Problem Set for Unit 3

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Here are some problems about **countability and uncountability**. Use these as practice problems to strengthen your understanding as you do the reading corresponding to this unit. Problems marked "HW" are to be submitted as part of your weekly written homework for HW2, which covers this unit and the previous one.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ have their usual meanings.

PS3-1 *HW*

Prove the following basic facts about odd and even integers. Remember that integers can be negative or zero, and that zero is even.

- ***a.*** The sum of two even integers is even. [2 points]
- ***b.*** The sum of two odd integers is even. [2 points]
- ***c.*** The sum of an odd integer and an even integer is odd. [2 points]
- ***d.*** The product of an even integer and an arbitrary integer is even. [2 points]
- ***e.*** The product of two odd integers is odd. [3 points]

PS3-2

Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(a, b) = 2^a 3^b$. In class, we proved that f is injective by using the powerful *unique factorization theorem* (UFT), a.k.a., the *fundamental theorem of arithmetic*. Give a different proof that uses just simple algebra and observations about odd and even numbers, without using UFT.

Suppose that we have arbitrary $a, b, c, d \in \mathbb{N}$ such that

$$2^a 3^b = 2^c 3^d \,. \tag{1}$$

- ***a.*** Consider the case when $b = d$. Prove that $(a, b) = (c, d)$.
- ***b.*** Now consider the case when $b \neq d$. Say $b < d$. Rewrite Eq. (1) in the form $2^p = 3^q$ with $q \in \mathbb{N}$.
- ***c.*** Based on ***PS3-1***, conclude that 3^q is odd.
- ***d.*** Based on ***PS3-1*** and the previous part, conclude that $p = 0$.
- ***e.*** Based on all of the above, conclude that $(a, b) = (c, d)$.
- ***f.*** Wrap up the proof that f is injective.

PS3-3

Prove that if A is a countable set and $B \subseteq A$, then B is countable.

PS3-4

Let A and B be two countable sets.

- ***a.*** Prove that $A \cup B$ is countable.
- ***b.*** Prove that $A \times B$ is countable.

PS3-5

Prove that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

PS3-6 *HW*

Let A be an infinite set. Prove that there exists a surjection $f : A \to \mathbb{N}$. [7 points]

Warning: A proof that tries to "list all the elements of A" is flawed, because A might not be countable.

PS3-7

Given a character set S (sometimes called an *alphabet*), we can consider *strings* formed from the characters in S. Formally:

- An *alphabet* is a nonempty finite set.
- Having chosen an alphabet S, each of its elements is called a character.
- A string is a finite-length sequence of zero or more characters.
- The set of all such strings (over the alphabet S) is denoted S^*.

Prove that S^* is countable.

Hint: Come up with a systematic scheme for listing all the strings in S^*.

PS3-8

Argue that the set of all conceivable Python programs is countable.

PS3-9

The open interval $(0,1)$ can be visualized as a line segment within the number line and the Cartesian product $(0,1) \times (0,1)$ can be visualized as a unit square in the 2-D plane. You might expect that the latter set, being two-dimensional, has way more elements than the former, but you would be wrong!

Construct an injection $f : (0,1) \times (0,1) \to (0,1)$.

PS3-10 ^{HW}

Construct a bijection $g : (0,1] \to (0,1)$. [7 points]

The notation $(0,1]$ denotes the half-open interval $\{x \in \mathbb{R} : 0 < x \leq 1\}$.

Hint: You can come up with a construction where $g(x) = x$ "most of the time" though obviously you can't do this for $x = 1$. So make $g(1) = \frac{1}{2}$. But now what should you do with $g(\frac{1}{2})$?

PS3-11 ^{EC}

Is the injection you constructed in ***PS3-9*** in fact a bijection? If so, prove it. If not, explain clearly why not and then construct a bijection $h : (0,1) \times (0,1) \to (0,1)$.