Solution for PS4-1.

Number	**Positive divisors**	**Prime?**
12	$1,2,3,4,6,12$	No
15	$1,3,5,15$	No
29	$1,29$	Yes
64	$1,2,4,8,16,32,64$	No
72	$1,2,3,4,6,8,9,12,18,24,36,72$	No
73	$1,73$	Yes
75	$1,3,5,15,25,75$	No

Solution for PS4-2. We claim that $n \in \mathbb{N}^+$ has an odd number of positive divisors iff n is a perfect square. Informally, this is because the positive divisors of n can be grouped into pairs of the form $\{d, n/d\}$, so there must be an even number of such divisors in total *except* when one of these "pairs" is in fact a singleton set, which happens when $d = n/d$ for some d, i.e., $n = d^2$.

Here is the formal proof. Let

$$A = \{d \in \mathbb{N}^+ : d \mid n \text{ and } d < \sqrt{n}\},$$
$$B = \{d \in \mathbb{N}^+ : d \mid n \text{ and } d > \sqrt{n}\},$$
$$D = \{d \in \mathbb{N}^+ : d \mid n\}.$$

The function $f : A \to B$ given by $f(a) = n/a$ is a bijection from A to B, because its inverse is the function $g : B \to A$ given by $g(b) = n/b$. Therefore, $|A| = |B|$. If $\sqrt{n}$ is not an integer, then $D = A \cup B$, so $|D| = |A| + |B| = 2|A|$, which is even. If $\sqrt{n}$ is an integer, then $D = A \cup B \cup \{\sqrt{n}\}$, so $|D| = |A| + |B| + 1 = 2|A| + 1$, which is odd. This completes the proof of our claim.

Using our claim, the desired set of integers is $\{1,4,9,16,25,36,49,64,81,100\}$.

Solution for PS4-3. For $\mathbb{Z}_{11}$:

$\otimes$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	1	3	5	7	9
3	0	3	6	9	1	4	7	10	2	5	8
4	0	4	8	1	5	9	2	6	10	3	7
5	0	5	10	4	9	3	8	2	7	1	6
6	0	6	1	7	2	8	3	9	4	10	5
7	0	7	3	10	6	2	9	5	1	8	4
8	0	8	5	2	10	7	4	1	9	6	3
9	0	9	7	5	3	1	10	8	6	4	2
10	0	10	9	8	7	6	5	4	3	2	1

For $\mathbb{Z}_{12}$:

$\otimes$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	1	6	11	4	9	2	7
6	0	6	0	6	0	6	0	6	0	6	0	6
7	0	7	2	9	4	11	6	1	8	3	10	5
8	0	8	4	0	8	4	0	8	4	0	8	4
9	0	9	6	3	0	9	6	3	0	9	6	3
10	0	10	8	6	4	2	0	10	8	6	4	2
11	0	11	10	9	8	7	6	5	4	3	2	1

Observe that in the table for $\mathbb{Z}_{11}$, the zero entries are confined to the zero row and the zero column, whereas in the table for $\mathbb{Z}_{12}$, there are some zero entries in addition to these "obvious" ones. In fact, for each $x \in \mathbb{Z}_{12}$ that is not relatively prime to 12, there is at least one zero entry in the row/column for x.

Solution for PS4-4. Using the definition of congruence, $d \mid a-b$ and $d \mid x-y$.

Therefore, $d \mid (a-b)+(x-y) = (a+x)-(b+y)$, i.e., $a+x \equiv b+y \pmod{d}$.

Further, $d \mid (a-b)x$ and $d \mid b(x-y)$. Therefore, $d \mid (a-b)x + b(x-y) = ax - by$, i.e., $ax \equiv by \pmod{d}$.

Solution for PS4-5. Using the result of ***PS4-4***, if $a \equiv b \pmod{d}$, then $a^2 \equiv b^2 \pmod{d}$. Using the same result again, $a^3 \equiv b^3 \pmod{d}$. Proceeding in this way, $a^n \equiv b^n \pmod{d}$ for all $n \in \mathbb{N}^+$.

Now take $d = a-b$. Trivially, $d \mid a-b$, so $a \equiv b \pmod{d}$. Therefore, $a^n \equiv b^n \pmod{d}$, which means $d \mid a^n - b^n$.

Solution for PS4-6. Take $d = a+b$. Then $a \equiv -b \pmod{d}$.

As in ***PS4-5***, $a^n \equiv (-b)^n \pmod{d}$. Since n is odd, $(-1)^n = -1$. Therefore, $(-b)^n = (-1)^n b^n = -b^n$. We conclude that $a^n \equiv -b^n \pmod{d}$, so $d \mid a^n + b^n$.

Solution for PS4-7. Start by observing that $2^4 = 16 \equiv -1 \pmod{17}$. Therefore, powers of 2^4 are going to be easy to figure out. We use this to simplify:

$$2^{2019} = 2^{4\times504+3} = 16^{504} \times 2^3 \equiv (-1)^{504} \times 8 = 8 \pmod{17}.$$

Solution for PS4-8. Let n^2 be a perfect square. To reason about the last digit of n^2, we consider arithmetic modulo 10. To reduce the number of cases to consider, we further use $9 \equiv -1 \pmod{10}$ and so on.

- If $n \equiv 0 \pmod{10}$, then $n^2 \equiv 0 \pmod{10}$.
- If $n \equiv \pm1 \pmod{10}$, then $n^2 \equiv 1 \pmod{10}$.
- If $n \equiv \pm2 \pmod{10}$, then $n^2 \equiv 4 \pmod{10}$.
- If $n \equiv \pm3 \pmod{10}$, then $n^2 \equiv 9 \pmod{10}$.
- If $n \equiv \pm4 \pmod{10}$, then $n^2 \equiv 16 \equiv 6 \pmod{10}$.
- If $n \equiv 5 \pmod{10}$, then $n^2 \equiv 25 \equiv 5 \pmod{10}$.

It follows that $n^2 \not\equiv 7 \pmod{10}$.

Alternate Solution for PS4-8. Let n^2 be a perfect square. If the last digit of n^2 is 7, then $n^2 \equiv 2 \pmod{5}$. However, the following exhaustive list of cases shows that this is not possible.

- If $n \equiv 0 \pmod 5$, then $n^2 \equiv 0 \pmod 5$.
- If $n \equiv \pm 1 \pmod 5$, then $n^2 \equiv 1 \pmod 5$.
- If $n \equiv \pm 2 \pmod 5$, then $n^2 \equiv 4 \pmod 5$.

Solution for PS4-9. Given that decimal representation, the value of n is given by

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^2 a_2 + 10 a_1 + a_0 = \sum_{j=0}^{k} 10^j a_j \,.$$

Now $10 \equiv 1 \pmod 3$, therefore $10^j \equiv 1^j = 1 \pmod 3$ for each j. Using this in the above equation, we obtain $n \equiv \sum_{j=0}^{k} a_j \pmod 3$, as desired.

Solution for PS4-10. Let $n = m(m+1)(m+2)$ be the product of three consecutive integers. Then, modulo 3, the integers m, $m+1$, and $m+1$ are congruent to 0, 1, and 2 in some order. Therefore $n \equiv 0 \times 1 \times 2 = 0 \pmod 3$, i.e., $n = 3k$ for some integer k.

At least one of m, $m+1$, and $m+2$ is even. Therefore, n is even. If k were odd, then $n = 3k$ would have been odd. Therefore, k is even. Let $k = 2\ell$ for some integer ℓ. Then $n = 3 \cdot 2\ell = 6\ell$, which is divisible by 6.

Note: It's not enough to say that n is divisible by 2 and by 3, *therefore* it is divisible by 6. That "therefore" needs to be justified.