ex}[Y] = (1/9)(9\times 1+3\times 2+1\times 6) = 21/9 = 7/3$.
- ***c.*** By similar reasoning, $\operatorname{Ex}[Y] = (1/15)(15\times 1+3\times 4+5\times 2+1\times 8) = 45/15 = 3$.
- ***d.*** Please solve this yourself. The final answer is $3-2/p$.
- ***e.*** Please solve this yourself. The final answer is $(2-1/p)(2-1/q)$.

Solution for PS13-3. Let the random variable X denote the amount (in dollars) that you win. Then $\operatorname{range}(X) = \{0, 10^7\}$ and $\Pr[X = 10^7] = 1/\binom{50}{6}$. Therefore,

$$\operatorname{Ex}[X] = 0\cdot\Pr[X=0] + 10^7\cdot\Pr[X=10^7] = \frac{10^7}{\binom{50}{6}} \approx 0.63\,.$$

Since this is below the ticket's price of \$1, the ticket is not worth its price.

Solution (Sketch) for PS13-4.

- ***a.*** Let X_j be the indicator r.v. for the event "$W_j = 6$." Then $\operatorname{Ex}[X_j] = \Pr[X_j = 1] = \Pr[W_j = 6] = 1/6$.

  The number of sixes seen is $Y := \sum_{j=1}^{24} X_j$, so by linearity of expectation,

  $$\operatorname{Ex}[Y] = \operatorname{Ex}\left[\sum_{j=1}^{24} X_j\right] = \sum_{j=1}^{24}\operatorname{Ex}[X_j] = 4\,.$$
- ***b.*** This doesn't affect our answer. Linearity of expectation always holds and has nothing to do with correlation (or independence).

Solution for PS13-6.

- ***a.*** Let's use the sample space $\{1,2,3,4,5,6\}$ for the experiment of rolling the red die. Upon conditioning on the event "X is a perfect square," the probability function is as follows:

  $$\Pr[1] = \frac{1}{2}\,,\quad \Pr[4] = \frac{1}{2}\,,\quad \Pr[2] = \Pr[3] = \Pr[5] = \Pr[6] = 0\,.$$

  Therefore, $\operatorname{Ex}[X^2 \mid X \text{ is a perfect square}] = 1^2\times\frac{1}{2} + 4^2\times\frac{1}{2} = 17/2$.
- ***b.*** We compute

  $$\begin{aligned}\operatorname{Ex}[WX] &= \operatorname{Ex}[X^2 + XY] = \operatorname{Ex}[X^2] + \operatorname{Ex}[X]\operatorname{Ex}[Y]\,,\\ \operatorname{Ex}[W]\operatorname{Ex}[X] &= \operatorname{Ex}[X+Y]\operatorname{Ex}[X] = \operatorname{Ex}[X]^2 + \operatorname{Ex}[X]\operatorname{Ex}[Y]\,.\end{aligned}$$

  The two are unequal because $\operatorname{Ex}[X^2] \neq \operatorname{Ex}[X]^2$, by direct computation.

Solution (Sketch) for PS13-7. Number the pairs of students as pair 1, pair 2, ..., pair $\binom{n}{2}$ in some manner.
Let Y be the number of bonds. Let X_i be the indicator r.v. for the event "pair i forms a bond."
Then $\operatorname{Ex}[X_i] = \Pr[X_i = 1] = 1/d$, so

$$\operatorname{Ex}[Y] = \operatorname{Ex}\left[\sum_{i=1}^{\binom{n}{2}} X_i\right] = \sum_{i=1}^{\binom{n}{2}}\operatorname{Ex}[X_i] = \frac{1}{d}\binom{n}{2}\,.$$

Selected Solutions for Unit 13

Solution (Sketch) for PS13-9. Let Y be the number of bins that remain empty. Let X_i be the indicator r.v. for the event "bin i remains empty."

Then $\mathrm{Ex}[X_i] = \Pr[X_i = 1] = (1-1/n)^n$, so $\mathrm{Ex}[Y] = \mathrm{Ex}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathrm{Ex}[X_i] = n(1-1/n)^n$.

If you've taken calculus, you know that $\lim_{n\to\infty}(1-1/n)^n = 1/e$, so $\mathrm{Ex}[Y] \approx n/e$ for large n.

Solution (Sketch) for PS13-11. Please do this computation. The final answer is $\dfrac{n^2-1}{3n}$.

Solution (Sketch) for PS13-12. The square of the previous expression, with $n = 1000$.