Problem Set for Unit 2

Here are some problems about **functions**. Use these as practice problems to strengthen your understanding as you do the reading corresponding to this unit. The topics you will want to read up on are listed at the end of the slides for this unit. Problems marked "HW" are to be submitted as part of your weekly written homework for HW2, which covers this unit and the next unit.

Problems marked "EC" are for extra credit. They are meant to provide a higher level for challenge for students who are *already comfortable* with the rest of the problem set. No one should feel pressured to submit solutions to these, as they won't count towards your grade in the course. However, if *after finishing the official homework* you are able to write up a *nice* solution to an extra credit problem, please submit it for my reading pleasure and to fuel interesting conversations outside of class.

Unmarked problems are for your own practice only and will not be graded.

PS2-1 *HW*

Suppose $f : A \to B$ is a function. Define the *relation* f^{-1} from B to A as follows:

$$f^{-1} \;=\; \{(y,x) \in B \times A :\, f(x) = y\}\,.$$

Prove the following statements.

a. If f is a bijection, then f^{-1} is a function. [4 points]

b. If f^{-1} is a function, then f is a bijection. [4 points]

We can combine the above two statements into one like this: f^{-1} is a function iff f is a bijection.

Or in symbols: f^{-1} is a function $\iff$ f is a bijection

The word "iff" and the symbol " $\iff$ " are pronounced "if and only if."

PS2-2

Let $f : A \to B$ be a function. Given subsets $S \subseteq A$ and $T \subseteq B$, we can extend the f and f^{-1} notations by making the following *definitions*:

$$f(S) \;=\; \{f(x) :\, x \in S\}\,,$$
$$f^{-1}(T) \;=\; \{x \in A :\, f(x) \in T\}\,.$$

The set $f(S) \subseteq B$ is called the *image* of S under f. The set $f^{-1}(T) \subseteq A$ is called the *preimage* of T under f.

Prove the following facts about images and preimages.

a. If $S_1, S_2 \subseteq A$, then $f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$.

b. If $T_1, T_2 \subseteq B$, then $f^{-1}(T_1 \cup T_2) = f^{-1}(T_1) \cup f^{-1}(T_2)$.

PS2-3

Let $f : B \to C$ and $g : A \to B$ be two bijections, where A, B, and C are arbitrary nonempty sets.

a. Prove that $f \circ g$ is a bijection.

b. According to ***PS2-1***, $f \circ g$ must have an inverse function. Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

PS2-4

Let S be a nonempty finite set. Consider the function $g : \mathscr{P}(S) \to \mathscr{P}(S)$ defined by $g(A) = S - A$ for all $A \subseteq S$. Give a careful proof that g is a bijection.

Use the definition of "bijection" and recall how you should prove that two sets are equal.

Problem Set for Unit 2

Prof. Amit Chakrabarti
Computer Science Department
Dartmouth College

PS2-5 *HW*

Let us denote

$$\mathscr{P}^{\text{odd}}(S) = \{A \subseteq S : |A| \text{ is an odd number}\},$$
$$\mathscr{P}^{\text{even}}(S) = \mathscr{P}(S) - \mathscr{P}^{\text{odd}}(S).$$

Make sure you understand what these notations mean by taking the *particular* 3-element set $T = \{1, 2, 3\}$ and writing out $\mathscr{P}^{\text{odd}}(T)$ and $\mathscr{P}^{\text{even}}(T)$ in roster notation. No need to turn this part in.

Now, let S be an *arbitrary* nonempty finite set. Construct a bijection $h\colon \mathscr{P}^{\text{odd}}(S) \to \mathscr{P}^{\text{even}}(S)$ and prove that your constructed function h is indeed a bijection. [7 points]

The above problem requires more thought than usual. It requires you to come up with a clever idea.

PS2-6

Let A and B be arbitrary finite sets. Explain to a friend why each of the following statements is true. Listen to your own explanation and based on that, give written proofs for each statement.

a. If there exists a surjection $f : A \to B$, then $|A| \geq |B|$.

b. If there exists an injection $g : A \to B$, then $|A| \leq |B|$.

c. If there exists a surjection $f : A \to B$ as well as an injection $g : A \to B$, then each of the functions f and g is, in fact, a bijection.

PS2-7

In class, we wrote down a bijection from $\mathbb{N}$ to $\mathbb{Z}$ by listing the integers in the following order:

$$0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \ldots\,. \tag{1}$$

Let's define the same bijection explicitly using algebraic formulas. First, define the function $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(m) = \begin{cases} (m+1)/2\,, & \text{if } m \text{ is odd,} \\ -m/2\,, & \text{if } m \text{ is even.} \end{cases}$$

Do a few computations to convince yourself that the list $f(0), f(1), f(2), \ldots$ is identical to the list in (1) above.

Now, prove that f is a bijection. Instead of using the definition of bijection, give an algebraic formula for a function $g : \mathbb{Z} \to \mathbb{N}$ such that $f \circ g = \mathrm{id}_{\mathbb{Z}}$ and $g \circ f = \mathrm{id}_{\mathbb{N}}$. Why does this prove that f is a bijection?

PS2-8

Let A be a set such that there exists an injection $f : A \to \mathbb{N}$. Prove that A is countable.

PS2-9

Prove that $\mathbb{N} \times \mathbb{N}$ is countable by constructing an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and invoking the result of ***PS2-8***.

PS2-10 *EC*

Let $\mathbb{N}^*$ denote the set of all finite-length lists (i.e., sequences) of non-negative integers. For example, here are four elements of $\mathbb{N}^*$:

$$(5, 93, 12, 0, 51)\,; \qquad (42, 42, 42)\,; \qquad (65)\,; \qquad (1, 2, 3, \ldots, 2019)$$

Give a detailed proof that $\mathbb{N}^*$ is countable.