# Moore-Pennrose Generalized Inverse

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## Disclaimer

This document collects a few facts about how to compute the Moore-Pennrose Generalized Inverse of a matrix.

## Introduction

What is shown here comes out of a review study of the book (Searle 1971). Chapter 1 of (Searle 1971) reviews many aspect of generalized inverse matrices pointing out that the Moore-Penrose generalized inverse is unique for any given matrix $A$.

## A Generalized Inverse

### Definition

We start with the definition of a generlized inverse matrix $G$ for any given matrix $A$. The definition is given as follows. Given any matrix $A$, a generalized inverse matrix $G$ of $A$ is defined as

$$AGA = A \tag{1}$$

### Computation

One possible solution for $G$ can be computed using the construction of a diagonal form of $A$. This diagonal form can be computed as

$$PAQ = \Delta = \left[ \begin{array}{cc} D_r & 0 \\ 0 & 0 \end{array} \right] \tag{2}$$

where $D_r$ is a diagonal matrix of order $r$ where $r$ is the rank of $A$. From the above diagonal form (2), we can also see that

$$A = P^{-1}\Delta Q^{-1} \tag{3}$$

The inverse matrices $P^{-1}$ and $Q^{-1}$ exist, because matrices $P$ and $Q$ are matrices of elementary operations. The matrix $\Delta^-$ defined as

$$\Delta^- = \left[ \begin{array}{cc} D_r^{-1} & 0 \\ 0 & 0 \end{array} \right] \tag{4}$$

is a generalized inverse of $\Delta$ satisfying the definition in (1). This is shown by writing the definition in (1) using the matrices $\Delta$ and $\Delta^-$, as shown below.

$$\Delta\Delta^{-}\Delta = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix} = \Delta$$

One possible instance of the matrix $G$ can be computed as

$$G = Q\Delta^{-}P \tag{5}$$

The matrix $G$ given in (6) is indeed a generalized inverse of $A$, because according to (1), (3) and (4), we can write

$$AGA = P^{-1}\Delta Q^{-1}Q\Delta^{-}PP^{-1}\Delta Q^{-1} = P^{-1}\Delta\Delta^{-}\Delta Q^{-1} = P^{-1}\Delta Q^{-1} = A \tag{6}$$

The results of (6) and (6) shows us how to come up with a generalized inverse $G$ for any matrix $A$. Now the question is how to compute such a matrix $G$ efficiently. For that reason it is well worth while to have a closer look at formula (3).

## Singular Value Decomposition (SVD)

In (3) the matrix $A$ is decomposed into the product of three matrices $P^{-1}$, $\Delta$ and $Q^{-1}$. What we know about the three matrices is that the matrix $\Delta$ is a diagonal matrix and that matrices $P^{-1}$ and $Q^{-1}$ are invertible. The structure of this decomposition is very similar to the **singular value decomposition** (SVD). The SVD of a matrix $A$ is defined as

$$A = U * D * V^T \tag{7}$$

where $D$ is a diagonal matrix and matrices $U$ and $V$ are orthogonal matrices. Orthogonal matrices are special because their transpose is equal to their inverse, hence $UU^T = U^TU = I$ and $VV^T = V^TV = I$. The diagonal elements in matrix $D$ correspond to the so called singular values of matrix $A$.

### Generalized inverse

When looking at the SVD in (7) and comparing that to the decomposition in (3), we can see that the former decomposition is a special case of the latter one. Hence, we can use the results of the SVD of $A$ to compute the generalized inverse $G$. As shown in (6), the matrix $G$ is

$$G = Q\Delta^{-}P$$

According to (4) $\Delta^{-}$ can be computed by inverting all the non-zero diagnoal elements in $\Delta$ and $\Delta$ corresponds to the matrix $D$ in the SVD of $A$. The matrices $P$ and $Q$ can also be taken from the SVD of $A$. The matrix $P^{-1}$ in (3) corresponds to the matrix $U$ in (7) and similarly the matrix $Q^{-1}$ corresponds to the matrix $V^T$. Taking into account that matrices $U$ and $V$ are orthogonal, we can write

$$G = Q\Delta^{-}P = VD^{-}U^T \tag{8}$$

## References

Searle, S.R. 1971. *Linear Models.* Wiley Classics.