Solutions for Unit 3

Solution for PS3-1.

- ***a.*** $(2m)+(2n)=2(m+n)$, which is even.
- ***b.*** $(2m+1)+(2n+1)=2(m+n+1)$, which is even.
- ***c.*** $(2m+1)+(2n)=2(m+n)+1$, which is odd.
- ***d.*** $(2m)\cdot(n)=2(mn)$, which is even.
- ***e.*** $(2m+1)\cdot(2n+1)=2(2mn+m+n)+1$, which is odd.

Solution for PS3-2.

- ***a.*** Since $3^b=3^d$, we can simplify the equation to $2^a=2^c$, which makes $a=c$. Thus, $(a,b)=(c,d)$.
- ***b.*** Rearranging the equation, $2^{a-c}=3^{d-b}$. So let $p=a-c, q=d-b$. By assumption, $q>0$, so $q\in\mathbb{N}$.
- ***c.*** Each time we multiply an odd integer by three, the result must be odd. Since 3^q is obtained by starting with one (an odd integer) and multiplying by three q times, the final result must be odd.
- ***d.*** We have that $2^p=3^q$, which is odd. If $p<0$, $2p$ is not even an integer. If $p>0$, $2^p=2\times 2^{p-1}$, which is two times an integer, so it is even. The only way that 2^p can be odd is if $p=0$.
- ***e.*** Since $p=a-c=0$, we have $a=c$. Thus, $2^a=2^c$ and the original equation simplifies to $3^b=3^d$, which implies $b=d$. Thus, $(a,b)=(c,d)$.
- ***f.*** We have shown that $f(a,b)=f(c,d) \implies (a,b)=(c,d)$, so by definition, f is injective.

Solution for PS3-3. If A is countable, then we know there exists and injection $g:A\to\mathbb{N}$ by the definition of countability. If B is a subset of A we can define a function $f:B\to\mathbb{N}$ with $f(x)=g(x)$. To prove that B is countable we will show that f is an injection.

Let x_1 and x_2 be any two elements of B. If $f(x_1)=f(x_2)$, then $g(x_1)=g(x_2)$. But g is injective, so this implies $x_1=x_2$. Therefore f is also injective.

Solution for PS3-4.

- ***a.*** If A and B are both countable then there are injections $g:A\to\mathbb{N}$ and $h:B\to\mathbb{N}$. We can define a function $f:(A\cup B)\to\mathbb{N}$ by the following:

$$f(x)=\begin{cases}2g(x)+1\,, & \text{if } x\in A,\\ 2h(x)\,, & \text{otherwise (i.e., } x\in(B-A))\,.\end{cases}$$

  We can show this function is an injection. Let x_1,x_2 be elements of $A\cup B$ with $f(x_1)=f(x_2)$. Either $f(x_1)$ will be odd, or it will be even.

  If $f(x_1)$ is odd, then $f(x_1)=2g(x_1)+1$ (note that $2h(x)$ can never be odd because $h(x)$ is a natural number). Similarly, $f(x_2)$ is odd (it is equal to $f(x_1$ after all) so we have $f(x_2)=2g(x_2)+1$. This means $2g(x_1)+1=2g(x_2)+1$ which implies $g(x_1)=g(x_2)$. But we know g is an injective function, so $x_1=x_2$.

  If $f(x_1)$ is even, then $f(x_1)=2h(x_1)$ (again note that $2g(x)+1$ can never be even because $g(x)$ is a natural number). We also know$f(x_2$ is even so it equals $2h(x_2)$, therefore $2h(x_1)=2h(x_2)$. This in turn implies $h(x_1)=h(x_2)$. We know h is an injection so $x_1=x_2$.

  So in either case we know if $f(x_1)=f(x_2)$ then $x_1=x_2$. Therefore f is an injective function from $A\cup B$ to $\mathbb{N}$ and there $A\cup B$ must be countable.
- ***b.*** If A and B are both countable then there are injections $g:A\to\mathbb{N}$ and $h:B\to\mathbb{N}$. As in part a, we can define a function $f:(A\times B)\to\mathbb{N}$ by $f(a,b)=2^{g(a)}3^{h(b)}$. As before we can prove this is an injection.

  Let $f(a_1,b_1)=f(a_2,b_2)$, so $2^{g(a_1)}3^{h(b_1)}=2^{g(a_2)}3^{h(b_2)}$. From the uniqueness of prime factorization we know that this implies $g(a_1)=g(a_2)$ and $h(a_1)=h(a_2)$. But g and h are both injections so $a_1=a_2$ and $b_1=b_2$. This means the ordered pairs (a_1,b_1) and (a_2,b_2) are equal to each other. Therefore f is an injective function and $A\times B$ is countable.

Solution for PS3-5. Let's repeatedly invoke the fact "A and B countable $\implies A \times B$ countable".

Using it with $A = \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N}$ is countable.

Now, using it with $A = \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

Next, using it with $A = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

Finally, using it with $A = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $B = \mathbb{N}$ tells us that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is countable.

Solution for PS3-6. Since A is infinite, there exists some element $a_0 \in A$.

Since A is infinite, it has at least two elements, so there exists some element $a_1 \in A$ distinct from a_0.

Since A is infinite, it has at least three elements, so there exists some element $a_3 \in A$ distinct from a_0, a_1.

Since A is infinite, it has at least four elements, so there exists some element $a_4 \in A$ distinct from a_0, a_1, a_2.

Proceeding in this fashion, for each $n \in \mathbb{N}$ we have an element a_n distinct from all elements a_m where $m < n$. Now define a function $f : A \to \mathbb{N}$ as follows.

$$f(x) = \begin{cases} n, & \text{if } x = a_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise (i.e., } x \text{ is not in the list } (a_0, a_1, a_2, \ldots)). \end{cases}$$

This function is surjective because, given any $n \in \mathbb{N}$, there exists the element $a_n \in A$ for which $f(a_n) = n$.

Solution for PS3-7. Since S is finite, given any particular length ℓ, there are only a finite number of ℓ-length strings in S^*. (To be precise, there are $|S|^\ell$ such strings, though we don't need this fact.) Therefore, we can list all the elements of S^* as follows:

> empty string, followed by
> all strings of length 1 in some arbitrary order, followed by
> all strings of length 2 in some arbitrary order, followed by
> all strings of length 3 in some arbitrary order, followed by ......

This listing implicitly defines a bijection $f : \mathbb{N} \to S^*$, proving that S^* is countable.

Solution for PS3-8. Every Python program is just a string over a certain alphabet: say, the alphabet of all Unicode characters.

Thus, the set of all Python programs is a subset of a countable set, so it is itself a countable set.

Solution for PS3-9. Let $I = (0, 1)$. We define $f(x, y)$ as follows, for $(x, y) \in I \times I$. Let

$$x = 0.a_1 a_2 a_3 \cdots,$$
$$y = 0.b_1 b_2 b_3 \cdots$$

be the unique decimal representations of x and y, as defined in class. Now construct the number

$$z = 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots.$$

The sequence of digits in this definition of z has infinitely many non-9s, so it is a legit decimal representation of a real number in I. Set $f(x, y) = z$.

Students: You should write up a formal proof that f is indeed an injection.

Solution for PS3-10. Consider the function $g : (0, 1] \to (0, 1)$ defined by:

$$g(x) = \begin{cases} \frac{x}{2}, & \text{if } \exists n \in \mathbb{Z} \text{ such that } x = 2^{-n}, \\ x, & \text{otherwise}. \end{cases}$$

To prove this is a bijection we will construct an inverse function. Define $h(x) : (0,1) \to (0,1]$ to be:

$$h(x) = \begin{cases} 2x\,, & \text{if } \exists n \in \mathbb{Z} \text{ such that } x = 2^{-n}\,, \\ x\,, & \text{otherwise}\,. \end{cases}$$

To prove that g and h are inverses of each other (which, in turn, shows that g is a bijection) we must show that $g \circ h = \mathrm{id}_{(0,1)}$ and $h \circ g = \mathrm{id}_{(0,1]}$.

To show $g \circ h = \mathrm{id}_{(0,1)}$, let x be any element of $(0,1)$. We must show $g(h(x)) = x$. If $x = 2^{-n}$ for some integer n, then $g(h(x)) = g(2x)$, but $2x = 2^{1-n}$ and $(n-1)$ is an integer, so $g(2x) = \frac{2x}{2} = x$. If instead $x \neq 2^{-n}$ for every integer n, we have $g(h(x)) = g(x) = x$. In either case $g(h(x)) = x$ so $g \circ h = \mathrm{id}_{(0,1)}$.

Similarly To show $h \circ g = \mathrm{id}_{(0,1]}$, we let $x \in (0,1]$. Once again we have two cases either $x = 2^{-n}$ for some integer n, or $x \neq 2^{-n}$ for every integer n. In the former case, $h(g(x)) = h(\frac{x}{2})$. However $\frac{x}{2} = 2^{-(n+1)}$ and $(n+1)$ is an integer so $h(\frac{x}{2}) = 2\frac{x}{2} = x$. In the latter case $h(g(x)) = h(x) = x$. Therefore $h(g(x))$ is always x and $h \circ g = \mathrm{id}_{(0,1]}$

So g and h are inverse functions of each other. Because g is a function with an inverse function it must be a bijection.