Problem Set for Unit 4

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Here are some problems about **divisors and modular arithmetic**.

By now you are familiar with what needs to be submitted towards graded homework and when.

The symbols $\mathbb{N}, \mathbb{N}^+, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ have their usual meanings. Additionally, for each $d \in \mathbb{N}^+$, we define $\mathbb{Z}_d = \{0, 1, 2, \ldots, d-1\}$, the set of possible remainders when an integer is divided by d.

PS4-1

List all positive divisors of each of the following integers:

$$12,\ 15,\ 29,\ 64,\ 72\ ,73\ ,75\,.$$

Which of the above integers are primes?

PS4-2 *HW*

Count the number of positive divisors of each integer in the above list. You'll notice that the counts are all even numbers, with one exception. Work out exactly which integers from 1 to 100 (inclusive) have an odd number of positive divisors. Prove your answer.

Hint: I'm not expecting a super-long proof that lists out all 100 cases. I want you to discover a pattern and use that to cut down your work a lot.

PS4-3 *HW*

Write out a multiplication table for $\mathbb{Z}_{11}$ using multiplication modulo 11 and another multiplication table for $\mathbb{Z}_{12}$ using multiplication modulo 12. What do you observe about the occurrences of zeroes in these tables?

PS4-4

Let $d \in \mathbb{N}^+$ and $a, b, x, y \in \mathbb{Z}$ be such that

$$a \equiv b \pmod{d}\,,$$
$$x \equiv y \pmod{d}\,.$$

Using the definition of congruence, prove that

$$a + x \equiv b + y \pmod{d}\,,$$
$$ax \equiv by \pmod{d}\,.$$

PS4-5

Prove that $\forall\, a, b \in \mathbb{Z}\ \forall\, n \in \mathbb{N}^+$, if $a \neq b$, then $a^n - b^n$ is divisible by $a - b$.

Hint: Think of arithmetic modulo $a - b$, assuming $a > b$.

PS4-6 *HW*

Prove that $\forall\, a, b, n \in \mathbb{N}^+$, if n is odd, then $a^n + b^n$ is divisible by $a + b$.

Hint: First figure out the square, cube, fourth power, etc. of -1.

PS4-7

Compute $2^{2019} \bmod 17$. Do not use a calculator. In fact, think of a way to compute this entirely in your head.

PS4-8

Prove that a perfect square cannot end in the digit 7 when written out in decimal representation.

Hint: For what value of d would arithmetic modulo d help you reason about the last digit of an integer?

PS4-9 *HW*

If you're given a somewhat large number such as 803411927792 and asked whether or not it's a multiple of 3, there's a nifty trick you can use. Simply add up all the digits and test whether the sum is divisible by 3. In the above example, $8+0+3+4+1+1+9+2+7+7+9+2=53$, which isn't divisible by 3, so we can answer "No."

Explain why this test works, by proving the following theorem. If the decimal representation of $n \in \mathbb{N}^+$ is $a_k a_{k-1} \cdots a_2 a_1 a_0$, where the a_is are the digits, then

$$n \equiv a_k + \cdots + a_1 + a_0 \pmod 3 .$$

PS4-10

Prove that the product of any three consecutive integers must be divisible by 6.

Write a careful proof using only the facts established in the course up to this point. Don't jump to conclusions.