***Solution for PS9-1.*** Let $P(n)$ be the statement " $\sum_{i=1}^{n}(2i-1) = n^2$. " We shall prove by induction on $n$ that $\forall\, n \in \mathbb{N} : P(n)$.

*Base case.* $P(0)$ states "$0 = 0^2$." This is obviously true.

*Induction step.* Assume that $P(k)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \left(\sum_{i=1}^{k}(2i-1)\right) + (2(k+1)-1) \\
&= k^2 + (2(k+1)-1) && \triangleleft \textit{ by the induction hypothesis} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2 .
\end{aligned}$$

Therefore, $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-2.*** Let $P(n)$ be the given statement. We shall prove it for all $n \in \mathbb{N}$ by induction on $n$.

*Base case.* $P(0)$ states "$0 = 1! - 1$." This is obviously true.

*Induction step.* Assume that $P(k)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} j \cdot j! &= \left(\sum_{j=1}^{k+1} j \cdot j!\right) + (k+1)\cdot(k+1)! \\
&= (k+1)! - 1 + (k+1)\cdot(k+1)! && \triangleleft \textit{ by the induction hypothesis} \\
&= (k+1)! \cdot (1 + (k+1)) - 1 \\
&= (k+2)! - 1 .
\end{aligned}$$

Therefore, $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-3.*** Fix a *particular*, though arbitrary, real number $x \in \mathbb{R} - \{1\}$.

(This is an important step! From here on, for the rest of this proof, $x$ is no longer a variable.)

Let $P_x(n)$ be the following statement:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1} .$$

We shall prove by induction on $n$ that $\forall\, n \in \mathbb{N} :\ P_x(n)$.

*Base case.* $P_x(0)$ states "$1 = (x-1)/(x-1)$." This is obviously true.

*Induction step.* Assume that $P_x(k)$ is true for some $k \geq 0$. Then

$$\begin{aligned}
1 + x + x^2 + \cdots + x^{k+1} &= \left(1 + x + x^2 + \cdots + x^k\right) + x^{k+1} \\
&= \frac{x^{k+1}-1}{x-1} + x^{k+1} && \triangleleft \textit{ by the induction hypothesis} \\
&= \frac{x^{k+1} - 1 + x^{k+2} - x^{k+1}}{x-1} \\
&= \frac{x^{k+2}-1}{x-1} + x^{k+1} .
\end{aligned}$$

Therefore, $P_x(k+1)$ is true. We have shown that $P_x(k) \implies P_x(k+1)$.

Thus, by the principle of mathematical induction, we've proved that $\forall\, n \in \mathbb{N} :\ P_x(n)$.

Since we did this for an arbitrary choice of $x$, we've in fact shown that $\forall\, x \in \mathbb{R} - \{1\}\ \forall\, n \in \mathbb{N} : P_x(n)$.

***Solution for PS9-4.*** Let $P(n)$ be the statement "$3 \mid n^3 + 2n$." We shall prove it for all $n \in \mathbb{N}$ by induction on $n$.

*Base case.* $P(0)$ states "$3 \mid 0^3 + 2 \times 0$," i.e., "$3 \mid 0$." This is obviously true.

*Induction step.* Assume that $P(k)$ is true for some $k \geq 0$. This implies $k^3 + 2k = 3m$, for some $m \in \mathbb{N}$. Notice that

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2(k+1) = (k^3 + 2k) + 3(k^2 + k + 1) = 3(m + k^2 + k + 1)\,.$$

Therefore, $3 \mid (k+1)^3 + 2(k+1)$, i.e., $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-5.*** Let $P(n)$ be the statement "$5 \mid 8^n - 3^n$." We shall prove it for all $n \in \mathbb{N}$ by induction on $n$.

*Base case.* $P(0)$ states "$5 \mid 8^0 - 3^0$," i.e., "$5 \mid 1 - 1 = 0$." This is obviously true.

*Induction step.* Assume that $P(k)$ is true for some $k \geq 0$. This implies $8^k - 3^k = 5m$, for some $m \in \mathbb{N}$. Notice that

$$8^{k+1} - 3^{k+1} = 8 \cdot 8^k - 3 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k = 5 \cdot 8^k + 3 \cdot 5m = 5(8^k + 3m)\,.$$

Therefore, $5 \mid 8^{k+1} - 3^{k+1}$, i.e., $P(k+1)$ is true. We have shown that $P(k) \implies P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-6.***

***a.*** Let $P(n)$ denote the statement " $2^n \geq n^3$ ". We shall prove that $P(n)$ holds for all $n \geq 10$, by induction on $n$.

*Base case* ($n = 10$). The following shows that the base case, i.e., $P(10)$, holds.

$$2^{10} = 1024 > 1000 = 10^3\,.$$

*Induction step* ($n \geq 10$). Assume that $P(k)$ is true for some $k \geq 10$. Then

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&\geq 2k^3 && \text{by the induction hypothesis} \\
&= k^3 + k^3 \\
&\geq k^3 + 10k^2 && \text{since } k \geq 10 \\
&\geq k^3 + 3k^2 + 7k && \text{since } k^2 \geq k \\
&\geq k^3 + 3k^2 + 3k + 1 && \text{since } 4k \geq 1 \\
&= (k+1)^3\,,
\end{aligned}$$

which is $P(k+1)$. This proves that $P(k) \implies P(k+1)$ for all $k \geq 10$.

Thus, by the principle of mathematical induction, the proof is complete.

***b.*** Consider the following for $n > 1$.

$$\begin{aligned}
\frac{1}{n} - \frac{1}{(n+1)^2} &= \frac{(n+1)^2 - n}{n(n+1)^2} = \frac{n^2 + 2n + 1 - n}{n(n+1)^2} \\
&= \frac{n^2 + n + 1}{n(n+1)^2} \\
&> \frac{n^2 + n}{n(n+1)^2} && \text{this is strictly greater because numerator is 1 less here,} \\
&= \frac{n(n+1)}{n(n+1)^2} \\
&= \frac{1}{n+1}\,.
\end{aligned}$$

Hence, $\frac{1}{n} - \frac{1}{(n+1)^2} > \frac{1}{n+1}$, and multiplying by $-1$ we get the reverse inequality,

$$-\left(\frac{1}{n} - \frac{1}{(n+1)^2}\right) < -\frac{1}{n+1}\,. \tag{1}$$

Let $P(n)$ denote the statement

$$\text{“}\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}\,.\text{”}$$

We shall prove that $P(n)$ holds for all $n > 1$, by induction on $n$.
*Base case* $(n = 2)$. The following shows that the base case, i.e., $P(2)$, holds.

$$\sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} = \frac{5}{4} < \frac{6}{4} = 2 - \frac{1}{2}\,.$$

*Induction step* $(n \geq 2)$. Assume $P(k)$ is true for some $k \geq 2$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{by induction hypothesis,} \\
&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) && \text{rearranging,} \\
&< 2 - \frac{1}{k+1} && \text{by (1),}
\end{aligned}$$

which is $P(k+1)$. This proves that $P(k) \implies P(k+1)$ for all $k \geq 2$.
Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-7.*** For this problem, we're given that the basic sum principle holds. Let $P(n)$ be the statement of the extended sum principle, i.e., the statement

"If the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$."

We shall prove it for all $n \geq 2$ by induction on $n$.

*Base case.* $P(2)$ is the basic sum principle, which does hold (we're given this).

*Induction step.* Assume that $P(k)$ is true for some $k \geq 2$. Towards proving $P(k+1)$, consider arbitrary pairwise disjoint sets $A_1, A_2, \ldots, A_{k+1}$. Let $C = A_1 \cup A_2 \cup \cdots \cup A_{k+1}$ and $B = A_1 \cup A_2 \cup \cdots \cup A_k$.

Since $A_{k+1}$ has no elements in common with any of $A_1, A_2, \ldots, A_k$, it has no elements in common with their union either. In other words, $B \cap A_{k+1} = \varnothing$. Therefore,

$$\begin{aligned}
|C| &= |B \cup A_{k+1}| \\
&= |B| + |A_{k+1}| \qquad = (k+1)! - 1 + (k+1) \cdot (k+1)! \qquad \triangleleft \textit{ by the induction hypothesis} \\
&= (k+1)! \cdot (1 + (k+1)) - 1 \\
&= (k+2)! - 1\,.
\end{aligned}$$

Since $P(k)$ is true (this is the induction hypothesis), $B$ is a countable set. Now $C = B \times A_{k+1}$ and we have proved earlier that the Cartesian product of two countable sets is countable. Therefore, $C$ is countable, establishing $P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-8.*** Let $P(n)$ be the statement

"For all sets $A_1, A_2, \ldots, A_n$, if each $A_i$ is countable, then so is $A_1 \times A_2 \times \cdots \times A_n$."

We shall prove it for all $n \in \mathbb{N}^+$ by induction on $n$.

*Base case.* $P(1)$ states "For all sets $A_1$, if $A_1$ is countable, then so is $A_1$." This is obviously true.

*Induction step.* Assume that $P(k)$ is true for some $k \in \mathbb{N}^+$. Towards proving $P(k+1)$, consider arbitrary sets $A_1, A_2, \ldots, A_{k+1}$ such that each $A_i$ is countable, and let $C = A_1 \times A_2 \times \cdots \times A_{k+1}$. Let $B = A_1 \times A_2 \times \cdots \times A_k$. Since $P(k)$ is true (this is the induction hypothesis), $B$ is a countable set. Now $C = B \times A_{k+1}$ and we have proved earlier that the Cartesian product of two countable sets is countable. Therefore, $C$ is countable, establishing $P(k+1)$.

Thus, by the principle of mathematical induction, the proof is complete.

***Solution for PS9-9.*** Let $P(n)$ be the statement "$n$ can be written as a sum of one or more distinct powers of 2." We shall prove it for all $n \in \mathbb{N}^+$ by induction on $n$.

*Base case.* $P(1)$ states "1 can be written as a sum of one or more distinct powers of 2."

This is true: we simply write $1 = 2^0$.

*Induction step.* Assume that $P(m)$ is true for all $m$ with $1 \le m < k$. We shall now prove $P(k)$.

Let $2^r$ (where $r \in \mathbb{N}$) be the largest power of 2 that is $\le k$. In other words, $2^r \le k < 2^{r+1}$. Two cases arise.

> *Case 1.* We have $k = 2^r$. In this case, $k$ can be written as the "sum" of a single power of 2, namely $2^r$. Therefore, $P(k)$ holds.
>
> *Case 2.* We have $k > 2^r$. In this case, let $\ell = k - 2^r \in \mathbb{N}^+$. By the induction hypothesis, $P(\ell)$ is true, so we can write $\ell = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_1 < a_2 < \cdots < a_s$. Therefore,
>
> $$k = \ell + 2^r = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s} + 2^s\,.$$
>
> We're almost there, but we have to show that $2^s$ is distinct from all the other powers of 2 appearing above. But $k < 2^{r+1}$, so $2^{a_s} \le \ell = k - 2^r < 2^r$. It follows that $a_s < r$, which proves the distinctness. Therefore, $P(k)$ holds.

We have now proved that $P(1) \wedge P(2) \wedge \cdots \wedge P(k-1) \implies P(k)$.

Thus, by the (strong version of the) principle of mathematical induction, the proof is complete.

***Alternate Solution for PS9-9.*** We can do the induction step differently. Let's jump right in to that part of the proof. We're trying to prove $P(k)$.

> *Case 1.* $k$ is even. By the induction hypothesis, $P(k/2)$ is true, so we can write $k/2 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_1 < a_2 < \cdots < a_s$. Therefore,
>
> $$k = 2^{1+a_1} + 2^{1+a_2} + \cdots + 2^{1+a_s}$$
>
> and these powers of 2 are in ascending order, so they are distinct. Therefore, $P(k)$ holds.
>
> *Case 2.* $k$ is odd. By the induction hypothesis, $P((k-1)/2)$ is true. This means that we can write $(k-1)/2 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_1 < a_2 < \cdots < a_s$. Therefore,
>
> $$k = 1 + 2\left(\frac{k-1}{2}\right) = 2^0 + 2^{1+a_1} + 2^{1+a_2} + \cdots + 2^{1+a_s}\,.$$
>
> Again, these powers of 2 are in ascending order, so they are distinct. Therefore, $P(k)$ holds.

The rest of the proof is the same as before.

***Solution for PS9-10.*** For all integers $r \in \mathbb{N}^+$, let $Q(r)$ be the statement that for all rectangular bars of chocolate with $r$ tiles, the number of snap operations used to break the bar into individual tiles must be exactly $r-1$. The proof is by strong induction on $Q(r)$.

*Base Case* $Q(1) = 0$, because a bar with one tile is already broken into individual tiles, and because the bar can not longer be broken into two distinct parts.

*Induction Step* For any $t \geq 2$, assume that for all $1 \leq k < t$, $Q(k)$ is true. To prove that $Q(t)$ follows, consider any rectangular bar with $t$ tiles. Any snap operation divides the rectangle into two rectangular fragments with $a$ and $b$ tiles, respectively, so that $a + b = t$. The number of snaps used to reduce the bar into individual tiles is one plus the number of snaps required to resolve each of the two fragments:

$$Q(t) = 1 + Q(a) + Q(b) = 1 + (a-1) + (b-1) = a + b - 1 = t - 1.$$

This is always the same value, no matter how the bar is partitioned into two parts.

By strong induction, $Q(t) = t - 1$ for all rectangular chocolate bars. As an $m \times n$ bar has $mn$ tiles, it uses exactly $mn - 1$ snaps to break up into individual tiles.

(The proof only uses the fact that each snap partitions the bar—a collection of tiles—into two (disjoint) smaller collections. The shape of the bar and manner of snapping do not matter.)

***Solution for PS9-11.*** Let $P(m)$ denote the statement "if a tournament has a cycle of length $m$, then it has a cycle of length 3". We shall prove that for all $m \geq 3$ $(P(m))$, by induction on $m$.

*Base case* $(m = 3)$. $P(3)$ is a statement of the form "if $X$, then $X$", so it is trivially true.

Assume $P(k)$ is true for some $k \geq 3$. We're going to prove $P(k+1)$.

First note that $P(k+1)$ is implicitly a "for all" statement. So, to prove it, we consider an arbitrary tournament that has a cycle of length $k+1$, consisting of the players $\{p_1, p_2, p_3, \ldots, p_k, p_{k+1}\}$, where $p_1$ beats $p_2$, who beats $p_3$, …, who beats $p_{k+1}$, who beats $p_1$. Now consider first 3 players in the cycle: $\{p_1, p_2, p_3\}$. Two cases arise.

*Case 1: $p_3$ beats $p_1$.* Then we have a cycle of length 3, consisting of the players $\{p_1, p_2, p_3\}$, where $p_1$ beats $p_2$, who beats $p_3$, who beats $p_1$.

*Case 2: $p_1$ beats $p_3$.* Then our tournament has a cycle of length $k$, consisting of the players $\{p_1, p_3, p_4, \ldots, p_k, p_{k+1}\}$, where $p_1$ beats $p_3$, who beats $p_4$, …, who beats $p_{k+1}$, who beats $p_1$. By the induction hypothesis, the existence of this cycle implies that the tournament has a cycle of length 3.

In both cases we concluded that our tournament has a cycle of length 3. Therefore $P(k+1)$ is true. This shows that $P(k) \implies P(k+1)$.

Thus, by mathematical induction, it follows that $\forall m \geq 3\ (p(m))$. □