Problem Set for Unit 14

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Here are some problems on **distributions and variance**. Use these as practice problems to strengthen your understanding as you do the reading corresponding to this unit. The topics you will want to read up on are listed at the end of the slides for this unit. You do not need to submit solutions to these problems. You are free to discuss these problems on Piazza.

We will use the following notation for the important probability distributions discussed in class.

- $\text{Bern}(p)$ denotes the Bernoulli distribution with parameter p.
- $\text{Bin}(n, p)$ denotes the binomial distribution with parameters n and p.
- $\text{Geom}(p)$ denotes the geometric distribution with parameter p.
- $\text{Pois}(\lambda)$ denotes the Poisson distribution with parameter λ.
- $\text{Unif}(A)$ denotes the uniform distribution over the set A.

We write $X \sim \text{Bern}(p)$ to denote that X is a random variable that has the Bernoulli distribution with parameter p, and so on.

PS14-1

Suppose R, S, and T are mutually independent random variables on the same probability space with uniform distribution on the range $\{1, 2, 3\}$. Let $M = \max\{R, S, T\}$. Compute the functions pdf_M and CDF_M.

PS14-2

A gambler bets \$10 on "red" at a roulette table (the odds of red are 18/38, slightly less than even) to win \$10. If he wins, he gets back twice the amount of his bet, and he quits. Otherwise, he doubles his previous bet and continues. For example, if he loses his first two bets but wins his third bet, the total spent on his three bets is $10 + 20 + 40$ dollars, but he gets back 2×40 dollars after his win on the third bet, for a net profit of \$10.

a. What is the expected number of bets the gambler makes before he wins?

b. What is his probability of winning?

c. What is his expected final profit (amount won minus amount lost)?

d. You can beat a biased game by bet doubling, but bet doubling is not feasible because it requires an infinite bankroll. Verify this by proving that the expected size of the gambler's last bet is infinite.

PS14-3 ^{HW}

Take a biased coin with heads probability p and flip it n times. Let the random variable J denote the number of heads obtained. Recall that, by definition, $J \sim \text{Bin}(n, p)$. Following our analysis from class (or Section 19.5.3 from [LLM]), we have $\text{Ex}[J] = np$.

Intuitively, the PDF of J should peak roughly at this expected value np. Here is what you can prove formally:

$$\begin{aligned} \text{pdf}_J(k-1) &< \text{pdf}_J(k) \qquad \text{for } k < np + p\,, \\ \text{pdf}_J(k-1) &> \text{pdf}_J(k) \qquad \text{for } k > np + p\,. \end{aligned}$$

Prove the above inequalities, using the formula for the PDF of a binomial distribution. Explain in words what this says about the "graph" of pdf_J.

PS14-4

Let C be the number of trials to first success, where a single trial success with probability p and the trials are mutually independent. Assume that $0 < p \leq 1$.

By definition, $C \sim \text{Geom}(p)$.

a. Let A be the event that the first trial succeeds. Conditioning on A and $\overline{A}$, using the law of total expectation, write an equation for $\text{Ex}[C^2]$.

b. Solve the equation, then use the solution to work out $\text{Var}[C]$. You should obtain $\text{Var}[C] = \dfrac{1-p}{p^2}$.

PS14-5
Solve [LLM] Problem 19.12 (about flipping a coin until certain patterns appear).

PS14-6 ^{HW}
Let R be a positive integer valued random variable.

a. If $\text{Ex}[R] = 2$, how large can $\text{Var}[R]$ be?

b. How large can $\text{Ex}[1/R]$ be? (Do not assume that $\text{Ex}[R] = 2$. That's only for the previous part.)

c. If $R \le 2$, that is, the only possible values of R are 1 and 2, then how large can $\text{Var}[R]$ be?

PS14-7 ^{HW}
Dr. Markov has a set of n keys, only one of which will fit the lock on the door to his apartment. He tries the keys until he finds the right one. Give the expectation and variance of the number of keys he has to try, when...

a. ...he tries the keys at random (possibly repeating a key tried earlier).

b. ...he chooses keys randomly among the ones that he has not yet tried.

PS14-8
Recall the theorem $\text{Var}[X] = \text{Ex}[X^2] - \text{Ex}[X]^2$, and the theorem that for independent RVs X and Y, $\text{Ex}[XY] = \text{Ex}[X]\,\text{Ex}[Y]$.

a. Using these two theorems and algebraic manipulation, prove that if X and Y are independent random variables, then

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]. \tag{1}$$

b. If $c \in \mathbb{R}$ is a constant and X is a random variable, prove that $\text{Var}[cX] = c^2\,\text{Var}[X]$. Explain why this shows that Eq. (1) does not hold for an *arbitrary* pair of RVs X, Y.

c. Extend Eq. (1) to show that if the random variables $X_1, \ldots, X_n$ are pairwise independent, then

$$\text{Var}[X_1 + \cdots + X_n] = \text{Var}[X_1] + \cdots + \text{Var}[X_n].$$

We saw in class that the sum of mutually independent Bernoulli random variables, all having the same parameter, is a binomial random variable: if $X_1 \sim \text{Bern}(p)$, $\ldots$, $X_n \sim \text{Bern}(p)$ and the X_is are mutually independent, then $X_1 + \cdots + X_n \sim \text{Bin}(n,p)$. The next few problems explore what happens when independent random variables from other important distributions are added.

PS14-9 ^{HW}
Consider adding two uniform distributions. Suppose that $X \sim \text{Unif}(A)$ and $Y \sim \text{Unif}(A)$ for some set $A \subseteq \mathbb{R}$ and that X and Y are independent. Is $X + Y \sim \text{Unif}(B)$ for some $B \subseteq \mathbb{R}$?

PS14-10
Show that if $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(q)$, with X and Y independent, then $X + Y$ need not have a geometric distribution. For the special case $p = q = 1/2$, work out what pdf_{X+Y} is.

PS14-11 ^{EC}
Let X and Y be independent nonnegative-integer-valued random variables. Let $f = \text{pdf}_X$, $g = \text{pdf}_Y$, and $h = \text{pdf}_{X+Y}$. Then prove that, for all integers $r \ge 0$,

$$h(r) = \sum_{t=0}^{r} f(t)g(r-t).$$

We say that h is the *convolution* of f and g.

Hint: Use the law of total probability. Study the event "$X + Y = r$" conditioned on the various values that X can take.

PS14-12 ^{EC}

Suppose that $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Pois}(\mu)$ are independent random variables, for some real numbers $\lambda, \mu \geq 0$. Prove that $X+Y \sim \mathrm{Pois}(\lambda+\mu)$. Does this result make intuitive sense to you, considering what a Poisson distribution is attempting to model?

Hint: Use the above convolution formula and the binomial theorem.

** I will add a couple of problems to this set **