Learning distributions on Riemannian manifolds

Autoencoder for SPD matrices

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Use of SPD matrices

- Covariances matrices
- Used in computer vision, brain imaging, brain-computer interface (EEG)

Existing solutions in Euclidean geometry

- Flatten to tangent space
- Distance approximation: Frobenius distance
  $L = ||A - B||_F = \sqrt{\sum_{i,j}(A_{ij} - B_{ij})^2}$

Problems

- Does not preserve the curvature of space
- Non-optimal results with euclidean distance
- Swelling effect : the determinants of the interpolation of flattened matrices

We therefore want to take into account the curvature and the non-linearity of the space of SPD matrices.

Riemannian geometry

Riemann metric *(AIRM)*: $\delta_R^2(X, Y) = ||\log(X^{-1/2} Y X^{-1/2})||_F^2$

- Measure the similarity between two SPD matrices while respecting the structure
- We will use it in our AE in the model, in the cost function and in the trustworthiness.

A Riemannian Network for SPD Matrix Learning

Introduction of a network architecture that preserves the properties of positive definite matrices for Deep Learning [6]

- 3 different layers: BiMap, ReEig, LogEig
- We will base ourselves on these layers for our autoencoder

Different works

DreamNet: A Deep Riemannian Manifold Network for SPD Matrix Learning [10]

- Methodology for creating deep networks
- Stacked Riemannian Autoencoder (SRAE) at the end of the network

Riemannian Multinomial Logistics Regression for SPD Neural Networks [3]

- Adapting logistic regression for SPD matrices
- New specific layer for classification
- Use of Log-Euclidean Metric or Log-Cholesky Metric

SPD domain-specific batch normalization to crack interpretable unsupervised domain adaptation in EEG [7]

- Specific batch normalization for SPD matrices

Different works

Riemannian batch normalization for SPD neural networks [1]

- Specific batch normalization for SPD matrices

A Riemannian Residual Learning Mechanism for SPD Network [2]

- Improves learning process for SPD networks

U-SPDNet: An SPD manifold learning-based neural network for visual classification [12]

- SPD matrices from visual data

Reducing the Dimensionality of SPD Matrices with Neural Networks in BCI [8]

- Simplification of complex data for a better interpretability and processing in BCI data

Schur's Positive Definite Network: Deep Learning in the SPD Cone With Structure [9]

- Shows that the use of the structure in the network improves the performances

Modeling Graphs Beyond Hyperbolic: Graph Neural Networks in Symmetric Positive Definite Matrices [13]

- Applies GNN to SPD matrices
- To model graph structures in SPD matrices

SymNet: A Simple Symmetric Positive Definite Manifold Deep Learning Method for Image Set Classification [11]

- Image set classification

From Manifold to Manifold: Geometry-Aware Dimensionality Reduction for SPD Matrices [4]

- Lower-dimensional and more discriminative SPD matrices from SPD matrices with orthonormal projection

Geometry-Aware Principal Component Analysis for Symmetric Positive Definite Matrices [5]

- PCA applied to SPD matrices
- Preserves more data variance
- Extends PCA from Euclidean to Riemannian geometries

Objectives

- Autoencoder for SPD matrices
- Layer to do the reverse operations of the autoencoder
- Comparison of different models
- Impact of distance for reconstruction error

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Autoencoder Basics

- Unsupervised learning: measurement of reconstruction error
- Dimension reduction
- Learn the underlying patterns
- Used for generative models

$$\phi : \mathcal{X} \rightarrow \mathcal{F} \text{ , encoder}$$
$$\psi : \mathcal{F} \rightarrow \mathcal{X} \text{ , decoder}$$
$$\phi, \psi = \arg\min_{\phi,\psi} ||X - (\psi \circ \phi)X||^2$$

Reconstruction error

- For each matrix, we calculate the Riemannian distance with its reconstruction.

$$\phi : \mathcal{X} \to \mathcal{F}$$
$$\psi : \mathcal{F} \to \mathcal{X}$$
$$\phi, \psi = \arg\min_{\phi,\psi} \delta_R^2(X, \psi(\phi(X))) = \arg\min_{\phi,\psi} ||\log(X^{-1/2}\psi(\phi(X))X^{-1/2})||_F^2$$

Trustworthiness

- For each matrix, we take its k closest matrices in the output space and its closest matrices in the input space.
- The distance is the same used to calculate our cost function.
- We penalize proportionally to the difference in ranks in the input space.
- We do not penalize matrices coming closer together.

$$T(k) = 1 - \frac{2}{nk(2n-3k-1)} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i^k} \max(0, (r(i,j) - k))$$

Accuracy

- We use MDM (Minimum Distance to Mean) to know the precision before reconstituting our matrices.
- For each class, a centroid is estimated according to our distance.
- We compare the initial precision with the final precision.

BiMap layer

- The function of this layer is to generate more compact and more discriminative SPD matrices.
- Layer which performs a bilinear map f_b to transform the initial matrices into new matrices of lower dimension.

$$X_k = f_b^{(k)}(X_{k-1}; W_k) = W_k X_{k-1} W_k^T$$

W_k is of full rank to guarantee that X_k remains SPD.

Network parameters

Number of input filters/channels *hi*, number of output filters/channels *ho*, size of input matrix *ni*, size of output matrix *no*

ReEig layer

- The function of this layer is to improve discriminative performance by introducing nonlinearity, in the same way as ReLU.
- Introduction of a non-linear function f_r which corrects the matrices by setting a threshold for low eigenvalues.

$$X_k = f_r^{(k)}(X_{k-1}) = U_{k-1} \max(\epsilon I, \Sigma_{k-1}) U_{k-1}^T$$

LogEig/ExpEig layers

LogEig

The function of this layer is to be able to apply Riemann geometry to the output matrix.

$$X_k = f_l^{(k)}(X_{k-1}) = \log(X_{k-1}) = U_{k-1} \log(\Sigma_{k-1}) U_{k-1}^T$$

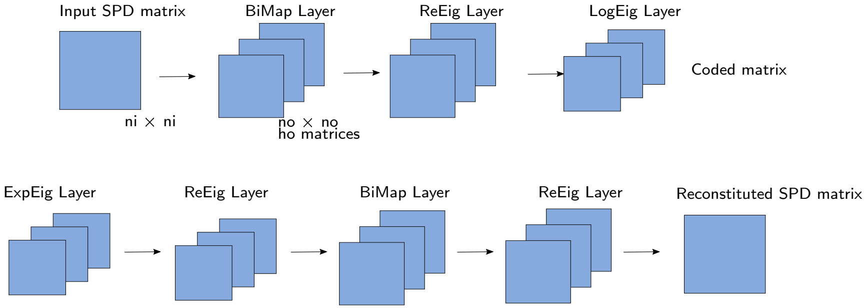
ExpEig

The function of this layer is to apply the inverse function of the LogEig layer.

$$X_k = f_e^{(k)}(X_{k-1}) = \exp(X_{k-1})$$

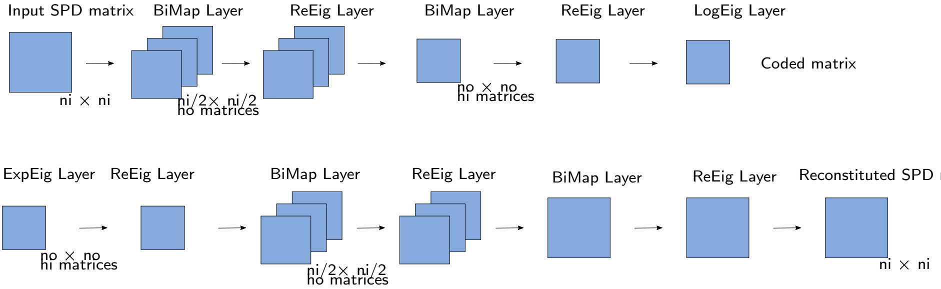
One layer

- Single BiMap layer for the encoder from $ni \to no$ and $hi \to ho$.
- We look at the influence of the output dimension and the output layer.
- The decoder does the opposite operation.



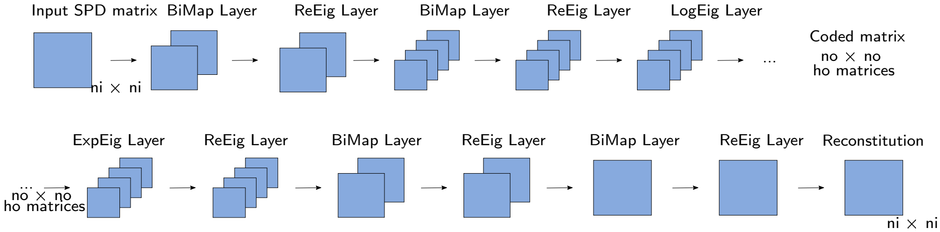
Two layers with funnel channels

- Two BiMap layers.
  - $ni \to ni/2$ and $hi \to ho$.
  - $ni/2 \to no$ and $ho \to hi$.
- We look at the influence of the number of intermediate channels and the output dimension.
- The decoder does the opposite operation.



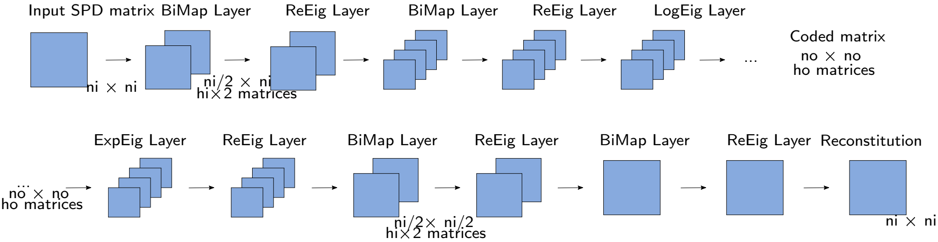
Multiple layers evenly distributed

- Number of BiMap layers set in parameters.
- Channels and intermediate matrix sizes based on the number of layers.
- The decoder does the opposite operation.



Multiple layers halved in dimension

- Number of BiMap layers and filters in layers depends on *ni* and *no*.
- Matrix size divided by two, number of filters multiplied by two at each layer.
- The decoder does the opposite operation.



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Synthetic data

Adding noises

Gaussian noise

Salt and pepper noise

Masking noise

BCI data

Results

- Prediction less accurate
- The more we preserve the neighborhood, the worse the accuracy becomes.
- Lossy compression

Accuracy is already good without the autoencoding. We should try with a dataset

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