Problem Set for Unit 9

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Here are some problems on **mathematical induction**. You may use either "ordinary" induction or "strong" induction, as described on the posted slides. To receive full credit, please do follow the template for proofs by induction as seen on the slides. In particular, we'll be looking for a clear definition of a *one-variable predicate* and a clear statement of what variable you are doing induction on.

If you need a review of sum and product notation (i.e., $\sum$ and $\prod$), bring this up in office hours with any of the course staff and we can help you.

PS9-1

Using mathematical induction, prove that $\forall\, n \in \mathbb{N}$: $\displaystyle\sum_{i=1}^{n}(2i-1) = n^2\,.$

PS9-2

Using mathematical induction, prove that $\forall\, n \in \mathbb{N}$: $\displaystyle\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1\,.$

PS9-3

Using mathematical induction, prove that the following identity holds for all $n \in \mathbb{N}$ and all $x \in \mathbb{R} - \{1\}$:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}\,,$$

Careful: the identity has two variables; your first step should be to write an appropriate one-variable predicate.

PS9-4

Using mathematical induction, prove that $\forall\, n \in \mathbb{N} : 3 \mid n^3 + 2n$. Pretend that you don't know modular arithmetic.

PS9-5 *HW*

Using mathematical induction, prove that $\forall\, n \in \mathbb{N} : 5 \mid 8^n - 3^n$. Pretend that you don't know modular arithmetic and don't use any number-theoretic results from previous problem sets. [5 points]

PS9-6 *HW*

Prove each of the following statements by mathematical induction.

a. For all integers $n \geq 10$, we have $2^n \geq n^3$. [5 points]

Hint: Consider the following calculation, used in the *inductive step* of a proof that $2^n \geq n^2$ for all $n \geq 4$:

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\geq 2n^2 && \text{(by the induction hypothesis)} \\ &\geq n^2 + 4n && \text{(since } n \geq 4) \\ &\geq n^2 + 2n + 1 && \text{(using } n \geq 4 \text{ again)} \\ &= (n+1)^2\,. \end{aligned}$$

b. For all integers $n > 1$: $\displaystyle\sum_{i=1}^{n}\frac{1}{i^2} < 2 - \frac{1}{n}$. [5 points]

Hint: First prove the following inequality for all $n > 1$ by direct algebraic manipulation (no induction!):

$$\frac{1}{n} - \frac{1}{(n+1)^2} > \frac{1}{n+1}\,.$$

Then use this inequality at the appropriate point in the inductive step of your main proof.

PS9-7

Recall that the basic sum principle applies to *exactly two* disjoint sets whereas the extended sum principle applies to $n \geq 2$ pairwise disjoint sets.

Use mathematical induction to prove that the extended sum principle follows from the basic sum principle.

PS9-8 *HW*

Using mathematical induction, prove that if each of the sets $A_1, A_2, \ldots, A_n$ is countable, then so is $A_1 \times A_2 \times \cdots \times A_n$.
[5 points]

PS9-9

Using mathematical induction, prove that every positive integer n can be written as a sum of one or more *distinct* powers of 2. For example, $42 = 2^5 + 2^3 + 2^1$ and $77 = 2^6 + 2^3 + 2^2 + 2^0$.

Hint: You might want to consider using strong induction.

PS9-10

You have a bar of chocolate with ridges dividing it into small pieces arranged in an $m \times n$ rectangular grid in the usual fashion (examples below). You'd like to break it down into its individual small pieces using as few "snap" operations as possible: a single snap occurs along a ridge line and breaks a larger rectangle down into two smaller rectangles.

Prove that no matter how you sequence your snaps, you'll always need exactly $mn - 1$ snaps. Use mathematical induction.

PS9-11 *HW*

Suppose a finite number of players play a round-robin tournament, with everyone playing everyone else exactly once. Each match has a winner and a loser (no ties). We say that the tournament has a *cycle of length m* if there exist m players $\{p_1, p_2, \ldots, p_m\}$ such that p_1 beats p_2, who beats p_3, ..., who beats p_m, who beats p_1. Clearly this is possible only for $m \geq 3$.

Using mathematical induction, prove that if such a tournament has a cycle of length m, for some $m \geq 3$, then it has a cycle of length 3.
[5 points]