Solutions to Problem Set 1

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PS1-1
Set-builder to roster notation.

a. $\{x : x \text{ is a multiple of 7 and } 0 < x < 50\}$.
Solution. $\{7, 14, 21, 28, 35, 42, 49\}$.

b. $\{x + y : x \in \mathbb{N}, y \in \mathbb{N}, \text{ and } xy = 12\}$.
Solution. $\{7, 8, 13\}$.

c. $\{S : S \subseteq \{1, 2, 3, 4\} \text{ and } |S| \text{ is odd}\}$.
Solution. $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.

PS1-2 *HW*
More set-builder to roster notation.

a. $\{x^3 : x \in \mathbb{Z} \text{ and } x^2 < 20\}$ [2 points]
Solution. $\{-64, -27, -8, -1, 0, 1, 8, 27, 64\}$.

b. $\{x \in \mathbb{R} : x = x^2\}$. [2 points]
Solution. $\{0, 1\}$.

c. $\{S : \{1, 2\} \subseteq S \subseteq \{1, 2, 3, 4\}\}$ [2 points]
Solution. $\{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$.

d. $\{S \subseteq \{1, 2, 3, 4\} : S \text{ is disjoint from } \{2, 3\}\}$ [2 points]
Solution. $\{\varnothing, \{1\}, \{4\}, \{1, 4\}\}$.

PS1-3
Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{0, 1, 5, 6, 9\}$.

a. What is $A \cup B$? What is $(A \cup B) \cup C$?
Solution. $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$; $(A \cup B) \cup C = \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10\}$.

b. What is $B \cup C$? What is $A \cup (B \cup C)$?
Solution. $B \cup C = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}$; $A \cup (B \cup C) = \{0, 1, 2, 3, 4, 5, 6, 8, 9, 10\}$.

c. What is $A \cap B \cap C$?
Solution. $\{6\}$.

d. Verify by direct computation that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Solution. We computed $A \cup B$ above. Using that, $(A \cup B) \cap C = \{1, 5, 6\}$.
Further, $A \cap C = \{1, 5, 6\}$ and $B \cap C = \{6\}$. So, $(A \cap C) \cup (B \cap C) = \{1, 5, 6\}$.
Hence, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

e. What is $A - B$? What is $B - C$?
Solution. $A - B = \{1, 3, 5\}$; $B - C = \{2, 4, 8, 10\}$.

f. What is $(A - B) - C$? What is $A - (B - C)$?
Solution. $(A - B) - C = \{3\}$; $A - (B - C) = \{1, 3, 5, 6\}$.

g. Verify by direct computation that $(A - B) - C = A - (B \cup C)$.
Solution. We already know $(A - B) - C = \{3\}$.
Further, $B \cup C = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}$ and so, $A - (B \cup C) = \{3\}$.
Hence, $(A - B) - C = A - (B \cup C)$.

h. Verify by direct computation that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.
Solution. We already know $A - (B - C) = \{1, 3, 5, 6\}$.
Further, $A - B = \{1, 3, 5\}$ and $A \cap B \cap C = \{6\}$. So, $(A - B) \cup (A \cap B \cap C) = \{1, 3, 5, 6\}$.
Hence, $A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

i. What is $(A \cap B) \times (B - C)$?

Solution. $\{(2,2),(2,4),(2,8),(2,10),\ (4,2),(4,4),(4,8),(4,10),\ (6,2),(6,4),(6,8),(6,10)\}$.

j. Verify by direct computation that $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.

Solution. We already computed $A \cup B \cup C = \{0,1,2,3,4,5,6,8,9,10\}$ above.
Now, $A - B = \{1,3,5\}$; $B - C = \{2,4,8,10\}$; $C - A = \{0,9\}$; $A \cap B \cap C = \{6\}$.
So, $(A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C) = \{0,1,2,3,4,5,6,8,9,10\}$.
Hence, $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.

PS1-4

Let A, B, and C be arbitrary sets. Prove each of the following.

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Solution. Consider an arbitrary element $(x, y) \in A \times (B \cup C)$.
Then $x \in A$ and $y \in B \cup C$, i.e., $y \in B$ or $y \in C$. Thus, $(x, y) \in A \times B$ or $(x, y) \in A \times C$.
So, $(x, y) \in (A \times B) \cup (A \times C)$. Hence, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. ... **(i)**

Again, consider any $(x, y) \in (A \times B) \cup (A \times C)$.
Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$. Thus, $x \in A$ and $y \in B$ or $y \in C$, i.e. $y \in B \cup C$.
So, $(x, y) \in A \times (B \cup C)$. Hence, $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. ... **(ii)**

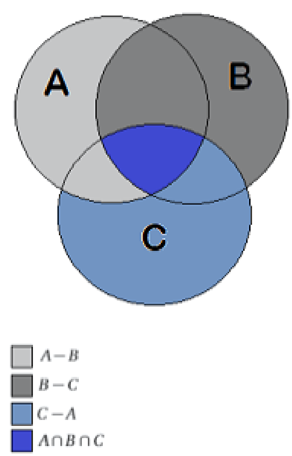
Thus, from **(i)** and **(ii)**, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

b. $(A - C) \cap (C - B) = \varnothing$.

Solution. Consider an arbitrary $x \in A - C$. Then $x \in A$ and $x \notin C$.
Now, $C - B = \{y : y \in C \text{ and } y \notin B\}$. Thus $x \notin C - B$.
Hence, $(A - C) \cap (C - B) = \varnothing$.

c. $A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$. ◁ It may help to draw a Venn diagram.

Solution. The following Venn diagram can help us in proving this.



Part (i) Consider any $x \in (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C)$.
Then $x \in A - B$ or $x \in B - C$ or $x \in C - A$ or $x \in A \cap B \cap C$.
In the first three cases, we have (respectively) $x \in A$ or $x \in B$ or $x \in C$. In the fourth case, we have all three things: $x \in A$, $x \in B$, and $x \in C$.
Thus, in any case, at least one of the following holds: $x \in A$, $x \in B$, or $x \in C$. Hence, $x \in A \cup B \cup C =$ LHS.
This proves that RHS $\subseteq$ LHS.

Part (ii) Now consider any $x \in A \cup B \cup C$. Then $x \in A$ or $x \in B$ or $x \in C$.

Case 1. x belongs to all three of A, B, and C.

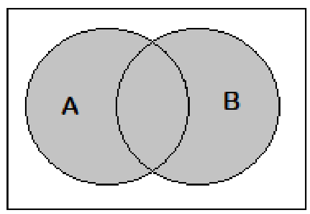
Then, $x \in A \cap B \cap C$. Therefore, $x \in (A-B) \cup (B-C) \cup (C-A) \cup (A \cap B \cap C)$.
Case 2. There is at least one set among A, B, and C to which x does not belong.
Without loss of generality, suppose that $x \notin B$. We now have two subcases.

Case 2.1. x doesn't belong to A.
In this case, $x \notin A$ and $x \notin B$, so we must have $x \in C$. So $x \in C - A$.
Therefore, $x \in (A-B) \cup (B-C) \cup (C-A) \cup (A \cap B \cap C)$.
Case 2.2. x does belong to A.
In this case, $x \in A$ and $x \notin B$, so we must have $x \in A - B$.
Therefore, $x \in (A-B) \cup (B-C) \cup (C-A) \cup (A \cap B \cap C)$.

Thus, in every case, $x \in$ RHS, so LHS $\subseteq$ RHS.

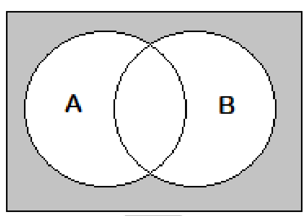
PS1-5 *HW*
Proofs of set equalities.

a. Proof that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. [4 points]
Solution.
The Venn diagram for $A \cup B$ is given by the following figure:



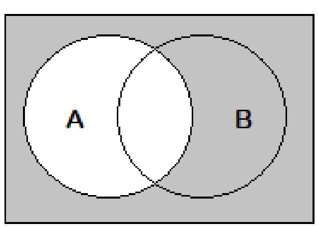
$A \cup B$

Hence, the Venn diagram for $\overline{A \cup B}$ is:

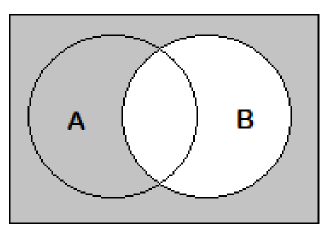


$\overline{A \cup B}$

Again, the Venn diagrams for $\overline{A}$ and $\overline{B}$ are:

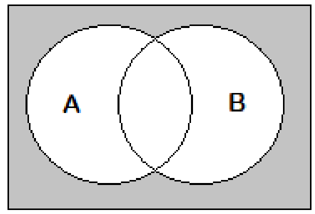


$\overline{A}$



$\overline{B}$

Hence, the Venn diagram for $\overline{A} \cap \overline{B}$ is given by:



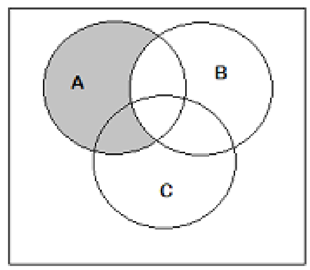
$\overline{A} \cap \overline{B}$.

So we see that the diagrams for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$ are the same. Hence $\overline{A \cup B} = \overline{A} \cap \overline{B}$

b. Similarly, prove that $(A - B) - C = (A - C) - (B - C)$. [4 points]

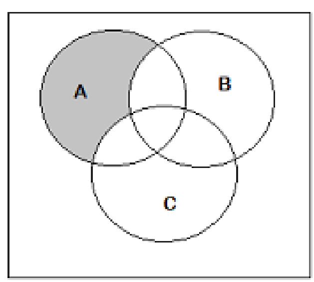
Solution.

The Venn diagram for $A - B$ is given by:



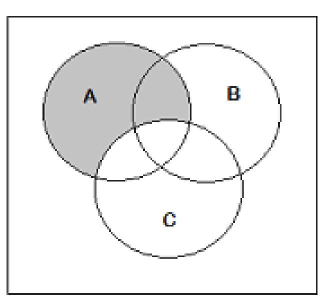
$A - B$

So, the Venn diagram for $(A - B) - C$ is:

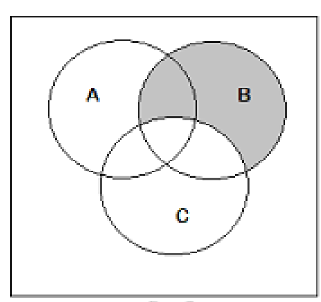


$(A - B) - C$

Again, the Venn diagrams for $A - C$ and $B - C$ are:

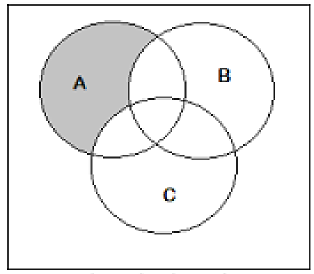


$A - C$



$B - C$

Thus, the Venn diagram for $(A - C) - (B - C)$ is given by:



$(A-C)-(B-C)$

So we see that the Venn diagrams for $(A-B)-C$ and $(A-C)-(B-C)$ are the same. Hence, $(A-B)-C = (A-C)-(B-C)$.

c. Algebra-style proof that $(A\cap B)\cup(A\cap\overline{B}) = A$. [4 points]

Solution. Consider an arbitrary $x \in A$.
If $x \in B$, then $x \in A\cap B$.
Otherwise, $x \notin B$, so $x \in \overline{B}$ and so $x \in A\cap\overline{B}$.
Combining the above two conclusions, $x \in (A\cap B)\cup(A\cap\overline{B})$.
Thus, $A \subseteq (A\cap B)\cup(A\cap\overline{B})$. ... **(i)**

Now consider an arbitrary $x \in (A\cap B)\cup(A\cap\overline{B})$.
Then $x \in A\cap B$ or $x \in A\cap\overline{B}$.
In the former case, $x \in A$ and $x \in B$. In the latter case, $x \in A$ and $x \in \overline{B}$.
In either case, $x \in A$.
Thus, $(A\cap B)\cup(A\cap\overline{B}) \subseteq A$. ... **(ii)**

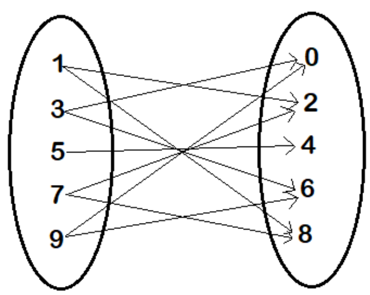
Combining **(i)** and **(ii)**, $(A\cap B)\cup(A\cap\overline{B}) = A$.

PS1-6 *HW*

The "completes" relation. [3 points]

Solution. As a set, the relation is $\{(1,2),(1,8),(3,0),(3,6),(5,4),(7,2),(7,8),(9,0),(9,6)\}$.

Below is a pictorial representation.



PS1-7

Are these relations (a) symmetric; (b) transitive?

a. The relation "divides", on $\mathbb{N}$ ("m divides n" means "n/m is an integer").
Solution. This relation is
(a) NOT symmetric [Reason: 1 divides 2, but 2 does not divide 1.]
(b) transitive

b. The relation "is disjoint from", on $\mathscr{P}(\mathbb{Z})$.

Solution. This relation is
(a) symmetric
(b) NOT transitive [Reason: $\{1\}$ disj from $\{2\}$, and $\{2\}$ disj from $\{1\}$, but $\{1\}$ is not disjoint from $\{1\}$.]

c. The relation "is no larger than", on $\mathscr{P}(\mathbb{Z})$. We say that A is no larger than B when one of the following holds:

- A and B are both finite sets, and $|A| \le |B|$.
- A is a finite set and B is an infinite set.
- A and B are both infinite sets.

Solution. This relation is:
(a) NOT symmetric [Reason: $(\{1\},\{1,2\}) \in$ "is no larger than", but $(\{1,2\},\{1\}) \notin$ "is no larger than".]
(b) transitive

PS1-8 *HW*

Same instructions as the previous problem, ***PS1-7***.

a. The relation "is a subset of", on $\mathscr{P}(\mathbb{Z})$. [4 points]

Solution. This relation is:
(a) NOT symmetric [Reason: $\{1\}$ is a subset of $\{1,2\}$ but $\{1,2\}$ is not a subset of $\{1\}$.]
(b) transitive

b. $\{(m,n) \in \mathbb{N} \times \mathbb{N}$: the sum of the digits of m equals the sum of the digits of $n\}$. [4 points]

Solution. This relation is (a) symmetric, (b) transitive.

c. The relation "overlapped" on the set of all US presidents. [4 points]

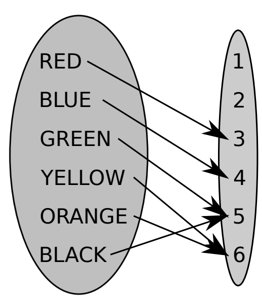
Solution. This relation is:
(a) symmetric
(b) NOT transitive [Reason: George Washington overlapped Thomas Jefferson and Thomas Jefferson overlapped Abraham Lincoln, but Washington did not overlap Lincoln.]

PS1-9

Let $S = \{$"RED", "BLUE", "GREEN", "YELLOW", "ORANGE", "BLACK"$\}$ and $T = \{1,2,3,4,5,6\}$. Consider the function len: $S \to T$ given by len(s) = the length of the string s (as in the Python programming language).

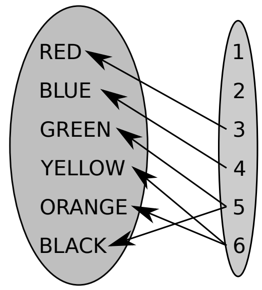
a. Describe the "len" function pictorially, using arrows, as done in class.

Solution.



b. Reverse the directions of all the arrows in your picture. Does this new picture represent a function $g : T \to S$. If not, why not?

Solution.



No: functions associate exactly one value (output) with each argument (input). In the picture above, there are multiple arrows leaving elements 5 and 6. Also, there are no arrows leaving elements 1 and 2.

PS1-10 Prove: if $S_1, S_2 \subseteq A$, then $f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$.

Solution. Consider an arbitrary element $y \in f(S_1 \cup S_2)$.

Then $y = f(x)$ for some $x \in S_1 \cup S_2$, i.e., $x \in S_1$ or $x \in S_2$. In the former case, $y = f(x) \in f(S_1)$. In the latter case, $y = f(x) \in f(S_2)$. Overall, $y \in f(S_1) \cup f(S_2)$. Thus, LHS $\subseteq$ RHS.

Next, consider an arbitrary element $y \in f(S_1) \cup f(S_2)$. Then $y \in f(S_1)$ or $y \in f(S_2)$.

In the former case, $y = f(x)$ for some $x \in S_1 \subseteq S_1 \cup S_2$. In the latter case, $y = f(x)$ for some $x \in S_2 \subseteq S_1 \cup S_2$. Thus, in each case, $y \in f(S_1 \cup S_2)$. Thus, RHS $\subseteq$ LHS.

PS1-11

The functions $f, g : \mathbb{R} \to \mathbb{R}$ are given by the formulas $f(x) = x^2 + 1$ and $g(x) = x + 2$. Find $f \circ g$ and $g \circ f$.

Solution. $(f \circ g)(x) = x^2 + 4x + 5; \quad (g \circ f)(x) = x^2 + 3$

PS1-12 ^{HW}

The functions $f, \mathrm{id} : \mathbb{R} \to \mathbb{R}$ are given by the formulas $f(x) = x^3 + 7$ and $\mathrm{id}(x) = x$.

a. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $f \circ g = \mathrm{id}$. [2 points]

Solution. Pick an arbitrary $x \in \mathbb{R}$ and let $y = g(x)$.
Then $f(y) = f(g(x)) = x$, since $f \circ g = \mathrm{id}$. Thus, $y^3 + 7 = x$, which implies $y = \sqrt[3]{x - 7}$.
We conclude that the required function $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = \sqrt[3]{x - 7}$.
(Note that this g is a well-defined function from $\mathbb{R}$ to $\mathbb{R}$ since every real number has a unique real cube root.)

b. For the function g you found above, find $g \circ f$. [2 points]

Solution. We compute: $(g \circ f)(x) = g(f(x)) = \sqrt[3]{f(x) - 7} = \sqrt[3]{x^3 + 7 - 7} = x$.
Hence, $g \circ f = \mathrm{id}$.