***Solution for PS11-1.***

***a.*** For this problem, let $D = \{0,1,2,\ldots,9\}$ and $P = \{1,2,\ldots,9\}$.

- *Sample space:* $S = \{(d_1,d_2,\ldots,d_k) : \text{each } d_i \text{ is a digit}\} = D^k$.
- *Events of interest:* $E = \{(d_1,d_2,\ldots,d_k) \in S : \text{each } d_i \neq 0\} = P^k$.
- *Outcome probabilities:* Each outcome is $S$ is given to be equally likely (see the words "independently and uniformly at random" in the problem statement), so the probability of each outcome is $1/|S|$.
- *Event Probabilities:* Because of the uniformity, $\Pr[E] = |E|/|S| = |P^k|/|D^k| = (9/10)^k$.

***b.*** Zephyr, please fill in details.
Final answer $= \binom{90}{10}/\binom{100}{10}$.

***c.***
- *Sample space:* $S = \{(a,b) : a \neq b \text{ and } 1 \leq a,b \leq 5\}$.
- *Events of interest:* $E = \{(2,1),(3,1),(4,1),(5,1),\ (1,3),(2,3),(4,3),(5,3),\ (1,5),(2,5),(3,5),(4,5)\}$.
- *Outcome probabilities:* Uniform, by the given info.
- *Event Probabilities:* $\Pr[E] = |E|/|S| = 12/(5 \times 4) = 3/5$.

***d.*** Zephyr, please fill in details.
Final answer $= 1/n$.

***e.***
- *Sample space:* $S = \{\mathrm{H},\mathrm{T}\}^n = \{\text{length } n \text{ sequences of letters H or T}\}$.
- *Events of interest:*

$$\begin{aligned} E &= \{\mathrm{HHH\cdots H},\ \mathrm{THH\cdots H},\ \mathrm{TTH\cdots H},\ \ldots,\ \mathrm{TT\cdots TH},\ \mathrm{TTT\cdots T}\} \\ &= \{\mathrm{T}^i\mathrm{H}^{n-i} : 0 \leq i \leq n\}\,. \end{aligned}$$

- *Outcome probabilities:* Uniform, as each character in a string of heads or tails has equal probability to be H or T. The probability of any specific string is the probability that $n$ independent coin flips produce the string, namely $(1/2)^n$, and is the same for all strings.
- *Event Probabilities:* $\Pr[E] = |E|/|S| = (n+1)/2^n$.

***f.*** Let $D$ be the set of all cards in the deck, so that $|D| = 52$, $S$ be the set of 13 spades, and $H$ be the initial hand of five cards.

- *Sample space:* $\Omega = \{\{a,b\} \subset D - H\}$, the set of groups of two cards drawn from the set of cards not currently in the hand. As the cards are drawn from "the rest of the deck", the two discarded cards are not eligible to be selected.
- *Events of interest:* $E = \{\{a,b\} \subset S - H\}$, the set of spades that are not currently in hand.
- *Outcome probabilities:* As the cards are selected uniformly at random, the probabilities corresponding to each set of two cards are also uniform.
- *Event Probabilities:* $\Pr[E] = |E|/|S| = \binom{|S-H|}{2}/\binom{|D-H|}{2} = \binom{10}{2}/\binom{47}{2}$.

***Solution for PS11-2.***

***a.*** Let $D$ be the standard set of 52 cards.

- *Sample space:* $S = \{H \subseteq D : |H| = 5\}$.
- *Events of interest:* $E = \{H \in S : H \text{ is a full house}\}$.
- *Outcome probabilities:* Uniform, according to the given information.
- *Event Probabilities:* Because of the uniformity, $\Pr[E] = |E|/|S|$. Clearly, $|S| = \binom{52}{5}$. To count $|E|$, break down the processing of choosing five cards to create a full house as follows:

  Step 1. Choose the rank of the triplet; there are 13 choices.
  Step 2. Choose the rank of the doublet; there are 12 choices, given the previous choice.
  Step 3. Choose the suits of the three cards in the triplet; there are $\binom{4}{3} = 4$ choices.
  Step 4. Choose the suits of the two cards in the doublet; there are $\binom{4}{2} = 6$ choices.

By the generalized product principle, $|E| = 13 \times 12 \times 4 \times 6$.
Therefore, $\Pr[E] = 3744/\binom{52}{5} \approx 0.0014405762304921968 \approx 0.14\%$.

***b.*** Let $B$ be the standard bag of 100 Scrabble tiles and let $C \subseteq B$ be the set of non-blank tiles: $|C| = 98$.

- *Sample space:* $S = \{R \subseteq B : |R| = 7\}$.
- *Events of interest:* $E = \{R \in S : R \nsubseteq C\}$ is the event that our rack doesn't consist only of non-blanks, i.e., that our rack contains a blank. We'll work instead with the complement $\overline{E} = \{R \subseteq C : |R| = 7\}$.
- *Outcome probabilities:* Uniform, according to the given information.
- *Event Probabilities:* Because of the uniformity,

$$\Pr[E] = \frac{|E|}{|S|} = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{\binom{98}{7}}{\binom{100}{7}} = 1 - \frac{93 \times 92}{100 \times 99} \approx 0.1357575758 \approx 13.58\%\,.$$

***c.*** The answer is the same as before.

The easiest way to see this by using a different sample space that makes a certain symmetry clear. First, let's give each of the 100 Scrabble tiles a unique index and assume that tiles #99 and #100 are the two blanks. Now let's model the experiment as choosing 14 tiles from the bag: the first seven for your opponent and the next seven for you.

- *Sample space:* $S = \{(x_1, \ldots, x_{14}) :$ each $x_i \in \{1, \ldots, 100\}$ and $x_i \neq x_j$ for $i \neq j\}$.
- *Events of interest:* $F = \{(x_1, \ldots, x_{14}) \in S : x_i \geq 99$ for some $i \in \{8, \ldots, 14\}\}$. Let's also consider another event $G = \{(x_1, \ldots, x_{14}) \in S : x_i \geq 99$ for some $i \in \{1, \ldots, 7\}\}$; we'll soon see why.
- *Outcome probabilities:* Uniform, according to the given information.
- *Event Probabilities:* We want to compute $\Pr[F]$. In ***PS11-2 b***, we computed $\Pr[G]$ (using a different sample space to model the experiment). But now consider the function $f : F \to G$ given by

$$f(x_1, \ldots, x_7, x_8, \ldots, x_{14}) = f(x_8, \ldots, x_{14}, x_1, \ldots, x_7)\,.$$

It is a bijection from $F$ to $G$ (notice that $f^{-1} = f$), proving that $|F| = |G|$. Therefore,

$$\Pr[F] = |F|/|S| = |G|/|S| = \Pr[G] \approx 13.58\%\,.$$

***Solution for PS11-3.***

*Step 1*: *Define the sample space.*

Let W denote a Boston Red Sox win in a particular game, and L denote a Boston Red Sox loss.

Then, $S = \{$WW, WLW, LL, LWL, LWW,WLL$\}$.

*Step 2*: *Define events of interest.*

Part (a): Let $A$ be the event that a total of 3 games are played.

So, $A = \{$WLW, LWL, LWW, WLL$\}$.

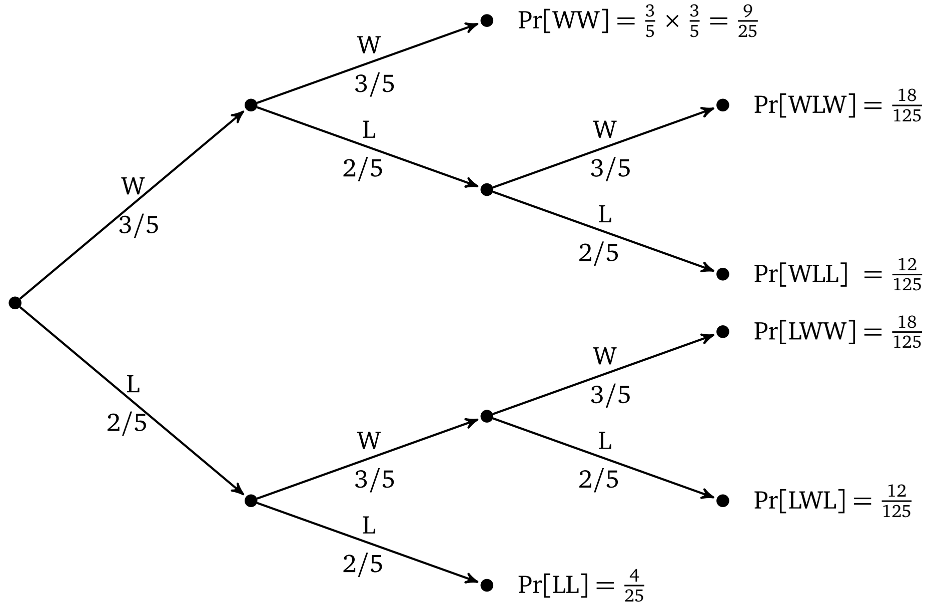
Part (b): Let $B$ be the event that the winner of the series loses the first game.

So, $B = \{$LWW, WLL$\}$.

Part (c): Let $C$ be the event that the *correct* team (which is obviously Red Sox) wins the series.

So, $C = \{$WW, WLW, LWW$\}$.

*Step 3*: *Figure out outcome probabilities.*




Sample space is set of leaves of the tree in the figure.

*Step 4*: *Compute event probabilities.*

$$\Pr[A] = \Pr[\{\text{WLW, LWL, LWW, WLL}\}] = \frac{18}{125} + \frac{12}{125} + \frac{18}{125} + \frac{12}{125} = \frac{12}{25},$$
$$\Pr[B] = \Pr[\{\text{LWW, WLL}\}] = \frac{18}{125} + \frac{12}{125} = \frac{6}{25},$$
$$\Pr[C] = \Pr[\{\text{WW,WLW, LWW}\}] = \frac{9}{25} + \frac{18}{125} + \frac{18}{125} = \frac{45+18+18}{125} = \frac{81}{125}.$$

***Solution for PS11-4.*** **Part (i)**. Rolling total is 8.

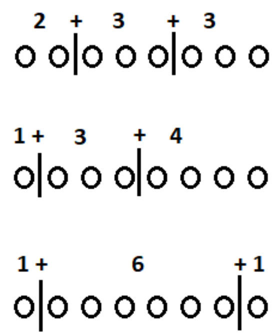
(a) When two dice are rolled, the sample space $S = \{(x, y) : 1 \le x \le 6; 1 \le y \le 6\}$
Our event of interest $A = \{(x, y) : x + y = 8; 1 \le x \le 6; 1 \le y \le 6\} = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$.
Each outcome is equally likely. Hence, $Pr[A] = |A|/|S| = 5/36$

(b) When three dice are rolled, the sample space $S = \{(x, y, z) : 1 \le x \le 6; 1 \le y \le 6; 1 \le z \le 6\}$
Our event of interest $B = \{(x, y, z) : x + y + z = 8; 1 \le x \le 6; 1 \le y \le 6; 1 \le z \le 6\}$
Each outcome is equally likely. Hence, $Pr[B] = |B|/|S| = |B|/216$.
So we need to find $|B|$. We shall check the possible unordered outcomes and then find the number of ways each of them can be permuted to get the number of ordered outcomes $(x, y, z)$.
$(1,1,6) \longrightarrow 3!/2! = 3$ ways.
$(1,2,5) \longrightarrow 3! = 6$ ways.
$(1,3,4) \longrightarrow 3! = 6$ ways.
$(2,2,4) \longrightarrow 3!/2! = 3$ ways.
$(2,3,3) \longrightarrow 3!/2! = 3$ ways.
Hence, $|B| = 3+6+6+3+3 = 21$. Hence $Pr[B] = 21/216 = 7/72$.

***Alternate Solution 1:*** Let $x = 1 + x_1$ and $y = 1 + x_2$. Then, $x_1, x_2 \ge 0$ and $x + y + z = 8 \Rightarrow x_1 + x_2 + z = 6$.
Since $z \ge 1$, we get $x_1 + x_2 \le 5$. Note that this ensures $0 \le x_1, x_2 \le 5$ and hence, $x, y, z \le 6$.
Thus, to count $|B|$, it is enough to find the number of possible non-negative integer solutions to the inequality $x_1 + x_2 \le 5$.

This is exactly $|S_{5,2}|$ as defined in ***PS6-4***. So, by Part **c** of ***PS6-4***, $|S_{5,2}| = \binom{5+2}{2} = 21$.

***Alternate Solution 2:*** Finding the value of $|B|$ is same as finding the number of possible positive integer solutions to the equation $x + y + z = 8$. (Note that $x, y, z$ being positive and summing up to 8 ensures that $x, y, z \leq 6$.) This is same as the number of ways of partitioning 8 identical objects into 3 groups. So this is same as arranging the 8 objects in a row and finding the number of ways of placing partition markers in any 2 gaps between the elements (so that it is partitioned into $2+1=3$ groups). There are 7 gaps between the 8 elements and we choose any 2 gaps to place the markers. So this can be done in $\binom{7}{2} = 21$ ways.




Therefore, $Pr[A] = 5/36 = 10/72 > 7/72 = Pr[B]$.

Hence it is more likely to get a total of 8 when two dice are rolled than when three dice are rolled.

**Part (ii)**. Rolling total is 9.

This is similar to **Part(i)** and we get that when two dice are rolled, the probability is $1/9$ and when three dice are rolled, it is $25/216$. So it is more likely to get a rolling total of 9 when three dice are rolled than when two dice are rolled.

**Note:** For **Part(ii)**, if you use one of the alternate methods mentioned in **Part(i)**, the equations will no longer ensure that $x, y, z \leq 6$. However, it will ensure that $x, y, z \leq 7$. So you have to eliminate the 3 cases $(1,1,7), (1,7,1), (7,1,1)$ in the end to get the correct number.

***Solution for PS11-5.***

*Step 1*: *Define the sample space.*

Let's think from the perspective of the first player. In each *round* (two tosses), let $W$ denote the first player wins; let $L$ denote the first player loses; let $T$ denote neither player wins (tie). Then

$$S = \{W, L, TW, TL, TTW, TTL, \ldots\}$$

*Step 2*: *Define events of interest.*

At the end of the game, let $E_W$ denote the first player wins, $E_L$ denote the first player loses, $E_T$ denote neither player wins. Note that the game will not stop until the winner is determined. Then

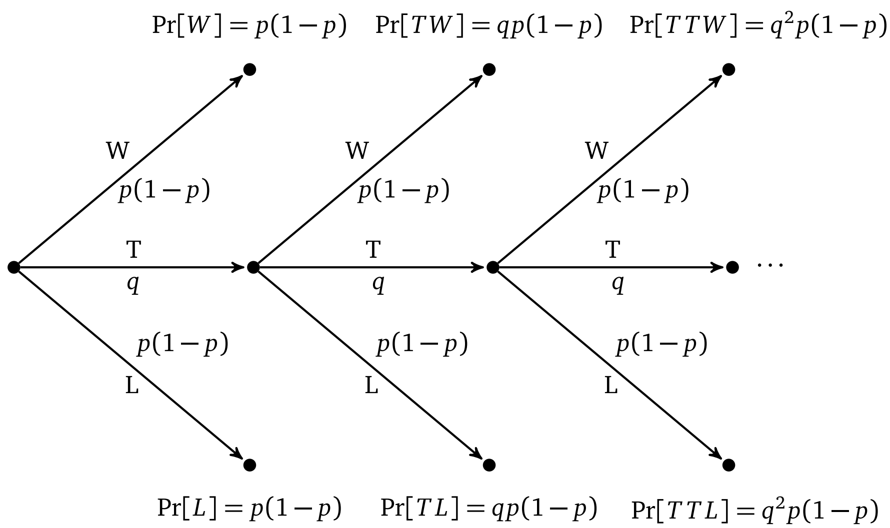
$$E_W = \{W, TW, TTW, \ldots\}$$
$$E_L = \{L, TL, TTL, \ldots\}$$
$$E_T = \phi$$

*Step 3*: *Figure out outcome probabilities.*

In each *round*,

$$\Pr[W] = p(1-p)$$
$$\Pr[L] = p(1-p)$$

$$Pr[T] = p^2 + (1-p)^2 = q \text{ (Say)}$$




*Step 4*: *Compute event probabilities.*

$$\Pr[E_W] = \Pr[\{W, TW, TTW, \ldots\}] = \sum_{i=0}^{\infty} q^i p(1-p)$$

$$\Pr[E_L] = \Pr[\{L, TL, TTL, \ldots\}] = \sum_{i=0}^{\infty} q^i p(1-p)$$

$$\Pr[E_T] = \Pr[\phi] = 0$$

We can obtain the probabilities by summing the infinite series. But let's use some neat trick here. Let $s = \Pr[E_W]$. We can observe that $\Pr[E_W] = \Pr[E_L]$, so, $s = \Pr[E_L]$. And because $E_W \cap E_L = \emptyset$, we can apply *Disjoint Sum Rule* here. So, $Pr[E_W \cup E_L] = \Pr[E_W] + \Pr[E_L] = 2s$. Besides, we know that $E_W \cup E_L = S$ is a certain event. So,

$$Pr[E_W \cup E_L] = \Pr[E_W] + \Pr[E_L] = 2s = 1$$

So,

$$s = \frac{1}{2}$$

**Alternate Solution:** We can see that the tree is repeating itself. In the beginning of every new round, the probability that the first player wins is always $s$, regardless of previous results. So, we can obtain an equation as follows:

$$s = \underbrace{p(1-p)}_{\text{wins in the first round}} + \underbrace{qs}_{\text{wins in the other rounds}}$$

$$s = p(1-p) + (p^2 + (1-p)^2)s$$

$$2p(p-1)s = p(p-1)$$

Because $0 < p < 1$

$$s = \frac{1}{2}$$

***Solution for PS11-6.*** Let's first clarify what a strategy is. A strategy is a *plan* for a game, which tells you what to do ("take" or "skip") under *all* circumstances in the course of a game until you reach the end of the game. Under a strategy, there is a probability to win the game. So we can define a function $q(n,k,S)$ as the probability of winning when we have $n$ cards with $k$ black ones and we use strategy S. If we let $S_0$ denote the strategy "take the top card", then

$$q(n,k,S_0) = \frac{k}{n}$$

Then let's define our predicate. Let $P(n)$ denote the statement "for any $k$ such that $0 \le k \le n$ and any strategy $S$, $q(n,k,S) \le q(n,k,S_0) = k/n$." We shall prove that for all $n \ge 1$ ($P(n)$), by induction on $n$.

Base case: ($n = 1$). As we have only one card, there is only one strategy, which is "take the top card". So, for $k = 0, 1$ and any strategy $S$, $q(n,k,S) \le q(n,k,S_0) = k/n$, i.e., $p(1)$ holds.

Induction step: ($n \ge 1$).

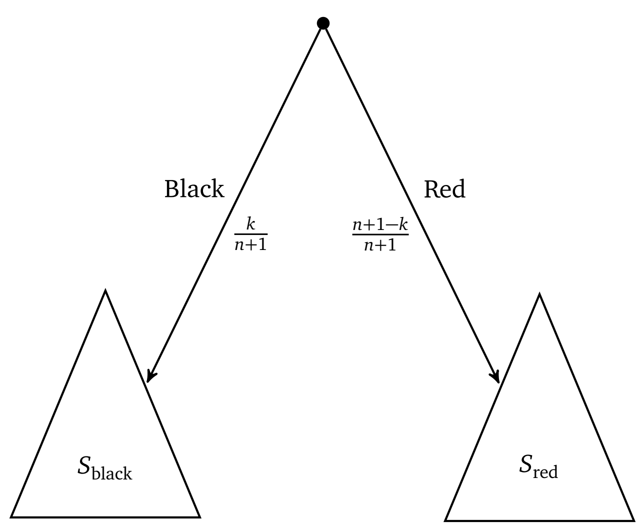
Assume $P(n)$. Consider $P(n+1)$.

if $k = 0$, whatever your strategy $S$ is, $q(n+1,k,S)$=0, because there are no black cards, which means we will never win the game. So, $q(n+1,k,S) \le q(n+1,k,S_0) = k/(n+1)$.

if $k = n+1$, whatever your strategy $S$ is, $q(n+1,k,S)$=1, because there are all black cards, which means we will always win the game. So, $q(n+1,k,S) \le q(n+1,k,S_0) = k/(n+1)$.

if $1 \le k \le n$, consider any other strategy but $S_0$. We shall skip the first card, otherwise it is $S_0$. After the first card is revealed: if it turns out to be black (in a probability of $k/(n+1)$), we know that $k-1$ black cards remain in the rest $n$ cards, and let $S_{\text{black}}$ denote the following sub-strategy; if it turns out to be red (in a probability of $(n+1-k)/(n+1)$), we know that $k$ black cards remain in the rest $n$ cards, and let $S_{\text{red}}$ denote the following sub-strategy. So,

$$\begin{aligned}
q(n+1,k,S) &= \frac{k}{n+1} \cdot q(n,k-1,S_{\text{black}}) + \frac{n+1-k}{n+1} \cdot q(n,k,S_{\text{red}}) \\
&\le \frac{k}{n+1} \cdot \frac{k-1}{n} + \frac{n+1-k}{n+1} \cdot \frac{k}{n} && \text{(by assumption } p(n)) \\
&= \frac{k^2-k+nk+k-k^2}{n(n+1)} \\
&= \frac{k}{n+1} \\
&= q(n+1,k,S_0)
\end{aligned}$$



So we proved that

$$q(n+1,k,S) \le q(n+1,k,S_0)$$

This is exactly $P(n+1)$. So we have shown that $P(n) \implies P(n+1)$. This completes the proof by induction.

***Solution for PS11-7.***

***a.*** $\Pr[A] = \Pr[(A-B) \cup (A \cap B)] = \Pr[A-B] + \Pr[A \cap B]$

***b.*** $1 = \Pr[S] = \Pr[A] + \Pr[\overline{A}]$, where $S$ is the sample space.

***c.*** $\Pr[A \cup B] = \Pr[(A-B) \cup B] = \Pr[A-B] + \Pr[B]$; now use Part **a**.

***d.*** Use Part **c** and the fact that $\Pr[A \cap B] \ge 0$.

***e.*** $\Pr[B] = \Pr[A \cup (B-A)] = \Pr[A] + \Pr[B-A] \ge \Pr[A]$.