

Inverse Numerator Relationship Matrix and Inbreeding

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Inverse of A Based on L and D

- ▶ Matrix A was decomposed into $A = L \cdot D \cdot L^T$
- ▶ Get A^{-1} as $A^{-1} = (L^T)^{-1} D^{-1} L^{-1}$
- ▶ D^{-1} is diagonal again with elements

$$(D^{-1})_{ii} = 1/(D)_{ii}$$

Inverse of L

- Compute m based on the two decompositions of a

$$a = P \cdot a + m \quad \text{and} \quad a = L \cdot m$$

- Solve both for m and set them equal

$$m = a - P \cdot a = (I - P) \cdot a \quad \text{and} \quad m = L^{-1} \cdot a$$

$$(I - P) \cdot a = L^{-1} \cdot a$$

and

$$L^{-1} = I - P$$

Example

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

Matrix D^{-1}

- ▶ Because D is diagonal

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- ▶ We get D^{-1} as

$$D^{-1} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

Matrix L^{-1}

- ▶ Use $L^{-1} = I - P$
- ▶ Matrix P from simple decomposition

$$P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{bmatrix}$$

- ▶ Therefore

$$L^{-1} = I - P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$

Decomposition of A^{-1} I

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$(L^{-1})^T \cdot D^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 & -0.5 & -0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \cdot \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

Decomposition of A^{-1} II

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix} \cdot \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$
$$= \begin{bmatrix} 1.5 & 0.5 & 0.0 & -1.0 & 0.0 \\ 0.5 & 2.0 & 0.5 & -1.0 & -1.0 \\ 0.0 & 0.5 & 1.5 & 0.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & -1.0 & 0.0 & 2.0 \end{bmatrix}$$

Henderson's Rules

- ▶ Both Parents Known
 - ▶ add 2 to the diagonal-element (i, i)
 - ▶ add -1 to off-diagonal elements (s, i) , (i, s) , (d, i) and (i, d)
 - ▶ add $\frac{1}{2}$ to elements (s, s) , (d, d) , (s, d) , (d, s)
- ▶ Only One Parent Known
 - ▶ add $\frac{4}{3}$ to diagonal-element (i, i)
 - ▶ add $-\frac{2}{3}$ to off-diagonal elements (s, i) , (i, s)
 - ▶ add $\frac{1}{3}$ to element (s, s)
- ▶ Both Parents Unknown
 - ▶ add 1 to diagonal-element (i, i)
- ▶ Valid without inbreeding

Inbreeding

- ▶ Elements in matrix D depend on coefficients of inbreeding
- ▶ Recap: From the simple decomposition of a , we derived

$$\begin{aligned} \text{var}(m_i) &= \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d) \right) \sigma_a^2 \\ &= \left(\frac{1}{2} - \frac{1}{4}(A_{ss} - 1 + A_{dd} - 1) \right) \sigma_a^2 \\ &= \left(1 - \frac{1}{4}(A_{ss} + A_{dd}) \right) \sigma_a^2 \\ &= (D)_{ii} \sigma_a^2 \end{aligned}$$

$$\rightarrow (D)_{ii} = \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d) \right) = \left(1 - \frac{1}{4}(A_{ss} + A_{dd}) \right)$$

Computation of Coefficients of Inbreeding

- ▶ Observation: Coefficients of inbreeding F_s and F_d can be read from $(A)_{ss}$ and $(A)_{dd}$ of A
- ▶ Cannot setup A to just get inbreeding coefficients
- ▶ More efficient method required
- ▶ **Cholesky** decomposition of A

$$A = R \cdot R^T$$

where R is a lower triangular matrix

Hint: Function `chol(A)` in R gives matrix R^T

Cholesky Decomposition

- ▶ Diagonal elements $(A)_{ii}$ of A are the sum of the squared elements of one row of R

$$(A)_{ii} = \sum_{j=1}^i (R)_{ij}^2$$

- ▶ Example

$$\begin{bmatrix} (A)_{11} & (A)_{12} & (A)_{13} \\ (A)_{21} & (A)_{22} & (A)_{23} \\ (A)_{31} & (A)_{32} & (A)_{33} \end{bmatrix} = \begin{bmatrix} (R)_{11} & 0 & 0 \\ (R)_{21} & (R)_{22} & 0 \\ (R)_{31} & (R)_{32} & (R)_{33} \end{bmatrix} \cdot \begin{bmatrix} (R)_{11} & (R)_{21} & (R)_{31} \\ 0 & (R)_{22} & (R)_{32} \\ 0 & 0 & (R)_{33} \end{bmatrix}$$

Recursive Computation of R

- ▶ Let us write the matrix R as a product of two matrices L and S :

$$R = L \cdot S$$

where L is the same matrix as in the LDL-decompositon and S is a diagonal matrix.

- ▶ Compute A as

$$A = R \cdot R^T = L \cdot S \cdot S \cdot L^T = L \cdot D \cdot L^T$$

- ▶ Hence

$$D = S \cdot S \quad \rightarrow \quad (S)_{ii} = \sqrt{(D)_{ii}}$$

Example

$$\begin{bmatrix} (R)_{11} & 0 & 0 \\ (R)_{21} & (R)_{22} & 0 \\ (R)_{31} & (R)_{32} & (R)_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (L)_{21} & 1 & 0 \\ (L)_{31} & (L)_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} (S)_{11} & 0 & 0 \\ 0 & (S)_{22} & 0 \\ 0 & 0 & (S)_{33} \end{bmatrix}$$

- ▶ Diagonal elements $(R)_{ii} = (S)_{ii}$
- ▶ Because $(S)_{ii} = \sqrt{(D)_{ii}}$, diagonal elements $(R)_{ii}$ of matrix R can be computed as

$$(R)_{ii} = (S)_{ii} = \sqrt{(D)_{ii}} = \sqrt{\left(1 - \frac{1}{4}(A_{ss} + A_{dd})\right)}$$

- ▶ A_{ss} and A_{dd} are
 - ▶ 0 if s and d are unknown (NA) or
 - ▶ have been computed before

Offdiagonal Elements of R

- ▶ Offdiagonal elements $(R)_{ij}$ of R are computed as

$$(R)_{ij} = (L)_{ij} * (S)_{jj}$$

- ▶ Use property of L : $L_{ij} = \frac{1}{2}((L)_{sj} + (L)_{dj})$ if s and d are parents of i

$$\begin{aligned}(R)_{ij} &= (L)_{ij} * (S)_{jj} \\&= \frac{1}{2} [(L)_{sj} + (L)_{dj}] * (S)_{jj} \\&= \frac{1}{2} [(L)_{sj} * (S)_{jj} + (L)_{dj} * (S)_{jj}] \\&= \frac{1}{2} [(R)_{sj} + (R)_{dj}]\end{aligned}$$

Example Pedigree

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2
6	4	5

Computations

- ▶ Compute diagonal elements $(A)_{ii}$ of A to get F_i
- ▶ Prerequisite: Pedigree sorted such that parents before progeny
- ▶ Start with $(A)_{11}$

$$(A)_{11} = (R)_{11}^2 = (D)_{11} = \left(1 - \frac{1}{4}(A_{NANA} + A_{NANA})\right) = 1$$

- ▶ $(A)_{22} = (R)_{21}^2 + (R)_{22}^2 = 0 + 1 = 1$
- ▶ $(A)_{33} = (R)_{31}^2 + (R)_{32}^2 + (R)_{33}^2 = 0 + 0 + 1 = 1$

Animals With Known Parents

$$\begin{aligned}(A)_{44} &= (R)_{41}^2 + (R)_{42}^2 + (R)_{43}^2 + (R)_{44}^2 \\&= \left(\frac{1}{2}(R_{11} + R_{21})\right)^2 + \left(\frac{1}{2}(R_{12} + R_{22})\right)^2 + \left(\frac{1}{2}(R_{13} + R_{23})\right)^2 \\&\quad + \left(1 - \frac{1}{4}(A_{11} + A_{11})\right) \\&= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1\end{aligned}$$

► $(A)_{55}$

► $(A)_{66}$