Class Exercises: 2019-09-24

Prof. Amit Chakrabarti
Computer Science Department
Dartmouth College

1. Let $f : B \to C$ and $g : A \to B$ be two bijections, where A, B, and C are arbitrary nonempty sets.

   1.1. Prove that $f \circ g$ is a bijection.

   1.2. Recall that every bijection has an inverse function. Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

2. Define the function $f : \mathbb{N} \to \mathbb{Z}$ by
$$f(m) = \begin{cases} (m+1)/2\,, & \text{if } m \text{ is odd,} \\ -m/2\,, & \text{if } m \text{ is even.} \end{cases}$$
   First, convince yourself that the infinite lists $(f(0), f(1), f(2), \ldots)$ and $(0, 1, -1, 2, -2, 3, -3, \ldots)$ are identical.

   Now, prove that f is a bijection. Instead of using the definition of bijection, give an algebraic formula for a function $g : \mathbb{Z} \to \mathbb{N}$ such that $f \circ g = \mathrm{id}_{\mathbb{Z}}$ and $g \circ f = \mathrm{id}_{\mathbb{N}}$. Why does this prove that f is a bijection?

3. Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

   Method 1: You can directly use the definition of a countably infinite set, i.e., give a bijection $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

   Method 2: Alternatively, you can construct an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and invoke the following result.
   "If there exists an injection $f : A \to \mathbb{N}$, then A is countable."