Livestock Breeding and Genomics - Solution 9

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Problem 1 Multivariate BLUP Animal Model

The table below contains data for pre-weaning gain (WWG) and post-weaning gain (PWG) for 5 beef calves.

Animal	Sex	Sire	Dam	WWG	PWG
4	Male	1	NA	4.5	6.8
5	Female	3	2	2.9	5.0
6	Female	1	2	3.9	6.8
7	Male	4	5	3.5	6.0
8	Male	3	6	5.0	7.5

The genetic variance-covariance matrix G_0 between the traits is

$$G_0 = \begin{bmatrix} 20 & 18 \\ 18 & 40 \end{bmatrix}$$

The residual variance-covariance matrix R_0 between the traits is

$$R_0 = \begin{bmatrix} 40 & 11 \\ 11 & 30 \end{bmatrix}$$

Your Task

Set up the mixed model equations for a multivariate BLUP analysis and compute the estimates for the fixed effects and the predictions for the breeding values.

Solution

The matrices X_1 and X_2 relate records of PWG and WWG to sex effects. For both traits, we have an effect for the male and female sex. Hence the vector β of fixed effects corresponds to

$$\beta = \begin{bmatrix} \beta_{M,WWG} \\ \beta_{F,WWG} \\ \beta_{M,PWG} \\ \beta_{F,PWG} \end{bmatrix}$$

The matrices X_1 and X_2 are the same and correspond to

$$
X_1 = X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}
$$

Combining them to the multivariate version leads to

$$
X = \left[\begin{array}{cc} X_1 & 0 \\ 0 & X_2 \end{array} \right]
$$

$$
X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

Using the matrix X together with matrix $R = I_n \otimes R_0$ to get

$$
X^T R^{-1} X = \left[\begin{array}{cccc} 0.0834105653382762 & 0 & -0.0305838739573679 & 0 \\ 0 & 0.0556070435588508 & 0 & -0.020389249304912 \\ -0.0305838739573679 & 0 & 0.111214087117702 & 0 \\ 0 & -0.020389249304912 & 0 & 0.0741427247451344 \end{array} \right]
$$

Similarly to the fixed effects, we can put together the vector of breeding values a.

$$
u = \begin{bmatrix} u_{1,WWG} \\ u_{2,WWG} \\ u_{3,WWG} \\ u_{4,WWG} \\ u_{5,WWG} \\ u_{6,WWG} \\ u_{7,WWG} \\ u_{8,WWG} \\ u_{1,PWG} \\ u_{2,PWG} \\ u_{3,PWG} \\ u_{4,PWG} \\ u_{5,PWG} \\ u_{6,PWG} \\ u_{7,PWG} \\ u_{8,PWG} \end{bmatrix}
$$

The design matrices Z_1 and Z_2 are equal and they link observations to breeding values.

$$
Z_1 = Z_2 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
Z = \left[\begin{array}{cc} Z_1 & 0 \\ 0 & Z_2 \end{array}\right]
$$

$$
Z = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Together with the numerator relationship matrix A we can get $G = G_0 \otimes A$ and from this $G^{-1} = G_0^{-1} \otimes A^{-1}$

$$
A^{-1} = \begin{bmatrix}
1.8333 & 0.5 & 0 & -0.6667 & 0 & -1 & 0 & 0 \\
0.5 & 2 & 0.5 & 0 & -1 & -1 & 0 & 0 \\
0 & 0.5 & 2 & 0 & -1 & 0.5 & 0 & -1 \\
-0.6667 & 0 & 0 & 1.8333 & 0.5 & 0 & -1 & 0 \\
0 & -1 & -1 & 0.5 & 2.5 & 0 & -1 & 0 \\
-1 & -1 & 0.5 & 0 & 0 & 2.5 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 2
\end{bmatrix}
$$

$$
G^{-1} = \begin{bmatrix}
0.154 & 0.042 & 0 & -0.056 & 0 & -0.084 & 0 & 0 & -0.069 & -0.019 & 0 & 0.025 & 0 & 0.038 & 0 & 0 \\
0.042 & 0.168 & 0.042 & 0 & -0.084 & -0.084 & 0 & 0 & -0.019 & -0.076 & -0.019 & 0 & 0.038 & 0.038 & 0 & 0 \\
0 & 0.042 & 0.168 & 0 & -0.084 & 0.042 & 0 & -0.084 & 0 & -0.019 & -0.076 & 0 & 0.038 & -0.019 & 0 & 0.038 \\
-0.056 & 0 & 0 & 0.154 & 0.042 & 0 & -0.084 & 0 & 0.025 & 0 & 0 & -0.069 & -0.019 & 0 & 0.038 & 0 \\
0 & -0.084 & -0.084 & 0.042 & 0.21 & 0 & -0.084 & 0 & 0 & 0.038 & 0.038 & -0.019 & -0.095 & 0 & 0.038 & 0 \\
-0.084 & -0.084 & 0.042 & 0 & 0 & 0.21 & 0 & -0.084 & 0.038 & 0.038 & -0.019 & 0 & 0 & -0.095 & 0 & 0.038 \\
0 & 0 & 0 & -0.084 & -0.084 & 0 & 0.168 & 0 & 0 & 0 & 0 & 0.038 & 0.038 & 0 & -0.076 & 0 \\
0 & 0 & -0.084 & 0 & 0 & -0.084 & 0 & 0.168 & 0 & 0 & 0.038 & 0 & 0 & 0.038 & 0 & -0.076 \\
-0.069 & -0.019 & 0 & 0.025 & 0 & 0.038 & 0 & 0 & 0.077 & 0.021 & 0 & -0.028 & 0 & -0.042 & 0 & 0 \\
-0.019 & -0.076 & -0.019 & 0 & 0.038 & 0.038 & 0 & 0 & 0.021 & 0.084 & 0.021 & 0 & -0.042 & -0.042 & 0 & 0 \\
0 & -0.019 & -0.076 & 0 & 0.038 & -0.019 & 0 & 0.038 & 0 & 0.021 & 0.084 & 0 & -0.042 & 0.021 & 0 & -0.042 \\
0.025 & 0 & 0 & -0.069 & -0.019 & 0 & 0.038 & 0 & -0.028 & 0 & 0 & 0.077 & 0.021 & 0 & -0.042 & 0 \\
0 & 0.038 & 0.038 & -0.019 & -0.095 & 0 & 0.038 & 0 & 0 & -0.042 & -0.042 & 0.021 & 0.105 & 0 & -0.042 & 0 \\
0.038 & 0.038 & -0.019 & 0 & 0 & -0.095 & 0 & 0.038 & -0.042 & -0.042 & 0.021 & 0 & 0 & 0.105 & 0 & -0.042 \\
0 & 0 & 0 & 0.038 & 0.038 & 0 & -0.076 & 0 & 0 & 0 & 0 & -0.042 & -0.042 & 0 & 0.084 & 0 \\
0 & 0 & 0.038 & 0 & 0 & 0.038 & 0 & -0.076 & 0 & 0 & -0.042 & 0 & 0 & -0.042 & 0 & 0.084
\end{bmatrix}
$$

Using the matrics X, Z, R^{-1} and G^{-1}, we can compute $Z^T R^{-1} X$ and $Z^T R^{-1} Z + G^{-1}$. These matrices define the coefficient matrix of the mixed model equations. But they are too be to be shown here.

The vector y of observations contains all observations of both traits

$$
y = \begin{bmatrix} 4.5 \\ 2.9 \\ 3.9 \\ 3.5 \\ 5 \\ 6.8 \\ 5 \\ 6.8 \\ 6 \\ 7.5 \end{bmatrix}
$$

The right-hand side is computed as

$$
\left[\begin{array}{c} X^T R^{-1} y \\ Z^T R^{-1} y \end{array} \right]
$$

The solutions are

$$
\begin{bmatrix} \widehat{\beta_{M,WWG}} \\ \widehat{\beta_{F,WWG}} \\ \widehat{\beta_{M,PWG}} \\ \widehat{\beta_{F,PWG}} \\ \widehat{u_{1,WWG}} \\ \widehat{u_{2,WWG}} \\ \widehat{u_{3,WWG}} \\ \widehat{u_{4,WWG}} \\ \widehat{u_{5,WWG}} \\ \widehat{u_{6,WWG}} \\ \widehat{u_{7,WWG}} \\ \widehat{u_{8,WWG}} \\ \widehat{u_{1,PWG}} \\ \widehat{u_{2,PWG}} \\ \widehat{u_{3,PWG}} \\ \widehat{u_{4,PWG}} \\ \widehat{u_{5,PWG}} \\ \widehat{u_{6,PWG}} \\ \widehat{u_{7,PWG}} \\ \widehat{u_{8,PWG}} \end{bmatrix} = \left[\begin{array}{c} 4.3609 \\ 3.3973 \\ 6.7999 \\ 5.8803 \\ 0.1509 \\ -0.0154 \\ -0.0784 \\ -0.0102 \\ -0.2703 \\ 0.2758 \\ -0.3161 \\ 0.2438 \\ 0.2796 \\ -0.0076 \\ -0.1703 \\ -0.0127 \\ -0.4778 \\ 0.5172 \\ -0.479 \\ 0.392 \end{array} \right]
$$

Problem 2 Comparison of Reliabilites

Compare the predicted breeding values and the reliabilites obtained as results of Problem 1 with results from two univariate analyses for the same traits are used in Problem 1. All parameters can be taken from Problem 1.

Solution

For a predicted breeding value $\hat{u}_i$, the reliability B_i is computed as

$$
B_i = r^2_{u,\hat{u}} = 1 - \frac{PEV(\hat{u}_i)}{var(u_i)} = 1 - \frac{C^{22}_{ii}}{var(u_i)}
$$

where C_{ii}^{22} are obtained from the inverse coefficient matrix of the mixed model equations. Just as a reminder, we can write the mixed model equations (MME) as

$$M \cdot s = r$$

with the vectors r and s corresponding to the right-hand side and to the unknowns of the MME. Hence

$$r = \left[\begin{array}{c} X^TR^{-1}y \\ Z^TR^{-1}y \end{array}\right]$$

and

$$s = \left[\begin{array}{c} \hat{\beta} \\ \hat{u} \end{array}\right]$$

The matrix C^{22} is taken from the inverse coefficient matrix.

$$M^{-1} = \left[\begin{array}{cc} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & Z^TR^{-1}Z + G^{-1} \end{array}\right]^{-1} = \left[\begin{array}{cc} C^{11} & C^{12} \\ C^{21} & C^{22} \end{array}\right]$$

For the two univariate analyses, we get the solutions for the fixed effects and the breeding values and their reliabilities as follows

- WWG: estimates s_{WWG} and reliabilites B_{WWG}
- PWG

The reliabilities from the bivariate analysis are obtained as