Problem Set for Unit 13

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Here are some problems on **random variables and expectation**.

PS13-1

Let $n \geq 2$ be an integer. We choose a random integer $X \in \mathbb{Z}_n$ uniformly. Let $Y = \gcd(X, n)$. Determine $\mathrm{Ex}[Y]$ in each of the following cases.

a. $n = 7$.

b. $n = 9$.

c. $n = 15$.

d. $n = p^2$, where p is a prime.

e. $n = pq$, where p and q are distinct primes.

PS13-2 ^{HW}

We flip a fair coin repeatedly until either it comes up heads twice or we have flipped it six times. What is the expected number of times we flip the coin?

PS13-3

A lottery ticket costs \$1. It contains six distinct integers, each from the set $S := \{1, 2, \ldots, 50\}$. The ticket wins and pays off \$10 million iff, on the day of the drawing, the six winning numbers chosen from S all appear on the ticket (the drawing is without replacement, so the same integer cannot be drawn more than once).

Perform an expected value analysis and answer this: is the lottery ticket worth its price?

PS13-4

We simultaneously roll 24 fair dice, and they show numbers $W_1, \ldots, W_{24}$.

a. How many sixes do we expect to see? In other words, compute $\mathrm{Ex}\big[|\{j : W_j = 6\}|\big]$.

b. Even though each die is fair, they have been connected together by very thin weightless threads and this causes $W_1, \ldots, W_{24}$ to be correlated in some unknown way. (For example, it may be that whenever W_1 is even, W_2 is more likely to be even than odd; or that whenever W_8 is a prime number, W_{20} is sure to be prime; or both of the above.) How does this affect your answer above?

PS13-5 ^{HW}

The final exam of a discrete mathematics course consists of 50 true/false questions, each worth two points, followed by 25 multiple-choice questions, each worth four points. Zoe answers each true/false question correctly with 90% probability and each multiple-choice question correctly with 80% probability.

Let the random variable X denote Zoe's score on the exam. Let Y_i be an indicator random variable for the event "Zoe gets question i correct," for $1 \leq i \leq 75$.

a. Express X in terms of $Y_1, \ldots, Y_{75}$.

b. Compute $\mathrm{Ex}[Y_i]$, for each i.

c. Use linearity of expectation to compute Zoe's expected score on the exam.

PS13-6

We roll two fair dice, a *red* die and a *blue* die, and they show numbers X and Y, respectively (these are therefore random variables). Let $W = X + Y$.

1. Compute $\mathrm{Ex}[X^2 \mid X \text{ is a perfect square}]$.
2. Show that $\mathrm{Ex}[WX] \neq \mathrm{Ex}[W]\,\mathrm{Ex}[X]$.

PS13-7

Two people that have the same birthday are said to form a *calendrical bond*. Assuming that the n students in a class have birthdays distributed uniformly among the d days in a year, and birthdays are mutually independent, what is the expected number of calendrical bonds among students in the class? Derive a formula in terms of n and d, then apply it to our CS30 class, using $n = 40$ and $d = 365$.

Hint: The random variable of interest here is a sum of $\binom{n}{2}$ indicator RVs.

PS13-8 *HW*

The following fragment of C code finds the maximum value in an array `arr` consisting of n integers:

```
1 max = arr[0];
2 for(i = 1; i < n; i++)
3    if(arr[i] > max)
4       max = arr[i];
```

In words, we pick the zeroth element of the array and store it in `max`, then for each successive element of the array, if it exceeds `max` then we update `max` with that element.

Determine the expected number of times that `max` is updated—i.e., Line 4 is executed—assuming that the elements of `arr` are all distinct and arranged in a uniformly random order. (Careful: I did not say that the *elements* are random, it's their *order* which is random.)

PS13-9

(This problem has special significance in Computer Science, for it is meant to model the process of inserting keys into a hash table.) There are n bins, initially all empty. Then n balls are thrown randomly (uniformly) and independently into the bins: "uniformly" means that each ball is equally likely to go into each of the bins. What is the expected number of bins that remain empty after this process?

PS13-10 *HW*

Prove these useful facts about expectation. Each proof can be written in a few lines of algebra.

a. Let I_A be an indicator random variable for an event A. Prove that $\mathrm{Ex}[I_A] = \Pr[A]$.

b. Prove the **law of total expectation**, which states that if X is a random variable on a probability space $(\mathscr{S}, \Pr)$ and $A_1, \ldots, A_n \subseteq \mathscr{S}$ are pairwise disjoint events such that $A_1 \cup \cdots \cup A_n = \mathscr{S}$, then

$$\mathrm{Ex}[X] = \sum_{j=1}^{n} \mathrm{Ex}[X \mid A_j] \Pr[A_j] .$$

c. Let X be a nonnegative integer valued random variable on a probability space $(\mathscr{S}, \Pr)$ where $\mathscr{S}$ is a finite set. For each integer $k \geq 0$, let A_k be the event "$X \geq k$." Prove that

$$\mathrm{Ex}[X] = \sum_{k=1}^{\infty} \Pr[A_k] .$$

PS13-11

Two independent random variables, X and Y, are each drawn uniformly from $\{1, 2, \ldots, n\}$, where $n \geq 1$ is an integer. What is $\mathrm{Ex}[|X - Y|]$?

PS13-12

A computer user, working on a 1000×1000 image (measured in pixels), makes a rectangular selection of a portion of the image by clicking on a pixel P and another pixel Q. The selected rectangle is the one that has P and Q as opposite corners. If P and Q are chosen at random, independently and uniformly, then what is the expected area of the selected rectangle?

Hint: What do you know about the expectation of the product of two independent random variables?

PS13-13 [EC]

Upon moving into her new house with one front door and one back door, Professor Random places three pairs of walking shoes at each door; all six pairs are distinct. From then on, she starts the following morning routine. Each morning she picks a door at random (uniformly), puts on a pair of shoes at that door, takes a walk outside and returns to a door chosen at random (uniformly), leaving the shoes at *that* door. What is the expected number of walks that Professor Random takes until one morning she finds that her chosen exit door has no walking shoes available?