Class Exercises: 2019-10-15

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These exercises are about basic counting. Some of them will continue the devlopment of ideas and methods we touched upon in the lecture. Do good work on them, understand them well, and seek help from your Ninja as needed.

Here are some important counting principles for *finite* sets.

- *Sum Principle.* If A and B are disjoint (i.e., $A \cap B = \varnothing$), then $|A \cup B| = |A| + |B|$.
- *Extended Sum Principle.* If $A_1, \ldots, A_k$ are pairwise disjoint, then $|A_1 \cup \cdots \cup A_k| = |A_1| + \cdots + |A_k|$.
- *Generalized Sum Principle.* For all sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$.
- *Product Principle.* For all sets A and B, $|A \times B| = |A| \cdot |B|$.
- *Extended Product Principle.* For all sets $A_1, A_2, \ldots, A_k$, $|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdots |A_k|$.
- *Generalized Product Principle.* If we have to make a sequence of k choices and
  - there are n_1 ways to make the first choice,
  - for each first choice, there are n_2 ways to make the second choice,
  - for each of the first two choices, there are n_3 ways to make the third choice,
  - $\cdots$
  - for each of the first $k-1$ choices, there are n_k ways to make the kth choice,

  then the overall number of ways to make the entire sequence of choices is $n_1 n_2 \cdots n_k$.
- *Bijection Principle.* If there exists a bijection $f : A \to B$, then $|A| = |B|$.
- *Division Principle.* If there exists an r-to-1 correspondence $f : A \to B$, then $|B| = |A|/r$.

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1. Solve each of the following counting problems. In each case, name the counting principle(s) you used.

   1.1. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

   1.2. Alice picks a card out of a standard 52-card deck. Then Bob picks a card from the ones that remain. Overall, how many different outcomes can there be?

   1.3. How many of the integers between 1 and 1000 (inclusive) are multiples of either 3 or 5?

2. Solve each of the following counting problems. In each case, name the counting principle(s) you used. You may need to further extend one of the given counting principles. Get help from your Ninja as needed.

   2.1. Each user on a certain computer system has a password, which is six to eight characters long, where each character is a letter (either uppercase or lowercase) or a digit. Each password must contain at least one digit. How many possible passwords are there?

   2.2. The password rules in the above system have been modified. Now each password must contain at least one uppercase letter, at least one lowercase letter, and at least one digit. How many possible passwords are there now?

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3. This part is warm-up. A *permutation* of a sequence is another sequence obtained by rearranging its terms. For instance, the three-term sequence (apple, pear, mango) has six permutations, shown below.

(apple, mango, pear)	(apple, pear, mango)	(mango, apple, pear)
(mango, pear, apple)	(pear, apple, mango)	(pear, mango, apple)

   Notice that a sequence is considered to be a permutation of itself.

   3.1. How many permutations does the four-term sequence $(1, 3, 8, 9)$ have?

   3.2. Generalize! How many permutations does an n-term sequence have, assuming the terms are all distinct?

   Now for the real problem. Suppose $|S| = n$. We're going to work out the number of k-element subsets of S. Define

$$\mathcal{T} = \{ (a_1, a_2, \ldots, a_k) \in S^k : \ a_i \neq a_j \text{ whenever } i \neq j \} ;$$
$$\mathcal{U} = \{ A \subseteq S : \ |A| = k \} .$$

   Study these definitions carefully!

   3.3. To solidify your understanding, redefine $\mathcal{T}$ in words like this: "$\mathcal{T}$ is the set of all k-tuples such that..."

   3.4. Determine $|\mathcal{T}|$ using the extended product principle.

   3.5. Define the function $\text{MAKESET} : \mathcal{T} \to \mathcal{U}$ by

$$\text{MAKESET}(a_1, a_2, \ldots, a_k) = \{a_1, a_2, \ldots, a_k\} .$$

   For what value of r is MAKESET an r-to-1 correspondence?

   3.6. Apply the division principle to determine $|\mathcal{U}|$.

4. Suppose that 13 people on a softball team show up for a game.

   4.1. How many ways are there to choose 10 players to take the field?

   4.2. How many ways are there to assign the 10 positions by selecting from the players who showed up?

   4.3. Of the 13 who showed up, 11 are students and the other 2 are professors. How many ways are there to choose 10 players to take the field if at least one of these players must be a professor?