Class Exercises: 2019-09-26

Prof. Amit Chakrabarti
Computer Science Department
Dartmouth College

1. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(a,b) = 2^a 3^b$. Prove that f is injective using just simple algebra and observations about odd and even numbers, without using the powerful Unique Factorization Theorem (UFT). You'll need the following facts:

   - The product of an even integer and an arbitrary integer is even.
   - The product of two odd integers is odd.

   Suppose that we have arbitrary $a, b, c, d \in \mathbb{N}$ such that

   $$2^a 3^b = 2^c 3^d \,. \tag{1}$$

   1.1. Consider the case when $b = d$. Prove that $(a, c) = (b, d)$.

   1.2. Now consider the case when $b \neq d$. Say $b < d$. Rewrite Eq. (1) in the form $2^p = 3^q$ with $q \in \mathbb{N}$.

   1.3. Based on the facts noted above, conclude that 3^q is odd.

   1.4. Based on the facts noted above and the previous part, conclude that $p = 0$.

   1.5. Based on all of the above, conclude that $(a, c) = (b, d)$.

   1.6. Wrap up the proof that f is injective.

2. Problem Set 3 asks you to prove the following fact:

   - If A and B are countable sets, then $A \times B$ is countable.

   Using the above fact, prove that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

3. Given a character set S (sometimes called an *alphabet*), we can consider *strings* formed from the characters in S. Formally:

   - An *alphabet* is a nonempty finite set.
   - Having chosen an alphabet S, each of its elements is called a character.
   - A string is a finite-length sequence of zero or more characters.
   - The set of all such strings (over the alphabet S) is denoted S^*.

   3.1. Prove that S^* is countable.
   Hint: Come up with a systematic scheme for listing all the strings in S^*.

   3.2. Argue that the set of all conceivable Python programs is countable.