Problem Set for Unit 10

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Here are some problems on **binomial coefficients** and **combinatorial proofs**.

PS10-1

We'll now study the fourth (and most complicated) case of the four-fold formulas. For starters, let's consider special cases.

You are in a candy store. There are six (6) kinds of candy on offer and the store has plenty of pieces of each kind in stock. You love all six kinds on offer. You just want to take home as much candy as your parent will allow!

a. You have been allowed to pick two (2) pieces of candy to take home. The two pieces may be of the same kind or of different kinds. In how many different ways can you make your picks?

b. Suppose, instead, that you have been allowed to pick three (3) pieces. How does the answer change? The new answer is $\binom{6}{1} + 2\binom{6}{2} + \binom{6}{3}$. How did I get this?

PS10-2

It's your lucky day: you have been allowed to pick 15 pieces of candy from the above candy store! In how many ways can you make your choice now?

It's going to be tedious to generalize the expressions you wrote in the previous problem, so you try another idea. Visualize a row of 15 books laid out on a bookshelf, to represent the 15 pieces of candy you'll pick. Now you want to assign a kind of candy to each book: remember that there are 6 kinds of candy in the store. To do so, visualize 5 separators placed on the same bookshelf, dividing up the row of books into 6 sections. The number of books in the jth section will correspond to the number of pieces of the jth kind of candy you'll pick.

a. Draw a picture showing two different bookshelf layouts with the 15 books and 5 separators. For each of the layouts, write down the candy choices they indicate.

b. In how many ways can you lay out 15 books and 5 separators on a bookshelf? The books are to be treated as indistinguishable from one another and so are the separators.

c. Generalize! Suppose the candy store had n kinds of candy on offer and you are allowed to take home t pieces (repetitions allowed, as usual). How many books and how many separators should you use to represent your possible picks? Based on this, what is the number of ways to pick t pieces of candy from a store than offers n kinds of candy?

PS10-3 *HW*

Let $n, k \in \mathbb{N}^+$.

a. Let $S_{n,k}$ be the possible nonnegative integer solutions to the inequality $x_1 + x_2 + \cdots + x_k \le n$. That is,

$$S_{n,k} = \{(x_1, x_2, \dots, x_k) \in \mathbb{N}^k : x_1 + x_2 + \cdots + x_k \le n\}.$$

Construct a bijection from $S_{n,k}$ to the set of bit strings that have exactly n zeroes and exactly k ones. You should define a function precisely, and prove that your function is indeed a bijection.

b. Let $L_{n,k}$ be the length-k non-decreasing sequences of nonnegative integers $\le n$. That is,

$$L_{n,k} = \{(y_1, y_2, \dots, y_k) \in \mathbb{N}^k : y_1 \le y_2 \le \cdots \le y_k \le n\}.$$

Construct a bijection from $L_{n,k}$ to $S_{n,k}$. As usual, you should define a function precisely, and prove that your function is indeed a bijection.

c. Based on your work above, determine the cardinalities $|S_{n,k}|$ and $|L_{n,k}|$. [5+5+2 points]

Do you see the connection between your answers above and the fourth of the four-fold formulas you developed in ***PS10-2 c***?

PS10-4

By using the factorials formula for binomial coefficients, give algebraic proofs of the following identities.

a. For all $k, n \in \mathbb{N}$ with $k \le n$: $\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$.

b. For all $k, m, n \in \mathbb{N}$ with $k \le m \le n$: $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

PS10-5

Write down the binomial theorem, i.e., expand $(x+y)^n$. Now, by plugging in appropriate values for x and y in the binomial theorem, give algebraic proofs of each of the following identities.

a. For all $n \in \mathbb{N}$: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

b. For all $n \in \mathbb{N}^+$: $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

PS10-6 ^{HW}

Give algebraic proofs of the following identities.

a. For all $n \in \mathbb{N}$: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$.

b. For all $n \in \mathbb{N}$: $\displaystyle\sum_{k=0}^{n} k \binom{n}{k} = n\,2^{n-1}$.

You can solve the latter problem by using ***PS10-7 c*** and ***PS10-7 a*** appropriately. Alternatively, if you know basic calculus, you can do it by expanding $(1+x)^n$ and taking a derivative. [3+5 points]

PS10-7

Give combinatorial proofs for the following identities, which you've already proved algebraically.

a. For all $n \in \mathbb{N}$: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

b. For all $n \in \mathbb{N}^+$: $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

c. For all $k, n \in \mathbb{N}$ with $k \le n$: $k\binom{n}{k} = n\binom{n-1}{k-1}$.

d. For all $k, m, n \in \mathbb{N}$ with $k \le m \le n$: $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

e. For all $n \in \mathbb{N}$: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$.

Hint: Consider choosing a set $A \subseteq \{1, 2, \ldots, n\}$ and then a subset $B \subseteq A$.

f. For all $n \in \mathbb{N}$: $\displaystyle\sum_{k=0}^{n} k \binom{n}{k} = n\,2^{n-1}$.

Hint: Consider choosing a committee and then a chairperson of that committee.

PS10-8 ^{HW}

Consider the following identity, which holds for all $k, n \in \mathbb{N}$ with $k \le n$:

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

a. Prove the above identity by induction, making use of Pascal's identity in the inductive step.

b. Give a combinatorial proof of the same identity. You will have to be creative! [5+5 points]

PS10-9

Prove that if p is a prime and $0 < k < p$, then p divides $\binom{p}{k}$.

PS10-10 ^{EC}

Give a combinatorial proof of the following identity, where n is a positive integer.

$$\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1\,.$$

PS10-11 ^{EC}

Using mathematical induction and the binomial theorem, give an alternate proof of the following version of Fermat's little theorem.

$$\forall\, a, p \in \mathbb{N} \text{ with } p \text{ prime}, \quad p \mid a^p - a\,.$$

Do not use anything about multiplicative inverses modulo p.