Problem Set for Unit 12

Here are some problems on **conditional probability**. Several of these problems are from the [LLM] book, but may be slightly modified, so read the wording carefully.

PS12-1
Two fair dice are rolled in another room, out of your sight. If the sum of the two dice values is seven, you *win*. Your friend is in the other room and can observe the dice.

a. Your friend calls out that one of the dice came up six. Given this information, what is the probability that you won?

b. Suppose, instead, that your friend tells you that you won. In this case, what is the probability that one of the dice came up five?

PS12-2
Outside of their humdrum duties as Discrete Mathematics Ninjas, Kim is trying to learn to levitate using only intense concentration and Liz is trying to become the world champion flaming torch juggler. Suppose that Kim's probability of success is 1/6, Liz's probability of success is 1/4, and these two events are independent.

a. If at least one of them succeeds, what is the probability that Kim learns to levitate?

b. If at most one of them succeeds, what is the probability that Liz becomes the world flaming torch juggling champion?

***PS12-3** HW*
When a test for steroids is given to soccer players, 98% of the players taking steroids test positive and 12% of the players not taking steroids test (falsely) positive. Suppose that exactly 5% of soccer players take steroids. What is the probability that a soccer player who tests positive takes steroids? Be systematic: use the four-step method.

***PS12-4** HW*
Solve Problem 18.2 ("Dirty Harry") from [LLM]. Be systematic: use the four-step method.

PS12-5
The Chain Rule for probability says that if $A_1, A_2, \ldots, A_n$ are events in a probability space $(\mathscr{S}, \Pr)$, then

$$\begin{aligned}\Pr[A_1 \cap A_2 \cap \cdots \cap A_n] &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \cdots \cdot \Pr[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}] \\ &= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1, A_2] \cdot \cdots \cdot \Pr[A_n \mid A_1, A_2, \ldots, A_{n-1}] \\ &= \prod_{j=1}^{n} \Pr[A_j \mid A_1, A_2, \ldots, A_{j-1}] .\end{aligned}$$

a. Prove this rule, i.e., prove the first equation. (The other two lines are just rewritings.)
Hint: You could use induction, but don't. Also, start with the right-hand side.

b. Use this rule to answer the following question. Suppose n passengers board a flight that has n seats and they each take a seat at random, ignoring their assigned seating. The passengers board one by one. What is the probability that passengers 1 through k (inclusive) all end up in their assigned seats?

***PS12-6** HW*
A 52-card deck is thoroughly shuffled and you are dealt a hand of 13 cards.

a. If you have one ace, what is the probability that you have a second ace?

b. If you have the ace of spades, what is the probability that you have a second ace? Remarkably, the answer is different from the previous one!

You may leave your answers in terms of binomial coefficients and/or factorials, but you must do enough work to convince your grader that the answers in Part (a) and Part (b) are different. [3+3 points]

PS12-7

Sally Smart just graduated from high school. She was accepted to three reputable colleges.

- With probability 4/12, she attends Brown.
- With probability 5/12, she attends Dartmouth.
- With probability 3/12, she attends Little Hoop Community College.

Sally is either happy or unhappy in college.

- If she attends Brown, she is happy with probability 4/12.
- If she attends Dartmouth, she is happy with probability 7/12.
- If she attends Little Hoop, she is happy with probability 11/12.

a. What is the probability that Sally is happy in college?

b. What is the probability that Sally attends Brown, given that she is happy in college?

c. Show that the events "Sally attends Brown" and "Sally is happy" **are not** independent.

d. Show that the events "Sally attends Dartmouth" and "Sally is happy" **are** independent.

PS12-8

Study Section 18.8 (Mutual Independence) of [LLM]. Then solve Problem 18.34, about non-independent events that nevertheless satisfy a "product rule".

PS12-9 ^{HW}

Let E and F be two independent events in some probability space $(\mathscr{S}, \Pr)$. Assume that $0 < \Pr[E] < 1$ and $0 < \Pr[F] < 1$.

a. Prove or disprove that $\overline{E}$ and $\overline{F}$ *must* be independent.

b. Prove or disprove that $\overline{E}$ and F *must* be independent.

PS12-10

Solve Problem 18.17 (variation of Monty Hall's game) from [LLM]. You will of course need to have done your earlier homework of reading Chapter 17, where Monty Hall's game is discussed. You will also need to review how to sum an *infinite* geometric series, which is discussed in Section 14.1.4 of [LLM].