Numerator Relationship Matrix

Peter von Rohr

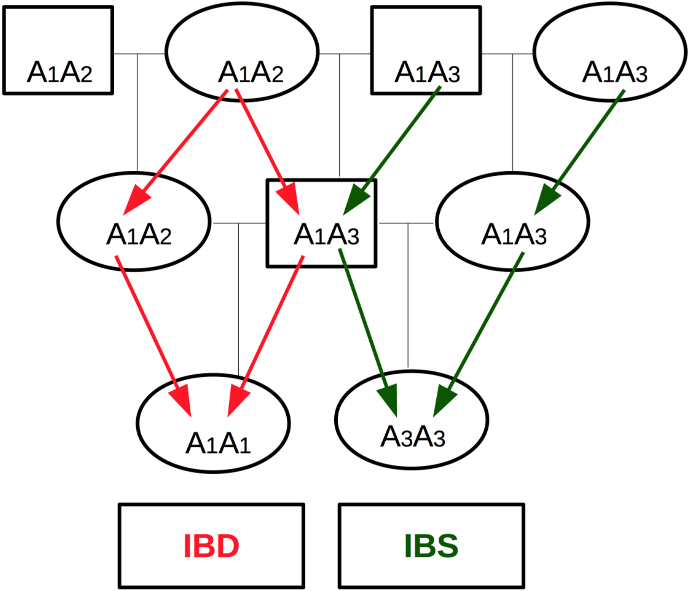
09 November 2018

Similarity Between Individuals

At the genetic level there are two different kinds of similarity

1. Identity by descent (IBD)
2. Identity by state

IBD versus IBS



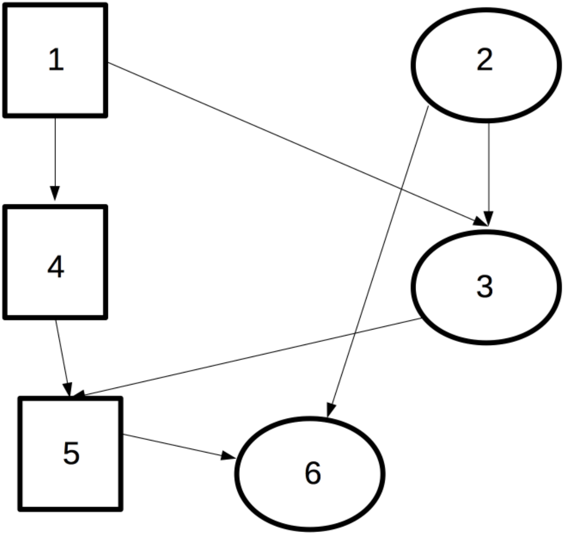
Numerator Relationship Matrix

- probability of IBD alleles in two individuals: **coancestry** or **coefficient of kinship**
- additive genetic relationship between two individuals is twice their coancestry
- matrix containing all additive genetic relationships in a population is called **numerator relationship matrix** (A)
- A is symmetric and contains on
  - diagonal: $(A)_{ii} = (1 + F_i)$
  - off-diagonal: $(A)_{ij} = cov(a_i, a_j)/\sigma_a^2$ (with $i \neq j$)

Recursive Computation of A

- If both parents s and d of animal i are known then
  - the diagonal element $(A)_{ii}$ corresponds to: $(A)_{ii} = 1 + F_i = 1 + \frac{1}{2}(A)_{sd}$ and
  - the offdiagonal element $(A)_{ji}$ is computed as: $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$
  - because A is symmetric $(A)_{ji} = (A)_{ij}$
- If only one parent s is known and assumed unrelated to the mate
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = \frac{1}{2}((A)_{js}$
- If both parents are unknown
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = 0$

Example



Tabular Representation of Pedigree

Example Pedigree To Compute Additive Genetic Relationship Matrix

Calf

Sire

Dam

3

1

2

4

1

NA

5

4

3

Stepwise Computation of A

- Start by extending pedigree with animals that do not have parents
- Order animals, such that parents before progeny

Animal

Sire

Dam

1

NA

NA

2

NA

NA

3

1

Initialize With Empty Matrix A

- Dimensions of A: number of rows and number of columns equal to the number of animals
- Our example: 6×6

$$A = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

First Diagonal Element

- Compute first element $(A)_{11} = 1 + F_1$
- Animal 1 has both parents unknown $\rightarrow F_1 = 0$

$$A = \begin{bmatrix} 1.00 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Off-diagonal Elements

- Assume animal i has parents s and d
- $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$

First Row of A

$$A = \begin{bmatrix} 1.00 & 0.00 & 0.50 & 0.50 & 0.50 & 0.25 \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

Use Symmetry of A

- Copy first row into first column

$$A = \begin{bmatrix} 1.00 & 0.00 & 0.50 & 0.50 & 0.50 & 0.25 \\ 0.00 & & & & & \\ 0.50 & & & & & \\ 0.50 & & & & & \\ 0.50 & & & & & \\ 0.25 & & & & & \end{bmatrix}$$

Remaining Elements of A

- Continue with rows and columns 2 to 6 using the same recipe

Final Result

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.5000 & 0.5000 & 0.5000 & 0.2500 \\ 0.0000 & 1.0000 & 0.5000 & 0.0000 & 0.2500 & 0.6250 \\ 0.5000 & 0.5000 & 1.0000 & 0.2500 & 0.6250 & 0.5625 \\ 0.5000 & 0.0000 & 0.2500 & 1.0000 & 0.6250 & 0.3125 \\ 0.5000 & 0.2500 & 0.6250 & 0.6250 & 1.1250 & 0.6875 \\ 0.2500 & 0.6250 & 0.5625 & 0.3125 & 0.6875 & 1.1250 \end{bmatrix}$$

The Inverse Numerator Relationship Matrix

- Recap: Henderson's mixed model equations depend on four matrices

1. Design matrix X for the fixed effects
2. Design matrix Z for the random effects
3. The inverse variance-covariance matrix R^{-1} for the residuals e and
4. The inverse variance-covariance matrix G^{-1} for the random breeding values a.

Animal Model

- Breeding values of all individuals as random effects
- Variance-Covariance matrix G corresponds to variance-covariance matrix of breeding values

$$G = A * \sigma_a^2$$

- We need: G^{-1}

$$G^{-1} = A^{-1} * \frac{1}{\sigma_a^2}$$

Need For Efficient Computation of $A-1$

- In practical livestock breeding evaluations A is very large
- Dimensions of A can be $10^7 \times 10^7$
- Explicit general inversion not possible
- Special structure of A^{-1} leads to efficient computation