Selected Solutions for Unit 13

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Solution for PS13-1. In each case, the sample space is $\mathbb{Z}_n$ and the probability function is $\Pr[j] = 1/n$ for all $j \in \mathbb{Z}_n$.

a. $\text{Ex}[Y] = \sum_{j=0}^{n-1} Y(j)\Pr[j] = (1/n)\sum_{j=0}^{n-1}\gcd(j,n) = (1/7)\left(7 + \sum_{j=1}^{6} 1\right) = 13/7$.

b. In this case, $\gcd(j,9)$ equals 9 for one value of j (namely, $j = 0$), equals 3 for two values of j (namely, $j = 3$ and $j = 6$), and equals 1 for the remaining six values of j. Therefore, $\text{Ex}[Y] = (1/9)(9\times 1 + 3\times 2 + 1\times 6) = 21/9 = 7/3$.

c. By similar reasoning, $\text{Ex}[Y] = (1/15)(15\times 1 + 3\times 4 + 5\times 2 + 1\times 8) = 45/15 = 3$.

d. Please solve this yourself. The final answer is $3 - 2/p$.

e. Please solve this yourself. The final answer is $(2 - 1/p)(2 - 1/q)$.

Solution for PS13-3. Let the random variable X denote the amount (in dollars) that you win. Then $\text{range}(X) = \{0, 10^7\}$ and $\Pr[X = 10^7] = 1\big/\binom{50}{6}$. Therefore,

$$\text{Ex}[X] = 0\cdot\Pr[X=0] + 10^7\cdot\Pr[X = 10^7] = \frac{10^7}{\binom{50}{6}} \approx 0.63\,.$$

Since this is below the ticket's price of \$1, the ticket is not worth its price.

Solution (Sketch) for PS13-4.

a. Let X_j be the indicator r.v. for the event "$W_j = 6$." Then $\text{Ex}[X_j] = \Pr[X_j = 1] = \Pr[W_j = 6] = 1/6$.
The number of sixes seen is $Y := \sum_{j=1}^{24} X_j$, so by linearity of expectation,

$$\text{Ex}[Y] = \text{Ex}\left[\sum_{j=1}^{24} X_j\right] = \sum_{j=1}^{24}\text{Ex}[X_j] = 4\,.$$

b. This doesn't affect our answer. Linearity of expectation always holds and has nothing to do with correlation (or independence).

Solution for PS13-6.

a. Let's use the sample space $\{1,2,3,4,5,6\}$ for the experiment of rolling the red die. Upon conditioning on the event "X is a perfect square," the probability function is as follows:

$$\Pr[1] = \frac{1}{2}\,, \quad \Pr[4] = \frac{1}{2}\,, \quad \Pr[2] = \Pr[3] = \Pr[5] = \Pr[6] = 0\,.$$

Therefore, $\text{Ex}[X^2 \mid X \text{ is a perfect square}] = 1^2 \times \frac{1}{2} + 4^2 \times \frac{1}{2} = 17/2$.

b. We compute

$$\text{Ex}[WX] = \text{Ex}[X^2 + XY] = \text{Ex}[X^2] + \text{Ex}[X]\,\text{Ex}[Y]\,,$$
$$\text{Ex}[W]\,\text{Ex}[X] = \text{Ex}[X+Y]\,\text{Ex}[X] = \text{Ex}[X]^2 + \text{Ex}[X]\,\text{Ex}[Y]\,.$$

The two are unequal because $\text{Ex}[X^2] \neq \text{Ex}[X]^2$, by direct computation.

Solution (Sketch) for PS13-7. Number the pairs of students as pair 1, pair 2, …, pair $\binom{n}{2}$ in some manner. Let Y be the number of bonds. Let X_i be the indicator r.v. for the event "pair i forms a bond."
Then $\text{Ex}[X_i] = \Pr[X_i = 1] = 1/d$, so

$$\text{Ex}[Y] = \text{Ex}\left[\sum_{i=1}^{\binom{n}{2}} X_i\right] = \sum_{i=1}^{\binom{n}{2}}\text{Ex}[X_i] = \frac{1}{d}\binom{n}{2}\,.$$

Solution (Sketch) for PS13-9. Let Y be the number of bins that remain empty. Let X_i be the indicator r.v. for the event "bin i remains empty."

Then $\mathrm{Ex}[X_i] = \Pr[X_i = 1] = (1-1/n)^n$, so $\mathrm{Ex}[Y] = \mathrm{Ex}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathrm{Ex}[X_i] = n(1-1/n)^n$.

If you've taken calculus, you know that $\lim_{n\to\infty}(1-1/n)^n = 1/e$, so $\mathrm{Ex}[Y] \approx n/e$ for large n.

Solution (Sketch) for PS13-11. Please do this computation. The final answer is $\dfrac{n^2-1}{3n}$.

Solution (Sketch) for PS13-12. The square of the previous expression, with $n = 1000$.