Inverse Numerator Relationship Matrix

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Structure of A^{-1}

▶ Look at a simple example of A and A^{-1}

Table 1: Pedigree Used To Compute Inverse Numerator Relationship Matrix

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

Numerator Relationship Matrix A

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.5000 \\ 0.5000 & 0.5000 & 0.0000 & 1.0000 & 0.2500 \\ 0.0000 & 0.5000 & 0.5000 & 0.2500 & 1.0000 \end{bmatrix}$$

Inverse Numerator Relationship Matrix A^{-1}

$$A^{-1} = \begin{bmatrix} 1.5000 & 0.5000 & 0.0000 & -1.0000 & 0.0000 \\ 0.5000 & 2.0000 & 0.5000 & -1.0000 & -1.0000 \\ 0.0000 & 0.5000 & 1.5000 & 0.0000 & -1.0000 \\ -1.0000 & -1.0000 & 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & -1.0000 & -1.0000 & 0.0000 & 2.0000 \end{bmatrix}$$

Conclusions

- A^{-1} has simpler structure than A itself
- Non-zero elements only at positions of parent-progeny and parent-mate positions
- Parent-mate positions are positive, parent-progeny are negative

Henderson's Rules

- Based on LDL-decomposition of A

$$A = L * D * L^T$$

where L Lower triangular matrix
        D Diagonal matrix

- Why?
  - matrices L and D can be inverted directly, we'll see how ...
  - construct $A^{-1} = (L^T)^{-1} * D^{-1} * L^{-1}$

Example

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 1.0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$\rightarrow$ Verify that $A = L * D * L^T$

Decomposition of True Breeding Value

- True breeding value (u_i) of animal i

$$u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$$

- Do that for all animals in pedigree

Decomposition for Example

$$u_1 = m_1$$
$$u_2 = m_2$$
$$u_3 = m_3$$
$$u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4$$
$$u_5 = \frac{1}{2}u_3 + \frac{1}{2}u_2 + m_5$$

Matrix Vector Notation

- ▶ Define vectors u and m as
- ▶ Coefficients of u_s and u_d into matrix P

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, \ m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix}, \ P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{bmatrix}$$

- ▶ Result: Decomposition of true breeding values

$$u = P \cdot u + m$$

Decomposition of Variance

▶ Analogous decomposition of $var(u_i)$

$$var(u_i) = var(1/2u_s + 1/2u_d + m_i)$$
$$= var(1/2u_s) + var(1/2u_d) + \frac{1}{2} * cov(u_s, u_d) + var(m_i)$$
$$= 1/4var(u_s) + 1/4var(u_d) + \frac{1}{2} * cov(u_s, u_d) + var(m_i)$$

▶ From the definition of A

$$var(u_i) = (1 + F_i)\sigma_u^2$$
$$var(u_s) = (1 + F_s)\sigma_u^2$$
$$var(u_d) = (1 + F_d)\sigma_u^2$$
$$cov(u_s, u_d) = (A)_{sd}\sigma_u^2 = 2F_i\sigma_u^2$$

Variance of Mendelian Sampling Terms

- What is $var(m_i)$?
- Solve equation for $var(u_i)$ for $var(m_i)$

$$var(m_i) = var(u_i) - 1/4 var(u_s) - 1/4 var(u_d) - 2 * cov(u_s, u_d)$$

- Insert definitions from A

$$var(m_i) = (1 + F_i)\sigma_u^2 - 1/4(1 + F_s)\sigma_u^2 - 1/4(1 + F_d)\sigma_u^2 - \frac{1}{2} * 2 * F_i\sigma_u^2$$

$$= \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d) \right) \sigma_u^2$$

- True, for both parents s and d of animal i are known

Unknown Parents

- Only parent s of animal i is known

$$u_i = \frac{1}{2}u_s + m_i$$
$$var(m_i) = \left(1 - \frac{1}{4}(1 + F_s)\right)\sigma_u^2$$
$$= \left(\frac{3}{4} - \frac{1}{4}F_s\right)\sigma_u^2$$

- Both parents are unknown

$$u_i = m_i$$
$$var(m_i) = \sigma_u^2$$

Recursive Decomposition

▶ True breeding values of s and d can be decomposed into

$$u_s = \frac{1}{2}u_{ss} + \frac{1}{2}u_{ds} + m_s$$
$$u_d = \frac{1}{2}u_{sd} + \frac{1}{2}u_{dd} + m_d$$

where ss sire of s
        ds dam of s
        sd sire of d
        dd dam of d

Example

▶ Add animal 6 with parents 4 and 5 to our example pedigree

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2
6	4	5

First Step Of Decomposition

$$u_1 = m_1$$
$$u_2 = m_2$$
$$u_3 = m_3$$
$$u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4$$
$$u_5 = \frac{1}{2}u_3 + \frac{1}{2}u_2 + m_5$$
$$u_6 = \frac{1}{2}u_4 + \frac{1}{2}u_5 + m_6$$

Decompose Parents

$$u_1 = m_1$$
$$u_2 = m_2$$
$$u_3 = m_3$$
$$u_4 = \frac{1}{2}m_1 + \frac{1}{2}m_2 + m_4$$
$$u_5 = \frac{1}{2}m_3 + \frac{1}{2}m_2 + m_5$$
$$u_6 = \frac{1}{2}\left(\frac{1}{2}(u_1 + u_2) + m_4\right) + \frac{1}{2}\left(\frac{1}{2}(u_3 + u_2) + m_5\right) + m_6$$
$$= \frac{1}{4}(u_1 + u_2) + \frac{1}{2}m_4 + \frac{1}{4}(u_3 + u_2) + \frac{1}{2}m_5 + m_6$$

Decompose Grand Parents

- Only animal 6 has true breeding values for grand parents

$$u_6 = \frac{1}{4}(u_1 + u_2) + \frac{1}{2}m_4 + \frac{1}{4}(u_3 + u_2) + \frac{1}{2}m_5 + m_6$$

$$= \frac{1}{4}m_1 + \frac{1}{4}m_2 + \frac{1}{4}m_3 + \frac{1}{4}m_2 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6$$

$$= \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6$$

Summary

$$u_1 = m_1$$
$$u_2 = m_2$$
$$u_3 = m_3$$
$$u_4 = \frac{1}{2}m_1 + \frac{1}{2}m_2 + m_4$$
$$u_5 = \frac{1}{2}m_3 + \frac{1}{2}m_2 + m_5$$
$$u_6 = \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6$$

Matrix-Vector Notation

- Use vectors u and m again

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \; m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}, \; L = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.50 & 0.50 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.00 & 1.00 & 0.00 \\ 0.25 & 0.50 & 0.25 & 0.50 & 0.50 & 1.00 \end{bmatrix}$$

- Result of recursive decomposition of u_i

$$u = L \cdot m$$

Variance From Recursive Decomposition

$$var(u) = var(L \cdot m)$$
$$= L \cdot var(m) \cdot L^T$$

where $var(m)$ is the variance-covariance matrix of all components in vector m.

- covariances of components m_i, $cov(m_i, m_j) = 0$ for $i \neq j$
- $var(m_i)$ computed as shown before

Result

- variance-covariance matrix $var(m)$ can be written as $D * \sigma_u^2$ where D is diagnoal

$$
\begin{aligned}
\rightarrow var(u) &= L \cdot var(m) \cdot L^T \\
&= L \cdot D * \sigma_u^2 \cdot L^T \\
&= L \cdot D \cdot L^T * \sigma_u^2 \\
&= A\sigma_u^2
\end{aligned}
$$

$$
\rightarrow A = L \cdot D \cdot L^T
$$

Inverse of A Based on L and D

- Matrix A was decomposed into $A = L \cdot D \cdot L^T$
- Get A^{-1} as $A^{-1} = (L^T)^{-1} D^{-1} L^{-1}$
- D^{-1} is diagonal again with elements

$$(D^{-1})_{ii} = 1/(D)_{ii}$$

Inverse of L

- Compute m based on the two decompositions of u

$$u = P \cdot u + m \quad \text{and} \quad u = L \cdot m$$

- Solve both for m and set them equal

$$m = u - P \cdot u = (I - P) \cdot u \quad \text{and} \quad m = L^{-1} \cdot u$$

$$(I - P) \cdot u = L^{-1} \cdot u$$

and

$$L^{-1} = I - P$$

Example

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

Matrix D^{-1}

- Because D is diagonal

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- We get D^{-1} as

$$D^{-1} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

Matrix L^{-1}

- Use $L^{-1} = I - P$
- Matrix P from simple decomposition

$$
P = \begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.5 & 0.5 & 0.0 & 0.0
\end{bmatrix}
$$

- Therefore

$$
L^{-1} = I - P = \begin{bmatrix}
1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
-0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\
0.0 & -0.5 & -0.5 & 0.0 & 1.0
\end{bmatrix}
$$

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$(L^{-1})^T \cdot D^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 & -0.5 & -0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \cdot \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix} \cdot \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 0.5 & 0.0 & -1.0 & 0.0 \\ 0.5 & 2.0 & 0.5 & -1.0 & -1.0 \\ 0.0 & 0.5 & 1.5 & 0.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & -1.0 & 0.0 & 2.0 \end{bmatrix}$$

Henderson's Rules

- ▶ Both Parents Known
  - ▶ add 2 to the diagonal-element (i, i)
  - ▶ add -1 to off-diagonal elements (s, i), (i, s), (d, i) and (i, d)
  - ▶ add $\frac{1}{2}$ to elements (s, s), (d, d), (s, d), (d, s)
- ▶ Only One Parent Known
  - ▶ add $\frac{4}{3}$ to diagonal-element (i, i)
  - ▶ add $-\frac{2}{3}$ to off-diagonal elements (s, i), (i, s)
  - ▶ add $\frac{1}{3}$ to element (s, s)
- ▶ Both Parents Unknown
  - ▶ add 1 to diagonal-element (i, i)
- ▶ Valid without inbreeding