

BLUP

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General Principle

- ▶ All methods to predict breeding values follow the same principle
 1. Correct information sources for some population mean
 2. Multiply corrected information source by an appropriate factor
- ▶ Regression Method

$$\hat{a} = b(y - \mu)$$

- ▶ Selection Index

$$\hat{a} = I = b^T y^*$$

where $b = P^{-1}Gw$ and y^* corrected information sources.

Problem with Correction

- ▶ Population mean is ideal as correction

$$y = \mu + a + e \quad \rightarrow \quad \bar{y} = \bar{\mu} + \bar{a} + \bar{e} = \mu$$

- ▶ Because performances are observed in different
 - ▶ environments and
 - ▶ time points
- ▶ Formation of comparison groups where animals are exposed to the same environments
- ▶ The more groups, the better the correction of environmental effects
- ▶ The more groups, the smaller the single groups

Bias

- ▶ With small comparison groups, it is more likely that mean breeding value of animals in a single group is not 0
- ▶ Average performance of all animals in a comparison group

$$\bar{y}_{CG} = \mu + \bar{a}_{CG} + \bar{e}_{CG}$$

* If \bar{a}_{CG} is not 0, the predicted breeding value \hat{a}_i of animal i is

$$\begin{aligned}\hat{a}_i &= I = b(y_i - (\mu + \bar{a}_{CG})) \\ &= b(y_i - \mu) - b\bar{a}_{CG} \\ &= \hat{a}_i - b\bar{a}_{CG}\end{aligned}$$

where $b\bar{a}_{CG}$ is called bias.

Solution - BLUP

- ▶ Solution to correction problem in selection index: BLUP
- ▶ Estimates environmental effects at the same time as breeding values are predicted
- ▶ Linear mixed effects model
- ▶ Meaning of BLUP
 - ▶ **B** stands for **best** which means that the correlation between the true (a) and the predicted breeding value (\hat{a}) is maximal or the prediction error variance ($\text{var}(a - \hat{a})$) is minimal.
 - ▶ **L** stands for **linear** which means the predicted breeding values are linear functions of the observations (y)
 - ▶ **U** stands for **unbiased** which means that the expected values of the predicted breeding values are equal to the true breeding values
 - ▶ **P** stands for **prediction**

Numeric Example

Animal	Sire	Dam	Herd	Weaning Weight
12	1	4	1	2.61
13	1	4	1	2.31
14	1	5	1	2.44
15	1	5	1	2.41
16	1	6	2	2.51
17	1	6	2	2.55
18	1	7	2	2.14
19	1	7	2	2.61
20	2	8	1	2.34
21	2	8	1	1.99
22	2	9	1	3.10
23	2	9	1	2.81
24	2	10	2	2.14
25	2	10	2	2.41
26	3	11	2	2.54
27	3	11	2	3.16

Linear Mixed Effects Model

- ▶ Simple linear model

$$y_{ij} = \mu + \text{herd}_j + e_{ij}$$

- ▶ Result: Estimate of effect of herd j
- ▶ What about breeding value a_i for animal i ?
 - ▶ Problem: breeding values have a variance σ_a^2
 - ▶ Cannot be specified in simple linear model

→ **Linear Mixed Effects Model (LME)**

$$y_{ijk} = \mu + \beta_j + u_i + e_{ijk}$$

Matrix-Vector Notation

- ▶ LME for all animals of a population

→ use matrix-vector notation

$$y = X\beta + Zu + e$$

where

- y vector of length n of all observations
- β vector of length p of all fixed effects
- X $n \times p$ design matrix linking the fixed effects to the observations
- u vector of length n_u of random effects
- Z $n \times n_u$ design matrix linking random effect to the observations
- e vector of length n of random residual effects.

Expected Values and Variances

- ▶ Expected values

$$E(u) = 0 \text{ and } E(e) = 0 \rightarrow E(y) = X\beta$$

- ▶ Variances

$$\text{var}(u) = G \text{ and } \text{var}(e) = R$$

with $\text{cov}(u, e^T) = 0$,

$$\text{var}(y) = Z * \text{var}(u) * Z^T + \text{var}(e) = ZGZ^T + R = V$$

The Solution

$$\hat{u} = GZ^T V^{-1}(y - X\hat{\beta})$$

$$\hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} y$$

Mixed Model Equations

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

Sire Model

- ▶ Breeding value of sire as random effect:

$$y = X\beta + Zs + e$$

Example

$$\begin{bmatrix} 2.61 \\ 2.31 \\ 2.44 \\ 2.41 \\ 2.51 \\ 2.55 \\ 2.14 \\ 2.61 \\ 2.34 \\ 1.99 \\ 3.10 \\ 2.81 \\ 2.14 \\ 2.41 \\ 2.54 \\ 3.16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \\ e_{14} \\ e_{15} \\ e_{16} \end{bmatrix}$$

Animal Model

- ▶ Breeding value for all animals as random effects

$$y = X\beta + Zu + e$$