

# Estimable Functions

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## Estimable Functions

Animal	Breed	Observation
1	Angus	16
2	Angus	10
3	Angus	19
4	Limousin	27
5	Simmental	11
6	Simmental	13

## Model

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$$

$$\mathbf{y} = \begin{bmatrix} 16 \\ 10 \\ 19 \\ 27 \\ 11 \\ 13 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

## Normal Equations

$$X^T X b^{(0)} = X^T y$$

$$\begin{bmatrix} 6 & 3 & 1 & 2 \\ 3 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \mu^0 \\ \alpha_1^0 \\ \alpha_2^0 \\ \alpha_3^0 \end{bmatrix} = \begin{bmatrix} 96 \\ 45 \\ 27 \\ 24 \end{bmatrix}$$

A solution

$$b^{(0)} = \begin{bmatrix} 13.5 \\ 1.5 \\ 13.5 \\ -1.5 \end{bmatrix}$$

## Solutions to Normal Equations

Elements of Solution	$b_1^0$	$b_3^0$	$b_4^0$
$\mu^0$	16	27	-2982
$\alpha_1^0$	-1	-12	2997
$\alpha_2^0$	-4	-15	2994
$\alpha_3^0$	11	0	3009

## Functions of Solutions

Linear Function	$b_1^0$	$b_2^0$	$b_3^0$	$b_4^0$
$\alpha_1^0 - \alpha_2^0$	3.0	3.0	3.0	3.0
$\mu^0 + \alpha_1^0$	15.0	15.0	15.0	15.0
$\mu^0 + 1/2(\alpha_2^0 + \alpha_3^0)$	19.5	19.5	19.5	19.5

- ▶  $\alpha_1^0 - \alpha_2^0$ : estimate of the difference between breed effects for Angus and Simmental
- ▶  $\mu^0 + \alpha_1^0$ : estimate of the general mean plus the breed effect of Angus
- ▶  $\mu^0 + 1/2(\alpha_2^0 + \alpha_3^0)$ : estimate of the general mean plus mean effect of breeds Simmental and Limousin

# Definition of Estimable Functions

$$\mathbf{q}^T \mathbf{b} = \mathbf{t}^T E(\mathbf{y})$$

- ▶ Why is  $\mathbf{q}^T \mathbf{b}$  estimable?
- ▶ Based on the definition of  $\mathbf{b}$  and  $E(\mathbf{y})$

$$\mathbf{q}^T \mathbf{b} = \mathbf{t}^T \mathbf{X} \mathbf{G} \mathbf{X}^T \mathbf{y}$$

where  $\mathbf{X} \mathbf{G} \mathbf{X}^T$  is the same for all choices of  $\mathbf{G}$

## Examples

$$E(y_{1j}) = \mu + \alpha_1$$

$$\text{with } \mathbf{t}^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{q}^T = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

$$E(y_{2j}) = \mu + \alpha_2$$

$$E(y_{3j}) = \mu + \alpha_3$$



## Property

Based on the definition, the following property can be derived

$$\mathbf{q}^t = \mathbf{t}^T \mathbf{X}$$

with the definition of an estimable function  $\mathbf{q}^T \mathbf{b}$ , we get

$$\mathbf{q}^T \mathbf{b} = \mathbf{t}^T E(\mathbf{y})$$

$$\mathbf{q}^T \mathbf{G} \mathbf{X}^T \mathbf{y} = \mathbf{t}^T \mathbf{X} \mathbf{G} \mathbf{X}^T \mathbf{y}$$

hence for any  $\mathbf{G}$ ,  $\mathbf{q}^t = \mathbf{t}^T \mathbf{X}$  which is helpful to find  $\mathbf{q}$  for a given  $\mathbf{t}$

# Test

When we want to test whether a certain vector  $\mathbf{q}$  can establish an estimable function, we can test whether

$$\mathbf{q}^T \mathbf{H} = \mathbf{q}^T$$

with  $\mathbf{H} = \mathbf{G}\mathbf{X}^T \mathbf{X}$

Setting  $\mathbf{q}^T = \mathbf{t}^T \mathbf{X}$ , we get

$$\mathbf{q}^T \mathbf{H} = \mathbf{t}^T \mathbf{X} \mathbf{H} = \mathbf{t}^T \mathbf{X} = \mathbf{q}^T$$