# Problem Set for Unit 8

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Here are some problems on **basic counting principles**. Do read the posted slides and relevant sections from the [LLM] book before you begin.

Reminder: You are expected to solve all the problems on every problem set, even though you only have to turn in a few for credit. This is especially important in the run up to the first midterm. A student who is thoroughly practiced will do well in the exam. The course staff is allowed to discuss the non-HW problems very thoroughly in the office hours, up to explaining solutions in full.

In all cases, you must demonstrate *how* you arrived at your final answers—i.e., you must show your steps—unless the problem statement makes an exception. Without such explanation, even a correct answer is worth nothing. You must also justify any steps that are not trivial. Please think carefully about how you are going to organize your answers *before* you begin writing.

***PS8-1***

Solve each of the following counting problems. In each case, explain how you obtained your answer (i.e., refer to the posted slides and name the counting principle(s) you used).

***a.*** An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

***b.*** Alice picks a card out of a standard 52-card deck. Then Bob picks a card from the ones that remain. Overall, how many different outcomes can there be?

***c.*** How many of the integers between 1 and 1000 (inclusive) are multiples of either 3 or 5?

***PS8-2*** $^{HW}$

Same instructions as above.

***a.*** A multiple-choice test contains 10 questions. There are 4 possible answers for each question. In how many ways can you answer the questions on the test if...

(a) ...you *must* answer every question?

(b) ...you can leave answers blank? [2+2 points]

***b.*** How many of the billion numbers in the range from 1 to $10^9$ contain the digit 1? [4 points]

***PS8-3*** $^{HW}$

Using only algebra and elementary arithmetic—i.e., without writing a computer program—determine the sum of all the integers between 1 and 1000 (inclusive) that are multiples of either 3 or 5. You must show all your steps of algebra clearly. You're allowed to use a calculator at the final step, where you'll need to multiply and add/subtract a handful of numbers. [5 points]

***PS8-4***

Each user on a certain computer system has a password, which is six to eight characters long, where each character is a letter (either uppercase or lowercase) or a digit. Each password must contain at least one digit. How many possible passwords are there?

***PS8-5***

The password rules in the above computer system have been modified and now each password must contain at least one uppercase letter, at least one lowercase letter, and at least one digit. How many possible passwords are there now?

***PS8-6*** $^{HW}$

Use the generalized product principle to solve the following counting problems.

***a.*** Let $D = \{n \in \mathbb{Z} : 0 \leq n \leq 9\}$. Determine $|T|$, where $T = \{(x, y, z) \in D^3 : x + y + z \text{ is even}\}$. [4 points]

***b.*** Let $n \geq 2$. How many $n$-digit natural numbers have the property that the sum of their digits is even? Note that the leftmost digit (i.e., most significant digit) cannot be zero. [4 points]

***PS8-7***

A *permutation* of a sequence is another sequence obtained by rearranging its terms. For instance, the three-term sequence (apple, pear, mango) has six permutations, shown below.

(apple, mango, pear) (apple, pear, mango) (mango, apple, pear)
(mango, pear, apple) (pear, apple, mango) (pear, mango, apple)

Notice that a sequence is considered to be a permutation of itself.

***a.*** How many permutations does the four-term sequence $(1, 3, 8, 9)$ have?

***b.*** How many permutations does an $n$-term sequence have, assuming the terms are all distinct?

***PS8-8** HW*

We have seen that an $n$-term sequence *with distinct terms* has exactly $n!$ permutations. But what if the terms are not distinct?

***a.*** How many permutations does the four-term sequence $(1, 1, 4, 9)$ have? Don't just write down a formula and calculate; explain why your formula is correct. [4 points]

Hint: For starters, pretend that one of two 1s is "red" and the other is "black" so that you can tell them apart. Now apply the division principle.

***b.*** How many anagrams does the word "CONDESCENDENCE" have? This is the same as asking how many permutations the sequence $(C, O, N, D, E, S, C, E, N, D, E, N, C, E)$ has. [6 points]

Hint: Generalize your reasoning in the previous problem. You may refer to results derived in either of the recommended textbooks. If you do so, point out precisely which results you are using.

***PS8-9***

Let $A$ and $B$ be finite sets with $|A| = m$ and $|B| = n$.

***a.*** How many relations are there from $A$ to $B$?

***b.*** How many functions are there from $A$ to $B$?

***c.*** How many injective functions are there from $A$ to $B$?

***d.*** How many bijections are there from $A$ to $B$?

***PS8-10** HW*

A *palindrome* is a string that is identical to its reversal (in other words, it reads the same backwards as forwards). How many $n$-bit strings are palindromes?

Express your answer succinctly (i.e., avoid multiple cases) by cleverly using the "ceiling" function, $\lceil x \rceil$, defined as the smallest integer $\geq x$. [4 points]

***PS8-11***

Suppose that 13 people on a softball team show up for a game.

***a.*** How many ways are there to choose 10 players to take the field?

***b.*** How many ways are there to assign the 10 positions by selecting from the players who showed up?

***c.*** Of the 13 who showed up, 11 are students and the other 2 are professors. How many ways are there to choose 10 players to take the field if at least one of these players must be a professor?

***PS8-12***

How many bit strings of length 10 contain...

***a.*** ...exactly four 1s?

***b.*** ...at most four 1s?

***c.*** ...at least four 1s?

***d.*** ...an equal number of 0s and 1s?

***PS8-13***

Solve all parts of Problem 15.12 (about seating 8 students around a circular table) from the [LLM] book. Make sure you are able to explain every step of your calculations.