Problem Set for Unit 1

Here are some problems about **sets, relations, and functions**. Use these as practice problems to strengthen your understanding as you do the reading corresponding to this unit. The topics you will want to read up on are listed at the end of the slides for this unit. Problems marked "HW" are to be submitted as your weekly written homework for HW1, which covers this unit.

Unmarked problems are for your own practice only and will not be graded.

As a reminder, here are our notations for some important sets.

$$\begin{aligned} \mathbb{Z} &= \text{the set of all integers}\,, \\ \mathbb{N} &= \text{the set of all non-negative integers}\,, \\ \mathbb{N}^+ &= \text{the set of all positive integers}\,, \\ \mathbb{R} &= \text{the set of all real numbers}\,. \end{aligned}$$

PS1-1

Here are some sets described in set-builder notation. Describe each of them in roster notation.

a. $\{x : x \text{ is a multiple of 7 and } 0 < x < 50\}$.

b. $\{x + y : x \in \mathbb{N}, y \in \mathbb{N}, \text{ and } xy = 12\}$.

c. $\{S : S \subseteq \{1, 2, 3, 4\} \text{ and } |S| \text{ is odd}\}$.

PS1-2 ^{HW}

Here are some sets described in set-builder notation. Describe each of them in roster notation. You can write each answer on a single line and you do not need to show any steps.

a. $\{x^3 : x \in \mathbb{Z} \text{ and } x^2 < 20\}$ [2 points]

b. $\{x \in \mathbb{R} : x = x^2\}$. [2 points]

c. $\{S : \{1, 2\} \subseteq S \subseteq \{1, 2, 3, 4\}\}$ [2 points]

d. $\{S \subseteq \{1, 2, 3, 4\} : S \text{ is disjoint from } \{2, 3\}\}$ [2 points]

PS1-3

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{0, 1, 5, 6, 9\}$. In the following subproblems, show your steps for those cases where the statement asks you to "verify" an equation. For the rest, you do not need to show any steps.

a. What is $A \cup B$? What is $(A \cup B) \cup C$?

b. What is $B \cup C$? What is $A \cup (B \cup C)$?

c. What is $A \cap B \cap C$?

d. Verify by direct computation that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

e. What is $A - B$? What is $B - C$?

f. What is $(A - B) - C$? What is $A - (B - C)$?

g. Verify by direct computation that $(A - B) - C = A - (B \cup C)$.

h. Verify by direct computation that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

i. What is $(A \cap B) \times (B - C)$?

j. Verify by direct computation that $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.

PS1-4

Let A, B, and C be arbitrary sets. Prove each of the following statements. Review the slides and be sure you understand how to prove that two sets are equal.

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

b. $(A - C) \cap (C - B) = \varnothing$.

c. $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$. ◁ It may help to draw a Venn diagram.

PS1-5 ^{HW}
Let A, B, and C be arbitrary sets, within some universal set. Prove each of the following statements as indicated.

a. Using Venn diagrams to justify your steps, prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. It's not enough to *just* draw diagrams: you must write down your steps of reasoning in full English sentences. [4 points]

b. Similarly, prove that $(A - B) - C = (A - C) - (B - C)$. [4 points]

c. Without diagrams, working algebraically, prove that $(A \cap B) \cup (A \cap \overline{B}) = A$. Don't *just* write some algebraic steps; your proof must use complete and grammatical English sentences. [4 points]

PS1-6 ^{HW}
Let $S = \{1, 3, 5, 7, 9\}$ and $T = \{0, 2, 4, 6, 8\}$. Let's say that an element $x \in S$ "completes" an element $y \in T$ if $x + y$ is divisible by 3. Describe the relation "completes" from S to T as a subset of $S \times T$ (i.e., write out all the pairs in this relation). Then describe the same relation pictorially, using arrows, as done in class. [4 points]

PS1-7
A relation R with the property that

whenever $(a, b) \in R$, we also have $(b, a) \in R$

is called a *symmetric relation*. A relation S with the property that

whenever $(a, b) \in R$ and $(b, c) \in R$, we also have $(a, c) \in R$

is called a *transitive relation*. For each of the following relations, state whether or not it is (a) symmetric; (b) transitive. Whenever your answer is "no", explain why. This means that if, for instance, you say that a relation R is not symmetric, you must exhibit a pair (a, b) such that $(a, b) \in R$ but $(b, a) \notin R$.

a. The relation "divides", on $\mathbb{N}$ ("m divides n" means "n/m is an integer").

b. The relation "is disjoint from", on $\mathscr{P}(\mathbb{Z})$.

c. The relation "is no larger than", on $\mathscr{P}(\mathbb{Z})$. We say that A is no larger than B when one of the following holds:

- A and B are both finite sets, and $|A| \le |B|$.
- A is a finite set and B is an infinite set.
- A and B are both infinite sets.

PS1-8 ^{HW}
Same instructions as the previous problem, ***PS1-7***.

a. The relation "is a subset of", on $\mathscr{P}(\mathbb{Z})$. [4 points]

b. $\{(m, n) \in \mathbb{N} \times \mathbb{N} :$ the sum of the digits of m equals the sum of the digits of $n\}$. [4 points]

c. The relation "overlapped" on the set of all US presidents. Two persons are said to "overlap" if there exists an instant in time when they were both alive. [4 points]

PS1-9 Let $S = \{$"RED", "BLUE", "GREEN", "YELLOW", "ORANGE", "BLACK"$\}$ and $T = \{1, 2, 3, 4, 5, 6\}$. Consider the function $\text{len}\colon S \to T$ given by $\text{len}(s) =$ the length of the string s (as in the Python programming language).

a. Describe the "len" function pictorially, using arrows, as done in class.

b. Reverse the directions of all the arrows in your picture. Does this new picture represent a function $g : T \to S$. If not, why not?

Problem Set for Unit 1

PS1-10 Let $f : A \to B$ be a function. Then the basic notation $f(x)$ applies to elements $x \in A$, but let's now extend the notation to subsets $S \subseteq A$, by *defining* $f(S) = \{f(x) : x \in S\}$.

The set $f(S) \subseteq B$ is called the *image* of S under f. Prove the following fact about images.

$$\text{If } S_1, S_2 \subseteq A, \text{ then } f(S_1 \cup S_2) = f(S_1) \cup f(S_2).$$

Write out the steps of your reasoning. Notice that you are being asked to prove equality between two sets (you know what to do, right?).

PS1-11

Suppose that $g : A \to B$ and $f : B \to C$ are two functions. Then we define the *composition* $f \circ g$ to be the function from A to C given by

$$(f \circ g)(x) = f(g(x)), \text{ for all } x \in A.$$

The functions $f, g : \mathbb{R} \to \mathbb{R}$ are given by the formulas $f(x) = x^2 + 1$ and $g(x) = x + 2$. Find $f \circ g$ and $g \circ f$.

PS1-12 ***HW***

The functions $f, \mathrm{id} : \mathbb{R} \to \mathbb{R}$ are given by the formulas $f(x) = x^3 + 7$ and $\mathrm{id}(x) = x$. You may recall that "id" is called the *identity function* on $\mathbb{R}$.

a. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = \mathrm{id}$. [2 points]

b. For the function g you found above, find $g \circ f$. [2 points]