Selected Solutions for Unit 12

Solution for PS12-1. There is a tempting intuitive approach to the first question: "Given the value of one die, the other die still equally likely to be any of the six possible values, so the probability of hitting the exact value required to win is $1/6$." This is incorrect!

Instead, let us do our usual four steps. Let $D = \{1,2,3,4,5,6\}$ be the possible values shown by one die.

- *Sample space:* $S = D \times D$.
- *Events of interest:*

$$W = \{(x,y) \in S : x + y = 7\} \text{ is the event that you won;}$$
$$E_6 = \{(x,y) \in S : x = 6 \lor y = 6\} \text{ is the event that one of the dice came up six;}$$
$$E_5 = \{(x,y) \in S : x = 5 \lor y = 5\} \text{ is the event that one of the dice came up five.}$$

- *Outcome probabilities:* Uniform on S, since the dice are fair.
- *Event Probabilities:* The two parts of the problem are asking for $\Pr[W \mid E_6]$ and $\Pr[E_5 \mid W]$, respectively.

a. Using the definition of conditional probability,

$$\Pr[W \mid E_6] = \frac{\Pr[W \cap E_6]}{\Pr[E_6]} = \frac{|W \cap E_6|/|S|}{|E_6|/|S|} = \frac{|\{(1,6),(6,1)\}|}{|\{(1,6),\ldots,(5,6),(6,6),(6,5),\ldots,(6,1)\}|} = \frac{2}{11}.$$

b. Similarly,

$$\Pr[E_5 \mid W] = \frac{\Pr[E_5 \cap W]}{\Pr[W]} = \frac{|E_5 \cap W|/|S|}{|W|/|S|} = \frac{|\{(2,5),(5,2)\}|}{|\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}|} = \frac{1}{3}.$$

Solution for PS12-2.

a. $4/9$.

b. $5/23$.

Solution for PS12-5.

a.

$$\begin{aligned}
&\Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \cdots \cdot \Pr[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}] \\
&= \Pr[A_1] \cdot \frac{\Pr[A_1 \cap A_2]}{\Pr[A_1]} \cdot \frac{\Pr[A_1 \cap A_2 \cap A_3]}{\Pr[A_1 \cap A_2]} \cdot \frac{\Pr[A_1 \cap A_2 \cap A_3 \cap A_4]}{\Pr[A_1 \cap A_2 \cap A_3]} \cdot \cdots \cdot \frac{\Pr[A_1 \cap A_2 \cap \cdots \cap A_n]}{\Pr[A_1 \cap A_2 \cap \cdots \cap A_{n-1}]} \\
&= \Pr[A_1 \cap A_2 \cap \cdots \cap A_n]
\end{aligned}$$

b. For $i = 1,2,\ldots,n$, let A_i be the event that the ith passenger (to board the flight) sits on his assigned seat.

$$\begin{aligned}
\Pr[A_1 \cap A_2 \cap \cdots \cap A_k] &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \cdots \cdot \Pr[A_k \mid A_1 \cap A_2 \cap \cdots \cap A_{k-1}] \\
&= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \cdots \cdot \frac{1}{n-k+1} \\
&= \frac{(n-k)!}{n!}
\end{aligned}$$

Solution for PS12-7. We'll denote the outcomes using two-letter strings: the first letter will be one of B (for Brown), D (for Dartmouth), or L (for Little Hoop); the second letter will be one of H (for happy) or U (for unhappy). Draw a tree diagram showing these outcomes: the root will have three children and each of those children will have two leaf children.

- *Sample space:* $S = \{\text{BH, BU, DH, DU, LH, LU}\}$.
- *Events of interest:*

$$\begin{aligned} \text{Happy} &= \{\text{BH, DH, LH}\}\,; \\ \text{Brown} &= \{\text{BH, BU}\}\,; \\ \text{Dartmouth} &= \{\text{DH, DU}\}\,. \end{aligned}$$

- *Outcome probabilities:* Label the tree diagram using the given numbers, then compute:

$$\begin{aligned} \Pr[\text{BH}] &= \frac{4}{12}\cdot\frac{4}{12} = \frac{16}{144}\,; & \Pr[\text{BU}] &= \frac{4}{12}\cdot\frac{8}{12} = \frac{32}{144}\,; \\ \Pr[\text{DH}] &= \frac{5}{12}\cdot\frac{7}{12} = \frac{35}{144}\,; & \Pr[\text{DU}] &= \frac{5}{12}\cdot\frac{5}{12} = \frac{25}{144}\,; \\ \Pr[\text{LH}] &= \frac{3}{12}\cdot\frac{11}{12} = \frac{33}{144}\,; & \Pr[\text{LU}] &= \frac{3}{12}\cdot\frac{1}{12} = \frac{3}{144}\,. \end{aligned}$$

- *Event Probabilities:* Computed below.

a. $\Pr[\text{Happy}] = \Pr[\text{BH,DH,LH}] = 84/144 = 7/12$.

b. $\Pr[\text{Brown} \mid \text{Happy}] = \dfrac{\Pr[\text{BH}]}{\Pr[\text{Happy}]} = \dfrac{16/144}{7/12} = \dfrac{4}{21}$.

c. Observe that $\Pr[\text{Brown} \mid \text{Happy}] \neq \Pr[\text{Brown}]$.

d. Observe that $\Pr[\text{Happy} \mid \text{Dartmouth}] = \Pr[\text{Happy}]$.

Solution for PS12-8. Uniform probability space on $\{1,2,3,4,5,6\}$; take $A = \{2,4,5,6\}$, $B = \{2,4,5\}$, $C = \{1,2,3\}$.

Solution for PS12-10.

a. Just as in the original Monty Hall game, $\Pr[\text{GP}] = 1/3$, because the prize is equally likely to be behind any particular door.

b. If the contestant does not pick the prize door, then the prize is behind one of the two remaining doors, both equally likely. When Carol picks a random door from among these two, she reveals the prize with probability $1/2$. Therefore, $\Pr[\text{OP} \mid \overline{\text{GP}}] = 1/2$.

c. We use the law of total probability:

$$\Pr[\text{OP}] = \Pr[\text{OP} \mid \text{GP}]\cdot\Pr[\text{GP}] + \Pr[\text{OP} \mid \overline{\text{GP}}]\cdot\Pr[\overline{\text{GP}}] = 0 \times \frac{1}{3} + \frac{1}{2}\times\frac{2}{3} = \frac{1}{3}\,.$$

d. In each round of the game, the probability that Carol will open the prize door (causing it to continue for at least one more round) is precisely $\Pr[\text{OP}]$, which we just calculated to be $1/3$. Since the rounds are independent,

$$\begin{aligned} \Pr[\text{game continues at least to round } n+1] &= \prod_{i=1}^{n} \Pr[\text{Carol opens prize door in } i\text{th round}] \\ &= \prod_{i=1}^{n} \Pr[\text{OP}] = \frac{1}{3^n}\,. \end{aligned}$$

Thus, the probability that the game will continue forever is $\lim_{n\to\infty} 1/3^n = 0$.

e. These probabilities are as follows.

i) When GP occurs, the contestant has chosen the prize door and the strategy of sticking with the choice is going to win. So $\Pr[W \mid \text{GP}] = 1$.

ii) When $\overline{\mathrm{GP}} \cap \mathrm{OP}$ occurs, the game gets restarted, which means we're back to square one. The probability of winning is again $\Pr[W]$. Thus, $\Pr[W \mid \overline{\mathrm{GP}} \cap \mathrm{OP}] = \Pr[W] = w$.

iii) When $\overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}$ occurs, the initial guess was wrong, the game ends in the first round, and by sticking with his initial choice, the contestant is guaranteed to lose. So, $\Pr[W \mid \overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] = 0$.

f. Using the law of total probability, we have

$$\begin{aligned} w &= \Pr[W] \\ &= \Pr[\mathrm{GP}] \cdot \Pr[W \mid \mathrm{GP}] + \Pr[\overline{\mathrm{GP}} \cap \mathrm{OP}] \cdot \Pr[W \mid \overline{\mathrm{GP}} \cap \mathrm{OP}] + \Pr[\overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] \cdot \Pr[W \mid \overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] \\ &= \frac{1}{3} \cdot 1 + \Pr[\mathrm{OP} \mid \overline{\mathrm{GP}}] \cdot \Pr[\overline{\mathrm{GP}}] \cdot w + \Pr[\overline{\mathrm{GP}} \cap \overline{\mathrm{OP}}] \cdot 0 \\ &= \frac{1}{3} + \frac{1}{2}\left(1 - \frac{1}{3}\right) w + 0 \\ &= \frac{1+w}{3}. \end{aligned}$$

Solving this equation gives us the answer: $w = 1/2$.

g. In the modified game, there are three possibilities for each outcome of the overall random experiment.

i) The contestant would win by using a "stick" strategy.

ii) The contestant would win by using a "switch" strategy.

iii) The game simply continues forever.

Because of the third possibility, we can't *immediately* conclude that $\Pr[\text{win using "switch" strategy}] = 1 - \Pr[W]$. Instead, we conclude that this probability equals $1 - \Pr[W] - \Pr[\text{game continues forever}]$.

However, we have computed $\Pr[\text{game continues forever}] = 0$, so in fact the desired probability is still equal to $1 - \Pr[W]$, i.e., the conclusion is still sound.